

# NON-PERSISTENCE OF STRONGLY ISOLATED SINGULARITIES, AND GEOMETRIC APPLICATIONS

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**ABSTRACT.** We obtain a generic regularity result for stationary integral  $n$ -varifolds assumed to have only strongly isolated singularities inside  $N$ -dimensional Riemannian manifolds, without any restriction on the dimension ( $n \geq 2$ ) and codimension. As a special case, we prove that for any  $n \geq 2$  and any compact  $(n + 1)$ -dimensional manifold  $M$  the following holds: for a generic choice of the background metric  $g$  all stationary integral  $n$ -varifolds in  $(M, g)$  will either be entirely smooth or have at least one singular point that is not strongly isolated. In other words, for a generic metric only “more complicated” singularities may possibly persist. This implies, for instance, a generic finiteness result for the class of all closed minimal hypersurfaces of area at most  $4\pi^2 - \varepsilon$  (for any  $\varepsilon > 0$ ) in nearly round four-spheres: we can thus give precise answers, in the negative, to the questions of persistence of the Clifford football and of Hsiang’s hyperspheres in nearly-round metrics. The aforementioned main regularity result is achieved as a consequence of the fine analysis of the Fredholm index of the Jacobi operator for such varifolds: we prove on the one hand an exact formula relating that number to the Morse indices of the conical links at the singular points, while on the other hand we show that such an index is non-negative for all such varifolds if the ambient metric is generic.

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## 1. INTRODUCTION

**1.1. Context and geometric motivations.** In 1983, Hsiang [22] solved Chern's spherical Bernstein conjecture by constructing a sequence of infinitely many, pairwise distinct, embedded minimal hyperspheres in the round four-dimensional sphere. This result is to be compared with what happens in one dimension less, as determined by the earlier rigidity theorems obtained by Almgren [4] and Calabi [10]. Such hyperspheres are constructed by means of equivariant methods and the analysis of the resulting singular ODE system in the quotient space. It turns out that their (unit multiplicity) varifold limit is the spherical suspension of a Clifford torus inside an equatorial three-sphere, thus a singular minimal subvariety (henceforth referred to as Clifford football), which is smooth at all points except the north and south pole.

It is an interesting question, brought to the attention of the first-named author about a decade ago by Neves, whether such a picture persists when considering nearly-round metrics on  $S^4$ . In particular, one may ask whether for *any* metric  $g$  sufficiently close to the round one - say in the smooth topology - the Riemannian manifold  $(S^4, g)$  still contains infinitely many (embedded) minimal hyperspheres. This question stems from the well-known link between rigidity phenomena characterizing special submanifolds in round spheres, and the resulting scarcity phenomena for slightly deformed metrics. Indeed, as a reflex of the well-known characterization of simple closed geodesics on the round two-sphere as equatorial circles it was proven by Morse that there are nearly-round metrics on  $S^2$  that have *only three* simple closed geodesics, and similarly (as a reflex of the aforementioned rigidity theorems by Almgren and Calabi) White [47] proved that there are nearly-round metrics on  $S^3$  that have *only four* minimal hyperspheres. Aiming for an understanding of the situation in ambient dimension more naturally leads to the question above.

In principle, one may attack this problem by means of a perturbative approach of essentially PDE-theoretic nature, by first showing (ideally) that any nearly-round metric allows for a - suitably defined - singular minimal subvariety modelled on the aforementioned Clifford football, and then desingularizing the football in question to obtain the desired minimal hyperspheres. Such an approach turns out to be at the very least challenging, because a direct application of the implicit function theorem is (unsurprisingly) obstructed by the large kernel of the Jacobi operator of the football; in suitable weighted Sobolev spaces such a kernel has actually dimension 18 (cf. [11], see later discussion in Section 3) and it is unclear how to possibly handle it by

means of a Lyapunov-Schmidt reduction; in fact, it is a consequence of the present work (see, specifically, the statements of Corollary 1.4 and Corollary 1.6 below), that such a program is inevitably doomed to fail.

**1.2. Main results.** For indeed, in this article we approach Neves' question from a totally different perspective, and answer it in strong negative terms. Such a conclusion ultimately descends from our main theorem, which can be stated as follows:

**Theorem 1.1.** *Given a closed manifold  $M$  of dimension  $N \geq 3$ , there exists a generic subset  $\mathcal{G}_0$  of the space of smooth metrics on  $M$  with the following property: for every  $g \in \mathcal{G}_0$ , any  $g$ -stationary integral  $n$ -varifold in  $(M, g)$ ,  $2 \leq n < N$ , will:*

- (i) *either be entirely smooth, or*
- (ii) *have at least one singular point that is not strongly isolated, or*
- (iii) *have only strongly isolated singular points all having Morse index equal to  $N$ .*

The precise notion of “strongly isolated singularity” is recalled in Definition 2.3. In particular, in the codimension one case (that is to say: when  $N = n + 1$ ) the third alternative may not possibly happen (see Remark 2.9) and so we conclude an unconditional generic regularity result.

**Corollary 1.2.** *Given a closed manifold  $M$  of dimension  $n + 1 \geq 3$ , there exists a generic subset  $\mathcal{G}_0$  of the space of smooth metrics on  $M$  with the following property: for every  $g \in \mathcal{G}_0$ , any  $g$ -stationary integral varifold will either be entirely smooth or have at least one singular point that is not strongly isolated.*

*Remark 1.3.* In the statements above and throughout this article the notion of “genericity” is understood in the sense of Baire; a well-known result by White (see [47, 49]) ensures that the set of smooth ( $C^\infty$ ) Riemannian metrics such that *any* closed, embedded minimal hypersurfaces are non-degenerate (namely: have no non-trivial Jacobi fields) is indeed generic. We can thus assume to have fixed, once and for all, a subset  $\mathcal{G}$  of the class of smooth Riemannian metrics enjoying such a property, and when we write - for instance - that  $g \in \mathcal{N}(\varepsilon)$  is a generic metric we mean  $g \in \mathcal{N}(\varepsilon) \cap \mathcal{G}$ . In particular, in the previous statement we shall tacitly agree that  $\mathcal{G}_0 \subset \mathcal{G}$ .

Now, the connection with the geometric question we started with, and more specifically with the problem of perturbing the Clifford football to nearby minimal varifolds lies in the fact that, if we impose an area bound - that is to say: a suitable upper bound on the mass of our integral varifold - then the same varifold necessarily has a very special singular set (all singularities are indeed strongly isolated) and thus has the structure of what we call an **MSI** (see Definition 2.3 and Proposition 5.1). Informally speaking, one can say that under such bound the Clifford football represents, in terms of regularity, the worst that can possibly happen. As a result, we have the following geometric consequence:

**Corollary 1.4.** *Given any  $\varepsilon > 0$ , there is a neighborhood  $\mathcal{N}(\varepsilon)$  of the round metric in the space of smooth metrics on  $S^4$  such that for a generic metric  $g \in \mathcal{N}(\varepsilon)$ , the following is true: Every mod 2 cyclic  $g$ -stationary integral varifold with mass less than  $4\pi^2 - \varepsilon$  is entirely smooth.*

This statement does not, in itself, give a direct response to Neves' question, although it indicates a definite obstruction to the possible "perturbation-desingularization" approach described above. But, in fact, Theorem 1.1 also implies a generic finiteness result under a pure area bound:

**Corollary 1.5.** *Given any  $\varepsilon > 0$  there exists a neighborhood  $\mathcal{N}(\varepsilon)$  of the round metric on  $S^4$  such that for a generic choice of  $g \in \mathcal{N}(\varepsilon)$  the Riemannian manifold  $(S^4, g)$  shall contain only finitely many closed, embedded minimal hypersurfaces of area less than  $4\pi^2 - \varepsilon$ .*

We recall that  $2\pi^2$  equals the area of an equatorial  $S^3$  in round  $S^4$ ; we expect the threshold  $4\pi^2$  to be sharp, based on the following argument. As will also be mentioned later (in Remark 1.7), any nearly-round metric on  $S^4$  will support (at least) five minimal hyperspheres, any pair of which shall intersect by virtue of the Frankel property. Now, one expects that the desingularization of (any) such pair would give rise to infinitely many closed minimal hypersurfaces with area arbitrarily close to  $4\pi^2$  (by virtue of the implicit function theorem).

A statement describing the implications at the level of the "perturbation" problem we posed above requires some notation and a brief digression. One can "parametrize" the Hsiang hyperspheres by an integer  $k \in \mathbb{N}$ , agreeing that  $M_k$  is the Hsiang hypersphere intersecting the Clifford football transversely along exactly  $2k + 1$  tori; in particular for  $k = 0$  one recovers the equatorial three-dimensional hypersphere. Of course, one can act (both on the the Clifford football, and on any of Hsiang's hypersurfaces) via isometries of round  $S^4$ ; considering such actions is already necessary if one attempts to perturb such hypersurfaces "one (minimal) hypersphere at a time" (see the sequel of this introduction for more on this matter).

For an open set  $\mathcal{N}$  of Riemannian metrics on  $S^4$  (with  $\mathcal{N} \ni g_0$ ) let us assume to have a continuous map

$$(1) \quad \Psi_k : \mathcal{N} \rightarrow O(5) \times C^1(M_k; \mathbb{R})$$

such that  $\Psi_k^{(2)}(g)$  defines a normal graph over the image through  $T^{(k)} =: \Psi_k^{(1)}(g)$  of  $M_k$  - a hypersurface henceforth simply denoted  $M_k(g)$  - that is minimal in metric  $g$ ; in particular, if this is the case we note that the "area" map  $\mathcal{N} \ni g \mapsto \|M_k(g)\|$  is itself continuous.

That being said, and recalling that the area of the Clifford football equals  $\pi^3 \simeq 31.00063$  it follows from the aforementioned varifold convergence that there exist at most finitely many (conjecturally none) Hsiang hyperspheres whose area exceeds the threshold value  $4\pi^2 = 39.47842$ ; thus note that the conclusion of Corollary 1.5 cannot possibly be true if we remove the genericity assumption (one counterexample being indeed the round metric). With all of this notation in place and keeping in mind all these remarks, the preceding statement implies this one:

**Corollary 1.6.** *For any  $\varepsilon > 0$  the following holds.*

*Given any neighborhood  $\mathcal{N} \ni g_0$  of smooth Riemannian metrics on  $S^4$  there are only finitely many integers  $k_1 < k_2 < \dots, k_\ell$  and, for each such integer, only a finite set  $I(k_j)$  such that a continuous map  $\Psi_{k_j}^{i_j}, (i_j \in I(k_j))$  as per (1) and satisfying*

$$\sup_{g \in \mathcal{N}} \|M_{k_j}^{i_j}(g)\| < 4\pi^2 - \varepsilon$$

*can possibly exist.*

This means that, no matter how small the neighborhood of nearly-round metrics we consider (and no matter how cleverly we choose the isometries  $T^{(k)}$ ), only finitely many Hsiang hyperspheres shall possibly persist.

*Remark 1.7.* We explicitly stress that the finiteness result given in Corollary 1.5 cannot possibly be upgraded to a uniform bound. For indeed, a standard application of the Lyapunov-Schmidt reduction (applied “one hypersphere at a time”) plus simple Morse-theoretic arguments (cf. [47]) allows to conclude, for any  $k \in \mathbb{N}$  the existence of a neighborhood  $\mathcal{N}_k \ni g_0$  such that for any  $g \in \mathcal{N}$  the Riemannian manifold  $(S^4, g)$  shall contain at least one minimal hypersphere defined by continuous deformations - understood in the sense above - of exactly as many isometric copies  $M_k$ ; in particular for any metric  $g \in \bigcap_{0 \leq k \leq \bar{k}} \mathcal{N}_k$  the manifold  $(S^4, g)$  shall contain a finite but arbitrarily large number of closed embedded minimal hypersurfaces, provided one takes  $\bar{k}$  big enough.

Let us add some comments on our “generic regularity” theorem, that is Theorem 1.1, without restricting to the codimension one case. When considering the class of  $n$ -dimensional stationary varifolds with only strongly isolated singularities (cf. Definition 2.3) in an  $N$ -dimensional Riemannian manifold, we can rule out, *generically*, all singularities whose link has Morse index strictly larger than  $N$  (note that the weak inequality is, in fact, always satisfied, see Remark 2.9). It is appropriate for us to remark that, when the link in question is disconnected, such a condition is, in fact, certainly fulfilled with the sole (well justified) exception of links consisting of the disjoint union of exactly two half-dimensional equatorial spheres (that is to say:  $2n = N$ ). The fact that there may (in fact: there should) be persistent isolated singularities in the case of connected link is less obvious, although e. g. we can note that the Veronese embedding of  $\mathbb{C}\mathbb{P}^2$  inside the 7-dimensional round sphere does not bound any smooth five-dimensional manifold (so that the cone over such  $\mathbb{C}\mathbb{P}^2$  cannot possibly be smoothed out by minimal submanifolds). This clearly provides partial albeit compelling evidence in that direction; we refer the reader to the striking recent work by Liu [28] for a thorough discussion of these aspects and the construction of a number of different examples of homologically area-minimizing submanifolds with non-smoothable singularities.

*Remark 1.8.* More generally, one can study (after Cartan) the following three examples of focal submanifolds of isoparametric minimal hypersurface in spheres. For  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  set  $m = \dim_{\mathbb{R}}(\mathbb{F})$  and consider the embedding map  $\mathbb{F}\mathbb{P}^2 \rightarrow \mathbb{S}^{3m+1} \subset \mathbb{R}^{3m+2} = \mathbb{F}^3 \times \mathbb{R}^2$  defined by

$$[u : v : w] \mapsto (|u|^2 + |v|^2 + |w|^2)^{-1} \cdot \left( \sqrt{3}v\bar{w}, \sqrt{3}w\bar{u}, \sqrt{3}u\bar{v}, \frac{3}{2}(|u|^2 - |v|^2), \frac{1}{2}(2|w|^2 - |u|^2 - |v|^2) \right)$$

This gives the classical Veronese minimal  $\mathbb{R}\mathbb{P}^2$  in  $\mathbb{S}^4$ , a minimal  $\mathbb{C}\mathbb{P}^2$  in  $\mathbb{S}^7$  and a minimal  $\mathbb{H}\mathbb{P}^2$  in  $\mathbb{S}^{13}$ . It is by now standard to prove that none of the three projective planes in question bounds a smooth manifold (of real dimension, respectively, 3, 5, 9) since they have odd Euler characteristic. We are led to believe that all of the corresponding minimal submanifolds have Morse index equal to  $3m + 2$  (i. e.  $N$  in the notation of Theorem 1.1) - which at the moment is only known for  $\mathbb{F} = \mathbb{R}$ , see Remark 2.9 - and provide models of persistent singularities.

**1.3. Previous work.** In recent years we have witnessed impressive advances on the theme of “generic regularity” of minimal hypersurfaces [12, 14, 15, 17, 25–27], not to mention the related (equally striking) advances on the study of generic mean curvature flows, which can simply

not properly be accounted for here. (For various significant results on generic properties of geodesics and geodesic nets the reader is instead referred to [13, 31] as well as references therein.) Besides being constrained to the codimension one case, many such results typically concern area-minimizing hypersurfaces, either with respect to a fixed boundary (i. e. in the framework of Plateau’s problem) or with respect to a fixed homology class (cf. [50, Problem 108] and [3, Problem 5.16]); this is the case, in particular, for [14, 15], which very significantly refined the pioneering theorem by Smale [42]. Instead [12, 17, 25–27, 43] rather deal (in various forms) with the somewhat more general case of minimal hypersurfaces arising from min-max techniques (in the spirit of Almgren-Pitts) thus not necessarily (and not typically) stable ones; however such works are constrained to ambient dimension 8 and allow for singularities that are modelled on stable minimal hypercones (as reflected by the assumption of local stability of the minimal hypersurfaces in question). (We wish to stress that, on the contrary, [14, 15] allow for more general singular sets, which in particular do not need to consist of isolated points.)

It is to be remarked that many of such contributions build upon the pioneering work by Hardt and Simon [21]: going beyond the results in Bombieri-De Giorgi-Giusti [8] (constructing minimal smoothings of the Simons cone in  $\mathbb{R}^{2n}$ ,  $2n \geq 8$ , by an ODE approach) they proved that, in fact, every regular minimizing hypercone can be uniquely (up to scaling) perturbed to one side to produce a smooth minimizing hypersurface asymptotic to it near infinity. Making use of this, they succeeded in showing that minimizing hypersurfaces with a generic boundary in  $\mathbb{R}^8$  are actually smooth. Such results about the minimal smoothing of a minimizing hypercone were generalized by the third-named author without assuming the hypercone in question being regular (see [44]), while the strong uniqueness (up to a scaling) of such one-sided smoothing was then established by Edelen-Szekelyhidi [18] for “cylindrical” hypercones under a pure density assumption.

All that said, there are two aspects of substantial novelty in the present work: on the one hand we deal here with isolated singularities that are modelled on regular yet not necessarily stable cones (which is in fact necessary to ultimately resolve the geometric questions we posed at the very beginning of this introduction, as already displayed by the aforementioned Clifford football) while on the other hand our analysis applies to minimal  $n$ -dimensional submanifolds in an  $N$ -dimensional ambient manifold, without restricting to the codimension one case. The case of surfaces ( $n = 2$  thus  $N \geq 4$ ) is somewhat special and can be tackled with tools that are quite different than the ones employed here; the reader should in particular compare our results with those of White [45, 46] (obtaining, respectively a full “generic regularity result” for surfaces minimizing area with respect to a fixed boundary, or in a fixed homology class), and the ones by Moore [33, 34] (much related to earlier work by Böhme and Tromba, [7]) which are casted in the language of (prime) parametric minimal surfaces.

**1.4. Approach.** Our main results here, while crucially building on previous work of the second and third author (see, in particular, [27]), and to some extent on the deep contributions by Edelen [17], are obtained by a blend of different techniques and ideas. In a nutshell, our approach consists in studying the Fredholm index of the Jacobi (stability) operator of minimal submanifolds with only strongly isolated singularities (see Definition 2.3 of **MSI**), acting on suitably defined weighted Sobolev spaces. Employing powerful tools in linear Analysis, we first obtain an exact formula giving the value of such an index in terms of the normalized Morse index of links of the cones at the singular points; we refer the reader to Theorem 3.2

for a precise statement. Roughly speaking, the presence of singularities perturbs the natural Fredholm index zero Schrödinger operator, in a way that is solely encoded by the structure of the singularities. Such a counting formula, in spite of its striking simplicity, appears to be new to the best of our knowledge. Since, by virtue of general lower bounds for the Morse index of minimal submanifolds in round spheres, the local contributions are all non-positive (i. e.  $\leq 0$ ) they force the Fredholm index of any **MSI** to be itself less or equal than zero, with strict inequality in many case of interest (for instance: unconditionally in the codimension one case). We then complement such a result by showing that, on the other hand, generically (in the sense of Baire) such an index must be non-negative (i. e.  $\geq 0$ ); the approach that we present (see Section 4) essentially follows - at least at a *conceptual* level - ideas going back to the fundamental work of White [47], where one obtained suitable local and global Sard’s lemmata for the natural projector of metric-**MSI** pairs onto the first factor; while in that context one could exploit a Banach manifold structure (and thus ultimately invoke Smale’s Sard’s theorem) here the use of (pseudo-)canonical neighborhoods, which are significantly wilder than metric balls, does not allow for the employment of such pre-existing methodologies, and we rather need to work harder to unwind such tools and recast/adapt them to our own setting. It is in that respect that we crucially exploit the tools of [27] and [43]; we warn the reader, however, that due to our necessity of dealing with unstable (infinite Morse index) cones, and without any codimensional restriction, several changes (and corresponding adaptations) had to be performed, thereby not allowing for a direct quote of technical lemmata (which would have shortened and simplified this manuscript).

**1.5. Organization of the paper.** Besides the present introduction, this manuscript consists of four sections plus five appendices, that contain technical yet *essential* material that we decided to separate from the main body of the paper with the sole scope of improving readability, thereby leading to a more direct path towards the main theorems. In Section 2 we present the general setup of this work and collect some of the basic definitions we employ in the sequel. Then, Section 3 combines some preliminary facts about analysis on regular minimal cones with the proof of our “counting formula” for the Fredholm index of the Jacobi operator of an **MSI**. Hence, we move to the detailed study of the generic behaviour of such an operator, which we carry through in Section 4, crucially building upon various results proven in the various appendices. We then capitalize such efforts in Section 5, devoted to the proofs of the main theorems, also getting back to the motivating geometric applications we presented above.

Concerning the “technical material”, we start in Appendix A by collecting certain key features of the minimal surface equation (system) in arbitrary codimension and then proving several useful facts about the “transfer of normal sections” between nearby subvarieties. It is already at this level that one can get an appreciation of the difference in complexity between the codimension one case and the case of higher codimension: while in the former setting the transfer of normal section is - at least at a local level - essentially trivial (since one can just pre-compose with the parametrization map), in the latter one needs to project back onto the normal bundle of the base submanifold, which poses the problem of discussing how natural geometric PDEs transform under this operation, and how one may handle the error terms in the resulting equations. We then move on to Appendix B, devoted to certain three-circle inequalities, which allow to compare the behaviour of an **MSI** at different scales thereby deriving  $L^2$ -decay (more generally  $L^2$ -non concentration) estimates that are key to analyzing the limit

behaviour of tame Jacobi fields (cf. Definition 4.12). In Appendix C we discuss the problem of quantitative (uniform) uniqueness of tangent cones at strongly isolated singularities, which in particular allows to obtain Corollary C.2 concerning the “uniform convergence at all scales” of a stationary integral varifold on approach to a strongly isolated singularity (crucially building on both [38] and, more closely and recently, on the advances in [17]). In turn, such a statement serves as input for Appendix D devoted to a fine analysis of the mutual parametrization of two (possibly three) nearby **MSI**, that comes into play in the core of the paper when we prove the fundamental limit equation (see Lemma 4.25). We expect that some of these tools will also be helpful for other related studies in the near future. Lastly, in Appendix E we discuss (following Edelen’s study of the codimension one case) how to effectively “parametrize” the space of all **MSI** inside a Riemannian manifold, and thus how to obtain a controlled covering theorem (namely: Theorem 4.23) for the space of metric-**MSI** pairs by means of countably many (carefully designed) canonical pseudo-neighborhoods. Note that, in comparison to [17], for submanifolds in codimension other than one one loses the natural “order structure” coming from the trivialization of the normal bundle, thereby requiring the introduction of new techniques, e.g. to handle the combinatorics of “cascades” of isolated conical singularities. The covering theorem in question is then one of the three crucial ingredients in the proof of the global Sard theorem, Theorem 4.1, that we alluded to before.

## 2. NOTATION, SETUP AND PRELIMINARY RESULTS

**2.1. MSI and general setup.** Let us start with some basic definitions; since we will be working with ambient metrics of possibly finite degree of regularity, say  $C^{k,\alpha}$ , it is appropriate, to avoid ambiguity, to state what we mean by regular set of an integral varifold.

**Definition 2.1.** Let  $(M, g)$  be a Riemannian manifold of dimension  $N \geq 3$ . Given an integral varifold  $V$ , of dimension  $2 \leq n < N$ , we say that  $p \in \text{spt}(\|V\|) \subset M$  belongs to its regular set if there exists  $r > 0$  such that  $\text{spt}(\|V\|) \cap \overline{B^g(x, r)}$  is a smooth, embedded, compact, connected  $n$ -dimensional submanifold of class  $C^2$  with boundary contained in  $\partial B^g(x, r)$ . The set of regular points of  $V$  shall be denoted by  $\text{Reg}(V)$ ; its complement in  $\text{spt}(\|V\|)$  will be referred to as singular set of  $V$ , i.e. we let  $\text{Sing } V = \text{spt}(\|V\|) \setminus \text{Reg}(V)$ .

*Remark 2.2.* Of course,  $\text{Reg}(V)$  (respectively:  $\text{Sing } V$ ) is open (respectively: closed) in  $\text{spt}(\|V\|)$ . More importantly, by standard elliptic regularity (Schauder theory applied to the minimal surface system) if - in the setting of the preceding definition -  $p \in \text{Reg}(V)$  and the ambient metric  $g$  is  $C^{k,\alpha}$  then, in fact,  $\text{spt}(\|V\|)$  is  $n$ -dimensional submanifold of class  $C^{k+1,\alpha}$  in an open neighborhood of the point in question (see Appendix A, specifically Proposition A.1 therein); in particular if the ambient metric is smooth (by which we shall mean  $C^\infty$ ) then so will be  $\text{spt}(\|V\|)$  away from the singular set of  $V$ .

Let us proceed and introduce the the objects that we study throughout this article:

**Definition 2.3.** Let  $(M, g)$  be a Riemannian manifold of dimension  $N \geq 3$ . For every stationary integral varifold  $V$ , of dimension  $2 \leq n < N$ , we call a point  $p \in \text{Sing}(V)$  **strongly isolated**, if *some* tangent cone of  $V$  at  $p$  is **regular**, i.e. it is of multiplicity one and has smooth link (equivalently: if the singular set of such a cone coincides with the origin). We will then say that a stationary integral varifold  $V$  **has only strongly isolated singularities** (and refer to it as **MSI**) if either  $\text{Sing}(V) = \emptyset$ , or every  $p \in \text{Sing}(V)$  is strongly isolated.

*Remark 2.4.* We remark that, in the setting of the preceding definition, at any singular point  $p \in \text{Sing}(V)$  there holds uniqueness of the tangent cone in question, thanks to a deep result of Simon [38] (see also [40]). Furthermore, the corollary stated after Theorem 5 therein implies that every such singular point has a neighborhood where there is exactly one singular point: hence, since  $\text{Sing}(V)$  is closed and thus compact whenever the ambient manifold is, the finite covering property tells us that there are only finitely many such singular points (thereby justifying the terminology we employ).

*Remark 2.5.* Relying on the content of the previous remark, we will switch to a somewhat more natural (or, possibly, more convenient) notation: given one such varifold, we will rather employ the letter  $\Sigma = \Sigma^n$  to denote it, meaning that  $V = |\Sigma|$  (that is to say:  $V$  is the *multiplicity 1* integral varifold associated to the submanifold  $\Sigma$ ) and in our work we will practically identify  $\Sigma$  with the regular part of  $\bar{\Sigma}$ . Thus  $\Sigma$  is treated as a regular (cf. Remark 2.2) but not necessarily closed submanifold. The (unique) tangent cone of  $\Sigma$  at a singular point  $p$  is denoted by  $\mathbf{C}_p(\Sigma)$ , or simply  $\mathbf{C}_p$  if there is no risk of confusion.

Based on White's natural homomorphism one can associate to an integral varifold a mod 2 flat chain (see [48]); if such object has zero boundary (in the sense of the standard boundary operator in the latter setting) we will simply refer to it as mod 2 cyclic varifold.

*Remark 2.6.* For later reference, let us note the following fact: an **MSI**  $\Sigma$  can always be regarded (in the sense of White's natural homomorphism [48]) as a mod 2 cyclic stationary varifold. This can be justified as follows. By Sard's theorem and the slicing theorem, we can choose a sequence  $r_i \rightarrow 0$  such that  $\Sigma_i := \Sigma \setminus B_{r_i}(\text{Sing } \Sigma)$  is a mod 2 flat chain with boundary  $\partial \Sigma_i$ , and, *as soon as*  $n \geq 2$ , the mass of  $\partial \Sigma_i$  converges to 0 as  $i \rightarrow \infty$ . By the Federer-Fleming compactness theorem ([20], see also [19]),  $\Sigma = \lim_i \Sigma_i$  is a mod 2 flat chain and  $\partial \Sigma = \lim_i \partial \Sigma_i = 0$ . Hence,  $\Sigma$  is a cycle.

To fix the notation, we then let  $\Sigma$  have only strongly isolated singularities  $p_1, \dots, p_\ell$  with (respectively) regular cones  $\mathbf{C}_1, \dots, \mathbf{C}_\ell$  and associated links  $S_1, \dots, S_\ell$  (that is to say:  $S_i := \mathbf{C}_i \cap \mathbb{S}^{N-1}$ ). When we wish to stress the role of the basepoint (rather than the label) we shall write  $\mathbf{C}_p$  and  $S_p$  instead. In this setting and under such assumptions, one can find a compact set  $\Sigma_0 \subset \Sigma$  such that  $\Sigma \setminus \Sigma_0 = \bigsqcup_{i=1}^\ell E_i$  and for each value of the index  $i$  we have that  $E_i$  is diffeomorphic to the product of the corresponding link with the interval  $(1, \infty)$ . That said, we let  $\rho_{\Sigma, g} : \Sigma \rightarrow \mathbb{R}$  denote a smooth ( $C^\infty$ ) positive function equal to the distance  $\rho_i$ , in metric  $g$ , from the singular point  $p_i$  along  $E_i$ .

**2.2. First and second variation.** For a smooth submanifold  $\Sigma = \Sigma^n$  (not necessarily closed, cf. Remark 2.5) in a Riemannian manifold  $(M, g)$  we shall denote by  $T\Sigma$  its tangent bundle and we let  $\mathbf{V} := \mathbf{V}(\Sigma, M)$  denote its normal bundle instead. For sections  $X, Y \in \Gamma(T\Sigma)$  the second fundamental form of  $\Sigma$  inside  $M$  is defined by

$$\mathbb{I}_{\Sigma, g}(X, Y) = (\nabla_X Y)^\perp$$

where  $\nabla$  is the Levi-Civita connection in metric  $g$ , and  $^\perp = {}^{\perp, g}$  stands for the orthogonal projection onto  $\mathbf{V}$ ; the trace of the second fundamental form is then the (vector-valued) second fundamental form

$$H_{\Sigma, g} = \text{tr}_\Sigma(\mathbb{I}_{\Sigma, g}).$$

In the setting of Definition 2.3, it is a standard consequence of the first variation formula that an **MSI** has mean curvature identically equal to zero on its regular part.

We will routinely work with sections of the bundle  $\mathbf{V}$  and corresponding differential operators; in particular we let  $\nabla^\perp := \nabla_{\Sigma, g}^\perp$  denote the metric connection on  $\mathbf{V}$  that is determined by  $\nabla$ , and shall further denote by  $\Delta_{\Sigma, g}^\perp$  the Laplace operator on the normal bundle of  $\Sigma$ , namely  $\mathbf{V}$ . If we then consider the second variation of the  $n$ -dimensional area functional at the  $g$ -critical point  $\Sigma$  we will find, for any section (here assumed smooth)  $u \in \Gamma(\mathbf{V})$

$$(2) \quad \delta^2 \Sigma[u] = - \int_{\Sigma} g(u, L_{\Sigma, g} u) d\|\Sigma\|$$

where  $L_{\Sigma, g}$  is the ‘‘Jacobi operator’’ of  $\Sigma$ , which takes the form (see e. g. [41])

$$(3) \quad L_{\Sigma, g} u = \Delta_{\Sigma, g}^\perp u + g(\mathbb{I}_{\Sigma, g}, u) \mathbb{I}_{\Sigma, g} + \text{tr}_{\Sigma} R_g(u, \cdot, \cdot)$$

with  $R_g$  denoting the  $(1, 3)$ -curvature tensor of the ambient manifold  $(M, g)$ ; if  $\{X_1, \dots, X_n\}$  is any local orthonormal frame of the tangent sub-bundle  $T\Sigma$  then we mean

$$g(\mathbb{I}_{\Sigma, g}, u) \mathbb{I}_{\Sigma, g} = \sum_{i, j=1}^n g(\mathbb{I}_{\Sigma, g}(X_i, X_j), u) \mathbb{I}_{\Sigma, g}(X_i, X_j), \text{ and } \text{tr}_{\Sigma} R_g(u, \cdot, \cdot) = \sum_{i=1}^n R_g(u, X_i, X_i).$$

*Remark 2.7.* For later reference, let us recall here the following basic fact. If  $u \in \Gamma(\mathbf{V})$  and  $X, Y$  are (any otherwise unspecified) sections of  $T\Sigma$ , in the setting above, then the identity  $g(Y, u) = 0$  implies, by covariant differentiation in the direction of  $X$ , that

$$g(Y, \nabla_X u) = -g(\mathbb{I}_{\Sigma, g}(X, Y), u).$$

By the arbitrariness of  $X, Y$  this identity allows to identify the ‘‘tangential’’ component of  $\nabla u$  as  $-g(\mathbb{I}_{\Sigma, g}, u)$  and thereby, to relate bounds for  $\nabla u$  to bounds for  $\nabla^\perp u$  by reference to the second fundamental form of the submanifold in question.

There is a special case that warrants further discussion (and partly special notation): for a regular minimal cone  $\mathbf{C}$ , of dimension  $n$ , in  $\mathbb{R}^N$ , we define the link  $S := \mathbf{C} \cap \mathbb{S}^{N-1}$  (note that  $\mathbb{S}^d$  shall henceforth denote the round unit sphere in Euclidean  $\mathbb{R}^{d+1}$ ); here we are not assuming  $S$  to be orientable, however recall that in the codimension one case ( $N = n + 1$ ) a simple topological argument ensures that any such  $S$  is two-sided hence orientable. At the link  $S$  we thus associate a Jacobi operator  $L_S$  that has the form

$$L_S u := \Delta_S^\perp u + (g(\mathbb{I}_S, \cdot) \mathbb{I}_S + (n - 1))u$$

where the background unit round metric is understood throughout, and whose spectrum is a discrete sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \dots \rightarrow +\infty,$$

under the sign convention that  $L_S u_j = -\lambda_j u_j$ . We shall now introduce the following - a posteriori very convenient - notion of ‘‘effective Morse index’’.

**Definition 2.8.** Let  $\Sigma$  denote a closed, embedded minimal submanifold in the round sphere of dimension  $N - 1$ , i. e.  $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ , and let  $\mathbf{C}$  denote the cone over  $\Sigma$  having vertex at the origin  $\mathbf{0} \in \mathbb{R}^N$  (that is:  $\mathbf{C} := \mathbf{0} \ast \Sigma$ ). Then we define the **effective Morse index** of  $\Sigma$  as

$$(4) \quad \mathbf{I}(\mathbf{C}) := (\text{index of the Jacobi operator of } \mathbf{C} \cap \mathbb{S}^{N-1}) - N.$$

(Here it is to be stressed that we are not assuming  $\Sigma$  to be connected.)

The following recollections will be repeatedly referred to in the sequel of this article.

- Remark 2.9.* (A) In the codimension one case it is well-known that the Morse index of any non-equatorial  $S$  is at least  $N + 1$ : there always holds  $I(\mathbf{C}) \geq 1$ .
- (B) In the case of higher (in fact: arbitrary) codimension Simons proved in [41] that the Morse index of the Jacobi operator of  $\mathbf{C} \cap \mathbb{S}^{N-1}$  is at least  $N$  for any connected  $n$ -dimensional minimal submanifold that is not a great sphere (and any equatorial  $n$ -sphere has of course index equal to  $N - n$ ). Hence, there *always* holds  $I(\mathbf{C}) \geq 0$  since if the link is disconnected then necessarily  $n \leq N/2$  and so the total Morse index of the link is at least  $2(N - n) \geq 2(N/2) = N$ . It is an open question to determine all  $(n - 1)$ -dimensional minimal submanifolds  $\Sigma$  in  $\mathbb{S}^{N-1}$  whose Morse index equals  $N$  (thereby saturating the preceding bound): as noted by Kusner-Wang [24, Theorem 4.8] recent work by Karpukhin [23] implies that the Veronese embedding  $\mathbb{R}\mathbb{P}^2 \rightarrow \mathbb{S}^4$  does provide such an example, whether by contrast all minimal 2-tori in  $\mathbb{S}^4$  have index at least 6.

**2.3. Functional spaces and augmentation by translation-like sections.** For any given ambient Riemannian manifold  $(M, g)$  and a smooth submanifold  $\Sigma = \Sigma^n$  (again: not necessarily closed) we will be working with functional spaces of maps that are either  $\mathbb{R}$ -valued, or  $\mathbf{V}$ -valued (that is to say: sections of the aforementioned normal vector bundle  $\mathbf{V} = \mathbf{V}(\Sigma, M)$ ). We will routinely write  $L^p(\Sigma)$  (respectively:  $L^p(\Sigma; \mathbf{V})$ ) for Lebesgue spaces of  $\mathbb{R}$ -valued (resp.:  $\mathbf{V}$ -valued) maps, as well as  $W^{k,p}(\Sigma)$  (and  $W^{k,p}(\Sigma; \mathbf{V})$ ) for standard Sobolev spaces instead. (Of course, such vector spaces may or may not have desirable functional-analytic properties depending on the actual, additional assumptions on  $\Sigma$ ). Furthermore, we will deal with the vector spaces of smooth sections, namely  $C^\infty(\Sigma)$  and  $C^\infty(\Sigma; \mathbf{V})$ , and with smooth *compactly supported* ones, that are  $C_c^\infty(\Sigma)$  and  $C_c^\infty(\Sigma; \mathbf{V})$ . The local counterparts of such spaces will also prominently come into play, in particular,  $L_{loc}^p, W_{loc}^{k,p}$  and  $C_{loc}^{k,\alpha}$  (with either target). Here we wish to stress and reiterate that, in the case of an **MSI**, as per Definition 2.3, we are working on the open manifold corresponding to the regular part of the integer varifold in question.

In the specific case when  $\Sigma$  is (the regular part of) an **MSI** we will further define weighted spaces, as follows. Given a multi-index  $\beta = (\beta_1, \dots, \beta_\ell) \in \mathbb{R}^\ell$ , we shall now define the functional spaces we need. To that aim, let us agree to denote by  $\rho^\beta$  a positive function that equals  $\rho_{\Sigma,g}^{\beta_i}$  along the end  $E_i \subset \Sigma$ ; in fact, without loss of generality, it is convenient for the purposes of the present paper to assume that  $\rho \leq 1$  at all points of  $\Sigma$ . For  $k \geq 0$  and finite  $p \geq 1$  we let  $W_\beta^{k,p}(\Sigma)$  to be the Banach space completion of  $C_c^\infty(\Sigma)$  with respect to the norm

$$\|u\|_{W_\beta^{k,p}} := \left( \sum_{j=0}^k \int_\Sigma |\rho^{(-\beta+j)} \nabla^{(j)} u|^p \rho^{-n} d\|\Sigma\| \right)^{1/p}.$$

It is also possible to define weighted Sobolev spaces  $W_\beta^{-k,p}(\Sigma)$  using the language of distributions; for manifolds with asymptotically cylindrical ends this is done e.g. in reference [32] and it is standard to adapt the treatment to manifolds with asymptotically conical or conically singular (CS) ends (see e.g. [35]), which in particular allows to cover the case under consideration. A posteriori, there holds a Banach space isomorphism  $(W_\beta^{k,p}(\Sigma))^* \simeq W_{-\beta-n}^{-k,p'}(\Sigma)$ .

Similarly, one defines  $C_{\beta}^{k,\alpha}(\Sigma)$  to be the Banach space completion of  $C^{\infty}(\Sigma)$  with respect to the norm

$$\|u\|_{C_{\beta}^{k,\alpha}} := \sum_{j=0}^k \sup_{\Sigma} \rho^{(-\beta+j)} |\nabla^{(j)} u| + \sup_{\substack{x \neq y \in \Sigma \\ \text{dist}_g(x,y) < \rho(x)/2}} \frac{|\rho^{(-\beta+k+\alpha)}(x) \nabla^{(k)} u(x) - \rho^{(-\beta+k+\alpha)}(y) \nabla^{(k)} u(y)|}{d_g(x,y)^{\alpha}}.$$

If  $\alpha = 0$ , we denote  $C_{\beta}^{k,0}(\Sigma)$  simply by  $C_{\beta}^k(\Sigma)$ .

Some fundamental facts about Analysis on manifolds with conical singularities or, more generally, on *conifolds* have been collected, for instance, in [35] (see also references therein); for our purposes we recall the following version of the Sobolev embedding theorem. In the setting above, assume  $k \in \mathbb{N}$ ,  $l \in \mathbb{N}^*$  and  $p \geq 1$ . Given a multi-index  $\beta$  for all  $\beta' \leq \beta$  the following statements hold:

- (i) If  $lp < m$  then there exists a continuous embedding  $W_{\beta}^{k+l,p}(\Sigma) \hookrightarrow W_{\beta'}^{k,p_l^*}(\Sigma)$ ;
- (ii) If  $lp = m$  then there exists a continuous embedding  $W_{\beta}^{k+l,p}(\Sigma) \hookrightarrow W_{\beta'}^{k,q}(\Sigma)$ ;
- (iii) If  $lp > m$  then there exists a continuous embedding  $W_{\beta}^{k+l,p}(\Sigma) \hookrightarrow C_{\beta'}^{k,\alpha}(\Sigma)$ .

Here we have denoted the Sobolev-dual exponent of  $p$  by  $p_l^*$ , namely  $p_l^* = \frac{mp}{m-\ell p}$ ; the second item holds true for all  $q \in [p, \infty)$ , and the third item for all  $\alpha \in [0, \min\{1, \ell - m/p\})$ .

Now, all the definition and basic results above can easily be adapted to the case of sections of any fixed (real) vector bundle over  $\Sigma$ , as opposed to the simplest case of scalar-valued maps; in particular, again in the setting above, we will be interested in  $\mathbf{V} := \mathbf{V}(\Sigma, M)$  and the corresponding spaces  $W_{\beta}^{k,p}(\Sigma; \mathbf{V})$  and  $C_{\beta}^{k,\alpha}(\Sigma; \mathbf{V})$ . That being said, we wish to single out a case that warrants special notation: when for some  $\tau \in \mathbb{R}$  we take  $\beta_i = \tau$  (for all  $i = 1, \dots, \ell$ ) we shall simply write  $W_{\tau}^{k,p}(\Sigma; \mathbf{V})$  or  $C_{\tau}^{k,\alpha}(\Sigma; \mathbf{V})$  in place of  $W_{\beta}^{k,p}(\Sigma; \mathbf{V})$  or, respectively,  $C_{\beta}^{k,\alpha}(\Sigma; \mathbf{V})$ .

Lastly, we will need to possibly ‘‘augment’’ such spaces in the terms that follow.

