ARITHMETIC UNIQUE ERGODICITY FOR INFINITE DIMENSIONAL FLAT BUNDLES

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ABSTRACT. In this paper, we prove a uniform version of quantum unique ergodicity for high-frequency eigensections of a certain series of unitary flat bundles over arithmetic surfaces.

1. INTRODUCTION

1.1. Backgrounds. Shnirelman's quantum ergodicity (QE) [18] states that on a compact Riemannian manifold whose geodesic flow is ergodic with respect to the *Liouville* measure, the eigenfunctions of the Laplacian tend to be equidistributed except a density zero subsequence.

The quantum unique ergodicity conjecture (QUE) of Rudnick-Sarnak [15] asserts that for Riemannian manifolds of negative sectional curvature, all eigenfunctions equidistribute. Lindenstrauss [9] solved the QUE for arithmetic surfaces (AQUE), which carry additional symmetries called *Hecke operators*. These operators commute with each other and with the Laplacian, so it is natural to consider the *joint* eigenfunctions of the Laplacian and all Hecke operators, called *Hecke-Maass forms*.

In [10, Theorems 1.3, 4.9], Ma-Ma established a *uniform version* of QE for a certain series of unitary flat bundles (UQE). Later, a similar but weaker, non-uniform version of QE was also obtained by Cekić-Thibault [6, Theorem 5.1.7]. In this paper, we prove the corresponding uniform QUE (UQUE) for arithmetic surfaces (AUQUE). Now we explain in detail.

1.2. Main results.

1.2.1. AQUE. To begin with, we consider the AQUE concerning a single flat bundle.

Let L be a totally real number field with $L \neq \mathbb{Q}$, and Γ a congruence subgroup derived from an \mathbb{R} -split quaternion algebra over L. Let $X = \Gamma \setminus \mathbb{H}^2$ be the associated hyperbolic surface with volume form dv_X .

Let ρ be a *nonidentity real* place of L, then it extends to a representation of the fundamental group $\pi_1(X)$ of X,

$$\rho \colon \pi_1(X) \cong \Gamma \to \mathrm{SU}_2. \tag{1.1}$$

Note that SU_2 acts unitarily on the complex linear space $(\mathbb{C}^2, h^{\mathbb{C}^2})$ as well as its *p*-th symmetric power $(Sym^p(\mathbb{C}^2), h^{Sym^p(\mathbb{C}^2)})$ for any $p \in \mathbb{N}$, and in fact, these are all irreducible representations of SU_2 . Then by regarding ρ in (1.1) as the holonomy of a flat principal SU_2 -bundle over X, we can construct a series of unitary flat vector bundles $\{F_p\}_{p\in\mathbb{N}}$ over X by

$$F_p = \Gamma \setminus \left(\mathbb{H}^2 \times \operatorname{Sym}^p(\mathbb{C}^2) \right)$$

= $\left\{ (x, v) \in \mathbb{H}^2 \times \operatorname{Sym}^p(\mathbb{C}^2) \right\} / \left((x, v) \sim (\gamma x, \rho(\gamma) v) \text{ for any } \gamma \in \Gamma \right).$ (1.2)

Let $\mathscr{C}^{\infty}(\mathbb{H}^2, \operatorname{Sym}^p(\mathbb{C}^2))$ be the space of smooth $\operatorname{Sym}^p(\mathbb{C}^2)$ -valued functions on \mathbb{H}^2 , and we have a Γ -action on it such that for any $\gamma \in \Gamma$ and $u \in \mathscr{C}^{\infty}(\mathbb{H}^2, \operatorname{Sym}^p(\mathbb{C}^2))$,

$$\gamma(u)(x) = \rho(\gamma)u(\gamma^{-1}x), \qquad (1.3)$$

then it follows from (1.2) that the space $\mathscr{C}^{\infty}(X, F_p)$ of smooth section of F_p on X is isomorphic to the Γ -invariant subspace of $\mathscr{C}^{\infty}(\mathbb{H}^2, \operatorname{Sym}^p(\mathbb{C}^2))$, that is,

$$\mathscr{C}^{\infty}(X, F_p) \cong \mathscr{C}^{\infty}_{\Gamma} (\mathbb{H}^2, \operatorname{Sym}^p(\mathbb{C}^2)).$$
(1.4)

Therefore, naturally we can define a flat connection ∇^{F_p} and a Hermitian metric h^{F_p} on F_p , using the derivative d on \mathbb{H}^2 and $h^{\operatorname{Sym}^p(\mathbb{C}^2)}$. Let $\langle \cdot, \cdot \rangle_{L^2(X,F_p)}$ be the L^2 -metric on $\mathscr{C}^{\infty}(X,F_p)$ induced by (h^{F_p}, dv_X) .

Let $\Delta^{\dot{F}_p}$ be the nonnegative Laplacian acting on $\mathscr{C}^{\infty}(X, F_p)$, which is also referred to as the *Schrödinger-Pauli* spin p/2 operator, and by (1.4), for local coordinates (x_1, x_2) of \mathbb{H}^2 , we have

$$\Delta^{F_p} = -x_2^2 \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \Big).$$
(1.5)

We list all the eigenvalues $0 \leq \lambda_{p,1} \leq \lambda_{p,2} \leq \cdots$ of Δ^{F_p} counted with multiplicity and associated orthonormal eigensections

$$\Delta^{F_p} u_{p,j} = \lambda_{p,j} u_{p,j}, \quad |u_{p,j}|_{L^2(X,F_p)} = 1.$$
(1.6)

There is an extra family of operators acting on $\mathscr{C}^{\infty}(X, F_p)$ called *Hecke operators*, see § 4.3. They commute with each other and Δ^{F_p} , therefore, we now assume further that each $u_{p,j}$ in (1.6) is also eigensections of all Hecke operators.

To simplify the notation, we write an additional overline on a trace operator or a measure to signify the normalized one in our subsequent discussions, for example,

$$\overline{\mathrm{Tr}}^{F_p} = \frac{1}{\dim F_p} \mathrm{Tr}^{F_p}, \quad d\overline{v}_X = \frac{1}{\mathrm{Vol}(X)} dv_X.$$
(1.7)

Now we state the AQUE for a flat vector bundle, and see Theorem 6.4 for a full version with momentum variables.

Theorem 1.1. For any $p \in \mathbb{N}$ and $A \in \mathscr{C}^{\infty}(X, \operatorname{End}(F_p))$, we have

$$\lim_{j \to \infty} \langle A u_{p,j}, u_{p,j} \rangle_{L^2(X, F_p)} = \int_X \overline{\mathrm{Tr}}^{F_p} [A] d\overline{v}_X.$$
(1.8)

Theorem 1.1 asserts that for each of $\{F_p\}_{p\in\mathbb{N}}$, high-frequency eigensections have the equidistribution property, then it is natural to ask about the *uniformity* of (1.8) concerning $p \in \mathbb{N}$. However, this question is subtle, as establishing uniformity requires defining a proper meaning for $\bigcap_{p\in\mathbb{N}} \mathscr{C}^{\infty}(X, \operatorname{End}(F_p))$ to test sections of different bundles. The first candidate, $\mathscr{C}^{\infty}(X) \subset \bigcap_{p\in\mathbb{N}} \mathscr{C}^{\infty}(X, \operatorname{End}(F_p))$, proves too limited because it can not capture the rich information of eigensections along the fiber. Instead, we shall consider a larger set capable of encompassing information about eigensections along the fiber. We now treat this in detail.

1.2.2. AUQUE. Let us begin by giving a geometric interpretation of $u_{p,j}$.

We form the 3-sphere

$$\mathbb{S}^3 = \{ z = (z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1 \}$$
(1.9)

and let $dv_{\mathbb{S}^3}$ be the Haar measure on \mathbb{S}^3 . Let $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^3)}$ be the L^2 -metric on $\mathscr{C}^{\infty}(\mathbb{S}^3)$ induced by $dv_{\mathbb{S}^3}$. Moreover, for any $g \in \mathrm{SU}_2, \alpha \in \mathscr{C}^{\infty}(\mathbb{S}^3)$, we define $gs \in \mathscr{C}^{\infty}(\mathbb{S}^3)$ by

$$(g\alpha)(z_0, z_1) = \alpha(g^{-1} \cdot (z_0, z_1)).$$
(1.10)

Now we view

$$\operatorname{Sym}^{p}(\mathbb{C}^{2}) \subset \mathscr{C}^{\infty}(\mathbb{S}^{3}), \tag{1.11}$$

as the space of homogeneous polynomial functions of degree p or

$$\operatorname{Sym}^{p}(\mathbb{C}^{2}) = \{ a_{0} z_{0}^{p} + a_{1} z_{0}^{p-1} z_{1} + \dots + a_{p} z_{1}^{p} \mid a_{0}, \dots, a_{p} \in \mathbb{C} \}.$$
(1.12)

Then $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^3)}$ restricts to the metric $h^{\operatorname{Sym}^p(\mathbb{C}^2)}$ on $\operatorname{Sym}^p(\mathbb{C}^2)$, and (1.10) restricts to the action of g on $\operatorname{Sym}^p(\mathbb{C}^2)$.

