# ON HILBERT SCHEME OF COMPLETE INTERSECTION ON THE BIPROJECTIVE

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ABSTRACT. The goal of this paper is to construct the Hilbert scheme of complete intersections in the biprojective space  $X = \mathbb{P}^m \times \mathbb{P}^n$  and for this, we define a partial order on the bidegrees of the bihomogeneous forms. As a consequence of this construction, we computer explicitly the Hilbert scheme for curves of genus 7 and 8 listed in [1] and [8] that are complete intersections. Finally, we construct the coarse moduli space of complete intersections in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

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## 1. INTRODUCTION

We recall that a variety  $Y \subset \mathbb{P}^n$  is a complete intersection in  $\mathbb{P}^n$  if Y generated by the homogeneous polynomials  $F_1, \dots, F_c$  and  $0 < \dim Y = n - c$ . We can generalize this definition in the following form,  $Y \subset X$  is a complete intersection in X if it is the intersection of  $c = \operatorname{codim}(Y, X)$  hypersurfaces of X, where X is a smooth arithmetically Cohen–Macaulay projective variety. Our interest is in complete intersection in biprojective spaces  $X = \mathbb{P}^m \times \mathbb{P}^n$ . The motivation is due to the remarkable works [1] and [8] of Mukai whose principal results we write in Example 10.

Our first goal is to understand the Hilbert scheme  $\mathcal{H}$ , of complete intersection in X, and for this, we will apply the tools of [6] where the Hilbert space is a tower of Grassmannians. Unlike projective space, on bi-projective space X we cannot define a total order on the bidegrees of the bihomogeneous forms but rather a partial order and so we reproduce the theorems of standard complete intersection in X. Furthermore, we computer the dimension of the Hilbert scheme of complete intersections listed by Mukai in [1] and [8]

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The second section of this paper will be dedicated to constructing the scheme X. We will review bigraded rings and modules as well as the definition of bihomogeneous elements and ideals. We will provide a construction of the scheme  $\operatorname{Proj}^2 S$ , where S is a bigraded ring. We will give a topology and a structural sheaf  $\mathcal{O}$  for  $\operatorname{Proj}^2 S$ .

The third section is devoted to studying the cohomology of complete intersections. The principal difference between the standard and biprojective cases X is that not every complete intersection of X is arithmetically Cohen-Macaulay (ACM). We say that a set of bi-homogeneous polynomials  $F_1, \dots, F_c$  is a regular sequence if any scheme generated by any subset  $F_{i_1}, \dots, F_{i_{\alpha}}$  is a complete intersection and ACM. We assume that the complete intersection  $Y \subset X$  is spanned by a regular sequence, when there is no confusion we just say Y is ACM.

Due to the difficulty of establishing an order on the bidegrees of the bihomogeneous, in Section 4 we restrict to the case where Y is ACM. Finally, we will discuss the Hilbert scheme of complete intersection in X. If it has codimension 1, then we can take any complete intersection and for codimension greater than 1 we will have to assume that the complete intersections are ACM.

To conclude we will talk about the coarse moduli of smooth complete intersection ACM curves on  $\mathbb{P}^m \times \mathbb{P}^n$ .

## 2. The Biprojective Space $\mathbb{P}^m \times \mathbb{P}^n$

#### 2.1. Bigraded Rings and Modules.

A bigraded ring is a ring S endowed with a direct sum decomposition

$$S = \bigoplus_{(i,j) \in \mathbb{N}^2} S_{i,j}$$

such that:

(1)  $S_{i,j}$  are additive subgroups of S;

(2)  $S_{i_1,j_1}S_{i_2,j_2} \subset S_{i_1+i_2,j_1+j_2}$  for all  $(i_1, j_1), (i_2, j_2) \in \mathbb{N}^2$ ; (3) S is a finitely generated  $S_{0,0}$ -algebra by elements of  $S_{0,1}$  and  $S_{1,0}$ .

A bigraded S-module is a S-module M endowed with a decomposition of the form M = $\bigoplus_{(i,j)\in\mathbb{N}^2} M_{i,j}$  such that  $S_{i_1,j_1}M_{i_2,j_2} \subset M_{i_1+i_2,j_1+j_2}$  for all  $(i_1,j_1), (i_2,j_2) \in \mathbb{N}^2$ . We call  $M_{i,j}$  the homogeneous component of M of bidegree (i,j). An element  $u \in M$  is bihomogeneous of bidegree (i, j) if  $u \in M_{i,j}$ . The bidegree of u is then denoted by deg u.

Let  $R := k[x_0, \dots, x_m, y_0, \dots, y_n]$  be the polynomial ring with coefficients in k. A monomial  $G = x_0^{a_0} \cdots x_m^{a_m} y_0^{b_0} \cdots y_n^{b_n} \in R$  has bidegree  $(\sum a_i, \sum b_i)$ . Let  $R_{i,j}$  be the *r*-dimensional vector space over *k* spanned by all monomial of bidegree (i, j). Let  $n_{i,j}$  be  $r = \dim R_{i,j} = \binom{m+i}{m} \binom{n+j}{n}$ , and thus *R* is a bigraded ring. A polynomial  $F \in R$  is bihomogeneous of bidegree  $(d_1, d_2)$  if *F* is a *k*-linear combination of monomials of bidegree  $(d_1, d_2)$ . We also say that F is a form of bi-degree  $(d_1, d_2)$ .

If  $F \in R_{d_1,d_2}$  and  $\lambda_1, \lambda_2 \in k$  then we have

$$F(\lambda_1 u, \lambda_2 v) = \lambda_1^{d_1} \lambda_2^{d_2} F(u, v), \tag{1}$$

for every  $(u, v) \in \mathbb{A}^{m+1} \times \mathbb{A}^{n+1}$  and conversely, it is easy to see that any polynomial of R satisfying the condition of the equation (1) is bihomogeneous of bidegree  $(d_1, d_2)$ .

Let  $I = (F_1, \dots, F_C) \subset R$  be an ideal. If every  $F_i$  is bihomogeneous, then I is considered a bihomogeneous ideal. For any ideal  $J \subset R$ , the (i, j) bihomogeneous component of J is the ideal generated by all the bihomogeneous elements of bidegree (i, j) of J, i.e

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 $J_{i,j} = J \cap R_{i,j}$ . If J is bihomogeneous, then  $J = \bigoplus_{(i,j) \in \mathbb{N}^2} J_{i,j}$  and conversely, every ideal that is the direct sum of its bihomogeneous components is also bihomogeneous.

**Example 1.** The ideals  $R_{1,0} = (x_0, \dots, x_m)$  and  $R_{0,1} = (y_0, \dots, y_n)$  are bihomogeneous, so called *irrelevant ideals*. The ideal  $I = (x_0x_1 + y_0^2)$  is not bihomogeneous, but homogeneous if R has the standard grading.

When I is a bihomogeneous ideal of R the quotient ring R/I also inherits a bigraded ring structure with  $(R/I)_{i,j} = R_{i,j}/I_{i,j}$ . In this manner R/I and I are bigraded Rmodules. For  $(a, b) \in \mathbb{N}^2$ , let us denote R(-a, -b) the polynomial ring R with a shifted grading, where  $R(-a, -b)_{i,j} = R_{i-a,j-b}$ .

**Example 2.** For a, b positive integers the  $\Delta = \{(na, nb) | n \in \mathbb{N}\}$  is called (a, b)-diagonal of  $\mathbb{N}^2$ . We may so define the diagonal of R along  $\Delta$  as the graded ring  $R_{\Delta} = \bigoplus_{n \in \mathbb{N}} R_{na,nb}$  and thus  $R_{\Delta}$  is a R-module. Analogously, we define  $M_{\Delta}$  for every R-module M.

2.2. **Biprojective Space.** When we talk about the projective variety  $X = \mathbb{P}^m \times \mathbb{P}^n$ , we always talk in terms of the Segree embedding, some literature refers to X as the Segree variety. In this session we will present a schematic construction of X and for a detailed construction see [7, Chapter-1]

Let S be a noetherian bigraded finitely generated ring over  $S_{0,0}$  by bihomogeneous elements  $x_0, \dots, x_m, y_0, \dots, y_n$ , such that deg  $x_i = (1,0)$  and deg  $y_j = (0,1)$ . Let us denote  $S_+ = (x_0, \dots, x_m)(y_0, \dots, y_n)$ , a prime ideal  $P \subset S$  is said relevant if P does not contain  $S_+$ . Then we define the set  $\operatorname{Proj}^2(S)$  to be the set of all relevant bihomogeneous prime ideals P.

For  $I \,\subset S$  a bihomogeneous ideal we define  $V_+(I) := \{P \in \operatorname{Proj}^2(S) \mid I \subset P\}$  and we see that if  $J = (\sqrt{I} : (x_0, \cdots, x_m)) + (\sqrt{I} : (y_0, \cdots, y_n))$ , then  $V_+(I) = V_+(J)$ , in particular  $V_+(I) = \emptyset$  if and only if every prime ideal  $P \subset S$  containing I also contains  $S_+$ , i.e.  $S_+ \subset \sqrt{I}$ . We have a topology on  $\operatorname{Proj}^2(S)$  where the closed subsets are the subsets of the form  $V_+(I)$ . We construct the structure sheaf  $\mathcal{O}$  on X in a manner equivalent to the structural sheaf of the projective space, but we take relevant prime homogeneous ideals making  $\operatorname{Proj}^2(S)$  a scheme. We call  $\operatorname{Proj}^2(S)$  the biprojective scheme associated to S. For  $R = k[x_0, \cdots, x_m, y_0, \cdots, y_n]$  we have  $\operatorname{Proj}^2(R) = \mathbb{P}^m_k \times \mathbb{P}^n_k = X$  and if  $\Delta$  is the (a, b)-diagonal, then the sheaf of ideals  $\mathcal{L} = (R_{a,b})\mathcal{O}_X$  defines an isomorphism between X and  $X_{\Delta} = \operatorname{Proj}(R_{\Delta})$ .

