# QUANTUM KKL-TYPE INEQUALITIES REVISITED

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ABSTRACT. In the present paper, we develop the random restriction method in the quantum framework. By applying this method, we establish the quantum Eldan-Gross inequality, the quantum Talagrand isoperimetric inequality, and related quantum KKL-type inequalities. Our results recover some recent results of Rouzé et al. [22] and Jiao et al. [11], which can be viewed as alternative answers to the quantum KKL conjecture proposed by Motanaro and Osborne in [18].

## 1. Introduction

Motivated by problems from complexity theory, geometric functional analysis, and computer science, numerous remarkable results have been developed in the hypercube framework, making Boolean analysis one of the most active areas in discrete Fourier analysis, combinatorial optimization, and related fields in the past decades. To state results, we begin with recalling basic concepts and notions in hypercube setting. For fixed $n \in \mathbb{N}$, let $\{-1,1\}^n$ be the hypercube equipped with the uniform probability measure $\mu_n$, and let $L_p(\{-1,1\}^n)$ be the associated $L_p$ space for $1 \le p \le \infty$. In the sequel, we will use the shorthand notation $[n] := \{1,2,\cdots,n\}$. For each $j \in [n]$, the $j$-th influence of $f : \{-1,1\}^n \to \mathbb{R}$ is given by

$$\mathrm{Inf}_j(f) = \mu_n(\{x \in \{-1,1\}^n | \, f(x) \neq f(x^{\oplus j})\}),$$

where $x^{\oplus j}$ means flipping the $j$-th variable of $x$, i.e. for $x = (x_1,\cdots,x_n) \in \{-1,1\}^n$,

$$x^{\oplus j} = (x_1,\cdots,x_{j-1},-x_j,x_{j+1},\cdots,x_n).$$

The total influence of $f$ is defined by $\mathrm{Inf}(f) = \sum_{j\in[n]} \mathrm{Inf}_j(f)$, which is often used to measure the complexity of the function $f$. To illustrate the analytic property of the $j$-th influence of $f$, we now recall the $j$-th partial derivative of the $f : \{-1,1\}^n \to \mathbb{R}$ as follows:

$$\mathbf{d}_j(f)(x) := \frac{f(x) - f(x^{\oplus j})}{2}, \quad x \in \{-1,1\}^n. \tag{1.1}$$

In particular, for each Boolean function $f : \{-1,1\}^n \to \{-1,1\}$, it is easy to compute that

$$\mathrm{Inf}_j(f) = \|\mathbf{d}_j(f)\|^p_{L_p(\{-1,1\}^n)}, \quad \forall 1 \le p < \infty, j \in [n]. \tag{1.2}$$

Bounding the (total) influence of $f$ in terms of the variance of $f$ is one of the essential themes of Boolean analysis, which is closely related to specific types of

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functional inequalities on the hypercube. In particular, the Poincaré inequality on the hypercube can be stated as follows (see e.g. [19, p. 36]):

$$\operatorname{var}(f) := \|f - \mathbb{E}_{\mu_n}(f)\|^2_{L_2(\{-1,1\}^n)} \le \sum_{j=1}^n \|\mathbf{d}_j(f)\|^2_{L_2(\{-1,1\}^n)}, \tag{1.3}$$

where $\mathbb{E}_{\mu_n}(f)$ is referred as the expectation of $f$ with respect to the uniform measure $\mu_n$. Hence, combining (1.2) with (1.3) leads to the following lower bound

$$\max_{j\in[n]} \operatorname{Inf}_j(f) \ge \frac{1}{n},$$

for each balanced Boolean function $f$, that is, a Boolean function with $\operatorname{var}(f) = 1$.

However, in many aspects, the Poincaré inequality far from be sharp. In the remarkable paper [12], Kahn, Kalai and Linial strengthened the Poincaré inequality in a fundamental way, which leads to the following inequality: there exists a universal constant $C > 0$ such that

$$\max_{j\in[n]} \operatorname{Inf}_j(f) \ge C\frac{\log(n)}{n}, \tag{1.4}$$

for every balanced Boolean function $f$. More precisely, Kahn, Kalai and Linial [12] established the following functional inequality elegantly.

**Theorem 1.1** (Kahn-Kalai-Linial)**.** *There exists a universal constant $C > 0$ such that the following holds*

$$\operatorname{var}(f) \le C\frac{\sum_{j=1}^n \|\mathbf{d}_j(f)\|^2_{L_2(\{-1,1\}^n)}}{\log\left(1/\max_{j\in[n]} \|\mathbf{d}_j(f)\|^2_{L_2(\{-1,1\}^n)}\right)}, \tag{1.5}$$

*for every function $f : \{-1,1\}^n \to \mathbb{R}$.*

Due to the fundamental role in Boolean analysis, (1.4) (or, (1.5)) is now known as the KKL inequality, and we refer to [19] to interesting applications of the inequality. One of significant improvements of the KKL inequality is the Talagrand ($L_1$-$L_2$-) influence inequality established in [24].

**Theorem 1.2** (Talagrand)**.** *There exists a universal constant $C > 0$ such that for each function $f : \{-1,1\}^n \to \mathbb{R}$ the following holds*

$$\operatorname{var}(f) \le C\sum_{j=1}^n \frac{\|\mathbf{d}_j(f)\|^2_{L_2(\{-1,1\}^n)}}{1 + \log(\|\mathbf{d}_j(f)\|_{L_2(\{-1,1\}^n)}/\|\mathbf{d}_j(f)\|_{L_1(\{-1,1\}^n)})}. \tag{1.6}$$

By (1.2), it is clear that the Talagrand influence inequality (1.6) implies the KKL inequality (1.4). Since then, the KKL inequality, the Talagrand influence inequality, and their extensions become one of the fundamental tools in Boolean analysis, geometric functional analysis, computer science, and related fields. We refer interested readers to [3, 5, 7, 14, 20] for further information and the extensive bibliographies therein.

More recently, motivated by a conjecture of Talagrand [25], Eldan and Gorss [6] (see also [7]) applied stochastic analysis techniques to prove the following result known as the Eldan-Gross inequality.

**Theorem 1.3** (Eldan-Gross). *There exists a universal constant $C>0$ such that the following inequality holds*

(1.7)

$$\mathrm{var}(f)\sqrt{\log\left(1+\frac{e}{\sum_{j=1}^n\|\mathbf{d}_j(f)\|^2_{L_1(\{-1,1\}^n)}}\right)}\le C\left\|\left(\sum_{j=1}^n|\mathbf{d}_j(f)|^2\right)^{1/2}\right\|_{L_1(\{-1,1\}^n)},$$

*for every Boolean function $f:\{-1,1\}^n\to\{-1,1\}$.*

Notably, the Eldan-Gross inequality (i.e., Theorem 1.3) unifies the KKL inequality (i.e., Theorem 1.1) and Talagrand's isoperimetric inequality [23, Theorem 1.1], making Theorem 1.3 into one of the most efficient tools in Boolean analysis. For some new proofs of (1.7), we refer the interested reader to [1, 8, 10, 21].

It is worthwhile to mention that the original proofs of the KKL inequality and the Talagrand influence inequality rely on the hypercontractivity principle and the heat semigroup theory on hypercubes, while the proof of the Eldan-Gross inequality utilizes methods from stochastic analysis which is of different nature of the semigroup approach. Recently, the random restriction method has been viewed as a valuable tool for proving functional inequalities in the hypercube setting. Specifically, Kelman et al. [14] applied this method to give a unified proof of the KKL inequality (1.4) and the Talagrand influence inequality (1.6), along with some extensions. Eldan et al. [7] reproved the Eldan-Gross inequality (1.7) and the Talagrand isoperimetric inequality (i.e., [23, Theorem 1.1]) via the random restriction technique. We refer the interested reader to [16, 15, 14, 8] for further recent developments of the Fourier random restriction method in Boolean analysis.

In the present paper, motivated by the quantum KKL conjecture and related problems, we aim to develop the random restriction method to the noncommutative (or quantum) settings and apply such method to establish some quantum analogies of (1.4), (1.6) and (1.7).

In the quantum setting, the $n$-folds tensor product of $M_{2\times 2}(\mathbb{C})$ equipped with normalized trace, denoted by $(\mathbb{M}_{2^n},\mathrm{tr})$ for short, is viewed as the noncommutative correspondence of $n$-dimensional hypercube $(\{-1,1\}^n,\mu_n)$. For each $j\in[n]$, let $\mathbf{d}_j$ be the $j$-th partial derivative operator on $\mathbb{M}_{2^n}$ (see Sect. 2 for the definition). Recall from [18, Definition 3.1] that an element $T\in\mathbb{M}_{2^n}$ is said to be Boolean if $T$ is self-adjoint and unitary, that is, $T^*=T$ and $T^*T=\mathbf{1}$. In [18, Proposition 11.1], Montanaro and Osborne derived a quantum analogy of the Talagrand influence inequality (1.6) via the quantum hypercontractivity principle. However, due to some intrinsic differences between classical and quantum hypercubes, the quantum Talagrand influence inequality can not generally lead to the quantum KKL inequality. On the other hand, Montanaro and Osborne [18, Proposition 11.5] applied some Fourier analysis techniques to derive a quantum KKL inequality for concrete quantum Boolean functions $T$ fulfilling $\|\mathbf{d}_j(T)\|_{L_1(\mathbb{M}_{2^n})}=\|\mathbf{d}_j(T)\|^2_{L_2(\mathbb{M}_{2^n})}$ for each $j\in[n]$. Such observations lead them conjecture the following problem, known as the quantum KKL conjecture.

**Conjecture 1.4** (Quantum KKL conjecture). *There exists a universal constant $C>0$ such that for each $n\in\mathbb{N}$ and quantum Boolean function $T$ the following holds*

$$\max_{j\in[n]}\|\mathbf{d}_j(T)\|^2_{L_2(\mathbb{M}_{2^n})}\ge\frac{C\mathrm{var}(T)\log(n)}{n}.$$

Recently, Rouzé, Wirth and Zhang [22] provided an alternative answer of the quantum KKL conjecture invoking the geometric influence of quantum Boolean functions. Notably, the main tools of their approach to the quantum KKL conjecture is the following quantum Talagrand-type inequality [22, Theorem 3.6], which was derived via the semigroup method.

**Theorem 1.5** (Rouzé-Wirth-Zhang)**.** *There exists a universal constant $C > 0$ such that for each $n \in \mathbb{N}$ and $1 \le p < 2$ the following holds*

$$\mathrm{var}(T) \le \left(\frac{C}{2-p}\right) \sum_{j=1}^{n} \frac{\|\mathbf{d}_j(T)\|_{L_p(\mathbb{M}_{2^n})}^p (1 + \|\mathbf{d}_j(T)\|_{L_p(\mathbb{M}_{2^n})}^p)}{1 + \log^+(1/\|\mathbf{d}_j(T)\|_{L_p(\mathbb{M}_{2^n})}^p)},$$

*for every self-adjoint $T \in \mathbb{M}_{2^n}$ with $\|T\|_{L_\infty(\mathbb{M}_{2^n})} \le 1$.*

By Theorem 1.5, one can easily derive the quantum KKL inequality invoking the $L_p$-influences in the following manner: There exists a universal constant $C > 0$ such that for each $1 \le p < 2$ and quantum Boolean function $T$, we have $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_p(\mathbb{M}_{2^n})}^p \ge \frac{(2-p)C\log(n)}{n}$ for every $n \in \mathbb{N}$. More recently, Jiao, Luo and Zhou [11] investigate the quantum KKL conjecture in the canonical anti-commuting (CAR) algebra framework. More precisely, we established the noncommutative Eldan-Gross inequality via the fermion oscillator semigroup theory and applied the noncommutative Eldan-Gross inequality to derive the following two types of noncommutative KKL inequalities. Let $\{Q_j\}_{j=1}^n$ be $n$-configuration observables and CAR algebra $\mathcal{A}_{car,n}$ be the $*$-algebra generated by $\{Q_j\}_{j=1}^n$.

**Theorem 1.6** (Jiao-Luo-Zhou)**.** *There exists a universal constant $C > 0$ such that, for each $\varepsilon \in (0,1)$ and each balanced Boolean function $T \in \mathcal{A}_{car,n}$, one of the following inequalities holds:*

(i) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2(\mathcal{A}_{car,n})}^2 \ge \frac{C\varepsilon\log(n)}{n}$;
(ii) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1(\mathcal{A}_{car,n})} \ge \frac{C}{n^{(1+\varepsilon)/2}}$.

To derive a noncommutative KKL inequality in the CAR algebra setting invoking $L_2$-influence, we introduce the index for balanced Boolean function in $\mathcal{A}_{car,n}$ and proved the following result; see [11, Theorem 6.7].

**Theorem 1.7.** *For each $n \in \mathbb{N}$ and every balanced Boolean function $T$ with $\mathrm{ind}(T) < 2$, there exists a constant $C_{\mathrm{ind}(T)} > 0$ (depending only on the index) such that*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2(\mathcal{A}_{car,n})}^2 \ge \frac{C_{\mathrm{ind}(T)}\log(n)}{n},$$

*where the definition of $\mathrm{ind}(T)$ will be given in Section 5.*

Furthermore, it has been shown in [11, Remark 6.5] that the CAR algebra counterpart of the KKL inequality (for $L_2$-influence) fails for general balanced Boolean functions. Precisely, let $T := \frac{1}{\sqrt{n}}\sum_{j=1}^n Q_j$, and it is easy to see that $T$ is a balanced Boolean function in $\mathcal{A}_{car,n}$ such that $\|\mathbf{d}_j(T)\|_{L_2(\mathcal{A}_{car,n})}^2 = \frac{1}{n}$ for each $j \in [n]$, which disproves the KKL conjecture in the CAR algebra setting. Nevertheless, the quantum KKL conjecture of Montanaro and Osborne remains open.

In the present paper, we continue to explore quantum functional inequalities which are closely related to the quantum KKL conjecture. On the one hand, due to the fundamental role of the random restriction method in hypercubes, we develop the random restriction technique in the quantum setting. On the other hand, inspired by Kelman et al. [14], Rouzé et al [22] and Jiao et al. [11], we will derive and recover all mentioned inequalities in the quantum setting via the quantum random restriction technique. The main results of the present paper are outlined as follows.

As the first main result of our random restriction technique, we establish the following dimension free quantum KKL inequality.

**Theorem 1.8.** *There exists a universal constant $K > 0$ such that for each $1 \leq p < 2$ and each $T \in \mathbb{M}_{2^n}$ with $0 \leq T \leq 1$, the following holds*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|^p_{L_p(\mathbb{M}_{2^n})} \geq \frac{1}{4}\exp\left\{-\left(\frac{K}{2-p}\right)\cdot\frac{\sum_{j=1}^n \|\mathbf{d}_j(T)\|^p_{L_p(\mathbb{M}_{2^n})}}{\mathrm{var}(T)}\right\}. \tag{1.8}$$

Analogous to the approach presented in [14], we derive the following quantum KKL inequality (invoking $L_p$-influence) via Theorem 1.8, which was recently proved by Rouzé et al. [22, Theorem 3.9]. And we show in Remark 4.4 that the following KKL-type inequality fails for $p = 2$ even in the commutative case. Hence, Theorem 1.9 may be the best possible quantum KKL-type inequality for bounded elements.

**Theorem 1.9.** *There exists a universal constant $C > 0$ such that for every $1 \leq p < 2$ and $T \in \mathbb{M}_{2^n}$ with $0 \leq T \leq 1$, the following holds*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|^p_{L_p(\mathbb{M}_{2^n})} \geq C\frac{(2-p)\mathrm{var}(T)\log(n)}{n}.$$

The second main ingredient of this paper consists of the following two quantum isoperimetric inequalities, which can be used to derive the quantum counterpart of KKL-type inequalities presented in [11, Sect. 6].

**Theorem 1.10** (Quantum Talagrand-type isoperimetric inequality)**.** *There exists a universal constant $K > 0$ such that for each projection $T \in \mathbb{M}_{2^n}$ the following holds*

$$\mathrm{var}(T)\sqrt{\log\left(\frac{1}{\mathrm{var}(T)}\right)} \leq K\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}\right\|_{L_1(\mathbb{M}_{2^n})}. \tag{1.9}$$

Combing Theorem 1.10 and estimations on the Fourier spectrum of projection $T \in \mathbb{M}_{2^n}$, we establish the quantum Eldan-Gross inequality as follows.

