ON A BIRCH AND SWINNERTON-DYER TYPE CONJECTURE FOR THE HASSE-WEIL-ARTIN *L*-FUNCTIONS IN CHARACTERISTIC p > 0

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ABSTRACT. Given an abelian variety *A* over a global function field *K* of characteristic p > 0 and an irreducible complex continuous representation ψ of the absolute Galois group of *K*, we obtain a BSD-type formula for the leading term of Hasse–Weil–Artin *L*-function for (A, ψ) at s = 1 under certain technical hypotheses. The formula we obtain can be applied quite generally; for example, it can be applied to the *p*-part of the leading term even when ψ is weakly wildly ramified at some place under additional hypotheses.

Our result is the function field analogue of the work of D. Burns and D. Macias Castillo [BMC24], built upon the work on the equivariant refinement of the BSD conjecture by D.Burns, M. Kakde and the first-named author [BKK]. To handle the *p*-part of the leading term, we need the Riemann–Roch theorem for equivariant vector bundles on a curve over a finite field generalising the work of S. Nakajima [Nak86], B. Köck [Köc04], and H. Fischbacher-Weitz and B. Köck [FWK09], which is of independent interest.

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1. INTRODUCTION

In [Tat68], Tate gave a uniform formulation of the conjecture of Birch and Swinnerton-Dyer (or the *BSD conjecture*) for abelian varieties over global fields of any characteristic. Furthermore, for a Jacobian of a curve over a global function field of characteristic p > 0, the "prime-to-p part" of the full conjecture (including the leading term formula up to ppower ambiguity) was obtained assuming finiteness of a certain object closely related to the Tate–Shafarevich group; see [Tat68, Theorem 5.2] for further details. It is now known that for an abelian variety A over a global function field K the full BSD conjecture follows from the finiteness of III(A/K){ ℓ }, the ℓ -primary part of the Tate–Shafarevich group for some prime ℓ ; see Kato–Trihan [KT03] for the precise result, and its introduction for the history. We note that the p-part of the argument in [KT03] heavily relies on the theory of p-adic cohomology, as anticipated by Tate [Tat68, p. 438].

In the work of D. Burns, M. Kakde and the first-named author [BKK], we formulated an *equivariant refinement* of the BSD conjecture for abelian varieties over a global function

²⁰²⁰ Mathematics Subject Classification. 11G40, 14G10, 11J95, 14C40.

Key words and phrases. Conjecture of Birch and Swinnerton-Dyer over global function fields, Hasse-Weil-Artin *L*-functions, Geometric equivariant Tamagawa number conjecture, equivariant Riemann-Roch theorem.

field of characteristic p > 0 in the spirit of the Equivariant Tamagawa Number Conjecture for motives over a number field. Given an abelian variety A over a global function field K and a finite Galois extension L/K, the equivariant BSD conjecture [BKK, Conjecture 4.3] refines the BSD conjecture for A/L by predicting that the "*derived* Galois module structure" of arithmetic invariants of A/L should encode the (suitably normalised) leading terms of Hasse–Weil–Artin L-functions at s = 1 attached to A and complex irreducible characters of G := Gal(L/K), as well as certain "algebraic relations" thereof.¹ (To forumlate the conjecture we need certain *perfect complexes* of integral Galois modules, which we refer to as the "*derived* Galois module structure".)

In this paper, we consider the following natural question.

Question 1.1. Let *A* be an abelian variety over a global function field *K*, and we choose a complex irreducible character ψ of the absolute Galois group of *K*. Assuming finiteness of the relevant Tate–Shafarevich group, can we get an *explicit* formula of the (suitably normalised) the leading term of the Hasse–Weil–Artin *L*-function attached to (A, ψ) at s = 1 (possibly imposing additional hypotheses that are not too restrictive)?

Note that the equivariant refinement of the BSD conjecture is known up to torsion in relative K_0 assuming a suitable finiteness condition on the Tate–Shafarevich group; *cf* [BKK, Theorem 4.10]. In fact, ignoring such torsion ambiguity does *not* affect individual leading terms, though we may lose algebraic relations among them. However, *loc. cit.* does not completely resolve the question; in fact, the resulting formula depends on the "derived Galois-module structure" of certain perfect complexes, so it is not explicit enough. (See Corollary 6.15 for the precise statement.) Nonetheless, one take [BKK, Theorem 4.10] and Corollary 6.15 as a starting point, and manage to extract some non-trivial and explicit formula on the leading term at s = 1 of *L*-function attached to (A, ψ) under some additional technical hypotheses; *cf.* Assumption 7.2.

Let us briefly indicate the nature of Assumption 7.2. Recall that the equivariant BSD conjecture [BKK, Conjecture 4.3] involves two perfect complexes: a kind of Selmer complex for A/L, and a coherent cohomology of a certain equivariant vector bundle. Our additional hypotheses are mainly to simplify the "Selmer complex term", clearly inspired by the number field analogue of our result obtained by D. Burns and D. Macias Castillo (*cf.* [BMC24], especially the set of hypotheses at the beginning of §6). Our main work is to control the "coherent cohomology term" (or rather, the *ramification correction* to the local volumes, so to speak) under a *mild* hypothesis – namely, Assumption 7.2(3) – and thereby obtain a formula for the *p*-*part* of the leading term in a satisfying generality. If *A* has semistable reduction at all places of *K*, then our main result can be applied if ψ has tame ramification at worst (or even, we allow "shallow wild ramification") assuming finiteness of a suitable Tate–Shafarevich group.

Let us set up the notation for more detailed introduction. In the setting of Question 1.1, let L/K be a finite Galois extension with G := Gal(L/K) such that ψ factors through G. Suppose that ψ can be defined over a number field $E \subset \mathbb{C}$, and we fix the underlying *E*vector space V_{ψ} for the representation ψ . Let *Z* be the set of places of *K* consisting exactly of the places ramified in L/K and the bad reduction places for *A*.

We consider the Hasse–Weil–Artin *L*-function $L_U(A, \psi, s)$ without Euler factors at *Z* as in (6.9), where *U* denotes the set of places of *K* away from *Z*. We normalise its leading term $\mathscr{L}_U(A, \psi)$ at s = 1 as (6.10) so that we have $\mathscr{L}_U(A, \psi) \in E^{\times}$. In particular, for any place λ of *E* it makes sense to consider the λ -adic valuation $v_{\lambda}(\mathscr{L}_U(A, \psi))$.

If λ is a place over ℓ coprime to |G| (which applies to all but finitely many places of *E*), then $v_{\lambda}(\mathscr{L}_U(A, \psi))$ is quite easy to describe since the group ring $\mathbb{Z}_{\ell}G$ is rather simple in terms of homological algebra; *cf.* Proposition 7.13. For a place λ of *E* over a prime

¹See [BKK, Proposition 4.8] for an example of algebraic relations implied by the equivariant BSD conjecture when L/K is a *p*-extension.

 ℓ dividing |G| however, one cannot really expect an explicit description of $v_{\lambda}(\mathscr{L}_U(A, \psi))$ without imposing an additional assumption for (A, ψ) to simplify the homological algebra involved. And unsurprisingly, it requires much harder extra work to handle places λ over p, the characteristic of K, when p divides |G|.

Theorem 1.2 (*Cf.* Theorem 7.12). In addition to the above setting, suppose that III(A/L) is finite. Choose a prime ℓ that does not divide any of $|A(L)_{tors}|$, $|A^t(L)_{tors}|$ and $|\mathcal{A}_L(k_w)|$, where \mathcal{A}_L is the Néron model of A/L and k_w is the residue field at a place w of L above some $v \in Z$. If $\ell = p$ then we assume that A has semistable reduction at all places of K and the extension L/K is at worst weakly ramified at each place in the sense of Definition 2.6. Then for any place λ of E above ℓ we have

$$\mathscr{L}_{U}(A,\psi)O_{E,\lambda} = \operatorname{vol}_{Z}(A/K)^{\operatorname{deg}\psi} \cdot \operatorname{loc}_{Z_{L}}(A,\psi) \cdot \frac{\operatorname{Reg}_{\lambda}^{\psi}}{|G|^{r_{\operatorname{alg}}(\psi)}} \cdot \operatorname{Char}_{\psi} \left(\operatorname{III}_{\psi,\lambda}^{\vee}(A/L) \right),$$

where $O_{E,\lambda}$ is the λ -adic completion of O_E , $\operatorname{Reg}_{\lambda}^{\psi}$ is the ψ -twisted regulator for A (cf. Definition 7.9) and $r_{\operatorname{alg}}(\psi)$ is the rank of the " ψ -part of A(L)" (6.12). Lastly, $\operatorname{Char}\left(\coprod_{\psi,\lambda}^{\vee}(A/L)\right)$ is the characteristic ideal (7.11) of $\coprod_{\psi,\lambda}^{\vee}(A/L)$ (7.1c), and $\operatorname{vol}_Z(A/K)$ and $\operatorname{loc}_{Z_L}(A,\psi)$ are p-power integers defined in Theorem 7.12.

We actually obtain a result in a more general setting where *A* has semistable reduction at all places of *L* (instead of *K*), L/K is at worst weakly ramified at each place, and L/Kis tamely ramified at all places of *K* where *A* has non-semistable reduction. The formula becomes more complicated in this generality, and we refer to the main body of the text.

If we choose L = K (so ψ is the trivial character and $E = \mathbb{Q}$), then Theorem 1.2 is compatible with the ℓ -part of the classical BSD formula [KT03, (1.8.1)]; indeed, in the setting of of Theorem 7.12 we have $\log_{Z_L}(A, \psi) = 1$ and $\operatorname{vol}_Z(A/K)$ is the *p*-part of vol $(\prod_{v \in Z} A(K_v))$, using the notation of *loc. cit.* In general, the *p*-power integer $\log_{Z_L}(A, \psi)$, given by an explicit local formula, can be thought of as the "ramification correction" to the volume term. In fact, we have $\log_{Z_L}(A, \psi) = 1$ if the ramification index of L/K at each place is a power of *p* (including the case where L/K is unramified everywhere). If the ramification index of L/K at each place divides p - 1 and $\deg \psi = 1$ then we have

$$\log_p\left(\log_{Z_L}(A,\psi)\right) = \frac{\dim A}{|G|} \sum_w j_{w,\psi}[k_w:\mathbb{F}_p],$$

where *w* runs through all places of *L* ramified over *K*, and j_w is determined so that the inertia subgroup at *w* acts on $\mathfrak{m}_w^{j_w}/\mathfrak{m}_w^{j_w+1}$ via the restriction of ψ , where \mathfrak{m}_w is the maximal ideal of O_{X_L} corresponding to *w*. (See Remarks 7.14 and 4.19 for further details.) We believe that the explicit formula for $\log_{Z_L}(A, \psi)$ is new even when L/K is cyclic and tame.

We note that for abelian varieties defined over a number field, an analogous result was obtained by D. Burns and D. Macias Castillo (*cf.* [BMC24, Proposition 7.3]), which clearly inspired our result.

Let us now list some of the main ingredients of the proof. Let $\pi: X_L \to X$ denote the covering of smooth projective curves corresponding to L/K, and let $Z_L \subset X_L$ be the closed subset consisting of places over Z. Let \mathcal{A}_L be the Néron model over X_L of A/L.

Let us specialise to the case where $\ell = p$, which is the main case of interest. To apply [BKK, Theorem 4.10] one needs to choose a suitable *G*-stable subbundle $\mathcal{L} \subseteq$ Lie $(\mathcal{A}_L)(-Z_L)$ such that $\mathbb{R}\Gamma(X_L, \mathcal{L})$ is a perfect $\mathbb{F}_p G$ -complex. The choice of such \mathcal{L} could a priori be very inexplicit, but we show that we can take $\mathcal{L} = \text{Lie}(\mathcal{A}_L)(-Z_L)$ provided that the following conditions are satisfied; *cf.* Assumption 7.2(3).

Assumption 1.3. • L/K is weakly ramified at all places (*cf.* Definition 2.6), and

• if L/K is wildly ramified at a place v of K then A has semistable reduction at v.

See Corollary 5.8(3) and Proposition 6.6(2) for the precise statement. Note that the main ingredient of the proof is Köck's local integral normal basis theorem for weakly ramified extensions [Köc04, Theorem 1.1], which we recall in Theorem 2.11.

Suppose that Assumption 1.3 is valid, and set $\mathcal{L} = \text{Lie}(\mathcal{A}_L)(-Z_L)$. Then we get a *p*-adic Selmer complex with support condition $\text{SC}_{Z_L,p}(A, L/K) \in D^{\text{perf}}(\mathbb{Z}_p G)$ following the construction of [BKK, Proposition 3.7(i)], and under the assumption of Theorem 1.2 one can compute its " ψ -isotypic parts" by the same argument as [BMC24, Proposition 7.3(ii)]; *cf.* Proposition 7.10.

By the equivariant BSD conjecture modulo torsion, it remains to compute the ψ -isotypic part of $R\Gamma(X_L, \text{Lie}(\mathcal{A}_L)(-Z_L))^{\vee}$. This can be achieved using the following result.

Theorem 1.4 (*Cf.* Theorem 3.2, Corollary 5.8). Set $\mathcal{L} := \text{Lie}(\mathcal{A}_L)(-Z_L)$ and view it as a *G*-equivariant vector bundle on X_L . If Assumption 1.3 is satisfied, then $\mathbb{R}\Gamma(X_L, \mathcal{L})$ can be represented by a two-term complex $[C^0 \to C^1]$ for some projective \mathbb{F}_pG -modules C^0, C^1 . Furthermore, we have an explicit formula for

$$[C^0] - [C^1] - \chi(\mathcal{L}^G)[\mathbb{F}_p G] \in \mathrm{K}_0(\mathbb{F}_p G)$$

in terms of the inertia action on the completed stalk $\widehat{\mathcal{L}}_w$ at each $w \in Z_L$. (Here, $\chi(\mathcal{L}^G) = \log_p(|\mathrm{H}^0(X, \mathcal{L}^G)|/|\mathrm{H}^1(X, \mathcal{L}^G)|)$.)

The precise formula is quite complicated, and we refer to the main body of the text.

The statement can be divided into two steps. By analysing the Néron models over X_L and X, we deduce a certain local property of \mathcal{L} in terms of ramification at each $w \in Z_L$ (*cf.* Corollary 5.8). And for *G*-equivariant vector bundles on X_L satisfying the same local property satisfied by \mathcal{L} , we prove a kind of "equivariant Riemann–Roch theorem"; *cf.* Theorem 3.2. When $X_L \to X$ is a *tame G*-cover of curves over an algebraically closed field k, then the Euler characteristic of a *G*-equivariant vector bundle \mathcal{L} in $K_0(kG)$ was computed modulo [kG] by S. Nakajima [Nak86, Theorem 2]. The rank-1 case of the equivariant Riemann–Roch theorem was obtained by Fischbacher-Weitz and Köck [FWK09, §3, Theorem 12] (built upon the case of curves over algebraically closed field [Köc04, Theorem 4.5]). We give a common generalisation of these arguments to obtain the equivariant Riemann–Roch theorem sufficient for the proof of our main result, Theorem 1.2.

By Theorem 1.4 we can compute the ψ -isotypic part of $R\Gamma(X_L, \mathcal{L})^{\vee}$, and compare it with the volume term and $\log_{Z_L}(A, \psi)$. In case where *A* admits non-semistable reduction at some place of *K*, the volume term needs to be corrected by analysing the behaviour of Néron models over tame extensions; *cf.* Proposition 5.5 and Theorem 7.12.

Let us outline the contents of the paper. In §2 we collect various results for semilinear representations of decomposition groups, including Köck's local integral normal basis theorem. In §3 we formulate and prove the "Riemann–Roch theorem for equivariant vector bundles" (*cf.* Theorem 3.2). In §4 we review the relative K_0 -groups and reinterpret Theorem 3.2 using relative K_0 -group. In §5 we collect various results on Néron models (including the behaviour under tame ramification) and show that the equivariant Riemann–Roch theorem can be applied to Lie(\mathcal{A}_L)($-Z_L$) under Assumption 1.3. In §6 we review the equivariant refinement of the BSD conjecture in [BKK], and in §7 we give a proof of the main theorem (*cf.* Theorem 7.12). In §8 we give some examples in which our main theorem can be applied unconditionally.

Notation and Conventions 1.5. For any commutative ring R (necessarily with 1) and for any group G, we let RG denote the group ring of G over R. We may write R[G] for RG if there is any risk of confusion.

By a *G*-representation ψ , we mean a finite-dimensional \mathbb{C} -linear *G*-representation. Let V_{ψ} denote the (left) $\mathbb{C}G$ -module underlying ψ . As a standard fact, there exists an *EG*-module $V_{\psi,E}$ for some number field $E \subset \mathbb{C}$ such that we have a $\mathbb{C}G$ -isomorphism $\mathbb{C} \otimes_E V_{\psi,E} \cong V_{\psi}$. We will also use V_{ψ} to refer to $V_{\psi,E}$ if there is no risk of confusion.

By $T_{\psi} = T_{\psi,O_E}$, we denote a (chosen) *G*-stable O_E -lattice in V_{ψ} . By abuse of notation, we also let ψ denote the ring homomorphisms $\mathbb{Z}G \to \operatorname{End}_{O_E}(T_{\psi})$ and $\mathbb{Q}G \to \operatorname{End}_E(V_{\psi})$ defined by the *G*-action on T_{ψ} and V_{ψ} .

We write $(-)^*$ for the linear dual, and $(-)^{\vee}$ for the Pontryagin dual. For T_{ψ} as above, we regard T^*_{ψ} and T^{\vee}_{ψ} as *right* O_EG -modules. We write $\check{\psi}$ for the contragredient of ψ .

For any ring *A* (necessarily with 1 but not necessarily commutative), we let D(A) denote the derived category of complexes of *A*-modules, and $D^{\text{perf}}(A)$ for the triangulated full subcategory of perfect complexes of *A*-modules. For any $C^{\bullet} \in D^{\text{perf}}(A)$, we define its *Euler characteristic* as follows:

(1.6)
$$\chi_A(C^{\bullet}) \coloneqq \sum_i (-1)^i [C^i] \in \mathcal{K}_0(A),$$

where $K_0(A)$ is the Grothendieck group of the category of finitely generated projective *A*-modules and $[C^i] \in K_0(A)$ denote the class of C^i .

2. Review of local integral normal basis theorems

In this section, we collect various standard results on lattices in semilinear Galois modules for finite extensions of local fields, following Chinburg [Chi94] and Köck [Köc04].

Let K_v be a complete discrete valuation field with perfect residue field k_v of characteristic p > 0. Let O_v and \mathfrak{m}_v respectively denote the valuation ring and its maximal ideal. We fix a finite Galois extension L_w/K_v with valuation ring O_w , maximal ideal \mathfrak{m}_w , and residue field k_w . Set $G_w := \operatorname{Gal}(L_w/K_v)$, and write I_w and P_w for the inertia and wild inertia subgroups, respectively. (Although the results in this section are purely local, we will later apply them in the setting where L_w/K_v arises from some global extension L/Kvia completing at $w \mid v$.)

By semilinear G_w -representation over O_w , we mean a finite free O_w -module W_w equipped with semilinear G_w -action.

Lemma 2.1. For a semilinear G_w -representation W_w over O_w the following are equivalent.

- (1) W_w is free as an $O_v[G_w]$ -module;
- (2) W_w is projective as an $O_v[G_w]$ -module;
- (3) W_w is cohomologically trivial for G_w (i.e., the Tate cohomology $\widehat{H}^i(G, W_w)$ is trivial for each degree *i*).

Proof. Note that $W_w \otimes_{O_w} L_w$ is free as an $K_v[G_w]$ -module by standard Galois descent, so the equivalence of (1) and (2) follows from [Swa60, Corollary 6.4]. The equivalence between (2) and (3) is standard as W_w is projective as O_w -module.²

Let us now recall the following "higher-rank version" of the local integral normal basis theorem in the tame setting, which is essentially due to Chinburg.

Proposition 2.2. Let W_w be a semilinear G_w -representation over O_w . Then W_w is free as an $O_v[G_w]$ -module if and only if it is cohomologically trivial for I_w . In particular, if L_w/K_v is tame then any semilinear G_w -representation over O_w is free as an $O_v[G_w]$ -module.

Proof. The case when L_w/K_v is *unramified* is standard; *cf.* [Nak84, §2, Lemma 1]. To handle the general case, it suffices, by Lemma 2.1, to show that W_w is cohomologically trivial for G_w if and only if it is cohomologically trivial for I_w . And since by the unramified case $W_w^{I_w}$ is cohomologically trivial for G_w/I_w (being a semilinear G_w/I_w -lattice over $O_w^{I_w}$), the desired claim follows from the inflation-restriction sequence for the Tate cohomology. The claim for the tame case now follows since any $O_v[I_w]$ -module cohomologically trivial for I_w when $|I_w|$ is prime to p.

²The proof of the $\mathbb{Z}G$ -projectivity criterion [Ser79, Chap IX, §5, Theorem 7] can be repeated to show the *RG*-projectivity criterion for any Dedekind domain *R*. This is also implicitly proved in [Chi94, Proposition 4.1].

We next classify the inertia action on any semilinear G_w -representation over O_w in the tame case. For this let us first describe all the mod p absolutely irreducible representations of I_w , without assuming tameness.

Definition 2.3. Let $\theta_w \colon I_w \to k_w^{\times}$ be the character corresponding to the natural I_w -action on $\mathfrak{m}_w/\mathfrak{m}_w^2$; in other words, choosing a uniformiser $\varpi_w \in \mathfrak{m}_w$ we have

$$\theta_w(g) \equiv g \varpi_w / \varpi_w \mod m_w \quad \forall g \in I_w.$$

The I_w -action on $\mathfrak{m}_w^n/\mathfrak{m}_w^{n+1}$ is given by θ_w^n .

Remark 2.4. Note that θ_w induces an inclusion $I_w/P_w \hookrightarrow k_w^{\times}$, so the order of θ_w is $|I_w/P_w|$. Furthermore, since P_w acts trivially on any simple $k_w[I_w]$ -module, any simple $k_w[I_w]$ -module is isomorphic to exactly one of $\mathfrak{m}_w^n/\mathfrak{m}_w^{n+1}$ for $n \in \mathbb{Z}/(|I_w/P_w|)$.

Lemma 2.5. Suppose that L_w/K_v is tame, and let W_w be a rank-d semilinear G_w -representation over O_w . Then there exist integers $n_{w,1}, \dots, n_{w,d} \in \{0, \dots, |I_w| - 1\}$, unique up to ordering, such that we have a $k_w[I_w]$ -module isomorphism

$$W_{w} \otimes_{\mathcal{O}_{w}} k_{w} \cong \bigoplus_{i=1}^{d} (\mathfrak{m}_{w}^{-n_{w,i}}/\mathfrak{m}_{w}^{-n_{w,i}+1}).$$

Furthermore, the above isomorphism can be lifted to an isomorphism

$$W_w \cong \bigoplus_{i=1}^d \mathfrak{m}_w^{-n_{w,i}}.$$

of semilinear I_w -representations over O_w .

Proof. By tameness, the group ring $k_w[I_w]$ is semi-simple and its simple modules are described in Remark 2.4. Therefore, one can find a k_w -basis $\bar{e}_1, \dots, \bar{e}_d$ of $W_w \otimes_{O_w} k_w$ such that I_w acts on \bar{e}_i via $\theta_w^{-n_{w,i}}$ for $0 \leq n_{w,i} < |I_w|$. We choose a lift $e_i \in W_w$ of \bar{e}_i for each i, and set

$$e'_i \coloneqq \frac{1}{|I_w|} \sum_{g \in I_w} \theta_w^{n_{w,i}}(g) \cdot (ge_i).$$

Then each $e'_i \in W_w$ lifts \bar{e}_i and satisfies $ge'_i = \theta_w^{-n_{w,i}}(g)e'_i$ for any $g \in I_w$. Therefore, W_w can be written as a direct sum of I_w -stable O_w -submodules $O_w e'_i$, which is isomorphic to $\mathfrak{m}_w^{-n_{w,i}}$ as a semilinear I_w representation over O_w .