Classically the (1, 1)-diagonal corresponds to the Segree embedding of X in  $\mathbb{P}^N$ , where N = (m + 1)(n + 1) - 1, we can see that the topology of X is induced by the standard Zariski topology of  $\mathbb{P}^N$ . The homogeneous coordinate ring of its image, via the Segre embedding, is  $R_{\Delta}$ .

On the other hand, we can make the classical construction of the Zarisk topology on X by repeating the construction of the Zarisk topology of the projective space by considering bihomogeneous ideals. We finish this section with the Theorem which can be found in [5, Theorem 1.8.1].

**Theorem 3** (Biprojective Nullstellensatz). A bihomogeneous ideal  $I \subset R$  is the vanishing ideal of V(I) if and only if I is radical and saturated concerning each "irrelevant" ideal. In other words  $I(V(I)) = (\sqrt{I} : (x_0, \dots, x_m)) + (\sqrt{I} : (y_0, \dots, y_n))$ 

**Remark 4.** From Theorem 3 we see that  $V(I) = \emptyset$ , if and only if,  $(x_0, \dots, x_m) \in \sqrt{I}$ or  $(y_0, \dots, y_n) \in \sqrt{I}$  or  $(x_0, \dots, x_m)(y_0, \dots, y_n) \in \sqrt{I}$ . Let  $I = (F_1, \dots, F_c)$  be a bihomogeneous ideal of R where c < n + m and we have  $V(I) = \emptyset$ , if and only if, there is a  $t \in \mathbb{N}$  such that  $(x_0, \dots, x_m)^t \in I$  or  $(y_0, \dots, y_n)^t \in I$ . Supposes, without loss of generality that  $(x_0, \dots, x_m)^t \in I$  for some t, in particular, for all i, the forms  $x_i^t$ of bidegree (t, 0) are elements of I, this implies that at last m + 1 generators of  $F_i$  has bidegree  $(a_i, 0)$ .

3. On the Cohomology of Complete Intersection in  $\mathbb{P}^m \times \mathbb{P}^n$ 

Let us consider the variety  $X = \mathbb{P}^m \times \mathbb{P}^n$ . Throughout this section, we will computer the cohomology of X and we provide some conditions to find the cohomology of complete intersections in X.

**Definition 5.** By considering  $X = \mathbb{P}^m \times \mathbb{P}^n$  and  $D_i = \mathcal{O}_X(a_i, b_i)$  be effective divisors of X, take  $F_i \in H^0(X, D_i)$ , where  $a_i + b_i > 1$ . We say that  $Y = \bigcap_{i=1}^c V(F_i)$  is a complete intersection of X if dim Y = m + n - c, where  $0 \le c < n + m$ .

We denote  $X := \mathbb{P}^m \times \mathbb{P}^n$ , where  $m \leq n$  and  $Y_j = \bigcap_{i=1}^j V(F_i)$ , where  $Y_0 = X$  and  $Y_c = Y$ , thus if Y is a complete intersection, then  $Y_j$  is a complete intersection. By Segree embedding we have  $X \subset \mathbb{P}^r$ , where r = (m+1)(n+1) - 1.

A scheme  $Y \subset \mathbb{P}^r$  is said to be Arithmetically Cohen-Macaulay (ACM) if its coordinate ring is a Cohen-Macauly ring. Unlike the projective case, not all of the complete intersection in X is ACM, for example, let  $Y = V(x_0^2)$  be the curve of bidegree (2,0) in  $\mathbb{P}^1 \times \mathbb{P}^1$  under the Segre embedding in  $\mathbb{P}^3$ . In fact, the coordinate ring of Y is  $R = k[z_0, z_1, z_2, z_3]/(z_0z_3 - z_1z_2, z_0^2, z_0z_1, z_1^2)$ , where dim<sub>Krull</sub> R = 2 and  $z_2$  is a maximal regular sequence of R, hence R is not Cohen-Macauly. An equivalent definition of ACM variety is

 $H^{i}(Y, \mathcal{O}_{Y}(d)) = 0$  for every  $1 \leq i \leq \dim Y - 1$  or  $H^{i}(Y, \mathcal{I}_{Y}(d)) = 0, \forall 1 \leq i \leq \dim Y$ .

**Definition 6.** Let  $Y = \bigcap_{i=1}^{c} V(F_i)$  be a complete intersection of X. We say that  $F_1, \dots, F_c$  is a regular sequence on X if for every index  $i_1 < i_2 < \dots < i_{\alpha}$  the scheme  $\bigcap_{r=1}^{\alpha} V(F_{i_r})$  is ACM.

Let D be the divisor  $\mathcal{O}_X(a, b)$  not necessarily effective. By Kunneth formula

$$h^{i}(X, \mathcal{O}_{X}(a, b)) = \sum_{r+s=i} h^{r}(\mathbb{P}^{m}, \mathcal{O}_{\mathbb{P}^{m}}(a)) \cdot h^{s}(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(b))$$

and we summarize the cohomology of D in the Table 1.

From Table 1, we have  $h^i(X, \mathcal{O}_X(d, d)) = 0$  for all  $d \in \mathbb{Z}$  and  $1 \le i \le m + n - 1$ , i.e X is ACM. Let  $Y_c = \bigcap_{i=1}^c V(F_i)$  be a complete intersection in X. For integers  $a_1, b_1$  and  $d \in \mathbb{Z}$ , take the exact sequence

$$0 \longrightarrow \mathcal{O}_X(d-a_1, d-b_1) \xrightarrow{\cdot F_1} \mathcal{O}_X(d, d) \longrightarrow \mathcal{O}_{Y_1}(d, d) \longrightarrow 0,$$
(2)

and we can observe that  $F_1$  is regular if, and only if,

$$h^{i}(X, \mathcal{O}_{X}(d-a_{1}, d-b_{1})) = 0$$
 for all  $1 \le i \le m+n-1$ 

which is equivalent to

$$h^{i}(X, \mathcal{O}_{X}(d-a_{1}, d-b_{1})) = 0$$
 for all  $i \in \{m, n\} \Rightarrow (a_{1}-b_{1}) < m+1$  and  $(b_{1}-a_{1}) < m+1$ .

	$h^0(\mathcal{O}_X(a,b))$	$h^m(\mathcal{O}_X(a,b))$	$h^n(\mathcal{O}_X(a,b))$	$h^{m+n}(\mathcal{O}_X(a,b))$
$a \ge 0, b \ge 0$	$\binom{m+a}{m} \cdot \binom{n+b}{n}$	0	0	0
$b \ge 0,$	0	$\binom{n+b}{2}$ , $\binom{-1-a}{2}$	0	0
$a \leq -m-1$	0	$\binom{n}{m}$	0	0
$a \ge 0,$	0	0	$(m+a) \cdot (-1-b)$	0
$b \leq -n-1$	0	0	(m) $(n)$	0
$a \leq -m-1,$	0	0	0	(-1-a), $(-1-b)$
$b \leq -n-1$	V	U	0	(n) $(n)$

TABLE 1. Cohomology of  $\mathbb{P}^m \times \mathbb{P}^n$ .

Now, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_1}(d-a_2, d-b_2) \longrightarrow \mathcal{O}_{Y_1}(d, d) \longrightarrow \mathcal{O}_{Y_2}(d, d) \longrightarrow 0,$$
(3)

and we can observe that  $F_1, F_2$  is a regular sequence if, and only if,

$$h^{i}(Y_{1}, \mathcal{O}_{Y_{1}}(d - a_{2}, d - b_{2})) = 0, \forall 1 \le i \le m + n - 2$$

which implies

$$h^{i}(X, \mathcal{O}_{X}(d-a_{j}, d-b_{j})) = 0 \text{ and } h^{i}(X, \mathcal{O}_{X}(d-a_{1}-a_{2}, d-b_{1}-b_{2})) = 0,$$

then

$$(a_j-b_j) < m+1, (b_j-a_j) < n+1$$
 and  $(a_1+a_2-b_1-b_2) < m+1, (b_1+b_2-a_1-a_2) < n+1$ .  
By applying the same process we can show that  $F_1, F_2, F_3$  is a regular sequence if only if

$$\begin{aligned} &(a_i - b_i) < m + 1, (b_i - a_i) < n + 1 \text{ for } i = 1, 2, 3\\ &(a_i + a_j - b_i - b_j) < m + 1, (b_i + b_j - a_i - a_j) < n + 1 \text{ for all } i < j,\\ &(a_1 + a_2 + a_3 - b_1 - b_2 - b_3) < m + 1 \text{ and } (b_1 + b_2 + b_3 - a_1 - a_2 - a_3) < m + 1. \end{aligned}$$

With an extension of the above arguments we get:

**Proposition 7.** Let Y be a complete intersection on X. Then, Y is generated by a regular sequence if and only if,

$$(a_{i_1} - b_{i_1} + \dots + a_{i_\alpha} - b_{i_\alpha}) < m + 1$$

and

$$(b_{j_1} - a_{j_1} + \dots + b_{j_\beta} - a_{j_\beta}) < n + 1$$

for every  $i_1 < i_2 < \cdots < i_{\alpha}$  and  $j_1 < j_2 < \cdots < j_{\beta}$ .

From now on, we will call a scheme Y a complete intersection ACM, or just ACM, when a regular sequence generates Y.

**Proposition 8.** Let  $Y = \bigcap_{i=1}^{c} V(F_i)$  be a closed subscheme of X. If  $F_i$  is a regular sequence, then:

(1) For all  $0 \le j \le c$ ,  $Y_j$  is a complete intersection of codimension j of X;

- (2) Y is ACM;
- (3) The dualizing sheaf of Y is  $\omega_Y \cong \mathcal{O}_Y(\sum a_i m 1, \sum b_i n 1)$ .

*Proof.* Every subsequence of a regular sequence is regular, so we need to show that Y is a complete intersection, but this is clear since the regularity guarantees that Y has the expected dimension.