**Definition 2.10.** We define a function (section)  $\phi \in C_{loc}^{k,\alpha}(\Sigma; \mathbf{V})$  **translation-like** if for every  $q \in \text{Sing}(\Sigma)$ , there is a neighborhood  $U \ni q$  in  $M$  and  $v \in T_q M$  (viewed as a vector field in  $U$  using the normal coordinates) such that under normal coordinates  $x^i$  centered at  $q$ ,

$$\phi|_U(x) = \Pi_x^{\perp}(v),$$

where

$$\Pi_x^{\perp} : T_x M \rightarrow T_x M$$

is the  $g$ -orthogonal projection map onto  $T_x^{\perp} \Sigma (= \mathbf{V}_x)$ .

Hence:

**Definition 2.11.** If  $W_{loc}^{k,p}(\Sigma; \mathbf{V})$  and  $C_{loc}^{k,\alpha}(\Sigma; \mathbf{V})$  are the spaces of sections that are locally  $W^{k,p}$  or, respectively,  $C^{k,\alpha}$  then one sets

$$\hat{W}_{\tau}^{k,p}(\Sigma; \mathbf{V}) := \{u \in W_{loc}^{k,p}(\Sigma) : \|u - \phi\|_{W_{\tau}^{k,p}} < +\infty \text{ for some translation-like function } \phi\}.$$

and

$$\hat{C}_{\tau}^{k,\alpha}(\Sigma; \mathbf{V}) := \{u \in C_{loc}^{k,\alpha}(\Sigma) : \|u - \phi\|_{C_{\tau}^{k,\alpha}} < +\infty \text{ for some translation-like function } \phi\}.$$

Let us discuss, very concretely, the effects of such an ‘‘augmentation’’; we will refer, for the sake of notational definiteness, to weighted Sobolev spaces, but the discussion would be

identical in the case of weighted Hölder spaces. In essence, for  $\tau > 0$  (thus imposing decay at the singular tips of the limit cones) we simply have that

$$(5) \quad \hat{W}_\tau^{k,p}(\Sigma; \mathbf{V}) = W_\tau^{k,p}(\Sigma; \mathbf{V}) \oplus X_{TS}$$

where the vector space  $X_{TS}$  has dimension exactly equal to  $N\ell$  (notation as in Section 2.1), with a basis being given by the collection of functions

$$(6) \quad \{(1 - \zeta_{\Sigma,g,r_0}) \times \Pi_x^\perp(v) : i = 1, \dots, \ell, v \in T_{p_i}M\}$$

where  $\zeta_{\Sigma,g,r_0}$  is the smooth cutoff function defined in the dedicated paragraph of Section 2.6 (so that  $1 - \zeta_{\Sigma,g,r_0}$  transitions from the value 1 for  $r_{\Sigma,g} \leq r_0$  and 0 for  $r_{\Sigma,g} \geq 2r_0$  in terms of a suitable scale  $r_0 > 0$ ).

*Remark 2.12.* Note that the vector space  $X_{TS}$  comes naturally equipped with a quotient norm; however, being finite-dimensional, such a norm is equivalent to any norm of our liking. For technical reasons that will emerge at a later stage (cf. Corollary B.6 and Lemma 4.26) we will assume to work with the  $L^\infty$  norm determined by the ambient Riemannian manifold  $(M, g)$ .

**2.4. Normal graphs and transfer of sections.** For each of the following definitions, we let  $(M, g)$  denote an ambient Riemannian manifold (not necessarily complete) and  $\Sigma_0, \Sigma_1$  denote smooth submanifolds, again possibly open. Furthermore, set  $\mathbf{V}_0$  (respectively:  $\mathbf{V}_1$ ) denote the normal bundle of  $\Sigma_0$  (respectively:  $\Sigma_1$ ) in  $(M, g)$ . We shall now present two related, yet different, notions of “graphicality”: the first notion is essentially local and has to do with the description of a (suitable portion of a) submanifold as a graph over its tangent space at a point, while the second rather pertains to viewing an **MSI** as a normal graph over a nearby reference **MSI**.

**Definition 2.13.** In the setting described above, we define the **regularity scale** of  $\Sigma_0$  to be the function

$$\mathfrak{r}_{\Sigma_0, g} : \Sigma_0 \rightarrow \mathbb{R}^+$$

determined by letting  $\mathfrak{r}_{\Sigma_0, g}(x)$  be the supremum (least upper bound) of all  $r > 0$  such that the following conditions hold:

- (i)  $r < \text{inrad}_{M, g}(x)$ , and  $r^{-2} \|(\exp_x^{-1})^*g - g_{\text{euc}}\|_{C^4(\mathbb{B}(r))} \leq 1$ ;
- (ii) there exists  $\phi : \text{dom}(\phi) \subset T_x\Sigma_0 \rightarrow T_x^\perp\Sigma_0$ , of class  $C^2$ , satisfying the inequality  $r^{-1}|\phi| + |\mathring{\nabla}\phi| + r|\mathring{\nabla}^2\phi| \leq 1$  and

$$(\exp_x)^{-1}(\Sigma_0 \cap B^g(x, r)) = \text{graph}_{T_x\Sigma_0}(\phi)$$

where we identify the pair  $(T_x\Sigma_0, T_xM)$  with  $(\mathbb{R}^n, \mathbb{R}^N)$  and set

$$\text{graph}_{T_x\Sigma_0}(\phi) = \{(x, \phi(x)) : x \in \text{dom}(\phi)\}.$$

*Remark 2.14.* Note that, by the very definition above “a bound on the regularity scale implies a curvature bound, as well as an area bound”, i. e. in the setting above

$$(7) \quad |\mathbb{I}_{\Sigma_0, g}(y)|_g \leq C/\mathfrak{r}_{\Sigma_0, g}(x), \quad \forall y \in B^g(x, \mathfrak{r}_{\Sigma_0, g}(x)/C)$$

and

$$(8) \quad \|\Sigma_0\|(B^g(x, \mathfrak{r}_{\Sigma_0, g}(x)/C)) \leq C\mathfrak{r}_{\Sigma_0, g}^n(x)$$

for a constant  $C = C(M, g) \geq 2$  only depending on the ambient manifold. Hence, if the ambient manifold is compact we can - by means of a standard covering argument - derive a

global, uniform area bound as soon as we are given a positive, uniform lower bound on the regularity scale of  $\Sigma$ . Therefore, it is by now standard to show (cf. e.g. [37]) that a uniform, positive lower bound on the regularity scale implies a compactness result with respect to graphical, smooth convergence (with multiplicity one); we will invoke this result for a sequence  $\{\Sigma_j\}_{j \geq 1}$  of smooth, closed minimal submanifolds of dimension  $n - 1$  in the unit round sphere of dimension  $N - 1$ .

*Remark 2.15.* We wish to explicitly stress that, in absence of restrictions on the range of  $n$  (with respect to  $N$ ), assuming a uniform bound on the second fundamental form and area *does not* imply smooth graphical convergence with multiplicity one. For instance, one may consider  $\Sigma = \Sigma(d)$  to be the disjoint union of the two spheres in  $\mathbb{S}^5$  obtained by intersecting with two three-dimensional subspaces only meeting at the origin, whose distance is equal to  $d$  in  $\mathbb{G}(3, 6)$ : as  $d \rightarrow 0^+$  clearly  $\Sigma(d)$  will converge to an equatorial two-sphere with multiplicity two. This phenomenon is ruled out by a positive lower bound on the regularity scale.

**Definition 2.16.** In the setting above we shall say that  $\Sigma_1$  is a  $\kappa$ - $C^k$  **graph** over  $\Sigma_0$  in an open subset  $U \subset M$  if there exist  $U \cap \Sigma_0 \subset \Omega_0 \subset \Sigma_0$  and  $\phi \in C^k(\Omega_0; \mathbf{V}_0)$ ,

$$\Sigma_1 \cap U \subset \text{graph}_{\Sigma_0}(\phi) := \{\exp_x(\phi(x)) : x \in \Omega_0\} \subset \Sigma_1$$

implicitly assuming such a definition is well-posed, depending on the injectivity radius of the ambient manifold, and moreover

$$\sum_{p=0, \dots, k} \sup_{x \in \Sigma_0} (\mathbf{r}_{\Sigma_0, g}^{p-1}(x) \cdot |(\nabla^\perp)^p \phi(x)|_g) \leq \kappa.$$

*Remark 2.17.* Let  $\delta_0 \in (1/4)$  be the dimensional constant determined by Lemma D.1. If for  $i \in \{1, 2\}$ ,  $\Sigma^i$  is a  $\delta_0^2$ - $C^2$  graph over  $\Sigma^0$  in  $U$ , then for every  $\check{x} \in \Sigma^0$  such that  $B^g(\check{x}, \mathbf{r}_{\Sigma^0, g}(\check{x})) \subset U$ , the assumptions of Lemma D.1 are satisfied for  $\check{\eta}^{-1}(\Sigma^j)$ ,  $\check{r}^{-2}\check{\eta}^*g$  in place of  $\Sigma^j$ ,  $g$  therein,  $j \in \{0, 1, 2\}$ , where  $\check{r} := \delta_0 \mathbf{r}_{\Sigma^0, g}(\check{x})$  and  $\check{\eta} : x \mapsto \exp_x^g(\check{r}x)$ . Therefore, Lemma D.1 allows us to quantitatively compare the graphical section of  $\Sigma^2$  over  $\Sigma^1$  with the difference of the graphical sections of  $\Sigma^i$  over  $\Sigma^0$ ,  $i \in \{1, 2\}$  as well as of slightly translated graphs. For this important technical reason, we introduce the following convention:

*In the setting of Definition 2.16, we call  $\Sigma^1$  a  $C^2$  graph if it is a  $\delta_0^2$ - $C^2$  graph.*

**Definition 2.18.** In the context of the preceding definition, under the specification that  $\Sigma_0$  be an **MSI** in  $(M, g)$ , we define the **graphical radius**  $\mathbf{r}_{\Sigma_0, g}^{\Sigma_1}(x)$  of  $\Sigma_1$  over  $\Sigma_0$  (in the precise sense just declared) under metric  $g$  as the infimum over all nonnegative function  $\mathbf{r}$  on  $\text{Sing}(\Sigma_0)$  so that  $\Sigma_1$  is a  $C^2$  graph over  $\Sigma_0$  in the open set

$$M \setminus \bigcup_{x \in \text{Sing}(\Sigma_0)} \overline{B^g(x, \mathbf{r}(x))}.$$

We use  $\mathbf{G}_{\Sigma_0, g}^{\Sigma_1}$  to denote the (uniquely defined) **graphical section** over  $\Sigma_0$  of  $\Sigma_1$  under metric  $g$  wherever defined, which we then extend by zero to an  $L^\infty$  section on the whole  $\Sigma_0$ , so that  $\mathbf{G}_{\Sigma_0, g}^{\Sigma_1} \in L^\infty(\Sigma_0; \mathbf{V}_0)$ .

*Remark 2.19.* We explicitly point out that the previous definition (like the following one) will sometimes be employed, as a special case, for pairs of  $n$ -dimensional (non-trivial, yet regular) minimal cones in Euclidean  $\mathbb{R}^N$ .

**Definition 2.20.** Let  $(M, g)$  be a Riemannian manifold of dimension  $N \geq 3$ , and let  $\{g_j\}_{j \geq 1}$  be a sequence of  $C^{k, \alpha}$  Riemannian metrics converging, in that Banach space, to  $g$  as  $j \rightarrow \infty$ . Assume further, to be given an **MSI**  $\Sigma$  in  $(M, g)$  as well as for every  $j \geq 1$  an **MSI**  $\Sigma_j$  in  $(M, g_j)$ . We shall say that the sequence  $\{\Sigma_j\}_{j \geq 1}$  **converges** to  $\Sigma$  in  $C_{loc}^{k', \alpha'}$  (for  $0 \leq k' \leq k + 1$  and  $0 \leq \alpha' \leq \alpha$ ) if, as one lets  $j \rightarrow \infty$ , the following two conditions hold:

$$\mathbf{r}_{\Sigma, g}^{\Sigma_j}(x) \rightarrow 0 \text{ for all } x \in \text{Sing}(\Sigma)$$

and for any  $r > 0$

$$\|\mathbf{G}_{\Sigma, g}^{\Sigma_j}\|_{C^{k', \alpha'}(\Sigma \setminus B^g(\text{Sing}(\Sigma), r))} \rightarrow 0.$$

That said, whenever we have “graphical convergence” we can in fact “transfer sections” employing the exponential map. In order to clarify this point, we give the following:

**Definition 2.21.** In the setting of Definition 2.16, set  $\Sigma_1 = \text{graph}_{\Sigma_0}(\phi)$ . Given any  $u : \Sigma_1 \rightarrow \mathbf{V}_1$  we let  $\mathbf{T}_{\Sigma_0, g}^{\Sigma_1}(u) : \Sigma_0 \rightarrow \mathbf{V}_0$  be defined as follows

$$\mathbf{T}_{\Sigma_0, g}^{\Sigma_1}(u)(x) = [(\exp_x(\phi(x)))_*^{-1}(u(\exp_x(\phi(x))))]^\perp.$$

In practice, in this paper we will deal with “transferring sections” in two special cases:

- a sequence of **MSI**  $\{\Sigma_j\}_{j \geq 1}$  converging (in **F**-metric, with multiplicity one) and in  $C_{loc}^{k', \alpha'}$  as per Definition 2.20 to a reference “base” **MSI**  $\Sigma$ ;
- an **MSI** that – based on [38] – can be written, locally around  $q \in \text{Sing}(\Sigma)$  as a normal graph over its unique tangent cone  $\mathbf{C}_q = \mathbf{C}_q(\Sigma)$  in  $T_q M$ .

Note that the second instance can also be phrased in terms of “convergence” on approach to  $q$  by the decay properties of the defining normal section.

The key analytic features of the transfer operators are collected in Proposition A.3, in a way that is equally well applicable to both circumstances. The second special case (normal graph over tangent cone) will also feature prominently in the second part of Appendix B, whose key results will in turn be employed in Section 4.

**2.5. An ancillary computation about translation-like sections.** The following result will be profitably used throughout the paper (for instance in the proof of Lemma 4.14).

**Lemma 2.22.** *Let  $(M, g)$  be a closed Riemannian manifold, and let  $\Sigma$  be a  $g$ -stationary varifold with only strongly isolated singularities (**MSI**). For any translation-like section  $\phi : \Sigma \rightarrow \mathbf{V} := \mathbf{V}(\Sigma, M)$  belonging to the space  $X_{TS}$  there holds  $L_{\Sigma, g} \phi \in L^\infty(\Sigma)$  and there exists a constant  $C = C(g, \Sigma)$  such that, if  $\phi = \mathbf{x}^\perp$  around  $q \in \text{Sing}(\Sigma)$  then (in that same neighborhood)*

$$\|L_{\Sigma, g} \mathbf{x}^\perp\|_{L^\infty} \leq C |\mathbf{x}|,$$

where  $|\mathbf{x}|$  denotes the norm of  $\mathbf{x} \in T_q M$  with respect to the metric  $g$ . In particular, there holds  $L_{\Sigma, g} \phi \in W_{-\varepsilon}^{0, p}(\Sigma; \mathbf{V})$  for any  $p \geq 1$  and any  $\varepsilon > 0$ .

*Proof.* We work in the neighborhood of  $q \in \text{Sing}(\Sigma)$  endowed with geodesic normal coordinates centered at the point in question. Thus, with slight notational abuse, we will assume  $\Sigma^n \subset (\mathbb{R}^N, g)$  and  $\phi = \mathbf{x}^\perp$ ; we shall employ the well-known asymptotic expansion

$$(9) \quad g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^\ell + O(|x|^3), \quad (x \rightarrow 0).$$

Hence, by letting as usual  $r := |x|$ , there holds in particular for the Christoffel symbols and their first derivatives

$$(10) \quad \Gamma_{ij}^k = O(r), \text{ and } \Gamma_{ij,\ell}^k = O(1), \text{ (} r \rightarrow 0^+ \text{)}.$$

Now, it suffices in fact to consider the case when  $g(\mathbf{x}, \mathbf{x}) = 1$  at  $q$  and prove the desired local  $L^\infty$  bound, for the claim then comes by a standard scaling argument.

That said, for a fixed vector field  $\mathbf{x} = \sum_{i=1}^N \mathbf{x}_i \partial^i$  we need to compute at any  $x \in \Sigma$

$$(11) \quad (L_{\Sigma,g}(\mathbf{x}^\perp))(x) = \left[ \frac{d}{ds} \right]_{s=0} H_{\Sigma+s\mathbf{x},g}(x_s)$$

where  $x_s$  is the unique element in the intersection  $\cup_{t \in [0,2s]} \exp_x(t\mathbf{x}^\perp) \cap (\Sigma + s\mathbf{x})$ , which is well-defined for  $s = s(x)$  sufficiently small. In fact, due to the assumed minimality of  $\Sigma$  we have

$$\left[ \frac{d}{ds} \right]_{s=0} H_{\Sigma+s\mathbf{x},g}(x_s - (x + s\mathbf{x})) = g(H_{\Sigma,g}, \mathbf{x}^\perp - \mathbf{x}) = 0$$

and so we are actually left with computing

$$\left[ \frac{d}{ds} \right]_{s=0} H_{\Sigma+s\mathbf{x},g}(x + s\mathbf{x}) = \left[ \frac{d}{ds} \right]_{s=0} H_{\Sigma,\tau_{s\mathbf{x}}^*g}(x)$$

where the right-hand side is now the variation of mean curvature of a fixed submanifold with respect to a deformation of the ambient metric, and we have set  $\tau_W(x) = x + W$ . To move further let us recall that, if we denote  $h := \left[ \frac{d}{dt} \right]_{t=0} g(t)$  then there holds in general, again under the minimality assumption

$$\left[ \frac{d}{dt} \right]_{t=0} H_{\Sigma,g(t)} = -g(\mathbb{I}_{\Sigma,g}, h) + ((\operatorname{div}_\Sigma h)^\#)^{\perp_g} - \frac{1}{2} \operatorname{tr}_\Sigma \nabla^{\perp_g} h;$$

working in local coordinates  $\{x^1, \dots, x^n, x^{n+1}, \dots, x^N\}$  where  $\{\partial_1, \dots, \partial_n\}$  (and, respectively:  $\{\partial_{n+1}, \dots, \partial_N\}$ ) form a local basis for the the tangent (resp. normal) space to  $\Sigma$  the three terms above need to be understood as follows :

$$-g(\mathbb{I}_{\Sigma,g}, h) = -g^{ip} g^{jq} h_{pq} \Gamma_{ij}^a \partial_a, \quad ((\operatorname{div}_\Sigma h)^\#)^{\perp_g} = g^{ij} g^{ab} h_{ib,j} \partial_a, \quad \operatorname{tr}_\Sigma \nabla^{\perp_g} h = -g^{ij} g^{ab} h_{ij,b} \partial_a$$

for indices  $1 \leq i, j, p, q \leq n, n+1 \leq a, b \leq N$  and summation over repeated indices. (This can indeed be checked, for instance, by means of a simple computation in such local coordinates.)

As a result, since in our case

$$\left[ \frac{d}{ds} \right]_{s=0} \tau_{s\mathbf{x}}^* g = \mathcal{L}_{\mathbf{x}} g = \operatorname{Sym}(\nabla \mathbf{x})$$

(that is: the Lie derivative of the metric  $g$  in the direction  $\mathbf{x}$ , which equals the ‘‘symmetrized covariant derivative’’ of  $\mathbf{x}$  with respect to the metric  $g$ ) we will have, in the end

$$(12) \quad L_{\Sigma,g}(\mathbf{x}^\perp) = -g(\mathbb{I}_{\Sigma,g}, \operatorname{Sym}(\nabla \mathbf{x})) + ((\operatorname{div}_\Sigma \operatorname{Sym}(\nabla \mathbf{x}))^\#)^{\perp_g} - \frac{1}{2} \operatorname{tr}_\Sigma \nabla^{\perp_g} \operatorname{Sym}(\nabla \mathbf{x}).$$

At this stage we simply need to note that the tensor  $h := \operatorname{Sym}(\nabla \mathbf{x})$  satisfies, in view of (10) and the constancy of  $\mathbf{x}$  the bound  $\|h\|_g = O(r)$  as  $r \rightarrow 0^+$ , and so, by inspecting all terms of the right-hand side of (12) we conclude that each of them is uniformly bounded as  $r \rightarrow 0^+$  and so will the same conclusion hold for  $L_{\Sigma,g}(\mathbf{x}^\perp)$ , as claimed.  $\square$

*Remark 2.23.* By inspecting the previous proof it is straightforward to see that, in fact, the constant  $C = C(g, \Sigma)$  can be chosen uniformly for all pairs (as per Definition 4.10 below)  $(g', \Sigma') \in \mathcal{M}_n^{k, \alpha}(M)$  satisfying the bounds

$$\|g'\|_{C^{k, \alpha}} \leq \Lambda, \quad \mathfrak{r}_{\Sigma', g'} \geq \Lambda^{-1} \rho_{\Sigma', g'}$$

where we recall that  $\rho_{\Sigma', g'}$  denotes the distance function to  $\text{Sing}(\Sigma')$  in  $(M, g')$ . That is to say: the claim of Lemma 2.22 holds true for all  $\phi \in X_{TS}$  for a constant  $C = C(\Lambda)$ .

**2.6. Conventions.** We shall collect here some more notation and conventions that are implicitly assumed throughout the paper.

**Cutoff functions.** First of all, let  $\zeta \in C^\infty(\mathbb{R}; [0, 1])$  be a cutoff function (fixed once and for all) such that  $\zeta \equiv 0$  on  $(-\infty, 1]$  and  $\zeta \equiv 1$  on  $[2, +\infty)$ . Hence, given as above an ambient Riemannian manifold  $(M, g)$ , an **MSI**  $\Sigma$  and  $r_0 > 0$ , we let

$$(13) \quad \zeta_{\Sigma, g, r_0}(x) := \zeta(\text{dist}_g(x, \text{Sing}(\Sigma))/r_0).$$

When convenient we will simplify this notation; for instance given a sequence of data of the type above we may write  $\zeta_j$  in place of  $\zeta_{\Sigma_j, g_j, r_j}$ . Specific conventions of this sort will however always be declared.

**Metric notions.** In  $\mathbb{R}^N$  we shall denote by  $g_{\text{euc}}$  the standard Euclidean metric, by  $\text{dist}_{g_{\text{euc}}}$  the associated distance and by  $\nabla$  its Levi-Civita connection. Furthermore, we let  $\mathbb{B}(x, r)$  denote the open Euclidean ball of center  $x \in \mathbb{R}^N$ ; in the special case of balls centered at the origin we shall simply write  $\mathbb{B}(r)$ . Lastly, for Euclidean annuli we write  $\mathbb{A}(x, r, s) = \mathbb{B}(x, s) \setminus \overline{\mathbb{B}(x, r)}$ ; when  $x$  is the origin we convene to write  $\mathbb{A}(r, s)$ . In a Riemannian manifold  $(M, g)$  we let  $\text{dist}_g$  denote the corresponding distance and by  $\nabla$  the Levi-Civita connection. (We agree that all metrics are smooth, i. e.  $C^\infty$ , unless otherwise stated.) It is convenient to explicitly indicate the background metric when talking about balls and annuli, so we will write  $B^g(x, r)$  (respectively:  $A^g(x, r, s)$ ) for the ball of center  $x \in M$  and radius  $r > 0$  (respectively: for the annulus of center  $x$  and radii  $0 < r < s$ ). At certain (relatively rare) points of the paper we will need to deal simultaneously with multiple metrics, in which case we will rather indicate the selected metric explicitly, like e.g. for  $\nabla^g$  in lieu of  $\nabla$  and for  $\exp_x^g$  in lieu of  $\exp_x$  when employing the exponential map; in particular, this will be the case in Appendix A.

**Use of constants.** Throughout this article we shall employ the letter  $C$  to denote a constant that is allowed to vary from line to line (or even within the same line); we shall stress the functional dependence of any such constant on geometric quantities by including them in brackets, writings things like  $C = C(\Lambda, \sigma)$  or similar. The dimension of the ambient manifold (that is:  $N$ ) and of the subvarieties we work with (namely:  $n$ ) are fixed (i. e. we do not need to ever vary them in our arguments) and so, for notational simplicity, we agree not to indicate them among the parameters our constants depend upon; the only (explicitly stated) exception to such a rule is Appendix D, where (for instance) we determine the constant  $\delta_0 = \delta_0(N)$  that was mentioned above after the definition of  $\kappa$ - $C^k$  graph (Definition 2.16). Lastly, in the rare cases when – within a certain proof – it is appropriate to keep track of different constants (for instance because they display different functional dependence, or need to be chosen in a

certain order) we avoid ambiguities by employing different labels or numbers to indicate such constants. This will always be explicitly remarked if appropriate.

### 3. AN INDEX-THEORETIC PERSPECTIVE ON GENERIC REGULARITY

As anticipated in the introduction, we shall prove in this section an “index-counting formula” of independent interest. To get there, it is convenient to open a short digression pertaining to the analysis of the Jacobi operator of regular minimal cones in Euclidean space, which will anyways come into play multiple times along the course of this work (for instance, most prominently in Appendix B).

**3.1. Analysis on a regular minimal cone.** Given  $n \geq 2$  and  $N > n$  let  $\mathcal{C}_{N,n}$  denote the collection of *non-trivial* regular  $n$ -dimensional minimal cones  $\mathbf{C}$  in Euclidean  $\mathbb{R}^N$ , always understood as *multiplicity one* varifolds.

Thus, for  $\mathbf{C} \in \mathcal{C}_{N,n}$  be a regular minimal cone, recall that  $S := \mathbf{C} \cap \mathbb{S}^{N-1}$  is a smooth minimal submanifold in  $\mathbb{S}^{N-1}$  that is not an  $(n-1)$ -dimensional equator; as a result, we stress in particular that  $\mathcal{C}_{N,n}$  is actually *empty* when  $N = 3$  and  $n = 2$ . We can parametrize  $\mathbf{C}$  by

$$(14) \quad (0, +\infty) \times S \rightarrow \mathbf{C}, \quad (r, \omega) \mapsto x = r\omega.$$

At the cone  $\mathbf{C} \subset \mathbb{R}^N$  one deals with the Jacobi operator ( $r := |x|$ )

$$L_{\mathbf{C}} = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} (L_S - (n-1)).$$

(Note that, compared to the general notational conventions stipulated in Section 2 here we omit the explicit indications of the background metric for  $L_{\mathbf{C}}$  and  $L_S$ .)

If one looks for homogenous Jacobi fields, arguing by separation of variables (cf. [9]) the only possible rates of growth/decay are those solving the algebraic equation (one for each choice of the label  $j$  parametrizing the eigenfunctions on the link, with the sign conventions declared above)

$$\gamma^2 + (n-2)\gamma - (\lambda_j + (n-1)) = 0$$

and so we get

$$(15) \quad \gamma_j^{\pm}(\mathbf{C}) = -\left(\frac{n-2}{2}\right) \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + (\lambda_j + (n-1))} \in \mathbb{C}.$$

We collect these numbers in the set (henceforth referred to as asymptotic spectrum of  $\mathbf{C}$ )

$$(16) \quad \Gamma(\mathbf{C}) := \{\deg(w) : w \text{ is a homogeneous Jacobi field on } \mathbf{C}\} \equiv \{\Re(\gamma_j^{\pm}) : j \geq 1\} \subset \mathbb{R}$$

and conveniently define

$$(17) \quad \gamma_-(\mathbf{C}) := \sup(\Gamma(\mathbf{C}) \cap \mathbb{R}_{<1}) < 1.$$

*Remark 3.1.* Note for later reference that the inequality  $\lambda_j < 0$  (that corresponds to the “index contributions” of the link) is true if and only if

$$(18) \quad \gamma_j^+ = -\left(\frac{n-2}{2}\right) + \sqrt{\left(\frac{n-2}{2}\right)^2 + (\lambda_j + (n-1))} < 1.$$

With this in mind, it is convenient for us to set  $\gamma_* = -(n-1)$  and  $\gamma^* = 1$  (that are the roots corresponding to setting  $\lambda_j = 0$ ). Furthermore, due to the presence of translations in Euclidean  $\mathbb{R}^N$ , it is always the case that  $0 \in \Gamma(\mathbf{C})$  and so it follows that  $\gamma_-(\mathbf{C}) \geq 0$  for any  $\mathbf{C} \in \mathcal{C}_{N,n}$ .

That said, and conveniently set  $\mu_j = \lambda_j + (n - 1)$  for any  $j \geq 1$ , a general real-valued Jacobi field  $v \in C_{loc}^\infty(\mathbf{C})$  is thus given by a linear combination of homogeneous Jacobi fields,

$$(19) \quad v(r, \omega) = \sum_{j \geq 1} (v_j^+(r) + v_j^-(r)) \varphi_j(\omega),$$

where

$$(20) \quad \begin{aligned} v_j^+(r) &= c_j^+ \cdot r^{\gamma_j^+}; \\ v_j^-(r) &= \begin{cases} c_j^- \cdot r^{\gamma_j^-}, & \text{if } \mu_j \neq -\frac{(n-2)^2}{4}; \\ c_j^- \cdot r^{\gamma_j^-} \log r, & \text{if } \mu_j = -\frac{(n-2)^2}{4}. \end{cases} \end{aligned}$$

for some  $c_j^\pm \in \mathbb{R}$  if  $\mu_j \geq -(n - 2)^2/4$ , while instead  $c_j^\pm \in \mathbb{C}$ ,  $c_j^- = \overline{c_j^+}$  when  $\mu_j < -(n - 2)^2/4$ .

**3.2. An index-counting formula.** Here is the main result of this section:

**Theorem 3.2.** *Let  $(M, g)$  be an  $N$ -dimensional closed Riemannian manifold and let  $\Sigma$  be a stationary integral  $n$ -varifold with only strongly isolated singularities (as per Definition 2.3 of MSI). Let*

$$(21) \quad \tau \in \left( \sup_{p \in \text{Sing}(\Sigma)} \gamma_-(\mathbf{C}_p), 1 \right).$$

Then for every integer  $k \geq 2$ , the Jacobi operator

$$L_{\Sigma, g} : \hat{W}_\tau^{k, 2}(\Sigma; \mathbf{V}) \rightarrow W_{\tau-2}^{k-2, 2}(\Sigma; \mathbf{V})$$

is a Fredholm operator of index

$$\widehat{\text{index}}_\tau(L_{\Sigma, g}) = - \sum_{p \in \text{Sing}(\Sigma)} \text{I}(\mathbf{C}_p).$$

Towards the proof of this result, let us first handle the effect of the aforementioned ‘‘augmentation’’. We will employ the following functional-analytic statement, whose proof is straightforward:

**Lemma 3.3.** *Let  $X, Y$  be Banach spaces and let  $T : X \rightarrow Y$  be a bounded linear operator. Assume  $X_0 \subset X$  be a closed subspace of finite codimension equal to  $d \in \mathbb{N}$  and let  $T_0$  denote the restriction of  $T$  to  $X_0$ . Then  $T$  is Fredholm if and only if  $T_0$  is, and the corresponding indices are related by the equation*

$$\text{index}(T) = \text{index}(T_0) + d.$$

*Proof of Theorem 3.2.* We claim that the operator  $L_{\Sigma, g} : W_\tau^{k, 2}(\Sigma; \mathbf{V}) \rightarrow W_{\tau-2}^{k-2, 2}(\Sigma; \mathbf{V})$  is Fredholm and that its index satisfies

$$(22) \quad \text{index}_\tau(L_{\Sigma, g}) = - \sum_{p \in \text{Sing}(\Sigma)} (\text{I}(\mathbf{C}_p) + N) = -N\ell - \sum_{p \in \text{Sing}(\Sigma)} \text{I}(\mathbf{C}_p)$$

where, we recall,  $\ell$  is the notation we employ to denote the number of singular points of  $\Sigma$ . If that is the case, then the desired conclusion comes at once from Lemma 3.3 since - like we discussed in Section 2.3 - as soon as  $\tau > 0$  (which is indeed the case, cf. Remark 3.1)  $W_\tau^{k, p}(\Sigma; \mathbf{V})$  is a closed subspace of  $\hat{W}_\tau^{k, p}(\Sigma; \mathbf{V})$  of codimension equal to  $N\ell$ .

That said, in order to prove (22) we will appeal to Lockhart-McOwen theory (specifically to Theorem 6.2 and Section 7 in [29]; see also Section 9 of [35], to be specified to the ‘‘conically

singular” (CS) case only). For this reason we shall consider the set  $\Gamma(\mathbf{C}_i)$  (as per (16)) and further set

$$(23) \quad \Gamma(\Sigma) := \{\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_\ell) \in \mathbb{R}^\ell : \gamma_i \in \Gamma(\mathbf{C}_i) \text{ for some } i \in \{1, \dots, \ell\}\}.$$

Let then  $L_{\Sigma,g}^*$  denote the formal adjoint of  $L_{\Sigma,g} : W_\tau^{k,2}(\Sigma; \mathbf{V}) \rightarrow W_{\tau-2}^{k-2,2}(\Sigma; \mathbf{V})$ , so (by the duality result recalled in Section 2.3) we will then have

$$(24) \quad L_{\Sigma,g}^* : W_{2-\tau-n}^{-k,2}(\Sigma; \mathbf{V}) \rightarrow W_{-\tau-n}^{-k,2}(\Sigma; \mathbf{V});$$

let  $\text{index}_\tau(L_{\Sigma,g}^*)$  denote the Fredholm index of such an adjoint. Since however  $L_{\Sigma,g}$  is formally self-adjoint (when acting on smooth, compactly supported sections) and by elliptic regularity the Fredholm index of  $L_{\Sigma,g}$  is independent of the choice of  $k$  within any connected component of  $\mathbb{R}^\ell \setminus \Gamma(\Sigma)$  (namely: in any connected component of the complement of the union of all hyperplanes determined by indicial roots) we have that  $\text{index}_\tau(L_{\Sigma,g}^*) = \text{index}_{2-\tau-n}(L_{\Sigma,g})$ . On the other hand, we can rely on the usual “orthogonality relations” for the adjoint, which (in the context under consideration) give  $\text{index}_\tau(L_{\Sigma,g}^*) = -\text{index}_\tau(L_{\Sigma,g})$ . Hence combining these two equations we finally get

$$(25) \quad \text{index}_{2-\tau-n}(L_{\Sigma,g}) = -\text{index}_\tau(L_{\Sigma,g}).$$

The next step is then to compute the *difference*  $\text{index}_\tau(L_{\Sigma,g}) - \text{index}_{2-\tau-n}(L_{\Sigma,g})$  at least for suitably chosen values of  $\tau$ ; it is here that the indicial roots (and their multiplicities) come into play. In order to relate this number to the Morse index of the links, keeping in mind Remark 3.1 we note that picking

$$\tau \in \left( \sup_{p \in \text{Sing}(\Sigma)} \gamma_-(\mathbf{C}_p), 1 \right)$$

there holds in fact  $-n + 1 < 2 - \tau - n < \tau$  and the interval  $(2 - \tau - n, \tau)$  intercepts all (and only) those indicial roots associated to negative eigenvalues of the links, namely to index contributions. That said, we further note that each such eigenvalue “generates” two indicial roots (possibly coinciding, i. e. one root with multiplicity two).

Hence, for any such fixed value of  $\tau$  the weight-crossing formula in [29] gives

$$\text{index}_\tau(L_{\Sigma,g}) - \text{index}_{2-\tau-n}(L_{\Sigma,g}) = -2 \sum_{j=1, \dots, \ell} (\mathbf{I}(\mathbf{C}_j) + N)$$

and so, by (25) we then get the claimed conclusion.  $\square$

**3.3. The case of the Clifford football in the four-sphere.** Let us now revisit the previous discussion by specializing to the case of our primary interest, that is the application of this result to the analysis of the Clifford football in  $\mathbb{S}^4$ .

**Example 3.4.** In this case we have  $N = 4$ ,  $n = 3$  and  $\ell = 2$ , each of two singularities being modelled on the “Clifford cone”, the cone over the Clifford torus in  $\mathbb{S}^3$ . The eigenvalues of the Jacobi operator of the link are computed - by separation of variables - to be

$$\lambda^{p,q} = 2[(p^2 + q^2) - 2], \quad (p, q \in \mathbb{N})$$

so that, in particular (properly accounting for the multiplicity of the associated eigenspaces)

$$\lambda_1 = -4, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -2, \quad \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = 0$$

and  $\lambda_k > 0$  for any  $k \geq 10$ . We then get the corresponding values of  $\gamma$ :

$$\begin{aligned} \gamma_1^\pm &= -\frac{1}{2} \pm \frac{\sqrt{-7}}{2}, \quad \gamma_2^+ = \gamma_3^+ = \gamma_4^+ = \gamma_5^+ = 0, \quad \gamma_2^- = \gamma_3^- = \gamma_4^- = \gamma_5^- = -1 \\ \gamma_6^+ &= \gamma_7^+ = \gamma_8^+ = \gamma_9^+ = 1, \quad \gamma_6^- = \gamma_7^- = \gamma_8^- = \gamma_9^- = -2 \end{aligned}$$

and for  $k \geq 10$  we will have instead  $\gamma_k^+ > 1$  and  $\gamma_k^- < -2$ .

Hence  $\gamma_-(\mathbf{C}_1) = \gamma_-(\mathbf{C}_2) = 0$  and so Theorem 3.2 is a statement about the Fredholm index of the Clifford football when each weight  $\beta_j$  (equivalently:  $\tau$  in that setting) lies in the open interval  $(0, 1)$ . Next, referring to the previous discussion, the counting of the indicial roots refers to the interval  $(-2, 1)$  and gives  $\text{index}_\tau(L_\Sigma) = -10$  for all  $\tau \in (0, 1)^2$ ; note that for  $\tau \in (1, 2)$  we get instead  $\text{index}_\tau(L_\Sigma) = -18$ .

#### 4. MEAGERNESS OF SINGULARITIES WITH POSITIVE NORMALIZED MORSE INDEX

In this section we will complement the previous, general index-counting formula with the following assertion about the generic sign of the Fredholm index of the Jacobi operator. We let, throughout,  $n \geq 2, N > 2$  and assume  $k \geq 4$  as well as  $\alpha \in (0, 1)$ .

**Theorem 4.1.** *Let  $(M, g)$  be an  $N$ -dimensional closed Riemannian manifold with a generic choice of metric  $g$ , understood either in  $C^{k, \alpha}$  or in  $C^\infty$ , and let  $\Sigma$  be a stationary integral  $n$ -varifold with only strongly isolated singularities. Let  $\tau$  satisfy equation (21). Then*

$$\widehat{\text{index}}_\tau(L_{\Sigma, g}) \geq 0.$$

*Remark 4.2.* As apparent from the preceding statement, there are in fact two versions of this theorem, depending on the (regularity of the) space of metric we work with. For the vast majority of this section we will in fact work with the space of  $C^{k, \alpha}$  metrics, and - at the very end - we will present the argument that allows to derive the smooth version of the theorem from that in finite regularity.

##### 4.1. Compact subclasses of regular minimal cones.

**Definition 4.3.** Recalling the definition of  $\mathcal{C}_{N, n}$  given at the beginning of Section 3, we shall set for any  $\Lambda > 0$

$$\mathcal{C}_{N, n}(\Lambda) := \{\mathbf{C} \in \mathcal{C}_{N, n} : \inf_{\mathbf{C} \cap \mathbb{S}^{N-1}} \mathbf{r}_{\mathbf{C}}(x) \geq \Lambda^{-1}\},$$

where, just by specializing Definition 2.13,

$$\mathbf{r}_{\mathbf{C}}(x) := \sup \left\{ r > 0 : \mathbf{C} \cap \mathbb{B}(x, r) = \text{graph}_{T_x \mathbf{C}}(u), \quad r^{-1}|u| + |\mathring{\nabla} u| + r|\mathring{\nabla}^2 u| \leq 1 \right\}$$

for  $\phi : \text{dom}(\phi) \subset T_x \mathbf{C} \rightarrow T_x^\perp \mathbf{C}$ , of class  $C^2$ .

*Remark 4.4.* In view of Remark 2.14 we note that a positive, uniform lower bound on the regularity scale of the link  $\mathbf{C} \cap \mathbb{S}^{N-1}$  implies an upper bound on its  $(n-1)$ -dimensional area, hence on the density of the cone  $\theta_{\mathbf{C}}(\mathbf{0})$ .

**Lemma 4.5.** *For every  $\Lambda \geq 1$ ,  $\mathcal{C}_{N, n}(\Lambda)$  is compact in  $\mathbf{F}$ -metric and in  $C^\infty$  topology along the cross-section.*

*Proof.* For every  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$ , its cross-section  $\Sigma := \mathbf{C} \cap \mathbb{S}^{N-1}$  is a minimal submanifold of dimension  $n-1$ , which (by definition) satisfies a positive, uniform lower bound on the regularity scale.

Therefore, in view of Remark 2.14, every sequence  $\{\mathbf{C}_i\}_{i \geq 1} \subset \mathcal{C}_{N,n}(\Lambda)$  has a subsequence, still denoted by  $\{\mathbf{C}_i\}$ , such that  $\Sigma_i := \mathbf{C}_i \cap \mathbb{S}^{N-1}$  converges to a minimal submanifold  $\Sigma_\infty$  of  $\mathbb{S}^{N-1}$  in  $C^\infty$ . Here, we stress once again that the convergence occurs with multiplicity one.

In particular,  $\mathbf{C}_i \rightarrow \mathbf{C}_\infty := 0 \ast \Sigma_\infty$  (the minimal cone over  $\Sigma_\infty$ ) under the  $\mathbf{F}$ -metric. Note that  $\Sigma_\infty$  cannot be a standard  $\mathbb{S}^n$ , because if it were,  $\mathbf{C}_\infty$  would be an  $n$ -dimensional subspace; thus, by Allard's regularity theorem [2], this would imply that  $\mathbf{C}_i$  is a smooth cone (for any large enough  $i$ ) and therefore also an  $n$ -dimensional subspace, which contradicts the assumption that  $\mathbf{C}_i \in \mathcal{C}_{N,n}$ . Lastly, as a direct consequence of the definition of regularity scale, one can verify that  $0 \ast \Sigma_\infty \in \mathcal{C}_{N,n}(\Lambda)$ .  $\square$

*Remark 4.6.* If  $\{\mathbf{C}_k\}_{k \geq 1}$  is a family of regular cones such that

$$\mathbf{F}(|\mathbf{C}_k|, |\mathbf{C}_\infty|) \rightarrow 0$$

(with multiplicity one) then the asymptotic spectrum also converges in the following sense: for every  $j \geq 1$ , as  $k \rightarrow \infty$ ,

$$\gamma_j^\pm(\mathbf{C}_k) \rightarrow \gamma_j^\pm(\mathbf{C}_\infty).$$

So, in particular this conclusion applies whenever we are within the range of applicability of the previous lemma.