If L_w/K_v is wildly ramified, then the Galois module structure of a semilinear G_w -representation over O_w could be quite complicated in general. Instead, we focus on the case where W_w is of rank 1. To proceed, we need the following definition.

Definition 2.6. We say that L_w/K_v is *weakly ramified* if the second lower-index ramification subgroup $I_{w,2}$ is trivial.

Recall that for any non-negative integer s, we set

$$I_{w,s} := \{ q \in I_w \mid q \varpi_w \equiv \varpi_w \mod \mathfrak{m}_w^{s+1} \}$$

for some (or equivalently, any) uniformiser $\varpi_w \in \mathfrak{m}_w$. Note that $I_w = I_{w,0}$ and $P_w = I_{w,1}$.

Clearly, unramified or tamely ramified extensions are weakly ramified. Much less obvious examples of weakly ramified extensions are those obtained by the completion of a finite Galois cover $\pi: X_L \to X$ of *ordinary*³ curves over a perfect field of characteristic p > 0; *cf.* [Nak87, Theorem 2(i)].

Being weakly ramified imposes a strong condition on the inertia group I_w as follows.

³A curve over a field of characteristic p > 0 is defined to be *ordinary* if the genus and the *p*-rank coincides.

Lemma 2.7. For any finite Galois extension L_w/K_v we have

$$I_w = P_w \rtimes C_w$$

where P_w is a p-group and C_w is a cyclic group of prime-to-p order. Furthermore, if L_w/K_v is weakly ramified, then P_w is an elementary p-group and the conjugation action of C_w on $P_w \setminus \{1\}$ is faithful.

Proof. The properties can be deduced from Proposition 9 and the corollaries of Proposition 7 in [Ser79, Chap IV, §2].

Remark 2.8. The choice of the lift C_w of I_w/P_w is far from canonical if P_w is a proper nontrivial subgroup of I_w , but different choices of C_w are conjugate to each other. Indeed, by direct computation we have

(2.9)
$$C_w \cap (gC_w g^{-1}) = \{1\} \quad \forall g \in P_w \setminus \{1\}\}$$

cf. the proof of Lemma 4.2 in [Köc04]. By simple counting we obtain $I_w \setminus P_w = \bigsqcup_{g \in P_w} (gC_w g^{-1} \setminus \{1\})$, so in particular any lift of I_w / P_w in I_w is of the form $gC_w g^{-1}$ for a unique $g \in P_w$.

Lemma 2.10. For any finite Galois extension L_w/K_v , any indecomposable projective $k_w[I_w]$ -module is isomorphic to exactly one of

$$M_{w}(j) \coloneqq \operatorname{Ind}_{C_{w}}^{I_{w}}\left((\mathfrak{m}_{w}^{j}/\mathfrak{m}_{w}^{j+1})|_{C_{w}}\right) \quad \text{for } j \in \mathbb{Z}/(|C_{w}|).$$

Furthermore, $M_w(j)$ is a $k_w[I_w]$ -projective cover of $\mathfrak{m}_w^j/\mathfrak{m}_w^{j+1}$, so it does not depend on the choice of C_w up to isomorphism.

If L_w/K_v is weakly ramified, then we have

$$M_{w}(j)|_{C_{w}} \cong (\mathfrak{m}_{w}^{j}/\mathfrak{m}_{w}^{j+1}) \oplus k_{w}[C_{w}]^{\oplus \frac{j+w-1}{|C_{w}|}}.$$

 $|P_{ij}| = 1$

Proof. Note that the radical $rad(k_w[I_w])$ of $k_w[I_w]$ is generated by the augmentation ideal of $k_w[P_w]$. Furthermore, we have an $k_w[I_w]$ -module isomorphism

$$M_{w}(j)/\operatorname{rad}(k_{w}[I_{w}]) = M_{w}(j)_{P_{w}} \cong \mathfrak{m}_{w}^{j}/\mathfrak{m}_{w}^{j+1}$$

Then essentially by the Nakayama lemma, $M_w(j)$ is a $k_w[I_w]$ -projective cover of $\mathfrak{m}_w^j/\mathfrak{m}_w^{j+1}$; *cf.* [CR81, Theorem (6.23)]. Indecomposability of $M_w(j)$ follows from being a projective cover of a simple $k_w[I_w]$ -module. Since $k_w[I_w] \cong \bigoplus_j M_w(j)$, any non-zero projective $k_w[I_w]$ -module contains a copy of some $M_w(j)$. Finally, the last claim is proved in [Köc04, Lemma 4.2].

Let us now recall the local integral normal basis theorem due to Köck:

Theorem 2.11 (Köck [Köc04, Theorem 1.1]). The local fractional ideal \mathfrak{m}_w^{-n} for $n \in \mathbb{Z}$ is free of rank 1 as an $O_v[G_w]$ -module if and only if L_w/K_v is weakly ramified and $n \equiv -1 \mod |P_w|$.

If L_w/K_v is tame (i.e., we have $|P_w| = 1$) then the theorem asserts that any fractional ideal \mathfrak{m}_w^{-n} is projective as an $O_v[G_w]$ -module, which is consistent with Proposition 2.2.

Remark 2.12. The higher-rank generalisation of Theorem 2.11 (or rather, the wildly ramified analogue of Lemma 2.5) could be quite complicated. To illustrate, let L_w/K_v be any finite Galois extension (not necessarily weakly ramified) and choose a semilinear G_w representation W'_w over O_w . (We do *not* require W'_w to be projective as $O_v[G_w]$ -module.) Then $O_w[G] \otimes_{O_w} W'_w$, with G acting diagonally, is a semilinear G_w -representation over O_w that is free as a $O_v[G_w]$ -module; indeed, the following O_w -linear isomorphism

$$\begin{split} \operatorname{Ind}_{1}^{G_{w}}W'_{w} &\cong O_{w}[G] \otimes_{O_{w}} W'_{w} \xrightarrow{\sim} O_{w}[G] \otimes_{O_{w}} W'_{w} \\ & (\sum_{g \in G_{w}} a_{g}g) \otimes x \mapsto \sum_{g \in G_{w}} (a_{g}g) \otimes gx \end{split}$$

is G_w -equivariant.

3. Equivariant Riemann-Roch for weakly ramified covering

Let *k* be a perfect field of characteristic p > 0. Let *X* be a smooth projective geometrically connected curve over *k*, with its function field denoted by *K*. For any finite extension *L* of *K*, we write X_L denote the normalisation of *X* in Spec *L* equipped with the covering map $\pi \colon X_L \to X$. (We do *not* require X_L to be geometrically connected over *k*.) Let |X|and $|X_L|$ respectively denote the set of closed points of *X* and X_L .

From now on, suppose that L/K is *Galois* with group G, so π is a G-covering. Choosing $v \in |X|$ and $w \in \pi^{-1}(v)$, we obtain the Galois extension L_w/K_v via completion. We employ the same notation as in §2.

We next study the "equivariant Euler characteristic" of the cohomology of *G*-equivariant vector bundles on X_L ; i.e., a locally free O_{X_L} -module with semilinear *G*-action. Given such \mathcal{E} , we may represent $R\Gamma(X_L, \mathcal{E})$ as a complex of kG-modules (eg, by choosing a *G*-stable Čech covering). If we have $R\Gamma(X_L, \mathcal{E}) \in D^{perf}(kG)$, then we write

(3.1)
$$\chi_{kG}(\mathcal{E}) \coloneqq \chi_{kG}(\mathrm{R}\Gamma(X_L, \mathcal{E})) \in \mathrm{K}_0(kG),$$

where the right hand side is defined in (1.6).

1.0

Given a finite Galois cover $\pi: X_L \to X$, we let Z_L^{ram} (respectively, Z_L^{wild}) denote the locus in X_L where π is ramified (respectively, wildly ramified).

We are now ready to state the *equivaiant Riemann–Roch theorem*, which generalises the results of Nakajima [Nak86], Köck [Köc04] and Fischbacher-Weitz and Köck [FWK09].

Theorem 3.2. Let $\pi: X_L \to X$ be a *G*-cover that is weakly ramified everywhere. Let \mathcal{E} be a *G*-equivariant vector bundle on X_L , and suppose that for any $w \in Z_L^{\text{wild}}$ we have

(3.2a)
$$\widehat{\mathcal{E}}_{w} \cong \bigoplus_{i=1}^{\mathsf{IKC}} \mathfrak{m}_{w}^{-n_{w,i}} \quad \text{where } n_{w,i} \equiv -1 \mod |P_{w}| \text{ for any } i$$

as a semilinear I_w -representation over O_w . Then the following properties hold.

- (1) We can represent $R\Gamma(X_L, \mathcal{E})$ by a complex of finitely generated projective kG-modules in degrees [0, 1]. In particular, we have $R\Gamma(X_L, \mathcal{E}) \in D^{perf}(kG)$.
- (2) For any $w \in Z_L^{\text{ram}}$ and $i \in \{1, \dots, \text{rk } \mathcal{E}\}$, define $l_{w,i}$ to be the unique integer satisfying

(3.2b)
$$l_{w,i} \equiv \frac{1+n_{w,i}}{|P_w|} - 1 \mod |I_w/P_w| \quad and \quad 0 \le l_{w,i} < |I_w/P_w|,$$

where $n_{w,i}$'s are as in (3.2a) for $w \in Z_L^{\text{wild}}$, and as in Lemma 2.5 if $w \notin Z_L^{\text{wild}}$.⁴ Then we have the following equality in $K_0(kG) \otimes \mathbb{Q}$

(3.2c)
$$\chi_{kG}(\mathcal{E}) = -(\operatorname{rk} \mathcal{E})[N(\pi)] + [W_G(\mathcal{E})] + \operatorname{Ind}_1^G(\chi_k(\mathcal{E}^G)),$$

where \mathcal{E}^G is the *G*-invariants of \mathcal{E} , which is a vector bundle on *X*, and

$$[N(\pi)] \coloneqq \frac{1}{|G|} \sum_{w \in Z_L^{\text{ram}}} |P_w| \sum_{j=1}^{|I_w/P_w|-1} j \cdot \left[\text{Ind}_{I_w}^G \left(M_w(j) \right) \right] \text{ and}$$
$$[W_G(\mathcal{E})] \coloneqq \sum_{w \in Z_L^{\text{ram}}} \frac{1}{|G:I_w|} \sum_{i=1}^{\text{rk} \mathcal{E}} \sum_{j=1}^{I_{w,i}} \left[\text{Ind}_{I_w}^G \left(M_w(-j) \right) \right].$$

Here, $M_w(j)$ is defined in Lemma 2.10.

Note that formula (3.2c) is generalises the rank-1 case stated in [FWK09, §3, Theorem 12], which is built upon [Köc04, Theorem 4.5]. When $k = \bar{k}$ and π is tame, then S. Nakajima [Nak86, Theorem 2] obtained (3.2c) modulo Ind₁^G ($\chi_k(\mathcal{E}^G)$).

Before we give a proof, let us make a few remarks.

⁴Note that for $w \in Z_L^{\text{ram}} \setminus Z_L^{\text{wild}}$, we have $l_{w,i} = n_{w,i}$ as we have $|P_w| = 1$ and $0 \le n_{w,i} < |I_w|$.

Remark 3.3. As the notation suggests, $[N(\pi)]$ and $[W_G(\mathcal{E})]$ respectively come from finitely generated projective kG-modules $N(\pi)$ and $W_G(\mathcal{E})$. Indeed, this is a byproduct of the rank-1 case of the formula; *cf*. Theorem 11 and Theorem 12(a) in [FWK09, §3]. In particular, formula (3.2c) holds in $K_0(kG)$ since $K_0(kG)$ is torsion-free (being a free abelian group). In the intended application, we only need formula (3.2c) in $K_0(kG) \otimes \mathbb{Q}$.

Remark 3.4. In the setting of Theorem 3.2, if we have $I_w = P_w$ for all $w \in Z_L^{\text{ram}}$ (e.g., if *G* is a *p*-group), then clearly both $[N(\pi)]$ and $[W_G(\mathcal{E})]$ are trivial so formula (3.2c) reduces to

$$\chi_{kG}(\mathcal{E}) = \operatorname{Ind}_{1}^{G} \left(\chi_{k}(\mathcal{E}^{G}) \right).$$

Remark 3.5. Applying the Hirzeburch–Riemann–Roch theorem to compute $\chi_k(\mathcal{E}^G)$ (*cf.* Theorem (4.11) and Exa 4.1.1 in [Har77, Appendix A]), we obtain

$$\operatorname{Ind}_{1}^{G}\left(\chi_{k}(\mathcal{E}^{G})\right) = \left((\operatorname{rk}\mathcal{E})\cdot(1-\operatorname{gen}_{K}) + \operatorname{deg}(\mathcal{E}^{G})\right)[kG],$$

where gen_K is the genus of *X*. (Note also that $\operatorname{rk}(\mathcal{E}^G) = \operatorname{rk} \mathcal{E}$.)

Example 3.6. The local assumption (3.2a) may look artificial, but it is a common generalisation of the tame case (*cf.* Lemma 2.5) and the following class of examples. Let \mathcal{F} be a rank-*d* vector bundle on *X*, and let $D_L = \sum_{w \in |X_L|} n_w w$ be a *G*-equivariant divisor of X_L . Then $\mathcal{E} := (\pi^* \mathcal{F})(D_L)$ is a *G*-equivariant vector bundle and we clearly have a G_w -equivariant O_w -linear isomorphism

(3.6a)
$$\widehat{\mathcal{E}}_{w} \cong (\mathfrak{m}_{w}^{-n_{w}})^{\oplus d},$$

as we have a natural G_w -equivariant isomorphism $(\pi^* \mathcal{F})_{\widehat{w}} \cong \widehat{\mathcal{F}}_{\pi(w)} \otimes_{\mathcal{O}_{\pi(w)}} \mathcal{O}_w$. Furthermore, we have $\mathcal{E}^G \cong \mathcal{F}(D_K)$ where D_K satisfies $(\mathcal{O}_{X_L}(D_L))^G = \mathcal{O}_X(D_K)$. More explicitly, we can apply [Köc04, Lemma 1.4(a)] to obtain $D_K = \sum_{v \in |X|} n_v v$ where for each $v \in |X|$ we set $n_v = -1 + \lceil \frac{n_v + 1}{|I_v|} \rceil$ for some (or equivalently, any) $\tilde{v} \in \pi^{-1}(v)$.

Now, assume that π is weakly ramified everywhere. Then Theorem 3.2 can be applied to $\mathcal{E} = (\pi^* \mathcal{F})(D_L)$ provided that the coefficient n_w at each $w \in Z_L^{\text{wild}}$ satisfies $n_w \equiv -1 \mod |P_w|$, in which case we have $n_{w,i} = n_w$ for all *i*.

Let us further specialise to the case when $\mathcal{E} = (\pi^* \mathcal{F})(-Z_L)$ for some reduced *G*-stable closed subscheme $Z_L \subset X_L$ containing Z_L^{ram} . In that case, we have $l_{w,i} = |I_w/P_w| - 1$ for any $w \in Z_L^{\text{ram}}$ and i, and $\mathcal{E}^G = \mathcal{F}(-Z)$ where $Z \subset X$ is the reduced image of Z_L . Therefore, formula (3.2c) reduces to the following

(3.6b)
$$\chi_{kG}\left((\pi^*\mathcal{F})(-Z_L)\right) = \frac{\operatorname{rk}\mathcal{F}}{|G|} \sum_{w \in Z_L^{\operatorname{ram}}} \sum_{j=1}^{|I_w/P_w|-1} j|P_w| \cdot \left[\operatorname{Ind}_{I_w}^G\left(M_w(-j)\right)\right] \\ + \left((\operatorname{rk}\mathcal{F}) \cdot (1 - \operatorname{gen}_K - \operatorname{deg}(Z)) + \operatorname{deg}(\mathcal{F})\right) [kG].$$

For the rest of the section we give a proof of Theorem 3.2.

Proof of Claim (1) of Theorem 3.2. The proof is essentially contained in the proof of Theorem 2.1(a) in [Köc04], which is the rank-1 case of our statement. Indeed, by Köck's theorem (Theorem 2.11) our assumption 3.2a implies that $\widehat{\mathcal{E}}_w$ is cohomologically trivial for I_w for any $w \in |X_L|$, so by Proposition 2.2 it follows that the following completed stalk

$$(\pi_*\mathcal{E})_{\widehat{v}} \cong \bigoplus_{w \in \pi^{-1}(v)} \widehat{\mathcal{E}}_w \cong \operatorname{Ind}_{G_{\widetilde{v}}}^G \widehat{\mathcal{E}}_{\widetilde{v}}$$

is $O_v[G]$ -free at any $v \in |X|$. (Here, $\tilde{v} \in \pi^{-1}(v)$ is any point above v.) By the standard result (*cf.* [CR06, Corollary (76.9)]), the Zariski stalk $(\pi_*\mathcal{E})_v$ is $O_{X,v}[G]$ -free, so we may apply [Chi94, Theorem 1.1] to obtain kG-perfectness of $\mathrm{R}\Gamma(X_L, \mathcal{E}) \cong \mathrm{R}\Gamma(X, \pi_*\mathcal{E})$. Finally, $\mathrm{R}\Gamma(X_L, \mathcal{E})$ can be represented by a two-term perfect kG-complex thanks to the cohomology vanishing outside degrees [0, 1].

Digressions to Brauer characters. Even if |G| is not invertible in k, there is a version of character theory for finitely generated $\bar{k}G$ -modules; namely, *Brauer characters.* (Here, \bar{k} denotes the algebraic closure of k.) We will recall the minimal background needed in the proof, and for further details we refer to [CR81, §18] or [Ser77, Ch 18].

We choose a complete discrete valuation field *E* of characteristic 0 with residue field \bar{k} . For any subset $S \subset G$ stable under conjugation, set

(3.7)
$$\operatorname{Cl}_{E}(S) := \{\phi : S \to E \mid \phi \text{ is invariant under conjugation}\}.$$

Let $G^{p\text{-reg}}$ denote the set of elements of G with prime-to-p order, which is stable under conjugation. Then for any finitely generated $\bar{k}G$ -module M, let $\operatorname{Bch}_M \in \operatorname{Cl}_E(G^{p\text{-reg}})$ denote the *Brauer character* in the sense of [CR81, Def (17.4)]. It is easy to check that $\operatorname{Bch}_M(1) = \dim_{\bar{k}}(M)$ and $\operatorname{Bch}_{\bar{k}G}(g) = 0$ for any $g \neq 1$. If M admits a lift to an EG-module \widetilde{M}_E (e.g., if M is projective), then we have $\operatorname{Bch}_M = \operatorname{ch}_{\widetilde{M}_E} |_{G^{p\text{-reg}}}$, which we take as a working definition; *cf.* [CR81, Proposition (17.5)(iv)].⁵

Let us now recall a few basic properties. The construction of Brauer characters naturally extends to the Grothendieck group $G_0(\bar{k}G)$ of finitely generated $\bar{k}G$ -modules, inducing an isomorphism $G_0(\bar{k}G) \otimes E \rightarrow \operatorname{Cl}_E(G^{p\text{-reg}})$; *cf.* [CR81, Theorem (17.9)]. There is a natural homomorphism $c: K_0(\bar{k}G) \rightarrow G_0(\bar{k}G)$, which turns out to be injective with finite cokernel; *cf.* [CR81, Theorem (21.22)]. By abuse of notation, for $[M] \in K_0(\bar{k}G)$ we let $\operatorname{Bch}_{[M]}$ denote $\operatorname{Bch}_{c([M])}$. As $K_0(\bar{k}G)$ and $G_0(\bar{k}G)$ are free abelian groups, we obtain the following:

Proposition 3.8. The homomorphism $\operatorname{Bch}_{(-)}$: $\operatorname{K}_0(\overline{k}G) \to \operatorname{Cl}_E(G^{p\operatorname{-reg}})$ is injective.

We record the following lemma, which should be well known.

Lemma 3.9. For finitely generated projective $\bar{k}G$ -modules M and M', we have $\operatorname{Bch}_M(g) = \operatorname{Bch}_{M'}(g)$ for any $g \neq 1$ if and only if [M] - [M'] is an integer multiple of $[\bar{k}G]$ in $K_0(\bar{k}G)$.

Proof. Suppose that an element $[N] \in K_0(\bar{k}G)$ satisfies $\operatorname{Bch}_{[N]}(g) = 0$ for any $g \neq 0$. We first claim that |G| divides $\operatorname{Bch}_{[N]}(1) = \dim_{\bar{k}}([N])$. The lemma easily follows from this claim for [N] = [M] - [M'] via Proposition 3.8.

For each prime divisor ℓ of |G|, choose a Sylow ℓ -subgroup G_{ℓ} of G. Note that the restriction $[N]|_{G_{\ell}}$ defines an element in $K_0(\bar{k}[G_{\ell}])$ as the restriction preserves projectivity, and we have $\operatorname{Bch}_{[N]|_{G_{\ell}}} = (\operatorname{Bch}_{[N]})|_{G_{\ell}}$.

If $\ell \neq p$ then we have $G_{\ell} \subset G^{p\text{-reg}}$ and $\operatorname{Bch}_{[N]}|_{G_{\ell}} = \operatorname{ch}_{[\widetilde{N}_E]}|_{G_{\ell}}$ where $[\widetilde{N}_E] \in \operatorname{K}_0(EG)$ is the lift of [N]. By the standard character theory in characteristic 0, we have $\dim_E([\widetilde{N}_E]^{G_{\ell}}) = \dim_{\widetilde{k}}([N])/|G_{\ell}|$, which is an integer. If $\ell = p$ then $\operatorname{K}_0(\widetilde{k}[G_p])$ is the free abelian group generated by $[\widetilde{k}[G_p]]$ as $\widetilde{k}[G_p]$ is a local ring. In particular, $|G_p|$ divides $\dim_{\widetilde{k}}([N])$. This shows that |G| divides $\dim_{\widetilde{k}}([N])$.

Remark 3.10. Lemma 3.9 cannot be extended to $G_0(\bar{k}G)$ in general. For example, if G admits a *normal* Sylow *p*-subgroup $P \neq \{1\}$, then we have $\operatorname{Bch}_{\bar{k}[G/P]}(g) = 0$ for any $g \in G^{p\operatorname{-reg}} \setminus \{1\}$. Then the Brauer characters of $\bar{k}[G/P]^{\oplus |P|}$ and $\bar{k}G$ coincide, but $\bar{k}[G/P]$ is not even projective as a $\bar{k}G$ -module.

Proof of Claim (2) of Theorem 3.2: The case where k **is algebraically closed.** In the setting of Theorem 3.2, assume that $k = \overline{k}$ so we have $G_w = I_w$. Choose $H = \langle g \rangle$ for some $g \in G^{p\text{-reg}} \setminus \{1\}$, a cyclic subgroup of prime-to-p order. Let $\pi_H \colon X_L \to X_{L^H}$ denote the natural projection, which is a everywhere tame H-covering.

The following lemma generalises the tame case [Nak86, p 120], and the main idea of proof can be read off from the proof of Theorem 3.1 and Theorem 4.3 of [Köc04].