From Koszul Complex we get:

$$\bigoplus_{i < j} \mathcal{O}_X(-a_i - a_j, -b_i - b_j) \longrightarrow \bigoplus_i \mathcal{O}_X(-a_i, -b_i) \longrightarrow \mathcal{I}_{Y,X} \longrightarrow 0,$$

where  $\mathcal{I}_{Y,X}$  is the ideal sheaf of Y in X. Therefore,

$$\omega_Y \cong \omega_X \otimes (\det \mathcal{I}_{Y,X})^{-1} \cong \mathcal{O}_Y(\sum a_i - m - 1, \sum b_i - n - 1).$$

**Corollary 9.** Under the hypothesis of Proposition 7, follows that

- **C.1** The map  $H^0(X, \mathcal{O}_X(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d))$  is surjective for every  $d \in \mathbb{Z}$ ;
- **C.2** The map  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \longrightarrow H^0(Y, \mathcal{O}_Y(d))$  is surjective for every  $d \in \mathbb{Z}$ ;
- **C.3** The kernel  $H^0(X, \mathcal{I}_{Y,X}(d))$  consists of bi-homogeneous polynomials  $F = \sum_i F_i H_i$ , where  $H_i \in H^0(X, \mathcal{O}_X(d a_i, d b_i))$ .

*Proof.* The items C.1 and C.3 follows the Proposition 7. Now we consider the exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where the map  $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d))$  is given by send a homogeneous monomial  $Z_{00}^{i_{00}} Z_{01}^{i_{01}} \cdots Z_{mn}^{i_{mn}}$  of degree d on a form  $(X_0Y_0)^{i_{00}} (X_0Y_1)^{i_{01}} \cdots (X_mY_m)^{i_{mn}}$ of bidegree (d, d). It's easy to see that the map is surjective, now applying item **C.1** we complete the proof.

It is important to comment that there are many cases of ACM complete intersections to be studied, the most simple example are the curves of bidegree (a, b) on a smooth quadric of  $\mathbb{P}^3$ , such that |a - b| < 2. Here are some interesting examples:

**Example 10.** Let C be a smooth curve of genus 8. From [8, (i) and (ii) of the Theorem] we have

- (1) If C is a general curve and has a  $g_7^2$  non-selfadjoint, then C is a complete intersection of divisors of bidegree (1, 1), (1, 2) and (2, 1) in  $\mathbb{P}^2 \times \mathbb{P}^2$ ;
- (2) If C has a g<sup>1</sup><sub>4</sub> but no g<sup>2</sup><sub>6</sub>, then C is the complete intersection of four divisors of bidegree (1, 1), (1, 1), (0, 2) and (1, 2) in P<sup>1</sup> × P<sup>4</sup>.

Let C be a smooth curve of genus 7. From [1, Table 1] we have

- If C is trigonal and has a g<sub>6</sub><sup>2</sup> non-selfadjoint, then C is a complete intersection of two divisors of bidegree (1, 1) and (3, 3) in P<sup>1</sup> × P<sup>2</sup>;
- (2) If C is tetragonal and has no  $g_6^2$ , then C is isomorphic to a complete intersection of a divisor of bidegree (1, 1) and two divisors of bidegree (1, 2) in  $\mathbb{P}^1 \times \mathbb{P}^3$ ;

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(3) Assume that C has gonality 4 and C has a g<sub>6</sub><sup>2</sup> non-selfadjoint, then C is isomorphic to a complete intersection of three divisors of bidegree (1, 1), (1, 1) and (2, 2) in P<sup>2</sup> × P<sup>2</sup>.

**Corollary 11.** Let Y be a complete intersection on  $X \subset \mathbb{P}^r$ . If Y is ACM, then:

- *a)* The scheme Y is connected;
- b)  $Y \subset \mathbb{P}^r$  is degenerate if, and only if,  $(a_i, b_i) = (1, 1)$  for some i;
- c) Y is not contained on a global section of neither  $\mathcal{O}_X(1,0)$  or  $\mathcal{O}_X(0,1)$ .

Proof. By considering the exact sequence

$$0 \longrightarrow \mathcal{I}_{Y,X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

follows from C.1 and C.3 for d = 0 that  $k = H^0(X, \mathcal{O}_X) \cong H^0(Y, \mathcal{O}_Y)$  which means Y connected, the second item just takes d = 1. For the third item, we take the exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_{j-1}}(-a_j, -b_j) \longrightarrow \mathcal{O}_{Y_{j-1}} \longrightarrow \mathcal{O}_{Y_j} \longrightarrow 0$$

and we tensor it by  $\mathcal{O}_X(1,0)$  or  $\mathcal{O}_X(0,1)$ . For all  $j \ge 1$  we going to show that

$$H^{0}(Y_{j-1}, \mathcal{O}_{Y_{j-1}}(1 - a_{j}, -b_{j})) = H^{0}(Y_{j-1}, \mathcal{O}_{Y_{j-1}}(-a_{j}, 1 - b_{j})) = 0$$

or equivalently:

$$h^q(X, \mathcal{F} \otimes \mathcal{O}_X(1, 0)) = h^q(X, \mathcal{F} \otimes \mathcal{O}_X(0, 1)) = 0,$$

for all  $q \in \{0, m, n\}$  and all  $i_1 < \cdots < j_{m+1}$ , where  $\mathcal{F} = \mathcal{O}_X(-a_{i_1} - \cdots - a_{i_{m+1}}, -b_{i_1} - \cdots - b_{i_{m+1}})$ . Since  $a_i + b_i > 1$ , for q = 0 is trivial. Lets show for m, we can assume without loss of generality that m < n, so take index  $i_1 < \cdots < i_m < i_{m+1}$ , one condition for  $h^m(X, \mathcal{F} \otimes \mathcal{O}_X(1, 0))$  not being zero is  $b_{i_1} = \cdots = b_{i_{m+1}} = 0$ , in particular  $a_{i_p} \ge 2$  for all  $p \in \{1, \cdots, m, m+1\}$ , thus

$$2m \le a_{i_1} + \dots + a_{i_m} < m+1$$

which is absurd.

Now one condition for  $h^m(X, \mathcal{F} \otimes \mathcal{O}_X(0, 1))$  not being zero is  $b_{i_1} + \cdots + b_{i_{m+1}} \leq 1$ , we can suppose  $b_{i_{m+1}} = 1$  and  $b_{i_1} = \cdots = b_{i_m} = 0$  in particular  $a_{i_p} \geq 2$  for all  $p \in \{1, \cdots, m\}$ , following as in the previous case, we will have an absurd. Similarly, the nth cohomology vanishes.

## 4. Complete Intersection Arithmetically Cohen-Macaulay on $\mathbb{P}^m \times \mathbb{P}^n$

In the study of the standard complete intersection in  $\mathbb{P}^n$ , we always order the degrees of the homogeneous forms such that  $d_i \leq d_{i+1}$ , unfortunately, we cannot get a total order for the bi-degrees in X. We need a partial order such that the intermediate cohomologies are null to replicate the theorems of standard complete intersection, but now in X, i.e there is an ordering  $Y = \bigcap_{i=1}^c V(F_i)$ , where the map  $\varphi_j : H^0(X, \mathcal{O}_X(a_j, b_j)) \longrightarrow$  $H^0(Y_{j-1}, \mathcal{O}_{Y_{j-1}}(a_j, b_j))$  is surjective for every j. To do this we only need to show that

$$h^{m}(X, \mathcal{O}_{X}(a_{i_{m+1}} - a_{i_{m}} - \dots - a_{i_{1}}, b_{i_{m+1}} - b_{i_{m}} - \dots - b_{i_{1}})) = 0$$
  
and (4)

$$h^{n}(X, \mathcal{O}_{X}(a_{j_{n+1}} - a_{j_{n}} - \dots - a_{j_{1}}, b_{j_{n+1}} - b_{j_{n}} - \dots - b_{j_{1}})) = 0,$$

for every  $i_1 < \cdots < i_m < i_{m+1}$  and for every  $j_1 < \cdots < j_n < j_{n+1}$ . For example, if  $a_i \le a_{i+1}$  and  $b_i \le b_{i+1}$ , then the theory will work similarly to the standard case, but we know that this is not always the case. The first attempt is the dictionary order, that is,  $a_i < a_{i+1}$  or  $a_i = a_{i+1}$  and  $b_i \le b_{i+1}$ , most of the time this order will be enough, but if it is not, we will present another order. If m = 1, the dictionary order is enough, therefore, without loss of generality, we can assume that m > 1.

If m = n, given an ACM complete intersection Y in  $\mathbb{P}^n \times \mathbb{P}^n$  generated by bihomogeneous polynomials of bidegree  $(a_i, b_i)$ , let us consider the order on the bi-degrees such that

$$|a_i - b_i| \ge |a_{i+1} - b_{i+1}|$$

and if  $|a_1 - b_1| = \cdots = |a_c - b_c| = 1$  and  $|a_j - b_j| = 0$  for all j > c, then we reorder, if possible, so that the first terms alternate signs.

**Example 12.** If  $X = \mathbb{P}^2 \times \mathbb{P}^2$  and  $a_i - b_i = 1$ ,  $a_j - b_j = 1$  and  $a_k - b_k = -1$ , then the order is  $(a_i, b_i) = (a_1, b_1)$ ,  $(a_k, b_k) = (a_2, b_2)$  and  $(a_j, b_j) = (a_3, b_3)$  or  $(a_k, b_k) = (a_1, b_1)$ ,  $(a_i, b_i) = (a_2, b_2)$  and  $(a_j, b_j) = (a_3, b_3)$ .