As a simple application of the previous compactness result we prove the following inequality, which we will employ in the sequel of this paper.

**Lemma 4.7.** *Given  $\Lambda > 0$  there exists a constant  $C = C(\Lambda)$  such that the following holds: for any cone  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$ , set  $\mathbf{A}(1, 2) := \mathbf{C} \cap \mathbb{A}(1, 2)$ , there holds*

$$(26) \quad |\mathbf{x}| \leq C(\Lambda) \|\mathbf{x}^\perp\|_{L^2(\mathbf{A}(1,2))}.$$

*Proof.* Assume, towards a contradiction, that there exists a sequence  $\{\mathbf{C}_k\}_{k \geq 1}$  in  $\mathcal{C}_{N,n}(\Lambda)$  and corresponding vectors  $\mathbf{x}_k \in \mathbb{R}^N$  (say normalized to have unit norm) such that

$$\|\mathbf{x}_k^\perp\|_{L^2(\mathbf{A}_k(1,2))} \leq 1/k,$$

where we have set (in analogy with the above)  $\mathbf{A}_k(1, 2) = \mathbf{C}_k \cap \mathbb{A}(1, 2)$ . Now, by Lemma 4.5, possibly extracting a subsequence (which we do not rename) there holds  $\mathbf{F}(|\mathbf{C}_k|, |\mathbf{C}_\infty|) \rightarrow 0$  for some  $\mathbf{C}_\infty \in \mathcal{C}_{N,n}(\Lambda)$ . (We stress that, in particular, this implies the non-triviality of the limit cone.) In addition, the sequence  $(\mathbf{x}_k)$  has itself a converging subsequence to a limit element  $\mathbf{x}_\infty \in \mathbb{R}^N$  with  $|\mathbf{x}_\infty| = 1$ .

Hence, passing to the limit in the previous inequality we would obtain  $\|\mathbf{x}_\infty^\perp\|_{L^2(\mathbf{A}_\infty(1,2))} = 0$ , which is only possible if the flow of translations generated by  $\mathbf{x}_\infty$  in  $\mathbb{R}^N$  preserves  $\mathbf{C}_\infty$ . However, this contradicts the fact that  $\text{Sing}(\mathbf{C}_\infty) = \{0\}$  (which is not kept still by any translation). Thereby the proof is complete.  $\square$

We proceed with a (by now well-known) result on the discreteness of the space of densities of minimal cones, which is key (among other things) to prove our Theorem 4.23, whose proof is the object of Appendix E.

**Lemma 4.8.**  *$\{\theta_{\mathbf{C}}(\mathbf{0}) : \mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)\}$  is a finite subset of  $\mathbb{R}$ .*

*Proof.* Suppose by contradiction that there exists a sequence of  $\{\mathbf{C}_i\}_{i \geq 1} \subset \mathcal{C}_{N,n}(\Lambda)$  with pairwise distinct densities at the origin.

By the preceding compactness result (Lemma 4.5),  $\Sigma_i := \mathbf{C}_i \cap \mathbb{S}^{N-1}$  converges in  $C^\infty$  to a minimal submanifold  $\Sigma_\infty$ . Let then  $\mathbf{C}_\infty := \mathbf{0} \ast \Sigma_\infty$ . Hence, for sufficiently large  $i$ ,  $\Sigma_i$  can be expressed as the graph of a smooth normal section

$$u_i : \Sigma_\infty \rightarrow \Sigma_\infty^\perp.$$

It follows from the Łojasiewicz-Simon inequality [38, Theorem 3] that for every  $\mu \in (0, 1)$ , there exist  $\vartheta(\Sigma_\infty) \in (0, \frac{1}{2})$ , and  $\varsigma(\Sigma_\infty, \mu) \in (0, \beta)$  such that every  $C^{2,\mu}$  normal section  $u : \Sigma_\infty \rightarrow \Sigma_\infty^\perp$  with  $|u|_{C^{2,\mu}} \leq \varsigma$  shall satisfy the bound

$$|\mathcal{H}^{n-1}(\text{graph}_{\Sigma_\infty}(u)) - \mathcal{H}^{n-1}(\text{graph}_{\Sigma_\infty}(0))|^{1-\vartheta} \leq \|\mathcal{M}(u)\|_{L^2(\Sigma_\infty)},$$

where  $\mathcal{M}(u) = 0$  corresponds to the minimal surface equation. In particular, for sufficiently large  $i$ ,  $|u_i|_{C^{2,\mu}} \leq \varsigma$  and thus,

$$|\mathcal{H}^{n-1}(\text{graph}_{\Sigma_\infty}(u_i)) - \mathcal{H}^{n-1}(\text{graph}_{\Sigma_\infty}(0))|^{1-\vartheta} = 0,$$

which implies that  $\theta_{\mathbf{C}_i}(\mathbf{0}) = \theta_{\mathbf{C}_\infty}(\mathbf{0})$ , a contradiction to our assumption.  $\square$

*Remark 4.9.* Following up on the preceding Remark 4.4, we note that Lemma 4.5 implies that we can define

$$(27) \quad \Theta(\Lambda) := \sup_{\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)} \theta_{\mathbf{C}}(\mathbf{0}) > 0$$

which does in fact determine a monotone function  $\Theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ .

**4.2. Key definitions and tools.** Now, we move on towards the proof of Theorem 4.1; we shall work with the following spaces.

**Definition 4.10.** Given a smooth, boundaryless compact manifold  $M$  of dimension  $N \geq 3$ , and fixed a reference (smooth) background metric  $g_0$ , we let (for  $k \geq 4, \alpha \in (0, 1)$ ):

- $\mathcal{G}^{k,\alpha}(M)$  denote the space of  $C^{k,\alpha}$  Riemannian metrics on  $M$ , endowed with its natural Banach manifold structure;
- $\mathcal{M}_n^{k,\alpha}(M)$  denote the space of pairs  $(g, \Sigma)$ , where  $g \in \mathcal{G}^{k,\alpha}(M)$  and  $\Sigma$  is a connected  $n$ -dimensional minimal submanifold with only strongly isolated singularities (**MSI**) in  $(M, g)$  for  $2 \leq n < N$ ;
- $\Pi : \mathcal{M}_n^{k,\alpha}(M) \rightarrow \mathcal{G}^{k,\alpha}(M)$  denote the projection onto the first factor.

*Remark 4.11.* The space  $\mathcal{M}_n^{k,\alpha}(M)$  is endowed with the topology induced by  $C^{k,\alpha}$  convergence in the  $g$  factor, where it is understood that the norms of all tensors are measured with respect to  $g_0$ , and multiplicity 1 **F**-convergence (following [36]) in the  $\Sigma$  factor.

**Definition 4.12.** For  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$ , we shall call a normal section  $u$  on  $\Sigma$  of  **$\tau$ -growth** if  $u \in \hat{W}_\tau^{0,2}(\Sigma; \mathbf{V})$ . When  $\tau$  satisfies (21), we shall also call such sections **tame**. We denote by  $\widehat{\text{Ker}}_\tau(L_{\Sigma,g})$  the space of all the Jacobi fields of  $\tau$ -growth on  $\Sigma$  in the ambient manifold  $(M, g)$ .

**Definition 4.13.** For an  $L_{loc}^2$ -normal section  $v$  defined near  $q$ , we introduce the notion of **asymptotic rate** of  $v$  at  $q$  as follows

$$(28) \quad \mathcal{AR}_q(v) := \sup \left\{ \gamma \in \mathbb{R} : \lim_{s \searrow 0} \int_{A^g(q,s,2s)} |v|^2 \cdot \rho^{-n-2\gamma} d\|\Sigma\| = 0 \right\},$$

where, as stipulated above,  $\rho(x) = \rho_{\Sigma,g}(x)$  is the distance function to the singular set (as discussed above). We use the convention that  $\sup \emptyset = -\infty$ .

Heuristically, if  $v$  grows like  $\rho^\gamma$  near  $q$ , then  $\mathcal{AR}_q(v) = \gamma$ . Note that in the setting of hypersurfaces, a similar (yet not identical) notion of asymptotic rate was introduced in [43] and studied in [26, 27].

**Lemma 4.14.** *Suppose  $p, k \geq 2$ ,  $\tau \in \mathbb{R}$  satisfies (21),  $u \in W_{loc}^{2,2}(\Sigma; \mathbf{V})$  is a non-zero normal section.*

- (i) *If  $L_{\Sigma,g}u \in W_{\tau-2}^{0,p}(\Sigma; \mathbf{V})$  then  $\mathcal{AR}_q(u) \in \{-\infty\} \cup \Gamma(\mathbf{C}_q) \cup \{-(n-2)/2\} \cup [\tau, +\infty]$  for every  $q \in \text{Sing}(\Sigma)$ , where  $\mathbf{C}_q := \mathbf{C}_q(\Sigma)$ .*
- (ii) *If  $u \in W_\tau^{0,p}(\Sigma; \mathbf{V})$  (or, respectively:  $\hat{W}_\tau^{0,p}(\Sigma; \mathbf{V})$ ), then  $\mathcal{AR}_q(u) \geq \tau$  (respectively:  $\mathcal{AR}_q(u) \geq 0$ ) for every  $q \in \text{Sing}(\Sigma)$ .*
- (iii) *If  $\mathcal{AR}_q(u) \geq \tau$  for every  $q \in \text{Sing}(\Sigma)$  and  $L_{\Sigma,g}u \in W_{\tau-2}^{k-2,p}(\Sigma; \mathbf{V})$ , then  $u \in W_{\tau'}^{k,p}(\Sigma; \mathbf{V})$  for every  $\tau' < \tau$ .*

*Remark 4.15.* If  $u_1, u_2$  are  $L_{loc}^2$ -normal sections of an **MSI**  $\Sigma$  in  $(M, g)$  both defined around  $q \in \text{Sing}(\Sigma)$  then, by the Minkowski (triangle) inequality there holds

$$(29) \quad \mathcal{AR}_q(u_1 + u_2) \geq \min_{i=1,2} \{\mathcal{AR}_q(u_i)\}.$$

*Remark 4.16.* We note the following weighted version of the Hölder inequality. If  $\Sigma = \Sigma^n$  is an **MSI** in a Riemannian manifold  $(M, g)$  then, and  $1/p_1 + 1/p_2 = 1$ , then

$$(30) \quad u_i \in W_{\tau_i}^{0,p_i}(\Sigma; \mathbf{V}), i = 1, 2 \Rightarrow \begin{cases} u_1 \cdot u_2 \in L^1(\Sigma; \mathbf{V}) & \text{for } \tau_1 + \tau_2 \geq -n \\ u_1 \cdot u_2 \in W_\tau^{0,1}(\Sigma; \mathbf{V}) & \text{for } \tau_1 + \tau_2 \geq \tau \end{cases}$$

and there hold the respective estimates (under the corresponding finiteness assumptions):

$$(31) \quad \|u_1 \cdot u_2\|_{L^1(\Sigma)} \leq \|u_1\|_{W_{\tau_1}^{0,p_1}(\Sigma)} \|u_2\|_{W_{\tau_2}^{0,p_2}(\Sigma)}, \quad \|u_1 \cdot u_2\|_{W_\tau^{0,1}(\Sigma)} \leq \|u_1\|_{W_{\tau_1}^{0,p_1}(\Sigma)} \|u_2\|_{W_{\tau_2}^{0,p_2}(\Sigma)}$$

where the unit constant descends from the fact that we had stipulated the convention  $\rho \leq 1$ ; else one would have to keep track of an inessential constant depending on the ambient Riemannian manifold. The proof is done by straightforward reduction to the unweighted (standard) Hölder inequality. (Of course, a fully analogous version of these results holds true for scalar-valued functions  $u_1, u_2$  in place of sections of the bundle  $\mathbf{V}$ ).

*Proof.* To prove (i), it suffices to show that if  $\gamma := \mathcal{AR}_q(u) \in (-\infty, \tau)$ , then  $\gamma \in \Gamma(\mathbf{C}_q) \cup \{-(n-2)/2\}$ . To see this, suppose for contradiction that  $\gamma$  does not belong to such a set, then there exist  $\sigma > 0$  and  $\gamma' < \gamma < \gamma''$  such that

$$[\gamma' - \sigma, \gamma'' + \sigma] \cap (\Gamma(\mathbf{C}_q) \cup \{-(n-2)/2\} \cup [\tau, +\infty)) = \emptyset.$$

Since the tangent cone of  $\Sigma$  at  $q$  is a regular cone of multiplicity 1, there exist  $s_0 > 0$  and  $\Lambda > 0$ , both depending on  $\Sigma$ , such that:

- after pulling back to  $T_q M$  using the exponential map,  $\Sigma$  is a graph over  $\mathbf{C} := \mathbf{C}_q$  in  $B^g(q, 2s_0)$ ; we use that graphical parametrization to define  $v := \mathbf{T}_{\mathbf{C},g_{\text{euc}}}^\Sigma(u)$ , that is a section defined on  $\mathbf{C}_q$ ;

- for every  $s \in (0, s_0]$ , (65) in Corollary B.5 holds for  $\Sigma$  with respect to the scaled metric  $s^{-2}g$  (for  $\Lambda, \sigma$  as defined above), (66) also holds for  $\gamma'$  in place of  $\gamma$ , and in addition (by our assumption on  $\mathcal{AR}_q(u)$ )

$$\limsup_{s \searrow 0} \int_{\mathbf{A}(s, 2s)} |v|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| = +\infty.$$

Hence, relying on such an equation, we can construct a monotonically decreasing sequence  $\{s_j\}_{j \geq 1}$  contained in the interval  $(0, s_0/2)$ , tending to 0 as  $j \rightarrow \infty$ , such that

$$\int_{\mathbf{A}(s_1, 2s_1)} |v|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| > 0$$

and for every  $j \geq 1$  and every  $s \in [s_j, s_1]$ , we have

$$\int_{\mathbf{A}(s_j, 2s_j)} |v|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| \geq \int_{\mathbf{A}(s, 2s)} |v|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\|.$$

Equivalently, by letting  $v_j(x) := v(s_j x)$ , there holds

$$(32) \quad \int_{\mathbf{A}(1, 2)} |v_j|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| \geq \int_{\mathbf{A}(s, 2s)} |v_j|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\|, \quad \forall s \in [1, s_j^{-1}s_1].$$

Note that this in particular implies, for every  $j \geq 1$ ,

$$(33) \quad \|v_j\|_{L^2(\mathbf{A}(1, 2))}^2 \geq C(\gamma'') \int_{\mathbf{A}(1, 2)} |v_j|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\|$$

$$(34) \quad \geq C(\gamma'') \int_{\mathbf{A}(s_j^{-1}s_1, 2s_j^{-1}s_1)} |v_j|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\|$$

$$(35) \quad = C(\gamma'') s_j^{2\gamma''} \int_{\mathbf{A}(s_1, 2s_1)} |v|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| > 0.$$

On the other hand, by applying Corollary B.5 to the data  $\Sigma, M, g_j := s_j^{-2}g, u_j$  as defined above and  $\gamma'$ , we see that for every  $j \geq 1$  and  $s \in (0, 1)$ , since  $\mathbf{A}(K^{-1}s, s)$  is contained in  $\mathbf{A}(K^{-\ell-2}, K^{-\ell})$  for some non-negative integer  $\ell$ , we have

$$(36) \quad \|v_j\|_{L^2(\mathbf{A}(K^{-1}s, s))} \leq C(\Lambda, \sigma) (F_j + \|v_j\|_{L^2(\mathbf{A}(K^{-1}, 1))}) s^{n/2+\gamma'},$$

where, letting  $f_j(x) := s_j^2 (\mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma (L_{\Sigma, g} u)) (s_j x)$ ,

$$F_j \leq \left\| |x|^{2-\gamma'-n/2} f_j \right\|_{L^2(\mathbf{B}(2))} \leq C(\sigma, \Lambda, \tau) \|L_{\Sigma, g} u\|_{L_{\tau-2}^2(B^g(q, 2s_0))} \cdot s_j^\tau.$$

Combined with (35), this estimate implies,

$$(37) \quad F_j \leq C(\sigma, \Lambda, \tau, \gamma'', u, L_{\Sigma, g} u, s_1) s_j^{\tau-\gamma''} \|v_j\|_{L^2(\mathbf{A}(1, 2))}.$$

By (32), (36), (37) and the Rellich Compactness Theorem for  $W^{2,2} \hookrightarrow L^2$ ,  $\|v_j\|_{L^2(\mathbf{A}(K^{-1}, 2))}^{-1} \cdot v_j$  subconverges to some  $\hat{v}_\infty$  in  $L_{loc}^2(\mathbf{C})$ , which is non-zero by the choice of normalization, satisfying  $L_{\mathbf{C}} \hat{v}_\infty = 0$  and the decay estimate near 0 and  $\infty$ :

$$\sup_{s>1} \int_{\mathbf{A}(s, 2s)} |\hat{v}_\infty|^2 \cdot |x|^{-n-2\gamma''} d\|\mathbf{C}\| < +\infty; \quad \sup_{s \in (0, 1)} \left( \int_{\mathbf{A}(K^{-1}s, s)} |\hat{v}_\infty|^2 \right) \cdot s^{-n-2\gamma'} d\|\mathbf{C}\| < +\infty.$$

But since in particular  $[\gamma', \gamma''] \cap \Gamma(\mathbf{C}) = \emptyset$ , there is no such non-zero Jacobi fields on  $\mathbf{C}$ , which is a contradiction.

For part (ii), let us first assume  $u \in W_\tau^{0,p}(\Sigma; \mathbf{V})$ ; let  $\tau' < \tau$  (without further restrictions). Then, due to the Hölder inequality in the annulus, since  $p \geq 2$  we have

$$\int_{A^g(q,s,2s)} |u|^2 \cdot \rho^{-n-2\tau'} d\|\Sigma\| \leq \left( \int_{A^g(q,s,2s)} |u|^p \cdot \rho^{-n-p\tau} d\|\Sigma\| \right)^{2/p} \left( \int_{A^g(q,s,2s)} \rho^{-n+\frac{2p(\tau-\tau')}{p-2}} d\|\Sigma\| \right)^{1-2/p}$$

therefore, as one lets  $s \searrow 0$  both factors on the right-hand side tend to zero and so will the left-hand side. But then, by the arbitrariness of  $\tau' < \tau$  this precisely means that  $\mathcal{AR}_q(u) \geq \tau$  for every  $q \in \text{Sing}(\Sigma)$ . When instead one works in the augmented spaces, namely if  $u \in \widehat{W}_\tau^{0,p}(\Sigma; \mathbf{V})$  then  $u = u_0 + \phi$  for  $u_0 \in W_\tau^{0,p}(\Sigma; \mathbf{V})$  and  $\phi \in X_{TS}$  (cf. Definition 2.10), and thus it suffices to note that, thanks to the Minkowski (triangle) inequality (see Remark 4.15) there holds

$$\mathcal{AR}_q(u) \geq \min_{i=1,2} \{ \mathcal{AR}_q(u_0); \mathcal{AR}_q(\phi) \}$$

and  $\mathcal{AR}_q(u_0) \geq \tau$  (as just shown) while trivially  $\mathcal{AR}_q(\phi) \geq 0$ ; since by (21) in particular  $\tau \in (0, 1)$  the conclusion follows at once.

To prove (iii), first notice that by applying Corollary B.5 in the same way as the proof of (i), for every  $\tau' < \tau$  and every  $q \in \text{Sing}(\Sigma)$ , there exists  $s_0 > 0$  small enough and  $K > 2$  such that for all  $s \in (0, s_0]$ ,

$$\|u\|_{W_{\tau'}^{0,2}(A^g(q,K^{-1}s,s))} \leq C(\Sigma, g, \tau, \tau') \left( \|L_{\Sigma,g}u\|_{W_{\tau'-2}^{0,2}(B^g(q,s_0))} + \|u\|_{L^2(A^g(q,K^{-1}s_0,s_0))} \right).$$

On the other hand, the interior elliptic estimate (in scaling-invariant form) reads for every  $s \in (0, s_0]$ ,

$$\|u\|_{W_{\tau'}^{k,p}(A^g(q,K^{-1}s,s))} \leq C(\Sigma, g) \left( \|L_{\Sigma,g}u\|_{W_{\tau'-2}^{k-2,p}(A^g(q,K^{-2}s,2s))} + \|u\|_{L_{\tau'}^2(A^g(q,K^{-2}s,2s))} \right).$$

Thus, if we plug-in the former in the latter we get for all  $s \leq s_0/2$  that

$$\|u\|_{W_{\tau'}^{k,p}(A^g(q,K^{-1}s,s))} \leq C(\Sigma, g, \tau, \tau') \left( \|L_{\Sigma,g}u\|_{W_{\tau'-2}^{k-2,p}(\Sigma)} + \|u\|_{L^2(A^g(q,K^{-1}s_0,2s_0))} \right)$$

that is a bound independent of  $s$ . As a result, if we take any  $\tau'' < \tau'$ , since there clearly holds

$$\|u\|_{W_{\tau''}^{k,p}(A^g(q,K^{-1}s,s))} \leq s^{(\tau'-\tau'')} \|u\|_{W_{\tau'}^{k,p}(A^g(q,K^{-1}s,s))}$$

we can bound

$$\|u\|_{W_{\tau''}^{k,p}(A^g(q,K^{-J}s,s))} \leq C(\Sigma, g, \tau, \tau', u) s^{(\tau'-\tau'')} \sum_{j=0}^{J-1} \left[ K^{p(\tau'-\tau'')} \right]^{-j}.$$

Letting  $J \rightarrow \infty$  and observing that the uniform bound on the right-hand side (which is a geometric series) implies the finiteness of  $\|u\|_{W_{\tau''}^{k,p}(B^g(q,s_0))}$ , by arbitrariness of  $\tau'', \tau'$  we readily get to the desired conclusion.  $\square$

**Corollary 4.17.** *For  $\tau \in \mathbb{R}$  satisfying (21),  $\widehat{\text{Ker}}_\tau(L_{\Sigma,g})$  is independent of  $\tau$  (hence, from now onward, we will simply denote it by  $\widehat{\text{Ker}}(L_{\Sigma,g})$ ); furthermore, any  $u \in \widehat{\text{Ker}}(L_{\Sigma,g})$  satisfies  $\mathcal{AR}_q(u) \geq 1$  for every  $q \in \text{Sing}(\Sigma)$ .*

*Proof.* Let  $u \in \widehat{\text{Ker}}_\tau(L_{\Sigma,g})$  for some  $\tau$  satisfying (21) and let  $\tau' \in (\tau, 1)$ . Recall that  $u = \phi + u_0$  for  $u_0 \in W_\tau^{0,2}$  and  $\phi$  a translation-like section; note, also, that such  $\phi$  is uniquely determined as soon as  $\tau > 0$  and, in addition, by direct computation there holds  $L_{\Sigma,g}\phi \in W_{-\varepsilon}^{0,2}(\Sigma; \mathbf{V})$  for all  $\varepsilon > 0$  (see Lemma 2.22).

The latter remark (together with the very definition of Jacobi field, which gives  $L_{\Sigma,g}u = 0$ ) implies in particular that  $L_{\Sigma,g}u_0 \in W_{\tau'-2}^{0,2}(\Sigma; \mathbf{V})$  and so, by items (i) and (ii) above there holds  $\mathcal{AR}_q(u_0) \geq \tau'$  for every  $q \in \text{Sing}(\Sigma)$ . At that stage, we appeal to item (iii) to conclude that  $u_0 \in W_{\tau''}^{0,2}(\Sigma; \mathbf{V})$  for any  $\tau'' < \tau'$ , which – by the arbitrariness of  $\tau'$  – completes the proof.  $\square$

**4.3. Canonical neighborhoods.** We shall begin here with a definition that will repeatedly be employed in the sequel of this section.

**Definition 4.18.** For  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$ ,  $\Lambda > 0$  such that  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)}\{\theta_{|\Sigma|}(p)\}, 1\}$  (based upon (27)), and  $\delta > 0$ , we set

$$\text{inrad}(g, \Sigma) := \min\{\text{inrad}(M, g), \min\{\text{dist}_g(p, p')/2 : p \neq p' \in \text{Sing}(\Sigma)\}\},$$

and define  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  to be the space of all pairs  $(g', \Sigma') \in \mathcal{M}_n^{k,\alpha}(M)$  such that:

- $\|g'\|_{C^{k,\alpha}} \leq \Lambda$ , and  $\|g - g'\|_{C^k} \leq \delta$ ;
- $\Sigma'$  is a connected **MSI** in  $(M, g')$  satisfying

$$(38) \quad \mathbf{r}_{\Sigma',g'} \geq \Lambda^{-1} \rho_{\Sigma',g'}, \quad \text{and} \quad \mathbf{F}(|\Sigma|_g, |\Sigma'|_{g'}) \leq \delta,$$

where  $\rho_{\Sigma',g'}$  denotes the distance function to  $\text{Sing}(\Sigma')$  in  $(M, g')$ .

- there exists a bijection  $\text{Sing}(\Sigma) \rightarrow \text{Sing}(\Sigma')$ , such that for every  $p \in \text{Sing}(\Sigma)$ , its image  $p' \in \text{Sing}(\Sigma')$  satisfies,

$$(39) \quad \text{dist}_g(p, p') \leq \text{inrad}(g, \Sigma)/2, \quad \theta_{|\Sigma'|}(p') = \theta_{|\Sigma|}(p).$$

We call such  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  a **canonical (pseudo-)neighborhood** of  $(g, \Sigma)$  in  $\mathcal{M}_n^{k,\alpha}(M)$ , and its topology is the one induced from  $\mathcal{M}_n^{k,\alpha}(M)$ .

Clearly, when  $g, \Sigma$  are fixed,  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  is monotonically increasing (with respect to set-theoretic inclusion) in both  $\Lambda$  and  $\delta$ . Also note that in general,  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  is not a genuine topological neighborhood of  $(g, \Sigma)$  in  $\mathcal{M}_n^{k,\alpha}(M)$ , as it is not necessarily an open subset (nor will it contain an open subset). Indeed, there might be  $(g', \Sigma') \in \mathcal{M}_n^{k,\alpha}(M)$  arbitrarily close to  $(g, \Sigma)$  but not contained in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ , which happens, for example, when multiple singular points collapse to a single one.

*Remark 4.19.* Note that for every  $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ , (38) implies (cf. Remark 4.9) that every tangent cone of  $\Sigma'$  at any of its singular points also belongs to  $\mathcal{C}_{N,n}(\Lambda)$ .

One of the main reasons behind this specific definition of canonical neighborhood lies in the associated compactness property, which we now state.

**Lemma 4.20** (Compactness of canonical neighborhoods). *For every  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$  and every  $\Lambda > 0$  such that  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)}\{\theta_{|\Sigma|}(p)\}, 1\}$ , there exists  $\delta_0(g, \Sigma, \Lambda, k, \alpha) \in (0, 1)$  with the following property.*

*For every  $\delta \in (0, \delta_0)$ ,  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  is compact under  $C^k$ -norm in the first factor and  $\mathbf{F}$ -distance in the second factor. Moreover, there holds  $C_{loc}^2$  convergence in the second factor.*

*Proof.* The first claim easily follows from the Arzelà-Ascoli compactness theorem, the definition of canonical neighborhood and the definition of regularity scale (see in particular Remark 2.14). The second relies on the first together with the fact that, again by virtue of the definition of canonical neighborhood, we are assuming a uniform bound on the regularity scale of all **MSI** in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  and thus Remark 2.14 applies.  $\square$

A second, important reason to consider such canonical neighborhoods is that the following local Sard-Smale-type theorem holds.

**Theorem 4.21.** *Let  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$  with  $\widehat{\text{index}}_\tau(L_{\Sigma,g}) < 0$ ,  $\tau \in (\sup_{p \in \text{Sing}(\Sigma)} \gamma_-(\mathbf{C}_p), 1)$  and  $\Lambda > 0$  such that  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)} \{\theta_{|\Sigma|}(p)\}, 1\}$ . Then there exists  $\kappa_0 = \kappa_0(g, \Sigma, \Lambda) > 0$  such that for every  $\kappa \in (0, \kappa_0)$ ,*

$$\mathcal{G}^{k,\alpha}(g, \Sigma; \Lambda, \kappa) := \mathcal{G}^{k,\alpha}(M) \setminus \Pi(\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa))$$

is an open dense subset of  $\mathcal{G}^{k,\alpha}(M)$ .

*Remark 4.22.* Going through spaces of metrics of finite regularity is forced by our necessity of having local compactness (i. e. compactness of balls in the space of metrics we work with). As mentioned before, the way to gain Theorem 4.1 as it is stated (namely for  $C^\infty$  metrics) will be discussed at the very end of this section.

Heuristically, recalling that one of the key points of [47],  $\widehat{\text{index}}_\tau(L_{\Sigma,g}) < 0$  (which implies the non-surjectivity of the Jacobi operator of  $\Sigma$  in  $(M, g)$ ) suggests that the ‘‘tangent map’’ of  $\Pi$  restricted to  $\mathcal{M}_n^{k,\alpha}$  is itself not surjective at  $(g, \Sigma)$ . Hence one may think of  $\mathcal{G}(g, \Sigma; \Lambda, \kappa_0)$  as the set of regular values of  $\Pi|_{\mathcal{L}(g, \Sigma; \Lambda, \kappa_0)}$ . When we only focus on regular minimal hypersurfaces, such a Sard-Smale Theorem was proved in [47] by showing that - working with spaces of metrics having finite, in fact  $C^{k,\alpha}$ , degree of regularity -  $\mathcal{M}^{k,\alpha}$  is a Banach manifold, and  $\Pi$  is a Fredholm map with Fredholm index 0. The discussion about how to derive the corresponding result for smooth metrics was then later presented in [49] (see also [5, Section 7] for a thorough study of this aspect, and the application to the free-boundary counterpart of such a result).

Here, however, it is hard to expect any Banach manifold structure on  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ . The way we prove Theorem 4.21 is rather by unwrapping the proof of the infinite dimensional Sard-Smale Theorem by hand. More precisely, we will prove the controlled behavior of tame Jacobi fields for a convergent sequence in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$ , and design a good way to slice  $\mathcal{G}^{k,\alpha}(M)$  into union of finite-dimensional subspaces  $\{\mathcal{F}_{g', \Sigma'}\}$  such that each  $\Pi^{-1}(\mathcal{F}_{g', \Sigma'}) \cap \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_0)$  is bi-Lipschitz to a compact subset in the finite dimensional vector space  $\widehat{\text{Ker}}(L_{\Sigma', g'})$ . The proof of Theorem 4.21 occupies most of the rest of this section.

To derive a global Sard-Smale type theorem on  $\mathcal{M}_n(M)$ , it thus suffices to show that it can be covered by countably many canonical neighborhoods.

**Theorem 4.23.** *Let  $\kappa : \mathcal{M}_n^{k,\alpha}(M) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive function, not necessarily continuous. Then there exists a countable number of triples  $\{(g_j, \Sigma_j; \Lambda_j)\} \in \mathcal{M}_n^{k,\alpha}(M) \times \mathbb{R}_+$  such that*

$$\mathcal{M}_n^{k,\alpha}(M) = \bigcup_{j \geq 1} \mathcal{L}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where  $\kappa_j := \kappa(g_j, \Sigma_j; \Lambda_j)$ .

Theorem 4.23 will be proved in Section E, based on decomposition arguments inspired by the work [17].

**4.4. Induced Jacobi fields for pairs.** Given a pair  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$ , for sufficiently small  $\delta$  and sufficiently large  $\Lambda$ , we can define a *semi-metric* on  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  by

$$\begin{aligned} \mathbf{D}^\mathcal{L}[(g_1, \Sigma_1), (g_2, \Sigma_2)] &:= \|g_1 - g_2\|_{C^{k,\alpha}(M)} \\ &+ \|\mathbf{G}_{\Sigma_1, g_1}^{\Sigma_2}\|_{L^2(\Sigma_1 \setminus B^{g_1}(\text{Sing}(\Sigma_1), C^{\mathcal{L}} \mathbf{r}_{\Sigma_1, g_1}^{\Sigma_2}))} + \|\Sigma_2\|_{g_2}(B^{g_1}(\text{Sing}(\Sigma_1), 2C^{\mathcal{L}} \mathbf{r}_{\Sigma_1, g_1}^{\Sigma_2})) \\ &+ \|\mathbf{G}_{\Sigma_2, g_2}^{\Sigma_1}\|_{L^2(\Sigma_2 \setminus B^{g_2}(\text{Sing}(\Sigma_2), C^{\mathcal{L}} \mathbf{r}_{\Sigma_2, g_2}^{\Sigma_1}))} + \|\Sigma_1\|_{g_1}(B^{g_2}(\text{Sing}(\Sigma_2), 2C^{\mathcal{L}} \mathbf{r}_{\Sigma_2, g_2}^{\Sigma_1})). \end{aligned}$$

Here, the constant  $C^\mathcal{L}$  is defined as follows. Since  $\Sigma$  is a **MSI** in  $(M, g)$ , by Definition 2.3 and Remark 2.5, there are finitely many singular points  $\text{Sing}(\Sigma) = \{p_i\}_{i=1}^Q$  and at each singular point  $p_i$   $\Sigma$  has a unique tangent cone  $\mathbf{C}_i := \mathbf{C}_{p_i}(\Sigma)$ . Using the fact that each set  $\Gamma(\mathbf{C}_i)$  is discrete, we can choose a small enough  $\sigma := \sigma(g, \Sigma) \in (0, 1)$  such that for every  $p_i \in \text{Sing}(\Sigma)$ ,

$$1 - 2\sigma > \gamma^-(\mathbf{C}_i), \quad \text{dist}_{\mathbb{R}}(1 - 2\sigma, \Gamma(\mathbf{C}_i) \cup \{-(n-2)/2\}) \geq 2\sigma.$$

We set

$$(40) \quad C^\mathcal{L} := C_2(1 - 2\sigma, \sigma, \Lambda) > 0$$

as in Corollary D.4. It is worth noting that  $C^\mathcal{L}$  depends on the choice of  $\sigma$ , but we will fix one admissible  $\sigma$  for each pair  $(g, \Sigma)$  from now on.

Recall that (consistently with Definition 4.10) norms of tensors are taken with respect to the reference metric  $g_0$  which we have fixed earlier in the discussion. The reason we call  $\mathbf{D}^\mathcal{L}$  a *semi-metric* rather than a *metric* is that it satisfies all the metric axioms except, possibly, for the triangle inequality.

*Remark 4.24.* By Allard's regularity theorem [2], a sequence  $(g_j, \Sigma_j)$  converges to  $(g_\infty, \Sigma_\infty)$  in  $\mathcal{L}^{k,\alpha}$  if and only if  $\mathbf{D}^\mathcal{L}[(g_j, \Sigma_j), (g_\infty, \Sigma_\infty)] \rightarrow 0$  when  $j \rightarrow \infty$ . And, as in Lemma 4.20, if that is the case then there holds  $C_{loc}^2$  convergence of  $\Sigma_j$  to  $\Sigma_\infty$  (graphical convergence with multiplicity one).

The following statement is an application of Corollary D.4 and Corollary D.5.

**Lemma 4.25.** *Given  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$ ,  $r \in (0, \text{inrad}(g, \Sigma))$ ,  $\Lambda > 0$ , and  $\delta > 0$ , suppose that:*

- (i)  $\mathcal{F}$  be a finite-dimensional subspace of  $C^{k,\alpha}(M)$ , consisting of functions supported in the complement of  $B^g(\text{Sing}(\Sigma), r) = \cup_{p \in \text{Sing}(\Sigma)} B^g(p, r)$ ;
- (ii)  $\{(\bar{g}_j, \bar{\Sigma}_j)\}_j$ ,  $\{(g_j^0, \Sigma_j^0)\}_j$  and  $\{(g_j^1, \Sigma_j^1)\}_j$  be three sequences in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  such that  $(\bar{g}_j, \bar{\Sigma}_j) \rightarrow (g, \Sigma)$  as  $j \rightarrow \infty$ , for every  $j$ ,  $(g_j^0, \Sigma_j^0)$  and  $(g_j^1, \Sigma_j^1)$  are distinct pairs, and both sequences  $(g_j^i, \Sigma_j^i) \rightarrow (g, \Sigma)$  as  $j \rightarrow \infty$ ;
- (iii) for  $i = 0, 1$  and  $j \geq 1$ ,  $g_j^i = (1 + f_j^i)\bar{g}_j$  for  $f_j^i \in \mathcal{F}$ , and either  $f_j^0 \equiv f_j^1$  (in which case we set  $f_\infty \equiv 0$ ), or

$$\frac{f_j^1 - f_j^0}{\|f_j^1 - f_j^0\|_{C^{k,\alpha}(M)}} \rightarrow f_\infty \quad \text{in } C^{k,\alpha}(M)$$

with the limit function satisfying the condition that  $\nabla_{\Sigma, g}^\perp(f_\infty) := (\nabla f_\infty)^\perp_{\Sigma, g}$  is not identically 0.

For all  $j \geq 0$  and  $i \in \{0, 1\}$ , we set

$$d_j := \mathbf{D}^\mathcal{L}[(g_j^0, \Sigma_j^0), (g_j^1, \Sigma_j^1)] > 0, \quad u_j^{(i)} := \mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^i} \in L^\infty(\bar{\Sigma}_j; \mathbf{V}_j), \quad v_j^{(i)} := \mathbf{T}_{\Sigma, g}^{\bar{\Sigma}_j}(u_j^{(i)}) \in L^\infty(\Sigma; \mathbf{V})$$

where  $\mathbf{V}_j$  (respectively:  $\mathbf{V}$ ) denotes the normal bundle to  $\bar{\Sigma}_j$  with respect to the metric  $\bar{g}_j$  (resp.: to  $\Sigma$  with respect to the metric  $g$ ). Then we have, after passing to a subsequence, as  $j \rightarrow \infty$ ,

$$d_j^{-1} \cdot (v_j^{(1)} - v_j^{(0)}) \rightarrow \hat{u}_\infty \quad \text{in } C_{loc}^2(\Sigma; \mathbf{V}), \quad d_j^{-1} \cdot (f_j^1 - f_j^0) \rightarrow \hat{f}_\infty \quad \text{in } C^{k,\alpha}(M),$$

where  $\mathbf{V}$  is the normal bundle to  $\Sigma$  with respect to the metric  $g$ ,  $\hat{f}_\infty$  is a non-negative multiple of  $f_\infty$  and  $\hat{u}_\infty$  is a non-zero solution of

$$L_{\Sigma,g} \hat{u}_\infty = \frac{n}{2} \nabla_{\Sigma,g}^\perp (\hat{f}_\infty).$$

Moreover,  $\hat{u}_\infty$  is a tame section.

*Proof.* For a fixed pair  $(g, \Sigma)$ , we choose  $\sigma$  as in the definition of  $C^\mathcal{L}$  in (40). Since  $\bar{\Sigma}_j, \Sigma_j^0, \Sigma_j^1 \rightarrow \Sigma$  in  $\mathcal{L}^{k,\alpha}$ , for all sufficiently large  $j$ , for any (necessarily non-trivial) tangent cone  $\mathbf{C}$  at points belonging to  $\text{Sing}(\bar{\Sigma}_j), \text{Sing}(\Sigma_j^0)$  or  $\text{Sing}(\Sigma_j^1)$ , we have

$$(41) \quad \text{dist}_{\mathbb{R}}(1 - 2\sigma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma.$$

Note that in  $B^g(\text{Sing}(\Sigma), r)$ , for all  $j$ ,

$$\bar{g}_j = g_j^0 = g_j^1$$

Therefore, together with (41), we can apply Corollary D.4 (keeping in mind the  $C_{loc}^2$  convergence of both  $\Sigma_j^0$  and  $\Sigma_j^1$  to  $\Sigma$ , cf. Remark 4.24) with  $\gamma = 1 - 2\sigma$ , so there exist  $C_0 = C_0(\sigma, \Lambda) > 0$ ,  $\tilde{C}_0 = \tilde{C}_0(\sigma, \Lambda) > 0$ , and  $r_0 = r_0(g, \Sigma; \Lambda, \sigma) \in (0, \min(r/(2\tilde{C}_0 + 1), 1/2))$  such that, for every  $\bar{\mathbf{x}}_j \in \text{Sing}(\bar{\Sigma}_j)$ , let  $\mathbf{x}_j^0$  and  $\mathbf{x}_j^1$  be the unique element in  $\text{Sing}(\Sigma_j^0) \cap B^{\bar{g}_j}(\bar{\mathbf{x}}_j, r_0)$  and  $\text{Sing}(\Sigma_j^1) \cap B^{\bar{g}_j}(\bar{\mathbf{x}}_j, r_0)$ , respectively, and for  $i = 0, 1$ , we have

$$(42) \quad |\mathbf{x}_j^1 - \mathbf{x}_j^0| \leq \mathbf{r}_j^{(i)} := \mathbf{r}_{\Sigma_j^{1-i}, \bar{g}_j}^{\Sigma_j^i} \leq C_0 \|\mathbf{G}_{\Sigma_j^{1-i}, \bar{g}_j}^{\Sigma_j^i}\|_{L^2(A^{\bar{g}_j}(\mathbf{x}_j^{1-i}, 2r_0, \tilde{C}_0 r_0))},$$

$$(43) \quad \|\mathbf{G}_{\Sigma_j^{1-i}, \bar{g}_j}^{\Sigma_j^i}\|_{C^0(A^{\bar{g}_j}(\mathbf{x}_j^{1-i}, 2r_0, \tilde{C}_0 r_0))} \leq C_0 \|\mathbf{G}_{\Sigma_j^{1-i}, \bar{g}_j}^{\Sigma_j^i}\|_{L^2(A^{\bar{g}_j}(\mathbf{x}_j^{1-i}, 2r_0, \tilde{C}_0 r_0))}.$$

In addition, we note that, letting  $j \rightarrow \infty$  in the equation  $g_j^i = (1 + f_j^i)\bar{g}_j$  we get (by virtue of the convergence assumptions (ii), based on the specification in (iii)) in fact

$$(44) \quad \|f_j^i\|_{C^2(M)} \rightarrow 0, \quad i = 0, 1 \quad (j \rightarrow \infty).$$

It follows from the minimal submanifold system (see Appendix A, cf. [27, Theorem B.1]) that, for sufficiently large  $j$ ,  $w_j := u_j^{(1)} - u_j^{(0)}$  satisfies the equation

$$(45) \quad L_{\bar{\Sigma}_j, \bar{g}_j} w_j - \frac{n}{2} \nabla_{\bar{\Sigma}_j, \bar{g}_j}^\perp (f_j^1 - f_j^0) + \nabla_{\bar{\Sigma}_j, \bar{g}_j}^\perp \cdot \tilde{b}_0 + \frac{\tilde{b}_1}{\rho_j} = 0$$

on  $\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\bar{\Sigma}_j))$ . Here,  $\rho_j := \rho_{\bar{\Sigma}_j, \bar{g}_j}$  is the distance function to  $\text{Sing}(\bar{\Sigma}_j)$ , and  $\tilde{b}_0$  and  $\tilde{b}_1$  are error terms satisfying pointwise estimates

$$(46) \quad \begin{aligned} |\tilde{b}_0|, |\tilde{b}_1| \leq C'_0 \cdot & \left( \sum_{i=0,1} \left( \frac{|u_j^{(i)}|}{\rho_j} + |\nabla^\perp u_j^{(i)}| + \sum_{\ell=0,1,2} |\nabla^{(\ell)} f_j^i| \right) \right) \\ & \cdot \left( \frac{|w_j|}{\rho_j} + |\nabla^\perp w_j| + \sum_{\ell=0,1,2} |\nabla^{(\ell)} (f_j^1 - f_j^0)| \right) \end{aligned}$$

where, for notational convenience, we have written  $\nabla^\perp$  in lieu of  $\nabla_{\bar{\Sigma}_j, \bar{g}_j}^\perp$ , and  $C'_0$  is a constant independent of  $j$ .