For any $u_{p,j} \in \mathscr{C}^{\infty}(X, F_p)$ given in (1.6), using (1.4), it lifts to a Γ -invariant Sym^{*p*}(\mathbb{C}^2)-valued function on \mathbb{H}^2 , and by (1.12), we can further regard $u_{p,j} \in \mathscr{C}^{\infty}(\mathbb{H}^2 \times \mathbb{S}^3)$ by

$$u_{p,j}\colon (x,z)\in \mathbb{H}^2\times \mathbb{S}^3\mapsto (u_{p,j}(x))(z), \tag{1.13}$$

By (1.10) and (1.13), and since $u_{p,j}$ is invariant with respect to the Γ -action given in (1.3), we have

$$u_{p,j}(\gamma x, \rho(\gamma)z) = \left(u_{p,j}(\gamma x)\right)(\rho(\gamma)z) = \left(\rho(\gamma)u_{p,j}(x)\right)\left(\rho(\gamma)z\right) = u_{p,j}(x)(z)$$
(1.14)

where $\rho(\gamma) \in SU_2$ acts naturally on $z \in S^3$. Therefore, if we define a flat S^3 -bundle \mathcal{M} over X by

$$\mathcal{M} = \Gamma \setminus \left(\mathbb{H}^2 \times \mathbb{S}^3 \right) = \{ (x, z) \in \mathbb{H}^2 \times \mathbb{S}^3 \} / ((x, z) \sim (\gamma x, \rho(\gamma) z) \text{ for any } \gamma \in \Gamma),$$
(1.15)

then $u_{p,j}$ passes to a smooth function in $\mathscr{C}^{\infty}(\mathscr{M})$. Let $dv_{\mathscr{M}}(x,z)$ be the measure on \mathscr{M} locally given by $dv_X(x)dv_{\mathbb{S}^3}(z)$, or equivalently, $dv_{\mathbb{H}^2}dv_{\mathbb{S}^3}$ is a volume form on $\mathbb{H}^2 \times \mathbb{S}^3$, which is invariant with respect to the diagonal Γ -action as in (1.15), so it passes to the volume form $dv_{\mathscr{M}}(x,z)$. Let $|\cdot|_{\mathbb{C}}$ be the modulus on \mathbb{C} . We end up with probability measures

$$\left\{ \left| u_{p,j} \right|_{\mathbb{C}}^{2} dv_{\mathscr{M}} \right\}_{p,j \in \mathbb{N}}$$

$$(1.16)$$

on \mathcal{M} . Now we can state the AUQUE, which gives the equidistribution property of the above measures, in a *uniform* manner with respect to $p \in \mathbb{N}$, and see Theorem 6.5 for a full version with momentum variables.

Theorem 1.2. For any $\mathscr{A} \in \mathscr{C}^{\infty}(\mathscr{M})$, we have

$$\lim_{\lambda \to \infty} \sup_{\substack{(p,j) \in \mathbb{N}^2, \\ \lambda_{p,j} \ge \lambda}} \left| \int_{\mathscr{M}} \mathscr{A} \left| u_{p,j} \right|_{\mathbb{C}}^2 dv_{\mathscr{M}} - \int_{\mathscr{M}} \mathscr{A} d\overline{v}_{\mathscr{M}} \right| = 0.$$
(1.17)

Theorem 1.2 suggests that $|u_{p,j}|_{\mathbb{C}}^2 dv_{\mathscr{M}}$ tends to be equidistributed as long as its corresponding eigenvalue $\lambda_{p,j}$ is large, regardless of $p \in \mathbb{N}$. We emphasize that it is this uniformity that makes Theorem 1.2 particularly noteworthy. By combining (1.8) and

[10, Remark 4.10], we can easily obtain a weaker non-uniform version of (1.17), that is, for any $p \in \mathbb{N}$,

$$\lim_{j \to \infty} \int_{\mathscr{M}} \mathscr{A} |u_{p,j}|_{\mathbb{C}}^{2} dv_{\mathscr{M}} = \int_{\mathscr{M}} \mathscr{A} d\overline{v}_{\mathscr{M}}.$$
(1.18)

However, (1.17) cannot be derived directly from (1.8), and these cases must be treated independently.

1.3. Main technique. The proofs of Theorems 1.1 and 1.2 follow Lindenstrauss's approach [9] for the AQUE conjecture and involve three steps, geodesic invariance, Hecke recurrence and strong positive entropy. It should be emphasized that in [9], the congruence lattice is over \mathbb{Q} , whereas we require a congruence lattice over a *totally real number field* $L \neq \mathbb{Q}$. Therefore, we should also follow Shem-Tov-Silberman [16], where they consider the QUE for

$$\Gamma_{m,n} \setminus \Big(\underbrace{\mathbb{H}^2 \times \cdots \times \mathbb{H}^2}_{m} \times \underbrace{\mathbb{H}^3 \times \cdots \times \mathbb{H}^3}_{n} \Big), \tag{1.19}$$

in which $m + n \ge 1$ and $\Gamma_{m,n} \subset \mathrm{SL}_2(\mathbb{R})^m \times \mathrm{SL}_2(\mathbb{C})^n$ is a congruence lattice over a general number field. The main focus in [16] is on dealing with extra difficulties posed by the \mathbb{H}^3 part, so we only need the relatively simpler \mathbb{H}^2 part. Additionally, we remark that Theorem 1.2 can be interpreted as the QUE of a partial Laplacian, which shares a similar form with Brooks-Lindenstrauss [5, Theorem 1.5], and see § 6.2 for more details.

Let us explain by providing a formal argument for Theorem 1.1, and Theorem 1.2 can be considered as an *infinite-dimensional extension*.

1.3.1. Finite dimensional case. Consider the unit tangent bundle U(X) of X given by

$$U(X) \cong \Gamma \backslash \mathrm{SL}_2(\mathbb{R}), \tag{1.20}$$

together with a natural projection map $\pi: U(X) \to X$ and the *Liouville volume form* $dv_{U(X)}$, passing from the Γ -quotient of the Haar volume form $dv_{\mathrm{SL}_2(\mathbb{R})}$ on $\mathrm{SL}_2(\mathbb{R})$. Let us fix a $p \in \mathbb{N}$, and put

$$\pi^* F_p = \Gamma \backslash (\mathrm{SL}_2(\mathbb{R}) \times \mathrm{Sym}^p(\mathbb{C}^2)), \qquad (1.21)$$

the pull-back flat vector bundle of F_p over U(X).

The geodesic flow (g_t) on U(X) is induced by the right action of

$$g_t = \begin{pmatrix} e^{t/2} & 0\\ 0 & e^{-t/2} \end{pmatrix}$$
(1.22)

on $SL_2(\mathbb{R})$. Using semiclassical analysis, any weak star limit L_X of functionals

$$\{A \in \mathscr{C}^{\infty}(X, \operatorname{End}(F_p)) \mapsto \langle Au_{p,j}, u_{p,j} \rangle_{L^2(X, F_p)} \}_{j \in \mathbb{N}}$$
(1.23)

when $\lambda_{p,j} \to \infty$ can be lifted to a *geodesic invariant* functional $L_{U(X)}$, acting on $\mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p))$. We shall derive (1.8) by proving for any $A \in \mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p))$,

$$L_{U(X)}(A) = \int_{U(X)} \overline{\mathrm{Tr}}^{\pi^* F_p}[A] d\overline{v}_{U(X)}.$$
(1.24)

To get (1.24), a key simplification is that due to the denseness of $\rho(\Gamma) \subset SU_2$, it is sufficient to show that the restriction of $L_{U(X)}$ to $\mathscr{C}^{\infty}(U(X)) \cdot \mathrm{Id}_{F_p}$ is the normalized Liouville measure $d\overline{v}_{U(X)}$. In other words, we shall prove that for any $f \in \mathscr{C}^{\infty}(U(X))$,

$$L_{U(X)}(f \cdot \mathrm{Id}_{F_p}) = \int_{U(X)} f d\overline{v}_{U(X)}.$$
(1.25)

Let Δ^X be the Laplacian acting on $\mathscr{C}^{\infty}(X)$. Since $\mathscr{C}^{\infty}(X)$ is also equipped with Hecke operator commutes with each other and Δ^X , we can let $\{u_j\}_{j\in\mathbb{N}}$ be orthonormal eigenfunctions of both Δ^X and the Hecke operators. We admit the fact that the restriction of $L_{U(X)}$ to $\mathscr{C}^{\infty}(U(X)) \cdot \mathrm{Id}_{F_p}$ shares properties similar to the lift of a weak star limit of $\{|u_j|^2 dv_X\}_{j\in\mathbb{N}}$. In particular, since the Hecke operators on $\mathscr{C}^{\infty}(U(X), \pi^*F_p)$ and $\mathscr{C}^{\infty}(U(X))$ are compatible in some sense, following Bourgain-Lindenstrauss [3, §3], Lindenstrauss [9, §8], Shem-Tov-Silberman [16] and Silberman-Venkatesh [17], we can prove Hecke recurrence and strong positive entropy respectively, from which we deduce (1.25) by the measure rigidity [9, Theorem 1.1].

