EXPLICIT CONSTRUCTION OF THE MAXIMAL SUBGROUPS OF THE MONSTER

HEIKO DIETRICH, MELISSA LEE, ANTHONY PISANI, AND TOMASZ POPIEL

Abstract. Seysen's Python package `mmgroup` provides functionality for fast computations within the sporadic simple group $\mathbb{M}$, the Monster. The aim of this work is to present an `mmgroup` database of maximal subgroups of $\mathbb{M}$: for each conjugacy class $\mathcal{C}$ of maximal subgroups in $\mathbb{M}$, we construct explicit group elements in `mmgroup` and prove that these elements generate a group in $\mathcal{C}$. Our generators and the computations verifying correctness are available in accompanying code. The maximal subgroups of $\mathbb{M}$ have been classified in a number of papers spanning several decades; our work constitutes an independent verification of these constructions. We also correct the claim that $\mathbb{M}$ has a maximal subgroup $\mathrm{PSL}_2(59)$, and hence identify a new maximal subgroup 59:29.

1. Introduction

The classification of the maximal subgroups of the Monster group $\mathbb{M}$, the largest sporadic simple group, is a result of decades of effort led primarily by Holmes, Norton, and Wilson. In 2017, Wilson [29] noted that over 15 papers had been dedicated to this classification, yet a few open cases cases seemed largely resistant to theoretical arguments and posed significant computational difficulties. These cases were recently settled by Dietrich, Lee, and Popiel [5] with the help of the newly available Python software package `mmgroup` developed by Seysen [21–23]. For more background information and a comprehensive discussion, we refer to [5] and the references therein.

While much of the early classification work was theoretical, extensive computations eventually became necessary to construct or rule out the existence of various almost simple maximal subgroups. Several constructions, in fact, rely on calculations using a non-standard computational model developed by Holmes and Wilson [9]. Although groundbreaking at the time, this model is not publicly available, making it difficult to verify or reproduce the results. Seysen's `mmgroup` package is a game changer in this regard, providing, for the first time, a publicly available framework for *fast* computations in $\mathbb{M}$. In the course of completing the maximal subgroup classification, the work in [5] yielded explicit generators in $\mathbb{M}$ (in `mmgroup` format) for the maximal subgroups $\mathrm{PGL}_2(13)$ and $\mathrm{PSU}_3(4)$, along with several auxiliary subgroups of $\mathbb{M}$. The aim of this paper is to consider the remaining maximal subgroups of $\mathbb{M}$, as described in the existing literature, and to provide explicit generators for these groups in `mmgroup` format with computationally reproducible proofs of correctness. Our generators (and computations verifying their correctness) are available in accompanying Python code [18]. Our explicit calculations in `mmgroup` confirm the maximal subgroup constructions available in the literature, with the exception of the construction of $\mathrm{PSL}_2(59)$ in [10]; this leads to our first result.

Theorem 1.1. *The Monster has no subgroup* $\mathrm{PSL}_2(59)$*, but a unique class of subgroups* $59{:}29$*, which are maximal subgroups.*

Theorem 1.1 is in conflict with the main result of [10] which claims that $\mathbb{M}$ *has* a maximal subgroup $\mathrm{PSL}_2(59)$. We prove Theorem 1.1 in Section 5.6 and illustrate in detail how the final conclusion of [10] is

(Dietrich, Lee, Pisani, Popiel) School of Mathematics, Monash University, Clayton VIC 3800, Australia
E-mail addresses: heiko.dietrich@monash.edu, melissa.lee@monash.edu, apis0001@student.monash.edu, , tomasz.popiel@monash.edu.

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$2{\cdot}\mathbb{B}$	$(\mathrm{D}_{10} \times \mathrm{HN})\,{\cdot}2$	$(\mathrm{A}_5 \times \mathrm{A}_{12})\,{:}2$
$2^{1+24}{\cdot}\mathrm{Co}_1$	$5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$	$(\mathrm{A}_6 \times \mathrm{A}_6 \times \mathrm{A}_6)\,.\,(2 \times \mathrm{S}_4)$
$2^{2\,.\,2}\mathrm{E}_6(2){:}\mathrm{S}_3$	$\left(5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)\right){:}\mathrm{S}_3$	$(\mathrm{A}_5 \times \mathrm{U}_3(8){:}3)\,{:}2$
$2^{2+11+22}{\cdot}\,(\mathrm{M}_{24} \times \mathrm{S}_3)$	$5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$	$(\mathrm{PSL}_3(2) \times \mathrm{S}_4(4){:}2)\,{\cdot}2$
$2^{3+6+12+18}{\cdot}\,(\mathrm{PSL}_3(2) \times 3.\mathrm{S}_6)$	$5^{3+3}{\cdot}\,(2 \times \mathrm{PSL}_3(5))$	$(\mathrm{PSL}_2(11) \times \mathrm{M}_{12})\,{:}2$
$2^{5+10+20}{\cdot}\,(\mathrm{S}_3 \times \mathrm{PSL}_5(2))$	$5^4{:}\,(3 \times 2{\cdot}\mathrm{PSL}_2(25))\,{:}2$	$\left(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5)\,{:}2^2\right){:}2$
$2^{10+16}{\cdot}\mathrm{O}_{10}^+(2)$	$(7{:}3 \times \mathrm{He})\,{:}2$	$\mathrm{M}_{11} \times \mathrm{A}_6{\cdot}2^2$
$3{\cdot}\mathrm{Fi}_{24}$	$7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)$	$(\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5)\,{:}\mathrm{S}_3$
$3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$	$\left(7^2{:}\,(3 \times 2.\mathrm{A}_4) \times \mathrm{PSL}_2(7)\right){:}2$	$(\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11))\,{:}4$
$\mathrm{S}_3 \times \mathrm{Th}$	$7^{2+1+2}{:}\mathrm{GL}_2(7)$	$\mathrm{U}_3(4){:}4$
$\left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$	$7^2{:}\mathrm{SL}_2(7)$	$\mathrm{PSL}_2(71)$
$3^{2+5+10}{:}\,(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$	$11^2{:}\,(5{\times}2.\mathrm{A}_5)$	$\cancel{\mathrm{PSL}_2(59)}$
$3^{3+2+6+6}{:}\,(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})$	$(13{:}6 \times \mathrm{PSL}_3(3))\,{\cdot}2$	$\mathrm{PSL}_2(41)$
$3^8{\cdot}\mathrm{O}_8^-(3).2$	$13^{1+2}{:}\,(3 \times 4.\mathrm{S}_4)$	$\mathrm{PGL}_2(29)$
	$13^2{:}\mathrm{SL}_2(13){:}4$	$\mathrm{PGL}_2(19)$
$59{:}29$	$41{:}40$	$\mathrm{PGL}_2(13)$

TABLE 1. The maximal subgroups of $\mathbb{M}$. For each group, we provide generators in `mmgroup` format; see Section 5.6 for comments on $\mathrm{PGL}_2(59)$ and $59{:}29$.

in contradiction with our calculations. Exhibiting elements in $\mathbb{M}$ generating $59{:}29$ presents significant computational challenges and is the subject of ongoing work, see Remark 5.7.

The main result of this paper can be summarised as follows.

Theorem 1.2. *For each maximal subgroup N of the Monster $\mathbb{M}$ listed in Table 1, with the exception of* $59{:}29$*, the accompanying code [18] provides explicit elements of $\mathbb{M}$ in* `mmgroup` *format that generate a maximal subgroup of $\mathbb{M}$ isomorphic to N.*

With the exception of the groups $\mathrm{PSL}_2(59)$ and $59{:}29$, which will be discussed in Section 5.6, the correctness of this table follows from [29] and [5], see in particular [5, Theorem 1.7]. Generators for the maximal subgroups $2{\cdot}\mathbb{B}$ and $2^{1+24}{\cdot}\mathrm{Co}_1$ are already provided in [5] and in `mmgroup`. For each remaining maximal subgroup $N \neq 59{:}29$, we prove that the associated elements provided in the accompanying code [18] generate a maximal subgroup of $\mathbb{M}$ that is isomorphic to N. Specifically, p-local maximal subgroups with p odd are discussed in Section 2. The 2-local maximal subgroups are discussed in Section 3. Section 4 provides some necessary details on alternating subgroups A_5 in $\mathbb{M}$. Non-local maximal subgroups are considered in Sections 5 and 6.

1.1. **How to read this paper.** This is a computational paper and the main result of our work is a database of maximal subgroups of $\mathbb{M}$ in `mmgroup` format, made available in a well-documented Jupyter Notebook [18]. This paper is concerned with the proof that the database is correct. Our proof relies on theoretical results from the classification of the maximal subgroups, and on explicit computations in `mmgroup`. We also often refer to data concerning the maximal subgroups of various simple groups listed in the Atlas [2, 3]. (For example, we establish that a group G is generated by given elements of orders n and m because, according to the Atlas, there is no maximal subgroup of G that has elements of these orders.) We recommend that our proofs are read alongside the information provided in the Notebook, because the latter often contains the calculations proving the claims made in the text.

For each maximal subgroup $N \neq 59{:}29$ of $\mathbb{M}$ listed in Table 1, there is a corresponding subsection in this paper and in the Jupyter Notebook. We define certain elements of $\mathbb{M}$ in `mmgroup` format and prove that the group generated by these elements is maximal in $\mathbb{M}$ and isomorphic to N. Depending on whether N is p-local or not, we employ different strategies that are outlined below.

Notation. Most of our notation is standard and typically aligns with the conventions established in the Atlas [3]. One significant exception is that we use $\mathrm{PSL}_d(q)$ in place of $\mathrm{L}_d(q)$, and apply similar conventions for other simple classical groups. Unless specified otherwise, we denote by 2A, 2B, 3A, etc. the conjugacy classes of $\mathbb{M}$, and for a conjugacy class pX we write $p\mathrm{X}^k$ for an elementary abelian group of size p^k whose non-identity elements all belong to pX; such a group is called pX-*pure*. We denote the dihedral group of order n by D_n, the alternating group of degree n by A_n, and the symmetric group of degree n by S_n. The symbol n is also used to denote a cyclic group of order n. An extension of a group B by a group A is denoted by $A.B$ (or occasionally AB), with A being the normal subgroup. To emphasise that an extension is split, we may use the notation $A{:}B$, whereas $A{}^{\cdot}B$ indicates a non-split extension. We call this description of a group the *shape* of the group, and stress that the shape does in general not define the group up to isomorphism. (For example, there are in general many non-isomorphic groups that are extensions of B by A, that is, of shape $A.B$.) We follow the convention that $A.B.C = (A.B).C$. An elementary abelian group of order p^k is denoted by p^k, where p is a prime number and k is a positive integer, while $p^{k+\ell}$ denotes an extension $p^k.p^\ell$. We often use a subscript to indicate the order of a group element; for instance, g_5 may refer to an element of order 5. Finally, we denote by $\mathbf{G}$ the centraliser $C_{\mathbb{M}}(z)$ of a distinguished 2B involution z. Extended functionality and faster computation are available in `mmgroup` for elements of $\mathbf{G}$; see [22] for details.

2. The Odd-Local Maximal Subgroups of $\mathbb{M}$

A subgroup U of $\mathbb{M}$ is p-*local* if $U = N_{\mathbb{M}}(E)$ for some p-subgroup E; it is *maximal* p-*local* if U is maximal among all p-local subgroups with respect to inclusion. We say that U is p-*local maximal* if U is p-local and also a maximal subgroup of $\mathbb{M}$. Incorporating earlier work of Norton, Wilson [27] classified the maximal p-local subgroups of $\mathbb{M}$ for odd p. However, questions about maximality of these groups were only resolved in all cases 20 years later through computational constructions of containing subgroups of the Monster, see for example [11]. The updated results from [27] pertaining to maximal subgroups of $\mathbb{M}$ are summarised in Table 2 and Proposition 2.3.

2.1. Preliminary lemmas.

We start with two preliminary results that are frequently used; the first is an easy observation.

Lemma 2.1. *Let G be a finite group with $H \leq G$ and $g \in G$. If $g \notin H$, then $|\langle H, g\rangle| \geq 2|H|$. Moreover, if the order of g is a prime power p^n and $g^{(p^{n-1})} \notin H$, then $|\langle H, g\rangle| \geq p^n|H|$.*

Proof. If $g \notin H$, then H and gH are distinct cosets in $\langle g, H\rangle$, so $|\langle g, H\rangle| \geqslant 2|H|$. If g has prime-power order p^n, then $\langle g\rangle \cap H = \langle g^{(p^i)}\rangle$ for some i. By assumption, $g^{(p^{n-1})} \notin H$, so $\langle g\rangle \cap H = 1$. Thus, the cosets $H, gH, g^2H, \ldots, g^{p^n-1}H$ are all distinct in $\langle g, H\rangle$, and therefore $|\langle g, H\rangle| \geqslant p^n|H|$. □

The next technical lemma plays an important role in our proofs to justify that we have constructed the full p-cores (i.e. maximal normal p-subgroups) $O_p(N)$ of the various p-local subgroups N of $\mathbb{M}$.

Lemma 2.2. *Let G be a finite group, $x \in G$ with prime order p, and $N \trianglelefteq N_G(\langle x\rangle)$ with $|N| = p^{2k+3}$ for some $k \geqslant 0$. Suppose $x \in N$ and there exist $y, \ell, g_1, \ldots, g_k, h_1, \ldots, h_k \in N \cap C_G(x)$ of order p and $\sigma \in N_G(\langle x, y\rangle)$ such that $y \notin \langle x\rangle$; all g_i, h_i commute with y, whereas ℓ does not; all g_j commute modulo $\langle x\rangle$ with all g_i, g_i^σ, h_i; all g_j commute with h_i^σ modulo $\langle x\rangle$ when $i < j$, but not when $i = j$. Then the following hold.*

a) *Every $S \subseteq \{x, y, g_1, \ldots, g_k, h_1, \ldots, h_k, g_1^\sigma, \ldots, g_k^\sigma\}$ generates a p-group of order at least $p^{|S|}$.*
b) *The group N is generated by $\{x, y, g_1, \ldots, g_k, h_1, \ldots, h_k, \ell\}$.*
c) *The group $\langle x, y, g_1, \ldots, g_k, h_1, \ldots, h_k, g_1^\sigma, \ldots g_k^\sigma, h_1^\sigma, \ldots, h_k^\sigma\rangle$ is a normal p-subgroup of $C_G(\langle x, y\rangle)$.*

Proof. a) Let $\phi\colon C_G(x) \to C_G(x)/\langle x\rangle$ be the natural homomorphism. We first show that S generates a group (not necessarily a p-group) of order at least $p^{|S|}$. Suppose, for a contradiction, that S is a counterexample of minimal size, that is, $|\langle S \setminus \{u\}\rangle| \geqslant p^{|S|-1}$ for all $u \in S$; note that $|S| \geqslant 1$ since $|\langle\emptyset\rangle| = p^0$. Let $u \in S$ be the element that occurs latest in the list $x, y, g_k, \ldots, g_1, h_1, \ldots, h_k, h_1^\sigma, \ldots, h_k^\sigma$; note the reversed labels for the g_i. The following case distinction shows that $u \notin \langle S \setminus \{u\}\rangle$.

- If $u = x$, then $S \setminus \{u\} = \emptyset$, and so $x \notin \langle S \setminus \{u\}\rangle$.

- If $u = y$, then $S \setminus \{u\} \subseteq \{x\}$, and $y \notin \langle x \rangle$ by assumption.
- If $u = g_i$ for some $1 \le i \le k$, then $S \setminus \{u\}$ consists of at most x, y, and g_j for various $j > i$. Note that g_m^σ, h_m^σ lie in $(C_G(x) \cap C_G(y))^\sigma = C_G(\langle x, y\rangle^\sigma) = C_G(\langle x, y\rangle)$, as do g_m, h_m for all $1 \le m \le k$. Thus $\phi(h_i^\sigma)$ commutes with $\phi(x)$, $\phi(y)$, and $\phi(g_j)$ for $j > i$, so $\phi(\langle S \setminus \{u\}\rangle)$ centralises $\phi(h_i^\sigma)$. Since $u = g_i$ does not commute with h_i^σ modulo $\langle x \rangle$ by assumption, we deduce $u \notin \langle S \setminus \{u\}\rangle$.
- If $u = h_i$ for some $1 \le i \le k$, then $S \setminus \{u\}$ consists of at most x, y, the g_m, and various h_j for $j < i$. Consider $\phi((S \setminus \{u\})^\sigma)$. Each $\phi(g_m^\sigma)$ and $\phi(h_j^\sigma)$ with $i > j$ commutes with $\phi(g_i)$ by assumption, as do $\phi(x^\sigma), \phi(y^\sigma) \in \phi(\langle x, y\rangle)$. It follows that $\phi(\langle (S \setminus \{u\})^\sigma \rangle)$ centralises $\phi(g_i)$. Since $\phi(u^\sigma) = \phi(h_i^\sigma)$ does not commute with $\phi(g_i)$ by assumption, we deduce $u \notin \langle S \setminus \{u\}\rangle$.
- If $u = h_i^\sigma$ for some $1 \le i \le k$, then $S \setminus \{u\}$ consists of at most x, y, the g_m, the h_m, and h_j^σ for $j < i$. All of $\phi(x)$, $\phi(y)$, $\phi(g_m)$, $\phi(h_m)$, and $\phi(h_j^\sigma)$ commute with $\phi(g_i)$ by assumption and, as before, $\phi(\langle S \setminus \{u\}\rangle)$ centralises $\phi(g_i)$. As before, $u = h_i^\sigma$ is not contained in $\langle S \setminus \{u\}\rangle$.