**Claim 1.** For any subsequence of  $\{j\}$ , we can find a further subsequence such that

$$\frac{(f_j^1 - f_j^0)}{d_j} \rightarrow \hat{f}_\infty = c \cdot f_\infty,$$

for some  $c \geq 0$ . The value of  $c$  could depend on the chosen subsequence.

*Proof.* If  $f_j^1 = f_j^0$ , then this is trivially true with  $c = 0$ .

Otherwise, by definition,

$$d_j \geq \|g_j^1 - g_j^0\|_{C^{k,\alpha}(M)} \geq \|g_j^1 - g_j^0\|_{L^\infty(M)} \geq C(g) \|f_j^1 - f_j^0\|_{L^\infty(M)},$$

and the existence of the limit (recalling the finite-dimensionality of  $\mathcal{F}$ ) follows immediately.  $\square$

**Claim 2.** Let  $d'_j := \|w_j\|_{L^2(\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\bar{\Sigma}_j), r_0))}$ . Then

$$\liminf_{j \rightarrow \infty} \frac{d_j}{d'_j} \in (0, +\infty).$$

*Proof.* Without loss generality, and without renaming we can restrict to a subsequence whose limit is equal to  $\liminf_{j \rightarrow \infty} \frac{d_j}{d'_j}$ . Suppose for the sake of contradiction that  $\lim_{j \rightarrow \infty} \frac{d_j}{d'_j} = 0$  or  $+\infty$ .

Case 1.  $\lim_{j \rightarrow \infty} \frac{d_j}{d'_j} = 0$ .

Note that by definition

$$d_j \equiv \mathbf{D}^{\mathcal{L}}[(g_j^0, \Sigma_j^0), (g_j^1, \Sigma_j^1)] \geq \|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), r_0/2))}.$$

Next, by the convergence assumptions in item (ii), for any  $\delta > 0$  there exists  $j_0 = j_0(\delta)$  such that  $j \geq j_0(\delta)$  implies that outside of  $B^{g_j^0}(\text{Sing}(\Sigma_j^0), r_0/8)$ , the submanifolds  $\Sigma_j^0$ ,  $\Sigma_j^1$  and  $\bar{\Sigma}_j$  are all  $\delta$ - $C^3$  graphs over  $\Sigma$ . Note, in particular, that for all sufficiently large  $j$ , there holds the inclusion  $B^{g_j^0}(\text{Sing}(\Sigma_j^0), r_0/2) \subset B^{\bar{g}_j}(\text{Sing}(\bar{\Sigma}_j), r_0)$ . It then follows from (76) in Lemma D.1 with a suitable covering that for all sufficiently large  $j$ , there exists a constant  $C_1 > 0$  independent of  $j$  such that

$$\begin{aligned} d'_j &= \|\mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^1} - \mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^0}\|_{L^2(\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\bar{\Sigma}_j), r_0))} \\ &\leq \|\mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^1} - \mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^0}\|_{L^2(\bar{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\Sigma_j^0), 2r_0/3))} \\ &\leq C_1 \|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), r_0/2))} \\ &\leq C_1 d_j. \end{aligned}$$

Therefore,  $\frac{d_j}{d'_j} \geq \frac{1}{C_1}$ . This contradicts our assumption that  $\lim_{j \rightarrow \infty} \frac{d_j}{d'_j} = 0$ .

Case 2.  $\lim_{j \rightarrow \infty} \frac{d_j}{d'_j} = +\infty$ . In this case, we can divide the PDE (45) by  $d_j$ , and then passing to a subsequence such that

$$\frac{\mathbf{T}_{\Sigma_j, g_j}^{\bar{\Sigma}_j} w_j}{d_j} \rightarrow \tilde{w}_\infty \text{ in } C_{loc}^2(\Sigma; \mathbf{V}), \quad \frac{(f_j^1 - f_j^0)}{d_j} \rightarrow \hat{f}_\infty \text{ in } C^2(M),$$

by (46), we obtain

$$(47) \quad -L_{\Sigma,g}\tilde{w}_\infty + \frac{n}{2}\nabla_{\Sigma,g}^\perp(\hat{f}_\infty) = 0.$$

We remark that the convergence of  $\mathbf{T}_{\Sigma,g}^{\tilde{\Sigma}_j}w_j/d_j$  follows again from (76) in Lemma D.1 and the standard elliptic estimates, whereas the convergence of  $(f_j^1 - f_j^0)/d_j$  follows from Claim 1. Note that in this case, we have

$$\|\tilde{w}_\infty\|_{L^2(\Sigma \setminus B^{\bar{g}}(\text{Sing}(\Sigma), r_0))} = \lim_{j \rightarrow \infty} \frac{\|w_j\|_{L^2(\tilde{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\tilde{\Sigma}_j), r_0))}}{d_j} = \lim_{j \rightarrow \infty} \frac{d'_j}{d_j} = 0.$$

Here, we use the fact that  $\text{Sing}(\Sigma_j) \rightarrow \text{Sing}(\Sigma_\infty)$  in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$ . Therefore,  $\tilde{w}_\infty \equiv 0$  in  $\Sigma \setminus B^g(\text{Sing}(\Sigma), r_0)$ . Since  $\hat{f}_\infty \equiv 0$  in  $\Sigma \cap B^g(\text{Sing}(\Sigma), 2r_0)$ , by the unique continuation property of the solution to (47) (for which [6] would suffice), we know that  $\tilde{w}_\infty \equiv 0$  on  $\Sigma$ .

Since  $\nabla_{\Sigma,g}^\perp(f_\infty)$  is not identically 0 and  $\hat{f}_\infty = cf_\infty$ , we have  $c = 0$ , i.e.,  $\hat{f}_\infty \equiv 0$ . As a result, as one lets  $j \rightarrow \infty$  there holds

$$\frac{\|g_j^1 - g_j^0\|_{C^{k,\alpha}(M)}}{d_j} \rightarrow 0.$$

Again by (76) in Lemma D.1, there exists a constant  $C_2$  independent of  $j$  such that

$$\begin{aligned} & \frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), 2r_0))} + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\Sigma_j^1 \setminus B^{g_j^1}(\text{Sing}(\Sigma_j^1), 2r_0))}}{d_j} \\ & \leq C_2 \frac{\|w_j\|_{L^2(\tilde{\Sigma}_j \setminus B^{\bar{g}_j}(\text{Sing}(\tilde{\Sigma}_j), r_0))}}{d_j} \rightarrow 0, \end{aligned}$$

as  $j \rightarrow \infty$ . By the pointwise estimates for  $\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}$  and  $\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}$  in (43), there exists  $\tilde{C}_2 > 0$  independent of  $j$  such that

$$\begin{aligned} & \frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\cup_{\mathbf{x} \in \text{Sing}(\Sigma_j^0)} A^{\bar{g}_j}(\mathbf{x}; C^{\mathcal{L}}\mathbf{r}_j^{(1)}, 2r_0))} + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\cup_{\mathbf{x} \in \text{Sing}(\Sigma_j^1)} A^{\bar{g}_j}(\mathbf{x}; C^{\mathcal{L}}\mathbf{r}_j^{(0)}, 2r_0))}}{d_j} \\ & \leq \tilde{C}_2 \cdot \frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\cup_{\mathbf{x} \in \text{Sing}(\Sigma_j^0)} A^{\bar{g}_j}(\mathbf{x}; 2r_0, \tilde{C}_0 r_0))} + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\cup_{\mathbf{x} \in \text{Sing}(\Sigma_j^1)} A^{\bar{g}_j}(\mathbf{x}; 2r_0, \tilde{C}_0 r_0))}}{d_j} \\ & \leq \tilde{C}_2 \cdot \frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), 2r_0))} + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\Sigma_j^1 \setminus B^{g_j^1}(\text{Sing}(\Sigma_j^1), 2r_0))}}{d_j} \rightarrow 0, \end{aligned}$$

as  $j \rightarrow \infty$ . Hence, we can conclude that

$$\frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\|\cdot\|_{g_j^0} \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), C^{\mathcal{L}}\mathbf{r}_j^{(1)}))} + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\|\cdot\|_{g_j^1} \setminus B^{g_j^1}(\text{Sing}(\Sigma_j^1), C^{\mathcal{L}}\mathbf{r}_j^{(0)}))}}{d_j} \rightarrow 0.$$

Moreover, since both  $\Sigma_j^0$  and  $\Sigma_j^1$  converge to  $\Sigma$  in  $\mathcal{L}^{k,\alpha}$ , by (42),

$$\mathbf{r}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}, \mathbf{r}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1} \rightarrow 0.$$

It follows from Corollary C.2 that for each  $\bar{\mathbf{x}}_j \in \text{Sing}(\bar{\Sigma}_j)$ , let  $\mathbf{x} \in \text{Sing}(\Sigma)$  be the unique element in  $\text{Sing}(\Sigma) \cap B^{g_j}(\bar{\mathbf{x}}_j, r_0)$  and we have, in the varifold sense,

$$\eta_{\bar{\mathbf{x}}_j^0, 2C\mathcal{L}\mathbf{r}_{\Sigma_1, g_1}^{\Sigma_0}}^{-1}(\Sigma_j^0) \rightarrow \mathbf{C}_{\mathbf{x}}(\Sigma), \quad \eta_{\bar{\mathbf{x}}_j^1, 2C\mathcal{L}\mathbf{r}_{\Sigma_0, g_0}^{\Sigma_1}}^{-1}(\Sigma_j^1) \rightarrow \mathbf{C}_{\mathbf{x}}(\Sigma).$$

Therefore, for sufficiently large  $j$ , there exists a constant  $C_3$  independent of  $j$  such that

$$\begin{aligned} & \frac{\|\Sigma_j^1\|_{g_j^1}(B^{g_j^0}(\text{Sing}(\Sigma_j^0), 2C\mathcal{L}\mathbf{r}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1})) + \|\Sigma_j^0\|_{g_j^0}(B^{g_j^1}(\text{Sing}(\Sigma_j^1), 2C\mathcal{L}\mathbf{r}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}))}{d_j} \\ & \leq (\#\text{Sing}(\Sigma))C_3 \frac{\binom{\Sigma_j^1}{\mathbf{r}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}}^n + \binom{\Sigma_j^0}{\mathbf{r}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}}^n}{d_j} \\ & \leq (\#\text{Sing}(\Sigma))C_3 C_0^n \frac{\|\mathbf{G}_{\Sigma_j^0, g_j^0}^{\Sigma_j^1}\|_{L^2(\Sigma_j^0 \setminus B^{g_j^0}(\text{Sing}(\Sigma_j^0), 2r_0))}^n + \|\mathbf{G}_{\Sigma_j^1, g_j^1}^{\Sigma_j^0}\|_{L^2(\Sigma_j^1 \setminus B^{g_j^1}(\text{Sing}(\Sigma_j^1), 2r_0))}^n}{d_j} \rightarrow 0 \end{aligned}$$

where this last conclusion follows from 42.

These three facts, that we just collected, together contradict the definition of  $d_j$ .  $\square$

By Claim 2, if we divide the PDE (45) by  $d_j$ , up to a subsequence of  $\{\frac{\mathbf{T}_{\Sigma_j, g_j}^{\Sigma_j} w_j}{d_j}\}_j$ , we obtain a non-zero solution  $\hat{u}_\infty$  to

$$-L_{\Sigma, g} \hat{u}_\infty + \frac{n}{2} \nabla_{\Sigma, g}^\perp(\hat{f}_\infty) = 0.$$

By Corollary D.5, for each  $\bar{\mathbf{x}}_j \in \text{Sing}(\bar{\Sigma}_j)$ ,  $\frac{\mathbf{x}_j^0 - \mathbf{x}_j^1}{d_j}$  converges, which induces a translation-like section  $\phi \in X_{TS} = X_{TS}(\Sigma)$ . Moreover, by the last inequality in Corollary D.5, for every  $p \in \text{Sing}(\Sigma)$ ,

$$\mathcal{AR}_p(\hat{u}_\infty - \phi) \geq \gamma > \gamma_j^-(\mathbf{C}_p),$$

so  $\hat{u}_\infty \in \hat{W}_\gamma^{0,2}(\Sigma; \mathbf{V})$  is a tame section.  $\square$

**4.5. The local Sard-Smale theorem.** Recalling the notion of canonical neighborhoods  $\mathcal{L}^{k,\alpha}$  presented in Definition 4.18, let us start with the first preparatory statement:

**Lemma 4.26** (Compactness of tame Jacobi fields). *Let  $(M, g)$  be a closed Riemannian manifold of dimension  $N \geq 3$ , let  $\Sigma$  be a connected,  $n$ -dimensional **MSI** therein, and let  $\delta_0 = \delta_0(g, \Sigma, \Lambda) > 0$  be as in Lemma 4.20. Assume the existence of a sequence  $\{(g_j, \Sigma_j)\}_{j \geq 1}$  of pairs in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_0)$  satisfying the following two conditions:*

- (i)  $(g_j, \Sigma_j) \rightarrow (g_\infty, \Sigma_\infty) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta_0)$  in  $\mathcal{M}_n^{k,\alpha}(M)$  as  $j \rightarrow \infty$ ;
- (ii) for every  $j \geq 1$  there exists a tame Jacobi field  $u_j \in \widehat{\text{Ker}}(L_{\Sigma_j, g_j})$  on  $\Sigma_j$  such that  $\|u_j\|_{L^2(\Sigma_j)} = 1$ .

Then, after passing to a subsequence,  $v_j := \mathbf{T}_{\Sigma_\infty, g_\infty}^{\Sigma_j}(u_j)$  converges to some tame Jacobi field  $u_\infty$  in  $C_{loc}^2(\Sigma_\infty; \mathbf{V})$ ; furthermore for any  $\varepsilon > 0$  there exist  $s = s(\varepsilon) \in (0, 1)$  and  $j_0 = j_0(\varepsilon)$  such that there holds

$$\sum_{p_j \in \text{Sing}(\Sigma_j)} \int_{B^{g_j}(p_j, s)} |u_j|^2 d\|\Sigma_j\| \leq \varepsilon, \quad \text{for all } j \geq j_0;$$

as a consequence, in particular  $\|u_\infty\|_{L^2(\Sigma_\infty)} = 1$ .

*Proof.* By classical (interior) elliptic estimates, the sequence  $\{v_j\}$  subconverges to some Jacobi field  $u_\infty$  on  $\Sigma_\infty$  in  $C_{loc}^2$ . The effort here is to prove the non-concentration claim, and to show that  $u_\infty$  is tame.

We first fix  $\sigma \in (0, 1)$  such that

$$\min_{p \in \text{Sing}(\Sigma_\infty)} \text{dist}_{\mathbb{R}}(1 - \sigma, \Gamma(\mathbf{C}_p \Sigma_\infty)) = \sigma.$$

For every  $p \in \text{Sing}(\Sigma_\infty)$ , there exists  $r_0 = r_0(\Sigma_\infty, g_\infty) > 0$  such that the assumptions in Corollary B.6 are satisfied for  $(\Sigma_\infty, r_0^{-2}g_\infty)$  near  $p$ . By the way we have defined the notion of convergence in  $\mathcal{L}^{k,\alpha}$  - if  $p_j \in \text{Sing}(\Sigma_j)$  are approaching  $p_\infty$ , thus with  $\theta_{\Sigma_j}(p_j) = \theta_{\Sigma_\infty}(p)$  - we know that for  $j$  large enough, the assumptions in Corollary B.6 are also satisfied for  $(\Sigma_j, r_0^{-2}g_j)$  near  $p_j$ . Therefore, Corollary B.6 provides an  $\varepsilon_1(\Lambda, \sigma) \in (0, 1)$  and a uniform  $C_{1-\sigma}^2$ -estimate on  $u_j$  near  $p_j$  of the form

$$(48) \quad \|\phi_j\|_{L^\infty(\Sigma_j)} + \|u_j - \phi_j\|_{C_{1-\sigma}^1(B^{g_j}(p_j, r_0) \cap \Sigma_j)} \leq C(\Lambda, \sigma) \|u_j\|_{L^2(A^{g_j}(p_j, \varepsilon'' r_0, 2r_0) \cap \Sigma_j)}$$

for some translation-like function  $\phi_j \in X_{TS} = X_{TS}(\Sigma_j)$ . In particular, this implies for every  $s \in (0, r_0)$

$$\|u_j\|_{L^2(B^{g_j}(p_j, s))} \leq C(\Lambda, \sigma) s^{n/2}$$

and from this inequality it is straightforward to derive the  $L^2$ -nonconcentration claim in the statement: specifically we get at once that

$$(49) \quad \int_{\Sigma_\infty} |u_\infty|^2 d\|\Sigma_\infty\| \geq 1.$$

Moreover, by passing to the limit as  $j \rightarrow \infty$  for the transferred sections, (48) also holds with  $u_\infty$  in place of  $u_j$  and some translation-like function  $\phi_\infty \in X_{TS}(\Sigma_\infty)$  on  $\Sigma_\infty$  in place of  $\phi_j$ . Then, by the choice of  $\sigma$  and item (iii) of Lemma 4.14,  $u_\infty$  is also a tame Jacobi field; as a result it will then satisfy the same non-concentration property near  $\text{Sing}(\Sigma_\infty)$  and so, at this stage, the  $C_{loc}^2$ -convergence implies that in fact equality must hold in (49), which completes the proof.  $\square$

**Corollary 4.27.** *(Setting as above.) There exists  $\kappa_1(g, \Sigma, \Lambda) \in (0, \delta_0)$  such that for every  $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_1)$ , we have*

$$\dim \widehat{\text{Ker}}(L_{\Sigma', g'}) \leq \dim \widehat{\text{Ker}}(L_{\Sigma, g}).$$

*Proof.* Let us argue by contradiction: if the assertion was false, then for any  $j \geq 1$  large enough one could find  $(g_j, \Sigma_j) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, 1/j)$  such that  $\dim \widehat{\text{Ker}}(L_{\Sigma_j, g_j}) > \dim \widehat{\text{Ker}}(L_{\Sigma, g})$ . In particular, such a sequence of pairs will converge to  $(g, \Sigma)$  in  $\mathcal{M}_n^{k,\alpha}$ . One can then take for each such  $j$  an  $L^2(\Sigma_j; \mathbf{V}_j)$ -orthonormal family, of cardinality exactly equal to  $d := \widehat{\text{Ker}}(L_{\Sigma, g}) + 1$ , in  $\widehat{\text{Ker}}(L_{\Sigma_j, g_j})$ , say  $\{u_j^{(1)}, \dots, u_j^{(d)}\}$ ; we then know, by appealing to Lemma 4.26 that one can extract, after passing to a subsequence, limit elements  $u_\infty^{(1)}, \dots, u_\infty^{(d)}$  in  $L^2(\Sigma; \mathbf{V})$  such that for each  $i = 1, \dots, d$  there holds  $\mathbf{T}_{\Sigma, g}^{\Sigma_j}(u_j^{(i)}) \rightarrow u_\infty^{(i)}$  in  $C_{loc}^2(\Sigma; \mathbf{V})$  as  $j \rightarrow \infty$ . But, in fact, it follows at once from the non-concentration claim proven above (also part of Lemma 4.26) that the family in question consists of pairwise orthogonal sections also having unit norm in  $L^2(\Sigma; \mathbf{V})$ ; hence  $\widehat{\text{Ker}}(L_{\Sigma, g})$  would have dimension no less than  $d$ , a contradiction.  $\square$

Let then

$$\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta) := \{(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta) : \dim \widehat{\text{Ker}}(L_{\Sigma',g'}) = \dim \widehat{\text{Ker}}(L_{\Sigma,g})\}.$$

Lemma 4.26 and Corollary 4.27 imply that  $\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  is closed in  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  for  $\delta \leq \kappa_1$ .

In the setting above, we further set for notational convenience

$$I := \dim \widehat{\text{Ker}}(L_{\Sigma,g}), \quad J := \dim \widehat{\text{Coker}}(L_{\Sigma,g}).$$

Note that by Theorem 3.2 there *always* holds  $J \geq I$ .

For  $(\bar{g}, \bar{\Sigma}) \in \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_1)$ , let

$$\pi_{\bar{\Sigma}, \bar{g}}^{L^2} : L^2(\bar{\Sigma}; \mathbf{V}) \rightarrow \widehat{\text{Ker}}(L_{\bar{\Sigma}, \bar{g}})$$

be the  $L^2$ -orthogonal projection to the finite dimensional (hence: closed) subspace. The following Lemma guarantees that we can parametrize slices of  $\mathcal{L}_{top}^{k,\alpha}$  by compact subset of  $\widehat{\text{Ker}}$ .

**Lemma 4.28.** *Let  $\kappa_1$  be as in Corollary 4.27. Then there exist constants  $\kappa_2(g, \Sigma, \Lambda) \in (0, \kappa_1)$ ,  $r_0(g, \Sigma, \Lambda) > 0$  and a linear subspace  $\mathcal{F} \subset C_c^{k,\alpha}(M \setminus B^g(\text{Sing}(\Sigma), 10r_0))$  of dimension  $J$ , also depending only on  $\Sigma, g, \Lambda$  with the following property.*

- (i) *For every  $(\bar{g}, \bar{\Sigma}) \in \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_2)$  and every non-zero  $f \in \mathcal{F}$ , there is no tame solution  $v \in \hat{W}_{\tau}^{k,2}(\bar{\Sigma}; \mathbf{V})$  to  $L_{\bar{\Sigma}, \bar{g}} v = \nabla_{\bar{\Sigma}, \bar{g}}^{\perp}(f)$ .*
- (ii) *Denote for simplicity  $\mathcal{F} \cdot \bar{g} := \{(1+f)\bar{g} : f \in \mathcal{F}\}$ . For every  $(\bar{g}, \bar{\Sigma}) \in \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_2)$ , the map*

$$\begin{aligned} \mathbf{P}_{\bar{g}, \bar{\Sigma}} : \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_2) \cap \Pi^{-1}(\mathcal{F} \cdot \bar{g}) &\rightarrow (\widehat{\text{Ker}} L_{\bar{\Sigma}, \bar{g}}), \\ (g', \Sigma') &\mapsto \pi_{\bar{\Sigma}, \bar{g}}^{L^2}(\mathbf{G}_{\bar{\Sigma}, \bar{g}}^{\Sigma'} \cdot \zeta_{\bar{\Sigma}, \bar{g}, r_0}), \end{aligned}$$

*is bi-Lipschitz onto its image, with bi-Lipschitz constant  $\leq C(g, \Sigma, \Lambda)$ , and thus injective. (Here we employ the definition of  $\zeta_{\bar{\Sigma}, \bar{g}, r_0}$  given in (13).)*

*Remark 4.29.* Note that, for what pertains to the second part of the statement above, it is understood that the semi-metric we use on  $\mathcal{L}_{top}^{k,\alpha}$  is  $\mathbf{D}^{\mathcal{L}}$ . Furthermore, it is convenient to assume  $\mathcal{F}$  endowed with the  $L^2$  metric determined by the ambient Riemannian manifold  $(M, g)$ , which is anyways equivalent to any other norm on the same  $J$ -dimensional vector space.

*Proof.* To prove (i), we start with specifying the choice of  $r_0$  and  $\mathcal{F}$ . Take  $\varphi_1, \dots, \varphi_J \in W_{\tau-2}^{k-2,2}(\Sigma; \mathbf{V})$  be a basis (after projecting to the quotient space) of  $\widehat{\text{Coker}}(L_{\Sigma,g})$ ; without loss of generality (by a cutoff) we can assume all such sections to be supported away from the singular set of  $\Sigma$ .

Set then

$$r_0(g, \Sigma, \zeta) \in (0, \min_{1 \leq i \leq J} \{\text{inrad}(g, \Sigma), \text{dist}_g(\text{spt}(\varphi_i), \text{Sing}(\Sigma))\}/40)$$

chosen small enough that the linear map

$$\widehat{\text{Ker}}(L_{\Sigma,g}) \rightarrow \widehat{\text{Ker}}(L_{\Sigma,g}), \quad v \mapsto \pi_{\Sigma,g}^{L^2}(\zeta_{\Sigma,g,r_0} \cdot v)$$

is an isomorphism, where  $\zeta_{\Sigma,g,r_0}$  is defined in (13). Then, by a standard regularization argument (via convolutions) we choose  $f_i \in C_c^{k,\alpha}(M \setminus B^g(\text{Sing}(\Sigma), 10r_0))$  such that after projecting to the quotient space  $\text{span}_{\mathbb{R}}\langle \nabla_{\Sigma,g}^{\perp} f_i : 1 \leq i \leq J \rangle = \text{span}_{\mathbb{R}}\langle \varphi_i : 1 \leq i \leq J \rangle$  along  $\Sigma$ , and thus define

$$\mathcal{F} := \text{span}_{\mathbb{R}}\langle f_i : 1 \leq i \leq J \rangle.$$

To see that for every non-zero  $f \in \mathcal{F}$ ,  $L_{\bar{\Sigma}, \bar{g}} v = \nabla_{\bar{\Sigma}, \bar{g}}^\perp(f)$  has no solution  $v \in \hat{W}_\tau^{k,2}(\bar{\Sigma}; \bar{\mathbf{V}})$  for every  $(\bar{g}, \bar{\Sigma}) \in \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  and  $\delta$  sufficiently small, suppose, for a contradiction, that there exist  $(g_j, \Sigma_j) \rightarrow (g, \Sigma)$  in  $\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_1)$ ,  $f_j \in \mathcal{F}$  non-zero and  $u_j \in \hat{W}_\tau^{k,2}(\Sigma_j; \mathbf{V}_j)$ , with unit  $L^2(\|\Sigma_j\|)$ -norm, such that

$$L_{\Sigma_j, g_j}(u_j) = \nabla_{\Sigma_j, g_j}^\perp f_j, \quad \pi_{\Sigma_j, g_j}^{L^2}(u_j) = 0$$

Note that since  $f_j = 0$  in  $B^g(\text{Sing}(\Sigma), 10r_0)$ ,  $u_j$  are in fact Jacobi fields in  $\Sigma_j \cap B^g(\text{Sing}(\Sigma_j), 10r_0)$ ; such Jacobi fields are tame by assumption. Now, after passing to a subsequence (which we do not rename), we may assume  $f_j \rightarrow f \in \mathcal{F}$  as  $j \rightarrow \infty$  and, on the other hand, since  $v_j := \mathbf{T}_{\Sigma_j, g}^{\Sigma_j}$  are tame Jacobi fields near  $\text{Sing}(\Sigma_j)$ , arguing as in the proof of the Lemma 4.26 (ultimately appealing to Corollary B.6), one shows that  $v_j$  has uniform growth upper bound near  $\text{Sing}(\Sigma_j)$ , hence subconverges to some  $L^2$ -unit  $v_\infty \in W_{loc}^{k,2}(\Sigma; \mathbf{V})$  (and in  $C_{loc}^2$ ) tame section near  $\text{Sing}(\Sigma)$  that still satisfies

$$L_{\Sigma, g} v_\infty = \nabla_{\Sigma, g}^\perp f_\infty, \quad \pi_{\Sigma, g}^{L^2}(v_\infty) = 0.$$

The aforementioned pointwise bound (which implies a bound on the asymptotic rate at each singular point, exactly as in the aforementioned previous proof) then implies, by item (iii) of Lemma 4.14, that  $v_\infty \in \hat{W}_\tau^{k,2}(\Sigma; \mathbf{V})$ . But by our choice of  $\mathcal{F}$ , this is impossible.

To prove (ii), we also argue by contradiction. Suppose  $\mathbf{P}_{\bar{g}_j, \bar{\Sigma}_j}$  is not uniformly bi-Lipschitz for  $(\bar{g}_j, \bar{\Sigma}_j) \in \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \kappa_1)$  with  $(\bar{g}_j, \bar{\Sigma}_j) \rightarrow (g, \Sigma)$  in  $\mathcal{L}^{k,\alpha}$ . Then there exists a sequence of pairs  $(g_j^0, \Sigma_j^0), (g_j^1, \Sigma_j^1) \rightarrow (g, \Sigma)$  in  $\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta)$  such that

- $g_j^i = (1 + f_j^i) \bar{g}_j$ ,  $i = 0, 1$ ,  $f_j^i \in \mathcal{F}$ ;
- set  $u_j^{(i)} := \mathbf{G}_{\bar{\Sigma}_j, \bar{g}_j}^{\Sigma_j^i}$ ,  $v_j^{(i)} = \mathbf{T}_{\Sigma_j, g}^{\Sigma_j^i}(u_j^{(i)})$ ;  $\zeta_j := \zeta_{\bar{\Sigma}_j, \bar{g}_j, r_0}$ ,  $d_j := \mathbf{D}^{\mathcal{L}}((g_j^0, \Sigma_j^0), (g_j^1, \Sigma_j^1))$  one of the following holds:

$$(50) \quad \text{either } \|\pi_{\bar{\Sigma}_j, \bar{g}_j}^{L^2}((u_j^{(1)} - u_j^{(0)}) \zeta_j)\|_{L^2(\bar{\Sigma}_j)} \leq \frac{d_j}{j},$$

$$(51) \quad \text{or } \|\pi_{\bar{\Sigma}_j, \bar{g}_j}^{L^2}((u_j^{(1)} - u_j^{(0)}) \zeta_j)\|_{L^2(\bar{\Sigma}_j)} \geq j \cdot d_j.$$

By Lemma 4.25 the normalized difference  $(f_j^{(1)} - f_j^{(0)})/d_j$  subconverges in  $C^2(M)$  to some  $\hat{f}_\infty \in \mathcal{F}$  and  $(v_j^{(1)} - v_j^{(0)})/d_j$  subconverges to some non-zero, tame section  $\hat{u}_\infty$  in  $C_{loc}^2$ , solving

$$(52) \quad L_{\Sigma, g} \hat{u}_\infty = \frac{n}{2} \nabla_{\Sigma, g}^\perp(\hat{f}_\infty).$$

Now, since  $\zeta_j$  is supported away from  $\text{Sing}(\Sigma)$ , by the convergence above it is straightforward to check that  $\zeta_j(v_j^{(1)} - v_j^{(0)})/d_j$  subconverges in  $L^2$  to  $\hat{u}_\infty \cdot \zeta_{\Sigma, g, r_0}$  and thus (51) cannot possibly hold for  $j$  large enough.

We then need to rule out the other possibility, namely (50). Towards that goal, by (52) and the result of item (i) above, we derive  $\hat{f}_\infty = 0$ , which means  $0 \neq \hat{u}_\infty \in \widehat{\text{Ker}}(L_{\Sigma, g})$ . Then since  $\zeta_j(v_j^{(1)} - v_j^{(0)})/d_j$  subconverges in  $L^2(\Sigma)$  to  $\hat{u}_\infty \cdot \zeta_{\Sigma, g, r_0}$ , passing to the limit as  $j \rightarrow \infty$  in (50) we would conclude  $\pi_{\Sigma, g}^{L^2}(\hat{u}_\infty \cdot \zeta_{\Sigma, g, r_0}) = 0$ . However, by the choice of  $r_0$  in item (i), this would force  $\hat{u}_\infty$  to be the trivial section, a contradiction.  $\square$

*Remark 4.30.* (Topological interpretation of Lemma 4.28) It may be enlightening to discuss, a posteriori, the topological content of part (ii) of the preceding statement. To do so we shall focus on the simplest possible case, namely when  $I = 0, J > 0$  and  $g = \bar{g}$ ,  $\Sigma = \bar{\Sigma}$ .

In that case, the statement in question implies that the domain of the map  $\mathbf{P}_{\bar{g}, \bar{\Sigma}}$ , which is  $\mathcal{L}_{top}^{k, \alpha}(g, \Sigma; \Lambda, \kappa_2) \cap \Pi^{-1}(\mathcal{F} \cdot \bar{g})$ , must consist of a single point. This corresponds to saying that (locally around  $(g, \Sigma)$ ) the two sets  $\Pi^{-1}(\mathcal{F} \cdot \bar{g})$  and  $\mathcal{L}_{top}^{k, \alpha}(g, \Sigma; \Lambda, \kappa_2)$  (that is necessarily the same as  $\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \kappa_2)$  when  $I = 0$ ) shall meet transversely.

Having established this, we can prove Theorem 4.21 by unwrapping the proof of the Sard-Smale Theorem, as we are now about to explain.

*Proof of Theorem 4.21.* Openness follows directly from Lemma 4.20: indeed, the image - via the continuous map  $\Pi$  - of the compact set  $\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$  is compact with respect to  $C^k$  convergence and therefore closed (in  $\mathcal{G}^{k, \alpha}(M)$ ).

That being said, we shall prove the denseness inductively on  $I := \dim \widehat{\text{Ker}}(L_{\Sigma, g}) \geq 0$ . By induction, we can assume that the denseness has been established for every pair  $(g, \Sigma)$  with  $\dim \widehat{\text{Ker}}(L_{\Sigma, g}) \leq I - 1$  (Note that when  $I = 0$ , the base case of induction, we are actually making no assumption.) The goal now is to prove it for  $I$ .

**Claim.** For each  $\bar{g} \in \mathcal{G}^{k, \alpha}(M) \setminus \mathcal{G}^{k, \alpha}(g, \Sigma; \Lambda, \delta) = \Pi(\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta))$ , there exists a sequence of metrics  $g_j \in \mathcal{F} \cdot \bar{g} \setminus \Pi(\mathcal{L}_{top}^{k, \alpha}(g, \Sigma; \Lambda, \delta))$  such that  $g_j \rightarrow \bar{g}$  in  $\mathcal{G}^{k, \alpha}(M)$  when  $j \rightarrow \infty$ .

We first finish the proof of denseness assuming this claim: it suffices to show that each  $g_j$  in the claim can be approximated by metrics in  $\mathcal{G}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ . Since  $\mathcal{L}_{top}^{k, \alpha}(g, \Sigma; \Lambda, \delta) = \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$  when  $I = 0$ , we only need to handle the case  $I \geq 1$ .

Let  $j \geq 1$  be fixed. Suppose, without loss of generality, that  $g_j \notin \mathcal{G}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ . Hence  $\Pi^{-1}(g_j) \cap \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta) \neq \emptyset$ . By definition of  $\mathcal{L}_{top}^{k, \alpha}$  and Corollary 4.27 we have, for every  $(g_j, \Sigma') \in \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ ,  $\dim \widehat{\text{Ker}}(L_{\Sigma', g_j}) \leq I - 1$ . Hence by the inductive assumption, there exists  $\delta_{g_j, \Sigma'} = \delta(g_j, \Sigma', \Lambda) > 0$  such that  $\mathcal{G}(g_j, \Sigma'; \Lambda, \delta_{g_j, \Sigma'})$  is open and dense in  $\mathcal{G}^{k, \alpha}(M)$ . Since  $\mathcal{L}^{k, \alpha}(g_j, \Sigma'; \Lambda, \delta_{g_j, \Sigma'})$  contains an open neighborhood of  $(g_j, \Sigma')$  in  $\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$  and, by Lemma 4.20,  $\Pi^{-1}(g_j) \cap \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$  is compact, we thus know that there exists a finite sequence of pairs  $(g_j, \Sigma^{(1)}), (g_j, \Sigma^{(2)}), \dots, (g_j, \Sigma^{(i)}) \in \Pi^{-1}(g_j) \cap \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ ,  $1 \leq i \leq i_0$ , such that

$$(53) \quad \bigcup_{i=1}^{i_0} \mathcal{L}^{k, \alpha}(g_j, \Sigma^{(i)}; \Lambda, \delta_{g_j, \Sigma^{(i)}}) \supset \Pi^{-1}(g_j) \cap \mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta).$$

Since  $\mathcal{G}_j := \bigcap_{i=1}^{i_0} \mathcal{G}(g_j, \Sigma^{(i)}; \Lambda, \delta_{g_j, \Sigma^{(i)}})$  is still open and dense in  $\mathcal{G}^{k, \alpha}(M)$ , there exists a sequence  $\{g'_m\}_{m \geq 1} \subset \mathcal{G}_j$  such that  $g'_m \rightarrow g_j$  in  $\mathcal{G}^{k, \alpha}$  as  $m \rightarrow \infty$ . Again by compactness of  $\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ , when  $m \geq m_0$  is large enough,

$$\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta) \cap \Pi^{-1}(g'_m) \subset \bigcup_{i=1}^{i_0} \mathcal{L}^{k, \alpha}(g_j, \Sigma^{(i)}; \Lambda, \delta_{g_j, \Sigma^{(i)}}).$$

By our choice of  $g'_m$  this is only possible if  $\mathcal{L}^{k, \alpha}(g, \Sigma; \Lambda, \delta) \cap \Pi^{-1}(g'_m) = \emptyset$ , and so  $g'_m \in \mathcal{G}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$  for  $m$  large enough. This concludes the proof of denseness assuming the Claim.

*Proof of Claim.* Suppose, without loss of generality, that  $(\bar{g}, \bar{\Sigma}) \in \mathcal{L}_{top}^{k, \alpha}(g, \Sigma; \Lambda, \delta)$ , otherwise just taking  $g_j \equiv \bar{g}$  proves the Claim.

Recall by Lemma 4.28,  $\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta) \cap \Pi^{-1}(\mathcal{F} \cdot \bar{g})$  is bi-Lipschitz embedded as a compact subset of finite dimensional Euclidean space  $\widehat{\text{Ker}}(L_{\Sigma, \bar{g}})$ ; we let, for the sake of notational convenience,  $\mathcal{Z} := \mathbf{P}_{\bar{g}, \Sigma}(\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta) \cap \Pi^{-1}(\mathcal{F} \cdot \bar{g}))$  be its image in  $\widehat{\text{Ker}}(L_{\Sigma, \bar{g}})$ . Consider the composite map,

$$\Pi : \mathcal{Z} \cong \mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta) \cap \Pi^{-1}(\mathcal{F} \cdot \bar{g}) \xrightarrow{\Pi} \mathcal{F} \cdot \bar{g} \cong \mathcal{F},$$

that is a Lipschitz map between (compact subsets of) vector spaces.

Since  $\widehat{\text{index}}_{\tau}(L_{\Sigma, g}) < 0$ , we know that  $I < J$  and therefore  $\Pi(\mathcal{Z})$  is an  $I$ -dimensional compact subset in  $\mathcal{F} \cdot \bar{g} \cong \mathcal{F}$  (which, by definition, has dimension  $J$ ). Hence it has dense complement and we can find  $g_j \in \mathcal{F} \cdot \bar{g} \setminus \Pi(\mathcal{L}_{top}^{k,\alpha}(g, \Sigma; \Lambda, \delta))$  to approximate  $\bar{g}$ , thereby finishing the proof of the claim.  $\square$

**4.6. Proof of Theorem 4.1.** We will proceed in two steps: first we prove the theorem for metrics of finite, in fact  $C^{k,\alpha}$ , regularity (which is the setting we have so far considered in this section) and then we shall discuss the simple argument that allows to derive the smooth version of the result (namely: Theorem 4.1 in the specific form we have stated). For that purpose we will employ the following lemma about nested metric spaces.

**Lemma 4.31.** *Let  $\{(X_k, d_k)\}_{k \geq k_0}$  be a sequence of complete metric spaces, with associated inclusions  $\iota_k : X_{k+1} \hookrightarrow X_k$  such that*

$$(54) \quad d_k(\iota_k(x), \iota_k(y)) \leq d_{k+1}(x, y), \quad \forall k \geq k_0.$$

*We identify, using such maps, each  $X_k$  as a subset of  $X_0$ , and define*

$$X_{\infty} := \bigcap_{k \geq k_0} X_k, \quad d_{\infty}(x, y) := \sum_{k \geq k_0} 2^{-k} \cdot \frac{d_k(x, y)}{1 + d_k(x, y)}, \quad \forall x, y \in X_{\infty}.$$

*Suppose that  $X_{\infty}$  is dense in  $(X_k, d_k)$  for all  $k \geq k_0$  and, in addition, that  $G \subset (X_{k_0}, d_{k_0})$  is an open (respectively  $G_{\delta}$ ) subset such that for every  $k \geq k_0$ ,  $G \cap X_k$  is dense in  $(X_k, d_k)$ . Then  $G \cap X_{\infty}$  is an open (resp.  $G_{\delta}$ ) dense subset in  $(X_{\infty}, d_{\infty})$ .*

*Proof.* The case when  $G$  is a  $G_{\delta}$  subset follows from the case when  $G$  is open and the Baire Category Theorem. So we shall prove the statement assuming  $G$  is open.