⁵One may refer to [Ser77, §18.1] instead, where Bch_M is called the *modular character*.

Lemma 3.11. In the above setting, we have the following elements in $K_0(kH)$

$$[N(\pi)]|_{H} - [N(\pi_{H})]$$
 and $[W_{G}(\mathcal{E})]|_{H} - [W_{H}(\mathcal{E})]$

are integer multiples of [kH].

Proof. For any $w \in Z_L^{\text{ram}}$, let $H_w := I_w \cap H$. Then by the Mackey formula [CR06, Theorem (44.2)⁶, we have

$$[\operatorname{Ind}_{I_{w}}^{G} M_{w}(j)]|_{H} = \sum_{s \in H \setminus G/I_{w}} \left[\operatorname{Ind}_{H_{sw}}^{H} \left(M_{sw}(j)|_{H_{sw}} \right) \right]$$
$$= \frac{1}{|H|} \sum_{s \in G/I_{w}} |H_{sw}| \cdot \left[\operatorname{Ind}_{H_{sw}}^{H} \left(M_{sw}(j)|_{H_{sw}} \right) \right].$$

Choosing C_{sw} to contain H_{sw} , which is possible by Remark 2.8, it follows from Lemma 2.10 that

(3.12)
$$M_{sw}(j)|_{H_{sw}} \cong (\theta_{sw}^H)^j \oplus (k[H_{sw}])^{\oplus \frac{|P_{sw}|-1}{|C_{sw}|}[C_{sw}:H_{sw}]},$$

where $\theta_{sw}^{H} = \theta_{sw}|_{H_{sw}}$. Let $n_{w,i}^{H} = l_{w,i}^{H}$ be the integers defined in Lemma 2.5 for \mathcal{E} viewed as an *H*-equivariant vector bundle. Then clearly we have

$$l_{w,i}^{H} \equiv l_{w,i} \equiv n_{w,i} \bmod |H_w|$$

possibly up to reordering $l_{w,i}^H$'s. Therefore we have

$$\begin{split} [W_{G}(\mathcal{E})]|_{H} &= \sum_{w \in Z_{L}^{ram}} \frac{1}{[G:I_{w}]} \sum_{i=1}^{rk \mathcal{E}} \sum_{j=1}^{l_{w,i}} \left[\operatorname{Ind}_{I_{w}}^{G} \left(M_{w}(-j) \right) \right] |_{H} \\ &= \sum_{w \in Z_{L}^{ram}} \frac{1}{[H:H_{w}]} \sum_{i=1}^{rk \mathcal{E}} \sum_{j=1}^{l_{w,i}} \left[\operatorname{Ind}_{H_{w}}^{H} \left(M_{w}(-j) |_{H_{w}} \right) \right] \\ &\equiv \sum_{w \in Z_{L}^{ram}} \frac{1}{[H:H_{w}]} \sum_{i=1}^{rk \mathcal{E}} \sum_{j=1}^{l_{w,i}} \left[\operatorname{Ind}_{H_{w}}^{H} \left((\theta_{w}^{H})^{-j} \right) \right] = W_{H}(\mathcal{E}) \bmod [kH], \end{split}$$

where the last congruence uses (3.12), (3.13) and $k[H_w] \cong \bigoplus_{j \in \mathbb{Z}/(|H_w|)} (\theta_w^H)^j$. The computation of $[N(\pi)]|_H$ is quite similar except that we use

$$\sum_{\substack{0 \leq j < |I_w/P_w| \\ j \equiv j_0 \mod |H_w|}} j|P_w| \cdot \left[\operatorname{Ind}_{I_w}^G(M_w(j)) \right]|_H \equiv j_0|I_w| \cdot \left[\operatorname{Ind}_{H_w}^H\left((\theta_w^H)^{j_0} \right) \right] \mod [kH]$$

for any $0 \leq j_0 < |H_w|$.

Corollary 3.14. Theorem 3.2(2) holds if $k = \bar{k}$.

Proof. We first show that $\chi_{kG}(\mathcal{E}) \equiv -\operatorname{rk}(\mathcal{E})[N(\pi)] + [W_G(\mathcal{E})] + c[kG]$ for some $c \in \mathbb{Z}$. By Lemma 3.9, we can proceed by comparing the values of Brauer characters at each $g \in G^{p-\text{reg}} \setminus \{1\}$. Now applying Lemma 3.11 to $H := \langle g \rangle$ for any $g \in G^{p-\text{reg}} \setminus \{1\}$, it suffices to prove the claim for $\pi_H \colon X_L \to X_{L^H}$ as we have $\chi_{kG}(\mathcal{E})|_H = \chi_{kH}(\mathcal{E})$. However, the claim for π_H is already obtained in [Nak86, Theorem 2].

Note that $(\chi_{kG}(\mathcal{E}))^G = \chi_k(\mathcal{E}^G)$ in $K_0(k)$. We next claim that $[N(\pi)]^G = [W_G(\mathcal{E})]^G =$ 0; indeed, we have by the Frobenius reciprocity that

$$\left(\mathrm{Ind}_{I_{w}}^{G}(M_{w}(j))\right)^{G} \cong \left(\mathrm{Ind}_{C_{w}}^{G}(\theta_{w}^{j}|_{C_{w}})\right)^{G} \cong \mathrm{Hom}_{C_{w}}(\mathbf{1}, \theta_{w}^{j}|_{C_{w}}),$$

 $^{^{6}}$ When all the modules involved are projective (as in our setting), one may alternatively obtain the mod pMackey formula by lifting to characteristic 0.

which is zero if and only if $j \neq 0 \mod |I_w/P_w|$. This shows that $c[kG] = [\text{Ind}_1^G(\chi_k(\mathcal{E}^G))]$, which implies formula (3.2c).

We are now ready to conclude the proof.

Proof of Claim (2) of Theorem 3.2: The general case. We now allow *k* to be any perfect field. By injectivity of the scalar extension map $K_0(kG) \rightarrow K_0(\bar{k}G)$, we may verify formula (3.2c) in $K_0(\bar{k}G)$.

Let $\bar{\pi} : \overline{X}_L \to \overline{X}$ denote the base change of π to \bar{k} , and let $\overline{\mathcal{E}}$ denote the pull back of \mathcal{E} to \overline{X}_L . We choose a connected component \overline{X}_L° of \overline{X}_L , with $\bar{\pi}^\circ := \bar{\pi}|_{\overline{X}_L^\circ}$. Let G° denote the stabiliser of \overline{X}_L° . By Corollary 3.14, we have the following formula in $K_0(\bar{k}[G^\circ])$

$$\chi_{\bar{k}[G^{\circ}]}(\overline{\mathcal{E}}|_{\overline{X}_{L}^{\circ}}) = -(\operatorname{rk} \mathcal{E}) \cdot [N(\bar{\pi}^{\circ})] + [W_{G^{\circ}}(\overline{\mathcal{E}}|_{\overline{X}_{L}^{\circ}})] + \operatorname{Ind}_{1}^{G^{\circ}}(\chi_{\bar{k}}(\overline{\mathcal{E}}^{G})).$$

(Note that $\overline{\mathcal{E}}^G = (\overline{\mathcal{E}}|_{\overline{X}^o_L})^{G^\circ}$.) We next claim that $\operatorname{Ind}_{G^\circ}^G$ of the above formula coincides with the scalar extension to \overline{k} of formula (3.2c) for \mathcal{E} .

Firstly, the following equality holds in $K_0(\bar{k}G)$

$$\chi_{kG}\big(\mathrm{R}\Gamma(X_L,\mathcal{E})\big)\otimes_k \bar{k} = \chi_{\bar{k}G}\big(\mathrm{R}\Gamma(\overline{X}_L,\overline{\mathcal{E}})\big) = \mathrm{Ind}_{G^\circ}^G\left(\chi_{\bar{k}G^\circ}\big(\mathrm{R}\Gamma(\overline{X}_L^\circ,\overline{\mathcal{E}}|_{\overline{X}_L^\circ})\big)\right).$$

Similarly, we have $\operatorname{Ind}_{G^{\circ}}^{G} \operatorname{Ind}_{1}^{G^{\circ}} \left(\chi_{\bar{k}}(\overline{\mathcal{E}}^{G}) \right) = \operatorname{Ind}_{1}^{G} \left(\chi_{k}(\mathcal{E}^{G}) \otimes_{k} \bar{k} \right)$ in $\operatorname{K}_{0}(\bar{k})$. Lastly, it remains to show

$$[N(\pi)] \otimes_k \bar{k} = \operatorname{Ind}_{G^\circ}^G [N(\bar{\pi}^\circ)] \quad \text{and} \quad [W_G(\mathcal{E})] \otimes_k \bar{k} = \operatorname{Ind}_{G^\circ}^G [W_{G^\circ}(\overline{\mathcal{E}}|_{\overline{X}^\circ_L})],$$

which can be deduced from $(\mathfrak{m}_{w}^{j}/\mathfrak{m}_{w}^{j+1}) \otimes_{k} \bar{k} \cong \bigoplus_{\bar{w}} \mathfrak{m}_{\bar{w}}^{j}/\mathfrak{m}_{\bar{w}}^{j+1}$ where \bar{w} runs through the closed points in \overline{X}_{L} over w. (More details can be found in the proof of Theorem 11 and Theorem 12 in [FWK09, §3].) Hence, formula (3.2c) is valid after the scalar extension to \bar{k} , as desired.

Remark 3.15. Assume that $k = \bar{k}$, and let $\pi: X_L \to X$ be a connected *G*-cover over k, not necessarily weakly ramified everywhere. Then for any *G*-equivariant vector bundle \mathcal{E} (for which $R\Gamma(X_L, \mathcal{E})$ may not be a perfect *kG*-complex), Köck [Köc04, Theorem 3.1] showed⁷

$$\sum_{i} (-1)^{i} \operatorname{Bch}_{\operatorname{H}^{i}(X_{L},\mathcal{E})} = c' \cdot \operatorname{Bch}_{kG} - \frac{1}{|G|} \sum_{w \in |X|} |P_{w}| \sum_{j=1}^{|I_{w}/P_{w}|-1} j \cdot \operatorname{Ind}_{I_{w}}^{G} \left(\operatorname{Bch}_{\mathfrak{m}_{w}^{j}\widehat{\mathcal{E}}_{w}/\mathfrak{m}_{w}^{j+1}\widehat{\mathcal{E}}_{w}} \right),$$

where

$$c' = 1 + \text{gen}_{K} + \frac{1}{|G|} \operatorname{deg}(\mathcal{E}) - \frac{\operatorname{rk} \mathcal{E}}{2|G|} \sum_{w \in |X_{L}|} \left(([I_{w} : P_{w}] - 1)(|P_{w}| + 1) + \sum_{s \ge 2} (|I_{w,s}| - 1) \right).$$

By Proposition 3.8, the verification of formula (3.2c) over \bar{k} reduces to comparing with the Brauer character of the right hand side of (3.2c) with the above formula. (This is how the rank-1 case of Theorem 3.2(2) is proved when $k = \bar{k}$ in [Köc04, Theorem 4.3].) In our proof of Theorem 3.2(2), we study the value of $\text{Bch}_{\chi_{kG}(\mathcal{E})}(g)$ at any $g \neq 1$ and concluded the proof via a simple and conceptual argument using $(\chi_{kG}(\mathcal{E}))^G = \chi_k(\mathcal{E}^G)$ and Lemma 3.9. This proof avoids evaluating the Brauer character at g = 1. Alternatively, one can explicitly compare the Brauer character value at q = 1 for both sides of (3.2c),

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⁷This formula is generalised to the case where k is any perfect field in [FWK09, §4, Theorem 16], but there is an obvious typo in the coefficient of Bch_k G that makes it inconsistent with [Köc04, Theorem 3.1] when $k = \bar{k}$.

using the theorems of Hirzeburch-Riemann-Roch and Riemann-Hurwitz, as well as the following computation

$$\deg(\mathcal{E}) - |G| \deg(\mathcal{E}^G) = \sum_{w \in Z_L^{ram}} \sum_{i=1}^{rk \mathcal{E}} (|P_w|(l_{w,i}+1) - 1))$$

which equals $\sum_{w \in Z_L^{\text{ram}}} \sum_{i=1}^{\text{rk} \mathcal{E}} n_{w,i}$ if we have chosen $0 \leq n_{w,i} < |I_w|$ for all $w \in Z_L^{\text{ram}}$ and *i*. This way, one can also show the compatibility of (3.2c) with [Köc04, Theorem 3.1] over \bar{k} . (We note that the coefficient *c'* of Bch_{kG} in the above formula may not coincide with the coefficient *c* of [kG] in K₀(*kG*) due to the subtlety explained in Remark 3.10.)

4. Relative K-theory of group rings

In this section, we review the Euler characteristics in the relative K_0 -group (*cf.* Def 4.8), and obtain some explicit computation for equivariant vector bundles (*cf.* Corollary 4.15).

Let us consider the following general setup. Let *E* be either a number field or a (possibly archimedean) local field of characteristic 0, and fix a Dedekind domain *R* contained in *E*. (We usually set $E = \operatorname{Frac}(R)$, but we also allow $E = \mathbb{R}$ and $R = \mathbb{Z}$.) We fix a finitedimensional semi-simple *E*-algebra *A* together with an *R*-order $\mathfrak{A} \subset A$. (In the intended setting, we consider $\mathfrak{A} = RG$ and A = EG for a finite group *G*.) Thanks to the assumption on *E*, the reduced norm map $K_1(A) \to \zeta(A)^{\times}$ is injective, where $\zeta(A)$ is the centre of *A*. We view $K_1(A)$ as a subgroup of $\zeta(A)^{\times}$; *cf*. [CR87, Theorem (45.3)].

We recall the explicit description of the abelian group $K_0(\mathfrak{A}, A)$ in terms of generators and relations, following [Swa68, p 215]:

generators: the isomorphism classes of triples $[P, \theta, Q]$, where *P* and *Q* are finitely generated projective \mathfrak{A} -modules, $\theta: E \otimes_R P \xrightarrow{\sim} E \otimes_R Q$ is an isomorphism of *A*-modules, with respect to the obvious notion of isomorphisms (*cf.* [Swa68, p 214]); **relations:** generated by the following

(4.1)
$$[P, \theta, Q] + [P', \theta', Q'] = [P \oplus P', \theta \oplus \theta', Q \oplus Q'] \text{ and} [P, \theta, P'] + [P', \theta', P''] = [P, \theta' \circ \theta, P''].$$

The relative K_0 -group fits into the natural *localisation sequence* (*cf.* [Swa68, Theorem 15.5]; in particular, we have the following connecting homomorphism

(4.2a)
$$\partial_{\mathfrak{A},E} \colon \mathrm{K}_{1}(A) \longrightarrow \mathrm{K}_{0}(\mathfrak{A},A)$$

which, in our setting, turns out to be the restriction of the following map:

(4.2b)
$$\delta_{\mathfrak{A},E} \colon \zeta(A)^{\times} \xrightarrow{a \mapsto [\mathfrak{A},a,\mathfrak{A}]} \mathsf{K}_{0}(\mathfrak{A},A) \,.$$

Example 4.3. Suppose that \mathfrak{A} is a PID and $A = \operatorname{Frac} \mathfrak{A}$ is a finite extension of E. Then the natural map $A^{\times} \to K_0(\mathfrak{A}, A)$, sending $a \in A^{\times}$ to $[\mathfrak{A}, a, \mathfrak{A}]$, induces an isomorphism

$$A^{\times}/\mathfrak{A}^{\times} \xrightarrow{\cong} K_0(\mathfrak{A}, A)$$

(This essentially follows from the structure theorem of finitely generated modules over PID. In fact, given $[P, \theta, Q]$ we choose an \mathfrak{A} -basis of P and Q so that θ can be represented by a diagonal matrix, and the relations (4.1) yield $[P, \theta, Q] = [\mathfrak{A}, \mathfrak{A}, \mathfrak{A}]$ where a is the determinant of the matrix representation of θ , showing surjectivity. Injectivity can be seen by keeping track of the effect of relations on "determinants"; *cf.* [Swa68, Lemma 15.8].)

Now, let *E*' be another global or local field containing *E*, and choose a Dedekind subdomain $R' \subset E'$ containing *R*. We choose an *E*-algebra homomorphism

$$\psi \colon A \to A' \coloneqq \operatorname{End}_{E'}(V_{\psi})$$

for some finite-dimensional E'-vector space V_{ψ} . Fix an R'-lattice T_{ψ} of V_{ψ} such that ψ restricts to a map $\mathfrak{A} \to \mathfrak{A}' := \operatorname{End}_{R'}(T_{\psi})$, which we also denote by ψ .

For a (left) \mathfrak{A} -module *M*, we define the following *R'*-module

$$(4.4) [M]_{\psi} \coloneqq T_{\psi}^* \otimes_{\mathfrak{A}} M$$

where we view the R'-linear dual T_{ψ}^* as a (R', \mathfrak{A}) -bimodule via ψ . Clearly, if P is a projective \mathfrak{A} -module then $[P]_{\psi}$ is also a projective R'-module, so we get a group homomorphism

(4.5)
$$\rho^{\psi} \colon \operatorname{K}_{0}(\mathfrak{A}, A) \longrightarrow \operatorname{K}_{0}(R', E');$$

$$[P, \theta, Q] \longmapsto [P, \theta, Q]_{\psi} = \left[[P]_{\psi}, [\theta]_{\psi}, [Q]_{\psi} \right].$$

Alternatively, this map can be obtained the following composition

$$\mathrm{K}_{0}(\mathfrak{A}, A) \xrightarrow{\psi_{*}} \mathrm{K}_{0}(\mathfrak{A}', A') \xrightarrow{\cong} \mathrm{K}_{0}(R', E')$$

where ψ_* is induced by the scalar extension $\mathfrak{A}' \otimes_{\mathfrak{A}} (-)$, and the second arrow by the Morita equivalence; *cf.* [BF01, §3.5].

Let us record the following basic properties of ρ^{ψ} .

Lemma 4.6. Choose an *E*-algebra map $\psi : A \to A' = \operatorname{End}_{E'}(V_{\psi})$ and an *R'*-lattice $T_{\psi} \subset V_{\psi}$ such that $\psi(\mathfrak{A})$ is contained in $\mathfrak{A}' := \operatorname{End}_{R'}(T_{\psi})$, as above.

- (1) The map $\rho^{\psi} \colon K_0(\mathfrak{A}, A) \to K_0(R', E')$ depends only on $\psi \colon A \to A' = \operatorname{End}_{E'}(V_{\psi})$, not on the choice of R'-lattice $T_{\psi} \subset V_{\psi}$.
- (2) We have the following commutative diagram

$$\begin{array}{cccc}
\mathrm{K}_{1}(A) & \xrightarrow{\operatorname{Nrd}^{\psi}} E'^{\times} & \stackrel{\cong}{\longleftarrow} \mathrm{K}_{1}(E') \\
 & & & & \downarrow \\
\partial_{\mathfrak{A},E} & & & \downarrow \\
\partial_{\mathfrak{R}',E'} & & & \downarrow \\
\mathrm{K}_{0}(\mathfrak{A},A) & \xrightarrow{\rho^{\psi}} \mathrm{K}_{0}(R',E')
\end{array}$$

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where $\operatorname{Nrd}^{\psi}$ is the map $\operatorname{K}_1(A) \hookrightarrow \zeta(A)^{\times} \to \zeta(A')^{\times} = E'^{\times}$ induced by ψ .

Proof. Let T'_{ψ} be another R'-lattice of V_{ψ} such that $\operatorname{End}_{R'}(T'_{\psi})$ contains $\psi(\mathfrak{A})$, and choose $\alpha \in A'^{\times}$ such that $T'_{\psi} = \alpha(T_{\psi})$. Then for any \mathfrak{A} -module M we have an isomorphism

$$T'^*_{\psi} \otimes_{\mathfrak{A}} M \xrightarrow{\alpha^* \otimes \mathrm{id}_M} T^*_{\psi} \otimes_{\mathfrak{A}} M ,$$

inducing an isomorphism of triples $T'^*_{\psi} \otimes_{\mathfrak{A}} (P, \theta, Q) \xrightarrow{\sim} T^*_{\psi} \otimes_{\mathfrak{A}} (P, \theta, Q)$ for any triple (P, θ, Q) representing an element of $K_0(\mathfrak{A}, A)$.

Claim (2) also follows from the straightforward diagram chasing.

For $C^{\bullet} \in D^{\text{perf}}(\mathfrak{A})$, a *trivialisation* over *E* (or an *E-trivialisation*) means an *A*-linear isomorphism

$$(4.7) h: \bigoplus_{i \text{ even}} \mathrm{H}^{i}(E \otimes_{R} C^{\bullet}) \xrightarrow{\sim} \bigoplus_{i \text{ odd}} \mathrm{H}^{i}(E \otimes_{R} C^{\bullet}).$$

By semi-simplicity of *A*, one can naturally extend *h* to an isomorphism $\bigoplus_{i \text{ even}} E \otimes_R C^i \xrightarrow{\sim} \bigoplus_{i \text{ odd}} E \otimes_R C^i$, also denoted by *h*.

Definition 4.8. Given a perfect \mathfrak{A} -complex with *E*-trivialisation (C^{\bullet}, h) , we define its *Euler characteristic* $\chi_{\mathfrak{A},E}(C^{\bullet}, h) \in \mathrm{K}_{0}(\mathfrak{A}, A)$ as follows:

$$\chi_{\mathfrak{A},E}(C^{\bullet},h) := \left[\left(\bigoplus_{i \text{ even}} C^{i} \right), h, \left(\bigoplus_{i \text{ odd}} C^{i} \right) \right].$$

Immediately, $\chi_{\mathfrak{A},E}(C^{\bullet},h)$ only depends on the isomorphism class of C^{\bullet} in $\mathbb{D}^{\text{perf}}(\mathfrak{A})$.

If C^{\bullet} is a perfect \mathfrak{A} -complex such that $E \otimes_R C^{\bullet}$ is acyclic, then we let $\chi_{\mathfrak{A},E}(C^{\bullet}, 0)$ denote the Euler characteristic with respect to the *unique E*-trivialisation on C^{\bullet} (i.e., the zero map between the zero modules).

The formation of $\chi_{\mathfrak{A},E}(-)$ is functorial with respect to ρ^{ψ} (4.5); indeed, given a perfect \mathfrak{A} -complex with *E*-trivialisation (C^{\bullet} , h), we have

(4.9)
$$\rho^{\psi}(\chi_{\mathfrak{A},E}(C^{\bullet},h)) = \chi_{R',E'}([C^{\bullet}]_{\psi},[h]_{\psi}),$$

where $[h]_{\psi}$ is the *E'*-trivialisation induced on $[C^{\bullet}]_{\psi} \coloneqq T^*_{\psi} \otimes_{\mathfrak{A}} C^{\bullet}$ via *h*.

Example 4.10. If \mathfrak{A} is a DVR with normalised valuation v_A , then let v_A also denote the following isomorphism

$$v_A \colon \mathrm{K}_0(\mathfrak{A}, A) \xrightarrow{\cong} \mathbb{Z}$$
 sending $[\mathfrak{A}, a, \mathfrak{A}] \mapsto v_A(a)$.