Taking into account this order, we suppose that 4 is not zero. Then we suppose, without loss of generality, that  $a_{i_n} + \cdots + a_{i_1} \ge a_{i_{n+1}} + n + 1$  and  $b_{i_{n+1}} \ge b_{i_n} + \cdots + b_{i_1}$ , which implies:

$$(a_{i_n} - b_{i_n}) + \dots + (a_{i_1} - b_{i_1}) \ge (a_{i_{n+1}} - b_{i_{n+1}}) + n + 1.$$

We claim that  $|a_{i_{n+1}} - b_{i_{n+1}}| \leq 1$ , indeed, if  $|a_{i_{n+1}} - b_{i_{n+1}}| > 1$ , then  $|a_{i_r} - b_{i_r}| > 1$  for all  $1 \leq r \leq n$ . Let the first terms be positive, so  $\sum_{k=1}^{\alpha} (a_{i_k} - b_{i_k}) \geq 2\alpha$  and  $\sum_{k=\alpha+1}^{n} (a_{i_k} - b_{i_k}) \leq -2(n - \alpha)$ , if  $\alpha \neq n/2$  then the norm of one of these sums will be greater than n, which contradicts the definition of ACM, if  $\alpha = n/2$ , suppose  $a_{i_{n+1}} - b_{i_{n+1}} > 0$ , then  $|\sum_{k=1}^{\alpha} (a_{i_k} - b_{i_k}) + (a_{i_{n+1}} - b_{i_{n+1}})| > n$ , another contradiction. Therefore  $|a_{i_r} - b_{i_r}| = 1$ , for all  $1 \leq r \leq n$  and from Proposition 7, we get

$$n \ge (a_{i_n} - b_{i_n}) + \dots + (a_{i_1} - b_{i_n}),$$

so  $a_{i_{n+1}} - b_{i_{n+1}} = -1$  and thus  $(a_{i_n} - b_{i_n}) = \cdots = (a_{i_1} - b_{i_n}) = 1$ . In this way, we will have at most n - 1 terms equal to -1, which prevents  $a_{i_{n+1}} - b_{i_{n+1}} = -1$ .

Now we suppose Y a complete intersection ACM on  $\mathbb{P}^m \times \mathbb{P}^n$ , where m < n. We will use the following order:  $|a_i - b_i| \ge |a_{i+1} - b_{i+1}|$  and if  $|a_{i_1} - b_{i_1}| = \cdots = |a_{i_c} - b_{i_c}| = 1$  and  $|a_j - b_j| \ne 1$  for all  $j \notin \{i_1, \cdots, i_c\}$ , then we reorder, if possible, so that  $(a_{i_1} - b_{i_1}) = \cdots = (a_{i_{m-1}} - b_{i_{m-1}}) = 1$  and  $(a_{i_r} - b_{i_r}) \le (a_{i_{r+1}} - b_{i_{r+1}})$  for all  $r \ge m$ .

Suppose that the first equation 4 is not zero. Then we get

$$m \ge (a_{i_m} - b_{i_m}) + \dots + (a_{i_1} - b_{i_1}) \ge (a_{i_{m+1}} - b_{i_{m+1}}) + m + 1,$$

in particular,  $(a_{i_{m+1}} - b_{i_{m+1}}) \leq -1$ , which makes  $(a_{i_k} - b_{i_k}) \neq 0$  for all k. If there exists a negative term in the above sum, let's assume that it is  $(a_{i_m} - b_{i_m})$ . In this form, we have  $(a_{i_m} - b_{i_m}) \leq (a_{i_{m+1}} - b_{i_{m+1}})$ , which implies  $\sum_{r=1}^{m-1} (a_{i_r} - b_{i_r}) \geq m + 1$  contradicting Proposition 7. Therefore, the only solution is  $(a_{i_m} - b_{i_m}) = \cdots = (a_{i_1} - b_{i_1}) = 1$  and  $(a_{i_{m+1}} - b_{i_{m+1}})$  which is not possible, given the order we are using. Similarly, we will have an absurdity if we assume that the second equation 4 is not zero.

**Remark 13.** In summary, an order satisfying the Equation 4 can be described as follows:

(1) Let Y be a complete intersection ACM on  $X = \mathbb{P}^n \times \mathbb{P}^n$ . We can order the bidegrees  $(a_1, b_1), \dots, (a_{2n-1}, b_{2n-1})$  such that  $|a_i - b_i| \ge |a_{i+1} - b_{i+1}|$  and if  $|a_{i_1} - b_{i_1}| = \dots = |a_{i_c} - b_{i_c}| = 1$  and  $|a_j - b_j| = 0$  for all  $j \notin \{i_1, \dots, i_c\}$ , then reorder, if possible, so that the first terms alternate signs.

(2) Let Y be a complete intersection ACM on  $X = \mathbb{P}^m \times \mathbb{P}^n$ , where m < n. We order the bi-degrees such that  $|a_i - b_i| \ge |a_{i+1} - b_{i+1}|$ . If  $|a_{i_1} - b_{i_1}| = \cdots = |a_{i_c} - b_{i_c}| = 1$  and  $|a_j - b_j| \ne 1$  for all  $j \ne \{i_1, \cdots, i_c\}$ , then reorder, if possible, so that  $(a_{i_1} - b_{i_1}) = \cdots = (a_{i_{m-1}} - b_{i_{m-1}}) = 1$  and  $(a_{i_r} - b_{i_r}) \le (a_{i_{r+1}} - b_{i_{r+1}})$  for all  $r \ge m$ .

**Proposition 14.** Let Y be an ACM complete intersection at X. Then there is a writing  $Y = \bigcap_{i=1}^{c} V(F_i)$  such that:

- (1) The map  $\varphi_j : H^0(X, \mathcal{O}_X(a_j, b_j)) \longrightarrow H^0(Y_{j-1}, \mathcal{O}_{Y_{j-1}}(a_j, b_j))$  is surjective for every j, where  $Y_j = \bigcap_{i=1}^j V(F_i)$  and  $Y_0 = X$ ;
- (2) The kernel  $H^0(X, \mathcal{I}_{Y_{j-1}, X}(a_j, b_j))$  of  $\varphi_j$  consists of bi-homogeneous polynomials  $F = \sum_i F_i H_i$ , where  $H_i \in H^0(X, \mathcal{O}_X(a_j a_i, b_j b_i))$ ;
- (3) The kernel  $H^0(X, \mathcal{I}_{Y,X}(d))$  depends only of the bidegrees  $(a_1, b_1), \dots, (a_c, b_c)$ and of the numbers d, c, m, n.

*Proof.* To prove the first and the second statement is sufficient to display an order on the bidegrees  $(a_i, b_i)$  such that  $H^1(Y_{j-1}, \mathcal{O}_{Y_{j-1}}(a_{j+1} - a_j, b_{j+1} - b_j)) = 0$  or equivalently that the Equation 4 is true. This is true from the above discussion and the Remark 4

Finally, to prove the last statement, we consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{Y_{j-1}}(d-a_j, d-b_j) \longrightarrow \mathcal{O}_{Y_{j-1}}(d) \longrightarrow \mathcal{O}_{Y_j}(d) \longrightarrow 0,$$

and the result follows from recurrence in j

**Example 15.** Let Y be the complete intersection of four divisors of bidegree (1, 1), (1, 1), (0, 2) and (1, 2) in  $\mathbb{P}^1 \times \mathbb{P}^4$ . If we consider the order used in Proposition 14 on the bidegrees, we get  $(a_1, b_1) = (0, 2), (a_2, b_2) = (1, 1), (a_3, b_3) = (1, 1)$  and  $(a_4, b_4) = (1, 2)$ . On the other hand, if we use the order  $(a_1, b_1) = (1, 1), (a_2, b_2) = (1, 1), (a_3, b_3) = (1, 2)$  and  $(a_4, b_4) = (0, 2)$ , then the Proposition 14 is also true which means that the order on the bidegrees is not unique,

From Proposition 14 we can order the values of the bidegrees conveniently. When Y is not ACM, it is a bit complicated, as shown in the following example.

**Example 16.** Let Y be the complete intersection of three divisors of bidegree (4, 1) in  $X = \mathbb{P}^2 \times \mathbb{P}^2$ . By taking any order on the bidegrees follows that  $h^2(X, \mathcal{O}_X(a_3 - a_2 - a_1, b_3 - b_2 - b_1)) = 9$ .

From Table 1, there are many cases in which Proposition 14 can be applied. For Y a scheme (not necessarily a complete intersection) of X, we say that Y has a *good order* if there is a writing  $Y = \bigcap_{i=1}^{c} V(F_i)$  such that for all j

$$H^{m}(X, \mathcal{O}_{X}(a_{j+1} - a_{i_{1}} - \dots - a_{i_{m}}, b_{j+1} - b_{i_{1}} - \dots - b_{i_{m}})) = 0$$
  
and

$$H^{n}(X, \mathcal{O}_{X}(a_{j+1} - a_{l_{1}} - \dots - a_{l_{n}}, b_{j+1} - b_{l_{1}} - \dots - b_{l_{n}})) = 0$$

for all  $i_1 < \cdots < i_m$  and  $l_1 < \cdots < l_n$ . We say that  $(a_{c+1}, b_{c+1})$  respects the relations of a complete intersection if the bidegrees  $(a_1, b_1), \cdots (a_c, b_c), (a_{c+1}, b_{c+1})$  are writing in a good order. For example, in every complete intersection  $Y = \bigcap_{i=1}^c V(F_i)$  the bidegree  $(a_j, b_j)$  respects the relations of  $Y_{j-1}$ . For every  $(a_{c+1}, b_{c+1})$  such that  $a_{c+1} > \sum_{i=1}^c a_i$ and  $b_{c+1} > \sum_{i=1}^c b_i$ , the pair  $(a_{c+1}, b_{c+1})$  respects the relations of Y.

