PROJECTIVE SMOOTH REPRESENTATIONS IN NATURAL CHARACTERISTIC

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Abstract. We investigate under which circumstances there exists nonzero *projective* smooth $F[G]$-modules, where F is a field of characteristic p and G is a locally pro-p group. We prove the non-existence of (non-trivial) projective objects for so-called *fair* groups – a family including $\mathbf{G}(\mathfrak{F})$ for a connected reductive group $\mathbf{G}$ defined over a non-archimedean local field $\mathfrak{F}$. This was proved in [SS24] for finite extensions $\mathfrak{F}/\mathbb{Q}_p$. The argument we present in this note has the benefit of being completely elementary and, perhaps more importantly, adaptable to $\mathfrak{F} = \mathbb{F}_q((t))$. Finally, we elucidate the fairness condition via a criterion in the Chabauty space of G.

1. Introduction

Projective objects play a prominent role in the modular representation theory of a finite group G. For instance, if F is a field of characteristic p dividing $|G|$, the Grothendieck group $K_0(F[G])$ is part of the Cartan-Brauer triangle [Sch13, p. 46]. Also, in the very definition of the all-important stable module category of $F[G]$ one kills morphisms which factor through a projective module.

One side of the p-adic Langlands correspondence (or rather its mod p counterpart) involves smooth $F[G]$-modules where G is now an infinite p-adic reductive group, and F is still a field of characteristic p. (A G-representation V is *smooth* if the action $G \times V \longrightarrow V$ is continuous for the discrete topology on V.) The smooth $F[G]$-modules form an abelian category $\mathrm{Mod}_F(G)$ and one could hope to recast modular representation theory in this generality. As was shown in [SS24] this situation is dramatically different from the case of finite groups: There are *no* projective objects in $\mathrm{Mod}_F(G)$ other than $V = \{0\}$.

We mention in passing that the complex case is dissimilar: If G is a p-adic reductive group, and we fix a character χ of the center, the category $\mathrm{Mod}_{\mathbb{C}}^{\chi}(G)$ of smooth $\mathbb{C}[G]$-modules with central character χ has lots of projective objects. In fact, any irreducible supercuspidal representation is projective (and injective). See [AR04] for more precise results in this direction – including a converse.

One of the goals of this note is to extend parts of [SS24] to groups $G = \mathbf{G}(\mathfrak{F})$ where $\mathbf{G}$ is a connected reductive group over *any* non-archimedean local field $\mathfrak{F}$ – possibly of positive characteristic. The arguments in [SS24] make heavy use of Poincaré subgroups, and therefore only apply for finite extensions of $\mathbb{Q}_p$. We stress that [SS24] had a different goal (to understand the derived functors of smooth induction) and the non-existence of nonzero projective objects was a byproduct. The methods of this note are more elementary and completely avoid cohomology.

We work in greater generality. We continue to let F denote a field of characteristic p, but we allow G to be any locally pro-p group (by which we mean it admits an open subgroup which is pro-p in the induced topology). If G is discrete, smoothness is automatic, and we obviously have plenty of projective $F[G]$-modules – such as $F[G]$ itself. For that reason we will always assume G is non-discrete.

The key hypothesis is the following: We assume G admits an open subgroup K such that for all open subgroups $H \subset K$ there exists an open subgroup $H' \subsetneq H$ for which we have *strict* inclusions

$$K \cap gH'g^{-1} \subsetneq K \cap gHg^{-1} \tag{1.1}$$

for all $g \in G$. This condition also appeared in [SS24] where it was shown to control the vanishing of the top derived functor $R^d\mathrm{Ind}_K^G(F)$ for a d-dimensional p-adic Lie group G. A pair (G, K) satisfying (1.1) is called *fair* in this note. Non-trivial reductive groups $G = \mathbf{G}(\mathfrak{F})$ as above always admit a subgroup K for which (G, K) is fair. In fact one can take *any* compact open subgroup $K \subset G$, as follows easily from Bruhat-Tits theory. We remark that in general (1.1) implies G is non-discrete (take $H = \{e\}$).

With this terminology our main result is the following:

Theorem 1.2. *Let F be a field of characteristic $p > 0$. Let G be a locally pro-p group which admits an open subgroup K such that (G, K) is fair, i.e. satisfies (1.1). Then the category of smooth $F[G]$-modules $Mod_F(G)$ has <u>no</u> nonzero projective objects.*

This extends [CK23, Thm. 3.1] to *locally* pro-p groups, and it partially generalizes [SS24] to local fields $\mathfrak{F} = \mathbb{F}_q(\!(t)\!)$ of characteristic p.

In fact our Theorem 4.4 gives a stronger result than Theorem 1.2: In Section 4 we consider the category of representations with a fixed central character. More precisely, we fix a closed central subgroup $C \subset G$, a continuous character $\chi : C \to F^\times$, and we show that the category $\mathrm{Mod}_F^\chi(G)$ (of smooth $F[G]$-modules on which C acts via χ) has no nonzero projective objects if $(G/C, KC/C)$ is fair ($\Rightarrow C$ is not open). For the sake of exposition we have emphasized the case where C is trivial here in the introduction.

Fixing the central character may seem like a nuance, but the categories appearing in the p-adic local Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$ consist of representations with a fixed central character. More precisely one considers locally admissible smooth $F[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules with central character χ (and similarly for more general coefficient rings $\mathcal{O}$ instead of F). See [Pas13] for example.

