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# Volume-law entanglement fragmentation of quasiparticles

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We study the entanglement entropy in quasiparticle states where certain unit patterns are excited repeatedly and sequentially in momentum space. We find that in the scaling limit, each unit pattern contributes independently and universally to the entanglement, leading to a volume-law scaling of the entanglement entropy. This characteristic of volume-law entanglement fragmentation is numerically confirmed in both fermionic and bosonic chains. We derive an analytical formula for fermions, which can also be applied to the spin-1/2 XXZ chain with appropriate identifications.

# I. INTRODUCTION

Quantum entanglement has been instrumental in various areas of physics, such as quantum information theory, condensed matter physics, and high-energy physics [\[1–](#page-4-0)[6\]](#page-4-1). Notably, entanglement adheres to distinct scaling laws in various phases of quantum systems, which allows it to serve as an indicator of quantum phase transitions [\[7,](#page-4-2) [8\]](#page-4-3). In the context of one-dimensional quantum spin chains with energy gaps, the ground state entanglement entropy typically conforms to the area law. Conversely, in critical spin chains, the ground state entropy adheres to a logarithmic law, with the proportionality constant being related to the central charge [\[9–](#page-4-4)[13\]](#page-4-5).

The study of the time evolution of entanglement entropy following a quantum quench in many-body systems [\[14\]](#page-4-6) has motivated research into entanglement entropy in excited states. This includes both low-energy states [\[15–](#page-4-7) [33\]](#page-5-0), where the entropy adheres to either the area law or logarithmic law, and high-energy states [\[34](#page-5-1)[–40\]](#page-5-2), where the entropy typically aligns with the volume law. In [\[41–](#page-5-3) [46\]](#page-5-4), the average entanglement entropy across the spectrum has been employed to characterize the universal properties of chaotic and integrable systems.

Entanglement of excited states in integrable models can often be understood in terms of quasiparticles [\[14,](#page-4-6) [47,](#page-5-5) [48\]](#page-5-6). When a finite number of quasiparticles with large energy and significant momentum differences are excited, a semiclassical picture can describe the entanglement entropy [\[25](#page-5-7)[–28\]](#page-5-8). For earlier related works, see [\[18,](#page-4-8) [20,](#page-4-9) [23\]](#page-5-9). On the other hand, when the momentum differences between the excited quasiparticles are small, there are strong coherence effects among the quasiparticles, which results in significant corrections to the semiclassical picture of the entanglement entropy [\[31–](#page-5-10)[33\]](#page-5-0).

In this paper, we study the entanglement entropy in states with a large number of quasiparticles, where certain unit patterns are excited sequentially and repeatedly in the momentum space. We consider a subsystem consisting of a consecutive block within circular quantum systems, including both free and interacting fermionic and bosonic chains, as well as spin-1/2 XXZ chains. Generally, the entanglement entropy in such states follows a volume law. In [\[15\]](#page-4-7), analytical expressions were obtained for fermionic chains in cases where the subsystem size is much smaller than the size of the entire system. In the case of free fermionic chains, we obtain analytical expressions for subsystems of arbitrary sizes, which, in the scaling limit, also apply to interacting fermionic chains and spin-1/2 XXZ chains.

We discover that the coherence among quasiparticles enables the entanglement entropy to be decomposed into separate contributions from distinct parts, each comprising a few sites in the coordinate space and the corresponding unit pattern quasiparticles in the momentum space. In essence, the volume-law entanglement entropy breaks down into components that correspond to the quasiparticle entanglement entropy of much smaller systems. This universal phenomenon of volume-law entanglement fragmentation could be useful for establishing robust entanglement, which is crucial for applications in quantum information processing and quantum computation.

# II. FREE FERMIONS

A chain of L free fermions has the Hamiltonian

$$
H = \sum_{j=1}^{L} \left( a_j^{\dagger} a_j - \frac{1}{2} \right), \tag{1}
$$

with the fermionic modes  $a_j$  and  $a_j^{\dagger}$  satisfying  ${a_{j_1}, a_{j_2}} = {a_{j_1}^{\dagger}, a_{j_2}^{\dagger}} = 0$  and  ${a_{j_1}, a_{j_2}^{\dagger}} = \delta_{j_1 j_2}$ . The excited states with translational invariance are generated by the global modes

$$
b_k^{\dagger} = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} a_j^{\dagger} e^{\frac{2\pi i j k}{L}}, \quad b_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} a_j e^{-\frac{2\pi i j k}{L}}, \quad (2)
$$

with the momentum  $k = 0, 1, \ldots, L - 1$ . We use the set of excited modes  $K = \{k_1, k_2, \ldots\}$  to denote the corresponding excited energy eigenstate  $|K\rangle$ .