**Theorem 1.11** (Quantum Eldan-Gross inequality)**.** *There exists a universal constant $K > 0$ such that for each projection $T \in \mathbb{M}_{2^n}$ the following holds*

$$\mathrm{var}(T)\sqrt{\log\left(1+\frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|^2_{L_1(\mathbb{M}_{2^n})}}\right)} \leq K\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}\right\|_{L_1(\mathbb{M}_{2^n})}.$$

The proofs of above theorems are provided in Section 4 and Section 5. As the applications of the quantum Eldan-Gross inequality, we establish quantum version of Theorem 1.6 and Theorem 1.7 in Section 5. Due to intrinsic differences between quantum and classical hypercubes, additional efforts must be made to overcome difficulties that arise from these differences when employing the random restriction method; see the proof of Proposition 5.2 for instance.

The rest of this paper is organized as follows. In Section 2, we present the necessary background and results in the quantum setting, including basic properties of the quantum Ornstein-Uhlenbeck semigroup, quantum hypercontractivity, and its equivalence to the quantum logarithmic Sobolev inequality. In Section 3, we introduce the quantum Fourier random restriction technique and provide some fundamental estimations on the Fourier spectrum for elements in $\mathbb{M}_{2^n}$. This method and the associated estimates are frequently used to derive inequalities throughout the paper. From Section 4 to Section 5, we provide proofs of the previously presented theorems by combining the quantum random restriction technique with the quantum semigroup method. Additionally, we derive the corresponding quantum KKL-type inequality in the respective sections.

**Remark 1.12.** *After completing this work, we learned that Blecher, Gao and Xu* [2] *developed a similar random restriction technique and applied it to investigate KKL inequality and high order extension of the Talagrand influence inequality in the quantum setting. Precisely, Theorem 1.8 with* $p = 1$ *is proved independently by Blecher et al. in* [2].

Throughout the paper, $n$ be a fixed positive integer and $[n] \coloneqq \{1, 2, 3 \ldots, n\}$. For a parameter $p$, we denote $K_p$ the positive constant depending only on the parameter $p$ (it may vary from line to line). We use the notation $A \approx_p B$ to stand that $K_p A \leq B \leq C_p A$ for some positive constants $K_p$ and $C_p$ (depending only on the parameter $p$), and we drop the subscript $p$ if the constants are universal. Notations $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively, and we let $(\mathbb{M}_2(\mathbb{C}), \mathrm{tr})$ be the algebra of all $2 \times 2$ complex matrices equipped with the normalized trace tr.

## 2. Preliminaries

In this section, we collect concepts and background that will be used throughout the paper.

### 2.1. The quantum hypercube.

Denote by $(\mathbb{M}_2(\mathbb{C}), \mathrm{tr})$ the algebra of $2 \times 2$ complex matrices equipped with the *normalized* trace tr and the unit $\mathbf{1}_2$ (i.e., the $2 \times 2$ identity matrix). The quantum analogue of the hypercube $\{-1, 1\}^n$ is $\mathbb{M}_2(\mathbb{C})^{\otimes n} \cong \mathbb{M}_{2^n}(\mathbb{C})$ equipped with the normalized trace $\mathrm{tr}_n \coloneqq \mathrm{tr}^{\otimes n}$ (simply denoted by $(\mathbb{M}_{2^n}, \mathrm{tr})$ if no confusion arise) and the unit $\mathbf{1} \coloneqq \mathbf{1}_2^{\otimes n}$. For every $1 \leq p \leq \infty$, the noncommutative $L_p$ space generated by $\mathbb{M}_{2^n}$, denoted by $L_p(\mathbb{M}_{2^n})$, is the space $\mathbb{M}_{2^n}$ equipped with the norm

$$\|T\|_{L_p} \coloneqq \begin{cases} \left(\mathrm{tr}(|T|^p)\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \max_{j \in [n]} s_j(T), & \text{if } p = \infty, \end{cases}$$

where $\{s_j(T)\}_{j=1}^n$ is the set of singular values of $T$. The *variance* of $T \in \mathbb{M}_{2^n}$ is defined by

$$\mathrm{var}(T) \coloneqq \mathrm{tr}(|T|^2) - |\mathrm{tr}(T)|^2 = \|T - \mathrm{tr}(T)\|_{L_2}^2 .$$

Following [18], we recall the concept of quantum Boolean functions in $\mathbb{M}_{2^n}$.

**Definition 2.1** ([18])**.** *An element* $T \in \mathbb{M}_{2^n}$ *is said to be a* quantum Boolean function *if* $T$ *is self-adjoint (i.e.,* $T^* = T$*) and unitary (i.e.,* $TT^* = T^*T = \mathbf{1}$*). A quantum Boolean function* $T \in \mathbb{M}_{2^n}$ *is said to be balanced if* $\mathrm{tr}(T) = 0$.

**Remark 2.2.** *It is clear that the concept of quantum Boolean functions and projections are equivalent in the following sense. For a given quantum Boolean function* $T \in \mathbb{M}_{2^n}$*,* $S := \frac{\mathbf{1}+T}{2}$ *is a projection in* $\mathbb{M}_{2^n}$*, that is,* $S^* = S = S^2$*. Conversely, for a given projection* $S \in \mathbb{M}_{2^n}$*, it follows that* $T := 2S - \mathbf{1}$ *is a quantum Boolean function in* $\mathbb{M}_{2^n}$*. Hence, we will ignore the difference between quantum Boolean functions and projections in* $\mathbb{M}_{2^n}$*.*

Using the Pauli matrices, we represent elements in $\mathbb{M}_{2^n}$ by their Fourier expansion, which is a quantum counterpart of the Walsh expansion for functions on $\{-1,1\}^n$. Recall that the Pauli matrices as follows:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

For $\mathbf{s} = (s_i)_{i=1}^n \in \{0,1,2,3\}^n$, set

$$\sigma_{\mathbf{s}} = \sigma_{s_1} \otimes \cdots \otimes \sigma_{s_n}.$$

Clearly, $\{\sigma_{\mathbf{s}}\}_{\mathbf{s}\in\{0,1,2,,3\}^n}$ are quantum Boolean functions which forms an orthonormal basis in $L_2(\mathbb{M}_{2^n})$. Hence, each $T \in \mathbb{M}_{2^n}$ can be uniquely represented by

$$T = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}},$$

where $\widehat{T}(\mathbf{s})$ is the Fourier coefficient defined by $\widehat{T}(\mathbf{s}) = \operatorname{tr}(\sigma_{\mathbf{s}}^* T)$.

For each $j \in [n]$ and $\alpha \in \{1,2,3\}$, we let $e_j^\alpha := (0,\ldots,0,\alpha,0\ldots,0)$ where $\alpha$ appears in the $j$-th position. For each $\mathbf{s} \in \{0,1,2,3\}^n$, $j \in [n]$ and $\alpha \in \{1,2,3\}$, define $\mathbf{s} \oplus e_j^\alpha$ (resp. $\mathbf{s} \ominus e_j^\alpha$) by

$$\mathbf{s} \oplus e_j^\alpha := (s_1,\ldots,s_{j-1}, s_j + \alpha, s_j, \ldots, s_n)$$
$$\big(\text{resp. } \mathbf{s} \ominus e_j^\alpha := (s_1,\ldots,s_{j-1}, s_j - \alpha, s_{j+1}, \ldots, s_n)\big).$$

For any $d \in [n]$, the *Rademacher projection* is defined by

$$\mathrm{Rad}_{\leq \mathrm{d}}(T) = \sum_{\substack{\mathbf{s}\in\{0,1,2,3\}^n \\ |\mathrm{supp}(\mathbf{s})|\leq d}} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}},$$

where $T = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}$.

2.2. **Hypercontractivity, influence and the modified Log-Sobolev inequality.** Here we collect analytic tools such as the hypercontractivity of the quantum Ornstein-Uhlenbeck semigroup, basic properties of influences, and the curvature condition in quantum hypercubes. For $\mathbf{s} \in \{0,1,2,3\}^n$, we define $\mathrm{supp}(\mathbf{s}) := \{j \in [n] : s_j \neq 0\}$ and $|\mathrm{supp}(\mathbf{s})|$ stand for the number of non-zero $s_j$'s, that is, $|\mathrm{supp}(\mathbf{s})| = \#\{j \in [n] : s_j \neq 0\}$. Let $L : \mathbb{M}_2 \to \mathbb{M}_2$ defined by

$$L(A) := A - \operatorname{tr}(A)\mathbf{1}_2, \tag{2.1}$$

and

$$e^{-tL}(A) := e^{-t}A + (1-e^{-t})\operatorname{tr}(A)\mathbf{1}_2. \tag{2.2}$$

In viewing of (2.1) and (2.2), we have

$$L(\sigma_j) = \begin{cases} \sigma_j, & \text{if } j \neq 0; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } e^{-tL}(\sigma_j) = \begin{cases} e^{-t}(\sigma_j), & \text{if } j \neq 0; \\ \sigma_0, & \text{otherwise.} \end{cases}$$

For each stabilizer operator $\sigma_{\mathbf{s}}$, let

$$\mathcal{P}_t(\sigma_{\mathbf{s}}) = e^{-tL}(\sigma_{s_1}) \otimes \cdots \otimes e^{-tL}(\sigma_{s_n}).$$

The infinitesimal generator of the semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ is given by

$$\mathcal{L}(\sigma_{\mathbf{s}}) := \lim_{t\to 0} \frac{\sigma_{\mathbf{s}} - \left(e^{-tL}\right)^{\otimes n}(\sigma_{\mathbf{s}})}{t} = |\mathbf{s}|\sigma_{\mathbf{s}},$$

for each stabilizer operator $\sigma_{\mathbf{s}}$. In the sequel, we denote $\{\mathcal{P}_t\}_{t\geq 0}$ by $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ to emphasize the generator $\mathcal{L}$. It is clear that for each $T \in \mathbb{M}_{2^n}$ we have

$$\mathcal{L}(T) = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} |\mathbf{s}|\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}$$

and

$$e^{-t\mathcal{L}}(T) = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} e^{-t|\mathbf{s}|}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}.$$

For $j \in [n]$, the $j$-th *partial differential operator* (or, quantum *bit-flip map*) is defined by

$$\mathbf{d}_j := \mathbf{1}_2^{\otimes(j-1)} \otimes (\mathbf{1}_2 - \mathrm{tr}) \otimes \mathbf{1}_2^{\otimes(n-j)} \tag{2.3}$$

Thanks to the Fourier expansion of $T \in \mathbb{M}_{2^n}$, we obtain the following explicit formula for partial differential operators. For each $j \in [n]$ and $T \in \mathbb{M}_{2^n}$, we have

$$\mathbf{d}_j(T) = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \widehat{T}(\mathbf{s})\mathbf{d}_j(\sigma_{\mathbf{s}}) = \sum_{\substack{\mathbf{s}\in\{0,1,2,3\}^n \\ s_j\neq 0}} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}. \tag{2.4}$$

Moreover, it is easy to verify that $\{\mathbf{d}_j\}_{j=1}^n$ are orthogonal projections on $L_2(\mathbb{M}_{2^n})$ such that $\mathcal{L} = \sum_{j=1}^n \mathbf{d}_j$. By [18] (or, [17, Corollary 2]), the quantum Ornstein-Uhlenbeck semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ fulfills the *optimal* hypercontractivity as follows: for each $1 < p \leq q < \infty$, we have

$$\left\|e^{-t\mathcal{L}}\right\|_{L_p\to L_q} = 1 \text{ if and only if } t \geq \frac{1}{2}\log\left(\frac{q-1}{p-1}\right). \tag{2.5}$$

The equivalence between the hypercontractivity of semigroup and the logarithmic Sobolev inequality has been established by Gorss in his seminal paper [9]. Hence, repeat the same treatments of Gorss, we can deduce the following ($L_2$-)logarithmic Sobolev inequality from (2.5) (the proof is also same to [4, Theorem 5.3]).

**Lemma 2.3** (Log-Sobolev)**.** *For each $T \in \mathbb{M}_{2^n}$, we have*

$$2\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \mathrm{tr}\left[|T|^2\log\left(|T|^2\right)\right] - \|T\|_{L_2}^2 \log\left(\|T\|_{L_2}^2\right).$$

Motivated by the quantum KKL-type inequalities invoking $L_p$-influences with $1 \leq p < \infty$, we derive the following ($L_p$-) *modified logarithmic Sobolev inequality.*

**Lemma 2.4** (Modified Log-Sobolev inequality)**.** *Let $1 \leq p < 2$. Then, for each $T \in \mathbb{M}_{2^n}$ with $|T| \leq 1$, we have*

$$2\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_2}^2 \geq -K_p\|T\|_{L_2}\|T\|_{L_p}^{\frac{p}{2}} - \|T\|_{L_2}^2\log\left(\|T\|_{L_2}^2\right),$$

*where $K_p = \frac{4}{(2-p)e}$.*

*Proof.* By the Cauchy-Schwarz inequality, we obtain

$$\operatorname{tr}\left[-|T|^2\log(|T|^2)\right] \leq \|T\|_{L_2}\left[\operatorname{tr}\left(|T|^2\log^2\left(|T|^2\right)\right)\right]^{\frac{1}{2}}. \tag{2.6}$$

Let $K_p := \frac{4}{(2-p)e}$. Then, it is clear that

$$K_p^2 t^p \geq t^2\log^2\left(t^2\right), \quad \forall t\in[0,1].$$

Hence, it follows from the functional calculus of $|T|$ that

$$K_p^2|T|^p \geq |T|^2\log^2\left(|T|^2\right), \quad T\in\mathbb{M}_{2^n}.$$

Therefore,

$$\left[\operatorname{tr}\left(|T|^2\log^2(|T|^2)\right)\right]^{\frac{1}{2}} \leq K_p\|T\|_{L_p}^{\frac{p}{2}}. \tag{2.7}$$

Combining (2.6), (2.7) and the Log-Sobolev inequality (i.e., Lemma 2.3), we get the desired result. □

To derive the isoperimetric inequality in Section 5, we need the following facts regarding as the curvature condition of the quantum Ornstein-Uhlenbeck semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$.

**Proposition 2.5.** *Keep the notations as previous subsection. Then, for each $T\in\mathbb{M}_{2^n}$, we have*

(i) $\mathcal{L}(T^*T)-\mathcal{L}(T)^*T-T^*\mathcal{L}(T) = -2\sum_{j=1}^n(\mathbf{d}_j(T))^*(\mathbf{d}_j(T))$;

(ii) $(\mathbf{d}_je^{-t\mathcal{L}}(T))^*(\mathbf{d}_je^{-t\mathcal{L}}(T) \leq e^{-2t}e^{-t\mathcal{L}}\left((\mathbf{d}_j(T))^*(\mathbf{d}_j(T))\right)$, *for each $j\in[n]$;*

(iii) $\sum_{j=1}^n(\mathbf{d}_je^{-t\mathcal{L}}(T))^*(\mathbf{d}_je^{-t\mathcal{L}}(T)) \leq e^{-2t}e^{-t\mathcal{L}}\left(\sum_{j=1}^n(\mathbf{d}_j(T))^*(\mathbf{d}_j(T))\right)$.

*Proof.* The proof is analogous to the classical case, which can be verified via the Fourier expansion and the Gronwall-type inequality. Hence, we provide the proof of (ii) for the reader's convenience, and leave the details of (i) and (iii) to the reader. For each $j\in[n]$ and $0\leq s\leq t$, we define

$$\Lambda(s) := e^{-(t-s)\mathcal{L}}\left[\left|\mathbf{d}_je^{-s\mathcal{L}}(T)\right|^2\right] = e^{-(t-s)\mathcal{L}}\left[\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right].$$

Differentiating $\Lambda(s)$ and applying (i) we obtain that

$$\begin{aligned}
\Lambda'(s) &= \mathcal{L}\left[e^{-(t-s)\mathcal{L}}\left(\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right)\right]\\
&\quad - e^{-(t-s)\mathcal{L}}\left[\left(\mathcal{L}\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right]\\
&\quad - e^{-(t-s)\mathcal{L}}\left[\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathcal{L}\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right]\\
&= e^{-(t-s)\mathcal{L}}\Big[\mathcal{L}\left(\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right)\\
&\quad - \left(\mathcal{L}\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\\
&\quad - \left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathcal{L}\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\Big]\\
&\leq -2e^{-(t-s)\mathcal{L}}\left[\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)^*\left(\mathbf{d}_je^{-s\mathcal{L}}(T)\right)\right]\\
&= -2\Lambda(s).
\end{aligned} \tag{2.8}$$

Define $F(s) \coloneqq e^{2s}\Lambda(s)$ and note that (2.8) entails $F'(s) \le 0$ for all $0 \le s \le t$. Therefore,

$$e^{2t}\Lambda(t) - \Lambda(0) = F(t) - F(0) = \int_0^t F'(s)\, ds \le 0.$$

Rearranging the inequality yields that $\Lambda(t) \le e^{-2t}\Lambda(0)$, that is,

$$\left|\mathbf{d}_j e^{-t\mathcal{L}}(T)\right|^2 \le e^{-2t} e^{-t\mathcal{L}}\left[|\mathbf{d}_j(T)|^2\right].$$

□

We conclude this subsection with the following well-known Paley-Zygmund inequality, and we include the proof for the reader's convenience.