1.3.2. Infinite dimensional case. We form the pull back \mathbb{S}^3 -bundle $\pi^* \mathscr{M}$ over U(X) by

$$\pi^* \mathscr{M} \cong \Gamma \backslash (\mathrm{SL}_2(\mathbb{R}) \times \mathbb{S}^3) \tag{1.26}$$

with a volume form $dv_{\pi^*\mathscr{M}}$ locally given by $dv_{U(X)}dv_{\mathbb{S}^3}$, passing from the Γ -quotient of $dv_{\mathrm{SL}_2(\mathbb{R})}dv_{\mathbb{S}^3}$.

For any weak star limit $\mu_{\mathscr{M}}$ of measures $\{|u_{p,j}|_{\mathbb{C}}^2 dv_{\mathscr{M}}\}_{p,j\in\mathbb{N}}$ on \mathscr{M} given in (1.16) when $\lambda_{p,j} \to \infty$, we can construct its microlocal lift measure $\mu_{\pi^*\mathscr{M}}$ on $\pi^*\mathscr{M}$, which is invariant with respect to the *horizontal geodesic flow* on $\pi^*\mathscr{M}$, denoted by (g_t) , defined by the right action of g_t given in (1.22) on the $\mathrm{SL}_2(\mathbb{R})$ component of $\pi^*\mathscr{M}$, that is,

$$\boldsymbol{g}_t \colon (\boldsymbol{y}, \boldsymbol{z}) \in \pi^* \mathscr{M} \to (\boldsymbol{y} \boldsymbol{g}_t, \boldsymbol{z}) \in \pi^* \mathscr{M}.$$
(1.27)

Compared with the semiclassical analysis on a single vector bundle in § 1.3.1, the main difference here is that p is not fixed, we shall be careful about uniformity for an infinite number of bundles. We achieve this by carrying on analysis on an infinite dimensional vector bundle using (1.11). We follow the representation-theoretic construction of Wolpert [21] and Lindenstrauss [8], which is an alternative to the original pseudo-differential approach of Schnirelman [18], Colin de Verdiére [7] and Zelditch [22].

We shall show that

$$\mu_{\pi^*\mathscr{M}} = d\overline{v}_{\pi^*\mathscr{M}}.\tag{1.28}$$

Similar to (1.24), to establish (1.28), it is sufficient to check that the restriction of $\mu_{\pi^*\mathscr{M}}$ to $\mathscr{C}^{\infty}(U(X))$ is $d\overline{v}_{U(X)}$, or for any $f \in \mathscr{C}^{\infty}(U(X))$,

$$\mu_{\pi^*\mathscr{M}}(f) = \int_{U(X)} f d\overline{v}_{U(X)}.$$
(1.29)

This can also be accomplished by proving Hecke recurrence and strong positive entropy.

1.4. Mixed quantization. The construction of the microlocal lift $\mu_{\pi^*\mathscr{M}}$ fits into the general *mixed quantization* framework studied in Ma-Ma [10, § 4], which was inspired by the asymptotic analytic torsions of Bismut-Ma-Zhang [1, 2]. For more asymptotic torsion results, see also Puchol [14] and Ma [11]. The mixed quantization is the composition of the following maps

$$\mathscr{C}^{\infty}(\pi^*\mathscr{M}) \xrightarrow{T_{p,\cdot}} \cap_{p \in \mathbb{N}} \mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p)) \xrightarrow{\operatorname{Op}_h(\cdot)} \cap_{p \in \mathbb{N}} \operatorname{End}(L^2(X, F_p)), \quad (1.30)$$

where $T_{p,\cdot}$ is the *Berezin-Toeplitz quantization* along the fiber S³, regulating the behavior of an infinite number of linear spaces, and Op_h is the *Weyl quantization* along the base manifold X, governing high-frequency eigensections. Combining these quantizations enables simultaneous control of the high-frequency eigensections of an infinite number of bundles. For more details on Berezin-Toeplitz and Weyl quantizations, we refer to Ma-Marinescu [12, § 7] and Zworski [23, § 4] respectively. The measures in (1.16) corresponds to $T_{p,\cdot}$, and the microlocal lift $\mu_{\pi^*\mathscr{M}}$ corresponds to $Op_h(\cdot)$. More precisely, $\mu_{\pi^*\mathscr{M}}$ is a weak-star limit of the following functionals

$$\left\{\mathscr{A}\in\mathscr{C}^{\infty}(\pi^{*}\mathscr{M})\mapsto\left\langle\operatorname{Op}_{\lambda_{p,j}^{-1/2}}(T_{\mathscr{A},p})u_{p,j},u_{p,j}\right\rangle_{L^{2}(X,F_{p})}\right\}_{p,j},$$
(1.31)

when $\lambda_{p,j} \to \infty$. Finally, we note that $T_{p,\cdot}$ in (1.30) is an abuse of notation, it is in fact along $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{CP}^1$ rather than \mathbb{S}^3 .

1.5. Organization of the paper. This paper is organized as follows. In § 2, we reinterpret the setup of Theorem 1.2 in an infinite dimensional bundle \mathscr{F} over X. In § 3, for the Laplacian $\Delta^{\mathscr{F}}$ acting on $\mathscr{C}^{\infty}(X, \mathscr{F})$, we construct the microlocal lift of its high frequency eigensections. In § 4, we define Hecke operators on \mathscr{F} . In § 5, we prove Hecke recurrence and strong positive entropy for the microlocal lifts. In § 6, we finish the proof by applying the measure rigidity theorem.

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2. Preliminaries

2.1. An infinite dimensional bundle. Recall the definition of \mathscr{M} in (1.15), then for $u \in \mathscr{C}^{\infty}(\mathscr{M})$, we regard it as a Γ -invariant function $u \in \mathscr{C}^{\infty}(\mathbb{H}^2 \times \mathbb{S}^3)$, that is,

$$u(\gamma x, \rho(\gamma)z) = u(x, z). \tag{2.1}$$

Now we view $u \in \mathscr{C}^{\infty}(\mathbb{H}^2, \mathscr{C}^{\infty}(\mathbb{S}^3))$ by $u(x) = u(x, \cdot) \in \mathscr{C}^{\infty}(\mathbb{S}^3)$, then from (2.1), u is Γ -invariant with respect to the action

$$\gamma(u)(x) = \rho(\gamma)u(\gamma^{-1}x), \qquad (2.2)$$

where $\rho(\gamma)$ acts on $u(\gamma^{-1}x)$ through (1.10). To verify this, we compute that

$$\left(\gamma \cdot u(x)\right)(z) = u(x)\left(\rho(\gamma)^{-1}z\right) = u\left(x,\rho(\gamma)^{-1}z\right) = u(\gamma x,z) = u(\gamma x)(z).$$
(2.3)

Therefore, let $\mathscr{C}^{\infty}_{\Gamma}(\mathbb{H}^2, \mathscr{C}^{\infty}(\mathbb{S}^3)) \subset \mathscr{C}^{\infty}(\mathbb{H}^2, \mathscr{C}^{\infty}(\mathbb{S}^3))$ denote the Γ -invariant subspace, then analogous to (1.4), we have the following isomorphism

$$\mathscr{C}^{\infty}(\mathscr{M}) \cong \mathscr{C}^{\infty}_{\Gamma} (\mathbb{H}^2, \mathscr{C}^{\infty}(\mathbb{S}^3)).$$
(2.4)

We form an infinite dimensional vector bundle \mathscr{F} over X by

$$\mathscr{F} = \Gamma \setminus \left(\mathbb{H}^2 \times \mathscr{C}^{\infty}(\mathbb{S}^3) \right)$$

= $\left\{ (x, \alpha) \in \mathbb{H}^2 \times \mathscr{C}^{\infty}(\mathbb{S}^3) \right\} / ((x, \alpha) \sim (\gamma x, \rho(\gamma) \alpha) \text{ for any } \gamma \in \Gamma),$ (2.5)

we get from (2.4)

$$\mathscr{C}^{\infty}(\mathscr{M}) \cong \mathscr{C}^{\infty}(X, \mathscr{F}).$$
(2.6)

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Let $\pi^*\mathscr{F}$ be the pull back bundle of \mathscr{F} over $\pi^*\mathscr{M}$. Similar to (2.6), we have

$$\mathscr{C}^{\infty}(\pi^*\mathscr{M}) \cong \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F}).$$
(2.7)

Now we can summarize all the geometric objects in the following diagram

2.2. Geometric data. Let $h^{\mathscr{F}}$ be the metric on \mathscr{F} induced by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^3)}$ and $\nabla^{\mathscr{F}}$ the natural flat connection on \mathscr{F} , then $h^{\mathscr{F}}$ is parallel with respect to \mathscr{F} . Let $\langle \cdot, \cdot \rangle_{L^2(X,\mathscr{F})}$ be the L^2 -metric on $\mathscr{C}^{\infty}(X, \mathscr{F})$ induced by $(h^{\mathscr{F}}, dv_X)$, and let $\langle \cdot, \cdot \rangle_{L^2(\mathscr{M})}$