Let Y be a complete intersection of  $X \subset \mathbb{P}^r$ . The Hilbert polynomial of a coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^r$  is

$$p_{\mathcal{F}}(d) = \sum_{i} h^{i}(\mathbb{P}^{r}, \mathcal{F}(d))$$

and the Hilbert polynomial  $p_Y$  of Y is defined as the Hilbert polynomial of  $\mathcal{O}_X$ . Denote by  $\mathcal{I}_Y$  the ideal sheaf of Y in  $\mathbb{P}^r$ ,  $\mathcal{I}_{Y,X}$  its restriction to X and from Serre's Vanishing Theorem, there is an integer  $d_0$  such that  $h^i(\mathbb{P}^r, \mathcal{O}_Y(d)) = 0$  for all i > 0 and  $d > d_0$  what means the Hilbert polynomial depends only of  $h^0(\mathbb{P}^r, \mathcal{O}_Y(d))$ . From exact sequence:

$$0 \longrightarrow \mathcal{I}_{Y,X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

we have  $p_Y(d) = h^0(X, \mathcal{O}_X(d)) - h^0(X, \mathcal{I}_X(d))$  for an integer sufficiently large d. From Proposition 14, for all d sufficiently large, the dimension  $h^0(X, \mathcal{I}_X(d))$  depends only of m, n, d, c and of the bidegrees, and if we take  $Y = \bigcap_{i=1}^c V(F_i)$  and  $Y' = \bigcap_{i=1}^c V(F'_i)$ , where  $F_i$  and  $F'_i$  are elements of  $H^0(X, \mathcal{O}_X(a_i, b_i))$ , then  $p_Y = p_{Y'}$ .

Conversely, we suppose  $Y = \bigcap_{i=1}^{c} V(F_i)$  a complete intersection of  $\mathbb{P}^m \times \mathbb{P}^n$ , where  $p_Y$  its Hilbert polynomial, and Y' is another complete intersection such that  $p_Y = p_{Y'}$ . If m < n, then there is a writing  $\bigcap_{i=1}^{c} V(F'_i)$  where  $(a_i, b_i) = (a'_i, b'_i)$  for all  $i \in \{1, \dots, c\}$ . This follows from recurrence in  $1 \le j \le c$  to the exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_{j-1}}(d-a_j, d-b_j) \longrightarrow \mathcal{O}_{Y_{j-1}}(d) \longrightarrow \mathcal{O}_{Y_j}(d) \longrightarrow 0.$$

By same sequence we can prove that, if m = n, then Y' is generated by bihomogeneous polynomial of bidegree  $(a_1, b_1), (a_2, b_2), \dots, (a_c, b_c)$  or by bihomogeneous polynomials of bidegree  $(b_1, a_1), (b_2, a_2), \dots, (b_c, a_c)$ .

**Remark 17.** With the above arguments, we have that the Hilbert polynomial of a complete intersection is (almost) determined by its bidegrees when m < n (m = n).

**Proposition 18.** Let  $k \subset K$  be an extension of field and Y a closed subscheme of  $\mathbb{P}^m \times \mathbb{P}^n$ . Then Y is an ACM subcheme if, and only if,  $Y_K$  is an ACM subcheme.

*Proof.* Denote  $Y = \bigcap_{i=1}^{c} V(F_i)$  and  $Y_K = \bigcap_{i=1}^{c} V(F_{i,K})$ . If Y is ACM, then, by flat propriety of change of base,  $Y_K$  is a complete intersection, since the bidegrees are preserved by base change, then  $Y_K$  is projectively normal.

On the other hand, let us suppose  $Y_K$  a ACM of  $X_K$  and we take  $G_1, \dots, G_c$  the regular sequence defining  $Y_K$ , writing in a good order. Let's prove by induction at  $0 \le r \le c$  that, even if that means changing the regular sequence, exists  $F_i \in H^0(X, \mathcal{I}_Y(a_i, b_i))$  such that  $G_i = F_{i,K}$ .

For r = 0 is trivial. We suppose that the assertion is true for some r = j. We denote  $Y_j = \bigcap_{i=1}^{j} V(F_i)$  and we consider the inclusion of k-vector spaces

$$H^0(X, \mathcal{I}_{Y_i}(a_{j+1}, b_{j+1})) \subset H^0(X, \mathcal{I}_Y(a_{j+1}, b_{j+1})).$$

If we extend by scalars to K then from the base change Theorem we get the inclusion of K-vector spaces  $H^0(X, \mathcal{I}_{Y_{j,K}}(a_{j+1}, b_{j+1})) \subset H^0(X, \mathcal{I}_{Y_K}(a_{j+1}, b_{j+1}))$ . Since  $G_{j+1}$  is an element of  $H^0(X, \mathcal{I}_{Y_K}(a_{j+1}, b_{j+1}))$  but it is not an element of  $H^0(X, \mathcal{I}_{Y_{j,K}}(a_{j+1}, b_{j+1}))$  the last inclusion is strict and by flatness the first inclusion is strict too. Therefore, we can take  $F_{j+1}$  such that

$$F_{j+1} \in H^0(X, \mathcal{I}_Y(a_{j+1}, b_{j+1}))$$
 but  $F_{j+1} \notin H^0(X, \mathcal{I}_{Y_j}(a_{j+1}, b_{j+1})).$ 

From Proposition 14,  $F_{j+1,K} \in H^0(X, \mathcal{I}_{Y_K}(a_{j+1}, b_{j+1}))$  can be represented as

$$F_{j+1,K} = \sum_{i=1}^{c} G_i H_i,$$

where  $H_i \in H^0(X_K, \mathcal{O}_{X_K}(a_{j+1} - a_i, b_{j+1} - b_i))$ . Since  $F_{j+1,K}$  is not an element of  $H^0(X, \mathcal{I}_{Y_{j,K}}(a_{j+1}, b_{j+1}))$  there is  $l \ge j + 1$  such that  $H_l \ne 0$  which implies  $(a_l, b_l) = (a_{j+1}, b_{j+1})$  and we can suppose l = j+1. Therefore, we construct a new regular sequence for  $Y_K$  by replacing  $G_{j+1}$  with  $F_{j+1,K}$ .

for  $Y_K$  by replacing  $G_{j+1}$  with  $F_{j+1,K}$ . Let's consider  $Y' := \bigcap_{i=1}^c V(F_i)$ . Since  $Y' \subset Y$  and  $Y'_K = Y_K$ , the flatness of the extension  $k \longrightarrow K$  ensures that Y' = Y. Finally, an ACM is defined by its bidegree and we can see in the induction process that the bidegrees are not changed (at most a change of order) because the flatness of the extension preserves the bidegree of bihomogeneous forms.  $\Box$ 

**Proposition 19.** Let T be a Noetherian scheme and  $Y \subset X_T$  a closed subscheme such that the projection  $\pi : Y \longrightarrow T$  is flat and its fibers are ACM. Then, for all  $d \in \mathbb{Z}$ ,  $\pi_*\mathcal{O}_Y(d)$  is locally free and its formation commutes at any change of basis. If these fibers has bidegree  $(a_1, b_1), \dots, (a_c, b_c)$  and suppose that  $(a_{c+1}, b_{c+1})$  respect the relation of these complete intersection, then  $\pi_*\mathcal{O}_Y(a_{c+1}, b_{c+1})$  is locally free and commutes with base change.

*Proof.* For every  $t \in T$ , we have the commutative diagram

From Proposition 14 or Corollary 9, the vertical arrow on the right is surjective, consequently, from [4, Proposition 12.11-a], the arrow on the left is also subjective. The map pris surjective by definition of polynomial, then pr' is surjective, thus from [6, Proposition 1.1.7] we have that  $\pi_*\mathcal{O}_Y(a_{c+1}, b_{c+1})$  is locally free and commutes with base change.  $\Box$ 

**Proposition 20.** Let T be the spec of a DVR and  $Y \subset X_T$  a flat closed subscheme over T. If the special fiber is an ACM complete intersection then its generic fiber is an ACM complete intersection. Moreover, any equation of the special fiber can be raised to an equation of Y.

*Proof.* Repeating the arguments of the proof of [6, Proposition 2.1.12].

## 5. HILBERT SCHEME OF COMPLETE INTERSECTION

In this section, we will construct the Hilbert scheme for complete intersections which are ACM. Throughout this section we consider X to be  $\mathbb{P}^m \times \mathbb{P}^n$ , where  $X \subset \mathbb{P}^r$  under Segre embedding. Let  $Y = \bigcap_{i=1}^c V(F_i)$  be an ACM and p(t) its Hilbert polynomial, where each form  $F_i$  has bidegree  $(a_i, b_i)$ . Fix a good order on bidegrees  $(a_1, b_1), \cdots, (a_c, b_c)$ . We denote by Hilb<sub>r</sub> the Hilbert scheme of  $\mathbb{P}^r$  and  $\operatorname{Hilb}_r^{p(t)} \subset \operatorname{Hilb}_r$  the Hilbert scheme associate to p(t). For a scheme S, we define

$$H_X^{p(t)}(S) = \begin{cases} \text{flat families } \mathcal{X} \subset X \times S \text{ of closed subschemes of } X, \text{ parametrized by } S, \\ \text{with fibers having Hilbert polynomial } p(t). \end{cases}$$

Since flatness is preserved by base change, this association defines a contravariant functor

$$\mathbf{H}_X^{p(t)}: Schemes^{op} \longrightarrow Set$$

called the Hilbert functor of X relative to p(t) where "Schemes" denotes the category of locally Noetherian separated k-schemes. Its known that  $H_X^{p(t)}$  is represented by a closed subscheme of  $\operatorname{Hilb}_r^{p(t)}$ .

Let  ${\cal F}$  be the contravariant functor of the category of schemes to the sets category such that for each scheme S

$$F(S) = \begin{cases} \text{flat families } \mathcal{X} \subset X \times S \text{ of closed subschemes of } X, \text{ parametrized by } S, \\ \text{with fibers ACM.} \end{cases}$$

The next subsections are devoted to providing an explicit construction of a scheme  $\mathcal{H}$  which will be the scheme that represents the contravariant functor F. We apply some arguments of [6, Section 1.2] to the construction of  $\mathcal{H}$ . The construction of  $\mathcal{H}$  Due to the difficulty of establishing an order in the bidegrees, we will divide the construction of  $\mathcal{H}$  in two cases which consist of the next 2 subsections.