**Lemma 2.6** (Paley-Zygmund inequality)**.** *For each positive $T \in \mathbb{M}_{2^n}$, we have*

$$\operatorname{tr}\left[\mathbb{1}_{[\delta\|T\|_{L_1},\infty)}(T)\right] \ge (1-\delta)^2 \frac{\|T\|_{L_1}^2}{\|T\|_{L_2}^2}, \quad 0 < \delta < 1. \tag{2.9}$$

*Proof.* Since the desired inequality only invokes one positive element, it follows from the spectral theory that the inequality is essentially the classical Paley-Zygmund inequality. For positive $T \in \mathbb{M}_{2^n}$, we have

$$\begin{aligned} \|T\|_{L_1} &= \operatorname{tr}\left[\mathbb{1}_{[\delta\|T\|_{L_1},\infty)}(T)\cdot T\right] + \operatorname{tr}\left[\mathbb{1}_{[0,\delta\|T\|_{L_1})}(T)\cdot T\right] \\ &\le \operatorname{tr}\left[\mathbb{1}_{[\delta\|T\|_{L_1},\infty)}(T)\right]^{1/2}\|T\|_{L_2} + \delta\|T\|_{L_1} \end{aligned} \tag{2.10}$$

where we used the the Cauchy-Schwarz inequality. Rearranging (2.10) yields the desired inequality. □

2.3. **$L_p$-influences and related basic properties.** For $j \in [n]$, $1 \le p < \infty$ and $T \in \mathbb{M}_{2^n}$, denote the $j$-th $L_p$-influence of $T$ by

$$\operatorname{Inf}_j^p(T) \coloneqq \|\mathbf{d}_j(T)\|_{L_p}^p,$$

and the total $L_p$-influence of $T$ by

$$\operatorname{Inf}^p \coloneqq \sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_p}^p.$$

The $L_1$-influence is usually called the *geometric influence* in some literature. For $p = 2$, we will simply denote the $j$-th $L_2$-influence and the total $L_2$-influence of $T$ by $\operatorname{Inf}_j(T)$ and $\operatorname{Inf}(T)$, respectively. Hence, by (2.4), it follows that for each $T \in \mathbb{M}_{2^n}$, we have

$$\operatorname{Inf}_j(T) = \sum_{\substack{\mathbf{s}\in\{0,1,2,3\}^n \\ s_j \ne 0}} |\widehat{T}(\mathbf{s})|^2,$$

and

$$\operatorname{Inf}(T) = \sum_{j\in[n]} \operatorname{Inf}_j(T) = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} |\operatorname{supp}(\mathbf{s})| \widehat{T}(\mathbf{s})^2.$$

The following elementary facts can be deduced from the contraction of conditional expectations and the noncommutative Hölder inequality.

**Proposition 2.7.** *For $1 \le p \le 2$ and $T \in \mathbb{M}_{2^n}$ with $\|T\|_{L_\infty} \le 1$ we have*

(i) *for each $j \in [n]$, we have $\|\mathbf{d}_j(T)\|_{L_\infty} \le 1$,*
(ii) *for each $j \in [n]$, we have $\operatorname{Inf}_j(T) \le \operatorname{Inf}_j^p(T)$.*

*Proof.* (i) For each $j \in [n]$, we define $S : \mathbb{M}_2 \to \mathbb{M}_2$ by

$$S(T) \coloneqq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} T^t \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

where $T^t$ stands for the transpose of $T$. It is clear that

$$S(\sigma_0) = \sigma_0 \quad \text{and} \quad S(\sigma_l) = -\sigma_l$$

for $l \in \{1, 2, 3\}$. We further define $S_j \coloneqq \mathbf{1}_2^{\otimes(j-1)} \otimes S \otimes \mathbf{1}_2^{\otimes(n-j)}$. Hence, it is clear that

$$\mathbf{d}_j(T) = \frac{1}{2}\left(T - S_j(T)\right), \quad \text{for } T \in \mathbb{M}_{2^n}.$$

Since the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is unitary and the norm $\|\cdot\|_{L_p}$ is unitary invariant for every $p \in (0, \infty]$, it follows that $\|S_j(T)\|_{L_\infty} = \|T\|_{L_\infty}$. Therefore, we have

$$\|\mathbf{d}_j(T)\|_{L_\infty} = \frac{1}{2}\|T - S_j(T)\|_{L_\infty} \leq \|T\|_{L_\infty},$$

which proves the first claim.

(ii) For each $j \in [n]$, we have

$$\mathrm{Inf}_j(T) = \|\mathbf{d}_j(T)\|_{L_2}^2 \leq \||\mathbf{d}_j(T)|^p\|_{L_1} \cdot \||\mathbf{d}_j(T)|^{2-p}\|_{L_\infty} \leq \|\mathbf{d}_j(T)\|_{L_p}^p,$$

where we used $1 \leq p < 2$ and $\|\mathbf{d}_j(T)\|_{L_\infty} \leq 1$. □

## 3. The Noncommutative Random Restrictions and Related Estimates

In this section, motivated by [14], we introduce a noncommutative random restriction technique, which is one of the efficient toolkits of establishing functional inequalities in the quantum hypercube.

For each subset $J = \{j_1, \cdots, j_k\} \subseteq [n]$, we order it in the increasing order, that is, $j_1 < j_2 < \cdots < j_k$, and let $\mathbb{M}_J$ be the $*$-sub-algebra of $\mathbb{M}_{2^n}$ such that $\mathbb{M}_2$ only appears in the $j_i$-th position for $1 \leq i \leq k$. Hence, there exists a conditional expectation $\mathcal{E}_{\mathbb{M}_J}$ from $\mathbb{M}_{2^n}$ onto $\mathbb{M}_J$. More precisely, $\mathcal{E}_{\mathbb{M}_J}$ has the following explicit formula.

**Proposition 3.1** (Conditional Expectation)**.** *For each $J \subseteq [n]$, we have*

$$\mathcal{E}_{\mathbb{M}_J}(T) \coloneqq \sum_{\mathrm{supp}(\mathbf{s}) \subseteq J} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}, \quad \forall\, T \in \mathbb{M}_{2^n}.$$

We now introduce the *random restriction operator* as follows.

**Definition 3.2** (Restrictions)**.** *For each $J \subseteq [n]$ and $j \in [n]$, let $J^c \coloneqq [n] \setminus J$ and define the* restriction operator *$R_j^J : \mathbb{M}_{2^n} \to \mathbb{M}_{2^n}$ by setting*

$$R_j^J(T) \coloneqq \begin{cases} \mathcal{E}_{\mathbb{M}_{J^c \cup \{j\}}}(\mathbf{d}_j(T)), & j \in J \\ 0, & j \in J^c. \end{cases}$$

The following essential property regarding the explicit formula for the restriction operator acting on an element is easily deduced from the formula of conditional expectations (i.e., Proposition 3.1) and the Fourier expansion of partial derivatives (2.4). The proof is left to the interested reader.

**Lemma 3.3.** *Let $J \subseteq [n]$ and $T \in \mathbb{M}_{2^n}$. For each $j \in [n]$, we have*

$$R_j^J(T) = \sum_{\substack{\mathrm{supp}(\mathbf{s}) \subseteq J^c \\ \alpha \in \{1,2,3\}}} \widehat{T}(\mathbf{s} \oplus e_j^\alpha) \sigma_{\mathbf{s} \oplus e_j^\alpha}.$$

Applying Lemma 3.3, we obtain the following corollary.

**Corollary 3.4.** *Let $T \in \mathbb{M}_{2^n}$ and a fixed subset $J \subseteq [n]$.*

(i) *For each $j \in J$, we have*

$$\sum_{k=1}^n \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 = \sum_{k \in J^c} \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 + \|\mathbf{d}_j(R_j^J(T))\|_{L_2}^2.$$

(ii) *We have that*

$$\sum_{j \in J} \|R_j^J(T)\|_{L_2}^2 \leq \mathrm{var}(T).$$

(iii) *For each $k \in J^c$, we have*

$$\sum_{j \in J} \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 \leq \|\mathbf{d}_k(T)\|_{L_2}^2 .$$

*Proof.* We only show item (iii), and leave the easy verification of (i) and (ii) to the reader. By Lemma 3.3, we have

$$\begin{aligned}
\sum_{j \in J} \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 &= \sum_{j \in J} \mathrm{tr} \left| \sum_{\substack{\mathrm{supp}(\mathbf{s}) \subseteq J^c, \\ s_k \neq 0, \alpha \in \{1,2,3\}}} \widehat{T}(\mathbf{s} \oplus e_j^\alpha) \sigma_{\mathbf{s} \oplus e_j^\alpha} \right|^2 \\
&\leq \sum_{\mathrm{supp}(\widetilde{\mathbf{s}}) \subseteq J} \mathrm{tr} \left| \sum_{\substack{s_k \neq 0 \\ \mathrm{supp}(\mathbf{s}) \subseteq J^c}} \widehat{T}(\mathbf{s} \oplus \widetilde{\mathbf{s}}) \sigma_{\mathbf{s} \oplus \widetilde{\mathbf{s}}} \right|^2 \\
&= \sum_{\mathrm{supp}(\widetilde{\mathbf{s}}) \subseteq J} \sum_{\substack{s_k \neq 0 \\ \mathrm{supp}(\mathbf{s}) \subseteq J^c}} \left| \widehat{T}(\mathbf{s} \oplus \widetilde{\mathbf{s}}) \right|^2 \\
&= \sum_{s_k \neq 0} \left| \widehat{T}(\mathbf{s}) \right|^2 = \|\mathbf{d}_k(T)\|_{L_2}^2.
\end{aligned}$$

□

Several basic properties regarding the $L_p$-norm of the restriction operator and its relation to the influence are collected as follows.

**Lemma 3.5.** *Let $J \subseteq [n]$ and $T \in \mathbb{M}_{2^n}$ with $\|T\|_{L_\infty} \leq 1$.*

(i) *For each $j \in [n]$, we have $\|R_j^J(T)\|_{L_p}^p \leq \|\mathbf{d}_j(T)\|_{L_p}^p$, $1 \leq p \leq \infty$.*

(ii) *For each $j \in [n]$, we have $\|R_j^J(T)\|_{L_2}^2 \leq \|R_j^J(T)\|_{L_p}^p$, $1 \leq p \leq 2$.*

(iii) *We have $\sum_{j \in J} \mathrm{Inf}\left(R_j^J(T)\right) \leq \mathrm{var}(T) + \mathrm{Inf}(T)$.*

*Proof.* (i) It follows from the definition of restriction operator that

$$\|R_j^J(T)\|_{L_p} = \|\mathbb{E}_{\mathbb{M}_{J^c \cup \{j\}}}(\mathbf{d}_j(T))\|_{L_p} \leq \|\mathbf{d}_j(T)\|_{L_p}. \tag{3.1}$$

For (ii), it suffices to note that

$$\begin{aligned}\|R_j^J(T)\|_{L_2}^2 &= \||R_j^J(T)|^p|R_j^J(T)|^{2-p}\|_{L_1} \\ &\leq \||R_j^J(T)|^p\|_{L_1} \cdot \||R_j^J(T)|^{2-p}\|_{L_\infty} \leq \|R_j^J(T)\|_{L_p}^p,\end{aligned}$$

where the last inequality follows from (3.1) and Proposition 2.7 (i).

(iii) By items (i) and (iii) in Corollary 3.4, we have

$$\begin{aligned}\mathrm{Inf}(T) &= \sum_{k\in[n]} \|\mathbf{d}_k(T)\|_{L_2}^2 \geq \sum_{k\in J^c} \|\mathbf{d}_k(T)\|_{L_2}^2 \geq \sum_{k\in J^c}\sum_{j\in J} \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 \\ &= \sum_{j\in J}\left(\sum_{k\in J^c} \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2\right) = \sum_{j\in J}\left(\sum_{k=1}^n \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 - \|\mathbf{d}_j(R_j^J(T))\|_{L_2}^2\right).\end{aligned}$$

Applying Proposition 2.7 (i) again, we get

$$\begin{aligned}\mathrm{Inf}(T) &\geq \sum_{j\in J}\left(\sum_{k=1}^n \|\mathbf{d}_k(R_j^J(T))\|_{L_2}^2 - \|R_j^J(T)\|_{L_2}^2\right) \\ &= \sum_{j\in J}\mathrm{Inf}(R_j^J(T)) - \sum_{j\in J}\|R_j^J(T)\|_{L_2}^2 \geq -\mathrm{var}(T) + \sum_{j\in J}\mathrm{Inf}(R_j^J(T)),\end{aligned}$$

where the last inequality follows from Corollary 3.4 (ii). □

To investigate the Fourier spectrum of a given $T \in \mathbb{M}_{2^n}$, we introduce the following notations, which are inspired from their Boolean counterparts.

**Definition 3.6.** *For each $T \in \mathbb{M}_{2^n}$ and for $d \in [n]$, define*

$$W_{=d}(T) := \sum_{|\mathrm{supp}(\mathbf{s})|=d} \widehat{T}(\mathbf{s})^2,$$

$$W_{\geq d}(T) := \sum_{|\mathrm{supp}(\mathbf{s})|\geq d} \widehat{T}(\mathbf{s})^2,$$

*and*

$$W_{\approx d}(T) := \sum_{d\leq|\mathrm{supp}(\mathbf{s})|<2d} \widehat{T}(\mathbf{s})^2.$$

**Remark 3.7.** *It is clear that every random set $J \subseteq [n]$, formed by each point selected with probability $\delta$, corresponds to a vector in $(\{0,1\}^n, \mu_\delta)$, where*

$$\mu_\delta(\{x\}) = \delta^{\sum_{j=1}^n x_j}(1-\delta)^{n-\sum_{j=1}^n x_j}, \quad x = (x_j)_{j=1}^n \in \{0,1\}^n.$$

*If there is no confusion arises, we will simply write $J \in (\{0,1\}^n, \mu_\delta)$ for a random set $J$ (with selecting probability $\delta$).*

**Lemma 3.8.** *Let $d \in \mathbb{Z}_+$ and $T \in \mathbb{M}_{2^n}$ with $\|T\|_{L_\infty} \leq 1$. Then*

$$\mathbb{E}_J\left[\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2\right] \geq \frac{1}{8} W_{\approx d}[T],$$

*where $\mathbb{E}_J$ is the expectation taking with respect to the random subset $J$ with selecting probability $\frac{1}{d}$. Hence, there exists $J_0 \subseteq [n]$ such that $\sum_{j\in J_0}\|R_j^J(T)\|_{L_2}^2 \geq \frac{1}{8}W_{\approx d}T$.*

*Proof.* Using Lemma 3.3 and the orthogonality of $\{\sigma_{\mathbf{s}}\}_{\mathbf{s}\in\{0,1,2,3\}^n}$, we have

$$\begin{aligned}
\mathbb{E}_J\left[\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2\right] &= \mathbb{E}_J\left[\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\\ \alpha\in\{1,2,3\}}}|\widehat{T}(\mathbf{s}\oplus e_j^\alpha)|^2\right]\\
&= \mathbb{E}_J\left[\sum_{\mathbf{s}\in\{0,1,2,3\}^n}|\widehat{T}(\mathbf{s})|^2\mathbb{1}_{\{J:|\mathrm{supp}(\mathbf{s})\cap J|=1\}}\right]\\
&= \sum_{\mathbf{s}\in\{0,1,2,3\}^n}|\widehat{T}(\mathbf{s})|^2\mu_{\frac{1}{d}}\{J:|\mathrm{supp}(\mathbf{s})\cap J|=1\}\\
&\geq \sum_{d\leq|\mathrm{supp}(\mathbf{s})|<2d}|\widehat{T}(\mathbf{s})|^2\mu_{\frac{1}{d}}\{J:|\mathrm{supp}(\mathbf{s})\cap J|=1\}.
\end{aligned}\tag{3.2}$$