In that case, for any $C^{\bullet} \in D^{\text{perf}}(\mathfrak{A})$ with all $H^{i}(C^{\bullet})$ torsion we have

$$v_A\left(\chi_{\mathfrak{A},E}(C^{\bullet},0)\right) = \sum_i (-1)^{i+1} \operatorname{length}_{\mathfrak{A}}\left(\operatorname{H}^i(C^{\bullet})\right)$$

Now, let \mathfrak{p} be a non-zero *principal* prime ideal of R, and write $k := R/\mathfrak{p}$. Given $C_k^{\bullet} \in D^{\text{perf}}(\mathfrak{A}/\mathfrak{p}\mathfrak{A})$, we abusively write

$$\chi_{\mathfrak{A},E}(C_k^{\bullet},0) \coloneqq \chi_{\mathfrak{A},E}(C_{\mathfrak{A}}^{\bullet},0)$$

where $C_{\mathfrak{A}}^{\bullet}$ is a perfect \mathfrak{A} -complex quasi-isomorphic to C_k^{\bullet} . (Note that $E \otimes_R C_{\mathfrak{A}}^{\bullet}$ is clearly acyclic.) In this case, we have another notion of Euler characteristic; namely, $\chi_{\mathfrak{A}/\mathfrak{P}\mathfrak{A}}(C_k^{\bullet}) \in K_0(\mathfrak{A}/\mathfrak{P}\mathfrak{A})$, which can be related to $\chi_{\mathfrak{A},E}(C_k^{\bullet}, 0)$ as follows. Define a homomorphism

$$J_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}} : \mathrm{K}_{0}(\mathfrak{A}/\mathfrak{p}\mathfrak{A}) \longrightarrow \mathrm{K}_{0}(\mathfrak{A}, A)$$

by sending a finitely generated projective $\mathfrak{A}/\mathfrak{pA}$ -module P_k , viewed also as a complex concentrated in degree 0, to

$$\mathfrak{J}_{\mathfrak{A}/\mathfrak{p}}(P_k) \coloneqq \chi_{\mathfrak{A},E}(P_k,0) = [P,\varpi^{-1},P],$$

where \widetilde{P} is a projective \mathfrak{A} -module lifting P_k and ϖ is a generator of \mathfrak{p} ; indeed, P_k is quasiisomorphic to a two-term complex $[\widetilde{P} \xrightarrow{\varpi} \widetilde{P}]$ concentrated in degree [-1, 0], so its Euler characteristic is as above. One can show (by straightforward computation) that this extends to any $C_k^{\bullet} \in D^{perf}(\mathfrak{A}/\mathfrak{p}\mathfrak{A})$; that is,

$$\chi_{\mathfrak{A},E}(C_k^{\bullet},0) = \jmath_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}(\chi_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}(C_k^{\bullet})).$$

Furthermore, given $\psi \colon \mathfrak{A} \to \operatorname{End}_{R'}(T_{\psi})$ as before such that R' is a DVR whose maximal ideal \mathfrak{p}' contains \mathfrak{p} , we also have

$$(4.11) \qquad \rho^{\psi} \left(\chi_{\mathfrak{A},E}(C_{k}^{\bullet},0) \right) = \chi_{R',E'}(T_{\psi}^{*} \otimes_{\mathfrak{A}}^{\mathbb{L}} C_{k}^{\bullet},0) = J_{R'/\mathfrak{p}'} \left(\chi_{R'/\mathfrak{p}'}(T_{\psi}^{*}/\mathfrak{p}T_{\psi}^{*} \otimes_{\mathfrak{A}/\mathfrak{p}} C_{k}^{\bullet}) \right).$$

Lemma 4.12. In the above setting, choose an algebraic closure \bar{k} of k and a k-embedding $R'/\mathfrak{p}' \hookrightarrow \bar{k}$. Set $\overline{T}_{\psi}^* \coloneqq T_{\psi}^* \otimes_{R'} \bar{k}$ and define

$$\mathbf{m}_{\psi} \colon \mathbf{K}_{0}(\mathfrak{A}/\mathfrak{p}\mathfrak{A}) \xrightarrow{\overline{T}_{\psi}^{*} \otimes_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}(-)} \mathbf{K}_{0}(\bar{k}) \xrightarrow{\dim_{\bar{k}}} \mathbb{Z}.$$

Then for any $C_k^{\bullet} \in D^{\text{perf}}(\mathfrak{A}/\mathfrak{p}\mathfrak{A})$ we have

$$v_{E'}\left(\rho^{\psi}(\chi_{\mathfrak{A},E}(C_{k}^{\bullet},0))\right) = -e_{\mathfrak{p}'|\mathfrak{p}}\cdot \mathbf{m}_{\psi}\left(\chi_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}(C_{k}^{\bullet})\right),$$

where $v_{E'}(-)$ is the normalised valuation on $E' := \operatorname{Frac} R'$ and $e_{\mathfrak{p}'|\mathfrak{p}}$ is the ramification index.

Proof. By Example 4.10 and (4.11), we have

$$\begin{split} v_{E'}\left(\rho^{\psi}\big(\chi_{\mathfrak{A},E}(C_{k}^{\bullet},0)\big)\right) &= \sum_{i}(-1)^{i+1}\operatorname{length}_{R'}\left(\operatorname{H}^{i}\left(T_{\psi}^{*}/\mathfrak{p}T_{\psi}^{*}\otimes_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}C_{k}^{\bullet}\right)\right) \\ &= \sum_{i}(-1)^{i+1}\operatorname{length}_{R'}\left(T_{\psi}^{*}/\mathfrak{p}T_{\psi}^{*}\otimes_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}C^{i}\right) \\ &= e_{\mathfrak{p}'|\mathfrak{p}}\cdot\sum_{i}(-1)^{i+1}\operatorname{dim}_{\bar{k}}\left(\overline{T}_{\psi}^{*}\otimes_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}C^{i}\right) = -e_{\mathfrak{p}'|\mathfrak{p}}\cdot\operatorname{m}_{\psi}\left(\chi_{\mathfrak{A}/\mathfrak{p}\mathfrak{A}}(C_{k}^{\bullet})\right), \end{split}$$

as desired.

Remark 4.13. In Lemma 4.12, the normalised valuation of $\rho^{\psi}(\chi_{\mathfrak{A},E}(C_k^{\bullet},0))$ changes if we replace T_{ψ} by the scalar extension under a finite ramified ring extension, but the formation of the fractional ideal \mathfrak{a} whose normalised valuation equals that of $\rho^{\psi}(\chi_{\mathfrak{A},E}(C_k^{\bullet},0))$ commutes with any finite scalar ring extensions.

Now consider a *G*-cover $\pi: X_L \to X$ and a *G*-equivariant vector bundle \mathcal{E} in X_L that satisfies the conditions in Theorem 3.2. We also assume that the constant field *k* of *X* is *finite* with characteristic *p*. For simplicity we write

(4.14)
$$\chi_p^G(\mathcal{E}) \coloneqq \chi_{\mathbb{Z}_p G, \mathbb{Q}_p}(\mathrm{R}\Gamma(X_L, \mathcal{E})^{\vee}, 0) \in \mathrm{K}_0(\mathbb{Z}_p G, \mathbb{Q}_p G),$$

which makes sense as $R\Gamma(X_L, \mathcal{E})^{\vee} \in D^{perf}(\mathbb{F}_p G)$. Here, $R\Gamma(X_L, \mathcal{E})^{\vee}$ denotes the Pontryagin dual (or equivalently, the \mathbb{F}_p -linear dual) with the contragredient *G*-action. To motivate this choice, see Theorem 6.14. For $\ell \neq p$, we set $\chi_{\ell}^G(\mathcal{E}) = 0$ in $K_0(\mathbb{Z}_{\ell}G, \mathbb{Q}_{\ell}G)$.

Let E' be a finite extension of \mathbb{Q}_p with valuation ring $R' = O_{E'}$ and residue field k'. Pick a *G*-stable $O_{E'}$ -lattice T_{ψ} in a finitely generated E'G-module V_{ψ} , so we get a map $\psi : \mathbb{Z}_p G \to \operatorname{End}_{R'}(T_{\psi})$. We now apply Lemma 4.12 to $\mathfrak{A} = \mathbb{Z}_p G$ and $\mathfrak{p} = (p)$ to obtain the following corollary.

Corollary 4.15. Let $\pi: X_L \to X$ be a *G*-cover that is weakly ramified everywhere. For each $w \in Z_L^{\text{ram}}$, choose $C_w \subseteq I_w$ so that we have $I_w = P_w \rtimes C_w$ (cf. Lemma 2.7). We write $\theta_{C_w} := \theta_w|_{C_w}$.

Let k be the finite constant field of X, and fix a k-embedding $k_w \hookrightarrow \bar{k}$ for each $w \in Z_L^{\text{ram}}$. Finally, fix an embedding $k' \hookrightarrow \bar{k}$ and set $\overline{T}_{\psi} := \bar{k} \otimes_{O_{F'}} T_{\psi}$.

(1) For any $w \in Z_L^{\text{ram}}$ and $j \in \mathbb{Z}/(|C_w|)$, we have

$$\mathbf{m}_{\psi,\mathbf{w}}(j) \coloneqq \mathbf{m}_{\psi} \left(\operatorname{Ind}_{I_{w}}^{G} M_{w}(j) \right) = \sum_{a=0}^{\left[k_{w}:\mathbb{F}_{p}\right]-1} \dim_{\bar{k}} \left(\overline{T}_{\psi} \left[\theta_{C_{w}}^{jp^{a}} \right] \right),$$

Here, $\overline{T}_{\psi}[\theta_{C_w}^s]$ for $s \in \mathbb{Z}$ is the maximal subspace of \overline{T}_{ψ} where C_w acts via $\theta_{C_w}^s$.

(2) Let \mathcal{E} be a G-equivariant vector bundle X_L that satisfies the condition (3.2a) for any $w \in Z_L^{\text{wild}}$. Then we have

$$-v_{E'}\left(\rho^{\psi}(\chi_p^G(\mathcal{E}))\right) = v_{E'}(p) \cdot \left[[k:\mathbb{F}_p] \cdot \dim_{\bar{k}}\left(\overline{T}_{\psi}^* \otimes_k \chi_k(\mathcal{E}^G)\right) + \operatorname{ram}_{\mathcal{E}}(\psi) \right]$$

where

$$\operatorname{ram}_{\mathcal{E}}(\psi) \coloneqq \frac{1}{|G|} \sum_{w \in Z_L^{\operatorname{ram}}} \sum_{i=1}^{\operatorname{rk} \mathcal{E}} \left(-|P_w| \cdot \sum_{j=1}^{|C_w|-1} j \cdot \mathbf{m}_{\psi,w}(-j) + |I_w| \cdot \sum_{j=1}^{l_{w,i}} \mathbf{m}_{\psi,w}(j) \right).$$

Here, $m_{\psi,w}(j)$ *is defined in (1), and l_{w,i}'s are as in (3.2b).*

Proof. By transitivity of inductions, we have $\operatorname{Ind}_{I_w}^G M_w(j) \cong \operatorname{Ind}_{C_w}^G \left((\mathfrak{m}_w^j/\mathfrak{m}^{j+1})|_{C_w} \right)$; *cf.* Lemma 2.10. Since $\overline{T}_{\psi}^* \otimes_{\mathbb{F}_p} \operatorname{Ind}_{C_w}^G (\theta_{C_w}^j)$ is a projective $\overline{k}G$ -module,⁸ we have

$$(4.16) \quad \overline{T}_{\psi}^{*} \otimes_{\mathbb{F}_{p}G} \operatorname{Ind}_{C_{w}}^{G} \left((\mathfrak{m}_{w}^{j}/\mathfrak{m}^{j+1})|_{C_{w}} \right) \cong \left(\overline{T}_{\psi}^{*} \otimes_{\mathbb{F}_{p}} \operatorname{Ind}_{C_{w}}^{G} \left((\mathfrak{m}_{w}^{j}/\mathfrak{m}^{j+1})|_{C_{w}} \right) \right)^{G} \\ \cong \operatorname{Hom}_{\bar{k}G} \left(\overline{T}_{\psi}, \, \bar{k} \otimes_{\mathbb{F}_{p}} \operatorname{Ind}_{C_{w}}^{G} \left((\mathfrak{m}_{w}^{j}/\mathfrak{m}^{j+1})|_{C_{w}} \right) \right) \cong \operatorname{Hom}_{\bar{k}[C_{w}]} \left(\overline{T}_{\psi}, \, \bar{k} \otimes_{\mathbb{F}_{p}} (\mathfrak{m}_{w}^{j}/\mathfrak{m}^{j+1}) \right),$$

where the last isomorphism is the Frobenius reciprocity. To conclude, observe that

$$\bar{k} \otimes_{\mathbb{F}_p} (\mathfrak{m}^j_w/\mathfrak{m}^{j+1}_w) \cong \bigoplus_{i=0}^{\lfloor k_w, \mathbb{F}_p \rfloor - 1} \bar{k} \otimes_{\mathrm{Fr}^i_p, k_w} (\mathfrak{m}^j_w/\mathfrak{m}^{j+1}_w),$$

where Fr_p is the *p*th power map and C_w acts on $\overline{k} \otimes_{\operatorname{Fr}_p^i, k_w} (\mathfrak{m}_w^j/\mathfrak{m}_w^{j+1})$ via $\theta_w^{jp^i}$. In particular, the last term in (4.16) is the direct sum of the multiplicity spaces for $\theta_w^{jp^i}$, and thus the displayed equation in claim (1) follows.

For any *k*-vector space *V*, we have

$$\overline{T}_{\psi}^* \otimes_{\mathbb{F}_{\rho}G} \operatorname{Ind}_1^G V \cong \overline{T}_{\psi}^* \otimes_{\mathbb{F}_{\rho}} V \cong (\overline{T}_{\psi}^* \otimes_k V)^{\oplus [k:\mathbb{F}_{\rho}]}$$

Note also that $M_w(j)^{\vee} \cong M_w(-j)$, where $M_w(j)^{\vee}$ is the Pontryagin dual with contragredient G_w -action. Claim (2) now follows from Theorem 3.2(2) and Lemma 4.12.

We conclude the section with some remarks on Corollary 4.15.

Remark 4.17. In Corollary 4.15(1), we have $\overline{T}_{\psi}[\theta_{C_w}^s] \cong \overline{T}_{\psi}^{ss}[\theta_{C_w}^s]$ as $k_w[C_w]$ is semi-simple, which shows that $m_{\psi,w}(j)$ depends only on $j \in \mathbb{Z}/(|C_w|)$ and V_{ψ} , not on the choice of $T_{\psi} \subset V_{\psi}$. In particular, the formula in Corollary 4.15(2) is independent of the choice of $T_{\psi} \subset V_{\psi}$.

Remark 4.18. The right hand side of the formula in Corollary 4.15(2) can be divided into two parts – the first terms involves deg $\psi := \dim_{E'} V_{\psi}$ and the Euler characteristic of \mathcal{E}^G , and the second term $\operatorname{ram}_{\mathcal{E}}(\psi)$ measures the "local ramification" of \overline{T}_{ψ}^{ss} and \mathcal{E} (that is, $\overline{T}_{\psi}^{ss}|_{I_w}$ and $l_{w,i}$'s for any $w \in Z_L^{\operatorname{ram}}$).

Let us now consider the special case of $\mathcal{E} = \pi^* \mathcal{F}(-Z_L)$ where \mathcal{F} is a vector bundle on X and $Z_L = \pi^{-1}(Z)$ for some closed subset $Z \subset |X|$ containing the ramification locus for π . Then as in (3.6b), the formula in Corollary 4.15(2) can be made more explicit using the following formula:

$$\begin{split} \dim_{\bar{k}} \left(\overline{T}_{\psi}^* \otimes_k \chi_k(\mathcal{E}^G) \right) &= (\deg \psi) \cdot \left((\operatorname{rk} \mathcal{F}) \cdot \left(1 - \operatorname{gen}_K - \deg(Z) \right) + \deg(\mathcal{F}) \right); \\ \operatorname{ram}_{\mathcal{E}}(\psi) &= \frac{\operatorname{rk} \mathcal{E}}{|G|} \sum_{w \in \mathbb{Z}_L^{\operatorname{ram}}} \sum_{j=1}^{|I_w/P_w| - 1} \sum_{a=0}^{[k_w:\mathbb{F}_p] - 1} j |P_w| \cdot \dim_{\bar{k}} \left(\overline{T}_{\psi} [\theta_{C_w}^{jp^a}] \right), \end{split}$$

where $\deg(Z) = \sum_{v \in Z} [k_v : k].$

Remark 4.19. Let us make $\operatorname{ram}_{\mathcal{E}}(\psi)$ more explicit in some special cases. Firstly, if we have $I_w = P_w$ for all $w \in Z_L^{\operatorname{ram}}$ (e.g., if *G* is a *p*-group or π is étale), then we have $\operatorname{ram}_{\mathcal{E}}(\psi) = 0$ for any ψ and \mathcal{E} (*cf.* Remark 3.4).

Now, suppose that π is tame everywhere and deg $\psi = 1$. We also specialise to the case when $\mathcal{E} = \pi^* \mathcal{F}(-Z_L)$ as in Remark 4.18. For any $w \in Z_L^{\text{ram}}$, let d_w denote the smallest positive integer such that $p^{d_w} \equiv 1 \mod |I_w|$. (Note that d_w divides $[k_w : \mathbb{F}_p]$.) For each

⁸One can show projectivity by realising $\overline{T}^*_{\psi} \otimes_{\mathbb{F}_p} \operatorname{Ind}^G_{C_w}(\theta^j_{C_w})$ as a direct summand of $\operatorname{Ind}^G_1(k_w \otimes_{\mathbb{F}_p} \overline{T}^*_{\psi})$.

 $a = 0, \dots, d_w - 1$, let $j_{\psi,w}^{(a)}$ denote the integer in $\{0, \dots, |I_w| - 1\}$ such that I_w acts on $\overline{T}_{\psi}|_{I_w}$ via $\theta_{\psi,w}^{j_{\psi,w}^{(a)};p^a}$. Then we have

(4.19a)
$$\operatorname{ram}_{\mathcal{E}}(\psi) = \frac{\operatorname{rk} \mathcal{E}}{|G|} \sum_{w \in Z_L^{\operatorname{ram}}} \sum_{a=0}^{d_w^{-1}} j_{\psi,w}^{(a)} \cdot \frac{[k_w : \mathbb{F}_p]}{d_w}$$

If furthermore $|I_w|$ divides p - 1 for any $w \in Z_L^{\text{ram}}$ (which follows if |G| divides p - 1), then setting $j_{\psi,w} = j_{\psi,w}^{(0)}$ we obtain

(4.19b)
$$\operatorname{ram}_{\mathcal{E}}(\psi) = \frac{\operatorname{rk} \mathcal{E}}{|G|} \sum_{w \in Z_{L}^{\operatorname{ram}}} j_{\psi,w} \cdot [k_{w} : \mathbb{F}_{p}].$$

5. Review of Néron models and base change

We use the setting of §3. Fix an abelian variety A over K, and let \mathcal{A} denote the Néron model of A over X. For the pull back A_L of A over L, we let \mathcal{A}_L denote the Néron model, and write $\mathcal{A}_{X_L} := \mathcal{A} \times_X X_L$. The connected Néron models are denoted as $\mathcal{A}^\circ, \mathcal{A}^\circ_L$, etc. Let A^t denote the dual abelian variety of A, and similarly define $\mathcal{A}^t, \mathcal{A}^{t,\circ}$, etc. We maintain this setting for the rest of the paper.

By the Néron mapping property, the *G*-action on X_L lifts to a *G*-action on \mathcal{A}_L , and we get a natural *G*-equivariant homomorphism $\mathcal{A}_{X_L} \to \mathcal{A}_L$ extending the identity map on the generic fibre. Furthermore, we have the following proposition.

Proposition 5.1. Let $U' \subset X$ be the maximal open subscheme such that the natural map $\mathcal{A}_{X_L}^{\circ}|_{U'_L} \to \mathcal{A}_L^{\circ}|_{U'_L}$ is an isomorphism, where $U'_L \coloneqq \pi^{-1}(U')$.

(1) The cokernel of the natural inclusion

$$\operatorname{Lie}(\mathcal{A}_{X_{L}}) \cong \pi^{*} \operatorname{Lie}(\mathcal{A}) \longrightarrow \operatorname{Lie}(\mathcal{A}_{L})$$

is supported exactly on $X_L \setminus U'_L$.

- (2) A closed point $v \in |X|$ lies in U' if either L/K is unramified at v or A has semistable reduction at v.
- (3) Suppose furthermore that L/K is tamely ramified at all places in $X' \setminus U'$. Then the natural map $\text{Lie}(\mathcal{A}) \to \text{Lie}(\mathcal{A}_L)^G$ is an isomorphism.

Proof. Fix a place $v \in |X|$, and choose a place $w \in |X_L|$ over v. We set

 $\mathcal{A}_{O_v} \coloneqq \mathcal{A} \times_X \operatorname{Spec} O_v, \quad \mathcal{A}_{L,O_w} \coloneqq \mathcal{A}_L \times_{X_L} \operatorname{Spec} O_w, \text{ and } \mathcal{A}_{O_w} \coloneqq \mathcal{A} \times_X \operatorname{Spec} O_w.$ We similarly define $\mathcal{A}_{O_v}^{\circ}$, etc. By standard properties of Néron models, we have $v \in U'$ if and only if the natural map $\mathcal{A}_{O_w}^{\circ} \to \mathcal{A}_{L,O_w}^{\circ}$ is an isomorphism, and U' contains the unramified locus for L/K; *cf.* §1.2, Proposition 4 and §7.2, Corollary 2 in [BLR90]. If $\mathcal{A}_{O_v}^{\circ}$ is a semi-abelian scheme, then $v \in U'$ by [BLR90, §7.4, Corollary 4]). This proves (2).

If L_w/K_v is tamely ramified, then by [Edi92, Theorem 4.2] we have a natural isomorphism

$$\mathcal{A}_{O_v} \xrightarrow{\cong} (\operatorname{Res}_{O_w/O_v} \mathcal{A}_{L,O_w})^{G_w}$$

of group schemes over O_v , where $\operatorname{Res}_{O_w/O_v} \mathcal{A}_{L,O_w}$ denotes the Weil restriction of scalars. Since the Lie algebra of $\operatorname{Res}_{O_w/O_v} \mathcal{A}_{L,O_w}$ coincides with $\operatorname{Lie}(\mathcal{A}_L)(O_w)$ viewed as an O_v -module, it follows that the natural map $\operatorname{Lie}(\mathcal{A}) \to \operatorname{Lie}(\mathcal{A}_L)^G$ induces an isomorphism on the completed stalks at all *tame* places $v \in |X|$. Since this map induces an isomorphism on the restriction to U' by (2), we obtain (3) by the standard descent argument.

To prove (1), we need to show that the natural map $\mathscr{R}^{\circ}_{O_{w}} \to \mathscr{R}^{\circ}_{L,O_{w}}$ is isomorphic if and only if $\operatorname{Lie}(\mathscr{R}_{X_{L}})(O_{w}) \hookrightarrow \operatorname{Lie}(\mathscr{R}_{L})(O_{w})$ is isomorphic. The "only if" direction is clear, so suppose that we have $\operatorname{Lie}(\mathscr{R}_{X_{L}})(O_{w}) \xrightarrow{\sim} \operatorname{Lie}(\mathscr{R}_{L})(O_{w})$. Then the natural map $\mathscr{R}^{\circ}_{O_{w}} \to \mathscr{R}^{\circ}_{L,O_{w}}$ is étale by smoothness of the source and the target (cf. [BLR90, §2.2, Corollary 10]), so it is an open immersion by Zariski's main theorem (*cf.* [BLR90, §2.3, Theorem 2']). Since all the fibres of $\mathcal{A}_{L,w}^{\circ}$ over Spec O_w is connected, the desired claim now follows.