First notice that for every  $k \geq k_0$ , since  $(X_k, d_k) \hookrightarrow (X_{k_0}, d_{k_0})$  is continuous (being a finite composition of continuous maps),  $G \cap X_k$  is open in  $(X_k, d_k)$ ; the openness of  $G \cap X_{\infty}$  in  $(X_{\infty}, d_{\infty})$  analogously follows from the continuity of the inclusion  $(X_{\infty}, d_{\infty}) \rightarrow (X_{k_0}, d_{k_0})$ , which can be checked at once by the way we have defined the distance  $d_{\infty}$ .

To prove the denseness of  $G \cap X_{\infty}$  in  $(X_{\infty}, d_{\infty})$ , let  $x \in X_{\infty}$  be an arbitrary point. By the denseness of  $G \cap X_k$  in  $(X_k, d_k)$ , for each  $k \in \mathbb{N}$  there exists  $x_k \in G \cap X_k$  such that  $d_k(x_k, x) \leq 1/k$ ; on the other hand, by the denseness of  $X_{\infty}$  in  $(X_k, d_k)$ , there exists a sequence  $\{x_k^j\}_{j \geq 0}$  in  $X_{\infty}$  such that

$$\lim_{j \rightarrow \infty} d_k(x_k^j, x_k) = 0.$$

Hence, by the openness of  $G \cap X_k$  in  $(X_k, d_k)$ , for every  $k \geq k_0$  we can find an index  $j_k$  large enough that

$$\bar{x}_k := x_k^{j_k} \in (G \cap X_k) \cap X_{\infty} = G \cap X_{\infty}, \quad d_k(\bar{x}_k, x) \leq 2/k.$$

Then, recalling the definition of  $d_\infty$ , we find,

$$d_\infty(x, \bar{x}_k) \leq \sum_{m=k_0}^k 2^{-m} \cdot \frac{2/k}{1+2/k} + \sum_{m=k+1}^{\infty} 2^{-m} \leq 4/k + 2^{-k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence  $x$  is in the closure of  $G \cap X_\infty$  in  $(X_\infty, d_\infty)$ .  $\square$

*Proof of Theorem 4.1.*

Step 1: metrics of finite regularity  $C^{k,\alpha}$

For every  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$  with  $\widehat{\text{index}}_\tau(L_{\Sigma,g}) < 0$  and  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)} \{\theta_{|\Sigma|}(p)\}, 1\}$ , by Theorem 3.2, we know that for some  $p \in \text{Sing}(\Sigma)$ ,

$$I(\mathbf{C}_p) > 0.$$

Note that there exists  $\delta(\mathbf{C}_p) > 0$  such that for any (regular minimal) cone  $\mathbf{C} \in \mathcal{C}_{N,n}$  satisfying  $\mathbf{F}(\mathbf{C}_p \cap \mathbb{S}^{N-1}, \mathbf{C} \cap \mathbb{S}^{N-1}) \leq \delta(\mathbf{C}_p)$ , we also have  $I(\mathbf{C}) > 0$ . Hence, by Definition 4.18 and the compactness Lemma 4.20, we can choose  $\kappa(g, \Sigma, \Lambda) \in (0, \kappa_0(g, \Sigma, \Lambda))$ , where  $\kappa_0$  is the threshold given by Theorem 4.21, with the following property:

*For every  $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa(g, \Sigma, \Lambda))$ , there exists  $p' \in \text{Sing}(\Sigma')$  such that*

$$\mathbf{F}(\mathbf{C}_p \cap \mathbb{S}^{N-1}, \mathbf{C}_{p'} \cap \mathbb{S}^{N-1}) \leq \delta(\mathbf{C}_p),$$

and thus,

$$I(\mathbf{C}_{p'}) > 0.$$

We can then extend the definition of the function  $\kappa$  by letting  $2\kappa(g, \Sigma, \Lambda) = \kappa_0(g, \Sigma, \Lambda)$  for all  $(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M)$  with  $\widehat{\text{index}}_\tau(L_{\Sigma,g}) \geq 0$  and  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)} \{\theta_{|\Sigma|}(p)\}, 1\}$ . In particular,  $\widehat{\text{index}}_\tau(L_{\Sigma',g'}) < 0$  holds for every  $(g', \Sigma') \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa(g, \Sigma, \Lambda))$ .

By Theorem 4.23, for such a function  $\kappa : \mathcal{M}_n^{k,\alpha}(M) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , we can obtain a countable cover consisting of canonical neighborhoods:

$$\mathcal{M}_n^{k,\alpha}(M) = \bigcup_{j \geq 1} \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j),$$

where  $\kappa_j := \kappa(g_j, \Sigma_j; \Lambda_j)$ .

For every integer  $j \geq 1$ , we let

$$\mathcal{L}_-^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j) := \{(g, \Sigma) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j) : \widehat{\text{index}}_\tau(L_{\Sigma,g}) < 0\}.$$

We claim that, for any fixed  $j \geq 1$ , the set  $\mathcal{L}_-^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$  can also be covered by countably many  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa(g, \Sigma, \Lambda))$  with  $\widehat{\text{index}}_\tau(L_{\Sigma,g}) < 0$ . Indeed, by the compactness Lemma 4.20 again (now considering a countable exhaustion of  $\mathcal{L}_-^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$  by closed - hence compact - sets, defined e. g. as  $1/i$ -sublevel sets of the distance function from the complement of such  $\mathcal{L}_-^{k,\alpha}$  in  $\mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$ ) it suffices to show that for every  $(g, \Sigma) \in \mathcal{L}_-^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$ , there exists a  $\Theta(\Lambda) > \max\{\sup_{p \in \text{Sing}(\Sigma)} \{\theta_{|\Sigma|}(p)\}, 1\}$  such that  $\mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda, \kappa(g, \Sigma, \Lambda))$  contains an open neighborhood of  $(g, \Sigma)$  in  $\mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$ .

Suppose for the sake of contradiction that there exists  $(g, \Sigma) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$  and a sequence  $\{(g_\ell, \Sigma_\ell)\} \subset \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j) \setminus \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda_j, \kappa(g, \Sigma, \Lambda_j))$ , so that

$$(g_\ell, \Sigma_\ell) \rightarrow (g, \Sigma) \text{ in } \mathcal{M}_n^{k,\alpha}(M).$$

(Note that here we have chosen  $\Lambda = \Lambda_j$ , which is legitimate by virtue of the very definition of canonical neighborhood.) For large enough  $\ell$ , based on the definition of  $\mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$  and the notion of convergence in  $\mathcal{M}_n^{k,\alpha}$ , we have

- $g_\ell$  is a  $C^{k,\alpha}$  metric on  $M$  with  $\|g_\ell\|_{C^{k,\alpha}} \leq \Lambda_j$  and  $\|g - g_\ell\|_{C^k} \leq \kappa(g, \Sigma, \Lambda_j)$ ;
- $\Sigma_\ell$  is a **MSI** in  $(M, g_\ell)$  satisfying

$$\mathbf{F}(|\Sigma_\ell|_{g_\ell}, |\Sigma|_g) \leq \kappa(g, \Sigma, \Lambda_j), \quad \mathbf{r}_{\Sigma_\ell, g_\ell} \geq \Lambda^{-1} \rho_{\Sigma_\ell, g_\ell}.$$

We further note that (since  $(g_\ell, \Sigma_\ell) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$ ) there is an ordered bijection between the points of  $\text{Sing}(\Sigma_\ell)$  and  $\text{Sing}(\Sigma_j)$ , and the densities of the corresponding cones are equal; also, since  $(g, \Sigma) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$  there is also a bijection between the points of  $\text{Sing}(\Sigma)$  and  $\text{Sing}(\Sigma_j)$ , and the densities of the corresponding cones are equal. Putting all information together, this implies that eventually  $(g_\ell, \Sigma_\ell) \in \mathcal{L}^{k,\alpha}(g, \Sigma; \Lambda_j, \kappa(g, \Sigma, \Lambda_j))$ , a contradiction.

Therefore, by now repeating the argument and construction above as one varies  $j \geq 1$ , we get that the set

$$\{(g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M) : \widehat{\text{index}}_\tau(L_{\Sigma, g}) < 0\} = \bigcup_{j \geq 1} \mathcal{L}_-^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$$

can be covered by a countable union of canonical neighborhoods  $\{\mathcal{L}^{k,\alpha}(g'_\ell, \Sigma'_\ell; \Lambda'_\ell, \kappa'_\ell)\}_{\ell=1}^\infty$  where for each  $\ell \geq 1$  there holds  $\widehat{\text{index}}_\tau(L_{\Sigma'_\ell, g'_\ell}) < 0$ ; here we have set  $\kappa'_\ell = \kappa(g'_\ell, \Sigma'_\ell, \Lambda'_\ell)$  for  $\Lambda'_\ell = \Lambda_j$  if  $(g'_\ell, \Sigma'_\ell) \in \mathcal{L}^{k,\alpha}(g_j, \Sigma_j; \Lambda_j, \kappa_j)$ .

Finally, by Theorem 4.21, the subset of metrics

$$\begin{aligned} & \{g \in \mathcal{G}^{k,\alpha} : \forall (g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M), \widehat{\text{index}}_\tau(L_{\Sigma, g}) \geq 0\} \\ &= \mathcal{G}^{k,\alpha} \setminus \{g \in \mathcal{G}^{k,\alpha} : \exists (g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M), \widehat{\text{index}}_\tau(L_{\Sigma, g}) < 0\} \\ &= \mathcal{G}^{k,\alpha} \setminus \Pi(\bigcup_{\ell \geq 1} \mathcal{L}^{k,\alpha}(g'_\ell, \Sigma'_\ell; \Lambda'_\ell, \kappa'_\ell)) = \bigcap_{\ell \geq 1} (\mathcal{G}^{k,\alpha}(g'_\ell, \Sigma'_\ell; \Lambda'_\ell, \kappa'_\ell)) \end{aligned}$$

is a residual set in the Baire category sense; equivalently, said otherwise, the set of Riemannian metrics  $\{g \in \mathcal{G}^{k,\alpha} : \exists (g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M), \widehat{\text{index}}_\tau(L_{\Sigma, g}) < 0\}$  is meagre. This completes the proof of Theorem 4.1 for  $C^{k,\alpha}$ -metrics.

Step 2: transition to smooth metrics

We apply Lemma 4.31 with  $k_0 = 4$ , taking for every  $k \geq k_0$  the metric space  $(X_k, d_k)$  to be  $\mathcal{G}^{k,\alpha}(M)$  and letting

$$G = \{g \in \mathcal{G}^{k_0,\alpha} : \forall (g, \Sigma) \in \mathcal{M}_n^{k_0,\alpha}(M), \widehat{\text{index}}_\tau(L_{\Sigma, g}) \geq 0\}$$

Note that Step 1 ensures that  $G$  is a  $G_\delta$  subset in  $\mathcal{G}^{k_0,\alpha}(M)$  and  $G \cap \mathcal{G}^{k,\alpha}(M)$  is dense in  $\mathcal{G}^{k,\alpha}$  for every  $k \geq k_0$ . Hence, the smooth version of the theorem follows at once.  $\square$

## 5. PROOF OF THE MAIN THEOREMS

*Proof of Theorem 1.1.* Straightforward by combining Theorem 3.2, Theorem 4.1 and the basic Morse index estimate recalled in Remark 2.9.  $\square$

Moving on, let us now see how a suitable area bound implies a definite structure of the singular set of a stationary integral varifold. With slight notational abuse, we let here  $\mathbb{S}^d$  denote the round unit sphere in  $\mathbb{R}^{d+1}$ , understood as a Riemannian submanifold,  $\omega_d$  denote the  $d$ -dimensional measure of the unit ball in  $\mathbb{R}^d$  and  $A_d$  be the  $d$ -dimensional measure of  $\mathbb{S}^d$ , hence in particular  $A_3 = 2\pi^2$ ; recall that  $\omega_d = A_{d-1}/d$ .

**Proposition 5.1.** *For every  $\varepsilon > 0$ , there exists a neighborhood  $\mathcal{N}(\varepsilon)$  of the round metric on  $S^4$  such that for every  $g \in \mathcal{N}(\varepsilon)$ , every mod 2 cyclic  $g$ -stationary integral 3-varifold  $V$  with total mass  $\leq 2A_3 - \varepsilon$  has only strongly isolated singularities.*

We must first prove the following simple density bound.

**Lemma 5.2.** *Let  $V \subset \mathbb{S}^n$  be a stationary integral  $(n-1)$ -varifold, let  $p \in \text{spt}(V)$  and let  $\Theta_V(p)$  denote the density of  $V$  at  $p$ ; that is to say:  $\Theta_V(p) = \Theta_V(p, 0)$ . Then there holds:*

$$\Theta_V(p) \leq \frac{\|V\|}{A_{n-1}}.$$

*Proof.* Let  $\mathbf{C}$  denote the stationary integral varifold corresponding to the ‘‘cone over  $V$ ’’ in  $\mathbb{R}^{n+1}$ , i. e.  $\mathbf{C} := \mathbf{0} \ast V$ . By the monotonicity formula for stationary integral varifolds one has:

$$\frac{\|V\|}{A_{n-1}} = \lim_{r \rightarrow \infty} \frac{\|\mathbf{C}\|(B_r(0))}{\omega_n r^n} = \lim_{r \rightarrow \infty} \frac{\|\mathbf{C}\|(B_r(p))}{\omega_n r^n} \geq \lim_{r \rightarrow 0} \frac{\|\mathbf{C}\|(B_r(p))}{\omega_n r^n} = \lim_{r \rightarrow 0} \frac{\|V\|(B_r(p))}{\omega_{n-1} r^{n-1}} = \Theta_V(p).$$

□

In the case when  $V$  is associated to a closed embedded minimal hypersurface in  $\mathbb{S}^{n+1}$  the density at each point is of course unitary. Hence:

**Corollary 5.3.** *For any  $n \geq 2$  the equatorial  $n$ -dimensional hypersphere is the (unique) element of least area among all closed embedded minimal hypersurfaces in  $\mathbb{S}^{n+1}$ .*

*Proof.* The (weak) inequality is implied by Lemma 5.2. By inspecting the previous proof, we see at once that equality can only occur if the cone in question splits off the line passing through the point  $p$  and the origin. Iterating the argument by downward induction we must conclude that  $\mathbf{C}$  is in fact an hyperplane through the origin in  $\mathbb{R}^{n+1}$ , hence the claim. □

**Lemma 5.4.** *Let  $\mathbf{C}$  be an irregular, mod 2 cyclic 3-dimensional stationary integral cone in  $\mathbb{R}^4$ . Then*

$$\Theta_{\mathbf{C}}(\mathbf{0}) \geq 2.$$

*Proof.* Let  $x \in \mathbb{S}^3 \cap \text{Sing}(\mathbf{C})$ ,  $\mathbf{C}_x$  be a tangent cone of  $\mathbf{C}$  at  $x$  that is *not* regular in the sense of Definition 2.3. The case when  $\mathbf{C}_x$  has multiplicity at least two is clear; so assume instead that  $\mathbf{C}_x$  has unit multiplicity but non-smooth link. It is well-known that, in this case,  $\mathbf{C}_x$  splits: there exists an isometry (say  $\Phi$ ) of  $\mathbb{R}^4$  sending  $x$  to  $(0, 0, 0, 1)$ , such that  $\Phi_*(\mathbf{C}_x) = \mathbb{R} \times \mathbf{C}'_x$  for some non-trivial 2-dimensional mod 2 cyclic stationary integral cone  $\mathbf{C}'_x$  in  $\mathbb{R}^3$ . Hence, appealing again to the standard monotonicity formula

$$\Theta_{\mathbf{C}}(\mathbf{0}) = \lim_{r \rightarrow \infty} \frac{\|\mathbf{C}\|(B_r(x))}{\omega_3 r^3} \geq \Theta_{\mathbf{C}}(x) = \Theta_{\mathbf{C}_x}(\mathbf{0}) = \Theta_{\mathbf{C}'_x}(\mathbf{0}) \geq 2.$$

where the last inequality follows from the classification of non-trivial 2-dimensional mod 2 cyclic stationary integral cones in  $\mathbb{R}^3$ ; ultimately that relies on the ‘‘structure theorem’’ for 1-dimensional stationary varifolds given in [1], plus an *ad hoc* argument ruling out triple junctions using the mod 2 cyclicity assumption, as can be found e. g. in [30, Lemma A.3]. □

*Proof of Proposition 5.1.* We assume for the sake of a contradiction that for some  $\varepsilon > 0$ , there exists smooth metrics  $g_j$  on  $S^4$  smoothly converging to the round metric and some mod 2 cyclic  $g_j$ -stationary integral varifolds  $V_j$  with not only strongly isolated singularities and total mass  $\leq 2A_3 - \varepsilon$ . Let  $p_j \in \text{Sing}(V_j)$  be a point having a tangent cone  $\mathbf{C}_j$  that is not regular. Then by Lemma 5.4,

$$\Theta_{V_j}(p_j) = \Theta_{\mathbf{C}_j}(\mathbf{0}) \geq 2.$$

By Allard Compactness Theorem (see e.g. [39, Theorem 42.7]),  $V_j$  subconverges to some stationary integral varifold  $V_\infty$  under the round metric, and clearly  $p_j$  subconverges to some  $p_\infty \in \text{Sing}(V_\infty)$ . In particular,

$$\|V_\infty\|(\mathbb{S}^4) \leq 2A_3 - \varepsilon.$$

while by the upper-semi-continuity of density under varifold convergence,

$$\Theta_{V_\infty}(p_\infty) \geq \Theta_{V_j}(p_j) \geq 2$$

Then by Lemma 5.2,

$$\|V_\infty\|(\mathbb{S}^4) \geq A_3 \Theta_{V_\infty}(p_\infty) \geq 2A_3$$

which contradicts the previous bound and thereby completes the proof.  $\square$

*Proof of Corollary 1.4.* Straightforward, by combining Theorem 1.1 with Proposition 5.1.  $\square$

*Proof of Corollary 1.5.* For given  $\varepsilon > 0$  let  $\mathcal{N}(\varepsilon)$  as afforded by Corollary 1.4 and let  $g \in \mathcal{N}(\varepsilon)$  be a generic metric (cf. Remark 1.3). Towards a contradiction, assume the class of closed, embedded minimal hypersurfaces of area less than  $4\pi^2 - \varepsilon$  contains a sequence  $\{\Sigma_k\}_{k \geq 1}$  with pairwise distinct elements.

Allard Compactness Theorem implies that we can extract a subsequence (which we shall not rename) converging to an integral stationary varifold  $V$ ; note that the mass of  $V$  is again bounded from above by the same threshold  $4\pi^2 - \varepsilon$ , and furthermore such a stationary varifold is mod 2 cyclic (by appealing e.g. to [48, Theorem 3.3]). Thanks to Lemma 5.2 we can thereby derive that the density of  $V$  at any point (of its support) is strictly below 2, in fact bounded from above by  $2 - \frac{\varepsilon}{2\pi^2}$ .

Hence, we have by Corollary 1.4 is actually smooth at all points, i.e. it is a smooth, closed, embedded minimal hypersurface in  $\mathbb{S}^{n+1}$ , say  $\Sigma$ , and the aforementioned density bound implies a posteriori that there does occur smooth graphical convergence of  $\Sigma_k$  to  $\Sigma$  with multiplicity one. But then, appealing to Sharp's analysis in [37], the minimal hypersurface  $\Sigma$  would be degenerate (i.e. it would come with at least a non-trivial Jacobi field), a contradiction.  $\square$

*Proof of Corollary 1.6.* Straightforward from Corollary 1.5.  $\square$

## APPENDIX A. THE MINIMAL SURFACE SYSTEM AND TRANSFER OF NORMAL SECTIONS

**Proposition A.1.** *There exist  $\kappa_1 = \kappa_1(N) \in (0, 1/4)$  and  $C = C(N) > 1$  such that the following statement holds. Let*

- (i)  $g, g'$  be  $C^4$  Riemannian metrics on  $\mathbb{B}^N(4)$  with  $\|g - g_{\text{euc}}\|_{C^4}, \|g' - g_{\text{euc}}\|_{C^4} \leq \kappa_1$ ;
- (ii)  $f_1, f_2 \in C^2(\mathbb{B}^N(4))$  be functions such that  $\|f_1 - 1\|_{C^4}, \|f_2 - 1\|_{C^4} \leq \kappa_1$ ;
- (iii)  $\Sigma$  be a  $g$ -minimal  $\kappa_1$ - $C^3$  graph in  $\mathbb{R}^N$  over  $\mathbb{B}^n(2) \times \{\mathbf{0}\}$ ;
- (iv)  $\mathbf{V}$  be the normal bundle of  $\Sigma$  in  $(\mathbb{R}^N, g)$  with induced connection  $\nabla^\perp$  from  $g$ ;
- (v)  $v, v_1, v_2 \in C^2(\Sigma, \mathbf{V})$  with  $\|v\|_{C^2}, \|v_1\|_{C^2}, \|v_2\|_{C^2} \leq \kappa_1$ .

Then we have

- (I)  $\Phi_v : x \mapsto \exp_x^g(v(x))$  is an embedding of  $\Sigma$  into  $\mathbb{R}^N$ ;  
 (II) there exists a  $C^3$  function  $\mathcal{A}^{g'} = \mathcal{A}^{g'}(x, z, \xi)$  such that for every  $x \in \Sigma$  the map  $x \mapsto \mathcal{A}^{g'}(x, \cdot, \cdot)$  on  $\mathbf{V} \oplus (T^*\Sigma \otimes \mathbf{V})$  satisfies

$$\|\Phi_v(\Sigma)\|_{g'} = \int_{\Sigma} \mathcal{A}^{g'}(x, v, \nabla^\perp v) d\|\Sigma\|_g(x),$$

and that under a local coordinate  $(x^i)$  of  $\Sigma$ , pointwisely,

$$\mathcal{A}^{g'}(x, v, \nabla^\perp v) = \sqrt{\frac{\det[(\Phi_v^* g')_{ij}]}{\det[g_{ij}]}}.$$

In particular,  $\Phi_v(\Sigma)$  is  $g'$  minimal if and only if

$$\nabla^\perp \cdot \partial_\xi \mathcal{A}^{g'}(x, v, \nabla^\perp v) - \partial_z \mathcal{A}^{g'}(x, v, \nabla^\perp v) = 0;$$

- (III) if  $\Phi_v(\Sigma)$  is minimal under metric  $g'$ , then  $v$  satisfies

$$L_{\Sigma, g} v + \nabla^\perp \cdot b_0(x) + b_1 = k,$$

where  $b_0 \in C^0(\Sigma; T^*\Sigma \otimes \mathbf{V})$ ,  $b_1, k \in C^0(\Sigma; \mathbf{V})$  satisfy the following estimates,

$$\begin{aligned} |b_0| + |b_1| &\leq C (|v| + |\nabla^\perp v|)^2; \\ \|k\|_{C^0(\Sigma)} &\leq C \|g' - g\|_{C^4}; \end{aligned}$$

- (IV) if for  $i = 1, 2$ ,  $\Phi_{v_i}(\Sigma)$  is minimal under metric  $g_i := f_i g$ , then  $w := v_2 - v_1$  satisfies

$$L_{\Sigma, g} w - \frac{n}{2} (\nabla(f_2 - f_1))^{\perp \Sigma, g} + \nabla^\perp \cdot \tilde{b}_0(x) + \tilde{b}_1 = 0,$$

where  $\tilde{b}_0 \in C^0(\Sigma; T^*\Sigma \otimes \mathbf{V})$ ,  $\tilde{b}_1 \in C^0(\Sigma; \mathbf{V})$  satisfy the following pointwise estimates,

$$|\tilde{b}_0| + |\tilde{b}_1| \leq C \left( \sum_{i=1}^2 |v_i| + |\nabla^\perp v_i| + \|f_i - 1\|_{C^2} \right) \cdot (|w| + |\nabla^\perp w| + \|f_2 - f_1\|_{C^2}).$$

*Remark A.2.* In general, given a Riemannian manifold  $(\Sigma, g)$  and a vector bundle  $\mathbf{V}$  with a fiberwise specified metric  $h$  and a metric compatible connection  $\nabla$ , for every  $b \in C^1(\Sigma, T^*\Sigma \otimes \mathbf{V})$ , we can define  $\nabla \cdot b \in C^0(\Sigma, \mathbf{V})$  to be such that for every  $\varphi \in C_c^1(\Sigma, \mathbf{V})$ , we have

$$\int_{\Sigma} \langle \nabla \cdot b, \varphi \rangle_h d\|\Sigma\| = - \int_{\Sigma} \langle b, \nabla \varphi \rangle_{g, h} d\|\Sigma\|_g.$$

It is easy to check that this is equivalent to the following explicit expression:

$$\langle \nabla \cdot b, \varphi \rangle_h := -d^* \langle \beta, \varphi \rangle_h - \langle \beta, \nabla \varphi \rangle_{g, h},$$

where  $d^*$  is the codifferential on 1-forms on  $(\Sigma, g)$ .

*Proof.* (I) follows from implicit function theorem, (II) follows from a direct, rather standard calculation. So we shall focus here on (III) and (IV).

To start with, consider a variation  $v_t := v + t\phi$ , where  $\phi \in C^2(\Sigma; \mathbf{V})$ . Let  $\vec{\phi}(x, t) := \partial_t \Phi_{v_t}(x) \in T_{\Phi_{v_t}(x)} \mathbb{R}^N$ . Note that when  $v(x) = 0$ ,  $\vec{\phi}(x, 0) = \phi(x)$ . Also, for each fixed  $x$ ,  $t \mapsto \Phi_{v_t}(x)$  is a geodesic under metric  $g$  with constant speed, and  $\vec{\phi}(x, t)$  is its velocity vector field, hence is parallel along  $t \mapsto \Phi_{v_t}(x)$ .

Now we work in local coordinates  $(x^i)$  of  $\Sigma$ , let  $g_{ij}^t := (\Phi_{v_t}^* g)_{ij}$  and  $\partial_i^t := \partial_i \Phi_{v_t}$ . Then we have

$$\begin{aligned} \partial_t g_{ij}^t &= g(\partial_t \partial_i \Phi_{v_t}, \partial_j \Phi_{v_t})|_{\Phi_{v_t}} + g(\partial_i \Phi_{v_t}, \partial_t \partial_j \Phi_{v_t})|_{\Phi_{v_t}} = g(\nabla_{\partial_i^t}^g \vec{\phi}, \partial_j^t)|_{\Phi_{v_t}} + g(\partial_i^t, \nabla_{\partial_j^t}^g \vec{\phi})|_{\Phi_{v_t}}; \\ \partial_{tt}^2 g_{ij}^t &= g(\nabla_{\vec{\phi}}^g \nabla_{\partial_i^t} \vec{\phi}, \partial_j^t) + g(\partial_i^t, \nabla_{\vec{\phi}}^g \nabla_{\partial_j^t} \vec{\phi}) + 2g(\nabla_{\partial_i^t}^g \vec{\phi}, \nabla_{\partial_j^t}^g \vec{\phi}) \\ &= -2 \operatorname{Riem}_g(\vec{\phi}, \partial_i^t, \partial_j^t, \vec{\phi}) + 2g(\nabla_{\partial_i^t}^g \vec{\phi}, \nabla_{\partial_j^t}^g \vec{\phi}). \end{aligned}$$

Denote for simplicity  $\mathcal{A}^g[v_t] := \mathcal{A}^g(x, v_t, \nabla^\perp v_t)$ ,  $h_{ij}^t := g(\nabla_{\partial_i^t} \vec{\phi}, \partial_j^t)$ . Then we derive,

$$\begin{aligned} \frac{d}{dt} \mathcal{A}^g[v_t] &= \frac{1}{2} \mathcal{A}^g[v_t] \cdot \partial_t g_{ij}^t \cdot (g^t)^{ij} = \mathcal{A}^g[v_t] \cdot h_{ij}^t \cdot (g^t)^{ij}; \\ \frac{d^2}{dt^2} \mathcal{A}^g[v_t] &= \mathcal{A}^g[v_t] \left[ (h_{ij}^t \cdot (g^t)^{ij})^2 - 2h_{ij}^t h_{kl}^t (g^t)^{ik} (g^t)^{jl} - \operatorname{Riem}_g(\vec{\phi}, \partial_i^t, \partial_j^t, \vec{\phi}) + g(\nabla_{\partial_i^t}^g \vec{\phi}, \nabla_{\partial_j^t}^g \vec{\phi}) \right] \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \mathcal{A}^g[v_t] &= \partial_z \mathcal{A}^g(x, v_t, \nabla^\perp v_t) \cdot \phi + \partial_\xi \mathcal{A}^g(x, v_t, \nabla^\perp v_t) \cdot \nabla^\perp \phi; \\ \frac{d^2}{dt^2} \mathcal{A}^g[v_t] &= \partial_{zz}^2 \mathcal{A}^g(x, v_t, \nabla^\perp v_t)(\phi, \phi) + 2\partial_{z\xi}^2 \mathcal{A}^g(x, v_t, \nabla^\perp v_t)(\phi, \nabla^\perp \phi) \\ &\quad + \partial_{\xi\xi}^2 \mathcal{A}^g(x, v_t, \nabla^\perp v_t)(\nabla^\perp \phi, \nabla^\perp \phi); \end{aligned}$$

By comparing the coefficients in front of  $\phi, \nabla^\perp \phi$  terms, we can find that,

$$(55) \quad \mathcal{A}^g(x, 0, 0) = 1, \quad \partial_z \mathcal{A}^g(x, 0, 0) = 0, \quad \partial_\xi \mathcal{A}^g(x, 0, 0) = 0;$$

$$(56) \quad \partial_{\xi\xi}^2 \mathcal{A}^g(x, 0, 0)(\zeta, \zeta) = |\zeta|^2, \quad \partial_{z\xi}^2 \mathcal{A}^g(x, 0, 0) = 0, \quad \partial_{zz}^2 \mathcal{A}^g(x, 0, 0) = -\mathbb{I}_{\Sigma, g}^2 - \mathcal{R}_{\Sigma, g}$$

where  $\mathbb{I}_{\Sigma, g}$  denotes the second fundamental form of  $\Sigma$  under  $g$ , and for  $\varphi \in \mathbf{V}$ ,

$$\mathbb{I}_{\Sigma, g}^2(\varphi, \varphi) := (\mathbb{I}_{ij} \cdot \varphi)(\mathbb{I}_{kl} \cdot \varphi) g^{ik} g^{jl}, \quad \mathcal{R}_{\Sigma, g}(\varphi, \varphi) := \operatorname{Riem}_g(\varphi, \partial_i, \partial_j, \varphi) g^{ij},$$

Therefore, recalling that for any  $y \in C^2([0, 1]; \mathbb{R})$  there holds

$$|y(1) - y(0) - y'(0)| \leq \frac{1}{2} \sup_{t \in [0, 1]} |y''(t)|,$$

we have pointwise error estimates,

$$|\partial_\xi \mathcal{A}^g[v] - \nabla^\perp v| + |\partial_z \mathcal{A}^g[v] + (\mathbb{I}_{\Sigma, g}^2 + \mathcal{R}_{\Sigma, g})(v, \cdot)| \leq C(|v| + |\nabla^\perp v|)^2.$$

Together with the following fact, which for  $\|v\|_{C^2} \leq \kappa_1$  and  $\kappa_1 = \kappa_1(N)$  sufficiently small:

$$\left| \nabla^\perp \cdot \left( \partial_\xi \mathcal{A}^g[v] - \partial_\xi \mathcal{A}^g[v] \right) \right| + \left| \partial_z \mathcal{A}^g[v] - \partial_z \mathcal{A}^g[v] \right| \leq C \|g' - g\|_{C^4}$$

this proves (III).

To prove (IV), first notice that by (II),

$$\mathcal{A}^{g_i}(x, v, \nabla^\perp v) = (f_i)^{n/2} \circ \Phi_v \cdot \mathcal{A}^g(x, v, \nabla^\perp v).$$

For every  $\varphi \in C^2(\Sigma; \mathbf{V})$ , if we let  $v_t^i := v_i + t\varphi$  and  $\vec{\varphi}_i(x, t) := \partial_t \Phi_{v_t^i}(x)$ , then,

$$\frac{d}{dt} \mathcal{A}^{g_i}[v_t^i] = (f_i)^{n/2} \circ \Phi_{v_t^i} \cdot \left( \partial_z \mathcal{A}^g[v_t^i] \cdot \varphi + \partial_\xi \mathcal{A}^g[v_t^i] \cdot \nabla^\perp \varphi \right) + \nabla^g(f_i^{n/2}) \cdot \vec{\varphi}_i(\cdot, t) \cdot \mathcal{A}^g[v_t^i],$$

hence we conclude that

$$\begin{aligned}\partial_\xi \mathcal{A}^{g_i}[v_i] &= (f_i)^{n/2} \circ \Phi_{v_i} \cdot \partial_\xi \mathcal{A}^g[v_i]; \\ \partial_z \mathcal{A}^{g_i}[v_i] \cdot \varphi &= (f_i)^{n/2} \circ \Phi_{v_i} \cdot \partial_z \mathcal{A}^g[v_i] \cdot \varphi + \nabla^g(f_i^{n/2}) \cdot \vec{\varphi}_i(x, 0) \cdot \mathcal{A}^g[v_i].\end{aligned}$$

By applying again (55) and (56), we find

$$\begin{aligned}& \left| \partial_\xi \mathcal{A}^{g_2}[v_2] - \partial_\xi \mathcal{A}^{g_1}[v_1] - \nabla^\perp w \right| \\ &+ \left| \partial_z \mathcal{A}^{g_2}[v_2] - \partial_z \mathcal{A}^{g_1}[v_1] - \frac{n}{2}(\nabla^g(f_2 - f_1))^{\perp \Sigma, g} + (\mathbb{I}_{\Sigma, g}^2 + \mathcal{R}_{\Sigma, g})(w, \cdot) \right| \\ &\leq C \left( \sum_{i=1}^2 |v_i| + |\nabla^\perp v_i| + \|f_i - 1\|_{C^2} \right) \cdot (|w| + |\nabla^\perp w| + \|f_2 - f_1\|_{C^2}).\end{aligned}$$

This proves (IV).  $\square$

**Proposition A.3.** *There exist  $\kappa_2 = \kappa_2(N) \in (0, 1/4)$  and  $C = C(N) > 1$  such that the following statement holds. Let  $\kappa \in (0, \kappa_2)$  and*

- (i)  $g, g'$  be  $C^4$  Riemannian metrics on  $\mathbb{B}^N(4)$  with  $\|g - g_{\text{euc}}\|_{C^4}, \|g' - g_{\text{euc}}\|_{C^4} \leq \kappa_2$  and  $\|g - g'\|_{C^4} \leq \kappa$ ;
- (ii)  $\Sigma$  be a  $g$ -minimal  $\kappa_2$ - $C^3$  graph in  $\mathbb{R}^N$  over  $\mathbb{B}^n(2) \times \{\mathbf{0}\}$ ;
- (iii)  $v \in C^2(\Sigma, \mathbf{V})$  with  $\|v\|_{C^2} \leq \kappa$ ;  $\Phi_v$  be defined in (I) of Proposition A.1;
- (iv)  $\mathbf{V}$  be the normal bundle of  $\Sigma$  in  $(\mathbb{R}^N, g)$  with induced connection  $\nabla^\perp$  from  $g$ ;  $\mathbf{V}'$  be the normal bundle of  $\Sigma' := \text{graph}_{\Sigma, g}(v)$  in  $(\mathbb{R}^N, g')$  with induced connection  $\nabla'^\perp$  from  $g'$ ;
- (v)  $\mathbf{T}_{\Sigma, g}^{\Sigma'}$  :  $\mathbf{V}' \rightarrow \mathbf{V}$  be the bundle maps defined in Definition 2.21, which induces linear maps (still denoted by  $\mathbf{T}_{\Sigma, g}^{\Sigma'}$ ) for every integer  $0 \leq k \leq n$ :

$$C^0(\Sigma', \Lambda^k \Sigma' \otimes \mathbf{V}') \rightarrow C^0(\Sigma, \Lambda^k \Sigma \otimes \mathbf{V}).$$

Then,

- (I) for every  $\beta' \in C^0(\Sigma'; \Lambda^k \Sigma' \otimes \mathbf{V}')$ , denote by  $\beta := \mathbf{T}_{\Sigma, g}^{\Sigma'}(\beta') \in C^0(\Sigma; \Lambda^k \Sigma \otimes \mathbf{V})$ . For every  $x \in \Sigma$ , we have,

$$(1 - C\kappa)|\beta(x)| \leq |\beta'(\Phi_v(x))| \leq (1 + C\kappa)|\beta(x)|;$$

- (II) for every  $u' \in C^1(\Sigma', \mathbf{V}')$  and  $u := \mathbf{T}_{\Sigma, g}^{\Sigma'}(u')$ ,

$$\left| \mathbf{T}_{\Sigma, g}^{\Sigma'}(\nabla'^\perp u')(x) - \nabla^\perp u(x) \right| \leq C\kappa|u(x)|;$$

- (III) there exist bundle endomorphisms  $\mathcal{S} : \mathbf{V} \rightarrow \mathbf{V}$ ,  $\mathcal{T} : T^*\Sigma \otimes \mathbf{V} \rightarrow T^*\Sigma \otimes \mathbf{V}$  such that, denoted by  $\mathbf{1}$  the identity map on either bundle in question, there holds

$$|\mathcal{S} - \mathbf{1}| + |\mathcal{T} - \mathbf{1}| \leq C\kappa$$

and for every  $\beta' \in C^1(\Sigma', T^*\Sigma' \otimes \mathbf{V}')$  and  $\beta := \mathbf{T}_{\Sigma, g}^{\Sigma'}(\beta')$ , we have the pointwise estimate

$$\left| \mathbf{T}_{\Sigma, g}^{\Sigma'}(\nabla'^\perp \cdot \beta') - \mathcal{S}(\nabla^\perp \cdot \mathcal{T}(\beta)) \right| \leq C\kappa|\beta|.$$

In particular, for every  $u' \in C^1(\Sigma', \mathbf{V}')$  and  $b'_0, b'_1 \in C^1(\Sigma', T^*\Sigma' \otimes \mathbf{V}')$  such that

$$-L_{\Sigma', g'}u' + \nabla'^\perp \cdot b'_0 + b'_1 = 0,$$

there exists  $b_0, b_1 \in C^1(\Sigma, T^*\Sigma \otimes \mathbf{V})$  such that  $u := \mathbf{T}_{\Sigma, g}^{\Sigma'}(u')$  solves,

$$(57) \quad -L_{\Sigma, g}u + \nabla^\perp \cdot b_0 + b_1 = 0,$$

and

$$(58) \quad \begin{aligned} & |b_0(x) - \mathbf{T}_{\Sigma, g}^{\Sigma'}(b'_0)(x)| + |b_1(x) - \mathbf{T}_{\Sigma, g}^{\Sigma'}(b'_1)(x)| \\ & \leq C\kappa (|b'_0(\Phi_v(x))| + |b'_1(\Phi_v(x))| + |u(x)| + |\nabla^\perp u(x)|). \end{aligned}$$

*Proof.* Denote for simplicity  $\mathbf{T} := \mathbf{T}_{\Sigma, g}^{\Sigma'}$ . Let

- $\mathbf{x}' : \Sigma' \rightarrow \mathbb{R}^n$  be the projection on to the first  $n$ -factors,  $\mathbf{x} := \mathbf{x}' \circ \Phi_v$ ;
- For every  $x' \in \Sigma'$  and  $1 \leq j \leq N - n$ , let

$$e'_j(x') := (0, \dots, 0, \underbrace{1}_{n+j\text{-th}}, \dots, 0)^{\perp_{T_{x'}\Sigma', g'}}.$$

And let  $e_j(x) := \mathbf{T}(e'_j)(x)$ .

It is straightforward to note that we can take  $\kappa_2(N)$  small enough that if (i)-(v) hold, then:

- $\mathbf{x}' = (\mathbf{x}'^1, \dots, \mathbf{x}'^m)$  and  $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^n)$  are coordinate systems of, respectively,  $\Sigma'$  and  $\Sigma$ , such that the metric  $g'$  and  $g$  restricted to  $\Sigma'$  and  $\Sigma$  and their Christoffel symbols satisfy

$$(59) \quad |g'_{ij} - \delta_{ij}|, |g_{ij} - \delta_{ij}|, |\Gamma'^k_{ij}|, |\Gamma^k_{ij}| \leq C\kappa_2; \quad |\Phi_v^* g'_{ij} - g_{ij}| \leq C\kappa;$$

- $\{e'_j\}$  and  $\{e_j\}$  are frames of, respectively,  $\mathbf{V}'$  and  $\mathbf{V}$ , such that the induced metrics  $h'_{ij} := g'(e'_i, e'_j)$  and  $h_{ij} := g(e_i, e_j)$  satisfy

$$(60) \quad |h'_{ij} - \delta_{ij}|, |h_{ij} - \delta_{ij}| \leq C\kappa_2; \quad |h'_{ij} \circ \Phi_v - h_{ij}| \leq C\kappa,$$

the connection forms  $\omega'^i_j$  and  $\omega^i_j$  of  $\nabla'^\perp$  and  $\nabla^\perp$  under these two frames satisfy

$$(61) \quad |\omega'^i_j|_{g'}, |\omega^i_j|_g \leq C\kappa_2; \quad |\Phi_v^* \omega'^i_j - \omega^i_j|_g \leq C\kappa.$$

Thus for every  $\beta' = \beta'^j \otimes e'_j \in C^0(\Sigma'; \Lambda^k \Sigma' \otimes \mathbf{V}')$ , we have

$$\beta := \mathbf{T}_{\Sigma, g}^{\Sigma'}(\beta') = \Phi_v^* \beta'^j \otimes e_j.$$

In particular, combined with (59) and (60), this proves (I).