## 5.1. On Codimension at Most 2.

**Proposition 21.** For hypersurfaces of bidegree (a, b) in  $X = \mathbb{P}^m \times \mathbb{P}^n$ , not necessarily ACM, let  $pr : X \longrightarrow \text{Spec}(\mathbb{Z})$  be the structure morphism. The scheme sought  $\mathcal{H}$  is  $\mathbb{P}(pr_*\mathcal{O}_X(a, b)^{\vee})$ .

*Proof.* We claim that  $pr_*\mathcal{O}_X(a, b)$  is a locally free sheaf on  $\operatorname{Spec}(\mathbb{Z})$  and it commutes with base change. Indeed, since  $a \ge 0$  and  $b \ge 0$ , we have  $H^1(X, \mathcal{O}_X(a, b)) = 0$ , in particular, for every  $y \in \operatorname{Spec}(\mathbb{Z})$  the natural map

$$R^1 pr_* \mathcal{O}_X(a,b) \otimes k(y) \longrightarrow H^1(X_y, \mathcal{O}_{X_y}(a,b)) = 0$$

is surjective and from [4, Proposition 12.11-a], follows that this map is an isomorphism. Now the map  $pr_*\mathcal{O}_X(a,b) \otimes k(y) \longrightarrow H^0(X_y, \mathcal{O}_{X_y}(a,b))$  is surjective, thus from [6, Proposition 1.1.7] we have the claim. The geometric fibers of  $pr_*\mathcal{O}_X(a,b)$  are identified with  $H^0(X_K, \mathcal{O}_{X_K}(a,b))$ , where K is the algebraically closed field on which it is defined each point. Denote by  $\mathcal{H}_{(a,b)}$ , or just  $\mathcal{H}$  when there is no confusion, the projective bundle  $\mathbb{P}(pr_*\mathcal{O}_X(a,b)^{\vee})$  and  $\pi : \mathcal{H} \longrightarrow \text{Spec}(\mathbb{Z})$  the projection. A geometric point of  $\mathcal{H}$  is the linear space  $\langle F \rangle$  of  $H^0(X_K, \mathcal{O}_{X_K}(a,b))$ . By construction, we have an injection of the tautological line bundle  $\mathcal{O}_{\mathcal{H}}(-1) \longrightarrow \pi^* pr_*\mathcal{O}_X(a,b)$ . We have a commutative diagram:

$$\begin{array}{ccc} X \times \mathcal{H} & \stackrel{\pi'}{\longrightarrow} X \\ pr' & \downarrow & \downarrow^{pr} \\ \mathcal{H} & \stackrel{\pi}{\longrightarrow} \operatorname{Spec} \left( \mathbb{Z} \right) \end{array}$$

and by base change  $\pi^* pr_* \mathcal{O}_X(a, b) = pr'_* \pi'^* \mathcal{O}_X(a, b) = pr'_* \mathcal{O}_{X \times \mathcal{H}}((a, b); 0)$ , so we rewrite the injection as  $\mathcal{O}_{\mathcal{H}}(-1) \longrightarrow pr'_* \mathcal{O}_{X \times \mathcal{H}}((a, b); 0)$ , applying the pullback  $pr'^*$  and by adjunction we get a morphism of line bundle  $\mathcal{O}_{X \times \mathcal{H}}((0, 0); -1) \longrightarrow \mathcal{O}_{X \times \mathcal{H}}((a, b); 0)$ , the zero locus of this morphism is a Cartier divisor  $\mathcal{X}$  (or  $\mathcal{X}_{(a,b)}$ ) of  $X \times \mathcal{H}$ . Let  $\mathcal{X} \longrightarrow \mathcal{H}$ be the projection pr' restrict to  $\mathcal{X}$ , by construction ever fiber of  $\langle F \rangle$  is given by the scheme of zeros of F in  $X_K$ . We can assume  $n \ge 2$  and let  $F_1, F_2$  be forms of bidegree  $(a_1, b_1)$  and  $(a_2, b_2)$ , respectively, and  $Y_1 = V(F_1)$  and  $Y_2 = V(F_2)$ . By Remark 4, the intersection  $Y_1 \cap Y_2$  is a proper closed subset of X, for all cases except when  $b_1 = b_2 = 0$ , m = 1 and  $gcd(F_1, F_2) = 1$ . If we take the relations of the bidegrees as described in Proposition 7, then by Remark 4, the set  $V(F_1, \dots, F_c)$  will be closed in X.

Now we will describe the Hilbert schemes of complete intersection on X, denoted by  $\mathcal{H}$ . Let's consider  $\alpha := \min\{a_1, a_2\}, \beta := \min\{b_1, b_2\}$ . Since the intersection is a closed set of X, for every irreducible component Z of  $Y_1 \cap Y_2$ , we have dim  $Z \ge \dim Y_1 + \dim Y_2 - \dim X = n + m - 2$ . If the intersection is not proper, then there is a form G of bidegree at most  $(\alpha, \beta)$  so that  $Y_1 \cap Y_2 = V(G)$ , in particular, G is a common factor of  $F_1$  and  $F_2$ .

**Proposition 22.** For complete intersection of codimension equal to 2, the scheme  $\mathcal{H}$  is  $\operatorname{Gr}(2, pr_*(\alpha, \beta)^{\vee})$  over  $\operatorname{Spec}(\mathbb{Z})$  or  $\mathbb{P}(\operatorname{pr}_* \mathcal{O}_{\mathcal{X}_1}(a_2, b_2)^{\vee})$  over  $\mathcal{H}_{(a_1,b_1)}$  or the complement of  $\overline{V_{1,2}}$ . Here the set  $\overline{V_{1,2}}$  is the closure of  $\mathcal{H}_{(a_1,b_1)} \times \mathcal{H}_{(a_2,b_2)}$  where its points  $(F_1, F_2)$  are such that  $F_1$  and  $F_2$  have a common component.

*Proof.* The first two cases are similar to the construction of towers of Grassmannian bundles in [3, Section 1.3].

**Case 1** - If  $(\alpha, \beta) = (a_i, b_i)$  for all *i*, then we take  $\mathcal{H}$  to be the Grassmannian bundle  $\operatorname{Gr}(2, pr_*(\alpha, \beta)^{\vee})$  over  $\operatorname{Spec}(\mathbb{Z})$ , the construction of  $\mathcal{X}$  is the same as the hypersurface case, the only difference is that we will take the tautological bundle of the Grassmannian instead of  $\mathcal{O}(-1)$ .

**Case 2** - If  $(\alpha, \beta) = (a_1, b_1)$  and  $(\alpha, \beta) \neq (a_2, b_2)$  we denote  $\pi : \mathcal{H}_1 \longrightarrow \text{Spec}(\mathbb{Z})$ and  $pr : X \longrightarrow \text{Spec}(\mathbb{Z})$ , where  $\mathcal{H}_1 := \mathcal{H}_{(a_1, b_1)}$  and  $\mathcal{X}_1 := \mathcal{X}_{(a_1, b_1)}$ . We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X \times \mathcal{H}_1}((-a_1, -b_1); -1) \longrightarrow \mathcal{O}_{X \times \mathcal{H}_1} \longrightarrow \mathcal{O}_{\mathcal{X}_1} \longrightarrow 0,$$

and if we tensor it by  $\mathcal{O}_{X \times \mathcal{H}_1}((a_2, b_2); 0)$  then we keep this sequence exact, since for every fiber the cohomology  $H^1(\mathcal{O}((a_2 - a_1, b_2 - b_1); -1)) = 0$ . From [4, Proposition 12.11-a] we have  $R^1 pr_* \mathcal{O}((a_2 - a_1, b_2 - b_1); -1) = 0$  and thus the push forward  $pr_*$  keeps the exact sequence as well. Finally, we apply the pullback  $\pi^*$ 

$$0 \longrightarrow \mathcal{F}(-a_1, -b_1) \otimes \mathcal{O}_{\mathcal{H}_1}(-1) \longrightarrow \mathcal{F} \longrightarrow \operatorname{pr}_* \mathcal{O}_{\mathcal{X}_1}(a_2, b_2) \longrightarrow 0,$$

where  $\mathcal{F} = \pi^* pr_* \mathcal{O}_X(a_2, b_2)$ , from Proposition 19, the sheaf  $pr_* \mathcal{O}_{\mathcal{X}_1}(a_2, b_2)$  is locally free. Thus  $\mathcal{H}$  will be the projective bundle  $\mathbb{P}(pr_* \mathcal{O}_{\mathcal{X}_1}(a_2, b_2)^{\vee})$  over  $\mathcal{H}_{(a_1, b_1)}$  whose fibers are

$$\mathbb{P}\left(\frac{H^0(X_K,\mathcal{O}_{X_K}(a_2,b_2))}{\langle \mathbf{F} \rangle}\right)$$

for every  $F \in \mathcal{H}_{(a_1,b_1)}$ , where  $\langle F \rangle$  is the K-space of forms of bidegree  $(a_2, b_2)$  which have F as component. The universal family  $\mathcal{X}$  will be the pre-image of  $\mathcal{X}_{(a,b)}$ .