Thus, $u \notin \langle S \setminus \{u\}\rangle$, and Lemma 2.1 yields $|\langle S\rangle| \ge p|\langle S \setminus \{u\}\rangle| \ge p^{1+|S|-1} = p^{|S|}$, which contradicts that S is counterexample. Thus, $|\langle S\rangle| \geqslant p^{|S|}$ for all S. It will follow from c) that $\langle S\rangle$ is a p-group.

b) The group $A = \langle x, y, g_1, \dots g_k, h_1, \dots h_k\rangle$ lies in $N \cap C_G(y)$ and has size $|A| \geqslant p^{2k+2}$ by a). The element $\ell \in N \setminus C_G(y)$ has order p, so $\langle N \cap C_G(y), \ell\rangle \leqslant N$ has size at least $p|N \cap C_G(y)| \ge p^{1+2k+2} = |N|$ by Lemma 2.1. This implies that $A = N \cap C_G(y)$ has order p^{2k+2} and $N = \langle A, \ell\rangle$, as claimed.

c) We continue with the previous notation. All generators of $A = N \cap C_G(y)$ lie in $C_G(x)$, so $A = N \cap C_G(\langle x, y\rangle)$. Since $N \trianglelefteq N_G(\langle x\rangle)$, we have $A \trianglelefteq C_G(\langle x, y\rangle)$. But then $A^\sigma \trianglelefteq C_G(\langle x, y\rangle)^\sigma = C_G(\langle x, y\rangle^\sigma) = C_G(\langle x, y\rangle)$, which proves the first claim of c), that is,

$$\langle x, y, g_1, \dots, g_k, h_1, \dots, h_k, g_1^\sigma, \dots g_k^\sigma, h_1^\sigma, \dots, h_k^\sigma\rangle = \langle A, A^\sigma\rangle \trianglelefteq C_G(\langle x, y\rangle);$$

note that the generators $x^\sigma, y^\sigma \in \langle x, y\rangle$ are redundant. Moreover, $A, A^\sigma \trianglelefteq \langle A, A^\sigma\rangle$, and so $|\langle A, A^\sigma\rangle| = |AA^\sigma| = |A||A^\sigma|/|A \cap A^\sigma|$ divides $|A||A^\sigma| = |A|^2 = p^{4k+4}$. Thus, $\langle A, A^\sigma\rangle$ is a p-group. □

2.2. **The p-local subgroups.** For completeness, we recall the classification of the odd-local maximal subgroups of $\mathbb{M}$, with the correction discussed in Section 5.6 (that is, with the addition of 59:29).

Proposition 2.3 ([27]; [29]; [11]; §5.6). *The odd-local maximal subgroups of $\mathbb{M}$ are, up to conjugacy, the normalisers $N_{\mathbb{M}}(E)$ listed in Table 2.*

The following subsections consider each of the odd-local maximal subgroups separately and prove that the generators in the accompanying code do indeed generate a maximal subgroup of the correct shape. For a more efficient exposition, we first outline the common steps of these proofs.

Procedure 2.4. Let $N = N_{\mathbb{M}}(E)$ be one of the p-local maximal subgroups in Table 2. The accompanying code provides two lists, L_E and L_N, of elements in $\mathbb{M}$ in `mmgroup` format. Let $\tilde{E} = \langle L_E\rangle$ and $\tilde{N} = N_{\mathbb{M}}(\tilde{E})$. We aim to prove that $E \cong \tilde{E}$ and $N \cong \tilde{N} = \langle L_N\rangle$. We usually proceed as follows, and explain alternative approaches in the relevant proofs.

(1) We first confirm that $\tilde{E}$ is an elementary abelian group of the correct type, see column “E” in Table 2. For this we confirm computationally that $\tilde{E}$ is elementary abelian of the correct size, and, if required, that all its non-identity elements lie in the correct conjugacy class of $\mathbb{M}$. We note that `mmgroup` provides the functionality to compute the values of the unique irreducible 198663-dimensional complex character $\chi_{\mathbb{M}}$ of $\mathbb{M}$ for elements that lie in the maximal subgroup $\mathbf{G} = N_{\mathbb{M}}(2\mathrm{B})$. Once the type of $\tilde{E}$ is established, the structure and maximality of $N_{\mathbb{M}}(\tilde{E}) \cong N$ is usually determined by Proposition 2.3.

(2) A direct calculation in `mmgroup` shows that L_N normalises $\tilde{E}$, and the next step is to show that $\langle L_N\rangle$ contains $C_{\mathbb{M}}(\tilde{E})$. This centraliser is usually an extension of the form $A.B$, where the structures of A and B are known. We exhibit words in the elements of L_N whose cosets with respect to A generate a group isomorphic to $C_{\mathbb{M}}(\tilde{E})/A \cong B$; if $C_{\mathbb{M}}(\tilde{E}) = A{:}B$ is known to be a split extension, then we often exhibit words in the elements of L_N that generate B. The last step is to construct, as words in elements of L_N, generators of A. Here it is useful that $\gcd(|A|, |B|)$ is usually small and that A contains low-index p-subgroups. In conclusion, at the end of this step we have shown that $C_{\mathbb{M}}(\tilde{E}) \leqslant \langle L_n\rangle \leqslant \tilde{N} = N_{\mathbb{M}}(\tilde{E})$.

$N_{\mathbb{M}}(E)$	E	Citations and Comments
$3{\cdot}\mathrm{Fi}_{24}$	3A	see [27, Theorem 3]; for $3\mathrm{A}^2$, see also the table above [27, Proposition 2.2]
$3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$	3B	
$\mathrm{S}_3 \times \mathrm{Th}$	3C	
$\left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$	$3\mathrm{A}^2$	
$3^{2+5+10}{:}\,(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$	$3\mathrm{B}^2$	see [27, Theorems 3, 6.5]; the structure of E is insufficient for the maximality and structure of $N_{\mathbb{M}}(E)$.
$3^{3+2+6+6}{:}\,(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})$	$3\mathrm{B}^3$	
$3^8{\cdot}\mathrm{O}_8^-(3).2$	3^8	see [27, Theorems 3, 7.1]; E extends $3^7 < 3^{1+12} \le N_{\mathbb{M}}(3\mathrm{B})$
$(\mathrm{D}_{10} \times \mathrm{HN}){\cdot}2$	5A	see [27, Theorem 5]; for $5\mathrm{A}^2$, $5\mathrm{B}^2$ see also the first two tables in [27, §9]
$5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$	5B	
$\left(5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)\right){:}\mathrm{S}_3$	$5\mathrm{A}^2$	
$5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$	$5\mathrm{B}^2$	
$5^{3+3}{\cdot}\,(2 \times \mathrm{PSL}_3(5))$	$5\mathrm{B}^3$	
$5^4{:}\,(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$	$5\mathrm{B}^4$	see [27, Theorem 5]; E extends $5^2 < 5^{1+6} \le N_{\mathbb{M}}(5\mathrm{B})$
$(7{:}3 \times \mathrm{He}){:}2$	7A	see [27, Theorem 7]; for $7\mathrm{A}^2$ see also the table at the start of [27, §10]
$7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)$	7B	
$\left(7^2{:}\,(3 \times 2.\mathrm{A}_4) \times \mathrm{PSL}_2(7)\right){:}2$	$7\mathrm{A}^2$	
$7^{2+1+2}{:}\mathrm{GL}_2(7)$	$7\mathrm{B}^2$	see [27, Theorem 7]; $E < 7^{1+4} \lhd N_{\mathbb{M}}(7\mathrm{B})$
$7^2{:}\mathrm{SL}_2(7)$	$7\mathrm{B}^2$	see [27, Theorem 7]; $E \not< 7^{1+4} \lhd N_{\mathbb{M}}(7\mathrm{B})$
$11^2{:}\,(5{\times}2.\mathrm{A}_5)$	11^2	see [27, §11]; note that $\mathbb{M}$ has unique conjugacy classes of elements of orders 11, 41, and that the two classes of elements of order 59 are inverses
$(13{:}6 \times \mathrm{PSL}_3(3)){\cdot}2$	13A	
$13^{1+2}{:}\,(3 \times 4.\mathrm{S}_4)$	13B	
$13^2{:}\mathrm{SL}_2(13){:}4$	$13\mathrm{B}^2$	
41:40	41	
59:29	59	

TABLE 2. The odd-local maximal subgroups $N_{\mathbb{M}}(E)$ of $\mathbb{M}$.

(3) Finally, we consider the homomorphism $\phi\colon \langle L_N\rangle \to \mathrm{Aut}(\tilde{E})$ induced by conjugation. Step (2) establishes that $C_{\mathbb{M}}(\tilde{E}) = \ker\phi$, so the image of ϕ is isomorphic to a subgroup of $N_{\mathbb{M}}(\tilde{E})/C_{\mathbb{M}}(\tilde{E})$. A direct calculation allows us to determine the size of the image of ϕ, and if this equals $|N_{\mathbb{M}}(\tilde{E})/C_{\mathbb{M}}(\tilde{E})|$, then $\langle L_N\rangle = \tilde{N}$ is established.

We use the notation and approach of Procedure 2.4 in each of the following proofs, sometimes with minor modifications. We always denote by L_E and L_N the lists of elements provided in the accompanying code, and always write

$$\tilde{E} = \langle L_E\rangle \quad \text{and} \quad \tilde{N} = N_{\mathbb{M}}(\tilde{E}).$$

We then prove that $\tilde{N} = \langle L_N\rangle$ is the required maximal subgroup. We usually do not comment on obvious verifications, such as checking that the elements in L_N normalise $\tilde{E}$, etc, but these verifications are done in the corresponding section of the accompanying Jupyter Notebook.

2.3. **The group $3{\cdot}\mathbf{Fi}_{24}$.** The next theorem is adapted from the accompanying code of [5].

Theorem 2.5. *The accompanying code defines $L_E = \{x_3\}$ and $L_N = \{g_3, h_3\}$. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $3{\cdot}\mathrm{Fi}_{24}$ of $\mathbb{M}$.*

Proof. The element $x_3 \in \mathbf{G}$ has order 3 and $\chi_{\mathbb{M}}(x_3) = 782$, so $x_3 \in 3\mathrm{A}$. Therefore, $N_{\mathbb{M}}(\tilde{E}) \cong 3{\cdot}\mathrm{Fi}_{24}$ and $C_{\mathbb{M}}(\tilde{E}) \cong \tilde{E}$. We confirm $x_3 = (g_3h_3g_3h_3^3g_3h_3^5)^{28} \in \langle L_N \rangle$. To ensure that all cosets of $N_{\mathbb{M}}(\tilde{E})/\tilde{E}$ have representatives in $\langle L_N \rangle$, we consider the elements $a = g_3h_3$ and $b = (g_3h_3)^5h_3$ of order 29 and 70, respectively. It follows that also $aC_{\mathbb{M}}(\tilde{E})$ and $bC_{\mathbb{M}}(\tilde{E})$ have order 29 and 70, respectively, and they are elements in $\mathrm{Fi}_{24} \cong N_{\mathbb{M}}(\tilde{E})/\tilde{E}$. Atlas information shows that Fi_{24} has a unique class of maximal subgroup whose order is divisible by 29 and 70, but this subgroup has no elements of order 70. We deduce that $aC_{\mathbb{M}}(\tilde{E})$ and $bC_{\mathbb{M}}(\tilde{E})$ generate a group isomorphic to Fi_{24}, and so $N_{\mathbb{M}}(\tilde{E}) = \langle L_N \rangle$. □

2.4. The group $3^{1+12}{\cdot}2{\cdot}\mathbf{Suz}{:}2$.

Theorem 2.6. *The accompanying code defines $L_E = \{x_{3b}\}$ and $L_N = \{g_{3b}, h_{3b}\}$. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ of $\mathbb{M}$.*

Proof. The element $x_{3b} \in \mathbf{G}$ has order 3 and $\chi_{\mathbb{M}}(x_{3b}) = 53$, so $x_{3b} \in 3\mathrm{B}$. Thus, $\tilde{E}$ has the correct type and $N_{\mathbb{M}}(\tilde{E}) \cong 3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ is a maximal subgroup. The elements g_{3b} and h_{3b} normalise $\tilde{E}$ and have orders 56 and 66, respectively, and therefore some of their powers project to elements of orders 28 and 11 in the factor group Suz.2 under the natural projection map φ. Atlas information shows that Suz.2 has a unique class of maximal subgroups of order divisible by both 28 and 11, namely Suz, but this subgroup has no element of order 28. This implies that $\langle L_N \rangle$ contains a complete set of coset representatives of $U = \ker\varphi \cong 3^{1+12}.2$ in $\tilde{N} \cong 3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. It remains to show that U lies in $\langle L_N \rangle$. Since Suz.2 contains no elements of order 56 and $\gcd(56, 3^{1+12} \cdot 2) = 2$, we have $g_{3b}^{28} \in U$. Since g_{3b}^{28} has order 2, it lies in $U \setminus 3^{1+12}$. Thus, it remains to show that $\langle L_N \rangle$ contains $V = 3^{1+12}$. The element $r = g_{3b}^{28}(g_{3b}^{28})^{h_{3b}}$ lies in V, and in the accompanying code we express x_{3b} as a word in conjugates of r. We also define elements y_{3b}, ℓ, $g_1, \dots, g_5$, $h_1, \dots, h_5$, σ as words in elements of L_N and apply Lemma 2.2 to show that these generate V. This completes the proof. □

2.5. The group $\mathrm{S}_3 \times \mathbf{Th}$.

The next theorem is proved in the with elements adapted from [5].

Theorem 2.7. *The accompanying code defines $L_E = \{x_{3c}\}$ and $L_N = \{g_{3c}, h_{3c}\}$. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $\mathrm{S}_3 \times \mathrm{Th}$ of $\mathbb{M}$.*

Proof. We exhibit an element $c \in \mathbb{M}$ such that $x_{3c}^c \in \mathbf{G}$ has order 3 and $\chi_{\mathbb{M}}(x_{3c}^c) = -1$, proving that $\tilde{E}$ has the correct form and $N_{\mathbb{M}}(\tilde{E}) \cong \mathrm{S}_3 \times \mathrm{Th}$. Let $a = g_{3c}^3$ and $b = h_{3c}^4$. The elements ab and $abab^2ab^2abab^2abababab^2ab^2abab$ have orders 19 and 31, respectively, and their cosets with respect to S_3 generate a subgroup of Th containing elements of these orders. Atlas information implies that this subgroup is the whole group Th. It remains to show that $\langle L_N \rangle$ contains S_3. Since $\tilde{E}$ is normal in $N_{\mathbb{M}}(\tilde{E})$, it must lie in S_3. Now confirming that $x_{3c} = g_{3c}^4$ and h_{3c}^3 generate S_3 completes the proof. □

2.6. The group $\left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$.

Theorem 2.8. *The accompanying code defines $L_E = \{x_3, y_3\}$ and $L_N = \{g_{3a2}, h_{3a2}\}$. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $\left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$ of $\mathbb{M}$.*

Proof. We confirm that $\tilde{E}$ is 3A-pure of size 3^2. There is a unique class of such subgroups, see [27, §2], so $\tilde{N} \cong \left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$. Using the auxiliary element $g = h_{3a2}^2(g_{3a2}h_{3a2})^2g_{3a2}^{-1}h_{3a2}^{-1}$, we confirm that $x_{13} = (g_{3a2}^2h_{3a2})^2g_{3a2}^4h_{3a2}^{-2}$, $x_2 = ((h_{3a2}g_{3a2})^2h_{3a2}g_{3a2}^{-1})^{15}$, and $x_{14} = (gx_2g^{-1}x_2g)^3$ have orders 13, 2, and 14, respectively, and centralise $\tilde{E}$. Thus, they lie in the factor $\mathrm{O}_8^+(3)$ of $C_{\mathbb{M}}(\tilde{E}) \cong 3^2 \times \mathrm{O}_8^+(3)$. By [3, p. 140], the only maximal subgroup of $\mathrm{O}_8^+(3)$ with order divisible by both 13 and 14 is $\mathrm{O}_7(3)$, in which the centraliser of an element of order 7 has order 14. Confirming that x_{14}^2 (of order 7) commutes with $x_2 (\neq x_{14}^7)$ as well as with x_{14}, we conclude that $\mathrm{O}_8^+(3)$ is contained in $\tilde{E}$. We then confirm that $x_3 = (h_{3a2}g_{3a2}^3(h_{3a2}g_{3a2})^2)^8$ and $y_3 = (h_{3a2}g_{3a2}h_{3a2}g_{3a2}^3h_{3a2}g_{3a2})^8$, and since $\mathrm{O}_8^+(3)$ has a trivial centre, $\tilde{E} \leqslant \langle L_N \rangle$, and so $C_{\mathbb{M}}(\tilde{E}) \leqslant \langle L_N \rangle$. Lastly, we obtain $48 = |2.\mathrm{S}_4|$ distinct automorphisms induced by the conjugation action of $\langle L_N \rangle$ on $\tilde{E}$, which proves that $\tilde{N} = \langle L_N \rangle$. □