Let $\langle \cdot, \cdot \rangle_{L^2(X,\mathscr{F})}$ be the L^2 -metric on $\mathscr{C}^{\infty}(X,\mathscr{F})$ induced by $(h^{\mathscr{F}}, dv_X)$, and let $\langle \cdot, \cdot \rangle_{L^2(\mathscr{M})}$ be the L^2 -metric on $\mathscr{C}^{\infty}(\mathscr{M})$ induced by $dv_{\mathscr{M}}$, then we have

$$\langle \cdot, \cdot \rangle_{L^2(X,\mathscr{F})} = \langle \cdot, \cdot \rangle_{L^2(\mathscr{M})}.$$
(2.9)

To see this, for any $u \in \mathscr{C}^{\infty}(X, \mathscr{F})$, using the two ways in (2.6) it is viewed, we can take pointwise norms with respect to \mathscr{F} and \mathbb{C} , obtaining smooth functions

$$|u|_{\pi^*\mathscr{F}}^2 \in \mathscr{C}^{\infty}(X), \quad |u|_{\mathbb{C}}^2 \in \mathscr{C}^{\infty}(\mathscr{M})$$
(2.10)

accordingly. We compute that locally

$$u|_{L^{2}(X,\mathscr{F})}^{2} = \int_{X} |u(x)|_{\mathscr{F}}^{2} dv_{X}(x) = \int_{X} \left(\int_{\mathbb{S}^{3}} |u(x,z)|^{2} dv_{\mathbb{S}^{3}}(z) \right) dv_{X}(x)$$

$$= \int_{\mathscr{M}} |u(x,z)|^{2} dv_{\mathscr{M}}(x,z) = |u|_{L^{2}(\mathscr{M})}^{2}.$$
 (2.11)

Similarly, from (2.7), for any $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$, by taking pointwise norms, we get

$$|s|^{2}_{\pi^{*}\mathscr{F}} \in \mathscr{C}^{\infty}(U(X)), \quad |s|^{2}_{\mathbb{C}} \in \mathscr{C}^{\infty}(\pi^{*}\mathscr{M})$$

$$(2.12)$$

accordingly, then calculate along (2.11), we get

$$\langle \cdot, \cdot \rangle_{L^2(U(X), \pi^*\mathscr{F})} = \langle \cdot, \cdot \rangle_{L^2(\pi^*\mathscr{M})}.$$
(2.13)

Let $\Delta^{\mathscr{F}}$ be the Laplacian acting on $\mathscr{C}^{\infty}(X,\mathscr{F})$, then for any $u \in \mathscr{C}^{\infty}(X,\mathscr{F})$, by viewing $u \in \mathscr{C}^{\infty}_{\Gamma}(\mathbb{H}^2, \mathscr{C}^{\infty}(\mathbb{S}^3))$ through (2.4), we have

$$\Delta^{\mathscr{F}}u = -x_2^2 \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)u. \tag{2.14}$$

From (1.2), (1.11), (2.5) and the Peter-Weyl theorem [4, Theorem 3.3.1], formally we have the following orthogonal decomposition

$$\mathscr{F} = \bigoplus_{p=0}^{\infty} \left(\underbrace{F_p \oplus \dots \oplus F_p}_{p+1} \right), \tag{2.15}$$

so $(\nabla^{\mathscr{F}}, h^{\mathscr{F}}, \Delta^{\mathscr{F}})$ restricts to $(\nabla^{F_p}, h^{F_p}, \Delta^{F_p})$. In the remainder of this paper, we shall place all discussions on the infinite-dimensional bundle \mathscr{F} , exploring the equidistribution property of eigensections of $\Delta^{\mathscr{F}}$.

This viewpoint offers the advantage that all the subsequent inequalities hold with constants that are *independent* of $p \in \mathbb{N}$, allowing for estimates that are uniform for $p \in \mathbb{N}$ as required in (1.17).

3. MICROLOCAL LIFTS OF EIGENSECTIONS

3.1. Universal enveloping algebra. Let $U(\mathfrak{sl}_2(\mathbb{C}))$ be the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, which can be viewed as differential operators acting on $\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$.

Now we list some operators for later use. Let E^+ and E^- be the raising and lowering operators defined by

$$E^{+} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E^{-} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad (3.1)$$

Let *H* be the derivative along (g_t) given in (1.22) and *W* a generator of the maximal compact subgroup $K = SO_2(\mathbb{R}) \subseteq SL_2(\mathbb{R})$, namely

$$H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
 (3.2)

Let Ω be the Casimir element defined as

$$\Omega = -\frac{1}{2} \left(E^+ E^- + E^- E^+ \right) + \frac{1}{4} W^2, \qquad (3.3)$$

which is in the center of $U(\mathfrak{sl}_2(\mathbb{C}))$ and agrees with $\Delta^{\mathscr{F}}$ on $\mathscr{C}^{\infty}(X, \mathscr{F})$.

3.2. Microlocal lift. Let $u_{\lambda} \in \mathscr{C}^{\infty}(X, \mathscr{F})$ be a normalized eigensection of $\Delta^{\mathscr{F}}$, called *Maass form*, which satisfies

$$\Delta^{\mathscr{F}} u_{\lambda} = \lambda u_{\lambda}, \quad |u_{\lambda}|_{L^{2}(X,\mathscr{F})} = 1, \tag{3.4}$$

and we set r > 0 by $\lambda = r^2 + \frac{1}{4}$. Our goal is to define a lifted section $s_{\lambda} \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ for each u_{λ} with the following result.

Theorem 3.1. Suppose that $\{u_{\lambda_i}\}_{i\in\mathbb{N}}$ is a sequence of Maass form with $\lim_i \lambda_i = \infty$, and $\{|u_{\lambda_i}|^2_{\mathbb{C}} dv_{\mathscr{M}}\}_{i\in\mathbb{N}}$ converges weak star to a measure $\mu_{\mathscr{M}}$ on \mathscr{M} . Then for any weak star limit $\mu_{\pi^*\mathscr{M}}$ of $\{|s_{\lambda_i}|^2_{\mathbb{C}} dv_{\pi^*\mathscr{M}}\}_{i\in\mathbb{N}}$ on $\pi^*\mathscr{M}$, $\mu_{\pi^*\mathscr{M}}$ projects to $\mu_{\mathscr{M}}$ and is invariant under the horizontal geodesic flow (g_t) .

For any $n \in \mathbb{Z}$, let A_n be the space of K-eigenfunction of weight n,

$$A_{n} = \left\{ \mathscr{A} \in \mathscr{C}^{\infty}(\pi^{*}\mathscr{M}) \mid \mathscr{A}(yk_{\theta}, z) = e^{\sqrt{-1}n\theta} \mathscr{A}(y, z) \right.$$
for any $(y, z) \in \pi^{*}\mathscr{M}, k_{\theta} = \begin{pmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta \end{pmatrix} \in K \right\},$

$$(3.5)$$

then A_0 is isomorphic to $\mathscr{C}^{\infty}(\mathscr{M})$. We say that \mathscr{A} is K-finite if for some $\ell \in \mathbb{N}$,

$$\mathscr{A} \in \sum_{|n| \le \ell} A_n. \tag{3.6}$$

Denote A_n with S_n when regard each function as being in $\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$,

$$S_n = \left\{ s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F}) \mid s(yk_{\theta}) = e^{\sqrt{-1}n\theta}s(y) \text{ for any } y \in U(X), k_{\theta} \in K \right\}, \quad (3.7)$$

then S_0 is isomorphic to $\mathscr{C}^{\infty}(X, \mathscr{F})$, $A_m \cdot S_n \subseteq S_{m+n}$, and $\{S_n\}_{n \in \mathbb{Z}}$ are pairwise orthogonal with respect to the $L^2(U(X), \pi^* \mathscr{F})$ -metric.

We define a series $\{s_{\lambda,2n} \in S_{2n}\}_{n \in \mathbb{Z}}$ inductively by setting $s_{\lambda,0} = u_{\lambda}$, and

$$s_{\lambda,2n+2} = \left(ir + \frac{1}{2} + n\right)^{-1} E^+ s_{\lambda,2n}, \quad s_{\lambda,2n-2} = \left(ir + \frac{1}{2} - n\right)^{-1} E^- s_{\lambda,2n}, \tag{3.8}$$

where the constant ensures $|s_{\lambda,2n}|_{L^2(U(X),\pi^*\mathscr{F})} = 1.$

Now we form the lift s_{λ} of u_{λ} by

$$s_{\lambda} = \frac{1}{\sqrt{2\lceil r^{1/2}\rceil + 1}} \sum_{|n| \leqslant \lceil r^{1/2}\rceil} s_{\lambda,2n}.$$
(3.9)

3.3. Pushforward and geodesic invariance.

Proposition 3.2. For any K-finite function $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$, there is C > 0 such that for any u_{λ} and its lifting s_{λ} , we have

$$\left| \langle \mathscr{A}s_{\lambda}, s_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} - \sum_{|n| \leq \lceil r^{1/2} \rceil} \langle \mathscr{A}s_{\lambda, 2n}, u_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} \right| \qquad (3.10)$$
$$\leq Cr^{-1/2} |\mathscr{A}|_{\mathscr{C}^{1}(\pi^{*}\mathscr{M})}.$$