For a subsystem  $A$  of  $L_A$  consecutive sites in a chain of total length L in state  $|K\rangle$ , one has the reduced density matrix (RDM)  $\rho_{A,K} = \text{tr}_{\bar{A}}(|K\rangle\langle K|)$ , from which one gets the Rényi entropy and entanglement entropy

$$
S_{L,L_A,K}^{(n)} = -\frac{1}{n-1} \log \text{tr}_A(\rho_{A,K}^n),
$$
  
\n
$$
S_{L,L_A,K} = -\text{tr}_A(\rho_{A,K} \log \rho_{A,K}).
$$
\n(3)

The entanglement entropy is just the  $n \to 1$  limit of the Rényi entropy  $S_{L,L_A,K}^{(1)} = S_{L,L_A,K}$ .

In the free fermionic chain, one has [\[10,](#page-4-10) [11,](#page-4-11) [49,](#page-5-11) [50\]](#page-5-12)

$$
S_{L,L_A,K}^{(n),\text{fer}} = -\frac{1}{n-1} \log \det[C_{L,L_A,K}^n + (1 - C_{L,L_A,K})^n],
$$
  
\n
$$
S_{L,L_A,K}^{\text{fer}} = \text{tr}_A[-C_{L,L_A,K} \log C_{L,L_A,K} - (1 - C_{L,L_A,K}) \log(1 - C_{L,L_A,K})],
$$
 (4)

where the  $L_A \times L_A$  matrix has entries

$$
[C_{L,L_A,K}]_{j_1j_2} = \langle a_{j_1}^{\dagger} a_{j_2} \rangle_K = h_{L,j_2-j_1,K},
$$
 (5)

with  $j_1, j_2 = 1, 2, \dots, L_A$  and the correlation function

<span id="page-1-0"></span>
$$
h_{L,j,K} = \frac{1}{L} \sum_{k \in K} e^{\frac{2\pi i jk}{L}}.
$$
 (6)

Note that  $S_{L,0,K}^{(n),\text{fer}} = S_{L,L,K}^{(n),\text{fer}} = 0.$ 

# A. Fully occupied states

We first consider the case that  $K$  is composed of the repetition of a unit pattern in the whole momentum space. The unit pattern has  $l$  consecutive sites in the momentum space, in which the excited modes are  $\kappa$ . The whole system has  $L = pl$  sites, and we denote the excited modes of the whole momentum space as K and use the shorthand  $K = p\kappa$ . Explicitly, we have  $K = \bigcup_{a=0}^{p-1} (\kappa + a l)$ . For instance, there are states

$$
l = 2, \ \kappa = \{0\}, \ p = 4: \ K = \{0, 2, 4, 6\},
$$
  

$$
l = 3, \ \kappa = \{0, 1\}, \ p = 2: \ K = \{0, 1, 3, 4\}, \tag{7}
$$

which can be denoted schematically as

•◦•◦•◦•◦, ••◦••◦, (8)

with the filled circles denoting the excited modes and the empty circles denoting the modes that are not excited.

From [\(6\)](#page-1-0) we get the correlation function

$$
h_{pl,j,p\kappa} = \delta_{(j \bmod p) = 0} h_{l,j/p,\kappa},\tag{9}
$$

which is possibly nonvanishing only when  $j$  is an integer multiple of p. The  $L \times L$  correlation matrix of the whole system  $C_{L,L,K}$  can be written as a matrix with  $l \times l$  blocks, with each block being proportional to the  $p \times p$  identity matrix. The  $L_A \times L_A$  correlation matrix  $C_{L,L_A,K}$  for the  $L_A$ -sized subsystem is just the first  $L_A$ rows and the first  $L_A$  columns of  $C_{L,L,K}$ .