2.7. The group $3^{2+5+10}{:}(\mathbf{M}_{11} \times 2.\mathrm{S}_4)$.

Theorem 2.9. *The accompanying code defines $L_E = \{x_{3b}, y_{3b}\}$ and $L_N = \{g_{3b2}, h_{3b2}\}$. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$ of $\mathbb{M}$.*

Proof. We check that $\tilde{E} \cong 3^2$ and $\langle L_N\rangle \leqslant \tilde{N}$. By [27, Theorem 5.2], the normaliser $\tilde{N}$ is contained in a maximal subgroup with structure $3{\cdot}\mathrm{Fi}_{24}$, $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$, $\left(3^2{:}2 \times \mathrm{O}_8^+(3)\right){\cdot}\mathrm{S}_4$, $3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$, or $3^8{\cdot}\mathrm{O}_8^-(3).2$; since only the fourth of these groups contains an element of order $|g_{3b2}| = 88$, we must have $\tilde{N} \leqslant 3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$. (We note that [27, Theorem 5.2] refers to maximal 3-local subgroups, but by Proposition 2.3, all such subgroups of $\mathbb{M}$ are maximal.) It remains to prove that $|\tilde{N}| \geqslant |3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)| = 3^{17}|\mathrm{M}_{11}| \cdot 48$. We first confirm that $\langle h_{3b2}^6, (g_{3b2}^8 h_{3b2}^3 g_{3b2}^{32})^2\rangle \cong \mathrm{M}_{11}$ by verifying a presentation for M_{11}. We then establish elements in $\langle L_N\rangle$ that satisfy the assumptions of Lemma 2.2 (with $x = x_{3b}$ and $y = y_{3b}$) and so generate a 3-group of size at least 3^{17} that is normal in $C_{\mathbb{M}}(\tilde{E})$. Together with the simple group $\mathrm{M}_{11} \leq C_{\mathbb{M}}(\tilde{E})$ exhibited above, we have generated a subgroup of $C_{\mathbb{M}}(\tilde{E}) \cap \langle L_N\rangle$ of order at least $3^{17}|\mathrm{M}_{11}|$. We conclude the proof by enumerating the 48 distinct automorphisms of $\tilde{E}$ induced by conjugation by $\langle L_N\rangle$. □

2.8. The group $3^{3+2+6+6}{:}(\mathrm{PSL}_3(3) \times \mathbf{SD}_{16})$.

Theorem 2.10. *The accompanying code defines $L_E = \{x_{3b}, y_{3b}, z_{3b}\}$ and $L_N = \{g_{3b3}, h_{3b3}\}$. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $3^{3+2+6+6}{:}(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})$.*

Proof. Theorem 2.9 shows that $\langle x_{3b}, y_{3b}\rangle \cong 3^2 < 3^{1+12} \trianglelefteq N_{\mathbb{M}}(\langle x_{3b}\rangle) \cong 3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. Setting $c = g_{3b3}h_{3b3}^3 g_{3b3}h_{3b3}g_{3b3}$, we verify that $c \in C_{\mathbb{M}}(x_{3b})$, that $z_{3b} = y_{3b}^c$ and z_{3b} commutes with x_{3b} and y_{3b}, so $\tilde{E} \cong 3^3$ is a subgroup of 3^{1+12} containing the centre. Now [27, Theorem 6.5] implies that $\tilde{N}$ belongs to a maximal subgroup of $\mathbb{M}$ of type $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$, $3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$, or $3^{3+2+6+6}{:}(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})$. We confirm $g_{3b3}, h_{3b3} \in \tilde{N}$ and $|h_{3b3}| = 104$, which forces $\tilde{N} \leqslant 3^{3+2+6+6}{:}(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})$. The elements $a = h_{3b2}^6$ and $b = (g_{3b2}^8 h_{3b2}^3 g_{3b2}^{32})^2$ from the proof of Theorem 2.9 generate a subgroup isomorphic to M_{11}, which is contained in $N_{\mathbb{M}}(\langle x_{3b}, y_{3b}\rangle) = 3^{2+5+10}{:}(\mathrm{M}_{11} \times 2.\mathrm{S}_4)$. We check that $u = (aba(b^2a)^2)^2$ and $v = ((ba)^3(b^2a)^2ba)^2$ generate a group of size 9 that centralises $\tilde{E}$. In the accompanying code, we establish 15 elements (as words in elements of L_N) that commute with all elements of L_E and lie in the subgroup $3^{2+5+10} \leqslant N_{\mathbb{M}}(\langle x_{3b}, y_{3b}\rangle)$. An application of Lemma 2.2 proves that these elements generate a 3-subgroup U of $C_{\mathbb{M}}(\tilde{E})$ of order at least 3^{15}. By construction, $|\mathrm{M}_{11} \cap U| = 1$, so U and $\{u, v\}$ together generate a 3-subgroup of $C_{\mathbb{M}}(\tilde{E})$ of size at least 3^{17}. We verify that g_{3b3}^{39} and h_{3b3}^{26} centralise $\tilde{E}$, and that the $\langle g_{3b3}^{39}, h_{3b3}^{26}\rangle$-class of a particular element $h_1 \in U$ of order 3 has size 72. By the Orbit–Stabiliser Theorem, the size of $\langle g_{3b3}^{39}, h_{3b3}^{26}\rangle$ is divisible by 72, which implies that $\tilde{N}$ contains a subgroup of $C_{\mathbb{M}}(\tilde{E})$ with order divisible by 8. This centraliser is therefore at least 8 times as large as the 3-subgroup of order 3^{17} found above. Accounting for the 11232 automorphisms of $\tilde{E}$ that we enumerate by considering the conjugation action of $\tilde{N}$, we have $|\tilde{N}| \geq 3^{17} \cdot 8 \cdot 11232 = |3^{3+2+6+6}{:}(\mathrm{PSL}_3(3) \times \mathrm{SD}_{16})|$, which completes the proof. □

2.9. The group $3^8{\cdot}\mathrm{O}_8^-(3).2$.

Theorem 2.11. *Let $L_E = \{x_{3b}, y_{3b}, g_1, \ldots, g_4, h_4, h_4^{\sigma}\}$ and $L_N = \{g_{3^8}, h_{3^8}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $3^8{\cdot}\mathrm{O}_8^-(3).2$.*

Proof. Lemma 2.2 implies that $\tilde{E}$ has size 3^8; we verify that the generators have order 3 and commute, hence $\tilde{E} \cong 3^8$. To prove that the elements in L_N normalise $\tilde{E}$ is tedious, hence the accompanying code provides explicit words in elements of L_E that illustrate how the elements in L_N act via conjugation. This proves that $\tilde{N} \leqslant N_{\mathbb{M}}(\tilde{E})$ and also yields an explicit matrix representation of the conjugation action of $\langle L_N\rangle$ on $\tilde{E}$. A computation in the computer algebra system GAP [6] proves that $\langle L_N\rangle$ acts on $\tilde{E}$ as the subgroup $\mathrm{O}_8^-(3).2 \leqslant \mathrm{GL}_8(3)$. The only maximal 3-local subgroup in Table 2 whose order is divisible by $|g_{3^8}| = 41$ is $3^8{\cdot}\mathrm{O}_8^-(3).2$, which tells us that $\tilde{N} \leqslant 3^8{\cdot}\mathrm{O}_8^-(3).2$. As for demonstrating that L_N generates $\tilde{N}$, it remains to prove that $L_E \leqslant \langle L_N\rangle$, which we establish by representing the elements in L_E as words in elements of L_N. □

2.10. **The group** $(\mathbf{D}_{10} \times \mathbf{HN}) \cdot 2$.

Theorem 2.12. *Let* $L_E = \{x_5\}$ *and* $L_N = \{g_5, h_5\}$ *be as in the accompanying code. The group* $N_{\mathbb{M}}(\langle L_E \rangle)$ *is generated by* L_N *and isomorphic to the maximal subgroup* $(\mathrm{D}_{10} \times \mathrm{HN}) \cdot 2$.

Proof. The element $x_5 \in \mathbf{G}$ has order 5 and $\chi_{\mathbb{M}}(x_5) = 133$, so $x_5 \in 5\mathrm{A}$ and $N_{\mathbb{M}}(\tilde{E}) \cong (\mathrm{D}_{10} \times \mathrm{HN}) \cdot 2$. We confirm that $L_N \leqslant N_{\mathbb{M}}(\tilde{E})$ and $h_5^{19} = x_5$. The elements g_5^4 and h_5^5 centralise x_5 and have orders 11 and 19, respectively, and it follows from the Atlas that they generate the subgroup $\mathrm{HN} \leqslant N_{\mathbb{M}}(\tilde{E})$. Lastly, $x_5^{g_5} = x_5^2$, so g_5 induces an automorphism of order 4 on $\langle x_5 \rangle$. Thus, $N_{\mathbb{M}}(\tilde{E}) = \langle L_N \rangle$. □

2.11. **The group** $5^{1+6}{:}2{\cdot}\mathbf{J}_2{:}4$. The next theorem is adapted from [5].

Theorem 2.13. *Let* $L_E = \{x_{5b}\}$ *and* $L_N = \{g_{5b}, h_{5b}\}$ *be as in the accompanying code. The group* $N_{\mathbb{M}}(\langle L_E \rangle)$ *is generated by* L_N *and isomorphic to the maximal subgroup* $5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$.

Proof. We find $c \in \mathbb{M}$ such that $x_{5b}^c \in \mathbf{G}$ and confirm $\chi_{\mathbb{M}}(x_{5b}) = 8$, so $x_{5b} \in 5\mathrm{B}$ and $N_{\mathbb{M}}(\tilde{E})$ is isomorphic to $5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$. We also confirm $L_N \leqslant N_{\mathbb{M}}(\tilde{E})$. The elements $a = h_{5b}^4$ and g_{5b} centralise x_{5b}, so $a, g_{5b} \in C_{\mathbb{M}}(x_{5b})$. They have orders 70 and 3, and a product ag_{5b} of order 5. It follows that a and g_{5b} project to elements of orders 7 and 3 in J_2. The different orders ensure they are not inverses; in particular, their product of order dividing $|g_{5b}a| = 5$ must project to an element of order 5 exactly. Atlas information now implies that $\langle g_{5b}, h_{5b} \rangle$ contains a representative for each coset of $5^{1+6}{:}2$ in $C_{\mathbb{M}}(\langle x_5 \rangle) \cong 5^{1+6}{:}2{\cdot}\mathrm{J}_2$. It follows from the above that g_{5b}^7 is an element of order 10 in the normal subgroup $5^{1+6}{:}2$ of $N_{\mathbb{M}}(\langle x_{5b} \rangle)$, so it and its conjugates lie in in $(5^{1+6}{:}2) \setminus 5^{1+6}$. It remains to show that 5^{1+6} is contained in $\langle L_N \rangle$. In the accompanying code, we define $\ell = g_{5b}^7 (g_{5b}^7)^{h_{5b}} \in 5^{1+6}$ and write x_{5b} as a word in conjugates of ℓ. We also define elements y_{5b}, and g_1, g_2 and h_1, h_2 that satisfy the criteria of Lemma 2.2, which allows us to conclude $5^{1+6} \leqslant \langle L_N \rangle$. Since h_{5b} induces an automorphism of order 4 on $\langle x_{5b} \rangle$, it follows that $\langle L_N \rangle \cong 5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$. □

2.12. **The group** $\left(5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)\right){:}\mathrm{S}_3$.

Theorem 2.14. *Let* $L_E = \{x_5, y_5\}$ *and* $L_N = \{g_{5a2}, h_{5a2}\}$ *be as in the accompanying code. The group* $N_{\mathbb{M}}(\langle L_E \rangle)$ *is generated by* L_N *and isomorphic to the maximal subgroup* $\left(5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)\right){:}\mathrm{S}_3$.

Proof. We verify $L_N \leqslant N_{\mathbb{M}}(\tilde{E})$ and show that $\tilde{E} = 5\mathrm{A}^2$ by checking that x_5, y_5 commute and that $x_5 \in 5\mathrm{A}$ is conjugate to $y_5 \notin \langle x_5 \rangle$ and to $y_5^i x_5$ for $1 \leq i \leq 4$ by h_{5a2}, g_{5a2}^4, $h_{5a2} g_{5a2}^2 h_{5a2}^2$, $g_{5a2}^2 h_{5a2} g_{5a2}$, and $h_{5a2} g_{5a2} h_{5a2}^2$, respectively. Thus, $L_N \subset N_{\mathbb{M}}(\tilde{E}) \cong \left(5^2{:}4{\cdot}2^2 \times \mathrm{U}_3(5)\right){:}\mathrm{S}_3$. Next, we show that $((g_{5a2} h_{5a2})^2 g_{5a2} h_{5a2}^{-1} g_{5a2}^{-2} h_{5a2}^{-1})^5$ and $(h_{5a2}^2 g_{5a2} h_{5a2} g_{5a2} h_{5a2}^{-2} g_{5a2}^{-1} h_{5a2}^{-1} g_{5a2}^{-1})^5$ lie in $C_{\mathbb{M}}(\langle x_5, y_5 \rangle)$ which is isomorphic to $5^2 \times \mathrm{U}_3(5)$. These elements have orders 7 and 8, and Atlas information implies that they generate the factor $\mathrm{U}_3(5)$. The factor 5^2 is generated by $y_5 = ((g_{5a2} h_{5a2}^2)^2 g_{5a2})^{18}$ and $x_5 = h_{5a2} y_5 h_{5a2}^{-1}$; note that $\mathrm{U}_3(5)$ has trivial centre. Finally, we verify that the conjugation action of $\langle g_{5a2}, h_{5a2} \rangle$ on $\langle x_5, y_5 \rangle$ produces $|4{\cdot}2^2||\mathrm{S}_3| = 16 \cdot 6 = 96$ automorphisms of $\langle x_5, y_5 \rangle$. Thus, $N_{\mathbb{M}}(\tilde{E}) = \langle L_N \rangle$. □

2.13. **The group** $5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$.

Theorem 2.15. *Let* $L_E = \{x_{5b}, y_{5b}\}$ *and* $L_N = \{g_{5b2}, h_{5b2}\}$ *be as in the accompanying code. The group* $N_{\mathbb{M}}(\langle L_E \rangle)$ *is generated by* L_N *and isomorphic to the maximal subgroup* $5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$.

Proof. We have shown in the proof of Theorem 2.13 that x_{5b} and $y_{5b} \notin \langle x_{5b} \rangle$ are commuting elements of order 5 in $5^{1+6} \trianglelefteq N_{\mathbb{M}}(\langle x_{5b} \rangle) \cong 5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$. They generate a group $5^2 < 5^{1+6}$ containing the centre $\langle x_{5b} \rangle$. By [27, §9], there are only two conjugacy classes of such subgroups in $5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$. One of these, whose centralisers have structure $(5 \times 5^{1+4}){:}5^2{:}\mathrm{S}_3$, have as normalisers the subgroups $5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$ sought; the other, the centralisers of which have structure $5 \times 5^{1+4}{:}2^{1+4}{:}5$, do not. In the accompanying code we exhibit $t \in C_{\mathbb{M}}(\langle x_{5b}, y_{5b} \rangle)$ of order 3, which proves that $\langle x_{5b}, y_{5b} \rangle$ is of the former kind. We have confirmed $L_N \leqslant N_{\mathbb{M}}(\tilde{E}) \cong 5^{2+2+4}{:}\,(\mathrm{S}_3 \times \mathrm{GL}_2(5))$. Hence we deduce $C_{\mathbb{M}}(\langle x_{5b}, y_{5b} \rangle) = (5 \times 5^{1+4}){:}5^2{:}\mathrm{S}_3$. All elements of order 2 or 3 therein project to elements of the same order in the factor S_3, and therefore generate it. We exhibit two such elements in the accompanying code. The 5-subgroup is constructed using Lemma 2.2 by exhibiting suitable elements $\ell, g_1, g_2, h_1, h_2, \sigma$.

These elements are expressed as words in elements of L_N, which proves that $\langle x_{5b}, y_{5b}\rangle \cap C_{\mathbb{M}}(\langle x_{5b}, y_{5b}\rangle)$ contains a 5-group of order at least $5^{2+2+2+2} = 5^8$; this can only be the normal 5-group in the centraliser. Finally, the conjugation action of $\langle g_{5b2}, h_{5b2}\rangle$ on $\tilde{E}$ induces $480 = |\mathrm{GL}_2(5)|$ automorphisms, which completes the proof of $N_{\mathbb{M}}(\tilde{E}) = \langle L_N\rangle$. □

2.14. **The group** $5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$.