Proof. Integrating by parts and using (3.8), we get

$$\langle \mathscr{A}s_{\lambda,2m}, s_{\lambda,2n} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})}$$

$$= \left(ir - \frac{1}{2} + m\right)^{-1} \langle \mathscr{A}E^{+}s_{\lambda,2m-2}, s_{\lambda,2n} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})}$$

$$= \left(ir - \frac{1}{2} + m\right)^{-1} \langle E^{+}(\mathscr{A}s_{\lambda,2m-2}) - E^{+}(\mathscr{A})s_{\lambda,2m}, s_{\lambda,2n} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})}.$$

$$(3.11)$$

This gives the following estimate together with (3.8) and the fact that $(E^+)^* = E^-$,

$$\left| \langle \mathscr{A}s_{\lambda,2m}, s_{\lambda,2n} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} - \langle \mathscr{A}s_{\lambda,2m-2}, s_{\lambda,2n-2} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} \right|$$

$$\leq C \left(\left| \mathscr{A} \right|_{\mathscr{C}^{1}(\pi^{*}\mathscr{M})} + \left| m - n \right| \left| \mathscr{A} \right|_{\mathscr{C}^{0}(\pi^{*}\mathscr{M})} \right) r^{-1}.$$

$$(3.12)$$

By (3.6), we see that $\langle \mathscr{A} s_{\lambda,2m}, s_{\lambda,2n} \rangle_{L^2(U(X),\pi^*\mathscr{F})} = 0$ whenever $|n-m| > \ell$, so we can replace |m - n| with C in (3.12). Repeating this, we get

$$\langle \mathscr{A}s_{\lambda,2m}, s_{\lambda,2n} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} - \langle \mathscr{A}s_{\lambda,2(m-n)}, u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} |$$

$$\leq C |\mathscr{A}|_{\mathscr{C}^{1}(\pi^{*}\mathscr{M})} nr^{-1}.$$

$$(3.13)$$

Summing (3.13) over $|n|, |m| \leq \lceil r^{1/2} \rceil$ with $|n-m| \leq 2\ell$, we get

$$\left| \langle \mathscr{A}s_{\lambda}, s_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} - \sum_{|n| \leq \ell} \frac{2\lceil r^{1/2} \rceil + 1 - |\ell|}{2\lceil r^{1/2} \rceil + 1} \langle \mathscr{A}s_{\lambda, 2n}, u_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} \right| \qquad (3.14)$$

$$\leq C |\mathscr{A}|_{\mathscr{C}^{1}(\pi^{*}\mathscr{M})} r^{-1/2},$$
which we get (3.10).

from which we get (3.10).

Applying (3.10) to $\mathscr{A} \in \mathscr{C}^{\infty}(\mathscr{M}) = A_0$, since $\mathscr{A}_{s_{\lambda,2n}} \in S_{2n}$ is orthogonal to $u_{\lambda} \in S_0$ for $n \neq 0$, we obtain

$$\left| \langle \mathscr{A}s_{\lambda}, s_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} - \langle \mathscr{A}u_{\lambda}, u_{\lambda} \rangle_{L^{2}(X, \mathscr{F})} \right| \leq Cr^{-1/2} \left| \mathscr{A} \right|_{\mathscr{C}^{1}(\mathscr{M})}, \tag{3.15}$$

from which we deduce the first statement of Theorem 3.1.

Proposition 3.3. For any K-finite function $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$, there is C > 0 such that for any u_{λ} , we have

$$\sum_{|n| \leq \lceil r^{1/2} \rceil} \langle H(\mathscr{A}) s_{\lambda,2n}, u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} \bigg| \leq Cr^{-1/2} |\mathscr{A}|_{\mathscr{C}^{2}(\pi^{*}\mathscr{M})}.$$
(3.16)

Proof. Since $u_{\lambda} \in S_0$, by (3.3) and (3.4), we compute that

$$\lambda \sum_{|n| \leq \lceil r^{1/2} \rceil} \langle \mathscr{A}s_{\lambda,2n}, u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} = \sum_{|n| \leq \lceil r^{1/2} \rceil} \langle \mathscr{A}s_{\lambda,2n}, E^{-}E^{+}u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})}$$

$$= \sum_{|n| \leq \lceil r^{1/2} \rceil} \langle E^{-}E^{+}(\mathscr{A}s_{\lambda,2n}), u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})},$$
(3.17)

then by the product formula, the last term in (3.17) is

$$\sum_{|n|\leqslant \lceil r^{1/2}\rceil} \langle E^{-}E^{+}(\mathscr{A})s_{\lambda,2n} + \mathscr{A}E^{-}E^{+}(s_{\lambda,2n}), u_{\lambda}\rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} + \langle E^{-}(\mathscr{A})E^{+}(s_{\lambda,2n}) + E^{+}(\mathscr{A})E^{-}(s_{\lambda,2n}), u_{\lambda}\rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})}.$$
(3.18)

Since $\sum_{|n| \leq \lceil r^{1/2} \rceil} E^{\pm} s_{\lambda,2n}$ and $(ir \mp \frac{1}{2}W - \frac{1}{2})s_{\lambda}$ only differ by terms orthogonal to $u_{\lambda} \in S_0$, integrating by parts for W in (3.18), we get

$$\sum_{|n| \leq \lceil r^{1/2} \rceil} 4ri \langle H(\mathscr{A}) s_{\lambda,2n}, u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} + \langle D(\mathscr{A}) s_{\lambda,2n}, u_{\lambda} \rangle_{L^{2}(U(X),\pi^{*}\mathscr{F})} = 0 \qquad (3.19)$$

for some $D \in U^2(\mathfrak{sl}_2(\mathbb{C}))$. The estimate (3.16) then follows immediately from (3.19). \Box

By (3.9), (3.10) and (3.16), for any K-finite $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$, we have

$$\left| \langle H(\mathscr{A}) s_{\lambda}, s_{\lambda} \rangle_{L^{2}(U(X), \pi^{*}\mathscr{F})} \right| \leq C \left| \mathscr{A} \right|_{\mathscr{C}^{2}(\pi^{*}\mathscr{M})} r^{-1/2}.$$

$$(3.20)$$

This implies the second statement of Theorem 3.1 for K-finite $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$, which extends to general $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$ by a K-finite approximation.

4. Hecke-Maass forms

4.1. Quaternionic arithmetic groups. Let L be a *totally real* algebraic number field with the ring of integers \mathcal{O}_L and the set of places Pl(L).

Suppose that $a, b \in L$ such that a, b > 0 and $\tau(a), \tau(b) < 0$ for every nonidentity real $\tau \in Pl(L)$. Let $D_{a,b}(L)$ denote the associated quaternion algebra over L, given by

$$D_{a,b}(L) = \{x_0 + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{ij} \mid x_0, \cdots, x_3 \in L\},$$
(4.1)

where $i^2 = a, j^2 = b, ij = -ji$.

We take \mathbb{G} to be the algebraic group of elements in $D_{a,b}(L)$ of reduced norm 1, then by [13, Proposition 6.2.5], $\mathbb{G}(\mathcal{O}_L) \subset \mathrm{SL}_2(\mathbb{R})$ is cocompact. In this section, without loss of generality, we assume that $\Gamma \subset \mathbb{G}(\mathcal{O}_L)$ is a principal congruence subgroup.

Let us now fix a *nonidentity real* $\rho \in Pl(L)$, then it extends to a representation

$$\rho \colon \mathbb{G}(L) \to \mathrm{SU}_2. \tag{4.2}$$

By composing with (1.10), we see that $\mathbb{G}(L)$ acts unitarily on $\mathscr{C}^{\infty}(\mathbb{S}^3)$.

Now, we suppose that the bundles $(F_p, \mathscr{F}, \pi^* F_p, \pi^* \mathscr{F})$ given in (2.8) are defined by the restriction of ρ to the subgroup $\Gamma \subseteq \mathbb{G}(\mathcal{O}_L) \subset \mathbb{G}(L)$. This is the key point that allows us to define Hecke operators.

4.2. Adelic quotients. For any finite $\wp \in Pl(L)$, let L_{\wp} be the completion of L at \wp and $\mathcal{O}_{\wp} \subset L_{\wp}$ the maximal compact subring. Let π_{\wp} be a uniformizer of L_{\wp} and $\ell_{\wp} = \mathcal{O}_{\wp}/\pi_{\wp}\mathcal{O}_{\wp}$ the residue field of \mathcal{O}_{\wp} , then we put $q_{\wp} = |\ell_{\wp}|$. For all but finite places \wp , we have $\mathbb{G}(L_{\wp}) \cong SL_2(L_{\wp})$ with a maximal compact subgroup $\mathbb{G}(\mathcal{O}_{\wp}) \cong SL_2(\mathcal{O}_{\wp})$, and in what follows, we shall implicitly exclude them when referring to places. Put

$$\mathbb{G}(\mathbb{A}_f) = \prod_{\nu < \infty}' \mathbb{G}(L_{\wp}), \tag{4.3}$$

the restricted direct product of $\mathbb{G}(L_{\wp})$ relative to $\mathbb{G}(\mathcal{O}_{\wp})$, and we set

$$\mathbb{G}(\mathbb{A}_L) = \mathrm{SL}_2(\mathbb{R}) \times \mathbb{G}(\mathbb{A}_f).$$
(4.4)