Set $\mathcal{B}_{O_v} \coloneqq \operatorname{Res}_{O_w/O_v} \mathcal{A}_{L,O_w}$, and denote its special fibre by \mathcal{B}_{k_v} . Let \mathcal{A}_{L,k_w} be the special fibres of \mathcal{A}_{L,O_w} . Then have a natural G_w -equivariant surjective map

$$(5.2) \qquad \qquad \mathcal{B}_{k_v} \longrightarrow \operatorname{Res}_{k_w/k_v} \mathcal{A}_{L,k_w}$$

with smooth connected unipotent kernel denoted by $F^1\mathcal{B}_{k_v}$; indeed, this can be seen by realising \mathcal{B}_{k_v} as the Weil restriction of scalars for a nilpotent thickening $\mathcal{O}_w \otimes_{\mathcal{O}_v} k_v \twoheadrightarrow k_w$; *cf.* [Edi92, §5.1] or [CGP15, Proposition A.5.12].

We retain the setting that X is defined over a *perfect* field k of characteristic p > 0, so k_v is perfect as well. Then $F^1\mathcal{B}_{k_v}$ is a vector group over k_v by [CGP15, Corollary B.2.7]; i.e., it is a direct product of copies of \mathbb{G}_a .

Proposition 5.3. In the above setting, if L_w/K_v is tame then the short exact sequence (5.2) induces the following short exact sequence

$$(5.3a) \quad 0 \longrightarrow (F^1 \mathcal{B}_{k_v})^{G_w} \longrightarrow (\mathcal{B}_{k_v})^{G_w} \cong \mathcal{A}_{k_v} \longrightarrow (\operatorname{Res}_{k_w/k_v} \mathcal{A}_{L,k_w})^{G_w} \longrightarrow 0 ,$$

which induce the following isomorphism

(5.3b)
$$\left(\operatorname{Lie}(\mathcal{A}_L)(k_w)\right)^{G_w} \cong \operatorname{coker}\left(\left(\operatorname{Lie}(\mathcal{A}_L)(\mathfrak{m}_w/\mathfrak{m}_w^{|I_w|})\right)^{G_w} \hookrightarrow \operatorname{Lie}(\mathcal{A})(k_v)\right).$$

Moreover, $(F^1\mathcal{B}_{k_v})^{G_w}$ is a vector group over k_v and the sequence (5.3a) remains exact on k_v -points.

Proof. Suppose that we know the sequence (5.3a) is exact and that $(F^1\mathcal{B}_{k_o})^{G_w}$ is a vector group. Then the isomorphism (5.3b) is a direct consequence of the short exact sequence of Lie algebras induced from (5.3a), and the sequence (5.3a) remains exact on k_v -points since $H^1(k_v, (F^1\mathcal{B}_{k_w})^{G_w})$ is trivial by the Hilbert normal basis theorem.

It remains to show that the sequence (5.3a) is exact and that $(F^1\mathcal{B}_{k_v})^{G_w}$ is a vector group, both of which can be checked after base change to k_w . Set $K'_w := (L_w)^{I_w}$, and let O'_w denote its valuation ring. Then as O'_w is a finite étale extension of O_v , we have

$$\mathcal{B}_{O_v} \times_{\operatorname{Spec} O_v} \operatorname{Spec} O'_w \cong \operatorname{Res}_{(O_w \otimes_{O_v} O'_w)/O'_w} \mathcal{A}_{L,O_w} \cong \prod_{G_w/I_w} (\operatorname{Res}_{O_w/O'_w} \mathcal{A}_{L,O_w}),$$

where the natural G_w -action is the extension of the natural I_w -action on $\operatorname{Res}_{O_w/O'_w} \mathcal{A}_{L,O_w}$ so that G_w acts transitively on the factors. Therefore, by taking G_w -invariants we get

$$(\mathcal{B}_{O_v})^{G_w} \times_{\operatorname{Spec} O_v} \operatorname{Spec} O'_w \cong \left(\operatorname{Res}_{O_w/O'_w} \mathcal{A}_{L,O_w}\right)^{I_v}$$

If we let \mathcal{B}_{k_w} denote the special fibre of $\operatorname{Res}_{O_w/O'_w} \mathcal{A}_{L,O_w}$, then we can also show that

$$(\mathbf{F}^{1}\mathcal{B}_{k_{v}})^{G_{w}} \times_{\operatorname{Spec} k_{v}} \operatorname{Spec} k_{w} \cong (\mathbf{F}^{1}\mathcal{B}_{k_{w}})^{I_{w}}$$

Therefore, to prove the proposition we may replace K_v with K'_w and suppose $G_w = I_w$.

Now, suppose that L_w/K_v is totally ramified, so we write \mathcal{B}_{k_w} and \mathcal{A}_{k_w} for \mathcal{B}_{k_v} and \mathcal{A}_{k_v} . Then by tameness, $|I_w|$ acts invertibly on the vector group $F^1\mathcal{B}_{k_w}$, which yields the following short exact sequence of smooth k_w -group schemes

$$0 \longrightarrow \left(\mathrm{F}^{1}\mathcal{B}_{k_{w}} \right)^{I_{w}} \longrightarrow \left(\mathcal{B}_{k_{w}} \right)^{I_{w}} \cong \mathcal{A}_{k_{w}} \longrightarrow \left(\mathcal{A}_{L,k_{w}} \right)^{I_{w}} \longrightarrow 0 \ .$$

Clearly, $(F^1 \mathcal{B}_{k_w})^{I_w}$ is still a vector group. This concludes the proof.

Corollary 5.4. In the same setting as in Proposition 5.3, if $\mathcal{A}_{L,O}^{\circ}$ is semi-abelian then we have $(F^1\mathcal{B}_{k_v})^{G_w} \cong \mathcal{R}_u(\mathcal{A}_{k_v}^{\circ})$, the unipotent radical of the neutral component of \mathcal{A}_{k_v} .

Proof. Since $\mathscr{A}_{L,k_{w}}^{\circ}$ is semi-abelian (hence, with trivial unipotent radical), the unipotent radical of the neutral component of $(\operatorname{Res}_{k_{w}/k_{v}} \mathscr{A}_{L,k_{w}})^{G_{w}}$ is also trivial, so the exact sequence (5.3a) identifies $(F^{1}\mathscr{B}_{k_{v}})^{G_{w}}$ with the unipotent radical of \mathscr{A}_{k}° .

Now, choose a dense open subscheme $U \subset X$ contained in both the good reduction locus for A/K and the unramified locus for π , and set $U_L := \pi^{-1}(U)$. Let $Z := X \setminus U$ and $Z_L := X_L \setminus U_L$ respectively denote the reduced complements.

Proposition 5.5. We make the same assumption as in Proposition 5.1(3), and let Z and Z_L be as above. Then we have natural inclusions

(5.5a)
$$\operatorname{Lie}(\mathcal{A})(-Z) \hookrightarrow (\operatorname{Lie}(\mathcal{A}_L)(-Z_L))^G \hookrightarrow \operatorname{Lie}(\mathcal{A})(-(Z \cap U'))$$
.

that restrict to isomorphisms over U', and the cokernel of the first inclusion is supported exactly on $X \setminus U'$. Furthermore, we have

(5.5b)
$$\frac{\operatorname{Lie}(\mathcal{A})(-(Z \cap U'))}{\left(\operatorname{Lie}(\mathcal{A}_L)(-Z_L)\right)^G} \cong \bigoplus_{v \notin U'} \left(\operatorname{Lie}(\mathcal{A}_L)(k_{\bar{v}})\right)^{G_{\bar{v}}},$$

where we choose a preimage $\tilde{v} \in \pi^{-1}(v)$ for each $v \notin U'$.

Remark 5.6. The last displayed equation is independent of the choice of \tilde{v} since we have

$$\left(\operatorname{Lie}(\mathcal{A}_L)(k_{\tilde{v}})\right)^{G_{\tilde{v}}} \cong \left(\bigoplus_{w|v} \operatorname{Lie}(\mathcal{A}_L)(k_w)\right)^{G}.$$

Proof of Proposition 5.5. We use the notation from Proposition 5.3 and its proof. By Proposition 5.1(3), we have a natural isomorphism $\text{Lie}(\mathcal{A})(-Z)|_{U'} \cong (\text{Lie}(\mathcal{A}_L)(-Z_L))^G|_{U'}$. For any $v \notin U'$ we have

$$\frac{\left(\operatorname{Lie}(\mathcal{A})(-(Z\cap U'))_{\widehat{v}}\right)}{\left(\operatorname{Lie}(\mathcal{A})(-Z))\right)_{\widehat{v}}} \cong \frac{\operatorname{Lie}(\mathcal{A})(O_v)}{\operatorname{Lie}(\mathcal{A})(\mathfrak{m}_v)} \cong \operatorname{Lie}(\mathcal{A})(k_v),$$

and the preimage of $\operatorname{Lie}(\mathbb{F}^1\mathcal{B}_{k_{\hat{v}}})^{G_{\hat{v}}}$ in $\operatorname{Lie}(\mathcal{A})(O_v)$ can be naturally identified with the completed stalk of $(\operatorname{Lie}(\mathcal{A}_L)(-Z_L))^G$ at v. Thus we get the desired inclusions of vector bundles. Furthermore, we have

$$\frac{\operatorname{Lie}(\mathcal{A})(-(Z\cap U'))}{\left(\operatorname{Lie}(\mathcal{A}_L)(-Z_L)\right)^G} \cong \bigoplus_{v\notin U'} \frac{\operatorname{Lie}(\mathcal{A})(k_v)}{\operatorname{Lie}(\mathrm{F}^1\mathcal{B}_{k_{\bar{v}}})^{G_{\bar{v}}}} \cong \bigoplus_{v\notin U'} \left(\operatorname{Lie}(\mathcal{A}_L)(k_{\bar{v}})\right)^{G_{\bar{v}}},$$

where the last isomorphism follows from Proposition 5.3.

Lastly, we show that $\operatorname{Lie}(\mathcal{A})(k_v) \cong \left(\operatorname{Lie}(\mathcal{A}_L)(k_{\tilde{v}})\right)^{G_{\tilde{v}}}$ for any $v \notin U'$ and $\tilde{v} \mid v$; i.e., the cokernel of $\operatorname{Lie}(\mathcal{A})(-Z) \hookrightarrow \left(\operatorname{Lie}(\mathcal{A}_L)(-Z_L)\right)^G$ is supported exactly on $X \setminus U'$. Indeed, if we have $\operatorname{Lie}(\mathcal{A})(k_v) \cong \left(\operatorname{Lie}(\mathcal{A}_L)(k_{\tilde{v}})\right)^{G_{\tilde{v}}}$, then it follows that the following composition

$$\operatorname{Lie}(\mathcal{A})(k_{v}) \otimes_{k_{v}} k_{\tilde{v}} \cong \left(\operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})\right)^{G_{\tilde{v}}} \otimes_{k_{v}} k_{\tilde{v}} \hookrightarrow \operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})$$

is an isomorphism for the dimension reason. Hence, by the Nakayama lemma, the natural map $\pi^* \operatorname{Lie}(\mathcal{A}) \to \operatorname{Lie}(\mathcal{A}_L)$ is isomorphic at \tilde{v} , so $v \in U'$ by Proposition 5.1(1). This concludes the proof.

Under the same setting as in Proposition 5.1(3), choose integers $r_{w,i} \in \{0, \dots, |I_w| - 1\}$ for any $w \in Z'_L$ so that $\text{Lie}(\mathcal{A}_L)(k_w) \cong \bigoplus_i (\mathfrak{m}_w^{-r_{w,i}}/\mathfrak{m}_w^{-r_{w,i}+1})$; *cf.* Lemma 2.5. Then one can shows that

(5.7)
$$\dim_{k_{\tilde{v}}} \left(\operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}}) \right)^{G_{\tilde{v}}} = d'_{\tilde{v}} = \left| \{ i \text{ such that } r_{w,i} = 0 \} \right|$$

Let us record the following immediate corollary.

Corollary 5.8. Under the same assumption as in Proposition 5.5, the following properties hold.

(1) If A is an elliptic curve, then we have $\left(\operatorname{Lie}(\mathcal{A}_L)(-Z_L)\right)^G \xrightarrow{\sim} \operatorname{Lie}(\mathcal{A})(-(Z \cap U'))$. (2) We have

$$\deg\left(\left(\operatorname{Lie}(\mathcal{A}_L)(-Z_L)\right)^G\right) = \deg\left(\operatorname{Lie}(\mathcal{A})(-(Z \cap U'))\right) \\ -\sum_{v \notin U'} [k_v : k] \cdot \dim_{k_v} \left(\operatorname{Lie}(\mathcal{A}_L)(k_{\tilde{v}})\right)^{G_{\tilde{v}}},$$

where we choose $\tilde{v} \in \pi^{-1}(v)$ for each $v \notin U'$.

(3) Suppose additionally that L/K is weakly ramified everywhere. Then the G-equivariant vector bundle Lie(A_L)(-Z_L) satisfies the condition (3.2a), so Theorem 3.2 holds. Furthermore, we can compute l_{w,i}'s defined in (3.2b) as follows:

$$l_{w,i} = \begin{cases} |I_w/P_w| - 1, & \forall i \text{ if } w \in Z_L^{\operatorname{ram}} \setminus Z_L';\\ r_{w,i} - 1 & if w \in Z_L' \text{ and } r_{w,i} \neq 0;\\ |I_w/P_w| - 1 & if w \in Z_L' \text{ and } r_{w,i} = 0, \end{cases}$$

where $r_{w,i}$'s are defined in (5.7).

Proof. Claim (1) can be deduced from the statement on the cokernels of the inclusions in (5.5a) in Proposition 5.5, noting that all the vector bundles involved are line bundles. Claim (2) is clear from Proposition 5.5.

Suppose that L/K is weakly ramified everywhere and we want to verify (3.2a) for $\mathcal{E} = \text{Lie}(\mathcal{A}_L)(-Z_L)$, which is a local condition at each $w \in Z_L^{\text{wild}}$. Since the natural inclusion $\text{Lie}(\mathcal{A}_{X_L})(-Z_L) \hookrightarrow \text{Lie}(\mathcal{A}_L)(-Z_L)$ restricts to an isomorphism over U'_L , the completed stalk of $\text{Lie}(\mathcal{A}_L)(-Z_L)$ at each $w \in U'_L$ is G_w -equivariantly isomorphic to $\mathfrak{m}_w^{\oplus d}$ with $d = \dim(A)$; *cf.* (3.6a). Now, the condition (3.2a) follows since the assumption of Proposition 5.5 implies that $Z_L^{\text{wild}} \subseteq Z_L \cap U'_L$. It also implies that $l_{w,i} \equiv -1 \mod |P_w|$ for any $w \in Z_L^{\text{ram}} \cap U'_L$. The computation of $l_{w,i}$ is clear for $w \notin U'_L$, so Claim (3) now follows.

Example 5.9. The assumption in Corollary 5.8(3) is satisfied if L/K is weakly ramified everywhere, *A* has semistable reduction at all places of *L*, and L/K is tame at all places in *K* where *A* does *not* have semistable reduction. In that case, $|X \setminus U'|$ is precisely the set of places of non-semistable reduction for *A*. As a special case, if *A* has semistable reduction at all places in *K* then by Proposition 5.5 we have $\text{Lie}(A)(-Z) \xrightarrow{\sim} (\text{Lie}(\mathcal{R}_L)(-Z_L))^G$.

6. REVIEW OF EQUIVARIANT BSD AND HASSE-WEIL-ARTIN L-VALUES

We introduce a certain perfect $\widehat{\mathbb{Z}}G$ -complex encoding the integral Galois module structure of the arithmetic invariants of A/L, and review the main result of [BKK] on the equivariant BSD conjecture. We maintain the setting of §5, and additionally assume that k is a *finite field* of characteristic p. In particular, L/K is an arbitrary finite Galois extension of global function fields.

For each $w \in Z_L$, we set

$$\mathcal{A}_{I}^{\circ}(\mathfrak{m}_{w}) \coloneqq \ker \left(\mathcal{A}_{I}^{\circ}(O_{w}) \to \mathcal{A}_{I}^{\circ}(k_{w}) \right),$$

which is a G_w -stable pro-p open subgroup of $A(L_w)$. Following [KT03, §2.2], we let $\mathrm{R}\Gamma_{\mathrm{ar},Z_L}(U_L, \mathcal{A}_{L,\mathrm{tors}}) \in D(\mathbb{Z}G)$ denote the mapping fibre of

(6.1)
$$\mathrm{R}\Gamma_{\mathrm{fl}}(U_L, \mathcal{A}_{L, \mathrm{tors}}) \oplus \left(\bigoplus_{w \in Z_L} \mathcal{A}_L^{\circ}(\mathfrak{m}_w) \overset{\mathrm{L}}{\otimes} \mathbb{Q}/\mathbb{Z} \right) [-1] \to \bigoplus_{w \in Z_L} \mathrm{R}\Gamma_{\mathrm{fl}}(\mathrm{Spec}\, L_w, \mathcal{A}_{L, \mathrm{tors}}) .$$

We will often write $\widehat{SC}_{Z_L} = \widehat{SC}_{Z_L}(A, L/K)$.

Definition 6.2. We set

 $\widehat{\mathrm{SC}}_{Z_L}(A,L/K) \coloneqq \left(\mathrm{R}\Gamma_{\mathrm{ar},Z_L}(U_L,\mathcal{A}_{L,\mathrm{tors}}) \right)^{\vee} [-2] \in \mathrm{D}(\widehat{\mathbb{Z}}G),$

where $(-)^{\vee}$ is the Pontryagin dual. For any prime ℓ (allowing $\ell = p$), we write

 $SC_{Z_L,\ell}(A, L/K) := \widehat{SC}_{Z_L}(A, L/K) \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_{\ell} \in D(\mathbb{Z}_{\ell}G).$

We will often write $\widehat{SC}_{Z_L} = \widehat{SC}_{Z_L}(A, L/K)$ and $SC_{Z_L,\ell} = SC_{Z_L,\ell}(A, L/K)$ for simplicity.

Proposition 6.3 (cf. [KT03, §2.5]). We have $H^i(\widehat{SC}_{Z_L}) = 0$ for $i \notin [0, 2]$. Furthermore, if III(A/L) is finite, then we have

$$\mathrm{H}^{0}(\widehat{\mathrm{SC}}_{Z_{L}})\cong A^{t}(L)\otimes\widehat{\mathbb{Z}}$$

and a long exact sequence

$$0 \rightarrow \operatorname{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/L)^{\vee} \rightarrow \operatorname{H}^{1}(\widehat{\operatorname{SC}}_{Z_{L}}) \rightarrow \bigoplus_{w \in Z_{L}} (\mathcal{A}_{L}(k_{w}))^{\vee} \rightarrow (A(L)_{\operatorname{tors}})^{\vee} \rightarrow \operatorname{H}^{2}(\widehat{\operatorname{SC}}_{Z_{L}}) \rightarrow 0.$$

Proof. Apply [KT03, §2.5, §2.3] to $V = (\mathcal{A}_L^{\circ}(\mathfrak{m}_w))_{w \in Z_L}$, noting that $A(L_w)/\mathcal{A}_L^{\circ}(\mathfrak{m}_w) \cong \mathcal{A}_L(O_w)/\mathcal{A}_L^{\circ}(\mathfrak{m}_w) \cong \mathcal{A}_L(k_w)$.

By Schneider's result [Sch82, p 509], we have a non-degenerate G-equivariant pairing

(6.4)
$$\langle , \rangle_{A/L} \coloneqq (\log p)^{-1} \cdot \langle , \rangle_{A/L,\mathrm{NT}} \colon A(L) \times A^t(L) \to \mathbb{Q},$$

where
$$\langle , \rangle_{A/L,NT}$$
 is the Néron–Tate height pairing

Definition 6.5. For any prime ℓ (allowing $\ell = p$), we write $h_{\ell} : A^{t}(L) \otimes \mathbb{Q}_{\ell} \to (A(L) \otimes \mathbb{Q}_{\ell})^{*}$ for the $\mathbb{Q}_{\ell}G$ -isomorphism induced by $\langle , \rangle_{A/L}$. If in addition $\mathrm{III}(A/L)$ is finite, then we interpret h_{ℓ} as a $\mathbb{Q}_{\ell}G$ -isomorphism

$$h_{\ell} \colon \mathrm{H}^{0}(\mathrm{SC}_{Z_{L},\ell} \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}) \xrightarrow{=} \mathrm{H}^{1}(\mathrm{SC}_{Z_{L},\ell} \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}).$$

Therefore, h_{ℓ} defines the \mathbb{Q}_{ℓ} -trivialisation in the sense of (4.7) if $\mathrm{III}(A/L)$ is finite and $\mathrm{SC}_{Z_{L},\ell}$ is a perfect $\mathbb{Z}_{\ell}G$ -complex. The following proposition gives a sufficient condition for the $\mathbb{Z}_{\ell}G$ -perfectness.

Proposition 6.6. (1) If $\ell \neq p$ then we have $SC_{Z_{\ell},\ell} \in D^{perf}(\mathbb{Z}_{\ell}G)$.

(2) Suppose that L/K is weakly ramified everywhere, and if L/K is wildly ramified at v ∈ Z then the natural map A° ×_X Spec O_w → A^o_L ×_{XL} Spec O_w is an isomorphism. (In other words, the condition for Corollary 5.8(3) is valid.) Then we have SC_{ZL}(A, L/K) ∈ D^{perf}(ZG), and hence, SC_{ZL} ∈ D^{perf}(Z_pG).

Before we prove the proposition, let us make the following remark.

Remark 6.7. Proposition 6.6 is a special case of [BKK, Proposition 3.7(i)], built upon the argument in [KT03, §6]. To explain, by [BKK, Proposition 3.4, Proposition 3.7(i)] one constructs a *perfect* $\widehat{\mathbb{Z}}G$ -complex \widehat{SC}_{V_L} using carefully chosen family of G_w -stable open compact subgroups $V_L := (V_w \subset \mathcal{A}_L^{\circ}(\mathfrak{m}_w))_{w \in Z_L}$, equipped with a distinguished triangle in $D(\widehat{\mathbb{Z}}G)$

$$\widehat{\mathrm{SC}}_{V_L} \longrightarrow \widehat{\mathrm{SC}}_{Z_L} \longrightarrow \bigoplus_{w \in Z_L} \left(\mathscr{R}_L^{\circ}(\mathfrak{m}_w) / V_w \right)^{\vee} [-1] \longrightarrow (+1).$$

(In [BKK, Proposition 3.7(i)], \widehat{SC}_{V_L} is denoted as $\operatorname{R}_{\operatorname{ar},V_L}(U_L, \mathcal{A}_{L,\operatorname{tors}})^{\vee}[-2]$.) As $\mathcal{A}_L^{\circ}(\mathfrak{m}_w)/V_w$ is a *p*-group for each $w \in Z_L$, it easily follows that the natural map $\widehat{SC}_{V_L} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_{\ell} \to \operatorname{SC}_{Z_{L,\ell}}$ is a quasi-isomorphism, proving Proposition 6.6(1). To prove Proposition 6.6(2), we have to show that the choice $V_L = (\mathcal{A}_L^{\circ}(\mathfrak{m}_w))_{w \in Z_L}$ makes \widehat{SC}_{V_L} a perfect $\widehat{\mathbb{Z}}G$ -complex under the additional assumption in the statement, which we explain in the proof.

Without the additional assumption in Proposition 6.6(2), the choice of V_L that make \widehat{SC}_{V_L} perfect is quite *inexplicit* and hard to work with. Therefore, we state Proposition 6.6

in a restrictive setting where there is a preferred explicit choice of V_L . See Remark 7.17 for further discussions.