Theorem 2.16. *Let $L_E = \{x_{5b}, y_{5b}, g_2\}$ and $L_N = \{g_{5b3}, h_{5b3}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$.*

Proof. We verify that $\langle x_{5b}, y_{5b}, g_2\rangle$ is an elementary abelian 5-group. By [27, Theorem 5], its normaliser lies inside a maximal subgroup $5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$, $5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$, $5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$, or $5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$. Only the third of these has order divisible by $31 = |h_{5b3}^2|$, so $\langle L_N\rangle \leqslant N_{\mathbb{M}}(\tilde{E}) \leqslant 5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$. We construct elements g_1, g_2, h_1, h_1^g and confirm that together with x_{5b} and y_{5b} they generate a 5-subgroup of $\langle L_N\rangle$ of order at least 5^6. These elements commute with g_2, so this 5-subgroup lies in $C_{\mathbb{M}}(\tilde{E})$. The element $(h_{5b3}g_{5b3}^2)^{15}$ is an involution that centralises x_{5b}, y_{5b}, and g_2, so it increases the size of the group generated by a factor of at least 2. Lastly, we enumerate $372000 = |\mathrm{PSL}_3(5)|$ automorphisms induced by the conjugation of $\langle g_{5b3}, h_{5b3}\rangle$ on $\tilde{E}$. This completes the proof that $\langle L_N\rangle = N_{\mathbb{M}}(\tilde{E}) \cong 5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$. □

2.15. **The group** $5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$.

Theorem 2.17. *Let $L_E = \{x_{5b4}, y_{5b4}, a, b\}$ and $L_N = \{g_{5b4}, h_{5b4}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$.*

Proof. We confirm that $\tilde{E}$ is an abelian 5-group of order 5^4, so by [27, Theorem 5] its normaliser is contained in a maximal subgroup of shape either $5^{1+6}{:}2{\cdot}\mathrm{J}_2{:}4$, $5^{2+2+4}{:}(\mathrm{S}_3 \times \mathrm{GL}_2(5))$, $5^{3+3\cdot}(2 \times \mathrm{PSL}_3(5))$, or $5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$. Exhibiting an element $[g_{5b4}, h_{5b4}]^6$ of order 13 in $N_{\mathbb{M}}(\langle L_E\rangle)$ rules out all but the last possibility, so we have confirmed that $L_N \leqslant N_{\mathbb{M}}(\tilde{E}) \leqslant 5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$. A calculation confirms that $x_{5b} = [h_{5b4}^{-2}, g_{5b4}^4]^3$, $y_{5b} = x_{5b}g_{5b4}[g_{5b4}^4, h_{5b4}^{-2}]g_{5b4}^{-1}$, $a = [h_{5b4}^{-1}, g_{5b4}^4]$, and $b = g_{5b4}h_{5b4}g_{5b4}^4h_{5b4}^{-1}g_{5b4}^3$, so $\tilde{E} \leqslant \langle L_N\rangle$. The conjugation action of $\langle L_N\rangle$ on $\tilde{E}$ induces $93600 = |3 \times 2 \cdot \mathrm{PSL}_2(25)|$ automorphisms, and it follows that $\langle L_N\rangle \cong 5^4{:}(3 \times 2{\cdot}\mathrm{PSL}_2(25)){:}2$. □

2.16. **The group** $(7{:}3 \times \mathbf{He}){:}2$.

Theorem 2.18. *Let $L_E = \{x_7\}$ and $L_N = \{g_7, h_7\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $(7{:}3 \times \mathrm{He}){:}2$.*

Proof. The argument is almost identical to that for Theorem 2.12, using the facts that $x_7 \in \mathbf{G}$ has order 7; moreover $\chi_{\mathbb{M}}(x_7) = 50$,so $x_7 \in 7\mathrm{A}$ and elements $g_7^6, h_7^7 \in C_{\mathbb{M}}(x_7) \cong 7 \times \mathrm{He}$ have orders 7 and 17, while $g_7^6 \notin \langle x_7\rangle$, $h_7^{85} = x_7$. Lastly, we note that the conjugation action of g_7 on $\tilde{E}$ induces an automorphism of order 7, in particular $x_7^{g_7} = x_7^3$. □

2.17. **The group** $7^{1+4}{:}(3 \times 2.\mathrm{S}_7)$. The next theorem is adapted from [5].

Theorem 2.19. *Let $L_E = \{x_{7b}\}$ and $L_N = \{g_{7b}, h_{7b}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $7^{1+4}{:}(3 \times 2.\mathrm{S}_7)$.*

Proof. We confirm that $x_{7b} \in \mathbf{G}$ has order 7, $\chi_{\mathbb{M}}(x_{7b}) = 1$, so $x_{7b} \in 7\mathrm{B}$ and $L_N \subset N_{\mathbb{M}}(\tilde{E})$. It follows that $N_{\mathbb{M}}(\tilde{E}) \cong 7^{1+4}{:}(3 \times 2.\mathrm{S}_7)$. We first demonstrate that $\langle L_N\rangle$ contains a representative of each coset of 7^{1+4}. For this define $a_0 = g_{7b}^6$, $a_1 = a_0^{h_{7b}}$, $a_3 = a_0^3a_1^{-2}(a_1^{h_{7b}}a_1^{h_{7b}^2})^{-1}$, $y_6 = a_3^{-1}g_{7b}$, and $t = (y_6h_{7b})^{14}$. We verify that t has order 3 and centralises $\langle y_6, h_{7b}\rangle$. Considering conjugates in $\mathbf{G}$, we verify that $|\langle y_6, h_{7b}\rangle| = 10080$ and that $t \notin \langle y_6, h_{7b}\rangle$, so $\langle t, y_6, h_{7b}\rangle \leq \langle L_N\rangle$ has $3 \cdot 10080 = |3 \times 2.\mathrm{S}_7|$ elements, which is precisely the number of cosets of 7^{1+4} in $7^{1+4}{:}(3 \times 2.\mathrm{S}_7)$. We show that each coset has a representative. Suppose, for a contradiction, that this is not true. In this case the intersection of $\langle t, y_6, h_{7b}\rangle$ with 7^{1+4} is a non-trivial 7-group. The image of $\langle t, y_6, h_{7b}\rangle$ under the canonical homomorphism $\phi\colon 7^{1+4}{:}(3 \times 2.\mathrm{S}_7) \to [7^{1+4}{:}(3 \times 2.\mathrm{S}_7)]/7^{1+4}$ then has order dividing $3 \cdot$

$10080/7 = 3 \cdot 1440$, so that the element $r_7 = (y_6 h_{7b})^6$ of order 7 must be mapped to the identity. On the other hand, the fact that $r_7 h_{7b}^{-1}$ and h_{7b} have orders 20 and 6 implies their images under ϕ are of those orders too; but this a contradiction to $|\phi(r_7 h_{7b}^{-1})| = |\phi(h_{7b}^{-1})| = |\phi(h_{7b})| = 6$. Thus, it remains to prove that $7^{1+4} \lhd 7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)$ is a subgroup of $\langle L_N \rangle$. Note firstly that g_{7b}^6 lies in this normal subgroup, for the only alternative is that g_{7b} (of order 42) belongs to a coset of order 42 in the quotient group $[7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)]/7^{1+4} \cong 3 \times 2.\mathrm{S}_7$. The absence of elements with this order in $2.\mathrm{S}_7$ means the 14th power of such a coset would lie in the factor 3, for which $\{e, t, t^2\}$ are coset representatives by the previous paragraph. But the orders 42, 21, and 21 of g_{7b}, $g_{7b}t^{-1}$ and $g_{7b}t^{-2}$ do not divide $|7^{1+4}|$. We therefore find that the elements a_0, a_1, a_3 defined above — which are products of conjugates of g_{7b}^6 in $\langle L_N \rangle$ — belong to 7^{1+4}, as does $a_2 = a_1 a_1^{h_{7b}}$. Noting that $x_{7b} = a_0^{-1} a_1^{-1} a_0 a_1$ and a_3 satisfy the hypotheses of Lemma 2.2 with $\ell = a_0$, suitable $g_1, h_1 \in \langle a_0, \ldots, a_3 \rangle$ and $\sigma \in N_{\mathbb{M}}(\langle x_{7b}, a_3 \rangle)$, we deduce that they generate a subgroup of size 7^5. It follows that $\langle L_N \rangle = N_{\mathbb{M}}(\tilde{E})$. ☐

2.18. **The group** $\big(7^2{:}\,(3 \times 2.\mathrm{A}_4) \times \mathrm{PSL}_2(7)\big){:}2$.

Theorem 2.20. *Let $L_E = \{x_7, y_7\}$ and $L_N = \{g_{7a2}, h_{7a2}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $\big(7^2{:}\,(3 \times 2.\mathrm{A}_4) \times \mathrm{PSL}_2(7)\big){:}2$.*

Proof. The proof is similar to that of Theorem 2.12. The element $x_7 \in 7\mathrm{A}$ commutes with $y_7 \notin \langle x_7 \rangle$ and is conjugate to y_7 and $x_7 y_7^i$ for $1 \le i \le 6$. Thus $\tilde{E} \cong 7^2$ is 7A-pure and we confirm $L_N \subseteq N_{\mathbb{M}}(\tilde{E}) \cong \big(7^2{:}\,(3 \times 2.\mathrm{A}_4) \times \mathrm{PSL}_2(7)\big){:}2$. The factor 7^2 of $C_{\mathbb{M}}(\langle x_7, y_7 \rangle) \cong 7^2 \times \mathrm{PSL}_2(7)$ can be handled in the same way as the 5^2 in Theorem 2.14 since $y_7 = (g_{7a2}^3 h_{7a2}^4 g_{7a2})^{18}$ and $x_7 \in y_7^{\langle g_{7a2}, h_{7a2} \rangle}$. We check that $s = (h_{7a2}^4 g_{7a2}^4)^7$ and $t = (g_{7a2}^3 h_{7a2} g_{7a2} h_{7a2}^3 g_{7a2} h_{7a2})^{14}$ have orders 3 and 2, so they lie in the factor $\mathrm{PSL}_2(7)$. Moreover, they generate this group since st and t satisfy Sunday's [25] presentation for $\mathrm{PSL}_2(7)$. The remaining factor of $|3 \times 2.\mathrm{A}_4||2| = 144$ in the order of the normaliser is then accounted for by enumerating the automorphisms induced by the conjugation action of $\langle L_N \rangle$ on $\tilde{E}$. ☐

2.19. **The group** $7^{2+1+2}{:}\mathrm{GL}_2(7)$.

Theorem 2.21. *Let $L_E = \{x_{7b}, a_3\}$ and $L_N = \{g_{7b2}, h_{7b2}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $7^{2+1+2}{:}\mathrm{GL}_2(7)$.*

Proof. We have seen in the proof of Theorem 2.19 that x_{7b} and $a_3 \notin \langle x_{7b} \rangle$ are commuting elements of order 7 such that $a_3 \in 7^{1+4} \trianglelefteq N_{\mathbb{M}}(\langle x_{7b} \rangle) \cong 7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)$. It follows that $\langle x_{7b}, a_3 \rangle \cong 7^2$ is a subgroup of 7^{1+4} containing the centre $x_{7b} \in 7\mathrm{B}$, so [27, §10] shows that $N_{\mathbb{M}}(\langle L_E \rangle)$ lies in a maximal subgroup $7^{1+4}{:}\,(3 \times 2.\mathrm{S}_7)$ or $7^{2+1+2}{:}\mathrm{GL}_2(7)$. We show that $|g_{7b2} h_{7b2}^2| = 48$, which implies that $\langle N_N \rangle \leqslant 7^{2+1+2}{:}\mathrm{GL}_2(7)$. We exhibit elements $x_{7b}, a_3, g_1, h_1, h_1^\sigma$ such that Lemma 2.2 implies that these elements generate a 7-group of order at least 7^5, centralising $\tilde{E}$. We demonstrate the presence of such a 7-group in $\langle L_N \rangle$ by expressing each of these five elements as words in elements of L_N. Noting that $(h_{7b2} g_{7b2}^3 h_{7b2} g_{7b2}^{-1} h_{7b2}^{-3} g_{7b2}^{-1})^7$ is an element of order 3 centralising the same 7^2, Lemma 2.1 shows that the 7-group extends to a subgroup of $\langle L_N \rangle \cap C_{\mathbb{M}}(\tilde{E})$ with order at least $7^5 \cdot 3$. Finally, enumerating the conjugation action, we obtain

$$|\langle L_N \rangle| \geq 672 |\langle L_N \rangle \cap C_{\mathbb{M}}(\tilde{E})| \geq 672 \cdot 7^5 \cdot 3 = |7^{2+1+2}{:}\mathrm{GL}_2(7)|.$$

Since $\langle L_N \rangle \leq N_{\mathbb{M}}(\langle L_E \rangle) \leq 7^{2+1+2}{:}\mathrm{GL}_2(7)$, equality follows. ☐

2.20. **The group** $7^2{:}\mathrm{SL}_2(7)$. The next theorem is adapted from [19].

Theorem 2.22. *Let $L_E = \{x_{7b}, y_{7b}\}$ and $L_N = \{x_{7b}, y_{7b}, x_4, x_{14}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E \rangle)$ is generated by L_N and isomorphic to the maximal subgroup $7^2{:}\mathrm{SL}_2(7)$.*

Proof. Arguing as in the proof of Theorem 2.20 establishes that $\tilde{E} \cong 7^2$ is 7B-pure; conjugating elements are provided in the code. This allows us to confirm that $L_N \leqslant N_{\mathbb{M}}(\tilde{E}) \leqslant 7^2{:}\mathrm{SL}_2(7)$. We verify that x_4 and x_{14} satisfy a presentation for the group $\mathrm{SL}_2(7)$, which implies the claim. ☐

2.21. **The group** $11^2\colon(5{\times}2.\mathrm{A}_5)$**.** The next theorem is adapted from [19].

Theorem 2.23. *Let $L_E = \{x_{11}, y_{11}\}$ and $L_N = \{x_{11}, y_{11}, x_3, x_4, x_5\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $11^2\colon(5{\times}2.\mathrm{A}_5)$.*

Proof. We confirm that $\tilde{E} \cong 11^2$ and $L_N \subset N_{\mathbb{M}}(\tilde{E}) \cong 11^2\colon(5{\times}2.\mathrm{A}_5)$, see Proposition 2.3; note that all elements of order 11 in $\mathbb{M}$ are conjugate. We now check that x_3, x_4 and x_5 have the orders indicated by their subscripts, and that x_5 commutes with x_3, x_4. Furthermore, the cosets of x_4 and x_3 modulo $\langle x_4^2\rangle$ satisfy the presentation $\langle a, b \mid a^2, b^3, (ab)^5\rangle$ for the simple group A_5, which by Von Dyck's Theorem [12, Theorem 2.53] ensures $\langle x_3, x_4\rangle/\langle x_4^2\rangle \cong \mathrm{A}_5$. Observing that x_4^2 commutes with x_3 and (of course) x_4, it must be that $\langle x_3, x_4, x_5\rangle \cong 5 \times 2.\mathrm{A}_5$, so $\langle L_N\rangle = N_{\mathbb{M}}(\tilde{E})$. □

2.22. **The group** $(13{:}6 \times \mathrm{PSL}_3(3))\cdot 2$**.** The next theorem is adapted from [5].