For the case $d=1$ with $d\leq|\mathrm{supp}(\mathbf{s})|<2d$, we have $|\mathrm{supp}(\mathbf{s})|=1$, and hence,

$$\mu_{\frac{1}{d}}\{J:|\mathrm{supp}(\mathbf{s})\cap J|=1\}=\frac{1}{d}=1.\tag{3.3}$$

For the case $d>1$ with $d\leq|\mathrm{supp}(\mathbf{s})|<2d$, we have

$$\begin{aligned}
\mu_{\frac{1}{d}}\{J:|\mathrm{supp}(\mathbf{s})\cap J|=1\} &= \left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|-1}\left(\frac{|\mathrm{supp}(\mathbf{s})|}{d}\right)\\
&\geq \inf_{d>1}\left(1-\frac{1}{d}\right)^{2d-1}\geq\frac{1}{8}.
\end{aligned}\tag{3.4}$$

Substituting (3.3) and (3.4) to (3.2) yields the desired result. $\square$

**Lemma 3.9.** *Let $T\in\mathbb{M}_{2^n}$ with $0\leq T\leq 1$. Then, for $1\leq p<2$ and $J\subseteq[n]$, we have*

$$\begin{aligned}
\mathrm{Inf}(T)+\mathrm{var}(T)\geq{}&\frac{1}{2}\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2\log\left(\frac{1}{\max_{j\in J}\|\mathbf{d}_j(T)\|_{L_p}^p}\right)\\
&-\frac{K_p}{2}\sqrt{\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2}\sqrt{\mathrm{Inf}^p(T)},
\end{aligned}$$

*where $K_p=\frac{4}{(2-p)e}$.*

*Proof.* For each $J\subseteq[n]$, by Lemma 3.5(iii) and Lemma 2.4, we have

$$\begin{aligned}
&\mathrm{Inf}(T)+\mathrm{var}(T)\\
\geq&\frac{1}{2}\sum_{j\in J}\left(\|R_j^J(T)\|_{L_2}^2\log\left(\frac{1}{\|R_j^J(T)\|_{L_2}^2}\right)-K_p\sqrt{\|R_j^J(T)\|_{L_2}^2}\sqrt{\|R_j^J(T)\|_{L_p}^p}\right)\\
\geq&\frac{1}{2}\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2\log\left(\frac{1}{\|R_j^J(T)\|_{L_2}^2}\right)-\frac{K_p}{2}\sqrt{\sum_{j\in J}\|R_j^J(T)\|_{L_2}^2}\sqrt{\sum_{j\in J}\|R_j^J(T)\|_{L_p}^p},
\end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz inequality. The desired assertion now follows from Lemma 3.5 (i) and (ii). $\square$

Choosing $J = J_0$ as in Lemma 3.8, we can relate the term $\sum_{j\in J} \left\|R_j^J(T)\right\|_{L_2}^2$ with $W_{\approx d}(T)$ and obtain the following corollary.

**Corollary 3.10.** *Let $1 \le p < 2$ and $T \in \mathbb{M}_{2^n}$ with $0 \le T \le 1$. Then the following holds*

$$\mathrm{Inf}(T) + \mathrm{var}(T) \ge \frac{1}{16} \log\left(\frac{1}{\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_p}^p}\right) W_{\approx d} T - \frac{K_p}{16}\sqrt{\mathrm{Inf}^p(T)}\sqrt{W_{\approx d}(T)},$$

*where $K_p = \frac{4}{(2-p)e}$.*

## 4. Proofs of Theorem 1.8 and Theorem 1.9

The proof of Theorem 1.8 is a little bit lengthy, which relies on sequence lemmas regarding decomposition of the Fourier spectrum. Hence, we will postpone the proof of Theorem 1.8 and show how it can be used to deduce Theorem 1.9 at first. Our method presented below is essential inspired by the approach in [14].

*Proof of Theorem 1.9.* Assume $\mathrm{Inf}^p(T) \ge \frac{\mathrm{var}(T)\log(n)}{192e^2K_p}$ ($K_p$ is the same as in Theorem 2.4). Noting that $\mathrm{Inf}^p(T) = \sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_p}^p$, it follows that

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_p}^p \ge \frac{\mathrm{var}(T)\log(n)}{192e^2K_pn}.$$

If $\mathrm{Inf}^p(T) < \frac{\log(n)\mathrm{var}(T)}{192e^2K_p}$, then, by Theorem 1.8, we have

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_p}^p \ge \frac{1}{4\sqrt{n}} \ge \frac{\log(n)}{4n} \ge \frac{\mathrm{var}(T)\log(n)}{4n} \ge \frac{\mathrm{var}(T)\log(n)}{4n}.$$

This completes the proof. □

To prove Theorem 1.8, we need a sequence of technical lemmas, which are necessary estimations on the Fourier spectrum. We now begin by introducing the operator $\delta^{\mathcal{L}}$ on $\mathbb{M}_{2^n}$ via functional calculus of the non-negative generator $\mathcal{L}$ for $\delta \in [0,1]$. More precisely, by the Fourier expansion, we have

$$\delta^{\mathcal{L}}(T) = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \delta^{|\mathrm{supp}(\mathbf{s})|}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}},$$

where $T = \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}} \in \mathbb{M}_{2^n}$. The operator $\delta^{\mathcal{L}}$ is one of the key ingredients of decomposing the Fourier spectrum of $T$. In the next proposition, we represent the operator $\delta^{\mathcal{L}}$ in terms of conditional expectation.

**Proposition 4.1.** *For each $T \in \mathbb{M}_{2^n}$, we have $\delta^{\mathcal{L}}(T) = \mathbb{E}_J[\mathcal{E}_{\mathbb{M}_J}(T)]$, where $J$ is a random set in $[n]$ corresponds to vector in $(\{0,1\}^n, \mu_\delta)$.*

*Proof.* It is clear that

$$\begin{aligned}\mathbb{E}_J(\mathcal{E}_{\mathbb{M}_J}(T)) &= \mathbb{E}_J\left(\sum_{\mathrm{supp}(\mathbf{s})\subseteq J} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}\right)\\ &= \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \mu_\delta\{J \in \{0,1\}^n : \mathrm{supp}(\mathbf{s}) \subseteq J\}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}\\ &= \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \delta^{|\mathrm{supp}(\mathbf{s})|}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}.\end{aligned}$$

□

**Lemma 4.2.** *Let $1 \le p < 2$, $T \in \mathbb{M}_{2^n}$ with $0 \le T \le 1$, and let $\mathbb{D} = \{2^k\}_{k\in\mathbb{Z}_+}$. For each $d \in \mathbb{D}$, we set $H_d(T) = (1-\frac{1}{2d})^{\mathcal{L}}(T) - (1-\frac{1}{d})^{\mathcal{L}}(T)$. Then we have*

(i) *for each $d \in \mathbb{D}$, $|H_d(T)| \le 1$;*
(ii) *for each $d \in \mathbb{D}$ and $j \in [n]$, $\|\mathbf{d}_j(H_d(T))\|_p^p \le 2^p \|\mathbf{d}_j(T)\|_{L_p}^p$;*
(iii) *for each $\mathbf{s} \in \{0,1,2,3\}^n$ with $d \le |\mathrm{supp}(\mathbf{s})| < 2d$, we have $|\widehat{T}(\mathbf{s})| \ge |\widehat{H}_d(\mathbf{s})| \ge \frac{1}{4}|\widehat{T}(\mathbf{s})|$;*
(iv) *$\sum_{d\in\mathbb{D}} \mathrm{var}(H_d(T)) = \sum_{d\in\mathbb{D}} \|H_d(T)\|_{L_2}^2 \le \mathrm{var}(T)$ and*

$$\sum_{d\in\mathbb{D}} \mathrm{Inf}(H_d(T)) \le \mathrm{Inf}(T).$$

*Proof.* (i) Note that by Proposition 4.1, for each $\delta \in [0,1]$, $\delta^{\mathcal{L}}(T) = \mathbb{E}_J[\mathcal{E}_{\mathbb{M}_J} T]$ which implies $0 \le \delta^{\mathcal{L}}(T) \le 1$. It follows that

$$|H_d(T)| = \left|(1-\frac{1}{2d})^{\mathcal{L}}(T) - (1-\frac{1}{d})^{\mathcal{L}}(T)\right| \le 1.$$

(ii) Using Proposition 4.1 and the Jensen inequality, we have

$$\begin{aligned}\|\mathbf{d}_j(\delta^{\mathcal{L}}(T))\|_{L_p}^p &= \|\mathbf{d}_j(\delta^{\mathcal{L}}(T))\|_{L_p}^p = \|\mathbf{d}_j(\mathbb{E}_J(\mathcal{E}_{\mathbb{M}_J}(T)))\|_{L_p}^p \\ &= \|\mathbb{E}_J(\mathcal{E}_{\mathbb{M}_J}(\mathbf{d}_j(T)))\|_{L_p}^p \\ &\le \mathbb{E}_J\|\mathbf{d}_j(T)\|_{L_p}^p = \|\mathbf{d}_j(T)\|_{L_p}^p.\end{aligned}$$

Moreover, we have

$$\|\mathbf{d}_j(H_d(T))\|_{L_p}^p = \left\|\mathbf{d}_j\left((1-\frac{1}{2d})^{\mathcal{L}}(T) - (1-\frac{1}{d})^{\mathcal{L}}(T)\right)\right\|_{L_p}^p \le 2^p\|\mathbf{d}_j(T)\|_{L_p}^p.$$

(iii) By the definition of $H_d$, we have

$$|\widehat{H}_d(\mathbf{s})| = \left(\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|} - \left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|}\right)|\widehat{T}(\mathbf{s})|.$$

It is clear that $|\widehat{H}_d(\mathbf{s})| \le |\widehat{T}(\mathbf{s})|$ for each $\mathbf{s} \in \{0,1,2,3\}^n$ with $d \le |\mathrm{supp}(\mathbf{s})| < 2d$. On the other hand side, we have

$$\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|} - \left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|} \ge \left(1-\frac{1}{2d}\right)^{2d} \ge \frac{1}{4},$$

where we used $d < |\mathrm{supp}(\mathbf{s})| < 2d$ in the last inequality.

(iv) By the definition of $H_d$ again, we have the following

$$\sum_{d\in\mathbb{D}} \|H_d(T)\|_{L_2}^2 = \sum_{d\in\mathbb{D}} \sum_{\mathbf{s}\in\{0,1,2,3\}^n} \left(\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|} - \left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|}\right)^2 |\widehat{T}(\mathbf{s})|^2,$$

and

$$\begin{aligned}&\sum_{d\in\mathbb{D}} \mathrm{Inf}(H_d(T)) \\ =&\sum_{d\in\mathbb{D}} \sum_{\mathbf{s}\in\{0,1,2,3\}^n} |\mathrm{supp}(\mathbf{s})| \left(\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|} - \left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|}\right)^2 |\widehat{T}(\mathbf{s})|^2.\end{aligned}$$

Note that

$$
\begin{aligned}
&\sum_{d\in\mathbb{D}}\left(\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|}-\left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|}\right)^2\\
\le&\left(\sum_{d\in\mathbb{D}}\left(1-\frac{1}{2d}\right)^{|\mathrm{supp}(\mathbf{s})|}-\left(1-\frac{1}{d}\right)^{|\mathrm{supp}(\mathbf{s})|}\right)^2\le 1,
\end{aligned}
$$

and we complete the proof. □

By Lemma 4.2 (iii), it follows that

$$
W_{\approx d}(H_d(T))\ge\frac{1}{16}W_{\approx d}(T). \tag{4.1}
$$

Before turning into the proof of the main theorem in this section, we provide the following lemma which is motivated by [14, Lemma 29].

**Lemma 4.3.** *Let $\mathbb{D}=\{2^k\}_{k\in\mathbb{Z}_+}$. For each $T\in\mathbb{M}_{2^n}$ with $0\le T\le 1$ and $1\le p<2$, define*

$$
\mathbb{D}_{\ge}:=\Big\{d\in\mathbb{D}:W_{\approx d}(T)\ge\frac{\mathrm{var}\,(T)^2}{16\mathrm{Inf}^p(T)}\Big\}.
$$

*Then we have*

$$
\sum_{d\in\mathbb{D}_{\ge}}W_{\approx d}(T)\ge\frac{1}{2}\mathrm{var}\,(T)\,.
$$

*Proof.* On the one hand, taking $d_0=\frac{4\mathrm{Inf}^p(T)}{\mathrm{var}(T)}$, we get that

$$
\begin{aligned}
\sum_{d\ge d_0}W_{\approx d}(T)&=\sum_{|\mathrm{supp}(\mathbf{s})|\ge d_0}|\widehat{T}(\mathbf{s})|^2\\
&\le\frac{1}{d_0}\sum_{|\mathrm{supp}(\mathbf{s})|\ge d_0}|\mathrm{supp}(\mathbf{s})||\widehat{T}(\mathbf{s})|^2\\
&\le\frac{1}{d_0}\sum_{\mathbf{s}\in\{0,1,2,3\}^n}|\mathrm{supp}(\mathbf{s})||\widehat{T}(\mathbf{s})|^2\\
&=\frac{\mathrm{Inf}(T)}{d_0}=\frac{\mathrm{Inf}(T)\mathrm{var}\,(T)}{4\mathrm{Inf}^p(T)}\le\frac{1}{4}\mathrm{var}\,(T)\,,
\end{aligned} \tag{4.2}
$$

where the last inequality is due to Proposition 2.7 (ii).

On the other hand, by the fact $|\{d\in\mathbb{D}:d\le d_0\}|\le\log(d_0)$ and the definition of $\mathbb{D}_{\ge}$, we get

$$
\sum_{d<d_0,d\notin\mathbb{D}_{\ge}}W_{\approx d}(T)\le\log(d_0)\frac{\mathrm{var}\,(T)^2}{16\mathrm{Inf}^p(T)}=\frac{\log(d_0)}{4d_0}\mathrm{var}(T)\le\frac{1}{4}\mathrm{var}\,(T)\,. \tag{4.3}
$$

Combining (4.2) and (4.3), we have

$$
\begin{aligned}
\sum_{d\in\mathbb{D}_{\ge}}W_{\approx d}(T)&=\sum_{d\in\mathbb{D}}W_{\approx d}(T)-\sum_{d\notin\mathbb{D}_{\ge}}W_{\approx d}(T)\\
&\ge\mathrm{var}(T)-\sum_{d\ge d_0}W_{\approx d}(T)-\sum_{d<d_0,d\notin\mathbb{D}_{\ge}}W_{\approx d}(T)\ge\frac{1}{2}\mathrm{var}(T).
\end{aligned}
$$

This completes the proof. □

Now we are at the position to give a proof of Theorem 1.8 in detail.

*Proof of Theorem 1.8.* Take $T \in \mathbb{M}_{2^n}$. We assume that

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_p}^p < e^{-1040K_p \frac{\mathrm{Inf}^p(T)}{\mathrm{var}(T)}}, \tag{4.4}$$

with $K_p = \frac{4}{(2-p)e}$. For $d \in \mathbb{D}_{\geq}$, we have

$$\frac{\mathrm{Inf}^p(T)}{\mathrm{var}\,(T)} W_{\approx d}(T) \geq \frac{1}{4}\sqrt{\mathrm{Inf}^p(T)}\sqrt{W_{\approx d}(T)}.$$

Using Corollary 3.10, (4.1), (4.4), and Lemma 4.2 (ii), we get that

$$\begin{aligned}
\mathrm{Inf}(H_d(T)) + \mathrm{var}(H_d(T)) &\geq \frac{1}{16}\log\left(\frac{1}{\max_{j\in[n]}\|\mathbf{d}_j(H_d(T))\|_{L_p}^p}\right) W_{\approx d}(H_d(T)) \\
&\quad - \frac{K_p}{16}\sqrt{\mathrm{Inf}^p(H_d(T))}\sqrt{W_{\approx d}(H_d(T))} \\
&\geq \frac{1}{256}\log\left(\frac{1}{\max_{j\in[n]}\|\mathbf{d}_j(H_d(T))\|_{L_p}^p}\right) W_{\approx d}(T) \\
&\quad - \frac{K_p}{64}\sqrt{\mathrm{Inf}^p(H_d(T))}\sqrt{W_{\approx d}(T)} \\
&\geq \frac{65K_p}{16}\frac{\mathrm{Inf}^p(T)}{\mathrm{var}\,(T)} W_{\approx d}(T) - \frac{K_p}{16}\sqrt{\mathrm{Inf}^p(T)}\sqrt{W_{\approx d}(T)} \\
&\geq 4K_p \frac{\mathrm{Inf}^p(T)}{\mathrm{var}\,(T)} W_{\approx d}(T).
\end{aligned} \tag{4.5}$$

Combining Lemma 4.2 (iv), (4.5) with Lemma 4.3 yields that

$$\begin{aligned}
2\mathrm{Inf}^p(T) \geq 2\mathrm{Inf}(T) \geq \mathrm{Inf}(T) + \mathrm{var}(T) &\geq \sum_{d\in\mathbb{D}_{\geq}} \mathrm{Inf}(H_d(T)) + \sum_{d\in\mathbb{D}_{\geq}} \mathrm{var}(H_d(T)) \\
\geq 4K_p \frac{\mathrm{Inf}^p(T)}{\mathrm{var}\,(T)} \sum_{d\in\mathbb{D}_{\geq}} W_{\approx d}(T) &\geq 2K_p \frac{\mathrm{Inf}^p(T)}{\mathrm{var}\,(T)} \mathrm{var}\,(T) \geq \frac{8}{e}\mathrm{Inf}^p(T),
\end{aligned}$$

which is a contradiction. □

The following counterexample demonstrates the failure of Theorem 1.8 for the case $p = 2$ even in the commutative hypercube setting, which means that, for bounded elements, the *dimension free* KKL inequality invoking $L_p$-influences with $1 \leq p < 2$ may be the best possible.