We parameterize the subsystem length as  $L_A = \alpha p + a$ , with  $\alpha = 0, 1, \dots, l-1$  and  $a = 0, 1, \dots, p-1$ , and the corresponding correlation matrix  $C_{L,L_A,K}$  is similar to

$$
C_{pl,\alpha p+a,p\kappa} \sim \underbrace{C_{l,\alpha+1,\kappa} \oplus \cdots}_{a} \oplus \underbrace{C_{l,\alpha,\kappa} \oplus \cdots}_{p-a} . \tag{10}
$$

This indicates that the RDM could be written as

<span id="page-1-2"></span>
$$
\rho_{pl,\alpha p+a,p\kappa} \sim \underbrace{\rho_{l,\alpha+1,\kappa} \otimes \cdots \otimes \rho_{l,\alpha,\kappa} \otimes \cdots}_{a}.
$$
 (11)

There are p copies of the unit pattern  $\kappa$ , and each copy is effectively confined in a circle of  $l$  sites. The subsystem has  $L_A = \alpha p + a$  sites, and they are effectively composed of subsystems of the p copies of the small systems. An example scheme of the entanglement fragmentation is shown in Fig. [1.](#page-1-1)



<span id="page-1-1"></span>FIG. 1. Example scheme of the entanglement fragmentation. Each pair of concentric circles symbolizes a system. The larger circle represents the spatial configuration, where red and blue points denote the complementary subsystems. The smaller circle depicts the momentum configuration, with filled purple points indicating excited modes and hollow points representing unexcited modes. For this illustration, we have set the parameters to  $L = 15$ ,  $L_A = 7$ ,  $\alpha = 1$ ,  $a = 2$ ,  $p = 5$ ,  $l = 3$ , and  $\kappa = \{0\}.$ 

From the RDM [\(11\)](#page-1-2), we get the decomposition of the Rényi entropy and entanglement entropy

<span id="page-1-3"></span>
$$
S_{pl,\alpha p+a,p\kappa}^{(n),\text{fer}} = a S_{l,\alpha+1,\kappa}^{(n),\text{fer}} + (p-a) S_{l,\alpha,\kappa}^{(n),\text{fer}}.
$$
 (12)

This inspired us to define the contributions to the entanglement from each unit pattern as

<span id="page-1-5"></span>
$$
s_{l,\kappa}^{(n),\text{fer}}(x) \equiv \frac{S_{pl, xpl, p\kappa}^{(n), \text{fer}}}{p},\tag{13}
$$

which is the building block for all the results in the paper. Explicitly, we have

<span id="page-1-4"></span>
$$
s_{l,\kappa}^{(n),\text{fer}}(x) = yS_{l,\alpha+1,\kappa}^{(n),\text{fer}} + (1-y)S_{l,\alpha,\kappa}^{(n),\text{fer}},\qquad(14)
$$

with the parameters

<span id="page-1-7"></span>
$$
\alpha \equiv \lfloor lx \rfloor \in \{0, 1, \cdots, l - 1\},
$$
  
\n
$$
y \equiv (lx \mod 1) \in [0, 1).
$$
 (15)

Note that these formulas are exact, valid for any  $p, l, \alpha$ , a, and  $\kappa$ . Furthermore, the function  $s_{l,\kappa}^{(n),\text{fer}}(x)$  is independent of p. For finite l and large p, i.e.  $l \ll p$ , the Rényi entropy and entanglement entropy  $(12)$  follow the volume law. Generally, the formula  $s_{l,\kappa}^{(n),\text{fer}}(x)$  is a piecewise function of  $x \in (0,1)$  with l pieces, each piece has length  $\frac{1}{l}$ , and within each piece the function is linear.

We use  $|\kappa|$  to denote the number of excited quasiparticles in the set  $\kappa$ . In the special limit,  $|\kappa| \ll l$ , the formula [\(14\)](#page-1-4) becomes

<span id="page-1-6"></span>
$$
\lim_{l \to +\infty} s_{l,\kappa}^{(n),\text{fer}}(x) = \lim_{l \to +\infty} S_{l,xl,\kappa}^{(n),\text{fer}}.
$$
 (16)

In the same limit, the formula [\(13\)](#page-1-5) becomes

<span id="page-2-0"></span>
$$
\lim_{l \to +\infty} s_{l,\kappa}^{(n),\text{fer}}(x) = \lim_{l \to +\infty} \frac{S_{pl,xpl,p\kappa}^{(n),\text{fer}}}{p}.\tag{17}
$$

The results of [\(16\)](#page-1-6) and [\(17\)](#page-2-0) should be the same and this is consistent with the fact, found in [\[31](#page-5-10)[–33\]](#page-5-0) that quasiparticles with large momentum differences have independent contributions to the entanglement.