Proof of Proposition 6.6. Claim (1) is already proven in Remark 6.7, so let us prove claim (2) in the setting of (2). By the proof of Proposition 3.7(i) in [BKK], we need to show that the continuous G_w -action on $\mathcal{R}_L^{\circ}(\mathfrak{m}_w)$ is cohomologically trivial, which is equivalent to the cohomological triviality of Lie(\mathcal{R}_L)(\mathfrak{m}_w) for G_w by the proof of Lemma 6.1 and Lemma 6.2 in [KT03]. (See also the proof of Lemma 3.4 in [BKK].) But by Corollary 5.8(3), we have G_w -equivariant isomorphisms of O_w -modules

$$\operatorname{Lie}(\mathcal{A}_L)(\mathfrak{m}_w) \cong \operatorname{Lie}(\mathcal{A})(\mathcal{O}_v) \otimes_{\mathcal{O}_v} \mathfrak{m}_w \cong \mathfrak{m}_w^{\oplus \dim(A)},$$

for any $w \in Z_L$ where L/K is weakly and wildly ramified, so the desired cohomological triviality follows from Köck's local integral normal basis theorem (*cf.* Theorem 2.11).

Notation 6.8. Let Ir(G) denote the set of isomorphism classes of (complex) irreducible *G*-representations, and choose a number field $E \subset \mathbb{C}$ over which any $\psi \in Ir(G)$ is defined. For each $\psi \in Ir(G)$, let V_{ψ} be the corresponding *EG*-module, and choose a *G*-stable O_E -lattice $T_{\psi} \subset V_{\psi}$. For any place λ of *E*, we write

$$V_{\psi,\lambda} := V_{\psi} \otimes_E E_{\lambda} \quad \& \quad T_{\psi,\lambda} := T_{\psi} \otimes_{O_E} O_{E,\lambda}.$$

We obviously extend the above notation for any *G*-representation ψ .

For any *G*-representation ψ , let $L_U(A, \psi, s)$ be the Hasse–Weil–Artin *L*-series for (A, ψ) without Euler factors away from U;⁹ i.e., choosing a place $\lambda \mid \ell$ of *E* with $\ell \neq p$ we have

(6.9)
$$L_U(A,\psi,s) \coloneqq \prod_{v \in |U|} \det_{E_{\lambda}} \left(1 - |k_v|^{1-s} \operatorname{Frob}_v | V_{\ell}(A) \otimes_{\mathbb{Q}_{\ell}} V_{\psi,\lambda} \right)^{-1},$$

where Frob_{v} is the *geometric* Frobenius at v. It is sometimes useful to apply the change of variable $t = p^{-s}$ and set $Z_U(A, \psi, t) = L_U(A, \psi, s)$.

Recall that by the Lefschetz trace formula we have

$$Z_U(A,\psi,t) = \prod_{i=0}^{2} \det_{E_{\lambda}} \left(1 - pt \cdot \operatorname{Frob}_{p} \mid \operatorname{H}^{i}_{\operatorname{\acute{e}t},c}(U \times_{\operatorname{Spec} \mathbb{F}_{p}} \operatorname{Spec} \overline{\mathbb{F}}_{p}, V_{\ell}(A) \otimes_{\mathbb{Q}_{\ell}} V_{\psi,\lambda}) \right)^{(-1)^{\ell+1}},$$

where Frob_p is the *geometric p*-Frobenius. Moreover, we have $Z_U(A, \psi, t) \in E(t)$ that is independent of the choice of λ , and there is an analogous formula for $\lambda \mid p$ recovering $Z_U(A, \psi, t)$ via rigid cohomology. (For more details, see [BKK, Theorem 8.2].)

For any *G*-representation ψ , set

(6.10)
$$r_{\mathrm{an}}(\psi) \coloneqq \mathrm{ord}_{s=1}L_{U}(A,\psi,s) = \mathrm{ord}_{t=p^{-1}}Z_{U}(A,\psi,t) \quad \text{and} \\ \mathscr{L}_{U}(A,\psi) \coloneqq \frac{L_{U}^{*}(A,\psi,1)}{(\log p)^{r_{\mathrm{an}}(\psi)}} = \lim_{t \to p^{-1}} \frac{Z_{U}(A,\psi,t)}{(1-pt)^{r_{\mathrm{an}}(\psi)}} \in E^{\times}.$$

We recall the following result; *cf.* [BKK, Proposition 5.6].

Proposition 6.11. (1) For any field automorphism τ of \mathbb{C} , we have

$$\tau(Z_U(A,\psi,t)) = Z_U(A,\tau\circ\psi,t) \quad and \quad \tau(\mathscr{L}_U(A,\psi)) = \mathscr{L}_U(A,\tau\circ\psi),$$

where we view $Z_U(A, \psi, t) \in \mathbb{C}(t)$.

(2) There exists an element $\mathscr{L}_U(A, L/K) \in K_1(\mathbb{Q}G)$ interpolating $\mathscr{L}_U(A, \psi)$'s in the following sense: for any *G*-representation ψ , we have

$$\operatorname{Nrd}^{\psi}(\mathscr{L}_U(A, L/K)) = \mathscr{L}_U(A, \psi).$$

⁹Sometimes it can be convenient to allow ψ to be reducible as we do (such as the regular representation), which is harmless as we have $L_U(A, \psi', \oplus \psi', s) = L_U(A, \psi', s) \cdot L_U(A, \psi'', s)$.

Proof. Claim (1) follows from Eq (2) in the proof of Proposition 2.2 in [BKK] (or alternatively, see the proof of Proposition 5.6 in [BKK]). Identifying $\zeta(\mathbb{C}G) \cong \prod_{\psi \in Ir(G)} \mathbb{C}$, we set

$$\mathscr{L}_U(A, L/K) := \left(\mathscr{L}_U(A, \psi)\right)_{\psi \in \mathrm{Ir}(G)} \in \zeta(\mathbb{C}G)^{\times}.$$

It follows from [BKK, Proposition 5.6] that $\mathscr{L}_U(A, L/K) \in K_1(\mathbb{Q}G) = \zeta(\mathbb{Q}G)^{\times} \cap K_1(\mathbb{R}G)$. This element clearly satisfies the interpolation property for any $\psi \in Ig(G)$, hence for any G- representation ψ .

For $\psi \in Ir(G)$, the *algebraic* ψ *-rank* of *A* is defined as follows:

(6.12)
$$r_{\text{alg}}(\psi) \coloneqq \dim_E \left(\operatorname{Hom}_{EG}(V_{\psi}, E \otimes A^t(L)) \right).$$

Recall the following standard result.

Theorem 6.13 (cf. [KT03], [BKK, Theorem 8.2]). The following are equivalent.

- (1) The ℓ_0 -primary part of $\operatorname{III}(A/L)$ is finite for some prime ℓ_0 .
- (2) III(A/L) is finite.
- (3) We have $r_{an}(\psi) = r_{alg}(\psi)$ for any $\psi \in Ir(G)$.

We conclude this section by recalling the following theorem from [BKK]. For any prime ℓ , we write $\partial_{\ell}^{G} \coloneqq \partial_{\mathbb{Z}_{\ell}G,\mathbb{Q}_{\ell}} \colon K_{1}(\mathbb{Q}_{\ell}G) \to K_{0}(\mathbb{Z}_{\ell}G,\mathbb{Q}_{\ell}G)$ and $\chi_{\ell}^{G}(C^{\bullet},h) \coloneqq \chi_{\mathbb{Z}_{\ell}G,\mathbb{Q}_{\ell}}(C^{\bullet},h) \in K_{0}(\mathbb{Z}_{\ell}G,\mathbb{Q}_{\ell}G)$; *cf.* (4.2a) and Def 4.8.

Theorem 6.14 (cf. [BKK, Theorem 4.9]). Suppose that the ℓ_0 -primary part of III(A/L) is finite for some prime ℓ_0 .

(1) For any prime $\ell \neq p$, the following formula

$$\partial_{\ell}^{G} (\mathscr{L}_{U}(A, L/K)) - \chi_{\ell}^{G} (SC_{Z_{L},\ell}, h_{\ell})$$

defines a torsion element in $K_0(\mathbb{Z}_\ell G, \mathbb{Q}_\ell G)$.

(2) Under the assumption as in Corollary 5.8(3), the following formula

 $\partial_{p}^{G} \left(\mathscr{L}_{U}(A, L/K) \right) - \chi_{p}^{G} \left(\operatorname{SC}_{Z_{L}, \ell}, h_{p} \right) + \chi_{p}^{G} \left(\operatorname{Lie}(\mathscr{A}_{L})(-Z_{L}) \right)$

defines a torsion element in $K_0(\mathbb{Z}_pG, \mathbb{Q}_pG)$. Here, $\chi_p^G(\mathcal{E})$ for a *G*-equivariant vector bundle \mathcal{E} is defined in (4.14).

Proof. This theorem essentially follows from [BKK, Theorem 4.9, Proposition 5.6], which we now explain in details. Using the notation of [BKK, Proposition 5.6], the image of $\chi_{G,\mathbb{Q}}^{\text{BSD}}(A, V_L)$ in $K_0(\mathbb{Z}_{\ell}G, \mathbb{Q}_{\ell}G)$ is equal to $\chi_{\ell}^G(\text{SC}_{Z_L,\ell}, h_{\ell})$ by Proposition 6.6 and Remark 6.7. (We allow $\ell = p$ under the addition assumption as in the statement.) The image of $\chi_G^{\text{coh}}(A, V_L)$ in $K_0(\mathbb{Z}_{\ell}G, \mathbb{Q}_{\ell}G)$ is equal to $\chi_{\ell}^G(\text{Lie}(\mathcal{A}_L)(-Z_L))$ since the vector bundle \mathcal{L}_L attached to $V_L = (\mathcal{A}_L^{\circ}(\mathfrak{m}_w))_{w \in Z_L}$ in [BKK, §3.5] is exactly Lie($\mathcal{A}_L)(-Z_L)$). (In *loc. cit.* we assumed that $V_w \subseteq \mathcal{A}_{X_L}^{\circ}(\mathfrak{m}_w)$ due to the way we construct cohomologically trivial V_w 's, but the same proof can be extended *verbatim* to $V_w = \mathcal{A}_L^{\circ}(\mathfrak{m}_w)$ and $\mathcal{L}_L = \text{Lie}(\mathcal{A}_L)(-Z_L)$.) Lastly, $\chi_G^{\text{sgn}}(A)$ is 2-torsion by definition. By [BKK, Theorem 4.9], the formula in [BKK, Proposition 5.6(ii)] holds up to torsion, and hence the theorem follows. □

We apply Theorem 6.14 to deduce a result on the normalised leading term $\mathscr{L}_U(A, \psi)$. To state it, let us introduce some notation, which is a slight adaptation of §4. Fix a place $\lambda \mid \ell$ of *E* and a *G*-representation ψ . Set

$$\rho_{\lambda}^{\psi} \colon \mathrm{K}_{0}(\mathbb{Z}_{\ell}G, \mathbb{Q}_{\ell}G) \longrightarrow \mathrm{K}_{0}(\mathcal{O}_{E,\lambda}, E_{\lambda}) \xrightarrow{v_{\lambda}} \mathbb{Z},$$

where the first map is induced by $\mathfrak{A} = \mathbb{Z}_{\ell}G \xrightarrow{\psi} \operatorname{End}_{O_{E,\lambda}}(T_{\psi,\lambda}).$

We also introduce the following integers

$$\chi_{Z_L,\lambda}^{\text{BSD}}(A,\psi) \coloneqq \rho_{\lambda}^{\psi} \left(\chi_{\ell}^G(\text{SC}_{Z_L,\ell}, h_{\ell}) \right) \quad \& \quad \chi_{Z_L,\lambda}^{\text{coh}}(A,\psi) \coloneqq \rho_{\lambda}^{\psi} \left(\chi_{\ell}^G\left(\text{Lie}(\mathcal{A}_L)(-Z_L) \right) \right),$$

where we set $\chi_{\ell}^G\left(\text{Lie}(\mathcal{A}_L)(-Z_L) \right) = 0$ for $\ell \neq p$.

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Corollary 6.15. Suppose that the ℓ_0 -primary part of III(A/L) is finite for some prime ℓ_0 . Fix a place $\lambda \mid \ell$ of E, and let ψ be a G-representation. Then we have

$$v_{\lambda} \big(\mathscr{L}_{U}(A, \psi) \big) = \chi_{Z_{L}, \lambda}^{\text{BSD}}(A, \psi) - \chi_{Z_{L}, \lambda}^{\text{coh}}(A, \psi)$$

if either $\ell \neq p$, or $\ell = p$ and the same assumption as in Corollary 5.8(3) holds.

Proof. By construction of $\mathscr{L}_U(A, L/K)$ and Lemma 4.6(2), we have

$$(\rho_{\lambda}^{\psi} \circ \partial_{\ell}^{G}) \big(\mathscr{L}_{U}(A, L/K) \big) = v_{\lambda} \big(\mathscr{L}_{U}(A, \psi) \big).$$

The corollary can now be obtained by applying ρ_{λ}^{ψ} to the formulae in Theorem 6.14. \Box

Remark 6.16. In the above setting, $\chi_{Z_L,\lambda}^{\text{coh}}(A, \psi)$ has already been computed in Corollary 4.15(2), so to make $v_{\lambda}(\mathscr{L}_U(A, \psi))$ explicit, it remains to compute $\chi_{Z_L,\lambda}^{\text{BSD}}(A, \psi)$ in terms of *arithmetic invariants* of A/K (as integral Galois modules). This step turns out to be quite subtle, and in the next section we carry it out under certain *simplifying assumptions* depending on ℓ ; *cf.* Assumptions 7.2. In particular, even though a more general version of Theorem 6.14(2) is obtained in [BKK, Theorem 4.9, Proposition 5.6] involving some inexplicit choice V_L as in Remark 6.7, the resulting general formula for $v_{\lambda}(\mathscr{L}_U(A, \psi))$ seems difficult to make explicit. See Remark 7.17 for further discussions.

7. The BSD-like formula for Hasse–Weil–Artin L-values

Assuming the finiteness of III(A/L), we shall express $\chi_{Z_L,\lambda}^{\text{BSD}}(A, \psi)$ in terms of the Galois module structure of A(L), $A^t(L)$ and III(A/L) under a certain set of assumptions satisfied for almost all primes ℓ under λ , and thereby obtain the formula for $v_\lambda(\mathscr{L}_U(A, \psi))$. We also introduce a stronger assumption to handle $\ell = p$. We closely follow the proof in Burns–Castillo [BMC24, Proposition 7.3], which proves an analogous result over a number field.

Notation 7.1. For a $\mathbb{Z}_{\ell}G$ -module *M* we set

(7.1a)
$$[M]_{\psi,\lambda} := \operatorname{Hom}_{O_{E,\lambda}}(T_{\psi,\lambda}, O_{E,\lambda} \otimes_{\mathbb{Z}_{\ell}} M)_G \cong T^*_{\psi,\lambda} \otimes_{\mathbb{Z}_{\ell} G} M, \quad \text{and}$$

(7.1b)
$$[M]^{\psi}_{\lambda} := \operatorname{Hom}_{\mathcal{O}_{E,\lambda}}(T_{\psi,\lambda}, \mathcal{O}_{E,\lambda} \otimes_{\mathbb{Z}_{\ell}} M)^{G} = \operatorname{Hom}_{\mathcal{O}_{E,\lambda}[G]}(T_{\psi,\lambda}, \mathcal{O}_{E,\lambda} \otimes_{\mathbb{Z}_{\ell}} M).$$

We extend these definitions to $\mathbb{Z}_{\ell}G$ -complexes. If *V* is a $\mathbb{Q}_{\ell}G$ -module and $\psi \in \operatorname{Ir}(G)$ then we have $[V]_{\psi,\lambda} \cong [V]_{\lambda}^{\psi}$ and its E_{λ} -dimension is the multiplicity of ψ in *V*.

If *M* is a finitely generated $\mathbb{Z}G$ -module, then we abusively write $[M]_{\psi,\lambda}$ for $[M \otimes \mathbb{Z}_{\ell}]_{\psi,\lambda}$ and similarly define $[M]_{\lambda}^{\psi}$.

Lastly, we set

(7.1c)
$$\operatorname{III}_{\psi,\lambda}^{\vee}(A/L) := \operatorname{ker}\left([\operatorname{Sel}_{\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}}(A/L)^{\vee}]_{\psi,\lambda} \twoheadrightarrow [(A(L) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{\vee}]_{\psi,\lambda} \right).$$

To motivate the notation, note that by right exactness of $[-]_{\psi,\lambda}$ we have a natural right exact sequence

$$[\operatorname{III}(A/L)^{\vee}]_{\psi,\lambda} \longrightarrow [\operatorname{Sel}_{\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}}(A/L)^{\vee}]_{\psi,\lambda} \longrightarrow [(A(L) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{\vee}]_{\psi,\lambda} \longrightarrow 0 ,$$

and $\amalg_{\psi,\lambda}^{\vee}(A/L)$ is the image of $[\amalg(A/L)^{\vee}]_{\psi,\lambda}$ in $[\operatorname{Sel}_{\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}}(A/L)^{\vee}]_{\psi,\lambda}$. In particular, if ℓ is prime to |G| then we have $[\amalg(A/L)^{\vee}]_{\psi,\lambda} \xrightarrow{\rightarrow} \amalg_{\psi,\lambda}^{\vee}(A/L)$. Note also that $\amalg_{\psi,\lambda}^{\vee}(A/L)$ is finite, in which case $\amalg_{\psi,\lambda}^{\vee}(A/L)$ is the torsion part of $[\operatorname{Sel}_{\mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}}(A/L)^{\vee}]_{\psi,\lambda}$.

To compute $\chi_{Z_L,\lambda}^{BSD}(A, \psi)$, we need to compute the cohomology of $[SC_{Z_L,\ell}]_{\psi,\lambda}$ in terms of the arithmetic invariants of A/L, which naturally involves some Hochschield–Serre-type spectral sequence. To make the spectral sequence *sufficiently degenerate*, we introduce the following conditions for $(A, L/K, Z_L)$ and a prime ℓ .

Assumption 7.2. For $(A, L/K, Z_L)$ and ℓ as above, suppose the following conditions hold.

- (1) Neither A(L) nor $A^{t}(L)$ have any non-trivial ℓ -torsion.
- (2) For any $w \in Z_L$, there is no non-trivial ℓ -torsion in $\mathcal{A}_L(k_w) \cong A(L_w)/\mathcal{A}_L^{\circ}(\mathfrak{m}_w)$.
- (3) If ℓ = p, then we assume that L/K is weakly ramified everywhere, A has semistable reduction at all places of L, and L/K is tamely ramified at all places v ∈ Z where A does not have semistable reduction. (Cf. Example 5.9.)

Condition (2) can be rephrased as ℓ dividing neither $|\mathcal{A}_L^{\circ}(k_w)|$ nor the local Tamagawa number $|A(L_w)/\mathcal{A}_L^{\circ}(O_w)|$.

Given $(A, L/K, Z_L)$, Assumption 7.2 is clearly satisfied for all but finitely many primes ℓ , but it is most interesting for $\ell = p$, especially when p divides [L : K]. In §8 we will present some non-trivial examples of $(A, L/K, Z_L)$ where Assumption 7.2 is satisfied for $\ell = p$ and III(A/L) is finite.

Condition (3) of Assumption 7.2 may look stronger than the assumption to ensure $SC_{Z_L,p} \in D^{perf}(\mathbb{Z}_pG)$ in Proposition 6.6(2), but the following lemma shows that these two conditions are equivalent under Assumption 7.2(2) for $\ell = p$.

- **Lemma 7.3.** (1) Assumption 7.2(2) for $\ell = p$ implies that A_L has semistable reduction at all places of L.
 - (2) For any torus T over a finite field k' of characteristic p, the order of T(k') is prime to p. In particular, if A_L has totally toric degeneration at all places in Z_L , then Assumption 7.2(2) is satisfied for $\ell = p$ if p does not divide $|A(L_w)/\mathcal{A}_L^{\circ}(O_w)|$ for any $w \in Z_L$.

Proof. Set $\mathcal{A}_{L,k_w}^{\circ} := \mathcal{A}_L^{\circ} \times_{X_K}$ Spec k_w , which is a semi-abelian variety if and only if the unipotent radical $\mathcal{R}_u(\mathcal{A}_{L,k_w}^{\circ})$ is trivial. Since any connected commutative unipotent algebraic group over a perfect field is a vector group (*cf.* [CGP15, Corollary B.2.7]), $\mathcal{R}_u(\mathcal{A}_{L,k_w}^{\circ})(k_w)$ is a non-trivial *p*-group whenever $\mathcal{R}_u(\mathcal{A}_{L,k_w}^{\circ})$ is non-trivial. This shows Claim (1).

To prove (2), recall that for any k'-torus T we have a short exact sequence of k'-tori

$$1 \to T' \to S \to T \to 1$$

where $S = \operatorname{Res}_{k''/k'} \mathbb{G}_m^d$ for some finite extension k''/k'. (This is a standard fact; see [HK21, pp 8–9] for the proof.) It now follows that T(k') is of prime-to-p order since it is a quotient of $S(k') = (k''^{\times})^d$ by surjectivity of the Lang isogeny. If A_L has totally toric degeneration at $w \in Z_L$, then we just showed that $p \nmid |\mathcal{R}_L^\circ(k_w)|$ since $\mathcal{R}_{Lk_w}^\circ$ is a torus. \Box

Let us now record the effect of Assumption 7.2 on the cohomology of $[SC_{Z_L,\ell}]_{\psi,\lambda}$.

Lemma 7.4. Suppose that the ℓ_0 -primary part of III(A/L) is finite for some ℓ_0 , and Assumption 7.2 is satisfied for ℓ . Then $SC_{Z_L,\ell}$ can be represented by a two-term complex $[P^0 \xrightarrow{d} P^1]$ of finitely generated projective $\mathbb{Z}_{\ell}G$ -modules concentrated in degrees [0, 1]. Furthermore, the following properties are valid for any G-representation ψ .

- (1) $\mathrm{H}^{0}([\mathrm{SC}_{Z_{L},\ell}]_{\psi,\lambda}) \cong [A^{t}(L)]_{\lambda}^{\psi}$, which is torsion-free.
- (2) $H^1([SC_{Z_L,\ell}]_{\psi,\lambda}) \cong [Sel_{\mathbb{Q}_\ell/\mathbb{Z}_\ell}(A/L)^{\vee}]_{\psi,\lambda}$, whose torsion part and maximal torsionfree quotient are respectively $\coprod_{\psi,\lambda}^{\vee}(A/L)$ and $[(A(L) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\vee}]_{\psi,\lambda}$; cf. (7.1c).

Proof. (Compare with the proof of Proposition 7.3(ii) in [BMC24].) By Proposition 6.3 and Assumption 7.2 for ℓ , we have $H^0(SC_{Z_L,\ell}) \cong A^t(L) \otimes \mathbb{Z}_{\ell}$, which is torsion-free, and $H^i(SC_{Z_L,\ell}) = 0$ for $i \neq 0, 1$. The $\mathbb{Z}_{\ell}G$ -perfectness (*cf.* Prop. 6.6) now implies that $SC_{Z_L,\ell}$ can be represented by a two-term perfect $\mathbb{Z}_{\ell}G$ -complex $[P^0 \xrightarrow{d} P^1]$.