By the strong approximation theorem [20, Corollary 28.6.8], let $K_f \subset \mathbb{G}(\mathbb{A}_f)$ be an open compact subgroup such that $\Gamma = \mathbb{G}(L) \cap K_f$, then we have an isomorphism

$$U(X) = \Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \cong \mathbb{G}(L) \backslash \mathbb{G}(\mathbb{A}_L) / K_f$$
(4.5)

by identifying the coset $\Gamma(g_{\infty})$ with the double coset $[(g_{\infty}, 1)] = \mathbb{G}(L)(g_{\infty}, 1)K_f$, where $\mathbb{G}(L)$ acts diagonally on $\mathbb{G}(\mathbb{A}_L)$. From (4.5), if we denoted by $\mathscr{C}^{\infty}_{\mathbb{G}(L)}(\mathbb{G}(\mathbb{A}_L)/K_f)$ the $\mathbb{G}(L)$ -invariant subspace of $\mathscr{C}^{\infty}(\mathbb{G}(\mathbb{A}_L)/K_f)$, then we have

$$\mathscr{C}^{\infty}(U(X)) \cong \mathscr{C}^{\infty}_{\Gamma}(\mathrm{SL}_{2}(\mathbb{R})) \cong \mathscr{C}^{\infty}_{\mathbb{G}(L)}(\mathbb{G}(\mathbb{A}_{L})/K_{f}).$$
(4.6)

Similarly, taking account of the $\mathbb{G}(L)$ -action on $\mathscr{C}^{\infty}(\mathbb{S}^3)$ given in (4.2), we see that

$$\pi^*\mathscr{F} = \Gamma \setminus \left(\mathrm{SL}_2(\mathbb{R}) \times \mathscr{C}^{\infty}(\mathbb{S}^3) \right) \cong \mathbb{G}(L) \setminus \left(\mathbb{G}(\mathbb{A}_L) \times \mathscr{C}^{\infty}(\mathbb{S}^3) \right) / K_f$$
(4.7)

by mapping $\Gamma(g_{\infty}, \alpha)$ to $[(g_{\infty}, 1), \alpha] = \mathbb{G}(L)((g_{\infty}, 1), \alpha)K_f$, where $\mathbb{G}(L)$ again acts diagonally. Moreover, $\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ is isomorphic to the $\mathbb{G}(L)$ -invariant subspace of $\mathscr{C}^{\infty}(\mathbb{G}(\mathbb{A}_L)/K_f, \mathscr{C}^{\infty}(\mathbb{S}^3))$, namely

$$\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F}) \cong \mathscr{C}^{\infty}_{\Gamma}(\mathrm{SL}_2(\mathbb{R}), \mathscr{C}^{\infty}(\mathbb{S}^3)) \cong \mathscr{C}^{\infty}_{\mathbb{G}(L)}(\mathbb{G}(\mathbb{A}_L)/K_f, \mathscr{C}^{\infty}(\mathbb{S}^3)).$$
(4.8)

4.3. Hecke operators. First, we discuss Hecke operators using the first isomorphism in (4.6) and (4.8).

4.3.1. Double coset. For any $\gamma \in \mathbb{G}(L)$, the set $[\gamma] = \Gamma \setminus \Gamma \gamma \Gamma$ is finite, and we can define the corresponding Hecke operator \mathcal{T}_{γ} acting on $\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ such that for any $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$,

$$\mathcal{T}_{\gamma}(s)(x) = \sum_{\gamma' \in [\gamma]} \rho(\gamma')^{-1} s(\gamma' x), \qquad (4.9)$$

where we view $s \in \mathscr{C}^{\infty}(\mathrm{SL}_2(\mathbb{R}), \mathscr{C}^{\infty}(\mathbb{S}^3))$ through (4.8). Also, \mathcal{T}_{γ} restricts to operators acting on $\mathscr{C}^{\infty}(X, \mathscr{F}), \mathscr{C}^{\infty}(U(X))$ and $\mathscr{C}^{\infty}(X)$.

In (4.9), the morphism $\rho(\gamma')$ is unitary, therefore, as we will see in the next section, estimates for eigenfunctions of $\mathcal{T}_{\gamma}|_{\mathscr{C}^{\infty}(U(X))}$ in [9, §8], [16, §5] and [17, §3, §5] also hold for eigensections of \mathcal{T}_{γ} , confirming again the uniformity discussed in § 2.2.

Now, we turn to a different adelic perspective of these operators using the second isomorphism in (4.8).

4.3.2. Adelic convolution. For any finite $\wp \in \operatorname{Pl}(L)$, let $\mathcal{H}_{\wp} = \mathscr{C}_{c}^{\infty}(\mathbb{G}(\mathcal{O}_{\wp})) \setminus \mathbb{G}(L_{\wp})/\mathbb{G}(\mathcal{O}_{\wp}))$, the convolution algebra of compactly supported K_{\wp} -biinvariant functions on G_{\wp} . Then \mathcal{H}_{\wp} acts on the right by convolution on $\mathscr{C}^{\infty}(\mathbb{G}(\mathbb{A}_{L}), \mathscr{C}^{\infty}(\mathbb{S}^{3}))$. This action preserves $\mathscr{C}^{\infty}(\mathbb{G}(\mathbb{A}_{L})/K_{f}, \mathscr{C}^{\infty}(\mathbb{S}^{3}))$ and commutes with the left $\mathbb{G}(L)$ -action. Therefore, it passes to an action on $\mathscr{C}^{\infty}(U(X), \pi^{*}\mathscr{F})$ by (4.6).

More precisely, consider the Bruhat-Tits tree $T_{\wp} = \operatorname{GL}_2(L_{\wp})/\operatorname{GL}_2(\mathcal{O}_{\wp})$, which is regular with degree $q_{\wp}+1$. Let $d_{T_{\wp}}(\cdot, \cdot)$ be the natural metric on T_{\wp} such that the distance between nearest neighbors is 1. Since there are two orbits of $\mathbb{G}(L_{\wp})$ among the vertices of T_{\wp} , we see that \mathcal{H}_{\wp} is generated by the square of the tree Laplacian.

For any $k \in \mathbb{N}$, we can define an operator $\mathcal{T}_{\omega^{2k}}$ acting on $\mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ by

$$\mathcal{T}_{\wp^{2k}}(s)(v) = \sum_{d_{T_{\wp}}(v,w)=2k} s(w),$$
(4.10)

where we regard $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ as an element of $\mathscr{C}^{\infty}(\mathbb{G}(\mathbb{A}_L), \mathscr{C}^{\infty}(\mathbb{S}^3))$ through (4.8). The operator $\mathcal{T}_{\wp^{2k}}$ also restricts to actions on $\mathscr{C}^{\infty}(X, \mathscr{F}), \mathscr{C}^{\infty}(U(X))$ and $\mathscr{C}^{\infty}(X)$. Also, we note that the cardinality of the summation set in (4.10) is given by

$$\left| \{ w \in T_{\wp} \mid d_{T_{\wp}}(v, w) = 2k \} \right| = q_{\wp}^{2k-1}(q_{\wp} + 1).$$
(4.11)

Since $\{\mathcal{T}_{\wp^{2k}}\}_{\wp,k}$ commutes with each other, and when restricting to $\mathscr{C}^{\infty}(X,\mathscr{F})$, they commutes with $\Delta^{\mathscr{F}}$, we now assume that u_{λ} in (3.4) is a *joint* eigensection of the Laplacian and all Hecke operators. By the construction in (3.8) and (3.9), the lift s_{λ} in (3.9) is also a joint eigensection of all $\{\mathcal{T}_{\wp^{2k}}\}_{\wp,k}$.

5. Recurrence and positive entropy

5.1. Recurrence. For a finite prime \wp and a finite measure $\mu_{U(X)}$ on U(X), we say that $\mu_{U(X)}$ is $\mathbb{G}(L_{\wp})$ -recurrent if for any Borel set $U \subset U(X)$ and $\mu_{U(X)}$ -almost every $x = [(g_{\infty}, 1)] \in U$, the set $\{g_{\wp} \in \mathbb{G}(L_{\wp}) \mid [(x, g_{\wp})] \in U\}$ in unbounded.

Our goal in this subsection is to prove the following recurrent property.

Theorem 5.1. Let $\{s_{\lambda_i} \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})\}_{i \in \mathbb{N}}$ be a sequence of normalized eigensections of $\{\mathcal{T}_{\wp^2}\}_{i \in \mathbb{N}}$. Then for any weak star limit $\mu_{\pi^*\mathscr{M}}$ of $\{|s_{\lambda_i}|_{\mathbb{C}}^2 dv_{\pi^*\mathscr{M}}\}_{i \in \mathbb{N}}$, its projection $\mu_{U(X)}$ to U(X) is $\mathbb{G}(L_{\wp})$ -recurrent.

First, we prove that the eigensection of a Hecke operator on a tree is not concentrated near a vertex.