**Remark 4.4.** *Define* $f : \{0,1\}^n \to [0,1]$ *as follows*

$$f := \frac{1}{2} + \frac{1}{2n}\sum_{j=1}^n r_j,$$

*where* $\{r_j\}_{j=1}^n$ *is an i.i.d. Rademacher sequence. Then, we have*

$$\|\mathbf{d}_j(f)\|_{L_2}^2 = \frac{1}{4n^2}, \quad \forall j \in [n],$$

*and*

$$\mathrm{var}(f) = \mathrm{Inf}(f) = \frac{1}{4n},$$

*which disproves Theorem 1.8 for the case $p = 2$ in the hypercube setting.*

We conclude this section with some comments on the quantum Talagrand influence inequality. Firstly, combining the idea in [14] with the random restriction treatment presented in Section 4, we can derive the following quantum Talagrand influence inequality.

**Theorem 4.5** (Quantum Talagrand influence inequality)**.** *For $1 \leq p < 2$, there exists a constant $C_p > 0$ depending only on $p$, such that for each $T \in \mathbb{M}_{2^n}$ with $0 \leq T \leq 1$, we have*

$$\mathrm{var}(T) \leq C_p \sum_{j=1}^{n} \frac{\|\mathbf{d}_j(T)\|_{L_p}^p}{\log\left(1/\|\mathbf{d}_j(T)\|_{L_p}^p\right)} \tag{4.6}$$

*where $C_p = O\left(\frac{1}{(2-p)^2}\right)$ as $p \to 2$.*

After completing our paper, we learned that Blecher et al. [2] have obtained a similar quantum Talagrand influence inequality with some interesting higher-order extensions. Therefore, we have chosen not to include the detailed proof of Theorem 4.5 here and instead refer interested readers to [2] for the proof. Secondly, the constant $C_p = O\left(\frac{1}{(2-p)^2}\right)$ appeared in the quantum Talagrand influence inequality is not the best possible. However, at this time of writing, we can not apply the random restriction method to achieve the constant $C_p = O\left(\frac{1}{2-p}\right)$ as stated in Theorem 1.5. Finally, we shall mention here that (4.6) is a straightforward consequence of Theorem 1.5 by using Proposition 2.7 (i).

## 5. Quantum Eldan-Gross inequality

In this section, we aim to derive a quantum Eldan-Gross inequality via the random restriction technique, and apply it to obtain several quantum KKL-type inequalities. Our approach of the quantum Eldan-Gross inequality is inspired by the techniques developed by Keller and Kinder [13] and Eldan et al. [8].

5.1. **Estimate on Fourier spectrum.** This subsection aims to establish the following noncommutative analogue of [13, Lemma 5]. For $T \in \mathbb{M}_{2^n}$ and $J \subseteq [n]$, we denote $M_J(T) \coloneqq \sum_{j \in J} \|\mathbf{d}_j(T)\|_{L_1}^2$ and $M(T) \coloneqq \sum_{j=1}^{n} \|\mathbf{d}_j(T)\|_{L_1}^2$.

**Theorem 5.1.** *For $d \geq 1$ and projection $T \in \mathbb{M}_{2^n}$, if $M(T) \leq e^{-2d}$, then we have*

$$\sum_{|\mathrm{supp}(\mathbf{s})|=d} \widehat{T}(\mathbf{s})^2 \leq \frac{6e}{d}\left(\frac{2e}{d}\right)^d M(T)\left(\log\left(\frac{d}{M(T)}\right)\right)^d.$$

The key ingredient in the proof of Theorem 5.1 is the following technical result, which is a noncommutative analogue of the estimate in [13, eq. (12)].

**Proposition 5.2.** *Let $T \in \mathbb{M}_{2^n}$ be a projection with $M(T) \leq e^{-2d}$. Then, for each $J \subseteq [n]$, we have*

$$\sum_{j \in J} \sum_{\substack{\mathrm{supp}(\mathbf{s}) \subseteq J^c, \\ |\mathrm{supp}(\mathbf{s})|=d-1, \alpha \in \{1,2,3\}}} |\widehat{T}\left(\mathbf{s} \oplus e_j^{\alpha}\right)|^2 \leq 6\left(\frac{2e}{d}\right)^d M_J(T)\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d.$$

Since the proof of Proposition 5.2 is a little bit lengthy, we will postpone its proof, and turn to demonstrate how it can be used to prove Theorem 5.1 at first. Recall that we identify random subset $J$ with the vector in $(\{0,1\}^n, \mu_{\frac{1}{d}})$, where

$$\mu_{\frac{1}{d}}(\{x\}) = \left(\frac{1}{d}\right)^{\sum_{j=1}^n x_j} \left(1-\frac{1}{d}\right)^{n-\sum_{j=1}^n x_j}, \quad x=(x_j)_{j=1}^n \in \{0,1\}^n.$$

*Proof of Theorem 5.1.* Firstly, note here that for each $\mathbf{s} \in \{0,1,2,3\}^n$ with $|\mathrm{supp}(\mathbf{s})| = d$, we know the probability of $J$ such that $\mathbf{s} = \mathbf{u} \oplus e_j^\alpha$ with $j \in J$ and $\mathrm{supp}(\mathbf{u}) \subseteq J^c$ is $\binom{d}{1} \cdot \frac{1}{d}(1-\frac{1}{d})^{d-1} = (1-\frac{1}{d})^{d-1} \geq \frac{1}{e}$. Hence,

$$\begin{aligned}
\text{(5.1)} \qquad & \mathbb{E}_J \left( \sum_{\substack{\mathrm{supp}(\mathbf{s}) \subseteq J^c, |\mathrm{supp}(\mathbf{s})| = d-1 \\ j \in J, \alpha \in \{1,2,3\}}} \widehat{T}\left(\mathbf{s} \oplus e_j^\alpha\right)^2 \right) \\
&= \left(1-\frac{1}{d}\right)^{d-1} \sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n, \\ |\mathrm{supp}(\mathbf{s})| = d}} \widehat{T}(\mathbf{s})^2 \geq \left(\frac{1}{e}\right) \sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n, \\ |\mathrm{supp}(\mathbf{s})| = d}} \widehat{T}(\mathbf{s})^2.
\end{aligned}$$

Secondly, assume Proposition 5.2 holds and note that $x \mapsto x \log(\frac{1}{x})^d$ is a concave function on $(0, e^d)$. Then it follows from the assumption $M(T) \leq e^{-2d}$ and the Jensen inequality that

$$\begin{aligned}
& \mathbb{E}_J \left[ 6 \left(\frac{2e}{d}\right)^d \left( \sum_{j \in J} \|\mathbf{d}_j(T)\|_{L_1}^2 \right) \left( \log \left( \frac{1}{\sum_{j \in J} \|\mathbf{d}_j(T)\|_{L_1}^2} \right) \right)^d \right] \\
\text{(5.2)} \qquad & \leq 6 \left(\frac{2e}{d}\right)^d \left( \mathbb{E}_J \left( \sum_{j \in J} \|\mathbf{d}_j(T)\|_{L_1}^2 \right) \right) \left( \log \left( \frac{1}{\mathbb{E}_J(\sum_{j \in J} \|\mathbf{d}_j(T)\|_{L_1}^2)} \right) \right)^d \\
& = 6 \left(\frac{2e}{d}\right)^d \left( \frac{M(T)}{d} \right) \left( \log \left( \frac{d}{M(T)} \right) \right)^d,
\end{aligned}$$

where we used $\mathbb{E}_J(\mathbb{1}_J(j)) = \frac{1}{d}$ in the last equality. Hence, combing (5.1), (5.2) with Proposition 5.2 yields the desired inequality. □

We conclude this subsection with the following noncommutative analogue of [8, Theorem 3.4].

**Theorem 5.3.** *Suppose that $T \in \mathbb{M}_{2^n}$ is a projection. Then*

$$\text{(5.3)} \qquad \sum_{1 \leq |\mathrm{supp}(\mathbf{s})| \leq \frac{1}{10} \log(1/M(T))} \widehat{T}(\mathbf{s})^2 \leq 12 e M(T)^{2/5}.$$

*Proof.* For the given projection $T$, recall here that

$$W_{=l}(T) = \sum_{|\mathrm{supp}(\mathbf{s})| = l} |\widehat{T}(\mathbf{s})|^2.$$

For $1 \leq d \leq \frac{1}{10}\log(1/M(T))$, we have $M(T) \leq e^{-10d}$, and then it follows from Theorem 5.1 that

$$\begin{aligned}
&W_{1\leq d\leq \frac{1}{10}\log(1/M(T))}(T)\\
=&\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil} W_{=l}(T)\\
\leq&6eM(T)\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil} \frac{1}{l}\left(\frac{2e}{l}\right)^l\left(\log\left(\frac{l}{M(T)}\right)\right)^l\\
=&6eM(T)\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil} \frac{1}{l}\left(\frac{2e}{l}\right)^l(\log(l)+\log(1/M(T)))^l\\
\leq&6eM(T)\left(\frac{4e\log(1/M(T))}{l}\right)^l\left(\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil}\frac{1}{l}\right),
\end{aligned}\tag{5.4}$$

where we used $\log(l) \leq \log(1/M(T))$ for $1 \leq l \leq \frac{1}{10}\log(1/M(T))$ in the last inequality.

Note that the function $l \mapsto \left(\frac{4e\log(1/M(T))}{l}\right)^l$ is increasing for $l \in [1, 4\log(1/M(T))]$. Hence, we estimate (5.4) as follows

$$\begin{aligned}
W_{1\leq d\leq \frac{1}{10}\log(1/M(T))}(T) &\leq 6eM(T)(40e)^{\frac{1}{10}\log(1/M(T))}\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil}\frac{1}{l}\\
&\leq 6eM(T)^{1/2}\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil}\frac{1}{l}\\
&\leq 6eM(T)^{1/2}\left(\log\log\left(\frac{1}{M(T)^{1/10}}\right)+1\right)\\
&\leq 12eM(T)^{2/5},
\end{aligned}\tag{5.5}$$

where we used $40e \leq e^5$ and $\sum_{l=1}^{\lceil \frac{1}{10}\log(1/M(T))\rceil}\frac{1}{l} \leq \log\log\left(\frac{1}{M(T)^{1/10}}\right)+1$ in the second and third inequalities, respectively. □

5.2. **The quantum Talagrand-type isoperimetric inequality.** In this subsection, we apply the semigroup technique to derive Theorem 1.10. The proof of the theorem relies on the following inequalities, which are of independent interest. The first key ingredient in our proof is the following quantum Buser-type inequality.

**Theorem 5.4** (Quantum Buser-type inequality). *For each $t \geq 0$ and $1 \leq p \leq 2$, the following inequality holds for every $T \in \mathbb{M}_{2^n}$*

$$\left\|T - e^{-t\mathcal{L}}(T)\right\|_{L_p} \leq \sqrt{2t}\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)^2|\right)^{1/2}\right\|_{L_p}.$$

To establish Theorem 5.4, we need the following sequence of lemmas, which are noncommutative analogies to their commutative correspondences.

**Theorem 5.5** (Local Reverse Poincaré inequality)**.** *For each $T \in \mathbb{M}_{2^n}$, we have*

$$(e^{2t}-1)\sum_{j=1}^{n} |e^{-t\mathcal{L}}\mathbf{d}_j(T)|^2 \leq e^{-t\mathcal{L}}(|T|^2) - |e^{-t\mathcal{L}}(T)|^2.$$

*Proof.* On the one hand, applying Proposition 2.5 (iii), we get

$$e^{-s\mathcal{L}}\left[\sum_{j=1}^{n}\left|\mathbf{d}_j e^{-(t-s)\mathcal{L}}(T)\right|^2\right] \geq e^{2s}\sum_{j=1}^{n}|\mathbf{d}_j e^{-t\mathcal{L}}(T)|^2, \tag{5.6}$$

for every $0 \leq s \leq t$.

On the other hand,

$$\begin{aligned}
& e^{-t\mathcal{L}}(T^*T) - (e^{-t\mathcal{L}}(T))^*(e^{-t\mathcal{L}}(T)) \\
=& \int_0^t \frac{\partial}{\partial s} e^{-s\mathcal{L}}\left[\left(e^{-(t-s)\mathcal{L}}(T)\right)^*\left(e^{-(t-s)\mathcal{L}}(T)\right)\right]\, ds \\
=& \int_0^t -\mathcal{L}e^{-s\mathcal{L}}\left(\left|e^{-(t-s)\mathcal{L}}(T)\right|^2\right) + e^{-s\mathcal{L}}\left(\left(e^{-(t-s)\mathcal{L}}(T)\right)^*\mathcal{L}e^{-(t-s)\mathcal{L}}(T)\right) \\
& \quad + e^{-s\mathcal{L}}\left(\left(\mathcal{L}e^{-(t-s)\mathcal{L}}(T)\right)^* e^{-(t-s)\mathcal{L}}(T)\right)\, ds \\
=& \int_0^t e^{-s\mathcal{L}}\Big[-\mathcal{L}\left(|e^{-(t-s)\mathcal{L}}(T)|^2\right) + e^{-(t-s)\mathcal{L}}(T)\left(\mathcal{L}e^{-(t-s)\mathcal{L}}(T)\right)^* \\
& \quad + \left(\mathcal{L}e^{-(t-s)\mathcal{L}}(T)\right)^* e^{-(t-s)\mathcal{L}}(T)\Big]\, ds \\
=& 2\int_0^t e^{-s\mathcal{L}}\left(\sum_{j=1}^{n}\left|\mathbf{d}_j e^{-(t-s)\mathcal{L}}(T)\right|^2\right)\, ds,
\end{aligned} \tag{5.7}$$

where the last line follows from Proposition 2.5 (i).

Combining (5.6) and (5.7) yields that

$$\begin{aligned}
e^{-t\mathcal{L}}(|T|^2) - |e^{-t\mathcal{L}}(T)|^2 =& 2\int_0^t e^{-s\mathcal{L}}\left(\sum_{j=1}^{n}\left|\mathbf{d}_j e^{-(t-s)\mathcal{L}}(T)\right|^2\right)\, ds \\
\geq& 2\int_0^t e^{2s}\sum_{j=1}^{n}\left|\mathbf{d}_j e^{-t\mathcal{L}}(T)\right|^2\, ds \\
=& \sum_{j=1}^{n}\left|\mathbf{d}_j e^{-t\mathcal{L}}(T)\right|^2 \int_0^{2t} e^s\, ds \\
=& (e^{2t}-1)\sum_{j=1}^{n}\left|\mathbf{d}_j e^{-t\mathcal{L}}(T)\right|^2,
\end{aligned}$$

which is the desired inequality. □

Thanks to the local reverse Poincaré inequality (i.e., Theorem 5.5), we obtain the following *gradient estimate* that is dual to the quantum Byser-type inequality.