For  $\kappa = \{0\}$ , with  $l = 2, 3, \dots$ , there are [\[18,](#page-4-8) [20,](#page-4-9) [23,](#page-5-9) [25,](#page-5-7) [26\]](#page-5-13)

$$
S_{l,l_A,\{0\}}^{(n),\text{fer}} = -\frac{1}{n-1} \log \left[ \left( \frac{l_A}{l} \right)^n + \left( 1 - \frac{l_A}{l} \right)^n \right],
$$
  

$$
S_{l,l_A,\{0\}}^{\text{fer}} = -\frac{l_A}{l} \log \frac{l_A}{l} - \left( 1 - \frac{l_A}{l} \right) \log \left( 1 - \frac{l_A}{l} \right), (18)
$$

with which the formula  $(12)$  is the same as the comb entropy defined in [\[51\]](#page-5-14) after one takes the position-momentum duality [\[52,](#page-5-15) [53\]](#page-5-16). For  $\kappa = \{k_1, k_2\}$  with  $l = 3, 4, \cdots$ , the analytical formulas for  $S_{l,l_A,\kappa}^{(n),\text{fer}}$  can be found in [\[31](#page-5-10)[–33\]](#page-5-0). Examples of the expression [\(14\)](#page-1-4) for the entanglement entropy from each unit pattern are shown in Fig. [2.](#page-2-1)



<span id="page-2-1"></span>FIG. 2. The formula [\(14\)](#page-1-4) of the contribution to the entanglement entropy from each unit pattern in the momentum space of a free fermionic chain.

### B. Partially occupied states

In the excited states, the unit pattern  $\kappa$  of length l can be repeated  $p$  times in the momentum space and these repeated patterns only occupy a finite ratio of the whole momentum space, leaving the rest of the momentum space unoccupied. We call such state partially occupied states. For example, we may have

$$
l = 2, \ \kappa = \{0\}, \ p = 2, \ L = 12: \ K = \{0, 2\}, \tag{19}
$$
\n
$$
l = 3, \ \kappa = \{0, 1\}, \ p = 2, \ L = 12: \ K = \{0, 1, 3, 4\},
$$

which may be denoted as

$$
\bullet \circ \bullet \circ \circ \circ \circ \circ \circ \circ \infty, \quad \bullet \bullet \circ \bullet \circ \circ \circ \circ \circ \circ \circ \circ.
$$
 (20)

In the scaling limit, we conjecture that each unit pattern still contributes independently to the entanglement entropy, and there is

$$
S_{L,xL,p\kappa}^{(n),\text{fer}} \approx p s_{l,\kappa}^{(n),\text{fer}}(x),\tag{21}
$$

with the universal function  $s_{l,\kappa}^{(n),\text{fer}}(x)$  being defined in [\(13\)](#page-1-5) and taking the form [\(14\)](#page-1-4). We further get

<span id="page-2-2"></span>
$$
\lim_{L \to +\infty} \frac{S_{L,xL,p\kappa}^{(n),\text{fer}}}{L} = z s_{l,\kappa}^{(n),\text{fer}}(x),\tag{22}
$$

with the definition

$$
z \equiv \lim_{L \to +\infty} \frac{p}{L} \in \left[0, \frac{1}{l}\right].\tag{23}
$$

It is easy to confirm the formula [\(22\)](#page-2-2) numerically.

### C. States with mixed occupancy

We also consider more general states, in which different unit patterns are excited in different parts of the momentum space. The parts with different unit patterns may be adjacent or disjoint in the momentum space, and all the excited parts together may or may not occupy the entire spectrum. Since the relative positions of the different patterns are not important, we denote such states shorthand as  $K = \bigcup_{i=1}^{r} (p_i \kappa_i)$ . Each excited part is characterized by the length of the unit pattern  $l_i$ , the mode of the unit pattern  $\kappa_i$ , and the number of repetitions  $p_i$ . In the scaling limit, we define

<span id="page-2-4"></span>
$$
z_i \equiv \lim_{L \to +\infty} \frac{p_i}{L}.\tag{24}
$$

Of course, there is  $\sum_{i=1}^r z_i l_i \in (0,1]$ .