**Case 3** - For  $(\alpha, \beta) \notin \{(a_1, b_1), (a_2, b_2)\}$ , let  $V_i$  be the closed (see Lorenzo [3, Subsection 1.3]) subset of  $\mathcal{H}_{(a_i,b_i)} \times \mathcal{H}_{(\alpha,\beta)}$  defined by

$$V_i := \{ (F_i, G) \mid F_i \in G.H^0(X, \mathcal{O}_X(a_i - \alpha, b_i - \beta)) \}.$$

In particular,  $\pi_1(V_i)$  is a closed set of  $\mathcal{H}_{(a_i,b_i)}$  generated by reducible forms of bidegree  $(a_i, b_i)$  and since  $a_i + b_i > 1$ , then  $\pi_1(V_i) = \overline{V_i}$  is proper. Similarly,

$$V_{1,2} = (V_1 \times \mathcal{H}_{(a_2,b_2)}) \cap (V_2 \times \mathcal{H}_{(a_1,b_1)})$$
(5)

$$= \{ (F_1, F_2, G) \mid F_i \in G.H^0(X, \mathcal{O}_X(a_i - \alpha, b_i - \beta)) \text{ for all } i \}$$
(6)

is a closed subset of  $\mathcal{H}_{(a_1,b_1)} \times \mathcal{H}_{(a_2,b_2)} \times \mathcal{H}_{(\alpha,\beta)}$  and its projection  $\overline{V_{1,2}}$  is a proper closed subset of  $\mathcal{H}_{(a_1,b_1)} \times \mathcal{H}_{(a_2,b_2)}$ . For every  $(\alpha',\beta') \neq (0,0)$  such that  $0 \leq \alpha' \leq \alpha$  and  $0 \leq \beta' \leq \beta$  we can construct closed sets of  $\mathcal{H}_{(a_1,b_1)} \times \mathcal{H}_{(a_2,b_2)}$  where its points  $(F_1,F_2)$ are such that  $F_1$  and  $F_2$  have a common component of bidegree  $(\alpha',\beta')$ . Let's consider  $\overline{V_{1,2}}$  the union of these closed sets. By considering  $\mathcal{H}$  be the complement of  $\overline{V_{1,2}}$ , we have a natural projection  $\pi : \mathcal{H} \longrightarrow \mathcal{H}_{(a_1,b_1)}$  and the universal family  $\mathcal{X}$  is the pre-image of  $\mathcal{X}_{(a_1,b_1)}$  under the morphism  $id \times \pi : X \times \mathcal{H} \longrightarrow X \times \mathcal{H}_{(a_1,b_1)}$ .

We have observed that in codimensions of at most 2, the Hilbert scheme is smooth and irreducible. Additionally, we could computer its dimension in both cases. As codimension grows, the problem becomes more complicated. Fortunately, when it is ACM, we have  $a_i - b_i < m + 1$ , which puts us in a privileged position. Now let's describe  $\mathcal{H}$  for the curves (not necessarily smooth) of genus 7 of the Example 10. For simplicity, we denote  $V(r,s) := H^0(X, \mathcal{O}_X(r,s))^{\vee}$  and  $\mathbb{P}(r,s) := \mathbb{P}(H^0(X, \mathcal{O}_X(r,s))^{\vee})$ :

- (1) For  $(a_1, b_1) = (1, 1)$ ,  $(a_2, b_2) = (3, 3)$  and  $X = \mathbb{P}^1 \times \mathbb{P}^2$  we are in case 2 and we have a projective bundle  $\pi : \mathcal{H} \longrightarrow \mathbb{P}(1, 1)$  whose fiber  $\pi^{-1}(F) = \mathbb{P}(\frac{V(3, 3)}{F.V(2, 2)})$  has dimension 21, which implying dim  $\mathcal{H} = 26$ .
- (2) Let  $(a_1, b_1) = (1, 1)$ ,  $(a_2, b_2) = (a_3, b_3) = (1, 2)$  and  $X = \mathbb{P}^1 \times \mathbb{P}^3$ . Combining Case 1 with Case 2, we have a Grassmannian bundle  $\pi : \mathcal{H} \longrightarrow \mathbb{P}(1, 1)$ , whose fibers  $\pi^{-1}(F) = G(2, \frac{V(1,2)}{F.V(0,1)})$  has dimension 28, which implying dim  $\mathcal{H} = 35$ .
- (3) Let  $(a_1, b_1) = (a_2, b_2) = (1, 1)$ ,  $(a_3, b_3) = (2, 2)$  and  $X = \mathbb{P}^2 \times \mathbb{P}^2$ . Combining Case 1 with Case 2, we have a projective bundle  $\pi : \mathcal{H} \longrightarrow G(2, V(1, 1))$ , whose fibers are  $\pi^{-1}(F, G) = \mathbb{P}(\frac{V(2, 2)}{F.V(1, 1) + G.V(1, 1)})$ . Thus,

 $\dim(\mathbf{F}.V(1,1) + \mathbf{G}.V(1,1)) = 9 + 9 - 1 = 17,$ 

which implies dim  $\pi^{-1}(F, G) = 17$ , thus dim  $\mathcal{H} = 32$ .

We hope to do the same for the remaining cases in example 10, but unfortunately, the bidegrees don't behave so well. In the next section, we will construct the Hilbert scheme for ACM complete intersections, the idea is derived from [6, Section 2.2.2].

5.2. On Complete Intersection ACM. In this subsection, we will find the scheme  $\mathcal{H}$  that represents the contravariant functor F when we consider complete intersections ACM of codimension greater than 2 in X. We will establish an order on the bidegrees of the bihomogeneous forms and we apply the tools of [6] to construct  $\mathcal{H}$ . Let's denote  $(\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s)$  the distinct bidegrees  $(a_i, b_i)$ , in other words

$$(a_1, b_1) = \cdots = (a_{m_1}, b_{m_1}) = (\alpha_1, \beta_1)$$
$$(a_{m_1+1}, b_{m_1+1}) = \cdots = (a_{m_1+m_2}, b_{m_1+m_2}) = (\alpha_2, \beta_2)$$

 $(a_{m_1+\dots+m_{s-1}+1}, b_{m_1\dots+m_{s-1}+1}) = \dots = (a_{m_1+\dots+m_s}, b_{m_1+\dots+m_s}) = (\alpha_s, \beta_s).$ 

We will show that F is representable by a scheme  $\mathcal{H}$  with a universal family  $\mathcal{X}$ . We construct this universal family  $\mathcal{X}_r$  and an integral smooth scheme  $\mathcal{H}_r$  on  $\text{Spec}(\mathbb{Z})$  by induction on  $0 \leq r \leq s$  such that the natural morphism  $p_r : \mathcal{X}_r \subset X \times \mathcal{H}_r \longrightarrow \mathcal{H}_r$  is flat and has fibers ACM defined by  $m_t$  equations of bidegree  $(\alpha_t, \beta_t), t = 1, \dots, r$ .

If r = 0 we take  $\mathcal{H}_0 = \operatorname{Spec}(\mathbb{Z})$  and  $\mathcal{X}_0 = X$ . Now we suppose, by induction hypothesis, that  $\mathcal{X}_{r-1}$  and  $\mathcal{H}_{r-1}$  are constructible. From Proposition 19,  $\mathcal{E}_r = p_{r-1,*}\mathcal{O}_{\mathcal{X}_{r-1}}(\alpha_r, \beta_r)$  is locally free sheaf and commutes with base change, hence we can consider the Grassmannian bundle  $G_r = G_r(m_r, \mathcal{E}_r^{\vee})$  and  $\pi_r : G_r \longrightarrow \mathcal{H}_{r-1}$  its natural projection. From this construction, we get the following commutative diagram

$$\begin{array}{ccc} \pi_r^* \mathcal{X}_{r-1} & \longrightarrow & \mathcal{X}_{r-1} \\ p_r & & & \downarrow^{p_{r-1}} \\ G_r & \xrightarrow{\pi_r} & \mathcal{H}_{r-1} \end{array}$$

We consider the inclusion of the tautological sheaf  $\mathcal{F}_r \longrightarrow \pi_r^* \mathcal{E}_r$  on the Grassmannian bundle  $G_r$  and by changing the basis of the morphism  $\pi_r$ , this injection is rewritten as  $\mathcal{F}_r \longrightarrow p_{r,*}\mathcal{O}_{\pi_r^*\mathcal{X}_{r-1}}(\alpha_r,\beta_r)$ . Pulling back on  $\pi_r^*\mathcal{X}_{r-1}$  and using adjunction we get a morphism of vector bundles  $p_r^*\mathcal{F}_r \longrightarrow \mathcal{O}_{\pi_r^*\mathcal{X}_{r-1}}(\alpha_r,\beta_r)$ . The locus of zeros of this morphism is a closed subscheme of  $\pi_r^*\mathcal{X}_{r-1}$  which we denote by  $Z_r$ .

This last part of the construction is the same one made by Olivier in 6 and we replicate it by considering bihomogeneous forms on X instead of homogeneous forms in  $\mathbb{P}^N$ . Let V be the  $m_r$ -dimensional vector subspace of  $\mathcal{E}_{r,x} = H^0(\mathcal{X}_{r-1,x}, \mathcal{O}_{\mathcal{X}_{r-1,x}}(\alpha_r, \beta_r))$  which corresponds  $y \in G_r$  with  $x = \pi_r(y)$ . Then, by construction,  $Z_{r,y}$  is identified by a subscheme of  $\mathcal{X}_{r-1}$  defined by  $\{F = 0 \mid F \in V\}$  and this implies that  $Z_{r,y}$  has codimension at most  $m_r$  in  $\mathcal{X}_{r-1}$ . Let  $\mathcal{H}_r$  be the open of  $G_r$  such that the  $Z_{r,y}$  has minimal dimension. The set  $Z_{r,y}$  is the locus of complete intersection defined by  $m_i$  equations of bidegree  $(\alpha_i, \beta_i)$  for all  $1 \leq i \leq r$ . Thus we consider  $\mathcal{X}_r$  to be the inverse image of  $\mathcal{H}_r$  restricted to  $Z_r$  and all hypotheses about  $\mathcal{X}_r$  and  $\mathcal{H}_r$  are satisfied as desired.

**Proposition 23.** The Hilbert scheme of ACM complete intersections is H, as constructed above.

Since ACM complete intersections in X are independent of change of basis c.f. Proposition 18 and are stable by generation, see Proposition 20, we can enunciate the following theorem whose proof is a simple replication of the arguments used by Olivier in [6, Proposition 2.2.5] and so we will omit it.

**Theorem 24.** The scheme  $\mathcal{H}$  constructed in the Propositions 21, 22 and 23 represents the functor F and the map  $pr : \mathcal{X} \to \mathcal{H}$  is the universal family.