**Lemma 5.6.** *For each $T \in \mathbb{M}_{2^n}$ and $2 \le p \le \infty$, we have*

$$\left\| \left( \sum_{j=1}^{n} \left| \mathbf{d}_j e^{-t\mathcal{L}}(T) \right|^2 \right)^{1/2} \right\|_{L_p} \le \frac{1}{\sqrt{2t}} \|T\|_{L_p}, \ t \ge 0.$$

*Proof.* Since $e^{2t} - 1 \ge 2t$, it follows from the reverse local Poincaré inequality (i.e., Theorem 5.5) that for each $t \ge 0$ we have

$$2t \sum_{j=1}^{n} |\mathbf{d}_j e^{-t\mathcal{L}}(T)|^2 \le e^{-t\mathcal{L}}(|T|^2) - |e^{-t\mathcal{L}}(T)|^2 \le e^{-t\mathcal{L}}(|T|^2),$$

which further implies

$$\left( \sum_{j=1}^{n} \left| \mathbf{d}_j e^{-t\mathcal{L}}(T) \right|^2 \right)^{1/2} \le \frac{1}{\sqrt{2t}} \left( e^{-t\mathcal{L}}(T^*T) \right)^{1/2}.$$

Therefore, we have

$$\left\| \left( \sum_{j=1}^{n} \left| \mathbf{d}_j e^{-t\mathcal{L}}(T) \right|^2 \right)^{1/2} \right\|_{L_p} \le \frac{1}{\sqrt{2t}} \left\| e^{-t\mathcal{L}}(|T|^2) \right\|_{L_{p/2}}^{1/2} \le \frac{1}{\sqrt{2t}} \|T\|_{L_p},$$

where the last inequality is due to the fact that $p/2 \ge 1$ and

$$\left\| e^{-t\mathcal{L}}(|T|^2) \right\|_{L_{p/2}}^{1/2} \le \left\| |T|^2 \right\|_{L_{p/2}}^{1/2} = \|T\|_{L_p}.$$

□

We now prove the quantum Buser-type inequality via duality.

*Proof of Theorem 5.4.* Since, for each $t \ge 0$,

$$T - e^{-t\mathcal{L}}(T) = -\int_0^t \frac{\partial}{\partial s} e^{-s\mathcal{L}}(T)\, ds = \sum_{j=1}^{n} \int_0^t \mathbf{d}_j e^{-s\mathcal{L}}(T)\, ds.$$

Hence, for every $1 \le p \le 2$, there exists $u \in L_{p'}$ with $\|u\|_{L_{p'}} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\begin{aligned}
\left\| T - e^{-t\mathcal{L}}(T) \right\|_{L_p} &= \operatorname{tr} \left( u \cdot \left( \sum_{j=1}^{n} \int_0^t \mathbf{d}_j e^{-s\mathcal{L}}(T)\, ds \right) \right) \\
&= \int_0^t \sum_{j=1}^{n} \operatorname{tr} \left( u \cdot \left( \mathbf{d}_j e^{-s\mathcal{L}}(T) \right) \right)\, ds \\
&= \int_0^t \sum_{j=1}^{n} \operatorname{tr} \left( \left( \mathbf{d}_j e^{-s\mathcal{L}}(u) \right) \cdot \left( \mathbf{d}_j(T) \right) \right)\, ds \\
&\le \int_0^t \left\| \left( \sum_{j=1}^{n} |\mathbf{d}_j e^{-s\mathcal{L}}(u^*)|^2 \right)^{1/2} \right\|_{L_{p'}} \cdot \left\| \left( \sum_{j=1}^{n} |\mathbf{d}_j(T)^2| \right)^{1/2} \right\|_{L_p} ds.
\end{aligned}$$

Thanks to Lemma 5.6, we obtain that

$$\left\|T-e^{-t\mathcal{L}}(T)\right\|_{L_p} \le \int_0^t \frac{1}{\sqrt{2s}}ds\ \|u\|_{L_{p'}} \left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)^2|\right)^{1/2}\right\|_{L_p}$$
$$\le \sqrt{2t}\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)^2|\right)^{1/2}\right\|_{L_p}.$$

□

Before turning to the Talagrand-type inequality, we show the following two elementary lemmas.

**Lemma 5.7.** *For each projection $T \in \mathbb{M}_{2^n}$ and $t \ge 0$, we have*

$$\operatorname{var}(T) \le \left\|T-e^{-t\mathcal{L}}(T)\right\|_{L_1} + \operatorname{var}\left(e^{-t\mathcal{L}/2}(T)\right).$$

*Proof.* Let $T \in \mathbb{M}_{2^n}$ be a projection. It follows from the trace preserving of $\{e^{-t\mathcal{L}}\}_{t\ge 0}$ that

$$\operatorname{var}\left(e^{-t\mathcal{L}/2}(T)\right) = \operatorname{tr}(T\cdot e^{-t\mathcal{L}}(T)) - \operatorname{tr}(T)^2,\ \forall\ t\ge 0.$$

Note that $\operatorname{var}(T) = \operatorname{tr}(T) - \operatorname{tr}(T)^2$. Therefore, we have

$$\begin{aligned}\operatorname{var}(T) - \operatorname{var}\left(e^{-t\mathcal{L}/2}(T)\right) &= \operatorname{tr}(T - T\cdot e^{-t\mathcal{L}}(T))\\ &= \operatorname{tr}\left(T(T-e^{-t\mathcal{L}}(T))\right)\\ &\le \left\|T(T-e^{-t\mathcal{L}}(T))\right\|_{L_1}\\ &\le \|T-e^{-t\mathcal{L}}(T)\|_{L_1}.\end{aligned}$$

The desired assertion follows. □

**Lemma 5.8.** *Let $T \in \mathbb{M}_{2^n}$ be a projection. For each $d\in\mathbb{N}$, we have*

$$\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}\right\|_{L_1} \ge \frac{1}{4}\sqrt{d}\,W_{\ge d}(T).$$

*Proof.* Fix $d\in\mathbb{N}$. Applying Lemma 5.7 (with $t=\frac{1}{d}$), we get

$$\begin{aligned}\left\|T-e^{-\frac{\mathcal{L}}{d}}(T)\right\|_{L_1} &\ge \operatorname{var}(T) - \operatorname{var}\left(e^{-\frac{\mathcal{L}}{2d}}(T)\right)\\ &= \sum_{|\mathrm{supp}(\mathbf{s})|\ge 1} (1-e^{-\frac{|\mathrm{supp}(\mathbf{s})|}{d}})\widehat{T}(\mathbf{s})^2\\ &\ge \sum_{|\mathrm{supp}(\mathbf{s})|\ge d} (1-e^{-\frac{|\mathrm{supp}(s)|}{d}})\widehat{T}(\mathbf{s})^2\\ &\ge (1-e^{-1})W_{\ge d}(T) \ge \frac{1}{2}W_{\ge d}(T).\end{aligned}$$

By Theorem 5.4, we have

$$\left\| \left( \sum_{j=1}^{n} |\mathbf{d}_j(T)|^2 \right)^{1/2} \right\|_{L_1} \geq \sqrt{\frac{d}{2}} \left\| T - e^{-\frac{\mathcal{L}}{d}}(T) \right\|_{L_1} \geq \frac{\sqrt{d}}{4} W_{\geq d}(T).$$

□

Before providing the proof of the quantum Talagrand-type isoperimetric inequality, i.e., Theorem 1.10, we derive the following moment comparison lemma via hypercontractivity of the semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ (see [18, Theorem 46]).

**Lemma 5.9** (moment comparison)**.** *Let $T \in \mathbb{M}_{2^n}$ with degree at most $k$. For every $r \geq 2$, we have*

$$\|T\|_{L_r} \leq (r-1)^{k/2} \|T\|_{L_2}.$$

*Proof.* Note that $T$ is of degree at most $k$, that is, $T = \sum_{|\mathrm{supp}(\mathbf{s})|\leq k} \widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}$. For each $r \geq 2$, take $t_0 = \frac{\log(r-1)}{2}$. Applying the hypercontractivity of $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ (see (2.5) above), we have

$$\left\| e^{-\frac{\log(r-1)}{2}\mathcal{L}}(T) \right\|_{L_r} \leq \|T\|_{L_2},$$

which implies

$$\|T\|_{L_r} = \left\| e^{-t\frac{\log(r-1)}{2}\mathcal{L}} \circ e^{t\frac{\log(r-1)}{2}\mathcal{L}}(T) \right\|_{L_r} \leq \left\| e^{t\frac{\log(r-1)}{2}\mathcal{L}}(T) \right\|_{L_2}.$$

Now the desired inequality follows from the fact

$$\left\| e^{\frac{\log(r-1)}{2}\mathcal{L}}(T) \right\|_{L_2}^2 = \sum_{|\mathrm{supp}(\mathbf{s})|\leq k} e^{\log(r-1)|\mathrm{supp}(\mathbf{s})|} |\widehat{T}(\mathbf{s})|^2 \leq (r-1)^k \|T\|_{L_2}^2.$$

□

*Proof of Theorem 1.10.* We first consider the case $\mathrm{var}(T) \geq e^{-16}$. Note that $\mathrm{var}(T) = W_{\geq 1}(T)$ and

$$\sqrt{\log\left(\frac{1}{\mathrm{var}(T)}\right)} \leq 4.$$

It follows from Lemma 5.8 (with $d = 1$ there) that

$$\mathrm{var}(T)\sqrt{\log\left(\frac{1}{\mathrm{var}(T)}\right)} \leq 4W_{\geq 1}(T) \leq 16 \left\| \left( \sum_{j=1}^{n} |\mathbf{d}_j(T)|^2 \right)^{1/2} \right\|_{L_1}.$$

Now we deal with the case $\mathrm{var}(T) \leq e^{-16}$. Let

$$d := \left\lceil \frac{1}{16} \log\left(\frac{1}{\mathrm{var}(T)}\right) \right\rceil.$$

For such $d$, we claim that

$$W_{\geq d}(T) \geq \frac{1}{2}\mathrm{var}(T). \tag{5.8}$$

Once the claim is proved, applying Lemma 5.8, we have

$$\left\| \left( \sum_{j=1}^{n} |\mathbf{d}_j(T)|^2 \right)^{1/2} \right\|_{L_1} \geq \frac{1}{4} \sqrt{\frac{1}{16} \log \left( \frac{1}{\mathrm{var}(T)} \right)} W_{\geq d}(T) \geq \frac{1}{32} \mathrm{var}(T) \sqrt{\log \left( \frac{1}{\mathrm{var}(T)} \right)}.$$

This is the desired assertion.

It remains to verify the claim. Since $T \in \mathbb{M}_{2^n}$ is a projection, it follows that $\mathrm{var}(T) = \mathrm{tr}(T)(1 - \mathrm{tr}(T)) \leq e^{-16}$. We assume without loss of generality that $\mathrm{tr}(T) < \frac{1}{2}$; otherwise, it suffices to replace $T$ by $\mathbf{1} - T$. Hence,

$$2\mathrm{var}(T) = 2\mathrm{tr}(T)(1 - \mathrm{tr}(T)) \geq \mathrm{tr}(T). \tag{5.9}$$

Write

$$\mathrm{Rad}_{\leq d}(T) = \sum_{|\mathrm{supp}(\mathbf{s})| \leq d} \widehat{T}(\mathbf{s}) \sigma_{\mathbf{s}}.$$

By the Hölder inequality, Lemma 5.9 and (5.9), we deduce that

$$\begin{aligned} W_{\leq d}(T) &= \mathrm{tr}\left(T \cdot \mathrm{Rad}_{\leq d}(T)\right) \leq \|T\|_{4/3} \cdot \|\mathrm{Rad}_{\leq d}(T)\|_{L_4} \\ &\leq 3^{d/2} \|\mathrm{Rad}_{\leq d}(T)\|_{L_2} \|T\|_{L_{4/3}} = 3^{d/2} \|\mathrm{Rad}_{\leq d}(T)\|_{L_2} \mathrm{tr}(T)^{3/4} \\ &\leq 3^{d/2} \cdot 2^{3/4} \|\mathrm{Rad}_{\leq d}(T)\|_{L_2} \mathrm{var}(T)^{3/4} = 3^{d/2} \cdot 2^{3/4} W_{\leq d}(T)^{1/2} \mathrm{var}(T)^{3/4}. \end{aligned}$$

Note that $3^{d/2} \leq e^d \leq \mathrm{var}(T)^{-1/16}$ and $\mathrm{var}(T) \leq e^{-16}$. We have

$$W_{\leq d}(T)^{1/2} \leq \frac{2^{3/4} \mathrm{var}(T)^{3/4}}{\mathrm{var}(T)^{1/16}} \leq \mathrm{var}(T)^{1/2} 2^{3/4} e^{-3} \leq \frac{1}{2} \mathrm{var}(T)^{1/2},$$

which further implies

$$W_{\geq d}(T) \geq \mathrm{var}(T) - W_{\leq d}(T) \geq \frac{3}{4} \mathrm{var}(T) \geq \frac{1}{2} \mathrm{var}(T).$$

We have verified the claim (5.8), and the proof is complete. □

5.3. **The quantum Eldan-Gross inequality and KKL-type inequalities.** In this subsection, we provide a details proof of the quantum Eldan-Gross inequality and apply it to deduce two quantum KKL-type inequalities and a stability result for the quantum KKL theorem with respect to the $L_1$-influences.

*Proof of Theorem 1.11.* Let $T$ be a projection in $\mathbb{M}_{2^n}$ and denote $M(T) := \sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2$ for simplicity. If $M(T) \geq \mathrm{var}(T)^{15}$, it follows that

$$\begin{aligned} \mathrm{var}(T) \sqrt{\log \left( 1 + \frac{1}{M(T)} \right)} &\leq \mathrm{var}(T) \sqrt{\log \left( 1 + \frac{1}{\mathrm{var}(T)^{10}} \right)} \\ &\leq 4 \mathrm{var}(T) \sqrt{\log \left( \frac{1}{\mathrm{var}(T)} \right)}, \end{aligned} \tag{5.10}$$

where we used $\mathrm{var}(T) \leq \frac{1}{4}$ (i.e., $\max_{\alpha \in [0,1]} \alpha - \alpha^2 \leq \frac{1}{4}$) in the second inequality. Apply Theorem 1.10 to (5.10) yields the desired quantum Eldan-Gross inequality for the case $M(T) \geq \mathrm{var}(T)^{15}$.

If $M(T) \leq \mathrm{var}(T)^{15}$, we let $d = \frac{1}{10}\log(1/M(T))$ and apply Theorem 5.3 to get that

(5.11)
$$\sum_{1\leq|\mathrm{supp}(\mathbf{s})|\leq\frac{1}{10}\log(1/M(T))} \widehat{T}(\mathbf{s})^2 \leq 12e\cdot M(T)^{2/5} \leq 12e\cdot \mathrm{var}(T)^6 \leq \left(\frac{12e}{4^5}\right)\mathrm{var}(T),$$

where we used $\mathrm{var}(T) \leq \frac{1}{4}$ in the last inequality. By (5.11), it follows that

$$W_{>d}(T) = \mathrm{var}(T) - \sum_{1\leq|\mathrm{supp}(\mathbf{s})|\leq d} \widehat{T}(\mathbf{s})^2 \geq \left(1-\frac{12e}{4^5}\right)\mathrm{var}(T) \geq \frac{1}{2}\mathrm{var}(T). \tag{5.12}$$

Combining Lemma 5.8 and (5.12), we get that

$$\mathrm{var}(T)\sqrt{\log\left(1+\frac{1}{\log(M(T))}\right)} \leq K\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}\right\|_{L_1},$$

where we used $1+\log(1/M(T)) \leq 2\log(1/M(T))$ when $M(T) \leq \mathrm{var}(T)^{15} \leq 1/2^{30}$. □

The following definition is motivated by Definition 2.1 and Remark 2.2.

**Definition 5.10.** *A projection $T \in \mathbb{M}_{2^n}$ is called balanced if* $\mathrm{var}(T) = \frac{1}{4}$.

As the first application of our noncommutative Eldan-Gross inequality, we derive the following KKL-type inequality in the CAR algebra, which is essentially due to Rouzé, Wirth and Zhang [22].

**Theorem 5.11** (Rouzé-Wirth-Zhang)**.** *For each balanced projection $T \in \mathbb{M}_{2^n}$, there exists a universal constant $C > 0$, such that*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1} \geq \frac{C\sqrt{\log(n)}}{n}.$$

*Proof.* For each balanced projection $T \in \mathbb{M}_{2^n}$, we apply Theorem 1.11 to get

$$\frac{1}{4}\sqrt{\log(1+\frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2})} \leq K\left\|\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right\|_{L_{1/2}}^{1/2} \leq K\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}, \tag{5.13}$$

where $K > 0$ is a universal constant. It follows from (5.13) that

$$\frac{1}{16} \leq \frac{(KnB)^2}{\log\left(1+\frac{1}{nB^2}\right)}, \tag{5.14}$$

where $B = \max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1}$.