As $T^*_{\psi,\lambda} \otimes_{\mathbb{Z}_{\ell}} P^i$ is also a cohomologically trivial $O_{E,\lambda}G$ -module, the norm map induces a natural $O_{E,\lambda}$ -linear isomorphism $N_G: [P^i]_{\psi,\lambda} \xrightarrow{\sim} [P^i]_{\lambda}^{\psi}$ for i = 0, 1. Therefore, we have

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the following commutative diagram with exact rows

(7.5)
$$[P^{0}]_{\psi,\lambda} \xrightarrow{[d]_{\psi,\lambda}} [P^{1}]_{\psi,\lambda} \longrightarrow [H^{1}]_{\psi,\lambda} \longrightarrow 0$$
$$N_{G} \downarrow \cong \qquad N_{G} \downarrow \cong$$
$$0 \longrightarrow [H^{0}]_{\lambda}^{\psi} \longrightarrow [P^{0}]_{\lambda}^{\psi} \xrightarrow{[d]_{\psi,\lambda}^{\psi}} [P^{1}]_{\lambda}^{\psi}$$

where $\mathrm{H}^{i} := \mathrm{H}^{i}(\mathrm{SC}_{Z_{L},\ell})$. Now the lemma follows, noting that $[(A(L) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})^{\vee}]_{\psi,\lambda} \cong ([A(L) \otimes \mathbb{Z}_{\ell}]_{\lambda}^{\check{\psi}})^{*}$ is torsion-free. \Box

Remark 7.6. For any $E_{\lambda}G$ -module V_{λ} and $\psi \in Ir(G)$, we have an isomorphism

(7.7)
$$(V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda})^G \xrightarrow{\cong} (V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda} \longrightarrow (V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda})_G$$

where the first map is the natural inclusion and the second the natural projection. Then the norm map $N_G : V_\lambda \to V_\lambda$ induces

$$(V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda})_G \xleftarrow{(7.7)} (V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda})^G$$

$$\xrightarrow{N_G} (V_{\psi,\lambda}^* \otimes_{E_{\lambda}} V_{\lambda})^G,$$

where the right diagonal map is multiplication by |G|.

Applying this observation to $V_{\lambda} = H^{i}(SC_{Z_{L},\ell}) \otimes_{\mathbb{Z}_{\ell}} E_{\lambda}$, we obtain the following commutative diagram

(7.8)
$$\begin{array}{c} \mathrm{H}^{i}([\mathrm{SC}_{Z_{L},\ell}]_{\psi,\lambda}) & \longleftrightarrow & \mathrm{H}^{i}([\mathrm{SC}_{Z_{L},\ell}]_{\psi,\lambda}) \otimes E_{\lambda} \xrightarrow{(7.7)} \mathrm{H}^{i}([\mathrm{SC}_{Z_{L},\ell}]_{\lambda}^{\psi}) \otimes E_{\lambda} \\ & \cong \bigvee N_{G} & \Longrightarrow & \bigvee N_{G} \\ & \mathrm{H}^{i}([\mathrm{SC}_{Z_{L},\ell}]_{\lambda}^{\psi}) & \longleftrightarrow & \mathrm{H}^{i}([\mathrm{SC}_{Z_{L},\ell}]_{\lambda}^{\psi}) \otimes E_{\lambda}. \end{array}$$

where the left vertical isomorphism is induced by the isomorphism

$$[\mathrm{SC}_{Z_L,\ell}]_{\psi,\lambda} \xrightarrow{N_G} [\mathrm{SC}_{Z_L,\ell}]_{\lambda}^{\psi}$$

given by (7.5). For i = 0 the left horizontal arrow in (7.8) coincides with the isomorphism

$$N_G: \operatorname{H}^0([\operatorname{SC}_{Z_L,\ell}]_{\psi,\lambda}) \xrightarrow{\cong} [A^t(L)]^{\psi}_{\lambda}$$

in Lemma 7.4(1). We use this observation in the computation of $\chi^{\text{BSD}}_{Z_L,\lambda}(A, \psi)$; *cf.* Proposition 7.10.

We now introduce the ψ -twisted regulator, following [BMC24, §7.2.2].

Definition 7.9. We maintain the setting of Lemma 7.4, and fix a place $\lambda \mid \ell$ of E. Given $\psi \in \operatorname{Ir}(G)$, choose $O_{E,\lambda}$ -bases $(e_i)_{i=1,\cdots,r_{\operatorname{alg}}(\psi)}$ of $[A(L)]^{\psi}_{\lambda}$, and $(\check{e}_j)_{j=1,\cdots,r_{\operatorname{alg}}(\psi)} [A^t(L)]^{\psi}_{\lambda}$, respectively. (We refer to §6.8 for the abuse of notation $[M]_{\psi,\lambda}$ and $[M]^{\psi}_{\lambda}$ when M is a finitely generated $\mathbb{Z}G$ -module.)

we define the ψ -twisted regulator to be

$$\operatorname{Reg}_{\lambda}^{\psi} \coloneqq \operatorname{det} \left(\langle e_i, \check{e}_j \rangle_{A/L} \right).$$

Note that $\operatorname{Reg}_{\lambda}^{\psi}$ is independent of the choice of $O_{E,\lambda}$ -bases only up to $O_{E,\lambda}^{\times}$ -multiple, so $v_{\lambda}(\operatorname{Reg}_{\lambda}^{\psi})$ is a well-defined integer.

Consider an E_{λ} -linear isomorphism

$$h^{\psi} \coloneqq \left[A^{t}(L)\right]_{\lambda}^{\psi} \otimes E_{\lambda} \xrightarrow{\sim} \left(\left[A(L)\right]_{\lambda}^{\psi}\right)^{*} \otimes E_{\lambda} \cong \left[A(L)^{*}\right]_{\psi,\lambda} \otimes E_{\lambda}$$

by sending \check{e}_j to the functional $\langle -, \check{e}_j \rangle_{A/L}$. If the ℓ -primary part of $\coprod(A/L)$ is finite, then we can interpret h^{ψ} as an E_{λ} -trivialisation of $[\operatorname{SC}_{Z_L,\ell}]_{\psi,\lambda}$ by Lemma 7.4.

Proposition 7.10. Suppose that the ℓ_0 -primary part of III(A/L) is finite for some ℓ_0 , and fix a place λ of E above a prime number ℓ that satisfies Assumption 7.2. Then for any $\psi \in Ir(G)$, we have

$$\chi_{Z_L,\lambda}^{\mathrm{BSD}}(A,\psi) = v_\lambda \big(\mathrm{Reg}_{\lambda}^{\psi} / |G|^{r_{\mathrm{alg}}(\psi)} \big) + \mathrm{length}_{\mathcal{O}_{E,\lambda}} \big(\mathrm{III}_{\psi,\lambda}^{\vee}(A/L) \big).$$

Proof. Recall that $\chi_{Z_L,\lambda}^{\text{BSD}}(A, \psi) = v_\lambda (\chi_{O_{E,\lambda},E_\lambda}([\text{SC}_{Z_L,\ell}]_{\psi,\lambda}, [h_\ell]_{\psi,\lambda}))$. So we proceed by making explicit $[\text{SC}_{Z_L,\ell}]_{\psi,\lambda}$ and $[h_\ell]_{\psi,\lambda}$.

By Lemma 7.4 we represent $SC_{Z_{L},\ell} \cong [P^0 \xrightarrow{d} P^1]$, and we have

$$[\mathrm{SC}_{Z_L,\ell}]_{\psi,\lambda} \cong \left[[P^0]_{\psi,\lambda} \xrightarrow{d_{\psi}} [P^1]_{\psi,\lambda} \right],$$

where $d_{\psi} = [d]_{\psi,\lambda}$. Write $H^i_{\psi} := H^i([SC_{Z_L,\ell}]_{\psi,\lambda})$, and set $H^1_{\psi,\text{tf}}$ to be the maximal torsion-free quotient of H^1_{ψ} . Choose a decomposition

$$[P^0]_{\psi,\lambda} = \mathbf{H}^0_{\psi} \oplus Q^0_{\psi} \quad \text{and} \quad [P^1]_{\psi,\lambda} = \mathbf{H}^1_{\mathrm{tf}} \oplus Q^1_{\psi},$$

so that d_{ψ} factorises as follows:

$$d_{\psi} \colon [P^0]_{\psi,\lambda} \longrightarrow Q^0_{\psi} \xrightarrow{d_{Q_{\psi}}} Q^1_{\psi} \longmapsto [P^1]_{\psi,\lambda},$$

where d_Q is an injective map with $\operatorname{coker}(d_{Q_{\psi}}) \cong \coprod_{\psi,\lambda}^{\vee}(A/L)$. Now, we can express $[h_{\ell}]_{\psi,\lambda}$ as follows

$$[h_{\ell}]_{\psi,\lambda} \colon \operatorname{H}^{0}_{\psi,E_{\lambda}} \oplus Q^{0}_{\psi,E_{\lambda}} \xrightarrow{(h_{\psi},d_{Q_{\psi}})} \operatorname{H}^{1}_{\psi,E_{\lambda}} \oplus Q^{1}_{\psi,E_{\lambda}}$$

for some E_{λ} -isomorphism \tilde{h}_{ψ} : $H^0_{\psi, E_{\lambda}} \xrightarrow{\sim} H^1_{\psi, E_{\lambda}}$, where the subscript E_{λ} stands for the scalar extension. Therefore, we have

$$\chi^{\text{BSD}}_{Z_L,\lambda}(A,\psi) = [\mathrm{H}^0_{\psi}, \tilde{h}_{\psi}, \mathrm{H}^1_{\psi,\text{tf}}] + [Q^0_{\psi}, d_{Q_{\psi}}, Q^1_{\psi}].$$

Recall that by Example 4.10 we have

$$v_{\lambda}([Q_{\psi}^{0}, d_{Q_{\psi}}, Q_{\psi}^{1}]) = \text{length}_{\mathcal{O}_{E,\lambda}} (\text{coker}(d_{Q_{\lambda}})) = \text{length}_{\mathcal{O}_{E,\lambda}} (\coprod_{\psi,\lambda}^{\vee}(A/L)),$$

so to prove the proposition it remains to compute $v_{\lambda}([H^0_{\psi}, \tilde{h}_{\psi}, H^1_{\psi,\text{tf}}])$. By Lemma 7.4 and Remark 7.6, we have the following commutative diagram of isomorphisms

$$\begin{bmatrix} A^{t}(L) \end{bmatrix}^{\psi}_{\lambda} \otimes E_{\lambda} \xleftarrow{(7.7)} H^{0}_{\psi, E_{\lambda}} \xrightarrow{h_{\psi}} H^{1}_{\psi, E_{\lambda}} \\ \downarrow & \downarrow \\ & \downarrow \\ & \downarrow \\ & [A^{t}(L)]^{\psi}_{\lambda} \otimes E_{\lambda} \xrightarrow{h_{\psi}} [A(L)^{*}]_{\psi, \lambda} \otimes E_{\lambda}.$$

Since (7.7) identifies H^0_{ψ} with $|G|^{-1} \cdot [A^t(L)]_{\psi,\lambda}$ in $[A^t(L)]_{\psi,\lambda} \otimes E_{\lambda}$, we have

$$\begin{split} \left[\mathbf{H}_{\psi}^{0}, \tilde{h}_{\psi}, \mathbf{H}_{\psi, \mathrm{tf}}^{1} \right] &= \left[\left[A^{t}(L) \right]_{\lambda}^{\psi}, |G|^{-1} \cdot h_{\psi}, \left[A(L)^{*} \right]_{\psi, \lambda} \right] \\ &= \left[O_{E, \lambda}, \left| G \right|^{-r_{\mathrm{alg}}(\psi)} \cdot \mathrm{Reg}_{\lambda}^{\psi}, O_{E, \lambda} \right], \end{split}$$

where the second equality uses the choice of $O_{E,\lambda}$ -bases as in Def 7.9.

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For any finite torsion $O_{E,\lambda}$ -module M_{λ} , we let $\text{Char}_{\lambda}(M_{\lambda})$ denote the characteristic ideal of M_{λ} . One can show that

(7.11)
$$\operatorname{Char}_{\lambda}(M_{\lambda}) = \mathfrak{p}_{\lambda}^{\operatorname{length}_{\mathcal{O}_{E,\lambda}}(M_{\lambda})},$$

where \mathfrak{p}_{λ} is the maximal ideal of $O_{E,\lambda}$.

Let us write $Z = Z_1 \sqcup Z_2$ where Z_2 is the reduced complement of U' defined in Proposition 5.1; i.e., $v \notin Z_2$ if and only if $\mathcal{A}_{X_L}^{\circ}$ and \mathcal{A}_L° are isomorphic at v. If A has semistable reduction at all places of L then Z_2 is the set of places of K where A has non-semistable reduction.

We are now ready to state our main result.

Theorem 7.12. Fix a place λ of E above a prime number ℓ , and suppose that the same assumption as in Corollary 6.15 is valid (that is, we assume that the ℓ_0 -primary part of III(A/L) is finite for some ℓ_0 , and if $\ell = p$ then we assume that L/K is weakly ramified everywhere and tamely ramified over Z_2). Then for any $\psi \in Ir(G)$, we have the following equality of fractional ideals:

(7.12a)
$$\mathscr{L}_{U}(A,\psi) \cdot \mathcal{O}_{E,\lambda} = \left(\frac{\operatorname{vol}_{Z_{1}}(A/K)}{\prod_{v \in Z_{2}} \left|\operatorname{Lie}(\mathscr{R}_{L})(k_{\tilde{v}})^{G_{\tilde{v}}}\right|}\right)^{\operatorname{deg}\psi} \cdot \operatorname{loc}_{Z_{L}}(A,\psi) \cdot \mathfrak{p}_{\lambda}^{\chi^{\operatorname{BSD}}_{Z_{L},\lambda}(A,\psi)}$$

where we choose $\tilde{v} \in \pi^{-1}(v)$ for each $v \in Z_2$. Here, $loc_{Z_L}(A, \psi), vol_{Z_1}(A/K) \in p^{\mathbb{Z}}$ are respectively defined by

(7.12b) $\log_p \left(\log_{Z_L}(A, \psi) \right) \coloneqq \operatorname{ram}_{\operatorname{Lie}(\mathcal{A}_L)(-Z_L)}(\psi) \text{ and }$

(7.12c)
$$\operatorname{vol}_{Z_1}(A/K) \coloneqq \mu \big(\operatorname{Lie}(A)(\mathbb{A}_K) / \operatorname{Lie}(A)(K) \big)^{-1} \cdot \prod_{v \in Z_1} \mu_v \big(\mathcal{A}^{\circ}(\mathfrak{m}_v) \big) \big)$$

with respect to the Haar measure μ_v on Lie(A)(K_v) and $\mu \coloneqq \prod \mu_v$ as in [KT03, §1.6, §1.7].

In particular, if Assumption 7.2 is valid for ℓ then we have

(7.12d)
$$\mathcal{L}_{U}(A,\psi) \cdot \mathcal{O}_{E,\lambda} = \left(\frac{\operatorname{vol}_{Z_{1}}(A/K)}{\prod_{v \in Z_{2}} \left|\operatorname{Lie}(\mathcal{A}_{L})(k_{\bar{v}})^{G_{\bar{v}}}\right|} \right)^{\operatorname{deg}\psi} \cdot \operatorname{loc}_{Z_{L}}(A,\psi) \cdot \frac{\operatorname{Reg}_{\lambda}^{\psi}}{|G|^{r_{\operatorname{alg}}(\psi)}} \cdot \operatorname{Char}_{\lambda}(\operatorname{III}_{\psi,\lambda}^{\vee}(A/L)).$$

Proof. By Corollary 6.15 we have $\mathscr{L}_U(A, \psi) \cdot \mathcal{O}_{E,\lambda} = \mathfrak{p}_{\lambda}^{\chi^{\text{BSD}}_{Z_L,\lambda}(A,\psi) - \chi^{\text{coh}}_{Z_L,\lambda}(A,\psi)}$. Since we have $\operatorname{vol}_{Z_1}(A/K) = \frac{|\mathrm{H}^0(X,\mathrm{Lie}\,\mathcal{R}(-Z_1))|}{|\mathrm{H}^1(X,\mathrm{Lie}\,\mathcal{R}(-Z_1))|}$ by [KT03, §3.7], it follows from Corollary 4.15 and Corollary 5.8(2) that

$$-\chi_{Z_L,\lambda}^{\mathrm{coh}}(A,\psi) = v_{\lambda} \left(\left(\frac{\mathrm{vol}_{Z_1}(A/K)}{\prod_{v \in Z_2} \left| \mathrm{Lie}(\mathcal{A}_L)(k_{\tilde{v}})^{G_{\tilde{v}}} \right|} \right)^{\deg \psi} \cdot \mathrm{loc}_{Z_L}(A,\psi) \right)$$

for any place λ of *E* over ℓ . If Assumption 7.2 is valid for ℓ then the formula (7.12d) immediately follows from the computation of $\chi_{Z_L,\lambda}^{\text{BSD}}(A, \psi)$ in Proposition 7.10.

If ℓ does not divide |G| then any $\mathbb{Z}_{\ell}G$ -module is cohomologically trivial so the functor $[-]_{\psi,\lambda}$ is exact for any place λ above ℓ . In particular, we have

$$\mathrm{H}^{i}\left([\mathrm{SC}_{Z_{L},\ell}]_{\psi,\lambda}\right) \cong [\mathrm{H}^{i}(\mathrm{SC}_{Z_{L},\ell})]_{\psi,\lambda} \quad \text{and} \quad \mathrm{III}_{\psi,\lambda}^{\vee}(A/L) \coloneqq [\mathrm{III}(A/L)^{\vee}]_{\psi,\lambda}$$

Therefore, we immediately obtain the following proposition even when Assumption 7.2 does not hold for ℓ .

Proposition 7.13. Suppose that the ℓ_0 -primary part of $\operatorname{III}(A/L)$ is finite for some ℓ_0 , and we fix a place λ of E over a prime ℓ not dividing |G|. Then for any $\psi \in \operatorname{Ir}(G)$, we have

$$(7.13a) \quad \chi_{Z_{L,\lambda}}^{\text{BSD}}(A,\psi) = v_{\lambda} \left(\text{Reg}_{\lambda}^{\psi} / |G|^{r_{\text{alg}}(\psi)} \right) + \text{length}_{\mathcal{O}_{E,\lambda}} \left((\text{III}_{\psi,\lambda}^{\vee}(A/L)) - \text{length}_{\mathcal{O}_{E,\lambda}} \left([A^{t}(L)_{\text{tors}}]_{\psi,\lambda} \right) + \sum_{v \in Z} \text{length}_{\mathcal{O}_{E,\lambda}} \left(\left[\bigoplus_{w \mid v} \mathcal{R}_{L}(k_{w})^{\vee} \right]_{\psi,\lambda} \right).$$

Furthermore, we have

$$(7.13b) \quad \mathscr{L}_{U}(A,\psi) \cdot \mathcal{O}_{E,\lambda} = \left(\frac{\operatorname{vol}_{Z_{1}}(A/K)}{\prod_{v \in Z_{2}} \left| \operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})^{G_{\tilde{v}}} \right|} \right)^{\operatorname{deg}\psi} \cdot \operatorname{loc}_{Z_{L}}(A,\psi) \cdot \frac{\operatorname{Reg}_{\lambda}^{\psi}}{|G|^{r_{\operatorname{alg}}(\psi)}} \\ \times \frac{\operatorname{Char}_{\lambda}(\operatorname{IIII}_{\psi,\lambda}^{\vee}(A/L)) \cdot \prod_{v \in Z} \operatorname{Char}_{\lambda}\left(\left[\bigoplus_{w \mid v} \mathcal{A}_{L}(k_{w})^{\vee} \right]_{\psi,\lambda} \right)}{\operatorname{Char}_{\lambda}([A(L)_{\operatorname{tors}}^{\vee}]_{\psi,\lambda}) \cdot \operatorname{Char}_{\lambda}([A^{t}(L)_{\operatorname{tors}}]_{\psi,\lambda})},$$

Proof. The formula (7.13a) is immediate from Proposition 6.3 by the exactness of $[-]_{\psi,\lambda}$, and (7.13b) follows from (7.12a) and (7.13a). Note that if p does not divide |G| then L/K is tame at all places so (7.12a) holds for any place $\lambda \mid p$ of E.

Remark 7.14. We can make $loc_{Z_L}(A, \psi)$ more explicit in some cases; *cf.* Remark 4.19. For example, if L/K is either a *p*-extension or unramified everywhere, then $loc_{Z_L}(A, \psi) = 1$ for any $\psi \in Ir(G)$. If L/K is cyclic and tame everywhere *A* has semistable reduction at all places of *K*, then one gets a simpler formula for $loc_{Z_L}(A, \psi)$. If *A* has semistable reduction at all places of *L* but admits non-semistable reduction at some place of *K* (so $Z_2 \neq \emptyset$), then we need to compute $r_{\tilde{v},i}$'s as in (5.7) for a preimage $\tilde{v} \in \pi^{-1}(v)$ of each $v \in Z_2$. In principle, $loc_{Z_L}(A, \psi)$ should be computable in any explicit examples.

Let us make a few remarks on the formulae in Theorem 7.12 and Proposition 7.13.

Remark 7.15. In the proof of Theorem 7.12 we used the interpretation of $\operatorname{vol}_{Z_1}(A/K)$ in terms of the Euler characteristic of $\operatorname{Lie}(\mathcal{A})(-Z_1)$, so we have

(7.15a)
$$\log_{|k|} (\operatorname{vol}_{Z_1}(A/K)) = \dim_k \chi_k (\operatorname{Lie} \mathcal{A}(-Z_1)) = \dim(A) \cdot (1 - \operatorname{gen}_K - \operatorname{deg}(Z_1)) + \operatorname{deg}(\operatorname{Lie} \mathcal{A}),$$

where gen_K is the genus of *X*. The same equality holds for $\text{vol}_Z(A/K)$ with *Z* in place of *Z*₁. If *A* is an elliptic curve, then we have deg(Lie \mathcal{A}) = $-\text{deg}(\Delta)/12$ where Δ is the global discriminant; *cf.* [Tan95, p 325, eq (9)].