Theorem 2.24. *Let $L_E = \{g_{13}\}$ and $L_N = \{g_{13}, y_{12}, c, d\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $(13{:}6 \times \mathrm{PSL}_3(3))\cdot 2$.*

Proof. We verify that $L_N \subset N_{\mathbb{M}}(\tilde{E})$. Enumerating a conjugate of $\langle c, d\rangle$ in $\mathbf{G}$ proves that $|\langle c, d\rangle| = 5616$, so g_{13} cannot be a 13B element: the size of the normaliser of the latter is not divisible by 5616. Thus, g_{13} is a 13A element, hence $N_{\mathbb{M}}(\tilde{E}) \cong (13{:}6 \times \mathrm{PSL}_3(3))\cdot 2$. We confirm that $C_{\mathbb{M}}(g_{13}) \cong 13 \times \mathrm{PSL}_3(3)$ is contained in $\langle g_{13}, y_{12}, c, d\rangle$ by verifying that c, d centralise the non-commuting elements g_{13} and y_{12}. Thus, g_{13} is an element of order 13 not belonging to $\langle c, d\rangle < C_{\mathbb{M}}(y_{12})$, and Lemma 2.1 implies that $|\langle g_{13}, c, d\rangle| \geq 13|\langle c, d\rangle| = 13 \cdot 5616 = |13 \times \mathrm{PSL}_3(3)|$. The observation that $|\langle g_{13}, c, d\rangle| \leqslant C_{\mathbb{M}}(g_{13})$ yields the desired result. The conjugation action of $\langle y_{12}\rangle$ on $\tilde{E}$ gives rise to 12 automorphisms; now $\langle L_N\rangle = N_{\mathbb{M}}(\tilde{E})$ follows. □

2.23. **The group** $13^{1+2}\colon(3 \times 4.\mathrm{S}_4)$**.**

Theorem 2.25. *Let $L_E = \{c\}$ and $L_N = \{g_{13}, c, c_2, x_1, x_2\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $13^{1+2}\colon(3 \times 4.\mathrm{S}_4)$.*

Proof. The element c has order 13 and a conjugate in $\mathbf{G}$. The Atlas implies that $\mathbf{G}$ has no 13A elements, so c lies in class 13B and we confirmed that $L_N \subset N_{\mathbb{M}}(\tilde{E}) \cong 13^{1+2}\colon(3 \times 4.\mathrm{S}_4)$. Enumerating a conjugate of $\langle x_1, x_2\rangle$ in $\mathbf{G}$ shows that $|\langle x_1, x_2\rangle| = 288 = |3 \times 4.\mathrm{A}_4|$, so this group must be $3 \times 4.\mathrm{A}_4$. The elements g_{13}, c, c_2 in turn have orders coprime to 288 and thus lie in the 13-group 13^{1+2}. An enumeration proves that g_{13}, c, c_2 generate 13^{1+2}. □

2.24. **The group** $13^2{:}\mathrm{SL}_2(13){:}4$**.**

Theorem 2.26. *Let $L_E = \{c, c_2\}$ and $L_N = \{c, c_2, x_1, x_2, x_4\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $13^2{:}\mathrm{SL}_2(13){:}4$.*

Proof. The proof of Theorem 2.25 shows that $\langle c, c_2\rangle \cong 13^2$; moreover, as $c \in$ 13B is conjugate to c_2 and cc_2^i for $1 \leq i \leq 13$ with conjugating elements x_2^3 and $x_2^3 x_1^{12i+13} x_2^3$, respectively, this group is 13B-pure. We confirm that $L_N \subset N_{\mathbb{M}}(\tilde{E}) \cong 13^2{:}\mathrm{SL}_2(13){:}4$. We verify that the conjugation action of $\langle x_1, x_2, x_4\rangle$ on $\tilde{E}$ induces $4|\mathrm{SL}_2(13)| = 8736$ automorphisms; this implies that $\langle L_N\rangle = N_{\mathbb{M}}(\tilde{E})$. □

2.25. **The group** $41{:}40$**.**

Theorem 2.27. *Let $L_E = \{g_{41}\}$ and $L_N = \{g_{41}, h_{41}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $41{:}40$.*

Proof. We show that g_{41} has order 41 and $g_{41}^{h_{41}} = g_{41}^{22}$ and 22 is a primitive root modulo 41. □

3. The Maximal 2-Local Subgroups of $\mathbb{M}$

The maximal 2-local subgroups of $\mathbb{M}$, all known to be maximal before the publication of the Atlas [3], were first classified by Meierfrankenfeld and Shpectorov, see [14, 24]. We introduce the necessary terminology before re-stating their results, which we need to do in order to justify our constructions. Recall that z denotes the central 2B involution in $\mathbf{G} = C_{\mathbb{M}}(z)$, and that $\mathbf{G} \cong \mathbf{Q}{}^{\cdot}\mathrm{Co}_1$ where $\mathbf{Q} = 2^{1+24}$. If $g_2 \in \mathbb{M}$ is a 2B involution and $u, v \in \mathbb{M}$ satisfy $g_2^u = z = g_2^v$, then $v^{-1}u \in C_{\mathbb{M}}(z)$ and therefore $\mathbf{Q}^{v^{-1}u} = \mathbf{Q}$. Thus, the group $\mathbf{Q}^{v^{-1}} = \mathbf{Q}^{u^{-1}}$ is independent of the choice of u or v, and we can define $\mathbf{Q}_{g_2}$ as $\mathbf{Q}^{u^{-1}}$ More generally, for a subgroup $U \leqslant \mathbb{M}$ we write

$$\mathbf{Q}_U = \bigcap_{x \in U \cap 2\mathrm{B}} \mathbf{Q}_x.$$

A 2B involution g_2 is *perpendicular* to g_1 if $g_1 \in \mathbf{Q}_{g_2}$. A *singular* subgroup of $\mathbb{M}$ is an elementary abelian 2-group in which all involutions are of class 2B and pairwise perpendicular. There are two conjugacy classes of singular subgroups of order 2^5 in $\mathbb{M}$, denoted as "type 1" and "type 2" in accordance with [24], see also Section 3.1.1. An *ark* A is a group generated by a type-1 $2\mathrm{B}^5$ singular subgroup U and all the type-2 $2\mathrm{B}^5$ subgroups that intersect U in an index 2 subgroup; it turns out that the size of an ark is always $|A| = 2^{10}$. With these definitions, Meierfrankenfeld and Shpectorov prove the following result.

Proposition 3.1 ([24, Theorem 1] and [14, Theorem A])**.** *The Monster contains exactly 7 conjugacy classes of maximal 2-local subgroups. They are:*

(1) *the normalisers of subgroups of types* 2A *and* $2\mathrm{A}^2$*, with structures* $2{}^{\cdot}\mathbb{B}$ *(where* $\mathbb{B}$ *is the Baby Monster) and* $2^{2}{}^{\cdot}{}^{2}\mathrm{E}_6(2){:}\mathrm{S}_3$ *respectively;*
(2) *the normalisers of singular subgroups of types* 2B, $2\mathrm{B}^2$, $2\mathrm{B}^3$*, and* $2\mathrm{B}^5$ *type-2, which have structures* $2^{1+24}{}^{\cdot}\mathrm{Co}_1$, $2^{2+11+22}{}^{\cdot}(\mathrm{M}_{24} \times \mathrm{S}_3)$, $2^{3+6+12+18}{}^{\cdot}(\mathrm{PSL}_3(2) \times 3.\mathrm{S}_6)$*, and* $2^{5+10+20}{}^{\cdot}(\mathrm{S}_3 \times \mathrm{PSL}_5(2))$ *respectively; and*
(3) *the normalisers of arks, with structure* $2^{10+16}{}^{\cdot}\mathrm{O}_{10}^{+}(2)$.

Remark 3.2. Generators for the centralisers of 2A and 2B involutions described in Proposition 3.1 already appear with proof in [5, §4] and [5, §2.5] respectively. For completeness, these generators are included in the accompanying code [18] along with the relevant arguments, but we do not reproduce them here.

3.1. **Preliminary results.**

3.1.1. **Checking singularity of 2B-pure subgroups.** The construction of several subgroups in Proposition 3.1 demands some means of identifying singular subgroups and classifying them up to conjugacy; otherwise, we have no means of ascertaining that what we produce are in fact maximal subgroups. Fortunately, Meierfrankenfeld and Shpectorov provide some simple tests for this.

Proposition 3.3. *The following hold.*

a) *There are exactly* 6 *conjugacy classes of non-trivial singular subgroups in* $\mathbb{M}$*; the corresponding orders are* $2, 2^2, 2^3, 2^4, 2^5$*, and* 2^5.
b) *The perpendicularity relation is symmetric.*
c) *A subgroup* U *is singular if and only if it is generated by a set* $\mathcal{U}$ *of pairwise perpendicular* 2B *involutions. Furthermore, the group* $\mathbf{Q}_U = \bigcap_{x \in U \cap 2\mathrm{B}} \mathbf{Q}_x$ *coincides with* $\bigcap_{x \in \mathcal{U}} \mathbf{Q}_x$.

Proof. Part a) is [24, Proposition 4.15], part b) is [24, Lemma 4.1], and part c) is [24, Lemma 4.3]. □

Part a) of Proposition 3.3 shows that, apart from the 2^5 case (see §3.1.2), the conjugacy class of a singular subgroup is determined by the group size. Singularity can be tested by straightforward enumeration of the group elements and ascertaining the non-identity elements are commuting perpendicular 2B involutions; this is a feasible task since `mmgroup` provides functionality to conjugate a 2B involution to $z \in \mathbf{G}$ and to test whether an element lies in $\mathbf{Q}$. Part c) provides some simplifications for this test. This approach suffices for the construction of the maximal $2\mathrm{A}^2$, $2\mathrm{B}^2$, and $2\mathrm{B}^3$ normalisers.

3.1.2. **The $2B^5$ and Ark Normalisers.** To construct the $2B^5$ and ark normalisers, the question of distinguishing the two conjugacy classes of singular subgroup with structure 2^5 must be addressed. The following result forms the basis of the method that we employ.

Proposition 3.4 ([24, Lemma 4.14]). *The following hold.*

a) *If $U \leqslant \mathbb{M}$ is singular 2^5 type-1, then $\mathbf{Q}_U/U$ is of order 2. Furthermore, all involutions in $\mathbf{Q}_U \setminus U$ are in class* 2A.
b) *If $U \leqslant \mathbb{M}$ is singular 2^5 type-2, then $\mathbf{Q}_U = U$.*

The task of determining $\mathbf{Q}_U$ can be simplified by the observation that, if U is generated by a subgroup $V < U$ and an involution t perpendicular to the generators of V, then Proposition 3.3 may be applied to yield $\mathbf{Q}_U = \mathbf{Q}_V \cap \mathbf{Q}_t$.

3.2. **The group** $2^{2\cdot 2}\mathrm{E}_6(2){:}\mathrm{S}_3$.

Theorem 3.5. *Let $L_E = \{y, y^{g_2}\}$ and $L_N = \{g_2, h_2\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $2^{2\cdot 2}\mathrm{E}_6(2){:}\mathrm{S}_3$.*

Proof. We confirm that $\tilde{E}$ is 2A-pure of size 2^2, and so $\tilde{N} \cong 2^{2\cdot 2}\mathrm{E}_6(2){:}\mathrm{S}_3$. Moreover, we confirm that $L_N \leqslant \tilde{N} = N_{\mathbb{M}}(\langle L_E\rangle)$. Define $a = h_2^2 g_2^9$ and $b = h_2^2 g_2^{12}$. We compute that $\langle a, b\rangle \le C_{\mathbb{M}}(\tilde{E}) \cong 2^{2\cdot 2}\mathrm{E}_6(2)$. Taking the image under the quotient by $\tilde{E}$, we have $\langle \tilde{E}a, \tilde{E}b\rangle \le {}^2\mathrm{E}_6(2)$. Now $a^2, b^2 \notin \tilde{E}$ satisfy $(a^2)^{13} = 1$ and $(b^2)^{19} = 1$, so that $\tilde{E}a^2$ and $\tilde{E}b^2$ must have orders 13 and 19, respectively. Since the sole maximal subgroup of ${}^2E_6(2)$ with order divisible by 19 has the form $U_3(8) : 3$ (see [28]), and the order of this group is not divisible by 13, the elements $\tilde{E}a, \tilde{E}b$ must in fact generate $C_{\mathbb{M}}(\tilde{E})/\tilde{E}$. Now $\tilde{E} \leqslant \langle L_N\rangle$ since $y^{g_2} = (h_2 g_2)^{14}$, and hence $C_{\mathbb{M}}(\tilde{E}) \leqslant \langle L_N\rangle$. Finally, we note that the conjugation action of $\langle L_N\rangle$ induces 6 = $|S_3|$ distinct automorphisms of $\tilde{E}$, implying that $N_{\mathbb{M}}(\tilde{E}) = \langle L_N\rangle$. □

3.3. **The group** $2^{2+11+22\cdot}(\mathbf{M}_{24} \times \mathrm{S}_3)$.

Theorem 3.6. *Let $L_E = \{z, z_1\}$ and $L_N = \{g_{2b}, h_{2b}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to $2^{2+11+22\cdot}(\mathrm{M}_{24} \times \mathrm{S}_3)$, a maximal subgroup of $\mathbb{M}$.*

Proof. We verify that $\tilde{E}$ is 2B-pure of size 2^2, and so $\tilde{N} \cong 2^{2+11+22\cdot}(\mathrm{M}_{24} \times \mathrm{S}_3)$. Moreover, $\tilde{E} \leqslant \mathbf{Q} \leqslant \mathbf{G}$ and $L_N \leqslant \tilde{N} = N_{\mathbb{M}}(\langle L_E\rangle)$. We now deviate from Procedure 2.4. To show that $\langle g_{2b}, h_{2b}\rangle$ is the complete normaliser of $\tilde{E}$, it suffices to prove that $|\langle g_{2b}, h_{2b}\rangle| \ge |N_{\mathbb{M}}(\tilde{E})|$. Define four subsets $\mathcal{F}_1 = L_N, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ of $\langle L_N\rangle$, and let $K_i = \langle \mathcal{F}_i, \ldots, \mathcal{F}_4\rangle$ for $i \in \{1, \ldots, 4\}$. We will describe homomorphisms $\varphi_1, \varphi_2, \varphi_3$ such that $K_{i+1} \le \ker\varphi_i$ and $\ker\varphi_3 < \ker\varphi_2 < \ker\varphi_1$ for $i = 1, 2, 3$. We can then find a lower bound for $|\langle g_{2b}, h_{2b}\rangle|$ using a repeated application of the Isomorphism Theorem.

First, let $\varphi_1\colon K_1 \to \mathrm{Aut}(\tilde{E})$ be the homomorphism induced by conjugation. By checking the generators of each K_i, we verify that $|\varphi_1(K_1)| = 6 = |\mathrm{S}_3|$, while $K_2 \leqslant C_{\mathbb{M}}(\tilde{E}) \leqslant \mathbf{G} = C_{\mathbb{M}}(z)$; this also shows that $K_2 \leqslant \ker\varphi_1$.

Now consider the action of K_2 on $\mathbf{Q}_{\tilde{E}}/\tilde{E}$, as defined in Lemma 3.3. By [24, Lemma 4.5], the image of this action is isomorphic to some subgroup of M_{24}. Let $\varphi_2\colon K_2 \to \mathrm{Aut}(\mathbf{Q}_{\tilde{E}}/\tilde{E})$ be the corresponding action homomorphism. In order to compute this homomorphism, we give (and verify) generators for $\mathbf{Q}_{\tilde{E}}$ in the accompanying code. It is then easily shown that $K_3 \le \ker\varphi_2$. Moreover, we claim that $\varphi_2(K_2) \cong \mathrm{M}_{24}$. We construct elements in $\varphi_2(K_2)$ with orders 5, 23, and 21. No proper subgroup of M_{24} has elements of all three of these orders, so the claim holds.