Choose a positive constant $C$ such that $16KC < 1$. If $M \geq \frac{C\sqrt{\log(n)}}{n}$, there is nothing to prove. We assume $B < \frac{C\sqrt{\log(n)}}{n}$ from now on. Choose a sufficient large $n$ such that

$$\frac{n}{C^2\log(n)} - n^{1/2} + 1 > 0.$$

It follows from $B < \frac{C\sqrt{\log(n)}}{n}$ that

$$\frac{(KnB)^2}{\log\left(1+\frac{1}{nB^2}\right)} < \frac{K^2C^2\log(n)}{\log\left(1+\frac{n}{C^2\log(n)}\right)}. \tag{5.15}$$

Note that $4\sqrt{2}KC < 1$ and $\frac{n}{C^2\log(n)} - n^{1/2} + 1 > 0$ imply that

$$(4KC)^2\log(n) < \frac{1}{2}\log(n) < \log\left(1+\frac{n}{C^2\log(n)}\right). \tag{5.16}$$

Substituting (5.16) to (5.15) entails that

$$\frac{(KnB)^2}{\log\left(1+\frac{1}{nB^2}\right)} < \frac{K^2C^2\log(n)}{\log\left(1+\frac{n}{C^2\log(n)}\right)} < \frac{1}{16},$$

which contradicts to (5.14). □

As the second application of the noncommutative Eldan-Gross inequality, we derive the following quantum KKL-type inequality, which is of independent interest.

**Theorem 5.12.** *There exists a universal constant $C > 0$ such that for every $\varepsilon \in (0,1)$ and balanced projection $T \in \mathbb{M}_{2^n}$, one of the following inequalities holds:*

(i) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \frac{C\varepsilon\log(n)}{n}$;
(ii) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1} \geq \frac{C}{n^{(1+\varepsilon)/2}}$.

*Proof.* For each $\varepsilon > 0$ and balanced projection $T \in \mathbb{M}_{2^n}$, by the quantum Eldan-Gross inequality (i.e., Theorem 1.11), there exists a universal constant $K > 0$ such that

$$\begin{aligned}\frac{1}{4}\sqrt{\log\left(1+\frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2}\right)} &\leq K\left\|\left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}\right\|_{L_1} \\ &\leq K\left(\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_2}^2\right)^{1/2}.\end{aligned} \tag{5.17}$$

We assume from now on that

$$\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_2}^2 \leq \frac{\varepsilon\log(n)}{16K^2} \tag{5.18}$$

for sufficient large $n$; otherwise, we obtain

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \frac{\varepsilon\log(n)}{16K^2n}.$$

Substituting the assumption (5.18) to (5.17) yields

$$\frac{1}{4}\sqrt{\log\left(1+\frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2}\right)} \leq \frac{\sqrt{\varepsilon\log(n)}}{4},$$

which further implies

$$\frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2} \leq n^{\varepsilon},$$

and consequently,

$$\frac{1}{n^{(1+\varepsilon)/2}} \leq \max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1}.$$

It remains to choose $C = \min\{\frac{1}{16K^2}, 1\}$ to obtain the desired result. □

In order to apply Theorem 5.12 to obtain another KKL-type inequality, we introduce the concept of index for elements in $\mathbb{M}_{2^n}$ as follows. For each $T \in \mathbb{M}_{2^n}$, we define

$$\mathrm{ind}(T) \coloneqq \inf\{\alpha \geq 0 : \|\mathbf{d}_j(T)\|_{L_1}^{\alpha} \leq \|\mathbf{d}_j(T)\|_{L_2}^2, \ \forall\, j \in [n]\}.$$

**Lemma 5.13.** *For each balanced projection $T \in \mathbb{M}_{2^n}$, we have* $\mathrm{ind}(T) \in [1, 2]$.

*Proof.* For balanced projection $T \in \mathbb{M}_{2^n}$, we note that

$$\|\mathbf{d}_j(T)\|_{L_1} \leq \|T\|_{L_1} = \frac{1}{2} < 1, \quad \forall\, j \in [n]. \tag{5.19}$$

If $\mathrm{ind}(T) > 2$, there exists $\mathrm{ind}(T) \geq \alpha > 2$ and some $j_0 \in [n]$ such that

$$\|\mathbf{d}_{j_0}(T)\|_{L_1}^{\alpha} > \|\mathbf{d}_{j_0}(T)\|_{L_2}^2 \geq \|\mathbf{d}_{j_0}(T)\|_{L_1}^2.$$

This implies that $\|\mathbf{d}_{j_0}(T)\|_{L_1}^{\alpha-2} > 1$, which contradicts to (5.19). Hence, $\mathrm{ind}(T) \leq 2$.

It remains to show that $\mathrm{ind}(T) \geq 1$. Indeed, since $\mathbf{d}_j$ is a contraction, it follows from (5.19) that

$$\|\mathbf{d}_j(T)\|_{L_2}^2 \leq \|\mathbf{d}_j(T)\|_{L_1} \|\mathbf{d}_j(T)\|_{L_\infty} \leq \|\mathbf{d}_j(T)\|_{L_1} < 1.$$

On the other hand, since $\|\mathbf{d}_j(T)\|_{L_1}^{\alpha}$ is decreasing on $\alpha$ by (5.19), we infer that $\mathrm{ind}(T) \geq 1$. □

We now conclude this subsection with the following quantum KKL-type inequality invoking $L_2$-influence.

**Theorem 5.14.** *For each $n \in \mathbb{N}$ and balanced projection $T \in \mathbb{M}_{2^n}$ with $\mathrm{ind}(T) < 2$, there exists a constant $C_{\mathrm{ind}(T)} > 0$ depending on the index of $T$ such that*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \frac{C_{\mathrm{ind}(T)} \log(n)}{n}. \tag{5.20}$$

*Proof.* Choose positive $\alpha$ such that $\mathrm{ind}(T) \leq \alpha < 2$ and set $\delta = \frac{2-\alpha}{4}$, $\varepsilon = \frac{2-\alpha}{2\alpha}$. By Theorem 5.12, there exists a universal constant $C > 0$ such that one of the following inequalities holds

(i) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \frac{C\varepsilon \log(n)}{n}$;
(ii) $\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1} \geq \frac{C}{n^{(1+\varepsilon)/2}}$.

If the first situation holds, there is nothing to prove. We assume from now on that item (ii) holds, that is,

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1} \geq \frac{C}{n^{(1-\delta)/\alpha}}, \tag{5.21}$$

where we used the fact $\varepsilon = \frac{2-2\delta-\alpha}{\alpha}$. Since $\|\mathbf{d}_j(T)\|_{L_1}^{\alpha} \leq \|\mathbf{d}_j(T)\|_{L_2}^2$, it follows from (5.21) that

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_2}^2 \geq \frac{C^\alpha}{n^{1-\delta}} \geq \frac{(2-\alpha)C^\alpha \log(n)}{4n}.$$

Choosing $C_{\mathrm{ind}(T)} = \min\{\frac{C(2-\alpha)}{2\alpha}, \frac{(2-\alpha)C^\alpha}{4}\}$ yields the desired inequality. □

**Remark 5.15.** *There exists a quantum Boolean function $T \in \mathbb{M}_{2^n}$ with* $\mathrm{ind}(T) = 2$ *such that the quantum KKL inequality holds for $L_2$-influence; see* [18, Proposition 11.5]*. However, as shown in* [11, Remark 6.5]*, this is not the case in the CAR algebra setting.*

Our final application of Theorem 1.11 is the following stability result for the quantum KKL-type inequality (invoking $L_1$-influences), which is quantum analogy of [8, Corollary 3.5]. For each $T \in \mathbb{M}_{2^n}$, we define that $|\nabla(T)| \coloneqq \left(\sum_{j=1}^n |\mathbf{d}_j(T)|^2\right)^{1/2}$.

**Corollary 5.16.** *Suppose that there exists a constant $C_1 > 0$ such that for each projection $T \in \mathbb{M}_{2^n}$ the following holds*

$$\max_{j\in[n]} \|\mathbf{d}_j(T)\|_{L_1} \leq \frac{C_1 \log(n)\mathrm{var}(T)}{n}. \tag{5.22}$$

*Then there exist constant $C_2 > 0$ such that*

$$\mathrm{tr}\left[\mathbb{1}_{(\frac{1}{2}\mathrm{var}(T)\sqrt{\log(n)},\infty)}(|\nabla(T)|)\right] \geq C_2 \mathrm{var}(T).$$

*Proof.* By assumption (5.22) and Proposition 2.7 (i), we have

$$\mathrm{tr}\left[|\nabla(T)|^2\right] = \sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_2}^2 \leq \sum_{j=1}^n \|\mathbf{d}_j T\|_{L_1} \leq C_1 \log(n)\mathrm{var}(T). \tag{5.23}$$

According to the assumption, there exists a universal constant $K_1 > 0$ such that

$$\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2 \leq \frac{C_1^2 \log^2(n)\mathrm{var}(T)^2}{n} \leq \frac{K_1}{\sqrt{n}}, \tag{5.24}$$

where we used $\mathrm{var}(T) \leq \frac{1}{4}$ for the projection $T \in \mathbb{M}_{2^n}$. Using Theorem 1.11 and (5.24), there exists $K_2 > 0$ such that

$$\|\nabla(T)\|_{L_1} \geq \frac{1}{K}\mathrm{var}(T)\sqrt{\log\left(1 + \frac{1}{\sum_{j=1}^n \|\mathbf{d}_j(T)\|_{L_1}^2}\right)} \geq K_2 \mathrm{var}(T)\sqrt{\log(n)}. \tag{5.25}$$

By the Paley-Zygmund inequality (2.9), there exists a constant $C_2 > 0$ such that

$$\mathrm{tr}\left[\mathbb{1}_{(\frac{1}{2}\mathrm{var}(T)\sqrt{\log(n)},\infty)}(|\nabla(T)|)\right] \geq \frac{\|\nabla(T)\|_{L_1}^2}{4\mathrm{tr}[|\nabla(T)|^2]} \geq C_2 \mathrm{var}(T),$$

where the last inequality follows from a combination of (5.23) and (5.25). □

5.4. **Proof of Proposition 5.2.** The proof of Proposition 5.2 relies on the following technical lemmas. Firstly, with the moment comparison lemma (i.e., Lemma 5.9) at hand, we can derive the following deviation inequality. Although this deviation inequality is well-known, we include its proof for the convenience of readers.

**Lemma 5.17.** *There exists universal constant $K > 0$ such that for each $T \in \mathbb{M}_{2^n}$ with degree at most $d \in \mathbb{N}$ and $\|T\|_{L_2} \leq 1$, we have*

$$\mathrm{tr}\left[\mathbb{1}_{[t,\infty)}(|T|)\right] \leq K \exp\left\{-\frac{d \cdot t^{2/d}}{4e}\right\}, \quad \textit{for all } t > 0.$$

*Proof.* Applying the assumption $\|T\|_{L_2} = 1$ and Lemma 5.9 it follows that

$$\|T\|_{L_r}^r \leq r^{dr/2} \quad \text{for each } r \geq 1. \tag{5.26}$$

Let $\alpha = \frac{d}{4e}$, and we now show that $\operatorname{tr}\left[\exp(\alpha|T|^{2/d})\right] < \infty$. Indeed, by the Taylor expansion and the Stirling formula $k! \sim \frac{k^k\sqrt{s\pi k}}{e^k}$, we get that

$$\begin{aligned}
\operatorname{tr}\left[\exp(\alpha|T|^{2/d})\right] &= \sum_{k=0}^{\infty} \frac{\alpha^k \|T\|_{L_{2k/d}}^{2k/d}}{k!} \\
&\leq \sum_{k=0}^{\infty} \frac{\left(\frac{2\alpha}{d}\right)^k \cdot k^k}{k!} \\
&\leq K_1 \sum_{k=0}^{\infty} \left(\frac{2e\alpha}{d}\right)^k = 2K_1 < \infty,
\end{aligned}$$

where we used (5.26) and the Stirling formula in the second and the third inequality, respectively. Therefore, by the Chebyshev inequality, we have that for each $t > 0$ the following holds

$$\operatorname{tr}[\mathbb{1}_{(t,\infty)}(|T|)] \leq e^{-\alpha t^{2/d}} \cdot \operatorname{tr}\left[\exp(\alpha|T|^{2/d})\right] \leq K \exp\left\{-\frac{d \cdot t^{2/d}}{4e}\right\}.$$

□

Applying the functional calculus for positive element in $\mathbb{M}_{2^n}$, we obtain the following integral representation lemma.

**Lemma 5.18.** *Let $S, T \in \mathbb{M}_{2^n}$. If $T$ is positive, then*

$$\operatorname{tr}(ST) = \int_0^{\infty} \operatorname{tr}\left(S \cdot \mathbb{1}_{(t,\infty)}(T)\right).$$

*Proof.* By the functional calculus of $T$, it is clear that $T = \int_0^\infty \mathbb{1}_T((t,\infty))dt$. We apply the integral representation of $T$ to the $\operatorname{tr}(ST)$ entails that

$$\operatorname{tr}(ST) = \operatorname{tr}\left(S \cdot \int_0^{\infty} \mathbb{1}_{(t,\infty)}(T)dt\right) = \int_0^{\infty} \operatorname{tr}\left(S \cdot \mathbb{1}_{(t,\infty)}(T)\right)dt,$$

where the last equality follows from the linearity of tr. □

We also need the following estimate from [13, Lemma 12].

**Lemma 5.19.** *Let $d \geq 1$ be a positive integer and $t_0 > (4e)^{\frac{d}{2}}$. Then we have*

$$\int_{t_0}^{\infty} t^2 \cdot \exp\left\{-\frac{d \cdot t^{2/d}}{2e}\right\} dt \leq 5et_0^{3-\frac{2}{d}} \exp\left\{-\frac{d \cdot t_0^{2/d}}{2e}\right\}.$$

We shall prove Proposition 5.2 in details. We here explain some notation we use below. For each $\mathbf{s} = (s_1, , \cdots, s_n) \in \{0,1,2,3\}^n$, it is viewed an element in $\{0,1,2,3\}^{n+1}$ via

$$\widetilde{\mathbf{s}} = (s_1, \cdots, s_n, 0).$$

To simplify symbols, we still write $\mathbf{s}$ instead of $\widetilde{\mathbf{s}}$. Hence, for each $\mathbf{s} \in \{0,1,2,3\}^n$, the summation $\mathbf{s} \oplus e_{n+1}^{\alpha}$ is read as follows

$$\mathbf{s} \oplus e_{n+1}^{\alpha} := \widetilde{\mathbf{s}} \oplus e_{n+1}^{\alpha} \in \{0,1,2,3\}^{n+1}, \quad \alpha \in \{0,1,2,3\}.$$

For each $\mathbf{s} \in \{0,1,2,3\}^n$, we set

$$\mathbf{s}^{j\frown} = (s_1, \cdots, s_{j-1}, 0, s_{j+1}, \cdots, s_n, s_j) \in \{0,1,2,3\}^{n+1},$$

i.e., we remove the original $s_j$ to the $n+1$-position and replace the original $s_j$ with 0. By the same spirit, for each $\mathbf{s} \in \{0,1,2,3\}^n$, $\sigma_{\widetilde{\mathbf{s}}} = \sigma_{\mathbf{s}} \otimes \mathbf{1}_2$ is viewed as an element in $\mathbb{M}_{2^{n+1}}$. For simplicity, we still write $\sigma_{\mathbf{s}}$ instead of $\sigma_{\widetilde{\mathbf{s}}}$.

In what follows, let $d \geq 1$ be fixed, $T \in \mathbb{M}_{2^n}$ and $J \subseteq [n]$. For $j \in J$, define

$$T_j := \sum_{\substack{\operatorname{supp}(\mathbf{s}) \subseteq J^c \\ |\operatorname{supp}(\mathbf{s})| = d-1}} \sum_{\alpha \in \{1,2,3\}} \widehat{T}(\mathbf{s} \oplus e_j^\alpha) \sigma_{\mathbf{s} \oplus e_{n+1}^\alpha}, \tag{5.27}$$

$$T_{\text{copy},j} := \sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n \\ s_j = 0}} \widehat{T}(\mathbf{s}) \sigma_{\mathbf{s} \oplus e_{n+1}^0} + \sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n \\ s_j \neq 0}} \widehat{T}(\mathbf{s}) \sigma_{\mathbf{s}^{j\frown}}, \tag{5.28}$$

and

$$\widetilde{T}_j := \underbrace{\sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n \\ s_j = 0}} \widehat{T}(\mathbf{s}) \sigma_{\mathbf{s} \oplus e_{n+1}^0}}_{\widetilde{T}_{j,L}} + \underbrace{\sum_{\substack{\mathbf{s} \in \{0,1,2,3\}^n \\ s_j \neq 0}} \widehat{T}(\mathbf{s}) \sigma_{\mathbf{s} \oplus e_{n+1}^{s_j}}}_{\widetilde{T}_{j,R}}. \tag{5.29}$$

Then, $T_j$, $T_{\text{copy},j}$ and $\widetilde{T}_j$ are elements in $\mathbb{M}_{2^{n+1}}$.