Let then  $u' = u'^j e'_j \in C^1(\Sigma', \mathbf{V}')$  and  $u^j := u'^j \circ \Phi_v$  (so that  $u := \mathbf{T}(u') = u^j e_j$ ). Then (II) follows directly from (59)-(61) and

$$\mathbf{T}(\nabla'^\perp u') = \mathbf{T}((du'^j + u'^k \omega'^j_k) \otimes e'_j) = (du^j + u^k \Phi_v^* \omega'^j_k) \otimes e_j = \nabla^\perp u + u^k \cdot (\Phi_v^* \omega'^j_k - \omega^j_k) \otimes e_j.$$

To prove (III), first recall that for  $\beta' = \beta'^j \otimes e'_j = \beta'^j_p d\mathbf{x}'^p \otimes e'_j \in C^1(\Sigma, T^*\Sigma' \otimes \mathbf{V}')$ ,

$$\begin{aligned} \nabla'^\perp \cdot \beta' &= - (d^*(\beta'^j h'_{jk}) + g'(\beta'^i, \omega'^j_k) h'_{ij}) h'^{kl} e'_l \\ &= (\partial_q(\beta'^j_p \cdot g'^{pq} h'_{jk}) - \beta'^j_p h'_{jk} \cdot \partial_q(g'^{pq}) - g'^{pq} \Gamma'^r_{pq}(\beta'^j_r h'_{jk}) - g'(\beta'^i, \omega'^j_k) h'_{ij}) h'^{kl} e'_l. \end{aligned}$$

Hence, having set  $\beta_p^j := \beta_p^j \circ \Phi_v$  (so that  $\beta^j := \beta_p^j d\mathbf{x}^p = \Phi_v^* \beta'^j$  and  $\beta := \beta^j \otimes e_j = \mathbf{T}(\beta')$ ), there holds

$$\begin{aligned} \mathbf{T}(\nabla'^\perp \cdot \beta') &= \partial_q \left( \beta_p^j \cdot (g'^{pq} h'_{jk}) \circ \Phi_v \right) (h'^{kl} \circ \Phi_v) e_l \\ &\quad - \left( \beta_p^j h_{jk} \cdot \partial_q(g'^{pq}) + g'^{pq} \Gamma_{pq}^r (\beta_r^j h_{jk}) + g(\beta^i, \omega_k^j) h_{ij} \right) h^{kl} e_l + O_N(\kappa|\beta|_g) \\ &= \partial_q \left( \beta_p^j \mathcal{T}_{j\bar{p}}^{\bar{j}p} \cdot g^{\bar{p}q} h_{\bar{j}k} \right) h^{k\bar{l}} \cdot \mathcal{S}_{\bar{l}}^l e_l \\ &\quad - \left( \beta_p^j h_{jk} \cdot \partial_q(g'^{pq}) + g'^{pq} \Gamma_{pq}^r (\beta_r^j h_{jk}) + g(\beta^i, \omega_k^j) h_{ij} \right) h^{kl} e_l + O_N(\kappa|\beta|_g) \\ &= \mathcal{S}(\nabla^\perp \cdot \mathcal{T}(\beta)) + O_N(\kappa|\beta|_g), \end{aligned}$$

where  $O_N(\kappa|\beta|_g)$  is a function that is pointwise bounded by  $C\kappa|\beta|$  for  $C = C(N)$  as throughout this appendix;  $\mathcal{T}_{j\bar{p}}^{\bar{j}p} := (g'^{pq} h'_{jk}) \circ \Phi_v \cdot g^{\bar{p}q} h_{\bar{j}k}$ ;  $\mathcal{S}_{\bar{l}}^l := (h'^{kl} \circ \Phi_v) \cdot h_{k\bar{l}}$ , so that there are indeed well-defined bundle endomorphisms:

$$\begin{aligned} \mathcal{T} : T^*\Sigma \otimes \mathbf{V} &\rightarrow T^*\Sigma \otimes \mathbf{V}, & \alpha_p^j d\mathbf{x}^p \otimes e_j &\mapsto \alpha_p^j \mathcal{T}_{j\bar{p}}^{\bar{j}p} d\mathbf{x}^{\bar{p}} \otimes e_{\bar{j}}; \\ \mathcal{S} : \mathbf{V} &\rightarrow \mathbf{V}, & w^l e_l &\mapsto w^l \mathcal{S}_{\bar{l}}^l e_{\bar{l}}. \end{aligned}$$

And the estimates above as well as the estimates for  $|\mathcal{T} - \mathbb{1}|, |\mathcal{S} - \mathbb{1}|$  follows both from (I) and (59)-(61). This finishes the proof of (III).

Finally, to prove (57) and (58), recall that using the notation in the proof of Proposition A.1,

$$L_{\Sigma, g} u = \nabla^\perp \cdot \nabla^\perp u + \mathbb{I}_{\Sigma, g}^2 u + \mathcal{R}_{\Sigma, g} u.$$

Hence, it suffices to combine (I)-(III) with the fact that

$$\|\Phi_v^* \mathbb{I}_{\Sigma', g'}^2 - \mathbb{I}_{\Sigma, g}\| + \|\Phi_v^* \mathcal{R}_{\Sigma', g'} - \mathcal{R}_{\Sigma, g}\| \leq C\kappa.$$

□

## APPENDIX B. A THREE-CIRCLE INEQUALITY AND RELATED DECAY ESTIMATES

The following growth rate monotonicity formula is a direct consequence of the decomposition (19), which was in fact first introduced in [38] in a different form.

**Lemma B.1.** *For any  $\sigma > 0$ , there exists  $K = K(\sigma) > 2$  with the following property.*

*For any  $\gamma \in \mathbb{R}$  and any minimal cone  $\mathbf{C} \in \mathcal{C}_{N, n}$  satisfying*

$$\text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma,$$

*if  $u \in C_{loc}^2(\mathbf{A}(K^{-3}, 1); \mathbf{V}) \cap L^2(\mathbf{A}(K^{-3}, 1); \mathbf{V})$  solves  $L_{\mathbf{C}} u = 0$  then its growth rate*

$$J_K^\gamma(u; r) := \int_{\mathbf{A}(K^{-1}r, r)} |u|^2 \cdot |x|^{-2\gamma-n} d\|\mathbf{C}\|,$$

*satisfies the inequality*

$$J_K^\gamma(u; K^{-2}) - 2J_K^\gamma(u; K^{-1}) + J_K^\gamma(u; 1) \geq 0.$$

*Moreover, equality holds if and only if  $u \equiv 0$ .*

Recall that, consistently with the general notational conventions we have stipulated in Section 2 we have denoted by  $\mathbf{V} = \mathbf{V}(\mathbf{C})$  the normal bundle to  $\mathbf{C}$  in  $\mathbb{R}^N$  and by  $\mathbf{A}(r, s) := \mathbb{A}(\mathbf{0}, r, s) \cap \mathbf{C}$  the annular region, on the cone, of radii  $r < s$ .

*Proof.* By (19)-(20), see Section 3.1, we have

$$J_K^\gamma(v; r) = \int_{K^{-1}r}^r t^{n-1-n-2\gamma} dt \int_{\mathbf{C} \cap \mathbb{S}^{N-1}} v(t, \omega)^2 d\omega = \sum_{j \geq 1} \underbrace{\int_{K^{-1}r}^r t^{-1-2\gamma} \cdot (v_j^+(t) + v_j^-(t))^2 dt}_{=: T_j},$$

where of course the second equality relies upon Parseval's identity, with respect to the orthonormal (Hilbertian) basis  $\varphi_1, \varphi_2, \dots, \varphi_k, \dots$  of the space  $L^2(\mathbf{C} \cap \mathbb{S}^{N-1}; \mathbf{V})$  of square-integrable normal sections to the link  $S := \mathbf{C} \cap \mathbb{S}^{N-1}$  in the unit sphere of Euclidean  $\mathbb{R}^N$ .

Recalling the definition of  $\mu_j = \lambda_j + (n-1)$  where  $\lambda_j$  is the  $j$ -th eigenvalue of the Jacobi operator of the link  $S$  in question (see the second part of Section 2.2), if  $\mu_j \geq -(n-2)^2/4$  then we have

$$\begin{aligned} T_j &= \int_{K^{-1}r}^r \left( c_j^+ t^{\gamma_j^+ - \gamma} + c_j^- t^{\gamma_j^- - \gamma} \right)^2 t^{-1} dt, & \text{if } \mu_j > -\frac{(n-2)^2}{4}; \\ T_j &= \int_{K^{-1}r}^r \left( c_j^+ t^{-(n-2)/2 - \gamma} + c_j^- t^{-(n-2)/2 - \gamma} \log t \right)^2 t^{-1} dt, & \text{if } \mu_j = -\frac{(n-2)^2}{4}; \end{aligned}$$

while if  $\mu_j < -(n-2)^2/4$ , set  $\alpha_j := \sqrt{-\mu_j - (n-2)^2/4}$ ,  $2c_j^\pm =: c_j e^{i\theta_j}$  (with  $c_j \in \mathbb{R}$ ), then

$$T_j = \int_{K^{-1}r}^r c_j^2 \left( \cos(\alpha_j \log(t) + \theta_j) t^{-(n-2)/2 - \gamma} \right)^2 t^{-1} dt.$$

At this stage it suffices to show that for  $K = K(\sigma)$  large enough, the desired inequality holds at the level of each  $T_j \geq 0$  with equality if and only if  $c_j^\pm = 0$ . When  $\mu_j \geq -(n-2)^2/4$ , this follows from [27, Lemma A.1 and Lemma A.2]; instead when  $\mu_j < -(n-2)^2/4$ , it is a consequence of the following statement.  $\square$

**Lemma B.2.** *Suppose  $\sigma > 0$ . Then there exists a real number  $K_0 = K_0(\sigma) > 2$  with the following property.*

*For any  $\alpha, \beta, \theta \in \mathbb{R}$  such that  $\lambda > 0, |\beta| \geq \sigma$  and any  $K \geq K_0$ , the integral*

$$\mathcal{I}_K(r) := \int_r^{Kr} \cos^2(\alpha \log(s) + \theta) s^{2\beta-1} ds,$$

*satisfies for every  $r > 0$ ,*

$$\mathcal{I}_K(K^2r) - 2\mathcal{I}_K(Kr) + \mathcal{I}_K(r) > 0.$$

*Proof.* Notice that

$$\int_r^{Kr} \cos^2(\alpha \log(s) + \theta) s^{2\beta-1} ds = \int_{\log(r)+\theta/\alpha}^{\log(r)+\theta/\alpha+\log(K)} \cos^2(\alpha\tau) e^{2\beta\tau-2\beta\theta/\alpha} d\tau.$$

Therefore, if we let  $r' := \log(r) + \theta/\alpha$ ,  $K' := \log(K)$ ,  $\beta' := 2\beta$ ,  $\alpha' = 2\alpha$  and

$$\tilde{\mathcal{I}}_{K'}(r') := \int_{r'}^{r'+K'} 2 \cos^2(\alpha\tau) e^{\beta'\tau} d\tau = \int_{r'}^{r'+K'} (1 + \cos(\alpha'\tau)) e^{\beta'\tau} d\tau$$

then it suffices to show that when  $K' \geq K'(\sigma)$  large enough for any  $r' \in \mathbb{R}$  there holds

$$\tilde{\mathcal{I}}_{K'}(r' + 2K') - 2\tilde{\mathcal{I}}_{K'}(r' + K') + \tilde{\mathcal{I}}_{K'}(r') > 0.$$

And by replacing  $r'$  by  $-r' - K'$  if necessary, we can assume without loss of generality that  $\beta' \geq 2\sigma > 0$ . Let  $\beta'' := \beta' + i\alpha' \in \mathbb{C}$ . Then,

$$\tilde{\mathcal{I}}_{K'}(r') = \Re \left( \int_{r'}^{r'+K'} e^{\beta'\tau} + e^{\beta''\tau} d\tau \right) = e^{\beta'r'} \cdot \frac{e^{\beta'K'} - 1}{\beta'} + \Re \left( e^{\beta''r'} \cdot \frac{e^{\beta''K'} - 1}{\beta''} \right).$$

We can employ this formula to rewrite the quantity we wish to prove positive; elementary manipulations allow to write:

$$\begin{aligned} & \tilde{\mathcal{I}}_{K'}(r' + 2K') - 2\tilde{\mathcal{I}}_{K'}(r' + K') + \tilde{\mathcal{I}}_{K'}(r') \\ &= e^{\beta'r'} \cdot \frac{(e^{\beta'K'} - 1)^3}{\beta'} + \Re \left( e^{\beta''r'} \cdot \frac{(e^{\beta''K'} - 1)^3}{\beta''} \right) \\ &= \frac{e^{\beta'r'} (e^{\beta'K'} - 1)^3}{|\beta''|} \left[ \frac{|\beta''|}{\beta'} + \Re \left( \frac{|\beta''| e^{i\alpha'(r'+3K')}}{\beta''} \cdot \left( \frac{e^{\beta'K'} - e^{-i\alpha'K'}}{e^{\beta'K'} - 1} \right)^3 \right) \right] \\ &\geq \frac{e^{\beta'r'} (e^{\beta'K'} - 1)^3}{|\beta''|} \left[ \frac{|\beta''|}{\beta'} - \left( \frac{|e^{\beta'K'} - e^{-i\alpha'K'}|}{e^{\beta'K'} - 1} \right)^3 \right]. \end{aligned}$$

Since

$$\begin{aligned} \left( \frac{|e^{\beta'K'} - e^{-i\alpha'K'}|}{e^{\beta'K'} - 1} \right)^2 &= \frac{(e^{\beta'K'} - \cos(\alpha'K'))^2 + \sin^2(\alpha'K')}{(e^{\beta'K'} - 1)^2} \\ &\leq \left( 1 + \frac{\min\{\alpha'K', 1\}^2}{e^{\beta'K'} - 1} \right)^2 + \frac{\min\{\alpha'K', 1\}^2}{(e^{\beta'K'} - 1)^2}. \end{aligned}$$

Thus when  $\beta'K'$  is large enough (arranged by requiring  $K \geq K_0(\sigma)$  for suitable  $K_0(\sigma)$ ), we have in fact

$$\left( \frac{|e^{\beta'K'} - e^{-i\alpha'K'}|}{e^{\beta'K'} - 1} \right)^6 \leq 1 + \frac{20 \min\{\alpha'K', 1\}^2}{e^{\beta'K'} - 1} < 1 + \frac{(\alpha'K')^2}{(\beta'K')^2} = \left( \frac{|\beta''|}{\beta'} \right)^2,$$

which implies the claim.  $\square$

We stress that the equality case in Lemma B.1 holds only when the section  $u$  is identically 0. Therefore, for non-zero sections we can suitably strengthen the lemma to an open condition, which allows us to perturb both the metrics and the coefficients in the PDE as described in the following statement. This refinement is particularly useful in our applications.

**Corollary B.3** (Perturbed version of Lemma B.1). *For  $\sigma > 0$  and  $\Lambda > 0$ , let  $K(\sigma) > 2$  be the same as in Lemma B.1. Then there exists  $\varepsilon(\sigma, \Lambda) > 0$  small enough satisfying the following property.*

*For  $\gamma \in [-\Lambda, \Lambda]$  and any  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$  satisfying*

$$\text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma,$$

*let  $0 \neq u \in W_{loc}^{1,2}(\mathbf{A}(K^{-3}, 1); \mathbf{V}) \cap L^2(\mathbf{A}(K^{-3}, 1); \mathbf{V})$  be a weak solution to*

$$(62) \quad \nabla^\perp \cdot (\nabla^\perp u + b_0(x)) + \langle A_{\mathbf{C}}, u \rangle A_{\mathbf{C}} + |x|^{-1} b_1(x) = 0,$$

*where  $\nabla^\perp = \nabla_{\mathbf{C}, \text{geuc}}^\perp$  is the Levi-Civita connection of the normal bundle  $\mathbf{V}$  to  $\mathbf{C}$  in Euclidean  $\mathbb{R}^N$ , and  $b_0, b_1 \in L^2(\Sigma; \mathbf{V})$  satisfy the following estimates a.e.,*

$$(63) \quad \|(|b_0| + |b_1| - \varepsilon |\nabla^\perp u|)_+\|_{L^2(\mathbf{A}(K^{-3}, 1))} \leq \varepsilon \|u\|_{L^2(\mathbf{A}(K^{-3}, 1))},$$

on  $\mathbf{A}(K^{-3}, 1)$ . (Here we use the notation  $v_+ := \max\{v, 0\}$ .) Then we have,

$$J_K^\gamma(u; K^{-2}) - 2(1 + \varepsilon)J_K^\gamma(u; K^{-1}) + J_K^\gamma(u; 1) > 0.$$

*Remark B.4.* When  $\Sigma := \text{graph}_{\mathbf{C}}(\phi) \cap \mathbb{B}(2)$  is a minimal submanifold, under a metric  $g$ , parametrized by the cone  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$  (that, let us stress the point, is minimal in Euclidean  $\mathbb{R}^N$  so under the metric  $g_{\text{euc}}$ ), and

$$\|g - g_{\text{euc}}\|_{C^4(\mathbb{B}(2))} + \|\phi\|_{C_1^2(\mathbb{B}(2))} \leq \tilde{\varepsilon},$$

where  $\tilde{\varepsilon} \leq \kappa_2$  and  $\kappa_2$  has been defined in Proposition A.3. Then for any Jacobi field  $u$  on  $\Sigma$ , the corresponding normal section  $v$  over  $\mathbf{C}$  under this parametrization (by which we mean  $v := \mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma(u)$  in the sense of Definition 2.21), solves an equation of form (62) with

$$(64) \quad |b_0(x)| + |b_1(x)| \leq C(\Lambda)\tilde{\varepsilon} \cdot (|x|^{-1}|v(x)| + |\nabla^\perp v(x)|).$$

To see this, for every  $\check{x} \in \mathbf{C}$ , let  $\check{r} := \kappa_2 \mathbf{r}_{\mathbf{C}, g_{\text{euc}}}(\check{x})/4$ , and  $\check{\eta} : x \mapsto \check{x} + \check{r}x$ , where  $\kappa_2$  is the dimensional constant in Proposition A.3 (III). Then, Proposition A.3 (III) applies with  $g_{\text{euc}}, \check{g} := \check{r}^{-2}\check{\eta}^*g, \check{\mathbf{C}} := \check{\eta}^{-1}(\mathbf{C}), \check{\Sigma} := \check{\eta}^{-1}(\Sigma), \check{v} := v \circ \check{\eta}, u \circ \check{\eta}$  in place of  $g, g', \Sigma, \Sigma', u, u'$  therein, which gives

$$-L_{\check{\mathbf{C}}, g_{\text{euc}}}(\check{v}) + \nabla^\perp \cdot \check{b}_0 + \check{b}_1 = 0,$$

where by there hold the estimates (58),

$$|\check{b}_0(0)| + |\check{b}_1(0)| \leq C(\Lambda)\varepsilon \cdot (|\check{v}(0)| + |\nabla^\perp \check{v}(0)|).$$

Now (64) at  $\check{x}$  follows by scaling back the equation and setting

$$b_0 = \check{b}_0 \circ \check{\eta}^{-1} \cdot \check{r}^{-1}, \quad b_1(x) = |x|\check{b}_0 \circ \check{\eta}^{-1} \cdot \check{r}^{-2}$$

so that an equation of the form (62) is satisfied. As a consequence of (64), given any  $\sigma > 0$  and  $\Lambda > 1$ , for  $\varepsilon(\sigma, \Lambda)$  determined in Corollary B.3, we can choose  $\tilde{\varepsilon}(\sigma, \Lambda)$  so small that the error estimate assumption (63) is satisfied with  $v$  in place of  $u$  therein. Lastly, going beyond the linear analysis of Jacobi fields, we note that (without much additional effort) a similar equation with analogous error estimates holds for  $\mathbf{T}_{\Sigma', g}^{\Sigma'}(w)$ , where  $w$  is a graphical section defining a minimal submanifold  $\Sigma'$  close to  $\Sigma$ , under any metric  $g'$  suitably close to  $g$  (possibly, but not necessarily,  $g$  itself).

*Proof.* Suppose, for a contradiction, that there exist sequences  $\mathbf{C}_j \in \mathcal{C}_{N,n}(\Lambda)$ , and  $u_j \in W_{loc}^{1,2}(\mathbf{A}_j(K^{-3}, 1); \mathbf{V}_j) \cap L^2(\mathbf{A}_j(K^{-3}, 1); \mathbf{V}_j)$  weak solutions to (62), where  $b_0, b_1$  are replaced by  $b_0^j, b_1^j$  satisfying (63) with  $\varepsilon = 1/j$  and the 3-circle inequality fails:

$$J_K^\gamma(u_j; K^{-2}) + J_K^\gamma(u_j; 1) \leq 2(1 + 1/j)J_K^\gamma(u_j; K^{-1}) =: 2c_j^2 > 0.$$

(Note that here we have employed the usual notation  $\mathbf{A}_j(r, s) = \mathbb{A}(r, s) \cap \mathbf{C}_j$ , understood for all  $j \geq 1$ .) Consider then  $\hat{u}_j := c_j^{-1}u_j, \hat{b}_i^j := c_j^{-1}b_i^j$ , where  $i = 0, 1$ ; of course (62) and (63) are also satisfied with such replacements throughout. Thanks to Lemma 4.5 we can extract a subsequence of cones converging smoothly (at the level of spherical section, hence on the whole annulus of radii  $K^{-3}/3$  and 3) to a limit cone  $\mathbf{C}_\infty$ , and for every  $p \geq 1$ , as  $j \rightarrow \infty$ ,

$$\gamma_p^\pm(\mathbf{C}_j) \rightarrow \gamma_p^\pm(\mathbf{C}_\infty).$$

Now, for  $j \geq 1$  large enough  $\mathbf{A}_j(K^{-3}, 1)$  is a subset of  $\text{graph}_{\mathbf{C}_\infty}(\phi_j) \cap \mathbb{A}(K^{-3}/2, 2)$  and one may consider

$$v_j := \mathbf{T}_{\mathbf{C}_\infty, g_{\text{euc}}}^{\mathbf{C}_j}(\hat{u}_j)$$

for a suitable sequence  $\{\phi_j\}_{j \geq j_0}$  satisfying  $\|\phi_j\|_{C_1^2(K^{-3/2,2})} \leq \tilde{\varepsilon}$  chosen so that condition (63) is fulfilled. Then, since (by its definition)  $\|\hat{u}_j\|_{L^2(\mathbf{A}_j(K^{-3},1))}$  is uniformly bounded in  $j \geq 1$ , by classical elliptic theory (interior estimates in Hilbertian Sobolev spaces) relying upon Remark B.4, up to extracting a subsequence (which we shall not rename),  $v_j$  converges, strongly in  $L_{loc}^2(\mathbf{A}_\infty(K^{-3},1); \mathbf{V}_\infty)$  and weakly in  $W_{loc}^{1,2}(\mathbf{A}_\infty(K^{-3},1); \mathbf{V}_\infty)$  to some non-zero section  $v_\infty$  in  $W_{loc}^{1,2}(\mathbf{A}_\infty(K^{-3},1); \mathbf{V}_\infty)$ . In particular, note that for any  $i = 0, 1, 2$  there holds

$$J_K^\gamma(v_\infty, K^{-i}) \leq \limsup_{j \rightarrow \infty} J_K^\gamma(v_j, K^{-i}) = \limsup_{j \rightarrow \infty} J_K^\gamma(\hat{u}_j, K^{-i})$$

with equality for  $i = 1$ , that is for the ‘‘intermediate annulus’’. (Also, note that the non-triviality of the limit section  $v_\infty$  follows from the  $L^2$  convergence in such an annulus.) Hence,

$$J_K^\gamma(v_\infty; K^{-2}) + J_K^\gamma(v_\infty; 1) \leq 2J_K^\gamma(v_\infty; K^{-1}).$$

Lastly, by (63) the dominated convergence theorem allows to conclude that  $v_\infty$  weakly solves  $L_{\mathbf{C}}v_\infty = 0$  on  $\mathbf{A}(K^{-3}, 1)$ . This contradicts Lemma B.1, thereby completing the proof.  $\square$

**Corollary B.5.** *For  $\sigma \in (0, 1)$ ,  $\Lambda > 0$ , let  $K = K(\sigma/2) > 2$  be given by Lemma B.1. Then there exists  $\varepsilon_0(\sigma, \Lambda) > 0$  small enough satisfying the following property.*

*Let  $\gamma \in [-\Lambda, \Lambda]$ ,  $\Sigma$  be an **MSI** in a Riemannian manifold  $(M, g)$ ,  $q \in \text{Sing}(\Sigma)$ ; suppose that, after pulling back to  $T_qM$  using the exponential map,  $\Sigma \cap B^g(q, 4) \subset \text{graph}_{\mathbf{C}}(\phi)$  for some  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$  and*

$$(65) \quad \|g - g_{\text{euc}}\|_{C^3(\mathbb{B}(4))} < \varepsilon_0, \quad \|\phi\|_{C_1^2(\mathbb{B}(4))} < \varepsilon_0.$$

*Assume further*

$$(66) \quad \text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}) \cup \{-(n-2)/2\}) \geq \sigma.$$

*Let  $u \in W_{loc}^{2,2}(\Sigma; \mathbf{V})$ , set  $v = \mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma(u)$  and  $f := \mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma(L_{\Sigma, g}u)$  be satisfying*

$$\mathcal{AR}_q(u) > \gamma, \quad F := \sup_{\ell \geq 0} \| |x|^{2-\gamma-n/2} f \|_{L^2(\mathbf{A}(K^{-\ell-1}, K^{-\ell}))} < +\infty.$$

*Then, we have for every  $\ell \geq 0$ ,*

$$(67) \quad \|v\|_{L^2(\mathbf{A}(K^{-\ell-1}, K^{-\ell}))} \leq C(\Lambda, \sigma) (F + \|v\|_{L^2(\mathbf{A}(K^{-1}, 1))}) \cdot K^{-\ell(n/2+\gamma)}.$$

*Proof.* Firstly, by taking  $\varepsilon_0$  small enough,  $v$  satisfies (62) and

$$|b_0|(x) + |b_1|(x) \leq \varepsilon (|x|^{-1}|v|(x) + |\nabla^\perp v|(x)) + |x||f(x)|$$

where  $\varepsilon$  is again as in the statement of Corollary B.3. Hence, for every  $\ell \in \mathbb{N}$ ,  $v_\ell(x) := v(K^{-\ell}x)$  also solves (62) with  $b_{i,\ell}$  in place of  $b_i$ ,  $i = 0, 1$ , satisfying

$$(68) \quad |b_{0,\ell}|(x) + |b_{1,\ell}|(x) \leq \varepsilon (|x|^{-1}|v_\ell|(x) + |\nabla^\perp v_\ell|(x)) + K^{-2\ell}|x||f_\ell(x)|,$$

on  $\mathbf{A}(\mathbf{0}, K^{-3}, 1)$ , where  $f_\ell(x) := f(K^{-\ell}x)$ .

Clearly, to prove (67), it suffices to show that there exists  $C_0(\Lambda, \sigma)$  so large that

$$(69) \quad \ell \mapsto J_\ell := \max\{K^{2\gamma\ell} J_K^\gamma(v_\ell, 1), C_0^2 F^2\}$$

is monotone non-increasing in  $\ell \geq 1$ . To show this, suppose for contradiction that for some  $\ell_0 \geq 1$ ,  $J_{\ell_0+1} > J_{\ell_0}$ , then

$$\begin{aligned} F &\leq C_0^{-1} C(\Lambda, \sigma) \|v_{\ell_0}\|_{L^2(\mathbf{A}(K^{-2}, K^{-1}))} K^{\gamma\ell_0}, \\ J_K^\gamma(v_{\ell_0}, K^{-1}) &= K^{2\gamma} J_K^\gamma(v_{\ell_0+1}, 1) \geq J_K^\gamma(v_{\ell_0}, 1). \end{aligned}$$

Combining the first inequality and (68), we can take  $C_0(\Lambda, \sigma)$  so big that (63) holds for  $b_{i, \ell_0}$  in place of  $b_i$  and  $v_{\ell_0}$  in place of  $u$ . Hence by Corollary B.3 and the second of such inequalities,

$$\begin{aligned} J_{\ell_0+2} &\geq K^{2(\ell_0+2)\gamma} J_K^\gamma(v_{\ell_0+2}, 1) = K^{2\ell_0\gamma} J_K^\gamma(v_{\ell_0}, K^{-2}) \\ &> K^{2\ell_0\gamma} J_K^\gamma(u_{\ell_0}, K^{-1}) = J_{\ell_0+1}. \end{aligned}$$

Inductively, we thus see  $J_\ell$  would be strictly increasing in  $\ell \geq \ell_0$  (and thus strictly greater than  $C_0^2 F^2$ ). But on the other hand, by definition of  $\mathcal{AR}_q(u)$ , since

$$K^{2\gamma\ell} J_K^\gamma(v_\ell, 1) = J_K^\gamma(v, K^{-\ell}) \rightarrow 0, \quad \text{as } \ell \rightarrow \infty.$$

which implies that when  $\ell$  is large enough,  $J_\ell = C_0^2 F^2$  unless  $F = 0$ . So if  $F > 0$  then we directly violate the aforementioned strict monotonicity, otherwise if  $F = 0$  we get a contradiction from the coexistence of such strict monotonicity with the fact that  $J_\ell \rightarrow 0$  as  $\ell \rightarrow \infty$ .  $\square$

We conclude this appendix with an important ‘‘non-concentration result’’ for Jacobi fields. In view of the applications in Section 4, this corollary is best stated in terms of sections of the normal bundle of  $\Sigma$ . It suffices to note that, when (65) is fulfilled, we can (for  $\varepsilon_0$  sufficiently small) assume that for any  $w \in L_{loc}^2(\Sigma; \mathbf{V})$  and for any  $0 < r, s \leq 2$  there holds

$$\frac{1}{2} \|\mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma(w)\|_{L^2(\mathbf{A}(r,s))} \leq \|w\|_{L^2(A^g(q,r,s))} \leq 2 \|\mathbf{T}_{\mathbf{C}, g_{\text{euc}}}^\Sigma(w)\|_{L^2(\mathbf{A}(r,s))}$$

Hence, we have - *mutatis mutandis* - in particular an estimate fully analogous to (67), but in terms of  $u$  and  $L_{\Sigma, g} u$ , which we will employ in the next proof.

**Corollary B.6.** *Let  $\sigma, \Lambda, K, \varepsilon_0, \Sigma, q$  be the same as Corollary B.5. Then there exists  $\varepsilon_1(\Lambda, \sigma) \in (0, \varepsilon_0)$  with the following property. Suppose (65) holds for  $\varepsilon_1$  in place of  $\varepsilon_0$ ,  $\gamma = 1 - \sigma$  satisfies (66) and  $u$  is a tame Jacobi field on  $\Sigma$ , then there exists  $\mathbf{x} \in T_q M$  such that*

$$(70) \quad |\mathbf{x}| + \|u - \mathbf{x}^\perp\|_{C_{1-\sigma}^1(B^g(q,1))} \leq C(\Lambda, \sigma) \|u\|_{L^2(A^g(q, \varepsilon_1, 2))}.$$

*Proof.* By definition of Jacobi field with slower growth and Corollary 4.17 there exists  $\mathbf{x} \in T_q M$  such that  $\mathcal{AR}_q(u - \mathbf{x}^\perp) \geq 1$ ; hence if we apply Corollary B.5 to  $\tilde{u} := u - \mathbf{x}^\perp$  (also keeping in mind the comments preceding the present proof, as well as the content of Lemma 2.22 and Remark 2.23), we find for every  $\ell \geq 1$

$$(71) \quad \|u - \mathbf{x}^\perp\|_{L^2(A^g(q, K^{-\ell-1}, K^{-\ell}))} \leq C(\Lambda, \sigma) (|\mathbf{x}| + \|u - \mathbf{x}^\perp\|_{L^2(A^g(q, K^{-1}, 1))}) \cdot K^{-\ell(n/2+1-\sigma)}$$

Hence, for every  $s \in (0, 1/2)$ , by virtue of Lemma 4.7 (that is: equation (26), suitably scaled), the Minkowski inequality (employed twice) and (71) there holds

$$\begin{aligned} |\mathbf{x}| &\leq C(\Lambda) s^{-n/2} \|\mathbf{x}^\perp\|_{L^2(A^g(q, s, 2s))} \\ &\leq C(\Lambda) s^{-n/2} (\|u - \mathbf{x}^\perp\|_{L^2(A^g(q, s, 2s))} + \|u\|_{L^2(A^g(q, s, 2s))}) \\ &\leq C(\Lambda, \sigma) s^{1-\sigma} (\|u - \mathbf{x}^\perp\|_{L^2(A^g(q, K^{-1}, 1))} + |\mathbf{x}|) + C(\Lambda) s^{-n/2} \|u\|_{L^2(A^g(q, s, 2s))} \\ &\leq C(\Lambda, \sigma) s^{1-\sigma} (\|u\|_{L^2(A^g(q, K^{-1}, 1))} + |\mathbf{x}|) + C(\Lambda) s^{-n/2} \|u\|_{L^2(A^g(q, s, 2s))}. \end{aligned}$$

Since  $\sigma \in (0, 1)$ , we can take  $s(\Lambda, \sigma) < K^{-1}$  such that  $C(\Lambda, \sigma) s^{1-\sigma} < 1/2$  (in order to allow for absorption on the left-hand side) and then  $\varepsilon_1 \leq s$  so that we eventually derive

$$(72) \quad |\mathbf{x}| \leq C(\Lambda, \sigma) \|u\|_{L^2(A^g(\varepsilon_1, 1))}.$$

Set  $\tilde{u}_\ell(x) := \tilde{u}(K^{-\ell}x)$ ,  $\ell \geq 1$  and recalling that  $|L_{\Sigma, g} \tilde{u}_\ell| \leq C(\Lambda)K^{-2\ell}|\mathbf{x}|$  on  $A^g(q, K^{-2}, 2)$ , if we combine (71) with the classical  $L^p$ -estimate (invoked for  $p$  large enough, depending on the dimension  $n \geq 2$ ) and Sobolev embedding we find

$$\begin{aligned} \|u - \mathbf{x}^\perp\|_{C_0^1(A^g(q, K^{-\ell-1}, K^{-\ell}))} &\leq \|\tilde{u}_\ell\|_{C^1(A^g(q, K^{-1}, 1))} \leq C(\Lambda, \sigma) (\|\tilde{u}_\ell\|_{L^2(A^g(q, K^{-2}, 2))} + K^{-2\ell}|\mathbf{x}|) \\ &\leq C(\Lambda, \sigma)K^{-\ell(1-\sigma)}(|\mathbf{x}| + \|u - \mathbf{x}^\perp\|_{L^2(A^g(q, K^{-1}, 2))}) \end{aligned}$$

for every  $\ell \geq 1$ . Together with (72) this inequality concludes the proof.  $\square$

## APPENDIX C. QUANTITATIVE UNIQUENESS OF TANGENT CONES IN ALL (CO-)DIMENSIONS

In [17, Theorem 6.3], Edelen proved a quantitative version of the uniqueness of tangent cones for minimal hypersurfaces in eight-dimensional manifolds by adapting Simon's argument [38], based on the Lojasiewicz inequality. That same argument can in fact be extended to general dimension and codimension, and leads to the following statement.

**Proposition C.1.** *Given  $0 < \varepsilon < 1$  and  $\Lambda > 0$ , there exists  $\delta = \delta(\varepsilon, \Lambda) > 0$  with the following property.*

*Let  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$  an  $n$ -dimensional minimal cone in  $\mathbb{R}^N$  with normal bundle  $\mathbf{V}$ ,  $g$  be a  $C^4$  metric on  $\mathbb{B}(4)$ ,  $V \in \mathcal{I}_n(\mathbb{B}(4))$  be a  $g$ -stationary integral varifold, and let  $r \in (0, 1/16)$ . Suppose:*

- (i)  $\|g - g_{\text{euc}}\|_{C^4(\mathbb{B}(4))} \leq \delta$ ;
- (ii)  $\theta_V(\mathbf{0}, 2) \leq \theta_V(\mathbf{0}, r) + \delta$ ;
- (iii) *there exists  $s \in (2r, 1/2)$  so that  $V$  is a  $\delta$ - $C^2$  graph over  $\mathbf{C}$  in  $A^g(\mathbf{0}, s, 2s)$ .*

*Then,  $V$  is an  $\varepsilon$ - $C^2$  graph over  $\mathbf{C}$  in  $A^g(\mathbf{0}, 2r, 1)$ .*

*Proof.* Suppose for contradiction, that there exist  $\varepsilon > 0$ ,  $\Lambda > 0$  and a sequence of metrics  $g_j$ , stationary integral varifolds  $V_j \in \mathcal{I}_n(\mathbb{B}(4))$ ,  $r_j \in (0, 1/16)$ ,  $s_j \in (2r_j, 1/2)$  and  $\mathbf{C}_j \in \mathcal{C}_{N,n}(\Lambda)$  such that for every  $j \geq 1$ , properties (i)-(ii)-(iii) hold for  $g_j, V_j, \mathbf{C}_j, r_j, s_j, 1/j$  in place of  $g, V, \mathbf{C}, r, s, \delta$ , but  $V_j$  is not an  $\varepsilon$ - $C^2$  graph over  $\mathbf{C}_j$  in  $A^{g_j}(\mathbf{0}, 2r_j, 1)$ .

By Lemma 4.5, after passing to a subsequence,  $\mathbf{C}_j$  smoothly converges to some  $\mathbf{C}_\infty \in \mathcal{C}_{N,n}(\Lambda)$  as  $j \rightarrow \infty$ ; then there exists a sequence  $\{\delta_j\}_{j \geq 1}$  with  $\delta_j \searrow 0$  such that  $V_j$  is a  $\delta_j$ - $C^2$  graph over  $\mathbf{C}_\infty$  in  $A^{g_j}(\mathbf{0}, s_j, 2s_j)$ . Let  $\bar{s}_j^- \leq s_j < \bar{s}_j^+$  be radii such that  $\mathbb{A}(\bar{s}_j^-, \bar{s}_j^+) \subset A^{g_j}(\mathbf{0}, r_j, 2)$  is the largest annulus centered at  $\mathbf{0}$  in which  $V_j$  is an  $\varepsilon'$ - $C^2$  graph over  $\mathbf{C}_\infty$ , where  $\varepsilon' = \varepsilon'(\mathbf{C}_\infty) < \varepsilon$  is to be determined at a later stage along the course of the argument (independently of  $j$ ).

Our goal is to show that for infinitely many indices  $j \geq 1$  there hold in fact the inequalities

$$(73) \quad \bar{s}_j^- \leq 3r_j/2, \quad \bar{s}_j^+ \geq 3/2.$$

Note that since  $\mathbf{C}_j \rightarrow \mathbf{C}_\infty$  smoothly away from  $\mathbf{0}$ , this immediately implies for all such indices (possibly with finitely many initial exceptions)  $V_j$  is an  $\varepsilon$ - $C^2$  graph over  $\mathbf{C}_j$  in  $A^g(\mathbf{0}, 2r_j, 1)$ , which is a contradiction.

To prove (73), we shall apply [17, Theorem 11.1]. Recalling the notation for the scaling map  $\eta_\lambda : x \mapsto \lambda x$ , we first observe that when  $j \rightarrow \infty$ ,  $s_j^{-2} \eta_{s_j}^* g_j$   $C^4$ -converges to  $g_{\text{euc}}$  and by Allard Compactness Theorem, after passing to a subsequence,  $V_j' := (\eta_{s_j}^{-1})_\# V_j$   $\mathbf{F}$ -converges to some  $g_{\text{euc}}$ -stationary integral varifold  $V_\infty'$  in  $\mathbb{A}(\hat{r}_-, \hat{r}^+)$ , where

$$\hat{r}^- := \lim_{j \rightarrow \infty} r_j/s_j, \quad \hat{r}^+ := \lim_{j \rightarrow \infty} 2/s_j$$

so that  $r \mapsto \theta_{V'_\infty}(\mathbf{0}, r)$  is constant in  $r \in (\hat{r}^-, \hat{r}^+)$ , and on the other hand  $V'_\infty$  coincides with the (multiplicity one) varifold associated to  $\mathbf{C}_\infty$  in  $\mathbb{A}(1, 2)$ . Note, in particular, that we allow  $\hat{r}^\pm$  being 0 or  $+\infty$ .

Hence by the rigidity case of monotonicity formula,  $V'_\infty$  is a regular cone with multiplicity one (namely  $\mathbf{C}_\infty$ ) in  $\mathbb{A}(\hat{r}^-, \hat{r}^+)$ ; then by Allard Regularity Theorem,  $V'_j$  converges smoothly, and with unit multiplicity, to  $\mathbf{C}_\infty$  in that same annulus. This implies

$$\lim_{j \rightarrow \infty} s_j^\pm = \hat{r}^\pm,$$

where  $s_j^\pm := \bar{s}_j^\pm / s_j$ , for either consistent choice of signs. Note that in particular, if  $\hat{r}^- > 0$ , then  $\bar{s}_j^- \leq 3r_j/2$  for all sufficiently large  $j$ ; and, analogously, if  $\hat{r}^+ < +\infty$ , then  $\bar{s}_j^+ \geq 3/2$  for all sufficiently large  $j$ . To prove (73), we are left to deal with the case when either  $\hat{r}^- = 0$  or  $\hat{r}^+ = +\infty$ .