In Section 6 we suggest one way out of the no projectives conundrum, which is to endow $\mathrm{Mod}_F(G)$ with a coarser exact structure relative to which there *are* enough projectives. We also discuss the corresponding stable category, following [Kel96]. This bears a resemblance to the relative homological approach of [DK23, Sect. 2, Sect. 5].

In Section 7 we give a topological criterion for (G, K) being fair, in terms of the Chabauty space $\mathcal{S}(G)$ of all closed subgroups. We reproduce an argument of Pierre-Emmanuel Caprace proving that (G, K) is fair if and only if the closure of the G-conjugacy class of K contains no discrete subgroups.

2. Preliminary remarks for profinite groups

Let K be an infinite profinite group. We let Ω denote the set of open subgroups of K.

Lemma 2.1. *The index $[K : U]$ becomes arbitrarily large as $U \in \Omega$ varies.*

Proof. Start with any $U \in \Omega$. Since K is infinite we may pick an element $u \in U \backslash \{e\}$. Since U is open there is a $U' \in \Omega$ such that $uU' \subset U$. By choosing U' small enough we can arrange that $e \notin uU'$. Clearly $U' \subsetneq U$, and consequently $[K : U'] > [K : U]$. Thus we can make the index arbitrarily large. □

We fix a field F and consider the category $\mathrm{Mod}_F(K)$ of smooth K-representations on F-vector spaces. Recall that a representation V is smooth if every $v \in V$ has an open stabilizer − in other words v is fixed by some $U \in \Omega$. In particular $\dim_F F[K]v < \infty$. We let V^U denote the subspace of U-fixed vectors in V.

Definition 2.2. For $v \in V$ as above, we let Ω_v denote the set of $U \in \Omega$ for which

(a) U fixes v, and

(b) $[K : U] > \dim_F F[K]v$.

Note that $\Omega_v \neq \varnothing$ by Lemma 2.1. By Frobenius reciprocity, each vector $v \in V^U$ corresponds to a morphism in $\mathrm{Mod}_F(K)$,

$$\begin{aligned}\varphi_{U,v} : \mathrm{ind}_U^K(F) &\longrightarrow V \\ f &\longmapsto \textstyle\sum_{\kappa \in U\backslash K} f(\kappa)\kappa^{-1}v.\end{aligned}$$

Here $\mathrm{ind}_U^K(F)$ is the space of functions $f : U\backslash K \to F$, on which K acts via right translations. This is a finite-dimensional smooth K-representation of dimension $[K : U]$. Obviously $\mathrm{im}(\varphi_{U,v}) = F[K]v$, so $\varphi_{U,v}$ is *not* injective when $U \in \Omega_v$.

We consider the sum of all these morphisms,

$$\begin{aligned}\varphi : S = \bigoplus_{v\in V} \bigoplus_{U\in\Omega_v} \mathrm{ind}_U^K(F) &\longrightarrow V \\ (f_{U,v})_{U,v} &\longmapsto \textstyle\sum_{U,v} \varphi_{U,v}(f_{U,v}).\end{aligned}$$

Clearly φ is surjective since $\varphi_{U,v}(\mathrm{char}_U) = v$ (and any $v \in V$ is fixed by some $U \in \Omega_v$). If V is a projective object of $\mathrm{Mod}_F(K)$ there exists a section $\sigma : V \to S$ of φ in $\mathrm{Mod}_F(K)$. Thus $\varphi \circ \sigma = \mathrm{Id}_V$.

Proposition 2.3. *Suppose F is a field of characteristic $p > 0$. If p^∞ divides $|K|$ there are no nonzero projective objects in $\mathrm{Mod}_F(K)$, and conversely.*

Proof. First assume p^∞ divides the pro-order $|K|$, and pick a Sylow pro-p-subgroup $K' \subset K$ (which is infinite by assumption). The restriction functor $\mathrm{Mod}_F(K) \to \mathrm{Mod}_F(K')$ preserves projective objects since $\mathrm{ind}_{K'}^K$ is an exact right adjoint functor (as K is compact). We may therefore assume that K is an infinite pro-p group.

If V is a projective object of $\mathrm{Mod}_F(K)$, we consider a section σ of φ as above. The section restricts to an embedding $\sigma : V^K \hookrightarrow S^K$. As noted earlier, $\ker(\varphi_{U,v}) \neq \{0\}$ when $U \in \Omega_v$. Therefore the inclusion

$$\{0\} \neq \ker(\varphi_{U,v})^K \subset \mathrm{ind}_U^K(F)^K = \{\text{constants}\}$$

is an equality. In particular $\varphi_{U,v}$ vanishes on the constant functions, and consequently φ vanishes on S^K. Since $\varphi \circ \sigma = \mathrm{Id}_V$ we deduce that $V^K = \{0\}$, which is equivalent to $V = \{0\}$. (See [AW67, Lem. 1, p. 111] for example.)

For the converse, suppose p has finite exponent in $|K|$. Then there exists a $U \in \Omega$ such that $p \nmid |U|$. A standard averaging argument shows the functor $(-)^U$ is exact on $\mathrm{Mod}_F(K)$. By Frobenius reciprocity this amounts to $\mathrm{ind}_U^K(F)$ being a projective object in $\mathrm{Mod}_F(K)$. □

This result (Proposition 2.3) was proved independently in [CK23, Thm. 3.1] using a different method.

3. The general case

We now take G to be a *locally* profinite group, by which we mean it has an open subgroup K which is profinite in the induced topology. We assume G is not discrete, i.e. any such K is infinite. We choose a K once and for all, and continue to let $\Omega = \{\text{open subgroups of } K\}$.