Proposition 5.2. There is C > 0 such that for any $n \in \mathbb{N}$ and eigensection $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$ of \mathcal{T}_{\wp^2} , we have for $|s|^2_{\pi^*\mathscr{F}} \in \mathscr{C}^{\infty}(U(X))$,

$$\sum_{k=0}^{n} \mathcal{T}_{\wp^{2k}}\left(\left|s\right|^{2}_{\pi^{*}\mathscr{F}}\right)(v) \ge Cn \left|s(v)\right|^{2}_{\pi^{*}\mathscr{F}}.$$
(5.1)

Proof. By (4.10) and Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left| \mathcal{T}_{\wp^{2k}}(s)(v) \right|_{\pi^*\mathscr{F}}^2 &\leqslant \Big(\sum_{d_{T_{\wp}}(v,w)=2k} |s(w)|_{\pi^*\mathscr{F}} \Big)^2 \\ &\leqslant \left| \left\{ w \in T_{\wp} \mid d_{T_{\wp}}(v,w)=2k \right\} \right| \cdot \mathcal{T}_{\wp^{2k}} \Big(\left| s \right|_{\pi^*\mathscr{F}}^2 \Big)(v) \\ &\leqslant C q_{\wp}^{2k} \mathcal{T}_{\wp^{2k}} \Big(\left| s \right|_{\pi^*\mathscr{F}}^2 \Big)(v), \end{aligned}$$
(5.2)

and similarly,

$$\left|\sum_{k=0}^{n} \mathcal{T}_{\wp^{2k}}(s)(v)\right|_{\pi^{*}\mathscr{F}}^{2} \leq \left(\sum_{k=0}^{n} \sum_{d_{T_{\wp}}(v,w)=2k} |s(w)|_{\pi^{*}\mathscr{F}}\right)^{2}$$
$$\leq \left|\bigcup_{k\leqslant n} \{w \in T_{\wp} \mid d_{T_{\wp}}(v,w)=2k\}\right| \cdot \sum_{k=0}^{n} \mathcal{T}_{\wp^{2k}}\left(|s|_{\pi^{*}\mathscr{F}}^{2}\right)(v)$$
$$\leq Cq_{\wp}^{n}\mathcal{T}_{\wp^{2k}}\left(|s|_{\pi^{*}\mathscr{F}}^{2}\right)(v).$$
(5.3)

Now suppose that $\mathcal{T}_{\wp^2}s = \lambda_{\wp^2}s$, then we set a virtual eigenvalue λ_{\wp} of \mathcal{T}_{\wp} by $\lambda_{\wp}^2 = \lambda_{\wp^2} + (q_{\wp} + 1)$. Following [9, Lemma 8.3], applying (5.2) and (5.3) to two cases, $|\lambda_{\wp}| \leq 2\sqrt{q_{\wp}}$ and $|\lambda_{\wp}| > 2\sqrt{q_{\wp}}$ respectively, we get (5.1).

For any open set $U \subseteq U(X)$, applying (5.1) to a sequence of continuous functions $\{f_j\}_{j\in\mathbb{N}}$ converge monotonically to $\mathbb{1}_U$, we get

$$\int_{U(X)} \sum_{k=0}^{n} \mathcal{T}_{\wp^{2k}}(\mathbb{1}_U) d\mu_{U(X)} \ge C n \mu_{U(X)}(U), \tag{5.4}$$

then parallel to the proof of [9, Lemma 8.3], we obtain Theorem 5.1 from (5.4).

5.2. Strong positive entropy. By (4.10), we have

$$\mathcal{T}_{\wp^2}^2 = \mathcal{T}_{\wp^4} + (q_\wp - 1)\mathcal{T}_{\wp^2} + q_\wp(q_\wp + 1)\mathrm{Id},$$
(5.5)

which, together with (4.11), clearly implies the following ampleness property.

Proposition 5.3. There is C > 0 such that for any Hecke eigensection $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$, if we denote the corresponding eigenvalue of \mathcal{T}_{\wp^2} and \mathcal{T}_{\wp^4} with λ_{\wp^2} and λ_{\wp^4} respectively, then we have either $|\lambda_{\wp^2}| \ge Cq_{\wp}$ or $|\lambda_{\wp^4}| \ge Cq_{\wp^2}^2$.

For a c > 0 to be determined later, we put for any small $\varepsilon > 0$

$$\mathscr{P}_{\varepsilon} = \{ \wp \in \operatorname{Pl}(L) \mid \varepsilon^{-c}/2 \leqslant q_{\wp} \leqslant \varepsilon^{-c} \}.$$
(5.6)

By Proposition (5.3), we can let ℓ be either 1 or 2 and choose a subset $\mathscr{P}'_{\varepsilon} \subseteq \mathscr{P}_{\varepsilon}$ with positive density and $|\lambda_{\wp^{2\ell}}| \ge Cq^{\ell}_{\wp}$ for all $\wp \in \mathscr{P}'_{\varepsilon}$. Following [17, Lemma 5.2] and [16, Proposition 30], we set a global amplifier h_{ε} by

$$h_{\varepsilon} = \sum_{\wp \in \mathscr{P}'_{\varepsilon}} \mathcal{T}_{\wp^{2\ell}}.$$
(5.7)

From Proposition 5.3 and (5.6), s is also an eigensection of h_{ε} , and we have the following estimate for the eigenvalue λ_{ε} , where $h_{\varepsilon}s = \lambda_{\varepsilon}s$,

$$\lambda_{\varepsilon} = \sum_{\wp \in \mathscr{P}'_{\varepsilon}} \lambda_{q_{\wp}^{2\ell}} \geqslant C \frac{\varepsilon^{-c}}{\log(\varepsilon^{-c})} \varepsilon^{-c\ell}.$$
(5.8)

Using (4.11), its follows that the size of the support of h_{ε} satisfies

$$\left|\operatorname{supp}(h_{\varepsilon})\right| = \sum_{\wp \in \mathscr{P}_{\varepsilon}'} q_{\wp}^{2\ell-1}(q_{\wp}+1) \leqslant C \frac{\varepsilon^{-c}}{\log(\varepsilon^{-c})} \varepsilon^{-2c\ell}.$$
(5.9)

We form two one-parameter subgroups

$$u_t^+ = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad u_t^- = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$
(5.10)

Recall (g_t) defined in (1.22), then for a fixed small $\delta > 0$, we define the tube neighborhood U_{ε} of Id $\in SL_2(\mathbb{R})$ by

$$U_{\varepsilon} = g_{(-\delta,\delta)} \cdot u_{(-\varepsilon,\varepsilon)}^{-} \cdot u_{(-\varepsilon,\varepsilon)}^{+}.$$
(5.11)

We have the following bounds on the mass of tubes.

Theorem 5.4. There are C, d > 0 such that for any $g_{\infty} \in SL_2(\mathbb{R})$, ε small enough and Hecke eigensection $s \in \mathscr{C}^{\infty}(U(X), \pi^*\mathscr{F})$, we have for $|s|^2_{\pi^*\mathscr{F}} \in \mathscr{C}^{\infty}(U(X))$,

$$\int_{\Gamma g_{\infty} U_{\varepsilon}} |s|^{2}_{\pi^{*}\mathscr{F}} dv_{X} \leqslant C\varepsilon^{d}.$$
(5.12)

Proof. Similar to [17, Lemma 3.3], applying the triangle inequality for (4.10), we have

$$\int_{U(X)} \mathbb{1}_{\Gamma g_{\infty} U_{\varepsilon}} \left| h_{\varepsilon} s \right|_{\pi^* \mathscr{F}}^2 dv_{U(X)} \\ \leqslant \left(\sum_{g_f \in \mathrm{supp}(h_{\varepsilon})} \left(\int_{U(X)} \mathbb{1}_{\Gamma g_{\infty} U_{\varepsilon} g_f} \left| s \right|_{\pi^* \mathscr{F}}^2 dv_{U(X)} \right)^{1/2} \right)^2.$$
(5.13)

Using [16, Proposition 10] and the argument of [17, Lemmas 3.4, 5.1], we can choose a c > 0 in (5.6) such that there exists C > 0 such that

$$\left(\sum_{g_f \in \operatorname{supp}(h_{\varepsilon})} \left(\int_{U(X)} \mathbb{1}_{\Gamma g_{\infty} U_{\varepsilon} g_f} \left|s\right|^2_{\pi^* \mathscr{F}} dv_{U(X)}\right)^{1/2}\right)^2 \leqslant C\left(\varepsilon^{-2c} + \left|\operatorname{supp}(h_{\varepsilon})\right|\right).$$
(5.14)

Combining (5.13) and (5.14), it follows that

$$\int_{U(X)} \mathbb{1}_{\Gamma g_{\infty} U_{\varepsilon}} |s|^{2}_{\pi^{*} \mathscr{F}} dv_{U(X)} \leqslant C |\lambda_{\varepsilon}|^{-2} \left(\varepsilon^{-2c} + |\operatorname{supp}(h_{\varepsilon})| \right).$$
(5.15)

Plugging (5.8) and (5.9) into (5.15), we get (5.12).

6. AUQUE

6.1. Measure rigidity. Combining Theorems 3.1, 5.1 and 5.4, we have verified the hypotheses of the measure rigidity of Lindenstrauss [9, Theorem 1.1]. Therefore, we get the following result.

Theorem 6.1. Let $\{u_{\lambda_i}\}_{i\in\mathbb{N}}$ be any sequence of Hecke-Maass form with $\lim_i \lambda_i = \infty$, then for any weak star limit $\mu_{\pi^*\mathscr{M}}$ of $\{|s_{\lambda_i}|^2 dv_{\pi^*\mathscr{M}}\}_{i\in\mathbb{N}}$, its projection $\mu_{U(X)}$ equals the normalized Liouville measure $d\overline{v}_{U(X)}$.