Now observe that $\mathcal{F}_3, \mathcal{F}_4 \subseteq \mathbf{G} \cong \mathbf{Q}{\cdot}\mathrm{Co}_1$ and consider $\varphi_3\colon K_3 \to \mathrm{Co}_1$, the restricted canonical homomorphism from $\mathbf{G}$ into Co_1. We find that $|\varphi_3(\langle\mathcal{F}_3\rangle)| = 2^{11}$ while $K_4 \le \ker\varphi_3$. Finally, with the observation that $\mathcal{F}_4 \subseteq \mathbf{Q}$, a direct calculation shows that $|K_4| = 2^{24}$. Lastly, $\langle L_N\rangle = |N_{\mathbb{M}}(\tilde{E})|$ follows from a repeated application of the Isomorphism Theorem, which reveals that

$$\langle L_N\rangle \geqslant |K_4| \prod_{i=1}^{3} |\operatorname{Im}\varphi_i| \geqslant 2^{24}.6.|\mathrm{M}_{24}|.2^{11} = |2^{2+11+22\cdot}(\mathrm{M}_{24} \times \mathrm{S}_3)| = |N_{\mathbb{M}}(\tilde{E})|. \qquad \square$$

3.4. The group $2^{3+6+12+18}\cdot(\mathrm{PSL}_3(2)\times 3.\mathrm{S}_6)$.

Theorem 3.7. *Let $L_E=\{z,z_1,w\}$ and $L_N=\{g_3,h_3\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $2^{3+6+12+18}\cdot(\mathrm{PSL}_3(2)\times 3.\mathrm{S}_6)$.*

Proof. We verify that $\tilde{E}$ is 2B-pure of size 2^3, and so $N_{\mathbb{M}}(\langle L_E\rangle)$ is of the required form. Moreover, $L_N\leqslant N_{\mathbb{M}}(\langle L_E\rangle)$. We proceed as in Theorem 3.6 to show $\langle L_N\rangle=N_{\mathbb{M}}(\langle L_E\rangle)$. Consider four subsets $\mathcal{F}_1=\{g_3,h_3\}$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ of $\langle g_3,h_3\rangle$, and, for each $1\leqslant i\leqslant 4$, let $K_i=\langle\mathcal{F}_i,\ldots,\mathcal{F}_4\rangle$. Let $\varphi_1\colon K_1\to\mathrm{Aut}(\tilde{E})$ be induced by conjugation; we compute that $|\varphi_1(\langle\mathcal{F}_1\rangle)|=168$ while $K_2\leq\ker\varphi_1$. In particular, $K_2\leq C_{\mathbb{M}}(\tilde{E})$. Now consider the action of K_2 on $\mathbf{Q}_{\tilde{E}}/\tilde{E}$. Since K_2 centralises $\tilde{E}$, it follows from [24, Lemma 4.8] that elements of K_2 permute $\mathbf{Q}_{\tilde{E}}/\tilde{E}$ by conjugation, inducing a group isomorphic to some subgroup of $3.\mathrm{S}_6$. Let $\varphi_2\colon K_2\to\mathrm{Aut}(\mathbf{Q}_{\tilde{E}}/\tilde{E})$ be the corresponding action homomorphism. To compute images under this homomorphism, verified generators for $\mathbf{Q}_{\tilde{E}}$ are given in the accompanying code. We find that $|\varphi_2(\langle\mathcal{F}_2\rangle)|=2160=|3.\mathrm{S}_6|$ while $K_3\leq\ker\varphi_2$. Noting that $\mathcal{F}_3,\mathcal{F}_4\subseteq\mathbf{G}\cong\mathbf{Q}{}^{\cdot}\mathrm{Co}_1$, define $\varphi_3\colon K_3\to\mathrm{Co}_1$ to be the restricted canonical homomorphism from $\mathbf{G}$ into Co_1. We find that $K_4\leq\ker\varphi_3$ and $|\varphi_3(\langle\mathcal{F}_3\rangle)|\geq 2^{16}$. Moreover by a direct calculation, $|\langle\mathcal{F}_4\rangle|=2^{23}$. The claim now follows from a repeated application of the Isomorphism Theorem, which shows that

$$\langle L_N\rangle\geqslant|K_4|\prod\nolimits_{i=1}^{3}|\operatorname{Im}\varphi_i|\geqslant 2^{23}.168.|3.\mathrm{S}_6|.2^{16}=|2^{3+6+12+18}\cdot(\mathrm{PSL}_3(2)\times 3.\mathrm{S}_6)|=|N_{\mathbb{M}}(\tilde{E})|.\ \square$$

3.5. The group $2^{5+10+20}\cdot(\mathrm{S}_3\times\mathrm{PSL}_5(2))$.

Theorem 3.8. *Let $L_E=\{z,z_1,w,k_4,k_5\}$ and $L_N=\{g_5,h_5\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $2^{5+10+20}\cdot(\mathrm{S}_3\times\mathrm{PSL}_5(2))$.*

Proof. We confirm that $\tilde{E}$ is 2B-pure of size 2^5, and that $|\mathbf{Q}_{\tilde{E}}|=|\tilde{E}|=2^5$. Hence, $\tilde{E}$ is a singular $2\mathrm{B}^5$ subgroup of type 2 by Proposition 3.4, so $N_{\mathbb{M}}(\langle L_E\rangle)$ is maximal of the required shape by Proposition 3.1. We also find that $L_N\leqslant N_{\mathbb{M}}(\langle L_E\rangle)$. To prove that $\langle L_N\rangle\leqslant N_{\mathbb{M}}(\tilde{E})$, we proceed as in Theorem 3.6. Define subsets $\mathcal{F}_1=\{g_5,h_5\}$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ of $\langle L_N\rangle$, and, for $1\leqslant i\leqslant 4$, write $K_i=\langle\mathcal{F}_i,\ldots,\mathcal{F}_4\rangle$. Let $\varphi_1\colon K_1\to\mathrm{Aut}(\tilde{E})$ be the action homomorphism for K_1 acting by conjugation on $\tilde{E}$. We verify that $K_2\leq\ker\varphi_1$, so $K_2\leq C_{\mathbb{M}}(\tilde{E})$. We further claim that $\varphi_1(\langle\mathcal{F}_1\rangle)\cong\mathrm{GL}_5(2)=\mathrm{PSL}_5(2)$. We find a pair of elements with images under φ_1 of orders 15 and 31 respectively. Since $\mathrm{PSL}_5(2)$ has no proper subgroup divisible by both of these orders, the claim follows. To introduce the next homomorphism φ_2, it is necessary to define the subgroup $V=\langle z,z_1,w,k_4\rangle<\tilde{E}$ of order 2^4. We verify that V is singular, and that $|\mathbf{Q}_V|=2^7$. Now $K_2\leqslant C_{\mathbb{M}}(\tilde{E})\leqslant C_{\mathbb{M}}(V)$, and so [24, Lemma 4.10] implies that K_2 permutes $\mathbf{Q}_V/V$, inducing a group isomorphic to some subgroup of S_3. Let $\varphi_2\colon K_2\to\mathrm{Aut}(\mathbf{Q}_V/V)$ be the corresponding action homomorphism. We compute that $|\varphi_2(\langle\mathcal{F}_2\rangle)|=6$ while $K_3\leq\ker\varphi_2$.

Now $\mathcal{F}_3,\mathcal{F}_4\subseteq\mathbf{G}\cong\mathbf{Q}{}^{\cdot}\mathrm{Co}_1$, so consider the restriction $\varphi_3\colon K_3\to\mathrm{Co}_1$ of the canonical homomorphism from $\mathbf{G}$ into Co_1. We calculate that $K_4\leq\ker\varphi_3$ and that $|\varphi_3(\langle\mathcal{F}_3\rangle)|\geq 2^{14}$. Moreover, a direct calculation shows that $|\langle\mathcal{F}_4\rangle|=2^{21}$. Applying the Isomorphism Theorem, we deduce that the following, which proves the claim.

$$\langle L_N\rangle\geqslant|K_4|\prod\nolimits_{i=1}^{3}|\operatorname{Im}\varphi_i|\geqslant 2^{21}.|\mathrm{PSL}_5(2)|.6.2^{14}=|2^{5+10+20}\cdot(\mathrm{S}_3\times\mathrm{PSL}_5(2))|=|N_{\mathbb{M}}(\tilde{E})|.\quad\square$$

3.6. The group $2^{10+16}\cdot\mathrm{O}_{10}^{+}(2)$.

Theorem 3.9. *Let $L_E=\{z,z_1,w,k_4,k_5,k_4^{\rho^{16}},k_5^{\rho},k_5^{\rho^{15}},k_5^{\rho^{16}},k_5^{\rho^{30}}\}$ and $L_N=\{g_{10},h_{10}\}$ be as in the accompanying code. The group $N_{\mathbb{M}}(\langle L_E\rangle)$ is generated by L_N and isomorphic to the maximal subgroup $2^{10+16}\cdot\mathrm{O}_{10}^{+}(2)$.*

Proof. We first verify that $\tilde{E}=\langle L_E\rangle$ is an *ark*, as in the definition preceding Proposition 3.1. We first verify that $T_1=\langle z,z^{\tau},w,k_4,k_4^{\rho^{16}}\rangle$ is a singular subgroup of order 2^5 and moreover that $|\mathbf{Q}_{T_1}|=64=2\cdot 2^5$. It follows from Proposition 3.4 that T_1 is a singular 2^5 subgroup of type 1.

To extend this to $\tilde{E}=\langle L_E\rangle$, recall from the proof of Theorem 3.8 that $\langle z,z^{\tau},w,k_4,k_5\rangle<\tilde{E}$ is a singular $2\mathrm{B}^5$ of type 2. It contains the index 2 subgroup $V=\langle z,z_1,w,k_4\rangle$ of $\tilde{E}$. Now [24, Corollary 4.12]

Type	Class Fusions	Centraliser	Normaliser
AAA	2A, 3A, 5A	A_{12}	$(A_5 \times A_{12}):2$
BAA	2B, 3A, 5A	$2.M_{22}.2$	
BBA	2B, 3B, 5A	M_{11}	$M_{11} \times S_5$
ACA	2A, 3C, 5A	$U_3(8){:}3$	$(A_5 \times U_3(8){:}3):2$
BCA	2B, 3C, 5A	$2^{1+4}(A_4 \times A_5)$	
BCB	2B, 3C, 5B	D_{10}	
B	2B, 3B, 5B	S_3	$A_5{:}4$
T	2B, 3B, 5B	2	$S_5 \times S_3$

TABLE 3. The conjugacy classes of A_5 in $\mathbb{M}$, the $\mathbb{M}$-classes to which their unique conjugacy classes of elements of orders 2, 3, and 5 fuse, their centralisers, and (where stated) their normalisers; this table is adapted from Norton [15], as well as [3, p. 234].

asserts that each singular $2B^4$ belongs to a unique $2B^5$ of type 2 (and two of type 1). On the other hand, there exists an element ρ of order 31 (defined in the accompanying code) which normalises both $\tilde{E}$ and T_1, but not $V < T_1$. We compute that ρ acts transitively on singular $2B^5$s of type 2 that meet T_1 in a singular $2B^4$. Hence $\tilde{E}$ is an ark, so $N_{\mathbb{M}}(\tilde{E})$ is maximal by Proposition 3.1.

We next confirm that $g_{10}, h_{10} \in N_{\mathbb{M}}(\tilde{E})$. To show that $\langle L_N \rangle = N_{\mathbb{M}}(\tilde{E})$, we proceed in a similar way to Theorem 3.6. Consider three subsets $\mathcal{F}_1 = \{g_{10}, h_{10}\}$, $\mathcal{F}_2$, and $\mathcal{F}_3$ of $\langle g_{10}, h_{10} \rangle$, and for $1 \leqslant i \leqslant 3$ define $K_i = \langle \mathcal{F}_i, \ldots, \mathcal{F}_3 \rangle$. Let $\varphi_1 \colon K_1 \to \mathrm{Aut}(\tilde{E})$ be induced by conjugation; we find that $K_2 \leq \ker \varphi_1$. Moreover, we claim that $\varphi_1(K_1) \cong \mathrm{O}_{10}^{+}(2)$. It follows from the structure of $N_{\mathbb{M}}(\tilde{E})$ (cf. [24, Lemma 5.18]) that $\varphi_1(K_1) \leq \mathrm{O}_{10}^{+}(2)$. We find permutations in $\varphi_1(K_1)$ of orders 17 and 31. Since $\mathrm{O}_{10}^{+}(2)$ has no maximal subgroups with order divisible by both 17 and 31, the claim follows.

At this point, we note that the elements x_{60}, r and s defined in the accompanying code belong to $N_{\mathbb{M}}(\tilde{E})$ and satisfy $g_{10} = (x_{60}^{30})^r$ and $h_{10} = (x_{60}^{3})^s$. Applying φ_1 reveals that x_{60} and $g_{10}h_{10}$ project to elements of order 60 and 21, respectively, in $\mathrm{O}_{10}^{+}(2)$. Since the GAP Character Table Library [2] shows that all elements of order 60 in this group power up to conjugacy classes 2A and 20A thereof, g_{10} and h_{10} are standard generators in the sense of [26].

Note that $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathbf{G} \cong \mathbf{Q}{\cdot}\mathrm{Co}_1$, and consider $\varphi_2 \colon K_2 \to \mathrm{Co}_1$ defined as the restriction of the canonical homomorphism from $\mathbf{G}$ into Co_1. It is easily verified that $|\varphi_2(\langle \mathcal{F}_2 \rangle)| = 2^9$ while $K_3 \leq \ker \varphi_2$. Finally, a direct calculation shows that $|\langle \mathcal{F}_3 \rangle| = 2^{17}$. Applying the Isomorphism Theorem, we deduce the following, which completes the proof.

$$\langle L_N \rangle \geqslant |K_3| \prod_{i=1}^{2} |\operatorname{Im} \varphi_i| \geqslant 2^{17}.|\mathrm{O}_{10}^{+}(2)|.2^9 = |2^{10+16}{\cdot}\mathrm{O}_{10}^{+}(2)| = |N_{\mathbb{M}}(\tilde{E})|. \qquad \square$$

4. A_5 in the Monster

The construction of the non-local subgroups of $\mathbb{M}$ requires some preliminary results on subgroups of $\mathbb{M}$ isomorphic to A_5. With the exception of $\mathrm{PGL}_2(13)$, all non-local maximal subgroups contain a subgroup of this shape, and considering how each A_5 can be extended to larger subgroups of $\mathbb{M}$ has conversely played a significant role in the existing classifications of some of the maximal subgroups.

The conjugacy classes of subgroups isomorphic to A_5 in $\mathbb{M}$ were classified by Norton [15]. There are eight conjugacy classes, as listed in Table 4. Six classes are uniquely identified by the (unique) $\mathbb{M}$-classes to which their elements of orders 2, 3, and 5 belong, while the two containing 2B, 3B, and 5B elements may be distinguished using their normalisers. We follow Norton in labelling the former sextet by their class fusions and the latter pair, which respectively occur as subgroups of the double cover of the Baby Monster and Thompson group, “B” and “T”. Instances of several of these types of A_5 are constructed in [5]. The following lemma reproduces these results, with the addition of a subgroup A_5 of type ACA and a simplification due to [19] in the proof for the type B case.

Lemma 4.1. *For each type* UVW $\in$ {AAA, BAA, BBA, ACA, BCA, BCB, B, T}*, the elements* g_{2UVW} *and* g_{3UVW} *as given in the accompanying code generate a subgroup* A_5 *of* $\mathbb{M}$ *of type* UVW.

Proof. In each case, we verify that the given generators satisfy a presentation for A_5. The generators are non-trivial, so by Von Dyck's Theorem and the simplicity of A_5, they generate a subgroup A_5. Since the conjugacy classes in $\mathbb{M}$ of elements of orders 2, 3, and 5 are distinguished by their values under $\chi_{\mathbb{M}}$, we can confirm the type of the A_5, with the exception of types B and T. To complete the proof, it must be shown that $\langle g_{2B}, g_{3B}\rangle$ is not of type T and $\langle g_{2T}, g_{3T}\rangle$ not of type B. The latter case is easier: in the code we exhibit an element y_3 that has order 3 and centralises $\langle g_{2T}, g_{3T}\rangle$, whereas all elements of order 3 in the normaliser $\mathrm{A}_5{:}4$ of an A_5 of type B lie in the A_5 and therefore do not centralise it. As for $\langle g_{2B}, g_{3B}\rangle$, the normaliser of an A_5 of type T has structure $\mathrm{S}_5 \times \mathrm{S}_3$ and cannot contain an element of order 4 whose square lies outside the A_5. In the code we exhibit an element y_4 with these properties: we verify that it has order 4, commutes with g_{2B}, and satisfies $g_{3B}^{y_4} = g_{2B}g_{3B}^2g_{2B}g_{3B}g_{2B}g_{3B}^2$; in particular, y_4 belongs to the normaliser of A_5. On the other hand, y_4^2 centralises $\langle g_{2B}, g_{3B}\rangle \cong \mathrm{A}_5$; since A_5 has a trivial centre, $y_4^2 \notin \langle g_{2B}, g_{3B}\rangle$, as required. □

Remark 4.2. Our subgroup of type T in Lemma 4.1 is different to that given in [5]; ours is chosen to intersect a conjugate of our A_5 of type B in a subgroup D_{10}, as this simplifies the proof of Theorem 5.6.

5. The Projective Linear Maximal Subgroups of the Monster

We discuss projective linear groups separately because confirmation of their construction is much easier than in most other cases due to the availability of presentations. Specifically, in the code we use Sunday's presentation [25] for $\mathrm{SL}_2(q)$ where q an odd prime power, and Robertson and Williams' presentation [20, Theorem 4] for $\mathrm{PGL}_2(p)$ with p a prime, with the correction of Hert [7, Theorem 3] who pointed out that the presentation for $\mathrm{PGL}_2(p)$ in [20, Theorem A] is in fact a presentation for $2 \times \mathrm{PSL}_2(p)$ whenever 2 is a quadratic residue modulo p. A common step in each of the following proofs is to verify a presentation; to keep the exposition short, we do not reiterate this every time, and only provide additional information on maximality if necessary.

5.1. The group $\mathrm{PSL}_2(71)$.

Theorem 5.1. *The elements* g_{71}, h_{71} *in the accompanying code generate a maximal subgroup* $\mathrm{PSL}_2(71)$.

Proof. It follows from [11, Theorem 1, §5.4] that all $\mathrm{PSL}_2(71) < \mathbb{M}$ are conjugate and maximal. □

5.2. The group $\mathrm{PSL}_2(41)$.

Theorem 5.2. *The elements* g_{41}, h_{41} *in the accompanying code generate a maximal subgroup* $\mathrm{PSL}_2(41)$.

Proof. It follows from [17, Theorem 1] that all $\mathrm{PSL}_2(41) < \mathbb{M}$ are conjugate and maximal. □

5.3. The group $\mathrm{PGL}_2(29)$.

Theorem 5.3. *The elements* g_{29}, h_{29} *in the accompanying code generate a maximal subgroup* $\mathrm{PGL}_2(29)$.

Proof. It follows from [8, Theorem 1] that all $\mathrm{PGL}_2(29) < \mathbb{M}$ are conjugate and maximal. □

5.4. The group $\mathrm{PGL}_2(19)$.

This theorem is adapted from [19, §6].