Define $\Psi_j : \mathbb{M}_{2^n} \otimes \mathbf{1}_2 \to \mathbb{M}_2^{\otimes j-1} \otimes \mathbf{1}_2 \otimes \mathbb{M}_2^{\otimes n-j}$ by setting: for each $\mathbf{s} = (s_j)_{j=1}^n \in \{0,1,2,3\}^n$,

$$\sigma_{\mathbf{s}} \otimes \mathbf{1}_2 \mapsto \sigma_{\mathbf{s}^{j\frown}}.$$

It is clear that $\Psi_j$ is an $*$-isomorphism from $\mathbb{M}_{2^n} \otimes \mathbf{1}_2$ onto $\mathbb{M}_2^{\otimes j-1} \otimes \mathbf{1}_2 \otimes \mathbb{M}_2^{\otimes n-j}$ with

$$\Psi_j(T \otimes \mathbf{1}_2) = T_{\text{copy},j}$$

and

$$\Psi_j\left(\mathbf{d}_j(T \otimes \mathbf{1}_2)\right) = \mathbf{d}_{n+1}(T_{\text{copy},j})$$

Hence, we have

$$\|\mathbf{d}_{n+1}(T_{\text{copy},j})\|_{L_1} = \|\Psi_j\left(\mathbf{d}_j(T \otimes \mathbf{1}_2)\right)\|_{L_1} = \|\mathbf{d}_j(T \otimes \mathbf{1}_2)\|_{L_1} = \|\mathbf{d}_j(T)\|_{L_1}. \tag{5.30}$$

The next several technical lemmas provide necessary information of $T_j$, $T_{\text{copy},j}$ and $\widetilde{T}_j$, which are key ingredients of proving Proposition 5.2.

**Lemma 5.20.** *Let $T \in \mathbb{M}_{2^n}$ and $J \subseteq [n]$. For each $j \in J$, we have*

$$\mathcal{E}_{J^c \cup \{n+1\}}\left(A_j \widetilde{T}_j\right) = \mathcal{E}_{J^c \cup \{n+1\}}\left(\mathbf{d}_{n+1}(T_{\text{copy},j})\right),$$

*where $\widetilde{T}_j$ and $T_{\text{copy},j}$ are as in (5.29) and (5.28), and $A_j$ is given by*

$$A_j = \sum_{\alpha \in \{1,2,3\}} \sigma_{e_j^\alpha}. \tag{5.31}$$

*Proof.* For fixed $j \in J$, according to the definition of conditional expectation, we note that for each $\widetilde{\mathbf{s}} \in \{0,1,2,3\}^{n+1}$, if $\widetilde{\mathbf{s}}_i \neq 0$ for some $i \in J$, then

$$\mathcal{E}_{J^c \cup \{n+1\}}(\sigma_{\widetilde{\mathbf{s}}}) = 0. \tag{5.32}$$

From this, we immediately deduce that $\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_{j,L}\right)=0$, where $\widetilde{T}_{j,L}$ is as in (5.29). Thus,

$$\begin{aligned}\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)&=\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_{j,R}\right)\\&=\sum_{\alpha\in\{1,2,3\}}\sum_{\substack{\mathbf{s}\{0,1,2,3\}^n\\ s_j\neq 0}}\widehat{T}(\mathbf{s})\mathcal{E}_{J^c\cup\{n+1\}}\left(\sigma_{e_j^\alpha}\cdot\sigma_{\mathbf{s}\oplus e_{n+1}^{s_j}}\right)\\&=\sum_{\alpha\in\{1,2,3\}}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}\widehat{T}(\mathbf{s})\mathcal{E}_{J^c\cup\{n+1\}}\left(\sigma_{e_j^\alpha}\cdot\sigma_{\mathbf{s}\oplus e_{n+1}^{s_j}}\right)\\&=\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}^{j\frown}},\end{aligned}\tag{5.33}$$

where we used (5.32) twice in last two equality.

On the other hand side, it follows from (2.4) that

$$\mathbf{d}_{n+1}(T_{\mathrm{copy},j})=\sum_{\substack{\mathbf{s}\in\{0,1,2,3\}^n\\ s_j\neq 0}}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}^{j\frown}}.$$

Hence, by (5.32) again,

$$\mathcal{E}_{J^c\cup\{n+1\}}\left(\mathbf{d}_{n+1}(T_{\mathrm{copy},j})\right)=\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}^{j\frown}}.$$

The desired assertion follows from the above argument. □

**Lemma 5.21.** *Let $T\in\mathbb{M}_{2^n}$ and $J\subset[n]$. For each $j\in J$, we have*

$$\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\\ |\mathrm{supp}(\mathbf{s})|=d-1,\alpha\in\{1,2,3\}}}|\widehat{T}(\mathbf{s}\oplus e_j^\alpha)|^2=\left(\mathrm{tr}\left[\overline{T}_j\cdot A_j\widetilde{T}_j\right]\right)^2,$$

*where $\overline{T}_j=T_j/\|T_j\|_{L_2}$, $T_j$, $\widetilde{T}_j$ and $A_j$ are referred to* (5.27), (5.29) *and* (5.31), *respectively.*

*Proof.* Take $j\in J$. From the orthogonality of $\{\sigma_\mathbf{s}\}_{\mathbf{s}\in\{0,1,2,3\}^{n+1}}$ in $L_2(\mathbb{M}_{2^{n+1}})$, it is not hard to see that for each $\mathbf{s}\in\{0,1,2,3\}^n$ and $\alpha\in\{1,2,3\}$,

$$\left\langle\sigma_{\mathbf{s}\oplus e_{n+1}^\alpha},A_j\widetilde{T}_{j,L}\right\rangle=0,$$

where $\widetilde{T}_{j,L}$ is as in (5.29). It follows that

$$\begin{aligned}&\left\langle T_j,A_j\widetilde{T}_j\right\rangle=\left\langle T_j,A_j(\widetilde{T}_{j,L}+\widetilde{T}_{j,R})\right\rangle=\left\langle T_j,A_j\widetilde{T}_{j,R}\right\rangle\\&=\sum_{\substack{\mathrm{supp}(\mathbf{s}')\subseteq J^c\\ |\mathrm{supp}(\mathbf{s}')|=d-1}}\sum_{\substack{\mathbf{s}\in\{0,1,2,3\}^n\\ s_j\neq 0}}\sum_{\alpha',\alpha\in\{1,2,3\}}\langle\widehat{T}(\mathbf{s}'\oplus e_j^{\alpha'})\sigma_{\mathbf{s}'\oplus e_{n+1}^{\alpha'}},\widehat{T}(\mathbf{s})\sigma_{e_j^\alpha}\cdot\sigma_{\mathbf{s}\oplus e_{n+1}^{s_j}}\rangle\\&=\sum_{\substack{\mathrm{supp}(\mathbf{s}')\subseteq J^c\\ |\mathrm{supp}(\mathbf{s}')|=d-1}}\sum_{\alpha'\in\{1,2,3\}}|\widehat{T}(\mathbf{s}'\oplus e_j^{\alpha'})|^2,\end{aligned}$$

where in the second equality we used the orthogonality of $\{\sigma_{\mathbf{s}}\}_{\mathbf{s}\in\{0,1,2,3\}^{n+1}}$ in $L_2(\mathbb{M}_{2^{n+1}})$. To verify the desired assertion, it suffices to note that

$$\|T_j\|_{L_2}^2 = \sum_{\substack{\mathrm{supp}(\mathbf{s}')\subseteq J^c \\ |\mathrm{supp}(\mathbf{s}')|=d-1}} \sum_{\alpha'\in\{1,2,3\}} |\widehat{T}(\mathbf{s}'\oplus e_j^{\alpha'})|^2.$$

□

**Lemma 5.22.** *Let $T\in\mathbb{M}_{2^n}$ and $J\subseteq[n]$. For each $j\in J$ and $t_0>0$, we have*

$$\int_0^{t_0} \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot\left|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right|\right]\, dt \le t_0\,\|\mathbf{d}_j(T)\|_{L_1}$$

*where $\overline{T}_j = T_j/\|T_j\|_2$, $T_j$, $\widetilde{T}_j$ and $A_j$ are referred to (5.27), (5.29) and (5.31), respectively.*

*Proof.* It follows from Lemma 5.20 that

$$\begin{aligned}&\int_0^{t_0} \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot\left|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right|\right]\, dt\\ \le& t_0\left\|\mathcal{E}_{J^c\cup\{n+1\}}(\mathbf{d}_{n+1}(T_{\mathrm{copy},j}))\right\|_{L_1} \le t_0\left\|\mathbf{d}_{n+1}(T_{\mathrm{copy},j})\right\|_{L_1},\end{aligned}$$

where the second inequality is due to the fact that conditional expectation is bounded on $L_p$, $1\le p\le\infty$. The desired inequality follows from (5.30). □

**Lemma 5.23.** *Let $T\in\mathbb{M}_{2^n}$ and $J\subseteq[n]$. For each $j\in J$ and $t_0>(2e)^{\frac{d}{2}}$, we have*

$$\begin{aligned}&\int_{t_0}^{\infty} \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot\left|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right|\right]\, dt\\ \le&\sqrt{5e}\,t_0^{1-\frac{1}{d}}\exp\left\{-\frac{d\cdot t_0^{2/d}}{4e}\right\}\left(\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}} |\widehat{T}(\mathbf{s})|^2\right)^{1/2},\end{aligned}$$

*where $\overline{T}_j = T_j/\|T_j\|_2$, $T_j$, $\widetilde{T}_j$ and $A_j$ are referred to (5.27), (5.29) and (5.31), respectively.*

*Proof.* Applying the Cauchy-Schwarz inequality twice, we get

$$\begin{aligned}&\int_{t_0}^{\infty} \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot\left|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right|\right] dt\\ \le&\left(\int_{t_0}^{\infty}\frac{1}{t^2}\,dt\right)^{1/2}\cdot\left(\int_{t_0}^{\infty} t^2\cdot\left(\mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot\left|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right|\right]\right)^2 dt\right)^{1/2}\\ \le&\frac{1}{\sqrt{t_0}}\left(\int_{t_0}^{\infty} t^2\mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(\overline{T}_j)\right] dt\right)^{1/2}\cdot\left\|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right\|_{L_2}.\end{aligned}$$

Combining Lemma 5.17 and Lemma 5.19, we have

$$\int_{t_0}^{\infty} t^2\mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(\overline{T}_j)\right] dt \le 5e t_0^{3-\frac{2}{d}}\exp\left\{-\frac{d\cdot t_0^{2/d}}{2e}\right\}.$$

Furthermore, according to (5.33), we have

$$\left\|\mathcal{E}_{J^c\cup\{n+1\}}\left(A_j\widetilde{T}_j\right)\right\|_{L_2}^2 = \sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}} |\widehat{T}(\mathbf{s})|^2, \quad \text{for each } j\in J.$$

The desired assertion follows. □

We now are ready to provide the proof of Proposition 5.2.

*Proof of Proposition 5.2.* For a given projection $T\in\mathbb{M}_{2^n}$, it is clear that $\|T\|_2\leq 1$. We assume without loss of generality that $T=\sum_{\mathbf{s}\in\{0,1,2,3\}^n}\widehat{T}(\mathbf{s})\sigma_{\mathbf{s}}$. For each $j\in J$, by Lemma 5.21, we have

$$\begin{aligned}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\\ |\mathrm{supp}(\mathbf{s})|=d-1,\alpha\in\{1,2,3\}}} |\widehat{T}(\mathbf{s}\oplus e_j^\alpha)|^2 &= \left(\mathrm{tr}\left[\overline{T}_j\cdot A_j\widetilde{T}_j\right]\right)^2\\ &= \left(\mathrm{tr}\left[\mathcal{E}_{J^c\cup\{n+1\}}(\overline{T}_j\cdot A_j\widetilde{T}_j)\right]\right)^2\\ &= \left(\mathrm{tr}\left[\overline{T}_j\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)\right]\right)^2,\end{aligned}$$

where we used that each conditional expectation preserves trace and $\mathcal{E}_{J^c\cup\{n+1\}}(T_j)=T_j$ (this follows from the definition of $T_j$ as in (5.27)). Using Lemma 5.18 we have

$$\begin{aligned}\left(\mathrm{tr}\left[\overline{T}_j\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)\right]\right)^2 &\leq \left(\mathrm{tr}\left[|\overline{T}_j|\cdot|\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)|\right]\right)^2\\ &= \left\{\int_0^\infty \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot|\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)|\right]\,dt\right\}^2\\ &\leq 2\left\{\int_0^{t_0} \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot|\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)|\right]\,dt\right\}^2\\ &\quad+2\left\{\int_{t_0}^\infty \mathrm{tr}\left[\mathbb{1}_{(t,\infty)}(|\overline{T}_j|)\cdot|\mathcal{E}_{J^c\cup\{n+1\}}(A_j\widetilde{T}_j)|\right]\,dt\right\}^2\\ &:= 2\mathsf{Y}_{1,j}(T)^2+2\mathsf{Y}_{2,j}(T)^2,\end{aligned}$$

where $t_0>0$ is chosen to satisfy

$$\exp\left\{-\frac{d\cdot t_0^{2/d}}{2e}\right\}=\sum_{j\in J}\|\mathbf{d}_j(T)\|_{L_1}^2=M_J(T). \tag{5.34}$$

Note here that (5.34) and the assumption $M(T)\leq e^{-2d}$ imply that

$$t_0^2=\left(\frac{2e}{d}\right)^d\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d \quad\text{and}\quad t_0\geq(4e)^{\frac{d}{2}}.$$

We conclude from the above argument that

$$\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\\ |\mathrm{supp}(\mathbf{s})|=d-1,\alpha\in\{1,2,3\}}} |\widehat{T}(\mathbf{s}\oplus e_j^\alpha)|^2\leq 2\sum_{j\in J}\mathsf{Y}_{1,j}(T)^2+2\sum_{j\in J}\mathsf{Y}_{2,j}(T)^2. \tag{5.35}$$

By Lemma 5.22, we have

$$\sum_{j\in J}\mathsf{Y}_{1,j}(T)^2\leq t_0^2\sum_{j\in J}\|\mathbf{d}_j(T)\|_{L_1}^2=\left(\frac{2e}{d}\right)^d M_J(T)\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d.$$

Using Lemma 5.23, we have

$$\begin{aligned}\sum_{j\in J}\mathsf{Y}_{2,j}(T)^2 &\le 5et_0^{2-\frac{2}{d}}\exp\left\{-\frac{d\cdot t_0^{2/d}}{2e}\right\}\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}|\widehat{T}(\mathbf{s})|^2\\ &= 5et_0^{-\frac{2}{d}}\left(\frac{2e}{d}\right)^d M_J(T)\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}|\widehat{T}(\mathbf{s})|^2\\ &\le 2\left(\frac{2e}{d}\right)^d M_J(T)\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d.\end{aligned}$$

Here we also used the fact (note that $T$ is a projection, and hence $\|T\|_{L_2}\le 1$)

$$\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\cup\{j\}\\ s_j\neq 0}}|\widehat{T}(\mathbf{s})|^2\le\|T\|_{L_2}^2\le 1.$$

Substituting the estimates of $\sum_{j\in J}\mathsf{Y}_{1,j}^2$ and $\sum_{j\in J}\mathsf{Y}_{2,j}^2$ to (5.35), we get

$$\sum_{j\in J}\sum_{\substack{\mathrm{supp}(\mathbf{s})\subseteq J^c\\ |\mathrm{supp}(\mathbf{s})|=d-1,\alpha\in\{1,2,3\}}}|\widehat{T}(\mathbf{s}\oplus e_j^\alpha)|^2\le 6\left(\frac{2e}{d}\right)^d M_J(T)\left(\log\left(\frac{1}{M_J(T)}\right)\right)^d.$$

This completes the proof of the proposition. □

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