Remark 6.2. Notice that all arguments leading to Theorem 6.1 hold for the weak star limit of functionals $\{A \in \mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p)) \mapsto \langle As_{p,j}, s_{p,j} \rangle_{L^2(U(X), \pi^*F)} \}_{j \in \mathbb{N}}$ with $\lambda_{p,j} \to \infty$, it follows that, for such a limit functional $L_{U(X)}$, its restriction to $\mathscr{C}^{\infty}(U(X))$ is $d\overline{v}_{U(X)}$.

Theorem 6.3. For the limit measure $\mu_{\pi^*\mathcal{M}}$ and functional $L_{U(X)}$ given in Theorem 6.1 and Remark 6.2, we have

$$L_{U(X)}(\cdot) = \int_{U(X)} \overline{\mathrm{Tr}}^{\pi^* F}[\cdot] d\overline{v}_{U(X)}, \quad \mu_{\pi^* \mathscr{M}} = d\overline{v}_{\pi^* \mathscr{M}}.$$
(6.1)

Proof. First, we easily check the semi-positivity of $L_{U(X)}$. In other words, for any $A \in \mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p))$ pointwisely positive semi-definite, we have $L_{U(X)}(A) \ge 0$. On the other hand $(|A|_{\operatorname{End}(F_p)} \cdot \operatorname{Id}_{F_p} - A)$ is also semi-positive defined, hence

$$0 \leq L_{U(X)}(A) \leq L_{U(X)}(|A|_{\operatorname{End}(F_p)} \cdot \operatorname{Id}_{F_p})$$

=
$$\int_{U(X)} |A(x)|_{\operatorname{End}(F_p)} d\overline{v}_{U(X)}(x),$$
 (6.2)

where the final equality follows from Remark 6.2. This, together with the complex linearity of $L_{U(X)}$, implies that $L_{U(X)}$ is absolutely continuous with respect to $d\overline{v}_{U(X)}$.

Using the Riesz representation theorem and lifting $L_{U(X)}$ to $SL_2(\mathbb{R})$ via (1.2), we can write $L_{U(X)}$ into the form of

$$L_{U(X)} = \int_{\mathrm{SL}_2(\mathbb{R})} L_x \cdot dv_{\mathrm{SL}_2(\mathbb{R})}, \qquad (6.3)$$

where $x \in \mathrm{SL}_2(\mathbb{R}) \mapsto L_x$ is a measurable function evaluating in $\mathrm{End}(\mathrm{Sym}^p(\mathbb{C}^2))^*$, the dual of $\mathrm{End}(\mathrm{Sym}^p(\mathbb{C}^2))$. Clearly L is Γ -invariant, namely that for any $x \in \mathbb{H}^2, \gamma \in \Gamma$ and $a \in \mathrm{End}(\mathrm{Sym}^p(\mathbb{C}^2))$, we have

$$\langle L_{\gamma \cdot x}, \rho(\gamma) a \rho(\gamma)^{-1} \rangle = \langle L_x, a \rangle.$$
 (6.4)

On the other hand, by Hopf argument, the geodesic invariance of $L_{U(X)}$ implies that $x \to L_x$ is constant. According to the Borel density theorem [13, Corollary 4.5.6], the image $\rho(\Gamma) \subset SU_2$ in (1.1) is dense, and so, by (6.4), L is SU₂-invariant. Using Schur's lemma, for any $a \in End(Sym^p(\mathbb{C}^2))$, we have

$$\int_{\mathrm{SU}_2} gag^{-1} d\overline{v}_{\mathrm{SU}_2}(g) = \overline{\mathrm{Tr}}^{\mathrm{Sym}^p(\mathbb{C}^2)}[a] \cdot \mathrm{Id}_{\mathrm{Sym}^p(\mathbb{C}^2)},\tag{6.5}$$

hence, for a constant c,

$$L = c \cdot \overline{\mathrm{Tr}}^{\mathrm{Sym}^p(\mathbb{C}^2)}.$$
(6.6)

Finally, since $L_{U(X)}$ is normalized, or $L_{U(X)}(\mathbb{1}_{U(X)}) = 1$, we deduce the first identity in (6.1) by (6.3) and (6.6).

Likewise, we can regard $\mu_{\pi^*\mathscr{M}}$ as a Γ -invariant measure on $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{S}^3$ by (2.8). Using Theorem 6.1 and disintegration, we then express $\mu_{\pi^*\mathscr{M}}$ by

$$\mu_{\pi^*\mathscr{M}} = \int_{\mathrm{SL}_2(\mathbb{R})} \mu_x dv_{\mathrm{SL}_2(\mathbb{R})}(x), \qquad (6.7)$$

where μ_x is a measure on $(x, \mathbb{S}^3) \subset \mathrm{SL}_2(\mathbb{R}) \times \mathbb{S}^3$. Again, Hopf argument shows that $x \to \mu_x$ is a constant measure-valued function. Then μ is a SU₂-invariant measure on \mathbb{S}^3 , by the Haar theorem [19, Proposition 4.2.4], we have

$$\mu = c \cdot dv_{\mathbb{S}^3}.\tag{6.8}$$

Since $\mu_{\pi^*\mathcal{M}}$ is normalized, we get the second identity in (6.1) from (6.7).

From Theorem 6.3, we end up with our main results.

Theorem 6.4. For any $p \in \mathbb{N}$ and $A \in \mathscr{C}^{\infty}(U(X), \pi^* \operatorname{End}(F_p))$, we have

$$\lim_{j \to \infty} \langle As_{p,j}, s_{p,j} \rangle_{L^2(U(X),\pi^*F_p)} = \int_{U(X)} \overline{\mathrm{Tr}}^{\pi^*F_p}[A] d\overline{v}_{U(X)}.$$
(6.9)

Theorem 6.5. For any $\mathscr{A} \in \mathscr{C}^{\infty}(\pi^*\mathscr{M})$, we have

$$\lim_{\lambda \to \infty} \left| \langle \mathscr{A}s_{\lambda}, s_{\lambda} \rangle_{L^{2}(\pi^{*}\mathscr{M})} - \int_{\pi^{*}\mathscr{M}} \mathscr{A}d\overline{v}_{\pi^{*}\mathscr{M}} \right| = 0.$$
(6.10)

Clearly Theorem 6.4 implies Theorems 1.1. Moreover, as discussed in § 2.2, applying Theorem 6.5 to eigensectors of the Laplacian on the finite dimensional subbundles $F_p \subset \mathscr{F}$ and restricting to $\mathscr{A} \in \mathscr{C}^{\infty}(\mathscr{M})$, we then derive Theorem 1.2.

6.2. Comparison with previous works. Theorem 6.5 can be compared with the QUE of Brooks-Lindenstrauss [5, Theorem 1.5] for partial Laplacian operators.

Consider

$$\Gamma_2 \setminus (\mathbb{H}^2 \times \mathbb{H}^2), \tag{6.11}$$

where $\Gamma_2 \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is a co-compact and *irreducible* lattice, then denote the two partial Laplacians by Δ_1 and Δ_2 . Let $\{\phi_j\}$ be a sequence of joint o(1)-quasimodes Δ_1 and Δ_2 with spectral parameters $\lambda_{1,j}$ and $\lambda_{2,j}$, where quasimode and spectral parameter are approximately eigenfunction and eigenvalue. Suppose that $\lim_{j\to\infty} \lambda_{1,j} \to \infty$ and $\lambda_{2,j}$ bounded. Then [5, Theorem 1.5] asserts that the sequence of lifts ν_j of $|\phi_j|^2 dv_{\Gamma_2 \setminus (\mathbb{H}^2 \times \mathbb{H}^2)}$ to $\Gamma_2 \setminus (\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H}^2)$ converges weak-star to the uniform measure on $\Gamma_2 \setminus (\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H}^2)$.

Recall \mathscr{M} defined in (1.15). We can form two partial Laplacians on \mathscr{M} , denoted by $\Delta_{\mathbb{H}^2}$ and $\Delta_{\mathbb{S}^3}$. Note that $\Delta_{\mathbb{H}^2}$ is precisely $\Delta^{\mathscr{F}}$ as defined in (2.14), the Laplacian on the infinite dimensional flat bundle \mathscr{F} . Hence, u_{λ} given in (3.4) are eigenfunctions of $\Delta_{\mathbb{H}}$. Therefore, from the perspective of [5, Theorem 1.5], Theorem 6.5 can also be reinterpreted as the QUE of the lifts $|s_{\lambda}|^2_{\mathbb{C}} dv_{\pi^*\mathscr{M}}$ of the eigenfunctions of a partial Laplacian.

The difference is that we do not assume that u_{λ} are eigenfunctions of $\Delta_{\mathbb{S}^3}$, nor do we impose any boundedness assumption on the corresponding eigenvalues. Furthermore, it is not known if Hecke operators necessarily exist on $\Gamma_2 \setminus (\mathbb{H}^2 \times \mathbb{H}^2)$. In the proof of [5, Theorem 1.5], they use the foliation given by varying the second coordinate, analogous to the use of the Hecke operators.

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