Theorem 5.4. *The elements* x_2, x_{19} *in the accompanying code generate a maximal subgroup* $\mathrm{PGL}_2(19)$.

Proof. It follows from [11, Theorem 1] that $\langle x_2, x_{19}\rangle \cong \mathrm{PSL}_2(19)$ is maximal in $\mathbb{M}$ if and only if it contains an A_5 of type B; the claim follows since $(x_{19}^2x_2)^2$ and $x_2x_{19}^2x_2x_{19}$ are the generators g_{2B} and g_{3B} of the A_5 of type B in Lemma 4.1. □

5.5. **The group** $\mathrm{PGL}_2(13)$**.** This theorem is adapted from [5, §3].

Theorem 5.5. *The elements* u, g_{13} *in the accompanying code generate a maximal subgroup* $\mathrm{PGL}_2(13)$.

Proof. To show that this subgroup is maximal, we first note that there are known to be exactly three conjugacy classes of $\mathrm{PSL}_2(13)$ in $\mathbb{M}$. Two of these, identified by Norton [15, §5], have centralisers of shapes $3^{1+2}.2^2$ and 3; the third, found by Dietrich et al. [5], has trivial centraliser and extends to a maximal subgroup $\mathrm{PGL}_2(13)$. It thus suffices to exhibit $\mathrm{PSL}_2(13) < \langle u, g_{13}\rangle$ which is not centralised by an element of order 3. As such, let $x_2 = (ug_{13}^2)^2 \in \langle u, g_{13}\rangle$; we verify $\langle x_2, g_{13}\rangle \cong \mathrm{PSL}_2(13)$. It is also clear that $C_{\mathbb{M}}(\langle x_2, g_{13}\rangle) \le C_{\mathbb{M}}(g_{13})$, One may recall from the proof of Theorem 2.24 that the latter is embedded in the direct product $13{:}6 \times \mathrm{PSL}_2(13)$ of $\langle g_{13}, y_{12}^2\rangle \cong 13{:}6$ and its centraliser. Writing $g_6 = g_{13}^{-1}x_2g_{13}^7x_2g_{13}^2x_2 \in \langle x_2, g_{13}\rangle$, the fact that $x_6 = y_{12}^{-2}g_6$ commutes with both g_{13} and y_{12}^2 then reveals that g_6 is the product of $y_{12}^2 \in 13 : 6$ and $x_6 \in \mathrm{PSL}_2(13)$. It follows that any element of $13{:}6 \times \mathrm{PSL}_2(13)$ which commutes with g_6 must also centralise both y_{12}^2 and x_6. Hence $C_{\mathbb{M}}(\langle g_{13}, x_2\rangle) \le C_{\mathbb{M}}(\langle g_{13}, g_6\rangle) \le C_{\mathbb{M}}(\langle g_{13}, y_{12}^2, x_6\rangle) = C_{\mathrm{PSL}_2(13)}(x_6)$, with x_6 an element of order 6 in $C_{\mathbb{M}}(\langle g_{13}, y_{12}^2\rangle) \cong \mathrm{PSL}_2(13)$. The Atlas implies that this centraliser is $\langle x_6\rangle$, and we show that x_6^2 (of order 3) does not commute with x_2. Thus, no element of order 3 centralises $\langle x_2, g_{13}\rangle$. □

5.6. **A correction for** $\mathrm{PSL}_2(59)$**.** Attempting to reproduce the methodology of Holmes and Wilson's [10] construction of a maximal $\mathrm{PSL}_2(59)$, we come to the conclusion that no such subgroup exists. Our argument is spelled out in the following proof; it reveals that there is instead a new maximal subgroup of $\mathbb{M}$, which is isomorphic to 59:29.

Theorem 5.6. *There is no subgroup* $\mathrm{PSL}_2(59)$ *of* $\mathbb{M}$*. Consequently, the normalisers* 59:29 *of elements of order* 59 *form a class of maximal subgroups of* $\mathbb{M}$.

Proof. The group $\mathrm{PSL}_2(59)$ has maximal subgroup A_5 and every element of order 5 has a normaliser $\mathrm{D}_{60} \cong 2 \times \mathrm{D}_{30}$. Norton and Wilson [16] have shown that elements of order 2, 3, or 5 in a maximal $G = \mathrm{PSL}_2(59)$ in $\mathbb{M}$ must lie in $\mathbb{M}$-classes 2B, 3B, or 5B, respectively. It follows from Table 4 that every subgroup A_5 of G is of type B or T. Let g_{2B}, g_{3B}, g_{2T}, and g_{3T} be as in Lemma 4.1, so that $\langle g_{2B}, g_{3B}\rangle$ and $\langle g_{2T}, g_{3T}\rangle$ are subgroups isomorphic to A_5s of the required types. In the code we fix an element c such that the putative G contains at least one of $\langle g_{2B}, g_{3B}\rangle^c$ and $\langle g_{2T}, g_{3T}\rangle$, see also Remark 4.2. Let $b_5 = g_{2T}g_{3T}$; a computation reveals that $g_{2T}g_{3T} = b_5 = g_{2B}^c g_{3B}^c$ and $g_{2T}^{g_{3T}g_{2T}g_{3T}^2} = (g_{2B}^{g_{3B}g_{2B}g_{3B}^2})^c = z$, where z is the central involution in $\mathbf{G}$. Our assumed $G \cong \mathrm{PSL}_2(59)$ therefore contains $\langle b_5, z\rangle$, which we confirm to be D_{10}. On the other hand, D_{10} is a group with trivial centre, normalising elements of order 5 in G; since such elements have normalisers $2\times\mathrm{D}_{30}$ in G, the group $\langle b_5, z\rangle$ must be centralised by an involution $x \in G$; in particular, we must have $x \in$ 2B as shown above. We check that no suitable x exists, and for this it suffices to consider all 2B involutions in the centraliser of the D_{10} we found above. For these subgroups D_{10} in A_5 of type B or T, Norton [15, Table 4] proves that the relevant centralisers have shape $5^3.(4\times\mathrm{A}_5)$; thus, any list of $|5^3.(4\times\mathrm{A}_5)| = 30000$ distinct elements commuting with b_5 and z generate the whole centraliser. In the code, we generate such a list from products of 6 pre-computed generators. Testing orders and $\chi_{\mathbb{M}}$ values yields that exactly 500 involutions of type 2B remain. This coincides with the computations reported by Holmes and Wilson [10, p. 13], and gives confidence that our construction so far is in line with the one in [10]. In [10] it is claimed that among these 500 involutions, some elements x extend A_5 to a subgroup $\mathrm{PSL}_2(59)$. However, we cannot confirm this: note that elements in $\mathrm{PSL}_2(59)$ have orders in $\mathcal{O} = \{1, 2, 3, 5, 6, 10, 15, 29, 30, 59\}$. In particular, if G contains a subgroup A_5 of type B, then all of $x(g_{2B}^{g_{3B}g_{2B}})^c$, xg_{3B}^c, and $g_{2B}^c x g_{3B}^c$ have orders in $\mathcal{O}$ for some involution x found above. Similarly, if G contains a subgroup A_5 of type T, then the orders of $xg_{2T}^{g_{3T}g_{2T}}$ and $g_{2T}xg_{3T}$ must belong to $\mathcal{O}$ for some such x. We run over all 500 involutions and conclude that none of them satisfies these conditions. This contradiction allows us to conclude that $\mathbb{M}$ has no subgroup $G = \mathrm{PSL}_2(59)$.

The existence of a subgroup $H = 59{:}29$ of $\mathbb{M}$ follows from [27]. It remains to prove maximality. If H is not maximal, then $H < M$ for some maximal $M < \mathbb{M}$. Note that $M = N_{\mathbb{M}}(T)$ where $T = S^m$ is a direct product of isomorphic simple groups T. If 59 divides $|T|$, then $m = 1$ and so $M = N_{\mathbb{M}}(59) = 59{:}29$ by [27], contradicting our assumption that H is not maximal. If 59 does not divide $|T|$, then the normal $59 < H$ normalises $T = S^m$. Since no prime power p^{59} divides $|\mathbb{M}|$, we deduce that $m < 59$ and therefore $59 < H$ does not permute the m factors of T. Thus this 59 must normalise each S, so induces an element of order 59 in $\mathrm{Aut}(S)/S$. Running over all simple subgroups S that could be involved in $\mathbb{M}$, we determine that 59 never divides $|\mathrm{Aut}(S)/S|$. Thus, this case is also not possible. Hence, H is maximal, as claimed. □

Remark 5.7. Establishing elements in `mmgroup` that generate a subgroup 59:29 constitutes ongoing work. While elements of orders 59 and 29 may be easily found by random search, constructing the normaliser of a given element of order 59 is very difficult – a naïve approach yields an approximately 1 in 10^{48} chance of success. An adaptation of a method developed by Bray et al. [1] for an analogous problem for the Baby Monster increases the probability of success to around 1 in 10^8.

6. Non-Local Maximal Subgroups of the Monster

We now consider the non-local maximal subgroups of $\mathbb{M}$.

Procedure 6.1. For many non-local subgroups of $\mathbb{M}$, it is convenient to use a modification of Procedure 2.4 when performing verifications; we usually proceed as follows.

(1) We exhibit generators L_E for a subgroup E such that $N_{\mathbb{M}}(E)$ is a maximal subgroup of $\mathbb{M}$ of the claimed structure; the results of Section 4, [3, p. 234] and [15, §4, §5] often provide the necessary data.
(2) We propose generators L_N for $N_{\mathbb{M}}(E)$ and first ascertain that L_N normalises E. Some generators in L_N can be written as a product gh where $g \in C_{\mathbb{M}}(E)$ and $h \in E$; for others, it may be directly verified that they map L_E to elements of E under conjugation.
(3) The remaining steps are then to confirm that L_N generates $N_{\mathbb{M}}(E)$. We usually begin by exhibiting words for the generators of E in elements of L_N, so $E \leqslant \langle L_N \rangle$, and then show that $C_{\mathbb{M}}(E) \leqslant \langle L_N \rangle$. The elements of $C_{\mathbb{M}}(E)$ used to write products in Step (2) belong to $\langle L_N \rangle$ since those of E do, so that it frequently suffices to show these elements generate $C_{\mathbb{M}}(E)$.
(4) Finally, we demonstrate that L_N extends $EC_{\mathbb{M}}(E)$ to $N_{\mathbb{M}}(E)$, usually by exhibiting elements which induce suitable outer automorphisms of E.

Throughout the following, we refer to the elements listed in Lemma 4.1.

6.1. The group $(\mathrm{A}_5 \times \mathrm{A}_{12}):2$.

Theorem 6.2. *Let $L_E = \{g_{2AAA}, g_{3AAA}\}$ and $L_N = \{g_{AAA}, h_{AAA}, n\}$ be as in the accompanying code. Then L_N generates the maximal subgroup $N_{\mathbb{M}}(E) \cong (\mathrm{A}_5 \times \mathrm{A}_{12}):2$.*

Proof. According to the Atlas, $N_{\mathbb{M}}(L_E) \cong (\mathrm{A}_5 \times \mathrm{A}_{12}):2$, and it remains to prove that $\langle L_N \rangle = N_{\mathbb{M}}(E)$, see also Lemma 4.1. We verify that $x_3 = g_{AAA}g_{2AAA}$ and $x_{10} = h_{AAA}g_{3AAA}^{-1}$ centralise E, and confirm that n centralises g_{2AAA} and that $g_{3AAA}^n \in E$. The latter is checked via an enumeration of E, which also confirms that conjugation by n induces an outer automorphism of A_5. We verify $g_{2AAA} = g_{AAA}^3$ and $g_{3AAA} = h_{AAA}^{10}$, so E and x_3, x_{10} lie in $\langle L_N \rangle$. We show that $\langle x_3, x_{10} \rangle$ is the full centraliser A_{12} of E by verifying that the generators x_3, x_{10} satisfy the group presentation of A_{12} obtained by Coxeter and Moser [4, §6.4]. Since n induces an outer automorphism, $\langle L_N \rangle \cong (\mathrm{A}_5 \times \mathrm{A}_{12}):2$. □

A subgroup $\mathrm{A}_5 \times \mathrm{A}_{12}$ has already been constructed in [5]; in our code we provide a conjugating element that maps A_5 into **G**.

6.2. The group $(\mathrm{A}_6 \times \mathrm{A}_6 \times \mathrm{A}_6).(2 \times \mathrm{S}_4)$.

Theorem 6.3. *Let $L_U = \{x_{A6}, y_{A6}\}$, t, and $L_N = \{g_{A6}, h_{A6}\}$ as in in accompanying code. Then L_N generates the maximal subgroup $N_{\mathbb{M}}(\langle E \rangle) \cong (\mathrm{A}_6 \times \mathrm{A}_6 \times \mathrm{A}_6).(2 \times \mathrm{S}_4)$ where $E = \langle U, U^t, U^{t^2} \rangle$ with $U = \langle L_U \rangle \cong \mathrm{A}_6$.*

Proof. The Atlas shows that the maximal subgroups $(\mathrm{A}_6 \times \mathrm{A}_6 \times \mathrm{A}_6).(2 \times \mathrm{S}_4)$ are precisely the normalisers of $\mathrm{A}_6 \times \mathrm{A}_6 \times \mathrm{A}_6$, where each A_6 contains 2A, 3A, 4B, and 5A elements. We verify that this holds for the subgroup $U = \langle x_{A6}, y_{A6}\rangle \cong \mathrm{A}_6$ using the presentation given in [4, §6.4]. Moreover, U contains elements of a subgroup A_5 of type AAA since it contains $g_{2AAA} = x_{A6}y_{A6}x_{A6}y_{A6}^2x_{A6}^2$ and $g_{3AAA} = x_{A6}y_{A6}^2x_{A6}^2y_{A6}x_{A6}$ as in Lemma 4.1. It follows from [15, Table 5] that there is a unique conjugacy class of groups $\mathrm{A}_6 < \mathbb{M}$ containing such elements. This shows that U, U^t, U^{t^2} are of the required type; we verify that they commute and that t has order 3. An explicit calculation shows that g_{A6} and h_{A6} permute the groups U, U^t, U^{t^2} cyclically, and we verify $E \leqslant \langle L_n\rangle \leqslant N_{\mathbb{M}}(E)$ by writing the generators L_U as words in elements of L_N, and confirming that $g_{A6}^4 \in \langle L_N\rangle$. It remains to show that $\langle L_N\rangle$ contains the factor $2\times \mathrm{S}_4$. The element $c_0 = (g_{A6}^3h_{A6}^2g_{A6}h_{A6}^3)^6$ centralises x_{A6}, x_{A6}^t and y_{A6}^t, but $y_{A6}^{c_0} \notin y_{A6}^{\langle x_{A6}\rangle}$. Any element of E which centralises x_{A6} is the product of an element centralising U and an element of $C_U(x_{A6}) = \langle x_{A6}\rangle$, so we deduce that $c_0 \notin E$, so $\langle E, c_0\rangle$ is at least twice as large as E by Lemma 2.1. The element $c_1 = (h_{A6}^2g_{A6}^3h_{A6}^3)^6$ centralises U and induces an outer automorphism of U^t which adjoining $c_0 \in C_{\mathbb{M}}(U^t)$ cannot produce; thus, $\langle E, c_0, c_1\rangle$ is at least $2^2 = 4$ times as large as E. We next check that U^t, U^{t^2}, and c_1 centralise U; on the other hand, since $c_0 \notin U$ is an involution normalising U, the Isomorphism Theorem shows $\langle x_{A6}, y_{A6}, c_0\rangle/U \cong \langle c_0, U\rangle/U \cong \langle c_0\rangle/(\langle c_0\rangle \cap U) = \langle c_0\rangle$, so that all automorphisms of U induced by the group generated so far arise from the product of an element of U and $\langle c_0\rangle$. Of these, $4\cdot 2 = 8$ elements centralise x_{A6}. Establishing that the conjugate of y_{A6} by $c_2 = g_{A6}^3y_{A6}^{-1} \in \langle g_{A6}, h_{A6}\rangle$ is not among the 8 elements in the $\langle x_{A6}, c_0\rangle$-class of y_{A6} therefore suffices to show that $\langle E, c_0, c_1, c_2\rangle$ is at least $2^3 = 8$ times as large as E. Moreover, recalling that conjugation by elements of $\langle g_{A6}, h_{A6}\rangle$ permutes the factors of E, checking that U^{tc_2} commutes with $U^{t^2} \cong \mathrm{A}_6$ and normalises U shows that $U^{tc_2} = U^t$ and hence that c_2, like all other generators of the extended group just exhibited, normalises U, U^t and U^{t^2}. Adjoining an element $c_3 = h_{A6}^{t^{-1}}$ which interchanges U and U^t thus produces a group at least $2\cdot 2^3 = 16$ times the size of E. Finally, since t does not normalise U^{t^2}, the automorphism of U induced by t is still unaccounted for after the introduction of c_3. Thus,

$$\langle E, c_0, c_1, c_2, h_{A6}^{t^{-1}}, t\rangle \leq \langle g_{A6}, h_{A6}\rangle$$

has order at least $3\cdot 2^4|\mathrm{A}_6|^3 = |\,(\mathrm{A}_6\times\mathrm{A}_6\times\mathrm{A}_6)\,.\,(2\times\mathrm{S}_4)\,|$, and the claim follows. ☐