For example, we may have the unit pattern with  $l_1 = 2$ and  $\kappa_1 = \{0\}$  excited in the momentum space  $[0, \frac{L}{3} - 1]$ , and the unit pattern with  $l_2 = 3$  and  $\kappa_2 = \{0, 1\}$  excited in the momentum space  $[\frac{L}{2}, \frac{3L}{4} - 1]$ . For  $L = 24$ , such a state has the excited modes

$$
K = (3\kappa_1) \bigcup (2\kappa_2) = \{0, 2, 4, 6, 12, 13, 15, 16\},\tag{25}
$$

and it can be denoted as

•◦•◦•◦•◦◦◦◦◦••◦••◦◦◦◦◦◦◦ . (26)

In such a state with mixed occupancy, we conjecture that each unit pattern still makes an independent and universal contribution to the entanglement entropy

$$
S_{L,xL,\bigcup_{i=1}^r(p_i\kappa_i)}^{(n),\text{fer}} \approx \sum_{i=1}^r p_i s_{l_i,\kappa_i}^{(n),\text{fer}}(x),\tag{27}
$$

which leads to

<span id="page-2-3"></span>
$$
\lim_{L \to +\infty} \frac{S_{L,xL,\bigcup_{i=1}^r(p_i \kappa_i)}^{(n), \text{fer}}}{L} = \sum_{i=1}^r z_i s_{l_i, \kappa_i}^{(n), \text{fer}}(x). \tag{28}
$$

This is the main result of this paper. We have expressed the volume-law many-body entanglement entropy as the sum of few-body entanglement entropies. It is easy to confirm the formula [\(28\)](#page-2-3) numerically. The results are also the same as those in the limit  $L_A \ll L$  found in [\[15\]](#page-4-7).

# III. INTERACTING FERMIONS

The numerical verification of formulas [\(22\)](#page-2-2) and [\(28\)](#page-2-3) in free fermionic chains motivates us to further conjecture that these formulas still hold true in integrable interacting fermionic chains. By applying the Jordan-Wigner transformation, the transverse field Ising chain, given by

$$
H = -\frac{1}{2} \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right), \tag{29}
$$

is transformed into a fermionic chain with nearestneighbor interactions. This model is solvable, as shown in [\[54,](#page-5-17) [55\]](#page-5-18), and the entanglement entropy can be calculated following the methods outlined in [\[10,](#page-4-10) [11,](#page-4-11) [15\]](#page-4-7). We have carried out extensive numerical verifications of formulas [\(22\)](#page-2-2) and [\(28\)](#page-2-3) in the Ising chain, thereby confirming the aforementioned conjecture.

### IV. SPIN-1/2 XXZ CHAIN

The spin-1/2 XXZ chain has the Hamiltonian

$$
H = -\frac{1}{4} \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right), \qquad (30)
$$

with periodic boundary conditions for the Pauli matrices  $\sigma_{L+1}^{\alpha} = \sigma_1^{\alpha}, \ \alpha = x, y, x.$  The excited states in the XXZ chain can be obtained from the coordinate Bethe ansatz [\[56–](#page-5-19)[59\]](#page-6-0).

We consider the energy eigenstates  $|\mathcal{I}\rangle$  with an extensive number of magnons, which are labeled by the Bethe numbers  $\mathcal{I} = \{I_1, I_2, \dots\}$ . For example, there may be a length l unit pattern  $\iota = \{0\}$  excited in the whole or part of the space of the Bethe numbers  $[0, L-1]$ , with the number of repetitions being p. When  $\Delta = 2$ , this corresponds to a length  $l-1$  unit pattern  $\kappa = \{0\}$  with repetition p in the momentum space. We obtain the Rényi entropy and entanglement entropy

$$
S_{L,xL,pu}^{(n),\text{XXZ}} \approx ps_{l-1,\kappa}^{(n),\text{fer}}(x),\tag{31}
$$

which is numerically checked in Fig. [3.](#page-3-0)



<span id="page-3-0"></span>FIG. 3. Entanglement entropy in states with an extensive number of magnons in the XXZ chain (symbols) is compared with the corresponding analytical formula from the fermionic chain (blue dashed lines).

# V. FREE BOSONS

A chain of free bosons has the Hamiltonian

$$
H_{\rm bos} = \sum_{j=1}^{L} \left( a_j^{\dagger} a_j + \frac{1}{2} \right),
$$
 (32)

with the bosonic modes  $a_j$  and  $a_j^{\dagger}$  satisfying  $[a_{j_1}, a_{j_2}] =$  $[a_{j_1}^{\dagger}, a_{j_2}^{\dagger}] = 0$  and  $[a_{j_1}, a_{j_2}^{\dagger}] = \delta_{j_1 j_2}$ . In an energy eigenstate, the entanglement entropy can be obtained from the subsystem mode method  $[33]$ , and the Rényi entropy could be obtained from the permanent formula derived in  $[32]$  using the wavefunction method  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$  $[25, 26, 60, 61]$ .