Definition 3.1. We say the pair (G, K) is fair if $\forall H \in \Omega$ there is an $H' \in \Omega$ such that

$$K \cap gH'g^{-1} \subsetneq K \cap gHg^{-1}$$

for all $g \in G$. The group G is fair if (G, K) is fair for some profinite open subgroup $K \subset G$.

In what follows Ind_K^G denotes the full smooth induction functor, i.e. the *right* adjoint to the restriction functor $\mathrm{Mod}_F(G) \to \mathrm{Mod}_F(K)$. Note that Ind_K^G is not exact in general; see [SS24]. To fix ideas we adopt the convention that G acts by right translations on induced representations.

Remark 3.2. The fairness condition (Df. 3.1) also appeared in [SS24]. For a d-dimensional p-adic Lie group G, and K a compact open subgroup, it is shown in [SS24] that (G, K) is fair if and only if $R^d\mathrm{Ind}_K^G(F) = 0$. (Here $R^i\mathrm{Ind}_K^G$ is the i^{th} right derived functor of Ind_K^G.)

Now V denotes an object of $\mathrm{Mod}_F(G)$. We will often denote its restriction $V|_K$ simply by V when there is no risk of confusion. From Section 2 we have the morphism $\varphi : S \twoheadrightarrow V$ in $\mathrm{Mod}_F(K)$.

Proposition 3.3. *The induced morphism* $\mathrm{Ind}_K^G(\varphi)$ *is surjective if* (G, K) *is fair (cf. Def. 3.1).*

Proof. Start with an arbitrary $F \in \mathrm{Ind}_K^G V$ and pick an $H \in \Omega$ fixing F. We simplify the notation by introducing $v_x := F(x) \in V^{K \cap xHx^{-1}}$ for all $x \in G$. If H' satisfies the condition in Def. 3.1 we see that

$$[K : K \cap xH'x^{-1}] > [K : K \cap xHx^{-1}] \geq \dim_F F[K]v_x.$$

Thus $K \cap xH'x^{-1} \in \Omega_{v_x}$, and it makes sense to consider the contribution to S indexed by $v = v_x$ and $U = K \cap xH'x^{-1}$.

Claim: $\forall x \in G$ there is an $f_x \in S^{K \cap xH'x^{-1}}$ such that $\varphi(f_x) = v_x$.

To see this, consider the morphism $\varphi_{K \cap xH'x^{-1}, v_x}$. It maps the characteristic function $\mathrm{char}_{K \cap xH'x^{-1}}$ to v_x. We take f_x to be $\mathrm{char}_{K \cap xH'x^{-1}}$ viewed as a vector in the summand $\mathrm{ind}_{K \cap xH'x^{-1}}^K(F)$ of S indexed by $v = v_x$ and $U = K \cap xH'x^{-1}$. This proves the claim.

Choose a set of representatives R' for $K \backslash G / H'$. This uniquely determines an $A \in (\mathrm{Ind}_K^G S)^{H'}$ such that $A(r') = f_{r'}$ for all $r' \in R'$. We check that $\mathrm{Ind}_K^G(\varphi)(A) = F$. Let $g \in G$ be arbitrary, and write $g = \kappa r' h'$ with $\kappa \in K$, $r' \in R'$, and $h' \in H'$. Then

$$\mathrm{Ind}_K^G(\varphi)(A)(g) = \varphi(A(g)) = \varphi(A(\kappa r')) = \kappa\varphi(A(r')) = \kappa\varphi(f_{r'}) = \kappa v_{r'}.$$

On the other hand, since F is fixed by $H' \subset H$,

$$F(g) = F(\kappa r') = \kappa F(r') = \kappa v_{r'}.$$

This shows that indeed $\mathrm{Ind}_K^G(\varphi)(A) = F$, and as F is arbitrary $\mathrm{Ind}_K^G(\varphi)$ is surjective. $\square$

Adjunction gives us the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Mod}_F(K)}(V, S) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Mod}_F(G)}(V, \mathrm{Ind}_K^G S) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \mathrm{Ind}_K^G(\varphi)_*} \\ \mathrm{Hom}_{\mathrm{Mod}_F(K)}(V, V) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Mod}_F(G)}(V, \mathrm{Ind}_K^G V). \end{array}$$

Assume (G, K) is fair. If V is a projective object of $\mathrm{Mod}_F(G)$, the morphism $\mathrm{Ind}_K^G(\varphi)_*$ in the diagram is surjective by Proposition 3.3. Hence so is the dashed morphism φ_*. In particular φ admits a section σ in $\mathrm{Mod}_F(K)$.

Theorem 3.4. *Let F be a field of characteristic $p > 0$. Suppose (G, K) is fair for some infinite pro-p open subgroup $K \subset G$. Then there are no nonzero projective objects in $\mathrm{Mod}_F(G)$.*

Proof. By the preliminary remarks leading up to the Theorem, φ admits a section σ in $\mathrm{Mod}_F(K)$. The (second paragraph of the) proof of Proposition 2.3 now applies verbatim. □

For future reference perhaps it is worth highlighting the following reformulation of Theorem 3.4.