Let  $\Sigma := \mathbf{C}_\infty \cap \mathbb{S}^{N-1}$ , and let here (with slight abuse of notation)  $\nabla^\perp$  denote the normal connection on  $T^\perp \Sigma$  in  $\mathbb{S}^{N-1}$ ; let  $u_j : \mathbf{A}(s_j^-, s_j^+) \rightarrow T^\perp \mathbf{C}_\infty$  be the graphical section defining  $V'_j$  over  $\mathbf{C}_\infty$  in  $\mathbb{A}(s_j^-, s_j^+)$ . Furthermore, set  $t := -\log r$ ,  $T_j^\pm := -\log(s_j^\pm)$ ,  $v_j(t, \theta) := r^{-1}u_j(r\theta)$  defined on  $\Sigma \times (T_j^-, T_j^+)$ . Note that for all sufficiently large  $j$  we have

$$|v_j|_2^*(t) := \sum_{i+j \leq 2, i, j \geq 0} |\partial_t^i (\nabla^\perp)^j v_j(\cdot, t)|_{C^0(\Sigma)} \leq C(\mathbf{C}_\infty) \delta_j, \quad \forall t \in [0, \ln 2],$$

and by taking  $\varepsilon'(\mathbf{C}_\infty)$  small enough, following the same argument as [17, Page 29 - 32], the assumptions in [17, Theorem 11.1] are satisfied by  $v_j$  on  $\Sigma \times [0, T_j^+)$  with  $n$  in place of  $m$ , which - in turn - implies that for some  $\alpha = \alpha(\mathbf{C}_\infty) \in (0, 1/2)$ ,

$$(74) \quad |v_j|_2^*(t) \leq C(\mathbf{C}_\infty) \delta_j^\alpha, \quad \forall t \in (0, T_j^+).$$

Also note that the assumption  $m > 0$  in [17, Theorem 11.1] can be replaced by  $|m| \neq 0$ . In fact, in [17] and [38], the only place where  $m > 0$  is used is in the proof of the stability inequality [38, Lemma 1], which lead to the  $L^1$  estimate [38, (6.34)] on the interval where the neutral mode dominates  $\partial_t v$ . This estimate can be still derived when  $m < 0$  following [40, 4.26-4.32]. By considering  $\check{v}_j(t, \theta) := v_j(\ln 2 - t, \theta)$  and repeating the argument above, the assumptions in [17, Theorem 11.1] are satisfied by  $\check{v}_j$  on  $\Sigma \times [0, \ln 2 - T_j^-)$  with  $-n$  in place of  $m$ , which again implies that for some  $\alpha' = \alpha'(\mathbf{C}_\infty) \in (0, 1/2)$  there holds

$$(75) \quad |\check{v}_j|_2^*(t) \leq C(\mathbf{C}_\infty) \delta_j^{\alpha'}, \quad \forall t \in (0, \ln 2 - T_j^-).$$

Combining (74) and (75), we conclude that  $V'_j$  is a  $\delta_j'$ - $C^2$  graph over  $\mathbf{C}_\infty$  in  $\mathbb{A}(s_j^-, s_j^+)$  for some sequence  $\delta_j' \rightarrow 0$ . Therefore, again by the Allard Compactness Theorem, the characterization of the rigidity case in the monotonicity formula and Allard Regularity Theorem, for any infinite subset  $I \subset \mathbb{N}$ ,  $V_j^\pm := (\eta_{s_j^\pm}^{-1})_\# V'_j = (\eta_{\bar{s}_j^\pm}^{-1})_\# V_j$  will (in both cases) still  $C_{loc}^2$ -converge to  $\mathbf{C}_\infty$  as  $I \ni j \rightarrow \infty$  in the annuli

$$\mathbb{A}^\pm(I) := \bigcup_{k \geq 1} \bigcap_{j \in I, j \geq k} \eta_{\bar{s}_j^\pm}^{-1}(A^{g_j}(\mathbf{0}, r_j, 2)).$$

If (73) fails for all sufficiently large  $j$ , then there exists an infinite subset  $\bar{I} \subset \mathbb{N}$  such that either  $\mathbb{A}^-(\bar{I}) \supset \mathbb{A}(2/3, 2)$ , or  $\mathbb{A}^+(\bar{I}) \supset \mathbb{A}(1/2, 4/3)$ . But by tracing back the scaling factor, in the former case,  $V_j$  is an  $\varepsilon_j^-$ - $C^2$  graph over  $\mathbf{C}_\infty$  in

$$A^{g_j}(\bar{s}_j^-, \bar{s}_j^+) \cup A^{g_j}(3\bar{s}_j^-/4, 2s_j^-) \supset A^{g_j}(3\bar{s}_j^-/4, \bar{s}_j^+);$$

for  $j \in \bar{I}$  and  $\varepsilon_j^- \rightarrow 0$ . Similarly,  $\mathbb{A}^+(\bar{I}) \supset \mathbb{A}(1/2, 4/3)$  implies that  $V_j$  is an  $\varepsilon_j^+$ - $C^2$  graph over  $\mathbf{C}_\infty$  in

$$A^{g_j}(\bar{s}_j^-, \bar{s}_j^+) \cup A^{g_j}(\bar{s}_j^+/4, 5\bar{s}_j^+/5) \supset A^{g_j}(\bar{s}_j^-, 5\bar{s}_j^+/4);$$

for  $j \in \bar{I}$  and  $\varepsilon_j^+ \rightarrow 0$ .

Thus, either way, when  $j$  is large enough  $\mathbb{A}(\bar{s}_j^-, \bar{s}_j^+)$  would not be the largest annulus in  $A^{g_j}(\mathbf{0}, r_j, 2)$  centered at  $\mathbf{0}$  in which  $V_j$  is an  $\varepsilon_j'$ - $C^2$  graph over  $\mathbf{C}_\infty$ . This is a contradiction.  $\square$

This in particular implies the following convergence result at all scales, which we will repeatedly employ in Appendix D.

**Corollary C.2.** *Let  $\Lambda > 0$ ,  $\{\Sigma_j\}_{j \geq 1}$  be a sequence of  $n$ -dimensional **MSI** in  $(\mathbb{B}(4), g_j)$  with  $\text{Sing}(\Sigma_j) = \{\mathbf{0}\}$  and  $\tau_{\Sigma_j, g_j} \geq \Lambda^{-1} \rho_j$  in  $\mathbb{B}(3)$ , where  $\rho_j$  denotes the distance to  $\mathbf{0}$  in metric  $g_j$ . Suppose that, as one lets  $j \rightarrow \infty$ , there holds*

- (i)  $\|g_j - g_{\text{euc}}\|_{C^4(\mathbb{B}(4))} \rightarrow 0$ ,
- (ii)  $\theta_{|\Sigma_j|}(\mathbf{0}, 2) - \theta_{|\Sigma_j|}(\mathbf{0}) \rightarrow 0$ .

*Then there exists some  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$  such that after passing to a subsequence (which we do not rename), both  $\Sigma_j$  and its tangent cone  $\mathbf{C}_j$  at  $\mathbf{0}$  are all  $\varepsilon_j$ - $C^2$  graphs over  $\mathbf{C}$  in  $\mathbb{B}(1)$  for some sequence  $\{\varepsilon_j\}$  such that  $\varepsilon_j \rightarrow 0$  as one lets  $j \rightarrow \infty$ .*

*Proof.* First note that, by the constraint on regularity scale function,  $\mathbf{C}_j \in \mathcal{C}_{N,n}(\Lambda)$ . By Lemma 4.5,  $\mathbf{C}_j$  subconverges to some  $\mathbf{C}$  smoothly away from  $\mathbf{0}$ . This implies, in particular, that for any  $j \geq j_0$  large enough  $\Sigma_j$  is a  $\kappa_j$ - $C^2$  graph over  $\mathbf{C}$  in  $A^{g_j}(\mathbf{0}, s_j, 2s_j)$  for suitable sequences  $\{\kappa_j\}_{j \geq j_0}$  and  $\{s_j\}_{j \geq j_0}$  both tending to zero as one lets  $j \rightarrow \infty$ .

Hence, set  $\varepsilon_j = 1/j$  let  $\delta_j := \delta(1/j, \Lambda)$  be as afforded by Proposition C.1. Thanks to the assumptions (i), (ii), for any  $j \geq j_0$  there exists  $i_0 = i_0(j)$  such that for any  $i \geq i_0(j)$  the hypotheses of Proposition C.1 are satisfied by  $\Sigma_i$  (and  $g_i$ ) for any  $i \geq i_0(j)$ . Thus the sequence  $\{\Sigma_{i_0(j)}\}_{j \geq 1}$  has the desired properties.  $\square$

#### APPENDIX D. COMPARING GRAPHICAL SECTIONS DEFINING SINGULAR MINIMAL SUBMANIFOLDS

**Lemma D.1.** *There exists  $\delta_0(N) \in (0, 1/4)$  and  $C = C(N) > 0$  such that the following statements hold true. Let  $\Sigma^0 \subset \mathbb{R}^N$  be a  $\delta_0$ - $C^3$  graph over  $\mathbb{B}^n(3) \times \{\mathbf{0}\}$ ,  $g$  be a  $C^4$  Riemannian metric on  $\mathbb{B}^N(4)$  with  $\|g - g_{\text{euc}}\|_{C^4} \leq \delta_0$ ; let  $\Sigma^i$  be  $\delta_0$ - $C^2$  graphs over  $\Sigma^0$  in  $(B^g(2), g)$  for  $i = 1, 2$ ; and let  $\mathbf{x} \in \mathbb{R}^N$  be a vector with  $|\mathbf{x}| \leq \delta_0$ . Then*

- (i) *for all  $i, j \in \{0, 1, 2\}$ , both  $\Sigma^j$  and  $\Sigma^j + \mathbf{x}$  are  $\sqrt{\delta_0}$ - $C^3$  graphs over  $\Sigma^i$  in  $B^g(1)$ ; we denote by  $u_i^j$  and  $\tilde{u}_i^j$  the graphical section of  $\Sigma^j$  and  $\Sigma^j + \mathbf{x}$  over  $\Sigma_i$ , respectively;*
- (ii) *for every  $i, j \in \{0, 1, 2\}$  there holds,*

$$(76) \quad \frac{1}{2} \|u_0^2 - u_0^1\|_{C^0(B^g(1/8) \cap \Sigma^0)} \leq \|u_1^2\|_{C^0(B^g(1/4) \cap \Sigma^1)} \leq 2 \|u_0^2 - u_0^1\|_{C^0(B^g(1/2) \cap \Sigma^0)};$$

$$(77) \quad \|\tilde{u}_i^j - u_i^j - \mathbf{x}^{\perp \Sigma^i, g}\|_{C^0(B^g(1/4) \cap \Sigma^i)} \leq C (\|u_i^j\|_{C^0(B^g(1/2) \cap \Sigma^i)} + |\mathbf{x}|) |\mathbf{x}|.$$

*Proof.* (i) follows directly from implicit function theorem. To prove the estimates in (ii), we use  $\Phi_i(x, v)$  to denote the ‘‘Fermi coordinates’’ of  $\Sigma^i$  under  $g$ : more precisely, given  $v \in C^2(\Sigma^i, \mathbf{V}_i)$  (where  $\mathbf{V}_i$  denotes the normal bundle of  $\Sigma^i$  with respect to the metric  $g$ , and  $x \in \Sigma^i$ , we write

$$\Phi_i(x, v) := \exp_x^g(v(x)).$$

To prove the left-hand side inequality of (76), consider the map

$$\phi : \Sigma^0 \times [0, 1] \rightarrow \mathbb{R}^N, \quad (x, s) \mapsto \Phi_1(\Phi_0(x, u_0^1), su_0^2).$$

Then by possibly taking  $\delta_0$  smaller, for every  $s \in [0, 1]$ ,  $\phi(\Sigma^0, s)$  is a  $\sqrt{\delta_0}$ - $C^2$  graph over  $\Sigma^0$  in  $B^g(1)$  with graphical section  $v_s$ . Moreover,  $v_1 = u_0^2$ ,  $v_0 = u_0^1$  and

$$\begin{aligned} \|u_0^2 - u_0^1\|_{C^0(B^g(1/8) \cap \Sigma^0)} &\leq \sup_{s \in [0, 1]} \|\partial_s v_s\|_{C^0(B^g(1/8) \cap \Sigma^0)} \leq (1 + C\delta_0) \|\partial_s \phi\|_{C^0((B^g(1/4) \cap \Sigma^0) \times [0, 1])} \\ &\leq (1 + C\delta_0) \|u_0^2\|_{C^0(B^g(1/2) \cap \Sigma^1)}. \end{aligned}$$

where the second inequality follows from the implicit function theorem and the last inequality follows from a direct computation. Taking  $\delta_0$  small enough finishes the proof.

To prove the right-hand side of (76), one can repeat the argument above for

$$\psi : \Sigma^1 \cap B^g(1) \times [0, 1] \rightarrow \mathbb{R}^N, \quad (x, s) \mapsto \Phi_0(\Pi_0(x), su_0^2 + (1-s)u_0^1),$$

where  $\Pi_0$  denotes the nearest point projection onto  $\Sigma^0$  under  $g$ .

Lastly, in order to prove (77), it suffices to only deal with the case when  $j = 1, i = 0$  since the other cases are similar. Consider

$$\varphi : \Sigma^0 \times [0, 1]^2 \rightarrow \mathbb{R}^N, \quad (x, s, t) \mapsto \Phi_0(x, su_0^1) + t\mathbf{x}.$$

Again, by possibly taking  $\delta_0$  even smaller, for every  $s, t \in [0, 1]$ ,  $\varphi(\Sigma^0, s, t)$  is a  $C^2$  graph over  $\Sigma^0$  in  $B^g(1)$  with graphical section  $w_{s,t}$ . Moreover,  $w_{1,1} = \tilde{u}_0^1$ ,  $w_{1,0} = u_0^1$  and  $\partial_t w_{0,0} = \mathbf{x}^{\perp \Sigma^0, g}$ . Since

$$w_{1,1} - w_{1,0} - \partial_t w_{0,0} = \int_0^1 \left( \int_0^1 \frac{d}{d\lambda} (\partial_t w_{\lambda, t\lambda}) d\lambda \right) dt = \int_0^1 \left( \int_0^1 \partial_{st}^2 w_{\lambda, t\lambda} + t \partial_{tt}^2 w_{\lambda, t\lambda} d\lambda \right) dt.$$

we then find by the same reasoning as above,

$$\begin{aligned} \|\tilde{u}_0^1 - u_0^1 - \mathbf{x}^{\perp \Sigma^0, g}\|_{C^0(B^g(1/4) \cap \Sigma^0)} &\leq \sup_{s, t \in [0, 1]} (\|\partial_{st}^2 w_{s,t}\|_{C^0(B^g(1/4) \cap \Sigma^0)} + \|\partial_{tt}^2 w_{s,t}\|_{C^0(B^g(1/4) \cap \Sigma^0)}) \\ &\leq (\|\partial_{st}^2 \varphi\|_{C^0((B^g(1/4) \cap \Sigma^0) \times [0, 1]^2)} + \|\partial_{tt}^2 \varphi\|_{C^0((B^g(1/4) \cap \Sigma^0) \times [0, 1]^2)}) \\ &\leq C (\|u_0^1\|_{C^0(B^g(1/2) \cap \Sigma^0)} + |\mathbf{x}|) |\mathbf{x}|, \end{aligned}$$

as claimed.  $\square$

Next, we show that if two **MSI** with a common singular point, each modeled on a minimal cone, are sufficiently close to each other, then each can be written as a graph of the other.

**Lemma D.2.** *Let  $\gamma, \sigma \in (0, 1)$  and  $\Lambda > 0$ . Then there exist constants  $\delta_1 = \delta_1(\gamma, \sigma, \Lambda) \in (0, 1)$ ,  $K = K(\sigma) > 2$  and  $C_1(\gamma, \sigma, \Lambda) > 0$  such that the following holds.*

*If  $\Sigma^0$  and  $\Sigma^1$  are **MSI** in  $(\mathbb{B}(4), g^0)$  and  $(\mathbb{B}(4), g^1)$ , respectively, satisfying:*

- (i) *for  $i \in \{0, 1\}$ ,  $\|g^i - g_{\text{euc}}\|_{C^4} \leq \delta_1$ ;*
- (ii) *for  $i \in \{0, 1\}$ ,  $\|\Sigma^i\|_{g^i}(\mathbb{B}(4)) \leq \Lambda$ ;*
- (iii) *for  $i \in \{0, 1\}$ ,  $\mathbf{0} \in \text{Sing}(\Sigma^i)$ , the cone densities satisfy:*

$$\theta_{|\Sigma^i|}(\mathbf{0}, 2) - \theta_{|\Sigma^j|}(\mathbf{0}) \leq \delta_1, \quad \theta_{|\Sigma^0|}(\mathbf{0}) = \theta_{|\Sigma^1|}(\mathbf{0}),$$

and

$$\text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}_0(\Sigma^i)) \cup \{-(n-2)/2\}) \geq \sigma;$$

- (iv) *for  $i \in \{0, 1\}$ , the regularity scale function, defined in Definition 2.13, satisfies*

$$\mathbf{r}_{\Sigma^i, g^i} \geq \Lambda^{-1} \text{dist}_{g^i}(\cdot, \mathbf{0});$$

(v) the graphical radius, defined in Definition 2.18, satisfies

$$\mathbf{r}_{\Sigma^0, g^0}^{\Sigma^1} \leq \delta_1;$$

(vi) the graphical section,  $u := \mathbf{G}_{\Sigma^0, g^0}^{\Sigma^1} \in L^\infty(\Sigma^0)$ , defined in Definition 2.18, satisfies

$$\|u\|_{L^2_\gamma(A^{g^0}(\mathbf{0}, 1, 2) \cap \Sigma^0)} \leq \delta_1.$$

Then in  $\mathbb{B}(1)$ ,  $\Sigma^1$  is a global  $C^2$  graph over  $(\Sigma^0, g^0)$ , i. e.,  $\mathbf{r}_{\Sigma^0, g^0}^{\Sigma^1} = 0$ , and the graphical function  $u$  satisfies the estimate

$$\|u\|_{C^1_\gamma(B^{g^0}(\mathbf{0}, 1) \cap \Sigma^0)} \leq C_1(\gamma, \sigma, \Lambda) \cdot \left( \|g^0 - g^1\|_{C^4} + \|u\|_{L^2_\gamma(A^{g^0}(\mathbf{0}, K^{-3}, 2) \cap \Sigma^0)} \right).$$

*Remark D.3.* Condition (v) is not redundant. Indeed, from (i) to (iv), with sufficiently small  $\delta_1$ , we can only conclude that  $\Sigma^0$  and  $\Sigma^1$  are close to minimal cones,  $\mathbf{C}_0$  and  $\mathbf{C}_1$ , respectively, which are centered at  $\mathbf{0}$  and have the same density. However, the links of  $\mathbf{C}_0$  and  $\mathbf{C}_1$  might be far apart. Therefore, condition (v) ensures that these two cones are, in fact, graphical with respect to each other.

*Proof.* Let  $K(\sigma)$  be as in Lemma B.1. It suffices to show that for any sequence of  $\{\gamma_j \in [-\Lambda, \Lambda]\}_{j \geq 1}$ , any sequence of pairs of metrics  $\{(g_j^0, g_j^1)\}_{j \geq 1}$  and any sequence of pairs  $\{(\Sigma_j^0, \Sigma_j^1)\}$ , where  $\Sigma_j^i \subset (\mathbb{B}(4), g_j^i)$  is an **MSI** for  $i = \{0, 1\}$ , if (i)-(vi) in the lemma hold for  $g_j^i, \Sigma_j^i, \gamma_j, 1/j$  in place of  $g^i, \Sigma^i, \gamma, \delta_1$ , then the conclusion of the lemma holds for all sufficiently large  $j$  and for some finite constant  $C > 0$ . Without loss of generality, we assume that  $\Sigma_j^0 \neq \Sigma_j^1$  for all  $j$ .

First notice that by Corollary C.2, as  $j \rightarrow \infty$ , up to subsequences,  $\Sigma_j^i$  are  $\varepsilon_j$ - $C^2$  graphs over  $\mathbf{C}^i$  for some  $\varepsilon_j \rightarrow 0$  and  $\mathbf{C}^i \in \mathcal{C}_{N,n}(\Lambda)$ , where  $i = 0, 1$ . And since by (vi),  $\Sigma_j^1$  and  $\Sigma_j^0$  are approaching each other in  $A^{g^0}(\mathbf{0}, 1, 2)$ , we must have  $\mathbf{C}^0 = \mathbf{C}^1 =: \mathbf{C}$ . Hence by (76) and standard elliptic estimates, for all  $j$  large enough  $\Sigma_j^1$  is a  $\tilde{\varepsilon}_j$ - $C^2$  graph over  $\Sigma_j^0$  in  $B^{g^0}(\mathbf{0}, 3/2)$  where  $\tilde{\varepsilon}_j \rightarrow 0$  as one lets  $j \rightarrow \infty$ . Said  $u_j$  the graphical section, it remains to estimate  $\|u_j\|_{C^1_\gamma(B^{g^0}(\mathbf{0}, 1) \cap \Sigma^0)}$ . By the graphical parametrization of  $\Sigma_j^0$  using  $\mathbf{C}$ , we view  $u_j$  as a section on a subdomain of  $\mathbf{C}$ . Then by suitably exploiting Proposition A.1 and Proposition A.3, the same argument as in the proof of Corollary B.5 leads to the desired conclusion.  $\square$

In general, we may consider the case where the two **MSI** have distinct singular points that are separated by a sufficiently small distance. Similarly, we can conclude that they are graphs to each other, modulo some translation-like functions.

**Corollary D.4.** *Given  $\gamma, \sigma \in (0, 1)$  and  $\Lambda > 0$ , let  $\delta_1 = \delta_1(\gamma, \sigma, \Lambda) > 0$  and  $K = K(\sigma) > 2$  be as in Lemma D.2. There exist constants  $\delta_2 = \delta_2(\gamma, \sigma, \Lambda) \in (0, \delta_1)$ ,  $\tilde{\delta}_2 = \tilde{\delta}_2(\gamma, \sigma, \Lambda) \in (0, \delta_2)$  and  $C_2 = C_2(\gamma, \sigma, \Lambda) > 0$  such that the following holds.*

*If  $\Sigma^0$  and  $\Sigma^1$  are both **MSI** in  $(\mathbb{B}(4), g)$  satisfying*

- (i)  $\|g - g_{\text{euc}}\|_{C^5} \leq \tilde{\delta}_2$ ;
- (ii) for  $i \in \{0, 1\}$ ,  $\|\Sigma^i\|_g(\mathbb{B}(4)) \leq \Lambda$ ;
- (iii) for  $i \in \{0, 1\}$ , there exist  $\mathbf{x}^i \in \text{Sing}(\Sigma^i) \cap \mathbb{B}_{\tilde{\delta}_2}$  such that the cone densities satisfy:

$$\theta_{|\Sigma^i|}(\mathbf{x}^i, 2) - \theta_{|\Sigma^i|}(\mathbf{x}^i) \leq \tilde{\delta}_2, \quad \theta_{|\Sigma^0|}(\mathbf{x}^0) = \theta_{|\Sigma^1|}(\mathbf{x}^1),$$

and

$$\text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}_{\mathbf{x}^i}(\Sigma^i)) \cup \{-(n-2)/2\}) \geq \sigma;$$

(iv) for  $i \in \{0, 1\}$ , the regularity scale function, defined in Definition 2.13, satisfies

$$\mathbf{r}_{\Sigma^i, g} \geq \Lambda^{-1} \operatorname{dist}_g(\cdot, \mathbf{x}^i);$$

(v) the graphical radius, defined in Definition 2.18, satisfies

$$\mathbf{r}_{\Sigma^0, g}^{\Sigma^1} \leq \tilde{\delta}_2;$$

(vi) the graphical section,  $u := \mathbf{G}_{\Sigma^0, g}^{\Sigma^1} \in L^\infty(\Sigma^0)$ , defined in Definition 2.18, satisfies

$$\|u\|_{L^2_\gamma(A^\theta(\mathbf{0}, 1, 2) \cap \Sigma^0)} \leq \tilde{\delta}_2,$$

then we have the following estimates,

$$(78) \quad |\mathbf{x}^1 - \mathbf{x}^0| \leq \mathbf{r} := \mathbf{r}_{\Sigma^0, g}^{\Sigma^1} \leq C_2 \|u\|_{L^2(A^\theta(\mathbf{x}^0, \delta_2, 2) \cap \Sigma^0)} < 2^{-2024},$$

$$(79) \quad \|u - (\mathbf{x}^1 - \mathbf{x}^0)^{\perp_{\Sigma^0, g}}\|_{C^0(A^\theta(\mathbf{x}^0, s, 2s) \cap \Sigma^0)} \leq C_2 \left( \|u\|_{L^2(A^\theta(\mathbf{x}^0, \delta_2, 2) \cap \Sigma^0)} \cdot s^\gamma + s^{-1} |\mathbf{x}^1 - \mathbf{x}^0|^2 \right).$$

for every  $s \in (C_2 \mathbf{r}, 1/2)$ .

In particular, for some  $\tilde{C}_2(\gamma, \sigma, \Lambda) = \tilde{C}_2(C_2) > 0$ ,

$$(80) \quad \|u\|_{C^0(A^\theta(\mathbf{x}^0, C_2 \mathbf{r}, 1) \cap \Sigma^0)} \leq \tilde{C}_2 \|u\|_{L^2(A^\theta(\mathbf{x}^0, \delta_2, 2) \cap \Sigma^0)}.$$

*Proof.* Fix  $\Lambda$  and  $\sigma$ , and the corresponding constants  $\delta_1$ ,  $C_1$  and  $K$  from Lemma D.2.

**Claim 1.** There exists  $\delta' > 0$  and  $C' > 1$  such that if  $\tilde{\delta}_2 \leq \delta'$  and  $r \in (0, 1)$ , then

$$(81) \quad |\mathbf{x}^1 - \mathbf{x}^0| \leq \mathbf{r}_{\Sigma^0, g}^{\Sigma^1} \leq C' \|(\mathbf{x}^1 - \mathbf{x}^0)^{\perp_{\Sigma^0, g}}\|_{C^0(A^\theta(\mathbf{x}^0; r, 2r) \cap \Sigma^0)}.$$

*Proof.* Since  $\mathbf{x}^i$  is a singular point of  $\Sigma^i$ , it is clear by definition that

$$|\mathbf{x}^1 - \mathbf{x}^0| \leq \mathbf{r}_{\Sigma^0, g}^{\Sigma^1}.$$

To prove the right-hand side inequality, suppose for the sake of contradiction, that there exist sequences  $\{\Sigma_j^0\}_{j \geq 1}$  and  $\{\Sigma_j^1\}_{j \geq 1}$  where  $\Sigma_j^0$  and  $\Sigma_j^1$  are **MSI** in  $(\mathbb{B}(4), g_j)$  satisfying conditions (i) - (vi) in the corollary with  $1/j$  in place of  $\tilde{\delta}_2$ , but

$$(82) \quad \mathbf{r}_{\Sigma_j^0, g_j}^{\Sigma_j^1} > j \|(\mathbf{x}_j^1 - \mathbf{x}_j^0)^{\perp_{\Sigma_j^0, g_j}}\|_{C^0(A^{g_j}(\mathbf{x}_j^0, s_j, 2s_j) \cap \Sigma_j^0)}.$$

for some  $s_j \in (0, 1)$ . First note that for any  $j$  sufficiently large, we must have

$$(83) \quad \sqrt{j} \|(\mathbf{x}_j^1 - \mathbf{x}_j^0)^{\perp_{\Sigma_j^0, g_j}}\|_{C^0(A^{g_j}(\mathbf{x}_j^0, s_j, 2s_j) \cap \Sigma_j^0)} \geq |\mathbf{x}_j^1 - \mathbf{x}_j^0|.$$

This is because otherwise, by (iii) and Corollary C.2,  $\eta_{\mathbf{x}_j^0, s_j}^{-1}(\Sigma_j^0)$  would subconverge to some nontrivial regular minimal cone  $\mathbf{C} \in \mathcal{C}_{N, n}(\Lambda)$ , but the failure of (83) implies that the subsequential limit  $\hat{\mathbf{x}} \in \mathbb{R}^N$  of  $(\mathbf{x}_j^1 - \mathbf{x}_j^0)/|\mathbf{x}_j^1 - \mathbf{x}_j^0|$  satisfies  $\hat{\mathbf{x}}^{\perp_{\mathbf{C}, g_{\text{euc}}}} = 0$ . Since  $\hat{\mathbf{x}} \neq 0$ , this means  $\mathbf{C}$  splits in  $\hat{\mathbf{x}}$  direction, contradicting the fact that  $\mathbf{C}$  is a regular cone.

Now let  $\mathbf{r}_j := \mathbf{r}_{\Sigma_j^0, g_j}^{\Sigma_j^1}$ : by (iii), Corollary C.2, (82) and (83),  $\hat{\Sigma}_j^i := \eta_{\mathbf{x}_j^0, \mathbf{r}_j}^{-1}(\Sigma_j^i)$  subconverges to some  $\mathbf{C} \in \mathcal{C}_{N, n}(\Lambda)$  for both  $i = 0, 1$ . It then follows from Allard's regularity theorem [2] that for sufficiently large  $j$ ,  $\hat{\Sigma}_j^1$  is a  $\delta_0^2/2$ - $C^2$  graph over  $\hat{\Sigma}_j^0$ , which contradicts the definition of  $\mathbf{r}_{\Sigma_j^0, g_j}^{\Sigma_j^1}$ . Therefore, we conclude the claim.  $\square$

In  $(\mathbb{B}(4), g_{\text{euc}})$ , let  $\tau_{\mathbf{x}^1 - \mathbf{x}^0}$  be the translation map which translates  $\mathbf{x}^0$  to  $\mathbf{x}^1$ . Define

$$\tilde{\Sigma}^1 := \tau_{\mathbf{x}^1 - \mathbf{x}^0}^{-1}(\Sigma^1), \quad \tilde{g} := (\tau_{\mathbf{x}^1 - \mathbf{x}^0})^*(g),$$

and let the corresponding graphical function be denoted by  $\tilde{u} := \mathbf{G}_{\Sigma^0, g}^{\tilde{\Sigma}^1}$ .

Clearly, there exists  $\delta'' \in (0, \delta')$  such that if we choose  $\tilde{\delta}_2 \leq \delta''$ , then the pair  $(\Sigma^0, \tilde{\Sigma}^1)$  satisfies conditions (i)-(vi) in Lemma D.2 for the pair  $(\Sigma^0, \Sigma^1)$ . In particular,  $\tilde{\Sigma}^1$  is a global  $C^2$  graph over  $(\Sigma^0, g)$  with graphical section  $\tilde{u}$  satisfying

$$(84) \quad \|\tilde{u}\|_{C_\gamma^1(B^g(\mathbf{x}^0, 1) \cap \Sigma^0)} \leq C''(\sigma, \Lambda) \left( |\mathbf{x}^1 - \mathbf{x}^0| + \|\tilde{u}\|_{L_\gamma^2(A^g(\mathbf{x}^0, K^{-3}/10, 1/5) \cap \Sigma^0)} \right),$$

since by (i) and the definition of  $\tilde{g}$ , we have  $\|\tilde{g} - g\|_{C^4} \leq C(N)|\mathbf{x}^1 - \mathbf{x}^0|$ .

**Claim 2.** There exist  $\delta'''(\Lambda, \sigma) \in (0, \delta'')$ ,  $C'''(\Lambda, \sigma), C''''(\Lambda, \sigma) > 1$  such that if  $\tilde{\delta}_2 \leq \delta'''$  and  $r \in ((C''')^2 \mathbf{r}_{\Sigma^0, g}^{\Sigma^1}, 1/4)$ , then we have,

$$(85) \quad \|u - \tilde{u} - (\mathbf{x}^1 - \mathbf{x}^0)^{\perp_{\Sigma^0, g}}\|_{C^0(A^g(\mathbf{x}^0, 2r, 3r) \cap \Sigma^0)} \leq C''' \left( \|\tilde{u}\|_{C^0(A^g(\mathbf{x}^0, r, 4r) \cap \Sigma^0)} + |\mathbf{x}^1 - \mathbf{x}^0| \right) r^{-1} \cdot |\mathbf{x}^1 - \mathbf{x}^0|,$$

$$(86) \quad \|u - \tilde{u} - (\mathbf{x}^1 - \mathbf{x}^0)^{\perp_{\Sigma^0, g}}\|_{C^0(A^g(\mathbf{x}^0, 2r, 3r) \cap \Sigma^0)} \leq C'''' \left( \|u\|_{C^0(A^g(\mathbf{x}^0, r, 4r) \cap \Sigma^0)} + |\mathbf{x}^1 - \mathbf{x}^0| \right) r^{-1} \cdot |\mathbf{x}^1 - \mathbf{x}^0|,$$

and

$$(87) \quad |\mathbf{x}^1 - \mathbf{x}^0| \leq C'''' \left( \|\tilde{u}\|_{C^0(A^g(\mathbf{x}^0, r, 4r) \cap \Sigma^0)} + \|u\|_{C^0(A^g(\mathbf{0}, 2r, 3r) \cap \Sigma^0)} \right).$$

*Proof.* In view Remark 2.17, by choosing sufficiently small  $\delta'''$  and large  $C'''$ , (85) and (86) follow directly by applying Lemma D.1 near each  $x \in A^g(\mathbf{x}^0, r, 2r)$ . At that stage, (87) then follows from (85), Claim 1 and the triangle inequality by taking  $C'''$  much larger than  $C'$ .  $\square$

We set  $\tilde{\delta}_2 \leq \delta'''$  be from the previous claim. Then for any  $r \in ((C''')^2 \mathbf{r}_{\Sigma^0, g}^{\Sigma^1}, 1/4)$ , we have

$$(88) \quad \begin{aligned} |\mathbf{x}^1 - \mathbf{x}^0| &\leq C'''' \left( \|\tilde{u}\|_{C^0(A^g(\mathbf{x}^0, r, 4r) \cap \Sigma^0)} + \|u\|_{C^0(A^g(\mathbf{0}, 2r, 3r) \cap \Sigma^0)} \right) \\ &\leq Cr^\gamma \|\tilde{u}\|_{C_\gamma^0(A^g(\mathbf{0}, r, 4r) \cap \Sigma^0)} + Cr^{-n/2} \|u\|_{L^2(A^g(\mathbf{x}^0, r, 4r) \cap \Sigma^0)} \\ &\leq Cr^\gamma \cdot \left( |\mathbf{x}^1 - \mathbf{x}^0| + \|\tilde{u}\|_{L^2(A^g(\mathbf{0}, K^{-3}/10, 1/5) \cap \Sigma^0)} \right) + Cr^{-n/2} \|u\|_{L^2(A^g(\mathbf{x}^0, r, 2) \cap \Sigma^0)} \\ &\leq \bar{C}r^\gamma |\mathbf{x}^1 - \mathbf{x}^0| + Cr^{-n/2} \|u\|_{L^2(A^g(\mathbf{x}^0, r, 2) \cap \Sigma^0)}. \end{aligned}$$

Here, the constants  $C, \bar{C}$  are all only depending on  $\Lambda, \sigma$  (and possibly on  $N$ ) and are changing from line to line; the first inequality follows from (87); the second inequality follows from the definition of  $C_\gamma^0$ , (iv) and classical elliptic estimates; the third inequality follows from (84) and the last inequality follows from triangle inequality and (86).

Now we fix the choice of  $\delta_2(\Lambda, \sigma)$  so small that  $\bar{C}\delta_2^\gamma < 1/2$ . In this way, (78) follows by combining (88) (with  $r = \delta_2$ ) and (81) with  $\tilde{\delta}_2 \in (0, \delta_2)$  small enough.

To prove (79), notice that for every  $s \in ((C''')^2 \mathbf{r}, 1/4)$  we have

$$\begin{aligned} \|u - (\mathbf{x}^1 - \mathbf{x}^0)^{\perp_{\Sigma^0, g}}\|_{C^0(A^g(\mathbf{x}^0, 2s, 3s) \cap \Sigma^0)} &\leq C \left( \|\tilde{u}\|_{C^0(A^g(\mathbf{x}^0, s, 4s) \cap \Sigma^0)} + s^{-1} \cdot |\mathbf{x}^1 - \mathbf{x}^0|^2 \right) \\ &\leq C \left( \|\tilde{u}\|_{C_\gamma^0(A^g(\mathbf{x}^0, s, 4s) \cap \Sigma^0)} \cdot s^\gamma + s^{-1} \cdot |\mathbf{x}^1 - \mathbf{x}^0|^2 \right) \\ &\leq \check{C} \left( \|u\|_{L^2(A^g(\mathbf{x}^0, \delta_2, 2) \cap \Sigma^0)} \cdot s^\gamma + s^{-1} \cdot |\mathbf{x}^1 - \mathbf{x}^0|^2 \right) \end{aligned}$$

Here, the constants  $C, \check{C}$  are again only depending on  $\Lambda, \sigma, N$  and are changing line by line; The first inequality follows from (85) and Claim 1, the second inequality follows by definition

of  $C_\gamma^0$  norm and (iv), while the last inequality follows by combining (84), (78) and (86). We can now choose  $C_2 := \max\{\check{C}, 2(C''')^2\}$  to obtain (79).  $\square$

For later applications, we also need to compare the graphical sections defining three different submanifolds.

**Corollary D.5.** *Let  $\gamma, \sigma \in (0, 1)$  and  $\Lambda > 0$ . Then there exists  $\delta_3 = \delta_3(\gamma, \sigma, \Lambda) \in (0, 1)$ ,  $\tilde{\delta}_3 = \tilde{\delta}_3(\gamma, \sigma, \Lambda) \in (0, \delta_3)$  and  $C_3 = C_3(\gamma, \sigma, \Lambda) > 2$  such that the following holds.*

*Let  $\Sigma^0, \Sigma^1, \Sigma^2$  be **MSI** in  $(\mathbb{B}(5), g)$  satisfying the following conditions*

- (i)  $\|g - g_{\text{euc}}\|_{C^4} \leq \tilde{\delta}_3$ ;
- (ii) for  $i \in \{0, 1, 2\}$ ,  $\|\Sigma^i\|_g(\mathbb{B}(5)) \leq \Lambda$ ;
- (iii) for  $i \in \{0, 1, 2\}$ , there exists  $\mathbf{x}^i \in \text{Sing}(\Sigma^i) \cap \mathbb{B}_{\tilde{\delta}_3}$  such that  $\mathbf{x}^0 = \mathbf{0}$  and the cone densities

$$\theta_{|\Sigma^i|}(\mathbf{x}^i, 2) - \theta_{|\Sigma^i|}(\mathbf{x}^i) \leq \tilde{\delta}_3, \quad \theta_{|\Sigma^0|}(\mathbf{x}^0) = \theta_{|\Sigma^1|}(\mathbf{x}^1) = \theta_{|\Sigma^2|}(\mathbf{x}^2),$$

and

$$\text{dist}_{\mathbb{R}}(\gamma, \Gamma(\mathbf{C}_{\mathbf{x}^i}(\Sigma^i)) \cup \{-(n-2)/2\}) \geq \sigma;$$

- (iv) For  $i \in \{0, 1, 2\}$ ,

$$\mathbf{r}_{\Sigma^i, g^i} \geq \Lambda^{-1} \text{dist}_g(\cdot, \mathbf{x}^i);$$

- (v) for  $i \in \{1, 2\}$ , the graphical radius satisfies

$$\mathbf{r}_{\Sigma^0, g}^{\Sigma^i} \leq \tilde{\delta}_3;$$

- (vi) for  $i \in \{1, 2\}$ ,  $\|u^{(i)}\|_{L^2(A^g(\mathbf{0}, 1/2, 2) \cap \Sigma^0)} \leq \tilde{\delta}_3$ , where  $u^{(i)} := \mathbf{G}_{\Sigma^0, g}^{\Sigma^i} \in L^\infty(\Sigma^0)$ ,  $i = 1, 2$ .

Then for  $i \in \{1, 2\}$ , we have

$$\mathbf{r}_{\Sigma^0, g}^{\Sigma^i} \leq |\mathbf{x}^i - \mathbf{x}^0| \leq C_3 \|u^{(i)}\|_{L^2(A^g(\mathbf{0}, \delta_3, 2) \cap \Sigma^0)} < 2^{-2024},$$

and, set  $\mathbf{r} := \max_i \mathbf{r}_{\Sigma^0, g}^{\Sigma^i}$ ,  $\mathbf{x} := \mathbf{x}^2 - \mathbf{x}^1$ , then for every  $s \in (C_3 \mathbf{r}, 1/4)$  there holds

$$\begin{aligned} & \| (u^{(1)} - u^{(2)}) + \mathbf{x}^{\perp \Sigma^0, g} \|_{C^0(A^g(\mathbf{0}, s, 2s) \cap \Sigma^0)} \\ & \leq C_3 \| \mathbf{G}_{\Sigma^1, g}^{\Sigma^2} \|_{L^2(A^g(\mathbf{0}, \delta_3, 2) \cap \Sigma^1)} \cdot \left( s^\gamma + s^{-1} (\|u^{(2)}\|_{L^2(A^g(\mathbf{0}, \delta_3, 2) \cap \Sigma^0)} + |\mathbf{x}^1| + |\mathbf{x}^2|) \right). \end{aligned}$$

*Proof.* In  $(\mathbb{B}(5), g_{\text{euc}})$ , let  $\tau_{\mathbf{x}}$  be the translation map which translates  $\mathbf{x}^1$  to  $\mathbf{x}^2$ . Define

$$\tilde{\Sigma}^2 := \tau_{\mathbf{x}}^{-1}(\Sigma^2), \quad \tilde{g} := (\tau_{\mathbf{x}})^*(g).$$

Note that  $\tilde{\Sigma}^2$  is singular at  $\mathbf{x}^1$ . Let  $\tilde{u}^{(2)} := \mathbf{G}_{\Sigma^0, g}^{\tilde{\Sigma}^2}$  be the graphical section of  $\tilde{\Sigma}^2$  over  $\Sigma^0$ , and  $v := \mathbf{G}_{\Sigma^1, g}^{\Sigma^2}$ ,  $\tilde{v} := \mathbf{G}_{\Sigma^1, g}^{\tilde{\Sigma}^2}$  be the graphical section of  $\Sigma^2$  and  $\tilde{\Sigma}^2$  over  $\Sigma^1$  respectively. By making  $\tilde{\delta}_3$  small enough, we see that the assumptions in Lemma D.2 hold for  $\Sigma^1, g, \tilde{\Sigma}^2, \tilde{g}$ . Therefore, since  $\tilde{\Sigma}^2$  is  $\tilde{g}$ -minimal and  $\|\tilde{g} - g\|_{C^4} \leq C|\mathbf{x}|$ , we have for every  $s \in (0, 1/4)$ ,

$$\begin{aligned} (89) \quad & \|\tilde{v}\|_{C^0(A^g(\mathbf{x}^1, s/2, 3s) \cap \Sigma^1)} \leq C (|\mathbf{x}| + \|\tilde{v}\|_{C^0(A^g(\mathbf{x}^1, K^{-4}, 1) \cap \Sigma^1)}) \cdot s^\gamma \\ & \leq C (|\mathbf{x}| + \|v\|_{C^0(A^g(\mathbf{x}^1, K^{-5}, 3/2) \cap \Sigma^1)}) \cdot s^\gamma \\ & \leq C \|v\|_{L^2(A^g(\mathbf{x}^1; \delta_3, 2) \cap \Sigma^1)} \cdot s^\gamma. \end{aligned}$$

where the first inequality follows by Lemma D.2; the second inequality follows from (77) in Lemma D.1 and the last follows from Corollary D.4 and classical elliptic estimate if  $\tilde{\delta}_3, \delta_3$  are taken small enough.