**Example 25.** For simplicity, we denote  $V(r,s) := H^0(X, \mathcal{O}_X(r,s))^{\vee}$  and  $\mathbb{P}(r,s) := \mathbb{P}(H^0(X, \mathcal{O}_X(r,s))^{\vee})$ . From Example 10 we have:

(1) Let  $(a_1, b_1) = (0, 2)$ ,  $(a_2, b_2) = (a_3, b_3) = (1, 1)$ ,  $(a_4, b_4) = (1, 2)$  and  $X = \mathbb{P}^1 \times \mathbb{P}^4$ . We have a projective bundle  $\pi : Y \longrightarrow U$ , where U is the locus of complete intersection of  $G(2, V(1, 1)) \times \mathbb{P}(0, 2)$ , whose fibers are

$$\pi^{-1}(H_1, H_2, F) = \mathbb{P}(\frac{V(1, 2)}{H_1 \cdot V(0, 1) + H_2 \cdot V(0, 1) + F \cdot V(1, 0)}).$$

Take  $\mathcal{H}$  to be the open set of complete intersection of Y.

(2) Let  $(a_1, b_1) = (1, 1), (a_2, b_2) = (1, 2), (a_3, b_3) = (2, 1)$  and  $X = \mathbb{P}^2 \times \mathbb{P}^2$ . Take  $\pi : Y \longrightarrow \mathbb{P}(1, 1)$ , whose fibers are  $\pi^{-1}(H) = \mathbb{P}(\frac{V(1, 2)}{F.V(0, 1)}) \times \mathbb{P}(\frac{V(2, 1)}{F.V(1, 0)})$ . Thus,  $\mathcal{H}$  is the open of complete intersection of Y.

## 6. Moduli of Complete Intersection in the Biprojective Space $\mathbb{P}^1 \times \mathbb{P}^1$

We will consider G one of the following groups  $PGL(2) \times PGL(2)$  or  $SL(n) \times SL(n)$ . Let  $Q := \mathbb{P}^1 \times \mathbb{P}^1$  be a smooth quadric in  $\mathbb{P}^3$  and denote by  $\pi_1$  and  $\pi_2$  the natural projections. We are interested in the case where  $\mathcal{C} = V(F)$  is a smooth curve. Without loss of generality, we can assume  $d_1 \leq d_2$  and F be a form of bi-degree  $(d_1, d_2)$ .

We recall some results about determinants which can be found in [10, pg 433]. The discriminant of a homogeneous polynomial  $f(x_0, \dots, x_n)$  of degree d is an irreducible polynomial  $\Delta(f)$  in the coefficients of f which vanishes if and only if all the partial derivatives  $\partial f/\partial x_0 \cdots \partial f/\partial x_n$  have a common zero in  $k^{n+1} - \{0\}$ . Under the geometric approach, the discriminant corresponds to the Veronese embedding

$$\mathbb{P}^n = \mathbb{P}(k^{n+1}) \longrightarrow \mathbb{P}(S^d k^{n+1}).$$

For  $f(x_0, x_1)$  homogeneous polynomial of degree d, we have  $\Delta(f)$  has degree 2d - 2. The action of GL(2) on  $\mathbb{P}^1$  induces an linear action on homogeneous polynomials via  $(A^*f)(x_0:x_1) := f(A^{-1}(x_0:x_1))$ , for every  $A \in GL(2)$  (similarly, we have an action of  $GL(2) \times GL(2)$  in an form F). We have that  $\Delta(A^*f) = (\det A^{-1})^{d(d-1)}\Delta(f)$ , for a proof see [10, pg 404].

Let

$$F(x_0, x_1, y_0, y_1) = \sum_{i+j=d} x_0^i x_1^j G_i(y_0, y_1)$$
(7)

be a form of bidegree  $(d_1, d_2)$ , where  $G_i(y_0, y_1)$  is a homogeneous polynomial of degree  $d_2$ . Now we define the discriminant  $(\Delta F)(y_0, y_1)$  or just  $\Delta(F)$ , where we take F as polynomials in the variables  $x_i$  and coefficients in  $G_i(y_0, y_1)$ . For example:

(1) If  $d_1 = 2$ , then

$$\Delta(F) = G_1^2 - 4G_0G_2$$

(2) If  $d_1 = 3$ , then

(3) If  $d_1 = 4$ , then

$$\Delta(F) = G_1^2 G_2^2 - 4G_0 G_2^3 - 4G_1^3 G_3 - 27G_0^2 G_3^2 + 18G_0 G_1 G_2 G_3$$

$$\begin{split} \Delta(F) &= 256G_0^3G_4^3 - 27G_0^2G_3^4 - 27G_4^2G_1^4 + 16G_0G_2^4G_4 - 4G_0G_2^3G_3^2 - 4G_1^2G_2^3G_4 \\ &- 4G_1^3G_3^3 + G_1^2G_2^2G_3^2 - 192G_0^2G_1G_4^2G_3 - 128G_0^2G_2^2G_4^2 \\ &+ 144G_0^2G_2G_4G_3^2 + 144G_0G_2G_4^2G_1^2 - 6G_0G_3^2G_4G_1^2 \\ &- 80G_0G_1G_2^2G_3G_4 + 18G_0G_1G_2G_3^3 + 18G_1^3G_2G_3G_4 \end{split}$$

**Remark 26.** Let F be a form of bidegree  $(d_1, d_2)$ , where  $d_i \ge 3$ . From [10, Proposition 1.2, p.g 445] we have that C = V(F) is smooth if, only if, every root of  $(\Delta F)(y_0, y_1)$  and  $(\Delta F)(x_0, x_1)$  is simple.

**Theorem 27.** Let  $F \in k[x_0, x_1, y_0, y_1]$  be a nonsingular form of bidegree  $(d_1, d_2)$ , where  $d_i \geq 3$ . Then F is invariant under at most finitely many  $g \in G$ .

*Proof.* Using the equation (7) we can write F with variables  $x_i$  where their coefficients are forms of degree  $d_2$  and its discriminant will be a homogeneous polynomial in the variables  $y_i$  which we denote by  $\Delta(y_0, y_1)$ . Since  $d_1 \ge 2$  we have

$$\deg(\Delta(y_0, y_1)) = (2d_1 - 2)(d_2) \ge 12.$$

Take  $g = (A, B) \in G$  such that g \* F = F. The discriminant we have

$$\Delta(g * F) = B^*((\det A^{-1})^{d_1(d_1 - 1)} \Delta(y_0, y_1)), \tag{8}$$

but F is smooth, which implies that the discriminant  $\Delta(y_0, y_1)$  has distinct roots, cf. Remark 26, hence the action of B on  $\Delta(y_0, y_1)$  will fix the set of its roots. As the set of roots of  $\Delta(y_0, y_1)$  It is finite and has at least 12 elements, so B is a scalar multiple of the identity matrix. Similarly, using  $\Delta(x_0, x_1)$  we have that A is a scalar multiple of the identity matrix.

Let  $B = b \cdot Id$  and  $A = a \cdot Id$  for some  $a, b \in \mathbb{C}^*$ . If  $G = PGL(2) \times PGL(2)$ , then g is the identity of G while for  $G = SL(2) \times SL(2)$  we get  $a, b \in \{-1, 1\}$  and in both cases there are finitely many  $g \in G$  such that g \* F = F.

**Remark 28.** The Theorem 27 is not true for  $GL(2) \times GL(2)$ . Indeed, if F is a form of bidegree  $(d_1, d_2)$ , then g \* F = F for every  $g \in H$ , where

$$H = \{ (a \cdot Id, b^{-1} \cdot Id) \in GL(2) \times GL(2) \mid a \in k^* \text{ an } b \text{ is an } (d_2) \text{-root of } a^{d_1} \}.$$

The Theorem 27 is true for any subgroup of  $K \subset GL(2) \times GL(2)$  where  $\#(K \cap H) < \infty$ .

Let  $p(t) = (d_1 + d_2)t + 1 - (d_1 - 1)(d_2 - 1)$  an Hilbert polynomial, if  $\mathcal{C} \in \mathcal{H}_Q^{p(t)}$ , then  $\mathcal{C}$  has bidegree  $(d_1, d_2)$  or  $(d_2, d_1)$ . To avoid confusion, we can take the action of  $G \rtimes \mathbb{Z}_2$  on  $\mathcal{H}_Q^{p(t)}$ , this is equivalent to the action of G on  $\mathbb{P}(H^0(Q, \mathcal{O}_Q(d_1, d_2)))$ .

Fix a bidegree  $(d_1, d_2)$ , every form F in this bidegree corresponds to a point of  $\mathbb{P}^N = \mathbb{P}(H^0(Q, \mathcal{O}_Q(d_1, d_2)))$ , where  $N = (d_1 + 1)(d_2 + 1) - 1$ . Denote by  $\mathcal{U}_{d_1, d_2} \subset \mathbb{P}^N$  the locus of smooth curve of bidegree  $(d_1, d_2)$ , it is an affine variety.

**Corollary 29.** Suppose  $d_i > 2$ , then every point of  $\mathcal{U}_{d_1,d_2}$  is stable for the action of G. *Therefore, there exists a good quotient*  $[\mathcal{U}_{d_1,d_2}/G]$ .

*Proof.* From equation 8, for  $g = (A, B) \in G$  and  $F \in U_{d_1,d_2}$ , the multiplicity of the zeros of the discriminant does not change by this action, and thus G acts on the projective space  $\mathbb{P}(H^0(Q, \mathcal{O}_Q(d_1, d_2)))$  preserving the open set  $\mathcal{U}_{d_1,d_2}$  of smooth curves. On the other hand, since every point of  $\mathcal{U}_{d_1,d_2}$  has a finite stabilizer, it follows from [2, Corollary 5.14, pg 166] that  $\mathcal{U}_{d_1,d_2}$  is stable. Since  $GL(2) \times GL(2)$  is linearly representable, then G is linearly representable, in particular, see [2, Corollary 5.17, pg 167], the quotient  $[\mathcal{U}/G]$  exists.

The Corollary 29 shows that the quotient  $[U_{d_1,d_2}/G]$  is a coarse moduli space for smooth curves on a fixed quadric of  $\mathbb{P}^3$ .

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