We consider the entanglement entropy in a state with repeated unit patterns in the bosonic chain. Following [\(13\)](#page-1-5) in the fermionic chain, we define

<span id="page-3-3"></span>
$$
s_{l,\kappa}^{(n),\text{bos}}(x) \equiv \lim_{p \to +\infty} \frac{S_{pl,xpl,p\kappa}^{(n),\text{bos}}}{pl}.\tag{33}
$$

Following [\(14\)](#page-1-4), one might naively expect that the function  $s_{l,\kappa}^{(n),\text{bos}}(x)$  takes the form

<span id="page-3-2"></span>
$$
s_{l,\kappa}^{(n),\text{naive}}(x) \equiv y S_{l,\alpha+1,\kappa}^{(n),\text{bos}} + (1-y) S_{l,\alpha,\kappa}^{(n),\text{bos}},\tag{34}
$$

with the definitions of  $\alpha$  and  $y$  given in [\(15\)](#page-1-7). However, numerical results show that while the function  $s_{l,\kappa}^{(n),\text{bos}}(x)$  is well-defined, it differs from the naive expectation  $s_{l,\kappa}^{(n),\text{naive}}(x)$ . We present numerical examples in Fig. [4.](#page-3-1)



<span id="page-3-1"></span>FIG. 4. The entanglement entropy in free bosonic chains (symbols) is contrasted with the naive expectation  $s_{l,\kappa}^{(n),\text{naive}}(x)$  [\(34\)](#page-3-2) (red dashed lines). In each panel, the blue dashed line represents the interpolation of the results corresponding to the maximum L.

### VI. INTERACTING BOSONS

Similarly to [\(28\)](#page-2-3) in the fermionic chain, we conjecture that in the free and interacting bosonic chains, there is

$$
\lim_{L \to +\infty} \frac{S_{L,xL,\bigcup_{i=1}^r(p_i \kappa_i)}^{(n),\text{bos}}}{L} = \sum_{i=1}^r z_i s_{l_i,\kappa_i}^{(n),\text{bos}}(x),\tag{35}
$$

with the definition of  $z_i$  in  $(24)$  and the definition of  $s_{l_1, \kappa_i}^{(n), \text{bos}}$  $\binom{n}{l_i,\kappa_i}(x)$  in [\(33\)](#page-3-3). We consider the harmonic chain

$$
H = \frac{1}{2} \sum_{j=1}^{L} \left[ p_j^2 + m^2 q_j^2 + (q_j - q_{j+1})^2 \right],\qquad(36)
$$

with the periodic boundary condition  $q_{L+1} = q_j$  and the commutation relations  $[q_{j_1}, q_{j_2}] = [p_{j_1}, p_{j_2}] = 0$  and  $[q_{j_1}, p_{j_2}] = i\delta_{j_1j_2}$ . The harmonic chain is essentially a chain of bosons with nearest-neighbor interactions. The above conjecture could be easily checked by numerically calculating the Rényi entropy in excited energy eigenstates following the wavefunction method [\[25,](#page-5-7) [26,](#page-5-13) [62,](#page-6-3) [63\]](#page-6-4).

### VII. DISCUSSIONS

We have studied the entanglement entropy in quasiparticle states that exhibit repeated and sequential unit patterns in momentum space. We found that each unit pattern contributes independently and universally to the entanglement in the scaling limit, leading to a volumelaw scaling of entanglement entropy. Numerical simulations confirmed these findings, demonstrating volumelaw entanglement fragmentation in both fermionic and bosonic chains. We derived an analytical formula for free fermions that is applicable not only to interacting fermions in the scaling limit but also, with certain identifications, to the spin-1/2 XXZ chain.

It was suggested in [\[25\]](#page-5-7) to harness the entanglement of quasiparticles for quantum information purposes. However, the entanglement among a small number of quasiparticles is delicate and susceptible to environmental disruptions. The results presented in this paper imply that by successively exciting repeated patterns in momentum space, it is possible to create a robust volume-law entanglement of quasiparticles.

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