Corollary 3.5. *Keep the setup and assumptions from Theorem 3.4. Suppose $\chi : \mathfrak{Z} \to F^\times$ is a continuous character of a profinite abelian subgroup $\mathfrak{Z} \subset G$ with pro-order prime-to-p. Then the subcategory $\mathrm{Mod}_F^\chi(G)$ (=smooth $F[G]$-modules on which $\mathfrak{Z}$ acts by χ) has no nonzero projective objects.*

Proof. For $V \in \mathrm{Mod}_F(G)$ we let $V^\chi := \{v \in V : zv = \chi(z)v, \forall z \in \mathfrak{Z}\}$ denote the χ-eigenspace. The resulting functor $(\cdot)^\chi$ is right adjoint to the inclusion functor $\iota : \mathrm{Mod}_F^\chi(G) \to \mathrm{Mod}_F(G)$. Once we observe $(\cdot)^\chi$ is exact, ι preserves projectives (and we are done by 3.4).

The usual averaging argument applies: For $v \in V$, fixed by some small enough open subgroup U, consider

$$\tilde{v} := \frac{1}{[\mathfrak{Z} : \mathfrak{Z} \cap U]} \cdot \sum\nolimits_{z \in \mathfrak{Z}/\mathfrak{Z} \cap U} \chi(z^{-1}) zv \in V^\chi.$$

If $\gamma : V \to V'$ is a morphism in $\mathrm{Mod}_F(G)$, and $v \in V$ is a vector for which $\gamma(v) \in V'^\chi$, then clearly $\gamma(\tilde{v}) = \gamma(v)$. Thus, if γ is surjective, then so is $\gamma^\chi : V^\chi \to V'^\chi$. □

4. Representations with a fixed central character

In this section we discuss how to adapt the previous arguments to the category of representations with a fixed central character. Our setup is the following: The group G is locally pro-p and we fix an open subgroup K which is pro-p in the induced topology. The field F has characteristic p. We pick a closed central subgroup $C \subset Z(G)$ along with a continuous character $\chi : C \to F^\times$ and consider the category $\mathrm{Mod}_F^\chi(G)$ of smooth $F[G]$-modules on which C acts by χ. (We observe that χ is automatically trivial on any pro-p subgroup of C such as $C \cap K$.)

We will assume (G, K) is C-fair in the following sense (cf. Df. 3.1, which is the case where C is the trivial subgroup):

Definition 4.1. We say the pair (G, K) is C-fair if $\forall H \in \Omega$ there is an $H' \in \Omega$ such that

$$K \cap gH'Cg^{-1} \subsetneq K \cap gHCg^{-1}$$

for all $g \in G$. (Here we keep the notation $\Omega := \{\text{open subgroups of } K\}$.) Equivalently, $(G/C, KC/C)$ is fair in the sense of Df. 3.1. (To see this note that every open subgroup of KC/C has the form HC/C for an open subgroup $H \subset K$, and vice versa.)

This implies that C is *not* open (by taking $H = C \cap K$ in Df. 4.1). As a result thereof, Lemma 2.1 generalizes:

Lemma 4.2. *The index $[K : U(C \cap K)]$ becomes arbitrarily large as $U \in \Omega$ varies.*

Proof. Start with any $U \in \Omega$. Pick an element $u \in U \backslash C$ (which is possible as C is not open). There is a $U' \in \Omega$ such that $uU' \subset U \backslash C$ (as C is closed). Clearly $U'(C \cap K) \subsetneq U(C \cap K)$ – otherwise one can write $u = u'c$ with $u' \in U'$ and $c \in C$ which leads to the contradiction $c \in uU'$. □

Instead of the set Ω_v from Df. 2.2 we consider the set $\Omega_{C,v}$ of all $U \in \Omega$ fixing v such that

$$[K : U(C \cap K)] > \dim_F F[K]v.$$

By Lemma 4.2 this set $\Omega_{C,v}$ is non-empty. Here v is a vector in an object V of $\mathrm{Mod}_F(K/C \cap K)$. Therefore, for $U \in \Omega_{C,v}$, we have a morphism $\varphi_{U,v} : \mathrm{ind}_{U(C\cap K)}^K(F) \to V$ in the latter category with nonzero kernel. We again consider their direct sum

$$\varphi : S = \bigoplus_{v \in V} \bigoplus_{U \in \Omega_{C,v}} \mathrm{ind}_{U(C\cap K)}^K(F) \longrightarrow V.$$

The right adjoint of the restriction functor $\mathrm{Mod}_F^\chi(G) \to \mathrm{Mod}_F(K/C \cap K)$ is given as follows: First we extend V to a representation of KC by letting C act via χ. This gives an object $V \boxtimes \chi$ of $\mathrm{Mod}_F^\chi(KC)$ which we induce to a representation $\mathrm{Ind}_{KC}^G(V \boxtimes \chi)$ in $\mathrm{Mod}_F^\chi(G)$.

Mimicking Proposition 3.3, we now start with an object V from $\mathrm{Mod}_F^\chi(G)$. We restrict V to K and consider the morphism $\varphi : S \twoheadrightarrow V$ constructed above.

Proposition 4.3. *The induced morphism $\mathrm{Ind}_{KC}^G(\varphi \boxtimes \chi)$ is surjective if (G, K) is C-fair (cf. Df. 4.1).*

Proof. Let $H \in \Omega$ and start with an $F \in \mathrm{Ind}_{KC}^G(V \boxtimes \chi)^H$. Now $v_x := F(x) \in (V \boxtimes \chi)^{KC \cap xHx^{-1}}$. Choose an H' as in 4.1. To run the proof of Proposition 3.3 in this context, it remains to note that

$$[K : (KC \cap xH'x^{-1})(C \cap K)] > [K : (KC \cap xHx^{-1})(C \cap K)].$$

If this inequality was an equality we would have

$$(KC \cap xH'x^{-1})C = (KC \cap xHx^{-1})C$$

which contradicts the strict inclusion in 4.1. □

The rest of the proof of Theorem 3.4 now extends word for word and gives:

Theorem 4.4. *Let F be a field of characteristic p. Suppose (G, K) is C-fair for some pro-p open subgroup $K \subset G$. Then there are no nonzero projective objects in $\mathrm{Mod}_F^\chi(G)$ for all characters $\chi : C \to F^\times$.*

Proof. In this setup we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Mod}_F(K/C\cap K)}(V, S) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Mod}_F^\chi(G)}(V, \mathrm{Ind}_{KC}^G(S \boxtimes \chi)) \\
\downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \mathrm{Ind}_{KC}^G(\varphi\boxtimes\chi)_*} \\
\mathrm{Hom}_{\mathrm{Mod}_F(K/C\cap K)}(V, V) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{Mod}_F^\chi(G)}(V, \mathrm{Ind}_{KC}^G(V \boxtimes \chi)).
\end{array}$$

If V is projective in $\mathrm{Mod}_F^\chi(G)$ we find that $\varphi : S \twoheadrightarrow V$ admits a section σ. Since $\varphi_{U,v}$ vanishes on the constant functions, φ must vanish on S^K which contains $\sigma(V^K)$. Thus $V^K = 0 \Longrightarrow V = 0$. □

5. Examples of fair pairs

A fair group is obviously not discrete (take $H = \{e\}$ in Def. 3.1). In this section we give some basic examples of fair groups.

5.1. **Groups with a non-discrete center.** Let G be a locally profinite group with non-discrete center $Z(G)$. Then (G,K) is fair for *any* profinite open subgroup $K \subset G$.

Proof. Let $H \in \Omega$. Pick a $z \in Z(G) \cap H$, $z \neq e$. There exists an $H' \in \Omega$, contained in H, such that

$$e \notin z(Z(G) \cap H').$$

Now, for the sake of contradiction, suppose there is a $g \in G$ for which $K \cap gH'g^{-1} = K \cap gHg^{-1}$. Intersecting both sides with $Z(G)$ yields the equality $Z(G) \cap H' = Z(G) \cap H$. For instance,

$$Z(G) \cap (K \cap gHg^{-1}) = K \cap g(Z(G) \cap H)g^{-1} = K \cap (Z(G) \cap H) = Z(G) \cap H.$$

However, the equality $Z(G) \cap H' = Z(G) \cap H$ implies $e \in z(Z(G) \cap H')$. This contradicts the properties of H'. □

5.2. **Reductive groups over local fields.** Let $G = \mathbf{G}(\mathfrak{F})$ for a connected reductive group $\mathbf{G}$ defined over a (non-archimedean) local field $\mathfrak{F}$. (We emphasize that we allow $\mathfrak{F}$ to have *positive* characteristic.) Then (G,K) is fair for any compact open subgroup K.

Proof. It suffices to show (G,K) is fair for a cofinal system of compact open subgroups, so we may assume K is a principal congruence subgroup. As in [SS24], we pick a special vertex x_0 in the Bruhat-Tits building and consider the principal congruence subgroups K_m of the special parahoric subgroup K_0. We have implicitly fixed a maximal $\mathfrak{F}$-split subtorus $\mathbf{S}$ such that x_0 lies in the associated apartment. We let $Z = \mathbf{Z}(\mathfrak{F})$ denote the $\mathfrak{F}$-points of the centralizer $\mathbf{Z}$ of $\mathbf{S}$, and Z^+ is the usual contracting monoid (see [SS24] for more details). By the Cartan decomposition $G = K_0 Z^+ K_0$ it suffices to show that $\forall n$ there is an $n' > n$ such that

$$K_m \cap zK_{n'}z^{-1} \subsetneq K_m \cap zK_n z^{-1}$$

for all $z \in Z^+$. This follows immediately from the Iwahori factorization of $K_m \cap zK_n z^{-1}$ as in [SS24]. □

In the next subsection we give an alternative proof which works in a more general setup.

5.3. **Groups with a weak Cartan decomposition.** We say G has a *weak Cartan decomposition* if there is a compact subset $\mathcal{C} \subset G$, and a non-discrete subgroup $S \subset G$, such that

$$G = \mathcal{C} \cdot Z_G(S) \cdot \mathcal{C}$$

where $Z_G(S)$ denotes the centralizer of S in G.

Theorem 5.1. *Suppose that G has a weak Cartan decomposition. Then (G,K) is fair for any compact open subgroup $K \subset G$.*

Remark 5.2. Such G are reminiscent of groups with a *Cartan-like decomposition*, cf. [CW23, Df. 3.1], which means $G = \mathcal{C} \cdot A \cdot \mathcal{C}$ for a compact open *subgroup* $\mathcal{C} \subset G$ and a set A of representatives for the double cosets $\mathcal{C} \backslash G / \mathcal{C}$.

Proof. We start by recalling the *tube lemma*: Let X, Y be topological spaces, and assume Y is compact. Suppose $\mathcal{V} \subset X \times Y$ is an open subset containing a slice $\{x\} \times Y$. Then there exists an open subset $U \subset X$ such that $\mathcal{V} \supset U \times Y \supset \{x\} \times Y$.

For lack of a reference we indicate a proof hereof: Write $\mathcal{V} = \bigcup_{i \in I} U_i \times V_i$ as a union of open boxes. Then $Y = \bigcup_{i \in I'} V_i$ where $I' = \{i \in I : x \in U_i\}$, which we refine to a finite subcovering $Y = V_{i_1} \cup \cdots \cup V_{i_N}$. One immediately checks that $U = U_{i_1} \cap \cdots \cap U_{i_N}$ satisfies the requirements.