Also, by applying (79) to  $\Sigma^0, \Sigma^2, g$  and choosing  $\delta_3 \leq \delta_2, \tilde{\delta}_3 \leq \tilde{\delta}_2$ , we have for every  $s \in (C_2\mathbf{r}, 1/2)$ ,

$$(90) \quad \|u^{(2)}\|_{C^0(A^g(\mathbf{0}, s, 2s) \cap \Sigma^0)} \leq C_2 \left( \|u^{(2)}\|_{L^2(A^g(\mathbf{0}, \delta_3, 2) \cap \Sigma^0)} \cdot s^\gamma + |\mathbf{x}^2| \right).$$

On the other hand, Lemma D.1 implies that for every  $s \in (2C_2\mathbf{r}, 1/4)$ ,

$$\begin{aligned} & \| (u^{(1)} - u^{(2)}) + \mathbf{x}^{\perp_{\Sigma^0, g}} \|_{C^0(A^g(\mathbf{0}, s, 2s) \cap \Sigma^0)} \\ & \leq \| (\tilde{u}^{(2)} - u^{(2)}) + \mathbf{x}^{\perp_{\Sigma^0, g}} \|_{C^0(A^g(\mathbf{0}, s, 2s) \cap \Sigma^0)} + \| u^{(1)} - \tilde{u}^{(2)} \|_{C^0(A^g(\mathbf{0}, s, 2s) \cap \Sigma^0)} \\ & \leq C \left( \| u^{(2)} \|_{C^0(A^g(\mathbf{0}, s/2, 3s) \cap \Sigma^0)} + |\mathbf{x}| \right) s^{-1} \cdot |\mathbf{x}| + C \| \tilde{v} \|_{C^0(A^g(\mathbf{x}^1, s/2, 3s) \cap \Sigma^1)}. \end{aligned}$$

Combining this with (89) and (90) allows to conclude the proof.  $\square$

## APPENDIX E. PARAMETRIZING THE SPACE OF MSI IN A RIEMANNIAN MANIFOLD

For  $N > n \geq 2$  be integers, recall that  $\mathcal{C}_{N,n}$  is the collection of *non-trivial* regular  $n$ -dimensional minimal cones  $\mathbf{C}$  in  $\mathbb{R}^N$ . For each  $\mathbf{C} \in \mathcal{C}_{N,n}$  and every  $x \in \mathbf{C} \setminus \{\mathbf{0}\}$ , we have (essentially as a specification of Definition 2.13)

$$\mathbf{r}_{\mathbf{C}}(x) := \sup \left\{ r > 0 : \mathbf{C} \cap \mathbb{B}(x, r) = \text{graph}_{T_x \mathbf{C}}(u), \ r^{-1}|u| + |\mathring{\nabla} u| + r|\mathring{\nabla}^2 u| \leq 1 \right\}$$

for  $\phi : \text{dom}(\phi) \subset T_x \mathbf{C} \rightarrow T_x^\perp \mathbf{C}$ , of class  $C^2$ . Note that  $\mathbf{r}_{\mathbf{C}}$  is 1-homogeneous in the radial direction. To obtain a compactness result, Lemma 4.5, for each  $\Lambda \geq 1$  we had set

$$\mathcal{C}_{N,n}(\Lambda) := \{ \mathbf{C} \in \mathcal{C}_{N,n} : \inf_{\mathbf{C} \cap \mathbb{S}^{N-1}} \mathbf{r}_{\mathbf{C}}(x) \geq \Lambda^{-1} \};$$

clearly, there holds  $\mathcal{C}_{N,n} = \bigcup_{\Lambda=1}^\infty \mathcal{C}_{N,n}(\Lambda)$ .

**Multiplicity-one cone decomposition.** In this appendix, we adapt a result of Edelen [17] to “parametrize” the class of all minimal submanifolds with strongly isolated singular points (MSI) inside a given Riemannian manifold. It is important to note that in his original work, Edelen considered minimal cones and varifolds with higher multiplicities, whereas for our applications it suffices to restrict to multiplicity-one objects. On the other hand Edelen focused on hypersurfaces, where there is a natural hierarchy among the minimal graphs over a given region; this hierarchy does not exist in the submanifold setting (namely: for general dimension and codimension), so we need to exploit partly different ideas.

**Definition E.1** (Strong-cone region). Let  $g$  be a  $C^2$  metric on  $\mathbb{B}(a, R) \subset \mathbb{R}^N$ , and  $V$  be an  $n$ -dimensional integral varifold on  $(\mathbb{B}(a, R), g)$ . Given  $\mathbf{C} \in \mathcal{C}_{N,n}$ ,  $\beta \in [0, 1/4]$ ,  $\rho \in [0, R]$ , we say that  $V \llcorner (\mathbb{A}(a, \rho, R), g)$  is a **( $\mathbf{C}, \beta$ )-strong-cone region** if there is a  $C^\infty$  normal section

$$u : (a + \mathbf{C}) \cap \mathbb{A}(a, \rho/8, R) \rightarrow \mathbf{C}^\perp$$

so that for any  $r \in [\rho, R] \cap (0, \infty)$  there holds  $V \llcorner \mathbb{A}(a, \rho/8, R) = |\text{graph}_{a+\mathbf{C}}(u)|$  (i. e.  $V$  coincides in that annulus with the multiplicity one varifold associated to the graph of  $u$ ) and we have

- (i) small  $C^2$  norms:  $r^{-1}|u| + |\nabla u| + r|\nabla^2 u| \leq \beta$ ;
- (ii) almost constant density ratios:  $\theta_{\mathbf{C}}(0) - \beta \leq \theta_V(a, r) \leq \theta_{\mathbf{C}}(0) + \beta$ .

In this case, for simplicity, we will also call  $\mathbb{A}(a, \rho, R)$  a strong-cone region for  $V$ .

*Remark E.2.* In [17, Definition 6.0.1, Definition 6.0.3], Edelen introduced notions of a *weak-cone region* and a *strong-cone region*, respectively. It is shown in [17, Definition 6.1(3)], when the parameters  $\beta$  and  $\tau$  are sufficiently small, depending on the cone  $\mathbf{C}$ , a weak-cone region is also a strong-cone region. This result relies on a compactness theorem (Theorem 5.1 therein), which can be replaced by Lemma 4.5 to obtain the same result in our (more general) setting. Therefore, for simplicity, in this manuscript we shall only employ the notion of strong-cone region.

As discussed prior to the definition, we consider only multiplicity-one objects, so we fix the multiplicity parameter  $m$  in [17, Definition 6.0.3] to be 1. This choice will remain fixed throughout this section.

**Definition E.3** (Smooth model). Given parameters  $\Lambda, \gamma \in (0, \infty), \sigma \in (0, 1/3)$ , a tuple  $(S, \mathbf{C}, \{(\mathbf{C}_\alpha, \mathbb{B}(y_\alpha, r_\alpha))\}_\alpha)$  is called a  $(\Lambda, \sigma, \gamma)$ -**smooth model** if

- $S$  is a stationary integral  $n$ -varifold in  $(\mathbb{R}^N, g_{\text{euc}})$  with

$$\theta_S(\mathbf{0}, \infty) \leq \Theta(\Lambda),$$

- $\mathbf{C}, \{\mathbf{C}_\alpha\}_\alpha \subset \mathcal{C}_{N,n}(\Lambda)$ ,
- $\{\mathbb{B}(y_\alpha, 2r_\alpha)\}$  is a finite collection of disjoint balls in  $\mathbb{B}(1 - 3\sigma)$ ,

such that the following conditions are satisfied:

- (i)  $\text{spt } S$  is a smooth, closed, embedded minimal submanifold  $\mathring{S}$  in  $\mathbb{R}^N \setminus \{y_\alpha\}_\alpha$  such that for every  $x \in \mathring{S}$ ,

$$\mathbf{r}_{S, g_{\text{euc}}}(x) \cdot \min\{1, \text{dist}_{g_{\text{euc}}}(x, \{y_\alpha\}_\alpha)\} \geq \Lambda^{-1},$$

where the regularity scale  $\mathbf{r}_{S, g_{\text{euc}}}$  is that of Definition 2.13;

- (ii)  $S_\perp(\mathbb{A}(\mathbf{0}, 1, \infty), g_{\text{euc}})$  is a  $(\mathbf{C}, \gamma)$ -strong-cone region;
- (iii) for each  $\alpha$ ,  $\text{spt } S \cap \mathbb{A}(y_\alpha, 0, 2r_\alpha)$  is a  $(\mathbf{C}_\alpha, \gamma)$ -strong-cone region.

(Note that the function  $\Theta(\Lambda)$  has been defined in Remark 4.9.) If there is no ambiguity, we will refer to the smooth model as  $S$  for simplicity.

*Remark E.4.* Intuitively, when a minimal submanifold is close to a cone outside some small ball, this approximation does not necessarily extend directly into the small ball. Fortunately, due to the volume monotonicity, on a small scale, if the minimal submanifold is modeled on another cone, the density of the new cone is controlled by that of the original one. The notion of smooth region, introduced below, aims at depicting, in  $\mathbb{B}(1) \setminus \bigcup_\alpha \mathbb{B}(y_\alpha, r_\alpha/4)$ , the region between strong-cone regions at two different scales.

**Definition E.5** (Smooth model scale constant). Given a  $(\Lambda, \sigma, \gamma)$ -smooth model  $S$ , we let  $\epsilon_S$  be the largest number  $\leq \min(1, \min_\alpha \{r_\alpha\})$  for which the graph map

$$\text{graph}_S : T^\perp(\mathring{S}) \rightarrow \mathbb{R}^N, \quad \text{graph}_S(x, v) := x + v,$$

is a diffeomorphism from  $\{(x, v) \in T^\perp(\mathring{S}) : x \in \mathbb{B}(2) \setminus \bigcup_\alpha \mathbb{B}(y_\alpha, r_\alpha/8), |v| < 2\epsilon_S\}$  onto its image, and satisfies

$$|D \text{graph}_S|_{(x,v)} - \mathbf{1}| \leq \epsilon_S^{-1} |v|.$$

(Here  $T^\perp(\mathring{S})$  denotes the normal bundle to  $S$  in Euclidean  $\mathbb{R}^N$ .)

**Definition E.6** (Smooth region). Given a smooth model  $S$ , a  $C^2$  metric  $g$  on  $\mathbb{B}(a, R) \subset \mathbb{R}^N$ , and  $\beta \in (0, 1)$ , we say that an integral varifold  $V$  in  $(\mathbb{B}(a, R), g)$  is an  $(S, \beta)$ -**smooth region** if there is a  $C^2$  function  $u : \mathring{S} \rightarrow \mathring{S}^\perp$  so that

$$((\eta_{a,R})_\# V) \llcorner (\mathbb{B}(1) \setminus \bigcup_{\alpha} \mathbb{B}(y_\alpha, r_\alpha/4)) = \left[ \text{graph}_{\mathring{S}}(u) \cap \mathbb{B}(1) \setminus \bigcup_{\alpha} \mathbb{B}(y_\alpha, r_\alpha/4) \right]_{R^{-2} \circ \eta_{a,R}^*(g)},$$

where  $\eta_{a,R}$  is the dilation of center  $a$  and scale  $1/R$ , and

$$|u|_{C^2(\mathring{S})} \leq \beta \epsilon_S,$$

where  $\epsilon_S$  is a scale constant in the previous definition.

In this case, for simplicity, we will also call  $\mathbb{B}(a, R)$  a smooth region for  $V$ .

**Definition E.7** (Local cone decomposition). Given  $\Lambda \geq 1$ ,  $\gamma, \beta \in \mathbb{R}$ ,  $\sigma \in (0, 1/3)$ , and  $N_R \in \mathbb{N}$ , we let

- $g$  be a  $C^2$  metric on  $\mathbb{B}(x, R) \subset \mathbb{R}^N$ ;
- $V$  be an integral varifold in  $(\mathbb{B}(x, R), g)$ ;
- $\mathcal{S} = \{S_s\}_s$  be a finite collection of  $(\Lambda, \sigma, \gamma)$ -smooth models.

A  $(\Lambda, \beta, \mathcal{S}, N_R)$ -**cone decomposition** of  $V$  consists of the following parameters:

- Integers  $N_C, N_S$  satisfying  $N_C + N_S \leq N_R$ , where  $N_C$  is the number of strong-cone regions and  $N_S$  the number of smooth regions;
- Points  $\{x_a\}_a, \{x_b\}_b \subset \mathbb{B}(x, R)$ , where  $\{x_a\}$  are henceforth referred to as centers of strong-cone regions and  $\{x_b\}$  as centers of smooth regions;
- Radii  $\{R_a, \rho_a \mid R_a \geq 2\rho_a\}_a, \{R_b\}_b$ , corresponding to radii of annuli in the definition of strong-cone regions and of balls in the definition of smooth regions, respectively;
- Cones  $\{\mathbf{C}_a\}_a \subset \mathcal{C}_{N,n}(\Lambda)$ ;
- Indices  $\{s_b\}_b$ , corresponding to the smooth model  $S_{s_b}$ ;

where  $a = 1, \dots, N_C$  and  $b = 1, \dots, N_S$ . Such parameters determine a covering of balls and annuli satisfying:

- (i) every  $V \llcorner (\mathbb{A}(x_a, \rho_a, R_a), g)$  is a  $(\mathbf{C}_a, \beta)$ -strong cone region and every  $V \llcorner (\mathbb{B}(x_b, R_b), g)$  is a  $(S_{s_b}, \beta)$ -smooth region;
- (ii) there is either a strong-cone region  $\mathbb{A}(x_a, \rho_a, R_a)$  for  $V$  with  $R_a = R$  and  $x_a = x$ , or a smooth region  $\mathbb{B}_{R_b}(x_b)$  for  $V$  with  $R_b = R$  and  $x_b = x$ ;
- (iii) if  $V \llcorner (\mathbb{A}(x_a, \rho_a, R_a), g)$  is a  $(\mathbf{C}_a, \beta)$ -strong-cone region and  $\rho_a > 0$ , then there exists either a smooth region  $\mathbb{B}(x_b, R_b)$  for  $V$  with  $R_b = \rho_a$ , or another cone region  $\mathbb{A}(x_{a'}, \rho_{a'}, R_{a'})$  for  $V$  with  $R_{a'} = \rho_a, x_{a'} = x_a$ . If  $\rho_a = 0$ , then  $\theta_{\mathbf{C}_a}(\mathbf{0}) > 1$ ;
- (iv) if  $V \llcorner (\mathbb{B}(x_b, R_b), g)$  is a smooth region with  $(S, \mathbf{C}, \{\mathbf{C}_{\hat{\alpha}}, \mathbb{B}(y_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha}}) \in \mathcal{S}$ , then for any  $\hat{\alpha}$ , there exists a point  $x_{b,\hat{\alpha}}$  and a radius  $R_{b,\hat{\alpha}}$  satisfying

$$|x_{b,\hat{\alpha}} - (x_b + R_b \cdot y_{\hat{\alpha}})| \leq \beta R_b r_{\hat{\alpha}}, \quad \frac{1}{2} \leq \frac{R_{b,\hat{\alpha}}}{R_b r_{\hat{\alpha}}} \leq 1 + \beta,$$

and either a strong-cone region  $\mathbb{A}(x_{a'}, \rho_{a'}, R_{a'})$  for  $V$  with  $R_{a'} = R_{b,\hat{\alpha}}, x_{a'} = x_{b,\hat{\alpha}}$ , or another smooth region  $\mathbb{B}(x_{b'}, R_{b'})$  with  $R_{b'} = R_{b,\hat{\alpha}}$  and  $x_{b'} = x_{b,\hat{\alpha}}$ .

*Remark E.8.* In these definitions, we do not assume the stability of minimal cones. Instead, we impose a regularity scale condition, in the definition of  $\mathcal{C}_{N,n}(\Lambda)$ , to have uniform control on the geometry of the link (cf. Section 4.1).

Given a stationary integral varifold  $V$  in  $(\mathbb{B}(1), g)$  with finitely many singular points  $\text{Sing } V$ , for every  $x \in \text{Reg } V$ , we define

$$\rho_V(x) := \min\{1, \inf_{p \in \text{Sing } V} \{\text{dist}_g(x, p)\}\}.$$

By Lemma 4.8, for each  $\Lambda > 0$ , there are only finitely many possible densities for regular minimal cones in  $\mathcal{C}_{N,n}(\Lambda)$ . Among various consequences, we have the following important local cone decomposition theorem.

**Theorem E.9** (Existence of local cone decomposition). *Given parameters  $I \in \mathbb{N}^+$ ,  $\Lambda \in \mathbb{N}^+$ ,  $\sigma \in (0, \frac{1}{100(\Lambda+1)})$ ,  $\gamma \in (0, 1)$ , there exist constants  $\delta$ ,  $N_R$  and a finite collection of  $(\Lambda, \sigma, \gamma)$ -smooth models  $\{S_s\}_s = \mathcal{S}$ , all depending only on  $(\Lambda, \sigma, \gamma)$  with the following property.*

*For any  $C^3$  metric  $g$  on  $\mathbb{B}(1)$  satisfying  $|g - g_{\text{euc}}|_{C^3(\mathbb{B}(1))} \leq \delta$ , any stationary integral varifold  $V$  in  $(\mathbb{B}(1), g)$  with  $\#\text{Sing } V \leq I$ ,  $\mathbf{C} \in \mathcal{C}_{N,n}(\Lambda)$ , if*

- (i)  $\theta_{\mathbf{C}}(0) \leq \Theta(\Lambda)$ ,
- (ii)  $\text{dist}_H(\text{spt } V \cap \mathbb{B}(1), \mathbf{C} \cap \mathbb{B}(1)) \leq \delta$ ,
- (iii)  $\frac{1}{2}\theta_{\mathbf{C}}(0) \leq \theta_V(0, 1/2)$  and  $\theta_V(0, 1) \leq \frac{3}{2}\theta_{\mathbf{C}}(0)$ ,
- (iv) for all  $x \in \text{Reg } V$ ,  $\mathbf{r}_{\text{Reg } V, g}(x) \geq \Lambda^{-1}\rho_V(x)$ ,

where  $\text{dist}_H$  stands for the Hausdorff distance, then there exists a radius  $r \in (1 - 20\sigma, 1)$  so that  $V \llcorner \mathbb{B}(x, r)$  admits a  $(\Lambda, \gamma, \mathcal{S}, N_R)$ -cone decomposition.

*Proof.* The proof follows verbatim from that of [17, Theorem 7.1]. Note that by Allard's regularity theorem [2], the regularity scale condition  $\mathbf{r}_{\text{Reg } V, g}(x) \geq \Lambda^{-1}\rho_V(x)$  and the assumption  $\#\text{Sing } V \leq I$  ensure that the convergence is smooth and of multiplicity one outside a finite set, and thus, only Case 1 ( $\mathcal{I} \subset \{0\}$ ) from the proof of [17, Theorem 7.1] will arise during the induction in our setting.  $\square$

**Tree representations of cone decomposition.** In the definition of the cone decomposition of a minimal submanifold near a cone, Definition E.7, there is a natural hierarchy among the strong-cone regions and smooth regions based on their respective scales due to the volume monotonicity. Therefore, we can use a tree structure to represent the cone decomposition.

**Definition E.10** (Tree representation of a local cone decomposition). Given a  $(\Lambda, \beta, \mathcal{S}, N_R)$ -cone decomposition of  $V \llcorner \mathbb{B}(x, R)$  as in Definition E.7 with parameters:

- Integers  $N_S, N_C$  satisfying  $N_S + N_C \leq N_R$ ;
- Points  $\{x_a\}_a, \{x_b\}_b \subset \mathbb{B}(x, R)$ ;
- Radii  $\{R_a, \rho_a : R_a \geq 2\rho_a\}_a, \{R_b\}_b$ ;
- Indices  $\{s_b\}_b$ ;
- Cones  $\{\mathbf{C}_a\}_a \subset \mathcal{C}_{N,n}(\Lambda)$ ,

where  $a = 1, \dots, N_C$  and  $b = 1, \dots, N_S$ . The corresponding **tree representation** of the cone decomposition is a rooted tree (in the sense of [16, Section B.5]) uniquely defined by the following requirements:

- (i) there are two types of nodes: every node of *type I* is labeled with  $(\mathbf{C}_a, x_a, R_a, \rho_a)$ , while every node of *type II* with  $(S_{s_b}, x_b, R_b)$ ;
- (ii) the root is labeled with either  $(\mathbf{C}_a, x_a = x, R_a = R, \rho_a)$  or  $(S_{s_b}, x_b = x, R_b = R)$ ;

- (iii) for any type I node  $(\mathbf{C}_a, x_a, R_a, \rho_a)$ , either  $\rho_a = 0$ ,  $\theta_{\mathbf{C}_a}(\mathbf{0}) > 1$  and it is a leaf; or  $\rho_a > 0$  and it has a unique child of either
- type I  $(\mathbf{C}_{a'}, x_{a'} = x_a, R_{a'} = \rho_a, \rho_{a'})$ , or
  - type II  $(S_{s_{b'}}, x_{b'} = x_a, R_{b'} = \rho_a)$ ;
- (iv) for any type II node  $(S_{s_b}, x_b, R_b)$  where  $S_{s_b} = (S, \mathbf{C}, \{\mathbf{C}_{\hat{\alpha}}, \mathbb{B}(y_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha} \in I_b})$ , it has  $\text{card}(I_b)$  child nodes such that for each  $\hat{\alpha}$ , there exists  $R_{b, \hat{\alpha}}$  and  $x_{b, \hat{\alpha}}$  such that

$$|x_{b, \hat{\alpha}} - (x_b + R_b \cdot y_{\hat{\alpha}})| \leq \beta R_b r_{\hat{\alpha}}, \quad \frac{1}{2} \leq \frac{R_{b, \hat{\alpha}}}{R_b r_{\hat{\alpha}}} \leq 1 + \beta,$$

so that the corresponding child node is either

- of type I  $(\mathbf{C}_{a'} = \mathbf{C}_{\hat{\alpha}}, x_{a'} = x_{b, \hat{\alpha}}, R_{a'} = R_{b, \hat{\alpha}}, \rho_{a'})$ , or
- of type II  $(S_{s_{b'}}, x_{b'} = x_{b, \hat{\alpha}}, R_{b'} = R_{b, \hat{\alpha}})$ .

The **coarse tree representation** of the cone decomposition is obtained by relabeling the rooted tree above, i.e., replacing the type I node  $(\mathbf{C}_a, x_a, R_a, \rho_a)$  by  $(\theta_{\mathbf{C}_a}(\mathbf{0}))$ , and the type II node  $(S_{s_b}, x_b, R_b)$  by  $S_{s_b}$ .

**Definition E.11.** For  $\gamma \in (0, 1/100)$ , two  $(\Lambda, \beta, \mathcal{S}, N_R)$ -tree representations of local cone decompositions with parameters

- $(N_S, N_C, \{x_a\}, \{x_b\}, \{R_a\}, \{\rho_a\}, \{R_b\}, \{\mathbf{C}_a\}, \{s_b\})$ ,
- $(N'_S, N'_C, \{x'_a\}, \{x'_b\}, \{R'_a\}, \{\rho'_a\}, \{R'_b\}, \{\mathbf{C}'_a\}, \{s'_b\})$ ,

are said to be  $\gamma$ -**close** if  $N'_S = N_S$ ,  $N'_C = N_C$ , they have the same coarse tree representations, and in addition:

- (i) if the corresponding two nodes are both of type I, then

- $\text{dist}_H(\mathbf{C}_a \cap \partial B_1, \mathbf{C}_{a'} \cap \partial B_1) \leq \gamma$ ,
- If  $\rho_a > 0$ , then
  - $|\rho_a - \rho_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$ ;
  - $|x_a - x_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$ ;
  - $|R_a - R_{a'}| \leq \gamma \min(\rho_a, \rho_{a'})$ ;

otherwise, if  $\rho_a = 0$ , then

- $\rho_{a'} = 0$ ;
- $|x_a - x_{a'}| \leq \gamma \min(R_a, R_{a'})$ ;
- $|R_a - R_{a'}| \leq \gamma \min(R_a, R_{a'})$ ;

- (ii) if the corresponding two nodes are both of type II, then

- $|x_b - x_{b'}| \leq \gamma \min(R_b, R'_b) \min_{\hat{\alpha} \in I_b}(r_{\hat{\alpha}})$ ,
- $|R_b - R_{b'}| \leq \gamma \min(R_b, R'_b) \min_{\hat{\alpha} \in I_b}(r_{\hat{\alpha}})$ .

In order to deal with minimal submanifolds in a closed Riemannian manifold directly, we also introduce a large-scale cone decomposition for an **MSI** in a closed Riemannian manifold  $M$  of dimension  $N$ .

**Definition E.12** (Large-scale cone decomposition). Given  $\Lambda, \gamma, \beta \in \mathbb{R}_+$ ,  $\sigma \in (0, 1/3)$ , and  $N_R \in \mathbb{N}$ , let

- $g_0, g$  be two  $C^3$  metrics on  $M$ ;
- $\Sigma_0, \Sigma$  be two **MSI** in  $(M, g_0)$  and  $(M, g)$  respectively;
- $\mathcal{S} = \{S_s\}_s$  be a finite collection of  $(\Lambda, \sigma, \gamma)$ -smooth models.

A **large-scale**  $(\Lambda, \beta, g_0, \Sigma_0, \mathcal{S}, N_R)$ -**cone decomposition** of  $\Sigma$  consists of:

- a collection of radii  $\{r_{\hat{\alpha}}\}_{\hat{\alpha}}$  corresponding to the singular set  $\text{Sing}(\Sigma_0) = \{p_{\hat{\alpha}}\}_{\hat{\alpha}}$ , such that the metric balls  $\{B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})\}_{\hat{\alpha}}$  are pairwise disjoint;
- a  $(\Lambda, \beta, \mathcal{S}, N_R)$ -cone decomposition for each  $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$ ;
- a  $C^2$  normal section  $u : \Sigma_0 \setminus \bigcup_{p_{\hat{\alpha}} \in \text{Sing}(\Sigma_0)} B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}}/2) \rightarrow \Sigma_0^\perp$  so that for  $r_0 = \min_{\hat{\alpha}} \{r_{\hat{\alpha}}\} > 0$ ,

$$r_0^{-1}|u| + |\nabla u| + r_0|\nabla^2 u| \leq \beta,$$

and  $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$  coincides with  $\text{graph}_{\Sigma_0}(u) \setminus B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$  (understood as varifold with multiplicity one);

Similarly, we can define the corresponding tree representation and the  $\gamma$ -closeness.

**Definition E.13** (Tree representation of large-scale cone decomposition). Given a large-scale  $(\Lambda, \beta, g_0, \Sigma_0, \mathcal{S}, N)$ -cone decomposition of an **MSI**  $\Sigma$  in  $(M, g)$  with parameters:

- $\text{Sing}(\Sigma_0) = \{p_{\hat{\alpha}}\}_{\hat{\alpha}}$ ;
- radii  $\{r_{\hat{\alpha}}\}_{\hat{\alpha}}$ ;
- $(\Lambda, \beta, \mathcal{S}, N)$ -cone decompositions for each  $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$ ;

The corresponding **tree representation** of the large-scale cone decomposition is a rooted tree uniquely defined by:

- (1) the root node is labeled by a tuple  $(\Sigma_0, g_0, \{p_{\hat{\alpha}}\}, \{r_{\hat{\alpha}}\})$ ;
- (2) the root node has  $\#\text{Sing}(\Sigma_0)$  children, indexed by  $\hat{\alpha}$ . The corresponding subtree rooted at the  $\hat{\alpha}$ -child is the tree representation of the  $(\Lambda, \beta, \mathcal{S}, N)$ -cone decomposition for each  $|\Sigma| \llcorner B^g(p_{\hat{\alpha}}, r_{\hat{\alpha}})$ .

Similarly, the **coarse tree representation** will be the same directed rooted tree with the subtrees above replaced by their corresponding coarse trees.

**Definition E.14.** For  $\gamma \in (0, 1/100)$ , two  $(\Lambda, \beta, g_0, \Sigma_0, \mathcal{S}, N)$ -tree representations are said to be  $\gamma$ -close if

- their root nodes have the same label;
- their subtrees corresponding to the  $\hat{\alpha}$ -child are  $\gamma$ -close for each  $\hat{\alpha}$ .

**Theorem E.15.** In a closed smooth manifold  $M$  of dimension  $N$ , for any given  $(g, \Sigma) \in \mathcal{M}_n^{k, \alpha}(M)$ ,  $\beta \in (0, 1/100)$ , and  $I, \Lambda \in \mathbb{N}$  with

$$\#\text{Sing}(\Sigma) \leq I, \quad \mathbf{r}_{\Sigma, g} \geq \Lambda^{-1} \rho_{\Sigma, g},$$

there exist  $\delta(g, \Sigma, \beta, \Lambda, I) > 0$  satisfying the following property.

Defined a  $\delta$ -“neighborhood” of  $(g, \Sigma)$  as

$$\mathcal{M}_n^{k, \alpha}(g, \Sigma; \Lambda, I, \beta) := \left\{ (g', \Sigma') \in \mathcal{M}_n^{k, \alpha}(M) : \|g'\|_{C^{k, \alpha}} \leq \Lambda, \mathbf{r}_{\Sigma', g'} \geq \Lambda^{-1} \rho_{\Sigma', g'}, \right. \\ \left. \#\text{Sing}(\Sigma') \leq I, \|g - g'\|_{C^k} + \mathbf{F}(|\Sigma|_g, |\Sigma'|_{g'}) \leq \delta(g, \Sigma, \beta, \Lambda, I) \right\}$$

- there exist a finite collection of  $(\Lambda, \sigma, \beta)$ -smooth models  $\mathcal{S}$  and an integer  $N_R$ , so that any  $(g', \Sigma') \in \mathcal{M}_n^{k, \alpha}(g, \Sigma; \Lambda, I, \beta)$  admits a large-scale  $(\Lambda, \beta, g, \Sigma, \mathcal{S}, N_R)$ -cone decomposition;
- there exists a countable collection  $\{(g_v, \Sigma_v)\}_{v \in \mathbb{N}} \subset \mathcal{M}_n^{k, \alpha}(g, \Sigma; \Lambda, I, \beta)$  with fixed large-scale  $(\Lambda, \beta, g, \Sigma, \mathcal{S}, N_R)$ -cone decompositions with the following property: every  $(g', \Sigma') \in \mathcal{M}_n^{k, \alpha}(g, \Sigma; \Lambda, I, \beta)$  admits a large-scale  $(\Lambda, \beta, g, \Sigma, \mathcal{S}, N_R)$ -cone decomposition whose tree representation is  $\beta$ -close to that of some  $(g_v, \Sigma_v)$ .

*Proof.* The proof is essentially the same as that of [27, Theorem 9.6].  $\square$

To proceed, define

$$\mathcal{M}_n^{k,\alpha}(M; \Lambda, I) := \left\{ (g, \Sigma) \in \mathcal{M}_n^{k,\alpha}(M) : \|g\|_{C^{k,\alpha}} \leq \Lambda, \mathbf{r}_{\Sigma, g} \geq \Lambda^{-1} \rho_\Sigma, \#\text{Sing}(\Sigma) \leq I \right\}.$$

For any  $\beta > 0$ , it follows from the corresponding compactness statement that there exists a sequence of  $\{(g_i, \Sigma_i)\}_i$  of  $\mathcal{M}_n^{k,\alpha}(M; \Lambda, I)$  such that

$$\mathcal{M}_n^{k,\alpha}(M; \Lambda, I) = \bigcup_{i=1}^{\infty} \mathcal{M}_n^{k,\alpha}(g_i, \Sigma_i; \Lambda, I, \beta).$$

With reference to the second point of Theorem E.15, for each  $v$ , we can define an “*intermediate canonical neighborhood*” by

$$\mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, I, \beta)$$

consisting of every pair  $(g', \Sigma') \in \mathcal{M}_n^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$  that admits a large-scale  $(\Lambda, \beta, g, \Sigma, \mathcal{S}, N_R)$ -cone decomposition whose tree representation is  $\beta$ -close to that of  $(g_v, \Sigma_v)$ .

Therefore, we have

$$\begin{aligned} \mathcal{M}_n^{k,\alpha}(M) &= \bigcup_{I=0}^{\infty} \bigcup_{\Lambda=1}^{\infty} \mathcal{M}_n^{k,\alpha}(M; \Lambda, I) \\ &= \bigcup_{I=0}^{\infty} \bigcup_{\Lambda=1}^{\infty} \bigcup_{i=1}^{\infty} \mathcal{M}_n^{k,\alpha}(g_i, \Sigma_i; \Lambda, I, \beta) \\ &= \bigcup_{I=0}^{\infty} \bigcup_{\Lambda=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{v=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_{i,v}, \Sigma_{i,v}; \Lambda, I, \beta). \end{aligned}$$

After relabeling the subscripts of the various parameters in play (each varying in a countable set), we obtain

$$(91) \quad \mathcal{M}_n^{k,\alpha}(M) = \bigcup_{i=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I_i, \beta).$$

**Second decomposition.** Following the arguments in [27, Subsection 9.2], we can prove that an intermediate canonical neighborhood  $\mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$  is sequentially compact, and thus, we have the following result.

**Proposition E.16** (Finite covering of  $\mathcal{L}_0^{k,\alpha}$ ). *For any  $g, \Sigma, \Lambda, I, \beta$  as in (91) (with subscripts omitted) and for any positive function  $\kappa : \mathcal{M}_n^{k,\alpha}(M) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  (not necessarily continuous), there exists a finite set of pairs  $\{(g_v, \Sigma_v)\}_v \subset \mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta)$  such that*

$$\mathcal{L}_0^{k,\alpha}(g, \Sigma; \Lambda, I, \beta) \subset \bigcup_{v=1}^V \mathcal{L}_0^{k,\alpha}(g_v, \Sigma_v; \Lambda, \kappa_v),$$

where  $\kappa_v = \kappa(g_v, \Sigma_v; \Lambda)$ .

**Proof of Theorem 4.23.** By (91) and the previous covering proposition, Proposition E.16, we have

$$\begin{aligned} \mathcal{M}^{k,\alpha}(M) &= \bigcup_{i=1}^{\infty} \mathcal{L}_0^{k,\alpha}(g_i, \Sigma_i; \Lambda_i, I_i, \beta) \\ &= \bigcup_{i=1}^{\infty} \bigcup_{v=1}^{V_i} \mathcal{L}^{k,\alpha}(g_{i,v}, \Sigma_{i,v}; \Lambda_i, \kappa_{i,v}), \end{aligned}$$

where  $\kappa_{i,v} = \kappa(g_{i,v}, \Sigma_{i,v}; \Lambda_i)$ .

Hence, Theorem 4.23 follows from relabeling the indices  $\{i, v\}$ .

#### ACKNOWLEDGEMENTS

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 947923). Parts of this article were finalized while A. C. was visiting the University of Chicago and then during various visits of the authors at the Simons Laufer Mathematical Sciences Institute (SLMath): the support and excellent working conditions of both institutions are gratefully acknowledged. Y. L. was partially supported by an AMS-Simons travel grant.

#### REFERENCES

- [1] W. K. Allard and F. J. Almgren Jr., *The structure of stationary one dimensional varifolds with positive density*, Invent. Math. **34** (1976), no. 2, 83–97.
- [2] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.
- [3] W. K. Allard and F. J. Almgren Jr. (eds.), *Geometric measure theory and the calculus of variations*, Proceedings of Symposia in Pure Mathematics, vol. 44, American Mathematical Society, Providence, RI, 1986.
- [4] F. J. Almgren Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein’s theorem*, Ann. of Math. (2) **84** (1966), 277–292.
- [5] L. Ambrozio, A. Carlotto, and B. Sharp, *Compactness analysis for free boundary minimal hypersurfaces*, Calc. Var. **57** (2018), no. article no. 22, 1–39.
- [6] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. (9) **36** (1957), 235–249.
- [7] R. Böhme and A. J. Tromba, *The index theorem for classical minimal surfaces*, Ann. of Math. (2) **113** (1981), no. 3, 447–499.
- [8] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
- [9] L. Caffarelli, R. Hardt, and L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. **48** (1984), no. 1–3, 1–18.
- [10] E. Calabi, *Minimal immersions of surfaces in euclidean spheres*, J. Differential Geom. **1** (1967), no. 1-2, 111–125.
- [11] A. Carlotto, *Minimal hyperspheres of arbitrarily large Morse index*, Comm. Anal. Geom. **27** (2019), no. 5, 991–1023.
- [12] O. Chodosh, Y. Liokumovich, and L. Spolaor, *Singular behavior and generic regularity of min-max minimal hypersurfaces*, Ars Inven. Anal. (2022), Paper No. 2, 27 pages.
- [13] O. Chodosh and C. Mantoulidis, *The  $p$ -widths of a surface*, Publ. Math. Inst. Hautes Études Sci. **137** (2023), 245–342.
- [14] O. Chodosh, C. Mantoulidis, and F. Schulze, *Generic regularity for minimizing hypersurfaces in dimensions 9 and 10*, arXiv preprint, available at [2302.02253](https://arxiv.org/abs/2302.02253).

- [15] ———, *Improved generic regularity of codimension-1 minimizing integral currents*, arXiv preprint, available at [2306.13191](#).
- [16] T. H. Corman, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*, MIT press, Cambridge, MA, 2022.
- [17] N. Edelen, *Degeneration of 7-dimensional minimal hypersurfaces which are stable or have a bounded index*, Arch. Ration. Mech. Anal. **248** (2024), no. 4, Paper No. 65, 73 pages.
- [18] N. Edelen and G. Székelyhidi, *A Liouville-type theorem for cylindrical cones*, Comm. Pure Appl. Math. **77** (2024), no. 8, 3557–3580.
- [19] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York, Inc., New York, 1969.
- [20] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520.
- [21] R. Hardt and L. Simon, *Area minimizing hypersurfaces with isolated singularities*, J. Reine Angew. Math. **362** (1985), 102–129.
- [22] W.-Y. Hsiang, *Minimal cones and the spherical Bernstein problem. I*, Ann. of Math. (2) **118** (1983), no. 1, 61–73.
- [23] M. Karpukhin, *Index of minimal spheres and isoperimetric eigenvalue inequalities*, Invent. Math. **223** (2021), no. 1, 335–377.
- [24] R. Kusner and P. Wang, *On the index of minimal 2-tori in the 4-sphere*, J. Reine Angew. Math. **806** (2024), 9–22.
- [25] Y. Li, *Existence of infinitely many minimal hypersurfaces in higher-dimensional closed manifolds with generic metrics*, J. Differential Geom. **124** (2023), no. 2, 381–395.
- [26] Y. Li and Z. Wang, *Generic regularity of minimal hypersurfaces in dimension 8*, arXiv preprint, available at [2012.05401](#).
- [27] ———, *Minimal hypersurfaces for generic metrics in dimension 8*, arXiv preprint, available at [2205.01047](#).
- [28] Z. Liu, *Homologically area-minimizing surfaces with non-smoothable singularities*, arXiv preprint, available at [2206.08315](#).
- [29] R. B. Lockhart and R. C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 409–447.
- [30] F. C. Marques and A. Neves, *Min-max theory and the Willmore conjecture*, Ann. of Math. (2) **179** (2014), no. 2, 683–782.
- [31] J. Marx-Kuo, L. Sarnataro, and D. Stryker, *Index, intersections, and multiplicity of min-max geodesics*, arXiv preprint, available at [2410.02580](#).
- [32] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, MA, 1993.
- [33] J. D. Moore, *Bumpy metrics and closed parametrized minimal surfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5193–5256.
- [34] ———, *Self-intersections of closed parametrized minimal surfaces in generic Riemannian manifolds*, Ann. Global Anal. Geom. **60** (2021), no. 1, 157–165.
- [35] T. Pacini, *Desingularizing isolated conical singularities: Uniform estimates via weighted sobolev spaces*, Comm. Anal. Geom. **21** (2013), no. 1, 105–170.
- [36] J. T. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds*, Mathematical Notes, vol. 27, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.
- [37] B. Sharp, *Compactness of minimal hypersurfaces with bounded index*, J. Differential Geom. **106** (2017), no. 2, 317–339.
- [38] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), no. 3, 525–571.
- [39] ———, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.
- [40] ———, *Isolated singularities of extrema of geometric variational problems*, Harmonic mappings and minimal immersions (Montecatini, 1984), 1985, pp. 206–277.
- [41] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

- [42] N. Smale, *Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds*, *Comm. Anal. Geom.* **1** (1993), no. 2, 217–228.
- [43] Z. Wang, *Deformations of singular minimal hypersurfaces I, isolated singularities.*, arXiv preprint, available at [2011.00548](#).
- [44] ———, *Mean convex smoothing of mean convex cones*, *Geom. Funct. Anal.* **34** (2024), no. 1, 263–301.
- [45] B. White, *Generic transversality of minimal submanifolds and generic regularity of two-dimensional area-minimizing integral currents*, arXiv preprint, available at [1901.05148](#).
- [46] ———, *Generic regularity of unoriented two-dimensional area minimizing surfaces*, *Ann. of Math. (2)* **121** (1985), no. 3, 595–603.
- [47] ———, *The space of minimal submanifolds for varying Riemannian metrics*, *Indiana Univ. Math. J.* **40** (1991), no. 1, 161–200.
- [48] ———, *Currents and flat chains associated to varifolds, with an application to mean curvature flow*, *Duke Math. J.* **148** (2009), no. 1, 41–62.
- [49] ———, *On the bumpy metrics theorem for minimal submanifolds*, *Amer. J. Math.* **139** (2017), no. 4, 1149–1155.
- [50] S. T. Yau, *Problem section*, *Seminar on Differential Geometry*, 1982, pp. 669–706.

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