6.3. The group $(\mathrm{A}_5 \times \mathrm{U}_3(8){:}3){:}2$.

Theorem 6.4. *Let $L_N = \{g_{3ACA}, g_{ACA}, h_{ACA}\}$ and $L_E = \{g_{2ACA}, g_{3ACA}\}$ be as in the accompanying code. Then $\langle L_E\rangle \cong \mathrm{A}_5$ and L_N generates the maximal subgroup $N_{\mathbb{M}}(E) \cong (\mathrm{A}_5 \times \mathrm{U}_3(8){:}3){:}2$.*

Proof. The group E is a subgroup A_5 of type ACA, and it follows from [3, p. 234] that $N_{\mathbb{M}}(E) \cong (\mathrm{A}_5 \times \mathrm{U}_3(8){:}3){:}2$ is maximal in $\mathbb{M}$. We verify that $L_E \subseteq \langle L_N\rangle$ and that $v = g_{ACA}g_{2ACA}$ centralises E. We have $g_{2ACA} = g_{ACA}^3$, so $v \in \langle L_N\rangle$. We claim that the centraliser $\mathrm{U}_3(8){:}3$ of E lies in $\langle L_N\rangle$ and is generated by $h_{ACA}vh_{ACA}v^2$, $(h_{ACA}v)^3(h_{ACA}v^2)^3$, and $(h_{ACA}v)^3h_{ACA}v^2$. These elements commute with L_E and have orders 19, 7, and 12, respectively. The former two must lie in $\mathrm{U}_3(8) < \mathrm{U}_3(8){:}3$. It follows from the Atlas that $\mathrm{U}_3(8)$ has no maximal subgroups of order divisible by 7 and 19, which combined with the fact that $\mathrm{U}_3(8){:}3/\mathrm{U}_3(8) \cong 3$ is cyclic of prime order and $\mathrm{U}_3(8)$ contains no elements of order 12 establishes that the three elements indeed generate the whole of $C_{\mathbb{M}}(E)$. Finally, we confirm (analogously to the proof of Theorem 6.2) that conjugation by h_{ACA} induces an outer automorphism of A_5; thus, $\langle L_N\rangle \cong (\mathrm{A}_5 \times \mathrm{U}_3(8){:}3){:}2$. ☐

6.4. The group $(\mathrm{PSL}_3(2) \times \mathrm{S}_4(4){:}2)\cdot 2$.

Theorem 6.5. *Let $L_E = \{x_{L2(7)}, y_{L2(7)}\}$ and $L_E = \{g_{L2(7)}, h_{L2(7)}, n\}$ be as in the accompanying code. Then $E \cong \mathrm{PSL}_2(7)$ and L_N generates the maximal subgroup $N_{\mathbb{M}}(E) \cong (\mathrm{PSL}_3(2) \times \mathrm{S}_4(4){:}2)\cdot 2$.*

Proof. By [15, §5, Case 1], the maximal subgroup we aim to construct is a normaliser of a subgroup $\mathrm{PSL}_3(2)$ with elements in $\mathbb{M}$-classes 2A, 3A, and 7A. We first check a presentation and verify that $E \cong \mathrm{PSL}_2(7)$; note that the latter is also isomorphic to $\mathrm{PSL}_3(2)$ by [13, Prop. 2.9.1, (xi)]. Our generators in L_E lie in $\mathbf{G}$ and we verify that $x_{L2(7)}, y_{L2(7)}, x_{L2(7)}y_{L2(7)} \in \mathbf{G}$ lie in classes 7A, 2A, 3A, respectively. We establish $g_{L2(7)}, h_{L2(7)}, n \in N_{\mathbb{M}}(E)$ by determining that $u = g_{L2(7)}x_{L2(7)}^{-1}$ and $v = h_{L2(7)}y_{L2(7)}$ centralise L_E, while n inverts $x_{L2(7)}$ and commutes with $y_{L2(7)}$. We have

$x_{L2(7)} = g_{L2(7)}^8$ and $y_{L2(7)} = h_{L2(7)}^{17}$, so $E \leqslant \langle L_N \rangle$. We know that u and v lie in $C_{\mathbb{M}}(E) \cong \mathrm{S}_4(4){:}2$, and show that they actually generate the centraliser. Since squares of all elements in $\mathrm{S}_4(4){:}2$ lie in the normal subgroup $\mathrm{S}_4(4)$, we conclude that the latter's intersection with $\langle u, v \rangle$ contains the element v^2 of order 17. Atlas information implies that the only maximal subgroup of $\mathrm{S}_4(4)$ containing such an element is $\mathrm{PSL}_2(16).2$. On the other hand, the orders of involution centralisers in $\mathrm{PSL}_2(16).2$ are not divisible by any power of 2 greater than 16; it will thus follow that $\langle u, v \rangle$ contains $\mathrm{S}_4(4)$ if it can be shown that it contains an involution centraliser of order a multiple of 32. To this end, we construct five elements as word in u, v that commute with the involution $(vuv)^2$, and generate a group of order 64 as sought. To complete the proof, we must verify that $\langle u, v \rangle$ properly contains $\mathrm{S}_4(4)$, so that $\langle u, v \rangle = C_{\mathbb{M}}(E)$, and that n extends $\mathrm{PSL}_3(2) \times \mathrm{S}_4(4){:}2$ to the full normaliser of E. The first assertion is demonstrated by an adaptation of above argument: the largest 2-power that divides the order of an involution centraliser in $\mathrm{S}_4(4)$ is 256, whereas the elements above together with with $(wvwu)^2w$, $(v^2w)^2v$ generate a group of order 512 centralising $(vuv)^2$. For the second result, an additional factor of 2 given by an outer automorphism of E is required. Since elements of order 7 in $\mathrm{PSL}_3(2)$ have normalisers 7:3, if follows that n (which inverts $x_{L2(7)}$) induces the required extension. ☐

6.5. **The group** $(\mathrm{PSL}_2(11) \times \mathbf{M}_{12}){:}2$.

Theorem 6.6. *Let $L_E = \{g_{2AAA}, x_{11}\}$ and $L_N = \{g_{11}, h_{11}, n\}$ as in the accompanying code. Then $E \cong \mathrm{PSL}_2(11)$ and L_N generates the maximal subgroup $N_{\mathbb{M}}(E) \cong (\mathrm{PSL}_2(11) \times \mathrm{M}_{12}){:}2$.*

Proof. By the Atlas, maximal subgroups $(\mathrm{PSL}_2(11) \times \mathrm{M}_{12}){:}2$ are the normalisers of subgroups isomorphic to $\mathrm{PSL}_2(11)$ containing elements of the $\mathbb{M}$-classes 2A, 3A, and 5A. We confirm a presentation and conjugacy class fusion for E, so $N_{\mathbb{M}}(E) \cong (\mathrm{PSL}_2(11) \times \mathrm{M}_{12}){:}2$; specifically, we show that E contains a subgroup A_5 of type AAA generated by g_{2AAA} and $g_{3AAA} = g_{2AAA}x_{11}g_{2AAA}x_{11}^3$. We also confirm that $u = g_{11}g_{2AAA}$ and $v = h_{11}x_{11}^{-1}$ both centralise g_{2AAA} and x_{11} (note that g_{2AAA} and x_{11} clearly both belong to the normaliser) while n commutes with the former and inverts the latter. We compute that $g_{2AAA} = g_{11}^3$ and $x_{11} = h_{11}^2$, so $E \leqslant \langle L_N \rangle$. Next, we use Atlas information to deduce that u and v generate the subgroup $C_{\mathbb{M}}(\langle g_{2AAA}, x_{11} \rangle) \cong \mathrm{M}_{12}$: the latter has three maximal subgroups of order divisible by 11, but none of these has elements of order 10, whereas vu and $(vu)^4uvu^2$ have orders 11 and 10, respectively. Thus, $E \times C_{\mathbb{M}}(E) \leqslant \langle L_N \rangle$. Elements of order 11 in $\mathrm{PSL}_2(11)$ have normalisers 11:5, whereas n inverts x_{11}. This proves that $\langle L_N \rangle \cong (\mathrm{PSL}_2(11) \times \mathrm{M}_{12}){:}2$. ☐

6.6. **The group** $\big(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2\big){:}2$.

Theorem 6.7. *Let $L_U = \{g_{2AAA}, g_{3AAA}\}$, σ, and $L_N = \{g_{A7}, h_{A7}\}$ as in in accompanying code. Then L_N generates the maximal subgroup $N_{\mathbb{M}}(E) \cong \big(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2\big){:}2$ where $E = \langle U, U^\sigma \rangle$ with $U = \langle L_U \rangle \cong \mathrm{A}_5$.*

Proof. By the Atlas, normalisers of the direct product of two A_5s of type AAA are exactly the maximal subgroups $\big(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2\big){:}2$. We verify that $E \cong \mathrm{A}_5 \times \mathrm{A}_5$ has the required type and that $L_N \subset N_{\mathbb{M}}(E) \cong \big(\mathrm{A}_7 \times (\mathrm{A}_5 \times \mathrm{A}_5){:}2^2\big){:}2$. We confirm that $E \leqslant \langle L_N \rangle$ by showing that σ and the elements of L_U can be written as words in the elements of L_N. Moreover, we verify that $x = (h_{A7}g_{A7}h_{A7}g_{A7}^2)^6$ and $y = (g_{A7}h_{A7}^3)^4$ commute with the generators of E and satisfy the presentation for A_7 given in [4, §6.4]; thus, $\langle L_N \rangle$ contains $P \cong \mathrm{A}_7 \times \mathrm{A}_5 \times \mathrm{A}_5$. We now show that $a = (h_{A7}g_{A7}^6)^{15}$ commutes with g_{2AAA}, g_{2AAA}^σ and g_{3AAA}^σ, while $g_{3AAA}^a \in U$. On the other hand, a cannot belong to P: since all elements thereof which centralise g_{2AAA} are the product of an element of $C_U(g_{2AAA})$ and an element centralising U, it would otherwise hold that g_{3AAA}^a lies in the $C_U(g_{2AAA})$-class of g_{3AAA}, which we disprove by an enumeration. Thus, P has index at least 2 in $\langle P, a \rangle \leqslant \langle L_N \rangle$. The analogous argument with a^σ yields a subgroup of $\langle L_N \rangle$ in which P has index at least 4. All generators so far normalise U and U^σ, so noting that σ swaps these factors produces a subgroup of $\langle L_N \rangle$ in which P has index at least 8. This proves that $\langle L_N \rangle = N_{\mathbb{M}}(E)$. ☐

6.7. **The group** $\mathbf{M}_{11} \times \mathrm{A}_6{\cdot}2^2$.

Theorem 6.8. *Let* $L_E = \{g_{2AAA}, y_4\}$ *and* $L_N = \{g_{M11}, h_{M11}\}$ *be as in the accompanying code. Then* L_N *generates the maximal subgroup* $N_{\mathbb{M}}(E) \cong \mathrm{M}_{11} \times \mathrm{A}_6{\cdot}2^2$ *of* $\mathbb{M}$.

Proof. By the Atlas, the maximal subgroup $\mathrm{M}_{11} \times \mathrm{A}_6{\cdot}2^2$ is the normaliser of a subgroup M_{11} containing 2A, 3A, and 5A elements. We check a presentation to prove $E \cong \mathrm{M}_{11}$. The group E contains a subgroup A_5 of type AAA meeting the correct conjugacy classes since the relevant generators defined in Lemma 4.1 can be written as words in elements of L_E. Since $u = g_{M11}y_4^{-1}g_{2AAA}$ and $v = h_{M11}y_4^{-1}(g_{2AAA}y_4)^{-3}$ centralise E, it follows that $L_N \leqslant N_{\mathbb{M}}(E)$, and $y_4 = g_{M11}^{30}h_{M11}^{-10}$ and $g_{2AAA} = g_{M11}^{-10}y_4^{-1}$ confirms that $E \leqslant \langle L_N \rangle$. Lastly, we confirm that $\langle v, u\rangle$ has size $|\mathrm{A}_5{\cdot}2^2|$, and therefore $\langle L_N \rangle = N_{\mathbb{M}}(E)$. □

6.8. **The group** $(\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5){:}\mathrm{S}_3$.

Theorem 6.9. *Let* $L_U = \{x_{S5}, y_{S5}\}$*,* t*, and* $L_N = \{g_{S5}, h_{S5}\}$ *as in in accompanying code. Then* L_N *generates the maximal subgroup* $N_{\mathbb{M}}(E) \cong (\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5){:}\mathrm{S}_3$ *where* $E = \langle U, U^t, U^{t^2}\rangle$ *with* $U = \langle L_U \rangle \cong \mathrm{S}_5$.

Proof. By the Atlas, maximal subgroups $(\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5){:}\mathrm{S}_3$ of $\mathbb{M}$ are the normalisers of direct products of three subgroups S_5 containing 2A, 3A, and 5A elements. We test that U has order 120 and that L_E satisfies a presentation for S_5, so $U \cong \mathrm{S}_5$. The fact that $g_{2AAA} = (x_{S5}y_{S5}^2)^2x_{S5}y_{S5}x_{S5}$ and $g_{3AAA} = x_{S5}y_{S5}x_{S5}y_{S5}^3$ guarantees that U meets the correct conjugacy classes. We verify that U, U^t, and U^{t^2} commute and intersect trivially (since the factors have trivial centre), so $E \cong \mathrm{S}_5{\times}\mathrm{S}_5{\times}\mathrm{S}_5$, as required. A direct computation also shows that $L_N \leqslant N_{\mathbb{M}}(E)$, and we verify $E \leqslant \langle L_N\rangle$ by writing the generators in words of elements of L_N. An explicit computation also shows that h_{S5} centralises $U = \langle x_{S5}, y_{S5}\rangle$ and interchanges U^t, U^{t^2} via conjugation, while conjugation by t permutes $\{U, U^t, U^{t^2}\}$ cyclically. This allows us to deduce that $\langle L_N\rangle \cong (\mathrm{S}_5 \times \mathrm{S}_5 \times \mathrm{S}_5){:}\mathrm{S}_3$. □

6.9. **The group** $(\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11)){:}4$. This is adapted from [19, §3].

Theorem 6.10. *Let* $L_U = \{g_{2AAA}, x_{11}\}$ *and* $L_N = \{g_{2AAA}, x_{11}, x_4\}$ *be as in the accompanying code. Then* L_N *generates a maximal subgroup* $N_{\mathbb{M}}(E) = (\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11)){:}4$ *where* $E = \langle U, U^{x_4}\rangle$ *with* $U = \langle L_U\rangle \cong \mathrm{PSL}_2(11)$.

Proof. According to the Atlas, the maximal subgroups $(\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11)){:}4$ of $\mathbb{M}$ are the normalisers of direct products $\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11)$ in which the elements of order 2, 3, and 5 in each factor belong to classes 2A, 3A, and 5A. It was shown in the proof of Theorem 6.6 that $U \cong \mathrm{PSL}_2(11)$ meets the correct conjugacy classes. We verify that U and U^{x_4} commute, so $E = \mathrm{PSL}_2(11){\times}\mathrm{PSL}_2(11)$ and $N_{\mathbb{M}}(E) \cong (\mathrm{PSL}_2(11) \times \mathrm{PSL}_2(11)){:}4$. Since x_4^2 centralises g_{2AAA} and inverts x_{11}, we deduce that $L_N \subset N_{\mathbb{M}}(E)$. As noted in the proof of Theorem 6.6, every automorphism of U that inverts x_{11} is an outer automorphism. Therefore $x_4^2 \notin E$, and the claim follows. □

6.10. **The group** $\mathrm{U}_3(4){:}4$. This is adapted from [5, §6].

Theorem 6.11. *Let* $L_N = \{j_2, a_{12}, g_{3BCB}\}$ *be as in the accompanying code. Then* L_N *generates a maximal subgroup* $\mathrm{U}_3(4){:}4$.

Proof. Per [5, §6], the maximal subgroups of $\mathbb{M}$ with the desired shape are the normalisers of subgroups $\mathrm{U}_3(4)$ containing A_5s of type BCB (as opposed to a second class, identified by Norton [15, Table 5], that contain subgroups A_5 of type BCA). We verify that $E = \langle j_2, g_{3BCB}\rangle \cong \mathrm{U}_3(4)$ by verifying a presentation. The presence of a subgroup $\langle g_{2BCB}, g_{3BCB}\rangle \cong \mathrm{A}_5$ of type BCB is established by verifying that we can write the relevant generators in Lemma 4.1 as words in elements of L_N. An explicit calculation in the accompanying code confirm that $L_N \leqslant N_{\mathbb{M}}(E)$. Now the claim follows since a_{12} has order 12 but $\mathrm{U}_3(4)$ has no elements of order 12 or 6. □

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