Next, let us show

(7.15b)
$$\frac{\operatorname{vol}_{Z_1}(A/K)}{\prod_{v \in Z_2} \left|\operatorname{Lie}(\mathcal{A}_L)(k_{\tilde{v}})^{G_{\tilde{v}}}\right|} = \mu \left(\frac{\operatorname{Lie}(A)(\mathbb{A}_K)}{\operatorname{Lie}(A)(K)}\right)^{-1} \cdot \prod_{v \in Z} \mu_v \left(\mathcal{A}_L^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}}\right)$$

for the Haar measure as in [KT03, §1.6, §1.7]; in other words, this expression roughly measures the volume of $(\prod_{w \in Z_L} \mathscr{R}^{\circ}_L(\mathfrak{m}_w))^G$. In fact, observe that

(7.15c)
$$\frac{\operatorname{vol}_{Z_1}(A/K)}{\prod_{v \in Z_2} \left|\operatorname{Lie}(\mathcal{A}_L)(k_{\bar{v}})^{G_{\bar{v}}}\right|} = \operatorname{vol}_Z(A/K) \prod_{v \in Z_2} \cdot \frac{\left|\operatorname{Lie}(\mathcal{A})(k_v)\right|}{\left|\operatorname{Lie}(\mathcal{A}_L)(k_{\bar{v}})^{G_{\bar{v}}}\right|}.$$

Now, for any place v of K where L/K is tamely ramified at worst, Proposition 5.3 yields

(7.15d)
$$\frac{\left|\operatorname{Lie}(\mathcal{A})(k_{v})\right|}{\left|\operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})^{G_{\tilde{v}}}\right|} = \left|\left(\operatorname{Lie}(\mathrm{F}^{1}\mathcal{B}_{k_{v}})(k_{v})\right)^{G_{\tilde{v}}}\right| = \left|\left(\mathrm{F}^{1}\mathcal{B}_{k_{v}}(k_{v})\right)^{G_{\tilde{v}}}\right| = \frac{\left|\mathcal{A}(k_{v})\right|}{\left|\left(\mathcal{A}_{L}(k_{\tilde{v}})\right)^{G_{\tilde{v}}}\right|},$$

where the first and last equalities follow from the short exact sequences induced by (5.3a) on the Lie algebras and k_v -rational points respectively, and the second equality holds since

 $(F^1\mathcal{B}_{k_v})^{G_{\bar{v}}}$ is a vector group; *cf.* Proposition 5.3. Note that (7.15d) applies to any $v \in Z_2$ in the setting where the formula (7.12d) or (7.13b) can be applied. If $v \notin Z_2$ then the leftmost ratio in (7.15d) is equal to 1 by Proposition 5.1. From this observation together with (7.15c) and (7.15d), we get the following equality

$$(7.15e) \quad \operatorname{vol}_{Z}(A/K) \cdot \prod_{v \in Z} \left| \mathcal{A}(k_{v}) \right| = \frac{\operatorname{vol}_{Z_{1}}(A/K)}{\prod_{v \in Z_{2}} \left| \operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})^{G_{\tilde{v}}} \right|} \cdot \prod_{v \in Z_{1}} \left| \mathcal{A}(k_{v}) \right| \cdot \prod_{v \in Z_{2}} \left| \mathcal{A}_{L}(k_{\tilde{v}})^{G_{\tilde{v}}} \right|$$
$$= \frac{\operatorname{vol}_{Z_{1}}(A/K)}{\prod_{v \in Z_{2}} \left| \operatorname{Lie}(\mathcal{A}_{L})(k_{\tilde{v}})^{G_{\tilde{v}}} \right|} \cdot \prod_{v \in Z} \left| \frac{A(K_{v})}{\mathcal{A}_{L}^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}}} \right|;$$

To see the last equality note that $\mathcal{A}_{L}^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}} = \mathcal{A}(\mathfrak{m}_{v})$ for $v \in Z_{1}$, as $\mathcal{A}_{L,O_{\tilde{v}}}^{\circ}$ is the base change of \mathcal{A}_{O}° . Now the formula (7.15b) follows.

Remark 7.16. Let us compare the formula (7.12d) for $\mathscr{L}_U(A, \mathbf{1}_G) \cdot \mathbb{Z}_p$ (with $\mathbf{1}_G$ denoting the trivial character of *G*) with the *p*-part of the classical BSD formula [KT03, (1.8.1)] when *p* satisfies Assumption 7.2 for $(A, L/K, Z_L)$. Since for any $P \in A(K)$ and $P \in A^t(K)$ we have

(7.16a)
$$\langle P, \dot{P} \rangle_{A/L} = |G| \cdot \langle P, \dot{P} \rangle_{A/K}$$

the discriminant of $\langle , \rangle_{A/K}$ coincides with $\operatorname{Reg}_p^1/|G|^{r_{\operatorname{alg}}(1)}$ up to \mathbb{Z}_p^{\times} -multiple.

Recall that $[SC_{Z_L,p}(A, L/K)]_{1_G,p} \cong (SC_{Z_L,p}(A, L/K))_G$, and we have the following distinguished triangle

(7.16b)
$$\operatorname{SC}_{Z,p}(A, K/K) \to \left(\operatorname{SC}_{Z_L,p}(A, L/K)\right)_G \to \bigoplus_{v \in Z} \left(\frac{\mathscr{R}^{\circ}_L(\mathfrak{m}_{\bar{v}})^{G_{\bar{v}}}}{\mathscr{R}^{\circ}(\mathfrak{m}_v)}\right)^{\vee} [-1] \to +1$$
,

where we choose $\tilde{v} \in \pi^{-1}(v)$ for each $v \in Z$. In fact, by [KT03, Lemma 6.1] and its proof $((SC_{ZL,p}(A, L/K))_G)^{\vee}[-2]$ is the mapping fibre of

$$\mathrm{Rr}_{\mathrm{fl}}(U,\mathcal{A}[p^{\infty}]) \oplus \left(\bigoplus_{v \in Z} \mathcal{A}_{L}^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}} \overset{\mathrm{L}}{\otimes} \mathbb{Q}_{p} / \mathbb{Z}_{p} \right) [-1] \longrightarrow \bigoplus_{v \in Z} \mathrm{Rr}_{\mathrm{fl}}(\mathrm{Spec}\, K_{v}, \mathcal{A}[p^{\infty}]) \ ,$$

so by comparing it with the *p*-primary part of (6.1) for L = K we obtain (7.16b).

Next we turn to the index $[\mathscr{R}_{L}^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}} : \mathscr{A}^{\circ}(\mathfrak{m}_{v})]$ for each $v \in Z$. If $v \in Z_{1}$ then this index is 1 as explained below (7.15e). Since L/K is tame at all places $v \in Z_{2}$ in the setting where the formula (7.12d) or (7.13b) can be applied, we have $\frac{\mathscr{R}_{L}^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}}}{\mathscr{A}^{\circ}(\mathfrak{m}_{v})} \cong \mathrm{F}^{1}\mathscr{B}_{k_{v}}(k_{v})^{G_{\tilde{v}}}$ for $v \in Z_{2}$ by the short exact sequence

$$0 \longrightarrow \mathrm{F}^{1}\mathcal{B}_{k_{v}}(k_{v})^{G_{\tilde{v}}} \longrightarrow \mathcal{A}(k_{v}) \longrightarrow \mathcal{A}_{L}(k_{\tilde{v}})^{G_{\tilde{v}}} \longrightarrow 0$$

induced by k_v -points of (5.3a). Therefore we get

(7.16c)
$$\prod_{v \in Z} \left[\mathcal{A}_L^{\circ}(\mathfrak{m}_{\tilde{v}})^{G_{\tilde{v}}} : \mathcal{A}^{\circ}(\mathfrak{m}_v) \right] = \prod_{v \in Z_2} \left[\mathcal{A}(k_v) : \mathcal{A}_L(k_{\tilde{v}})^{G_{\tilde{v}}} \right]$$

From this together with (7.16b) and (7.15e), the formula (7.12d) for $\mathscr{L}_U(A, \mathbf{1}_G) \cdot \mathbb{Z}_p$ can be reduced to the *p*-part of the the classical BSD formula [KT03, (1.8.1)].

Remark 7.17. This remark is a continuation of Remark 6.16. Even if L/K is *not* weakly ramified at some place, one can still obtain a formula for $v_{\lambda}(\mathscr{L}_U(A, \psi))$ at $\lambda \mid p$ analogous to Corollary 6.15, at the cost of replacing $(\mathscr{R}_L^{\circ}(\mathfrak{m}_w))_{w \in Z_L}$ with some (usually inexplicit) family G_w -stable open compact subgroups $V_L := (V_w)_{w \in Z_L}$ where each V_w as "cohomologically trivial" G_w -action. But then, it would be quite unlikely that $\widehat{SC}_{V_L} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p$, introduced in Remark 6.7, can be represented by a perfect 2-term complex. Indeed, at any place $w \in Z_L$ where Lie $\mathscr{R}_L(\mathfrak{m}_w)$ is not cohomologically trivial for G_w (with $v = \pi(w)$), we should choose V_w to be a proper subgroup of $\mathcal{A}_L^{\circ}(\mathfrak{m}_w)$, and $\mathcal{A}_L^{\circ}(\mathfrak{m}_w)/V_w$ is *not* cohomologically trivial as a $\mathbb{Z}_p[G_w]$ -module; *cf.* [BKK, Lemma 3.4, Proposition 3.7]. In particular, it is difficult to control

$$\rho_{\lambda}^{\psi} \big(\chi_p^G (\widehat{\mathrm{SC}}_{V_L} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_p, h_p) \big),$$

as $A(L_w)/V_w$ is neither cohomologically trivial for G_w nor prime to p. One should also note that the formula for $v_\lambda(\mathscr{L}_U(A, \psi))$ also involves the *G*-equivariant Euler characteristic of some (usually inexplicit) proper *G*-stable subbundle of Lie $(\mathcal{A}_L)(-Z_L)$.

8. Examples

In Theorem 7.12 we computed the ℓ -part of the normalised leading term $\mathcal{L}_U(A, \psi)$ under Assumption 7.2, which was imposed to simplify the homological algebra especially when ℓ divides |G|. The main novelty lies in obtaining the *p*-part of the normalised leading term when *p* divides |G|. We also have to assume the finiteness of III(*A*/*L*), which is still wide open in the general setting.

In this section, we present examples of $(A, L/K, Z_L)$ that satisfy Assumption 7.2 for $\ell = p$. For all but the last example, III(A/L) is known to be finite so Theorem 7.12 can be applied unconditionally.

Example 8.1. Suppose that L/K is weakly ramified everywhere, and A is a *constant* abelian variety over K; that is, there is an abelian variety A_0 over a finite subfield k_0 of K such that $A = A_0 \times_{\text{Spec } k_0} \text{Spec } K$. Then, III(A/L) is finite by [Mil68] and Assumption 7.2(3) is automatic for $\ell = p$. Since the torsion points of A(L) and $A^t(L)$ are defined over a finite subfield of L, to check Assumption 7.2 for $\ell = p$ it suffices to show that there is no place $w \in Z_L$ where k_w contains the field of definition of any non-trivial point in $A_0[p](\bar{k}_0)$. In particular, Assumption 7.2 holds for $\ell = p$ (with any Z_L) if A is a constant supersingular abelian variety.

Let us now focus on the case where *A* is a non-constant elliptic curve over *K*. Observe that Assumption 7.2 can be check locally at places $w \in Z_L$ except Assumption 7.2(1), which is on the ℓ -torsion of the Mordell–Weil groups of *A* and A^t . Let us give a convenient sufficient condition for Assumption 7.2(1) for $\ell = p$.

Lemma 8.2. Let A be an ordinary elliptic curve over L. If there is a non-trivial p-torsion point of A defined over a finite separable extension of L, then A can be defined over L^p so the *j*-invariant of A lies in L^p . In particular, if the *j*-invariant of A does not lie in L^p then A(L) has no non-trivial p-torsion.

Proof. If there is a non-trivial *p*-torsion point of *A* defined over a separable extension of *L*, then one can split the connected-étale sequence $0 \to A[p]^{\circ} \to A[p] \to A[p]^{\text{ét}} \to 0$, which in turn enables one to factorise $[p]: A \to A$ as

$$A \xrightarrow{\rho^{\operatorname{et}}} B \xrightarrow{\sigma_B} A$$
,

where $\rho^{\text{ét}}$ is a degree-*p* étale isogeny and σ_B is a degree-*p* purely inseparable isogeny. Therefore, σ_B can be identified with the Frobenius isogeny $B \to B^{(p)} \cong A$, so *A* can be defined over L^p .

Remark 8.3. Indeed, the converse of Lemma 8.2 also holds since the connected-étale sequence for A[p] splits after the Frobenius pullback. We do not need this property.

Example 8.4. Let $L = \mathbb{F}_q(t)$ be a rational function field of characteristic p > 3, and let A be an *Ulmer elliptic curve*; i.e., the elliptic curve over L given by the following equation:

(8.4a)
$$y^2 + xy = x^3 - t^a$$

for some *d* coprime to *p*. This elliptic curve has been studied by Ulmer [Ulm02]; namely, he showed that III(A/L) is finite [Ulm02, Proposition 6.4] and computed the rank of A(L) [Ulm02, Theorems 1.5, 9.2].

Since *A* is defined over $\mathbb{F}_p(t^d)$, we can choose an intermediate extension $K = \mathbb{F}_{q'}(t^{d'})$ so that L/K is Galois. (For example, we choose q' and q large enough so that e := d/d' divides q' - 1.) Let w_0 and w_∞ respectively denote the places of *L* corresponding to t = 0 and $t = \infty$, and write $v_0 = \pi(w_0)$ and $v_\infty = \pi(w_\infty)$. Then L/K is unramified away from $\{v_0, v_\infty\}$, and admits tame ramification at worst. We now show that Assumption 7.2 for $\ell = p$ is satisfied for *many* Ulmer elliptic curves *A* and L/K with "minimal" choice of Z_L .

Let us first recall various invariants and local properties of A/L, following [Ulm02, §2]. The discriminant of the model (8.4a) is

$$(8.4b) \qquad \qquad \Delta \coloneqq t^d (1 - 2^4 3^3 t^d)$$

and at each place dividing Δ the reduction of *A* is semistable with prime-to-*p* local Tamagawa number (as p > 3). Lastly, *A* has good reduction at w_{∞} if 6 | *d*, and additive reduction at w_{∞} otherwise.

Since the *j*-invariant of *A* is $1/\Delta \notin L^p$, Assumption 7.2(1) is satisfied for $\ell = p$ by Lemma 8.2. From now on, assume $d \mid 6$ so that *A* has semistable reduction at all places of *L*, and hence Assumption 7.2(3) is satisfied. Let Z_L be the union of the zeroes of Δ and w_{∞} , which is the *minimal* choice if $d' \neq d$. Since the local Tamagawa number at each place of *L* is prime to *p*, to verify Assumption 7.2(2) for $\ell = p$ it remains to show that $\mathcal{A}_L(k_{w_{\infty}})$ is *p*-torsion free (*cf.* Lemma 7.3). Indeed, the fibre of \mathcal{A}_L at w_{∞} is given by the following Weierstraß equation

(8.4c)
$$y^2 = x^3 - 1$$
,

which is the mod *p* reduction of an elliptic curve over \mathbb{Q} with complex multiplication by $\mathbb{Q}(\sqrt{-3})$. In particular, $\mathcal{A}_{L,k_{w_{\infty}}}$ is supersingular if $p \equiv 2 \mod 3$ (and p > 3), in which case $\mathcal{A}_L(\overline{k}_{w_{\infty}})$ is trivial.

To summarise, suppose that p > 3 and $p \equiv 2 \mod 3$. Let A be an elliptic curve defined by the equation (8.4a) with $6 \mid d$ and $p \nmid d$. We choose a finite extension $\mathbb{F}_q/\mathbb{F}_{q'}$ of finite fields of characteristic p and a positive integer e dividing gcd(d, q' - 1), and set $L := \mathbb{F}_q(t)$ and $K := \mathbb{F}_{q'}(t^{\frac{d}{e}})$. (We can arrange so that p divides [L : K] by manipulating $\mathbb{F}_q/\mathbb{F}_{q'}$.) Let Z_L be the disjoint union of the zeroes of $\Delta := t^d(1-2^43^3t^d)$ and w_{∞} . For such $(A, L/K, Z_L)$, Assumption 7.2 holds for $\ell = p$. We also note that III(A/K) is finite, and we can arrange so that A(L) has large rank by the work of Ulmer [Ulm02].

Remark 8.5. Let \mathcal{A}_{∞} be the elliptic curve over $\mathbb{Z}[1/6]$ defined by the equation (8.4c). We want to show that there is no non-trivial *p*-torsion in $\mathcal{A}_{\infty}(\mathbb{F}_p)$ for any $p \ge 5$. If $p \equiv 2 \mod 3$, then we already observed in Example 8.4 that \mathcal{A}_{∞} has supersingular good reduction so the assertion is obvious. So we may assume that $p \equiv 1 \mod 3$, in which case \mathcal{A}_{∞} has good *ordinary* reduction at *p*. Then there exists $\omega \in \mathbb{F}_p^{\times} \setminus \{1\}$ such that $\omega^3 = 1$. Now over \mathbb{F}_p , we can rewrite (8.4c) as follows

$$y^2 = (x-1)(x-\omega)(x-\omega^2),$$

so $\mathcal{A}_{\infty}(\mathbb{F}_p)[2]$ has order 4. Now suppose by contrary that $\mathcal{A}_{\infty}(\mathbb{F}_p)[p]$ is non-trivial, so 4p divides $|\mathcal{A}_{\infty}(\mathbb{F}_p)|$. Then we have

$$1+p-|\mathcal{A}_{\infty}(\mathbb{F}_p)| \leq 1-3p,$$

which clearly violates the Weil bound. Therefore, $\mathcal{A}_{\infty}(\mathbb{F}_p)[p]$ should be trivial.

Let *p* be a prime satisfying $p \equiv 1 \mod 3$. Then we just showed that the field of definition of the *p*-torsion points of $\mathcal{A}_{\infty,\mathbb{F}_p}$ is a non-trivial extension of \mathbb{F}_p . Returning to the setting of Example 8.4, choose a positive integer *d* such that $p \nmid d$ and $6 \mid d$. Set $L = \mathbb{F}_q(t)$ for some finite field $\mathbb{F}_q/\mathbb{F}_p$ and consider *A* and L/K as in Example 8.4. Then $(A, L/K, Z_L)$ as in Example 8.4 satisfies Assumption 7.2 for $\ell = p$ provided that $[k_0 : \mathbb{F}_p]$ does not divide $[\mathbb{F}_q : \mathbb{F}_p]$. Since $[k_0 : \mathbb{F}_p]$ divides p - 1, one can still produce examples where p divides [L : K] and $p \equiv 1 \mod 3$.

Example 8.6. Let us now give an example of a non-constant elliptic curve A/K and a finite Galois extension L/K with *wild* ramification where Assumption 7.2 holds for $\ell = p$. Let $K = \mathbb{F}_q(t)$ be a rational function field of characteristic of characteristic p, and consider an Artin–Schreier extension L := K(u) with $u^p - u = t$. Then L/K is a cyclic extension of degree p ramified only at the place v_∞ corresponding to $t = \infty$. For the unique place w_∞ above v_∞ one can check that $L_{w_\infty}/K_{v_\infty}$ is weakly wildly ramified.

Let *A* be an elliptic curve defined by the equation (8.4a) with $6 \mid d$ and $p \nmid d$, and suppose that p > 3 with $p \equiv 2 \mod 3$. Let Z_L be the disjoint union of the bad reduction places for *A* and $\{w_\infty\}$. Let us now verify Assumption 7.2 for $(A, L/K, Z_L)$ and $\ell = p$, using the properties of A/K obtained in Example 8.4.

Since *A* has semistable reduction at all places of *K* and *L/K* is weakly ramified everywhere, Assumption 7.2(3) holds. Assumption 7.2(1) follows from Lemma 8.2 since the *j*-invariant of *A/K* does not lie in K^p . Since *A* has good supersingular reduction at v_{∞} , $\mathcal{A}_L(k_{w_{\infty}})$ is trivial. And since the fibre of \mathcal{A} at each bad reduction place for *A/K* is either Néron *d*-gon (at t = 0) or of type I_1 (away of t = 0), each of its components is rationally defined and the local Tamagawa number remains prime to *p* under any unramified extension; *cf*. [Ulm02, §2.2]. By Lemma 7.3, it follows that $\mathcal{A}_L(k_w)$ has no non-trivial *p*-torsion for any $w \in Z_L$, verifying Assumption 7.2(2). We are not able to check if III(*A/L*) is finite.

Acknowledgement. We thank David Burns and Daniel Macias Castillo for explaining their result [BMC24] to us and clarified on the ψ -twisted regulators. The first named author (WK) was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT). (No. RS-2023-00208018). The third named author (FT) was supported by the Japanese Society for Promotion of Sciences (JSPS) (Research grant C/21K03186).

References

- [BF01] D. Burns and M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients, Doc. Math. 6 (2001), 501–570. MR 1884523
- [BKK] David Burns, Mahesh Kakde, and Wansu Kim, On a refinement of the Birch and Swinnerton-Dyer Conjecture in positive characteristic, Preprint.
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, Néron models, Springer-Verlag, Berlin, 1990. MR 91i:14034
- [BMC24] David Burns and Daniel Macias Castillo, On refined conjectures of Birch and Swinnerton-Dyer type for Hasse-Weil-Artin L-series, Mem. Amer. Math. Soc. 297 (2024), no. 1482, v+156. MR 4756396
- [CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad, Pseudo-reductive groups, second ed., New Mathematical Monographs, vol. 26, Cambridge University Press, Cambridge, 2015. MR 3362817
- [Chi94] Ted Chinburg, Galois structure of de Rham cohomology of tame covers of schemes, Ann. of Math. (2) 139 (1994), no. 2, 443–490. MR 1274097
- [CR81] Charles W. Curtis and Irving Reiner, Methods of representation theory. Vol. I, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1981, With applications to finite groups and orders, A Wiley-Interscience Publication. MR 632548
- [CR87] _____, Methods of representation theory. Vol. II, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1987, With applications to finite groups and orders, A Wiley-Interscience Publication. MR 892316
- [CR06] _____, Representation theory of finite groups and associative algebras, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1962 original. MR 2215618
- [Edi92] Bas Edixhoven, Néron models and tame ramification, Compositio Math. 81 (1992), no. 3, 291–306. MR 1149171
- [FWK09] Helena Fischbacher-Weitz and Bernhard Köck, Equivariant Riemann-Roch theorems for curves over perfect fields, Manuscripta Math. 128 (2009), no. 1, 89–105. MR 2470189
- [Har77] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

REFINED BSD CONJECTURE

- [HK21] Paul Hamacher and Wansu Kim, On G-isoshtukas over function fields, Selecta Math. (N.S.) 27 (2021), no. 4, Paper No. 75, 34. MR 4292785
- [Köc04] Bernhard Köck, Galois structure of Zariski cohomology for weakly ramified covers of curves, Amer. J. Math. 126 (2004), no. 5, 1085–1107. MR 2089083
- [KT03] Kazuya Kato and Fabien Trihan, On the conjectures of Birch and Swinnerton-Dyer in characteristic p > 0, Invent. Math. **153** (2003), no. 3, 537–592. MR 2000469
- [Mil68] J. S. Milne, The Tate-šafarevič group of a constant abelian variety, Invent. Math. 6 (1968), 91–105. MR 0244264
- [Nak84] Shoichi Nakajima, On Galois module structure of the cohomology groups of an algebraic variety, Invent. Math. 75 (1984), no. 1, 1–8. MR 728135
- [Nak86] _____, Galois module structure of cohomology groups for tamely ramified coverings of algebraic varieties, J. Number Theory 22 (1986), no. 1, 115–123. MR 821138
- [Nak87] _____, p-ranks and automorphism groups of algebraic curves, Trans. Amer. Math. Soc. 303 (1987), no. 2, 595–607. MR 902787
- [Sch82] Peter Schneider, Zur Vermutung von Birch und Swinnerton-Dyer über globalen Funktionenkörpern, Math. Ann. 260 (1982), no. 4, 495–510. MR 670197
- [Ser77] Jean-Pierre Serre, Linear representations of finite groups, french ed., Graduate Texts in Mathematics, vol. Vol. 42, Springer-Verlag, New York-Heidelberg, 1977. MR 450380
- [Ser79] _____, Local fields, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg.
- [Swa60] Richard G. Swan, Induced representations and projective modules, Ann. of Math. (2) 71 (1960), 552–578. MR 138688
- [Swa68] _____, *Algebraic K-theory*, Lecture Notes in Mathematics, vol. No. 76, Springer-Verlag, Berlin-New York, 1968. MR 245634
- [Tan95] Ki-Seng Tan, Refined theorems of the Birch and Swinnerton-Dyer type, Ann. Inst. Fourier (Grenoble) 45 (1995), no. 2, 317–374. MR 1343554
- [Tat68] John Tate, On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 189– 214. MR 3202555
- [Ulm02] Douglas Ulmer, Elliptic curves with large rank over function fields, Ann. of Math. (2) **155** (2002), no. 1, 295–315. MR 1888802

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