Consider the continuous map

$$\xi : S \times \mathcal{C} \longrightarrow G$$
$$(s, c) \longmapsto csc^{-1}.$$

It maps the slice $\{e\} \times \mathcal{C}$ to $\{e\}$. In particular $\xi^{-1}(K) \supset \{e\} \times \mathcal{C}$, and therefore $\xi^{-1}(K)$ contains a tube $U \times \mathcal{C}$ for some open neighborhood $U \subset S$ of e. This means:

$$\{csc^{-1} : s \in U,\ c \in \mathcal{C}\} \subset K. \tag{5.3}$$

Similarly, given an H as in Definition 3.1, the same argument applied to the map $(s, c) \mapsto c^{-1}sc$ yields an open neighborhood $V \subset S$ of e such that

$$\{c^{-1}sc : s \in V,\ c \in \mathcal{C}\} \subset H. \tag{5.4}$$

We may assume $V \subset U$.

Once and for all we pick an element $\sigma \in V - \{e\}$ (which exists since S is non-discrete) and introduce the compact subset

$$\Sigma := \{c^{-1}\sigma c : c \in \mathcal{C}\} \subset G.$$

Since the complement $G - \Sigma$ is an open neighborhood of e there is an open subgroup $H' \subsetneq H$ with $H' \cap \Sigma = \varnothing$. We claim this H' works in 3.1. If not, there is a $g \in G$ for which we get an equality

$$K \cap gH'g^{-1} = K \cap gHg^{-1}.$$

Write $g = czc'$ according to the weak Cartan decomposition ($c, c' \in \mathcal{C}$ and $z \in Z_G(S)$). Then:

i. $c\sigma c^{-1} \in K$ by (5.3);
ii. $c\sigma c^{-1} = gc'^{-1}z^{-1}\sigma zc'g^{-1} = gc'^{-1}\sigma c'g^{-1} \in gHg^{-1}$ by (5.4).

(In part ii we used the fact that z and σ commute.) In summary $c\sigma c^{-1} \in K \cap gHg^{-1}$. By our hypothesis on g this element $c\sigma c^{-1}$ must therefore lie in $K \cap gH'g^{-1}$. Consequently

$$c'^{-1}\sigma c' = g^{-1}cz\sigma z^{-1}c^{-1}g = g^{-1}c\sigma c^{-1}g \in g^{-1}(K \cap gH'g^{-1})g = g^{-1}Kg \cap H' \subset H'.$$

On the other hand $c'^{-1}\sigma c' \in \Sigma$. This contradicts the assumption that $H' \cap \Sigma = \varnothing$. □

This applies in particular to *covering* groups of $\mathbf{G}(\mathfrak{F})$, as discussed in [FP22, Sect. 3.1] for example. They consider central extensions $\widetilde{\mathbf{G}(\mathfrak{F})}$ of $\mathbf{G}(\mathfrak{F})$ by a finite abelian group. In [FP22, Sect. 5.4] the Cartan decomposition of $\mathbf{G}(\mathfrak{F})$ is lifted to a Cartan decomposition of $\widetilde{\mathbf{G}(\mathfrak{F})}$. See [FP22, Thm. 5.3] for instance.

Theorem 5.5. *Suppose that G has a weak Cartan decomposition $G = \mathcal{C} \cdot Z_G(S) \cdot \mathcal{C}$. Let A be a closed central subgroup of G such that $A \cap S$ is not open in S, and let $\chi : A \to F^\times$ be a smooth character. Then there are no nonzero projectives in $\mathrm{Mod}_F^\chi(G)$.*

Proof. By Theorem 5.1 and Theorem 4.4, it is enough to show that the weak Cartan decomposition of G induces a weak Cartan decomposition on G/A. Given a set $B \subset G$, we denote by $\overline{B} \subset G/A$ its image under the quotient map. Clearly, $G/A = \overline{\mathcal{C}} \cdot \overline{Z_G(S)} \cdot \overline{\mathcal{C}}$, and $\overline{\mathcal{C}}$ is compact. Since $\overline{Z_G(S)} \subset Z_{G/A}(\overline{S})$, we have $G/A = \overline{\mathcal{C}} \cdot Z_{G/A}(\overline{S}) \cdot \overline{\mathcal{C}}$. By assumption, $A \cap S$ is not open in S, hence $\overline{S}$ is not discrete in G/A. Therefore, $G/A = \overline{\mathcal{C}} \cdot Z_{G/A}(\overline{S}) \cdot \overline{\mathcal{C}}$ is a weak Cartan decomposition of G/A. □

6. Other exact structures

One solution addressing the lack of projective objects in $\mathrm{Mod}_F(G)$ is to endow this category with other exact structures. The survey [Buh10] serves as our main reference for the basic notions and properties of exact categories.

Let $\mathcal{E}_{\max}$ be the class of all short exact sequences $0 \to V' \to V \to V'' \to 0$ in $\mathrm{Mod}_F(G)$. For a fixed open subgroup $U \subset G$ we consider the class $\mathcal{E}_U \subset \mathcal{E}_{\max}$ of all such sequences which split in $\mathrm{Mod}_F(U)$. The axioms in [Buh10, Df. 2.1] are easy to verify. This is precisely [Buh10, Exc. 5.3] applied to the restriction functor $\mathrm{Mod}_F(G) \to \mathrm{Mod}_F(U)$ (with split exact sequences). Thus $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ is an exact category.

Remark 6.1. If $U' \subset U$ are compact open subgroups of G with index $[U : U'] \in F^\times$ an immediate averaging argument shows that $\mathcal{E}_{U'} = \mathcal{E}_U$. (If $V \to V''$ admits a U'-equivariant section σ then $\widetilde{\sigma}(v) = \frac{1}{[U:U']}\sum_{u \in U' \backslash U} u^{-1}\sigma(uv)$ defines a U-equivariant section.)

An admissible monic is a morphism $\alpha : V' \to V$ in $\mathrm{Mod}_F(G)$ which admits a U-equivariant retraction (a morphism $\rho : V \to V'$ in $\mathrm{Mod}_F(U)$ such that $\rho \circ \alpha = \mathrm{Id}_{V'}$). We use the notation $\rightarrowtail$ for admissible monics.

Similarly, an admissible epic is a morphism $\beta : V \to V''$ in $\mathrm{Mod}_F(G)$ which admits a U-equivariant section (a morphism $\sigma : V'' \to V$ in $\mathrm{Mod}_F(U)$ such that $\beta \circ \sigma = \mathrm{Id}_{V''}$). We use the notation $\twoheadrightarrow$ for admissible epics.

Projective and injective objects of $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ are defined in [Buh10, Df. 11.1]. For example, P is projective if every admissible epic $\beta : V \twoheadrightarrow V''$ induces a surjective map $\beta_* : \mathrm{Hom}_{\mathrm{Mod}_F(G)}(P, V) \to \mathrm{Hom}_{\mathrm{Mod}_F(G)}(P, V'')$. (See also [Buh10, Prop. 11.3].)

Proposition 6.2. *Let U be an open subgroup of G. Then the following holds:*

(a) *The exact category $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ has enough projectives (see* [Buh10, Df. 11.9]*) and enough injectives.*
(b) *The projective objects are precisely the direct summands of representations of the form $\mathrm{ind}_U^G(W)$ with $W \in \mathrm{Mod}_F(U)$. The injective objects are the summands of $\mathrm{Ind}_U^G(W)$ as W varies.*
(c) *If G is compact, $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ is a Frobenius category (see* [Buh10, Sect. 13.4]*).*

Proof. For part (a) let X be an arbitrary object of $\mathrm{Mod}_F(G)$ and consider the counit of adjunction

$$B : \mathrm{ind}_U^G(X|_U) \longrightarrow X$$
$$f \longmapsto \sum\nolimits_{g \in U \backslash G} g^{-1} f(g).$$

This is an admissible epic. Indeed B has a U-equivariant section $x \mapsto f_x$, where f_x denotes the function in $\mathrm{ind}_U^G(X|_U)$ supported on U and sending the identity to x. Also, $\mathrm{ind}_U^G(X|_U)$ is projective in $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ for the following reason. Any admissible epic $\beta : V \twoheadrightarrow V''$ as above induces a surjective map

$$\mathrm{Hom}_{\mathrm{Mod}_F(U)}(X|_U, V) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}_F(U)}(X|_U, V'')$$

since $\sigma \circ (-)$ is a right inverse. By Frobenius reciprocity this amounts to β_* being surjective. We conclude that for any X there is an admissible epic $P \twoheadrightarrow X$ with P projective.

To show $(\mathrm{Mod}_F(G), \mathcal{E}_U)$ also has enough injectives we consider the unit of adjunction

$$A : X \longrightarrow \mathrm{Ind}_U^G(X|_U)$$
$$x \longmapsto [g \mapsto gx].$$

This is an admissible monic. A U-equivariant retraction is given by evaluation at the identity. It remains to note that $\mathrm{Ind}_U^G(X|_U)$ is injective. So, let $\alpha : V' \rightarrowtail V$ be an admissible monic as above. Again, by Frobenius reciprocity for the full induction, it suffices to observe that pulling back via α induces a surjective map

$$\mathrm{Hom}_{\mathrm{Mod}_F(U)}(V, X|_U) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}_F(U)}(V', X|_U)$$

since $(-) \circ \rho$ is a left inverse. Hence any X admits an admissible monic $X \rightarrowtail I$ into an injective I.

In the previous proofs of the projectivity of $\mathrm{ind}_U^G(X|_U)$ and the injectivity of $\mathrm{Ind}_U^G(X|_U)$ there was nothing special about $X|_U$. We can run the exact same arguments for any U-representation W instead of $X|_U$. Altogether this proves parts (a) and (b).

Part (c) follows immediately from (b) since $\mathrm{ind}_U^G(W) = \mathrm{Ind}_U^G(W)$ when G is compact. □

Any Frobenius category has an associated stable category, which is triangulated. As described in [Kel96, Sect. 6] this construction can be mimicked in greater generality.

In our setup the injectively stable category $S_{\mathrm{in}}(G, U)$ has the same objects as $\mathrm{Mod}_F(G)$ but the morphisms are

$$\mathrm{Hom}_{S_{\mathrm{in}}(G,U)}(V_1, V_2) = \mathrm{Hom}_{\mathrm{Mod}_F(G)}(V_1, V_2)/\mathcal{I}(V_1, V_2)$$

where $\mathcal{I}(V_1, V_2)$ is the space of morphisms $V_1 \to V_2$ which factor through an injective object (with respect to the exact structure $\mathcal{E}_U$).

The suspension functor $T : S_{\mathrm{in}}(G, U) \longrightarrow S_{\mathrm{in}}(G, U)$ has the property that there is a short exact sequence

$$0 \to X \xrightarrow{A} \mathrm{Ind}_U^G(X|_U) \to T(X) \to 0$$

in $\mathcal{E}_U$ for all X (where A is the adjunction map which appeared in the proof of Proposition 6.2). In other words $T(X)$ is the cokernel of A. By [Kel96, Thm. 6.2] this gives $S_{\mathrm{in}}(G, U)$ the structure of a *suspended* category (see [Kel96, Sect. 7]). Essentially what this means is it satisfies all the axioms for a triangulated category except that the suspension functor need not be an equivalence.

Similarly, the projectively stable category $S_{\mathrm{pr}}(G, U)$ is defined by modding out morphisms which factor through a projective object. In this case there is a functor $\Omega : S_{\mathrm{pr}}(G, U) \longrightarrow S_{\mathrm{pr}}(G, U)$ such that there is an exact sequence

$$0 \to \Omega(X) \to \mathrm{ind}_U^G(X|_U) \xrightarrow{B} X \to 0$$

in $\mathcal{E}_U$ for all X (with B as in the proof of Proposition 6.2). Thus $\Omega(X)$ is the kernel of B.

When G is compact we have a Frobenius category. In this case $S_{\mathrm{in}}(G, U) = S_{\mathrm{pr}}(G, U)$ and T, Ω are mutually quasi-inverse equivalences of categories. This gives a triangulated category $S(G, U)$.

When G is finite and $U = \{e\}$ the above construction yields the stable module category $S(G)$ which is of pivotal importance in modular representation theory (when $|G|$ is divisible by the characteristic of F). See [BIK12] for example.

We are optimistic that $S_{\mathrm{in}}(G, U)$ and $S_{\mathrm{pr}}(G, U)$ will likewise play a central role in modular representation theory for non-compact groups, and we hope to explore this in continuation of this paper.

7. An interpretation of fairness in the Chabauty space

The set $\mathcal{S}(G) = \{\text{closed subgroups of } G\}$ carries a natural topology which makes it a compact Hausdorff space; see [Cha50]. The group G acts on $\mathcal{S}(G)$ by conjugation, and for a $K \in \mathcal{S}(G)$ we will consider the G-orbit $\{gKg^{-1} : g \in G\} \subset \mathcal{S}(G)$ and its closure.

We owe the following observation, and its proof, entirely to Pierre-Emmanuel Caprace. We are very grateful to him for allowing us to include his argument here.

Proposition 7.1. *The pair (G, K) is fair if and only if*

$$\overline{\{gKg^{-1} : g \in G\}} \subset \{\textit{non-discrete closed subgroups of } G\}.$$

Proof. First we assume (G, K) is *not* fair. This means there is some $H \in \Omega$ such that for all open subgroups $H' \subset H$ there exists a $g \in G$ for which we have an equality

$$K \cap gH'g^{-1} = K \cap gHg^{-1}.$$

We pick a neighborhood basis at the identity $\{H_i\}_{i\in I}$, for some directed set I, consisting of open subgroups $H_i \subset H$. For each $i \in I$ we select a $g_i \in G$ with the property that

$$K \cap g_iH_ig_i^{-1} = K \cap g_iHg_i^{-1}.$$

Equivalently, $g_i^{-1}Kg_i \cap H \subset H_i$. Consider the net $(g_i^{-1}Kg_i)_{i\in I}$ in $\mathcal{S}(G)$. Since $\mathcal{S}(G)$ is compact we can extract a convergent subnet $(g_{f(j)}^{-1}Kg_{f(j)})_{j\in J}$ for some reindexing function $f : J \to I$. Call the limit Δ. We claim Δ is discrete, which will finish the proof of the *if* part (by contraposition).

Intersection with H gives a continuous map (see [HS14, Prop. 2.2] for instance)

$$\mathcal{S}(G) \longrightarrow \mathcal{S}(H)$$
$$C \longmapsto C \cap H.$$

We deduce that $g_{f(j)}^{-1}Kg_{f(j)} \cap H$ converges to $\Delta \cap H$. On the other hand this net converges to $\{e\}$ since $g_i^{-1}Kg_i \cap H \subset H_i$ and $H_i \to \{e\}$. As $\mathcal{S}(G)$ is Hausdorff we conclude that $\Delta \cap H = \{e\}$, and in particular Δ is discrete.

To prove the *only if* part, assume there is some discrete group Δ in the closure of $\{g^{-1}Kg : g \in G\}$. We write Δ as a limit of a net $(y_j^{-1}Ky_j)_{j\in J}$ (for some possibly new directed set J). Since Δ is discrete, $\Delta \cap H = \{e\}$ for some open subgroup $H \subset K$. For every open subgroup $H' \subsetneq H$ there is a j for which

$$y_j^{-1}Ky_j \cap H \subset H',$$

using that $y_j^{-1}Ky_j \cap H \to \{e\}$. We infer that the inclusions below are equalities:

$$y_j^{-1}Ky_j \cap H \subset y_j^{-1}Ky_j \cap H' \subset y_j^{-1}Ky_j \cap H.$$

Conjugation by y_j shows that

$$K \cap y_jHy_j^{-1} = K \cap y_jH'y_j^{-1}.$$

Therefore (G, K) is not fair. □

In the previous proof we used the topology of geometric convergence on $\mathcal{S}(G)$. This is identical to the Chabauty topology by [GR06, Lem. 2, p. 880], for example.

Remark 7.2. Caprace has informed us that the so-called Neretin groups *fail* to satisfy Proposition 7.1: For such groups there are choices of K for which $\{gKg^{-1} : g \in G\}$ has $\{e\}$ as an accumulation point.

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