

The unification in an $\widehat{\mathfrak{su}}(8)_{k_U=1}$ affine Lie algebra

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Abstract

A flavor-unified theory based on the simple Lie algebra of $\mathfrak{su}(8)$ was previously proposed to generate the observed Standard Model quark/lepton mass hierarchies and the Cabibbo-Kobayashi-Maskawa mixing pattern due to their non-universal symmetry properties. A level-1 affine Lie algebra of $\widehat{\mathfrak{su}}(8)_{k_U=1}$ with the $\mathcal{N} = 1$ supersymmetric extension is found to unify three gauge couplings through the maximally symmetry breaking pattern.

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Contents

1	Introduction	1
1.1	Historical review and recent progresses	1
1.2	The main results	4
2	The conformal embedding of the $\widehat{\mathfrak{su}}(8)_{k_U}$ affine Lie algebra	8
3	The RGEs of the SUSY $\widehat{\mathfrak{su}}(8)_{k_U=1}$	11
4	Summary	15
A	Some results of the affine Lie algebra	18
A.1	The Lie algebra: Cartan-Weyl basis, simple roots, and the highest weight	18
A.2	The affine Lie algebra: Cartan-Weyl basis, simple roots, and the highest weight	21
B	The U(1) Cartan discontinuities	23

1 Introduction

1.1 Historical review and recent progresses

Grand Unified Theories (GUTs) were proposed to not only unify all three fundamental symmetries into one simple Lie group of either SU(5) [1] or SO(10) [2], but to also unify all Standard Model (SM) fermions into some anomaly-free irreps of the \mathcal{G}_U . The Georgi-Glashow SU(5) GUT [1] and the Fritzsch-Minkowski SO(10) GUT [2] contain the chiral fermions of $3 \times [\mathbf{5}_F \oplus \mathbf{10}_F]$ and $3 \times \mathbf{16}_F$, respectively. Neither framework can fully explain the three-generational SM fermion mass hierarchies and the Cabibbo-Kobayashi-Maskawa (CKM) mixing pattern [3, 4] due to their trivially repetitive flavor structures.

In a pioneering work by Georgi [5], an extension to the gauge group of SU($N > 5$) was proposed to embed three-generational SM fermions non-trivially. In his original proposal, any anti-symmetric chiral fermion irrep can appear at most once with the anomaly-free condition. Physically, this leads to a non-repetitive structure for both left-handed and right-handed components of chiral fermions based on an SU(11) gauge group. Given the current LHC measurements of one single SM Higgs boson so far, which have already confirmed the hierarchical Yukawa couplings of the third-generational SM quarks/leptons and displayed the evidence for the muon [6, 7], Georgi’s original proposal is insightful and can be further refined with the concept of chiral irreducible anomaly-free fermion sets (IRAFFSs) [8] as follows

Definition A chiral IRAFFS is a set of left-handed anti-symmetric fermions of $\sum_{\mathcal{R}} m_{\mathcal{R}} \mathcal{F}_L(\mathcal{R})$, with $m_{\mathcal{R}}$ being the multiplicities of a particular fermion irrep of \mathcal{R} . Obviously, the anomaly-free condition reads $\sum_{\mathcal{R}} m_{\mathcal{R}} \text{Anom}(\mathcal{F}_L(\mathcal{R})) = 0$. We also require the following conditions to be satisfied for a chiral IRAFFS:

- the greatest common divisor (GCD) of the $\{m_{\mathcal{R}}\}$ should satisfy that $\text{GCD}\{m_{\mathcal{R}}\} = 1$;
- the fermions in a chiral IRAFFS can no longer be removed, which would otherwise bring non-vanishing gauge anomalies;

- *there should not be any singlet, self-conjugate, adjoint fermions, or vectorial fermion pairs in a chiral IRAFFS.*

A minimal SU(8) theory [8–10] was thus proposed to include two following chiral IRAFFSs at the GUT scale

$$\{f_L\}_{\text{SU}(8)}^{n_g=3} = \left[\overline{\mathbf{8}}_{\mathbf{F}}^\omega \oplus \mathbf{28}_{\mathbf{F}} \right] \oplus \left[\overline{\mathbf{8}}_{\mathbf{F}}^{\dot{\omega}} \oplus \mathbf{56}_{\mathbf{F}} \right], \quad \dim_{\mathbf{F}} = 156, \\ \Omega \equiv (\omega, \dot{\omega}), \quad \omega = (3, \text{IV}, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VII}, \text{VIII}, \text{IX}), \quad (1)$$

with undotted/dotted indices for the $\overline{\mathbf{8}}_{\mathbf{F}}$'s in the rank-2 chiral IRAFFS and the rank-3 chiral IRAFFS¹, respectively. We also distinguish the heavy partner fermions and the SM fermions in terms of the Roman numbers and the Arabic numbers. To count the SM generations by decomposing the SU(8) fermion irreps into the SU(5) irreps [5], one finds three identical SM $\overline{\mathbf{5}}_{\mathbf{F}}$'s and three distinctive $\mathbf{10}_{\mathbf{F}}$'s [9]. Hence, it is sufficient to obtain three distinctive SM generations in the UV setup of the SU(8) theory.

Fermions	$\overline{\mathbf{8}}_{\mathbf{F}}^\Omega$	$\mathbf{28}_{\mathbf{F}}$	$\mathbf{56}_{\mathbf{F}}$	
$\tilde{\text{U}}(1)_T$	$-3t$	$+2t$	$+t$	
$\tilde{\text{U}}(1)_{\text{PQ}}$	p	q_2	q_3	
Higgs	$\overline{\mathbf{8}}_{\mathbf{H},\omega}$	$\mathbf{28}_{\mathbf{H},\dot{\omega}}$	$\mathbf{70}_{\mathbf{H}}$	$\mathbf{63}_{\mathbf{H}}$
$\tilde{\text{U}}(1)_T$	$+t$	$+2t$	$-4t$	0
$\tilde{\text{U}}(1)_{\text{PQ}}$	$-(p+q_2)$	$-(p+q_3)$	$-2q_2$	0

Table 1: The non-anomalous $\tilde{\text{U}}(1)_T$ charges and the anomalous global $\tilde{\text{U}}(1)_{\text{PQ}}$ charges for the SU(8) fermions and Higgs fields.

With the concept of the chiral IRAFFSs in Eq. (1), one identifies the global flavor symmetries of the chiral fermions in Eq. (1) to be

$$\tilde{\mathcal{G}}_{\text{flavor}}[\text{SU}(8), n_g = 3] = \left[\widetilde{\text{SU}}(4)_\omega \otimes \tilde{\text{U}}(1)_{T_2} \otimes \tilde{\text{U}}(1)_{\text{PQ}_2} \right] \otimes \left[\widetilde{\text{SU}}(5)_{\dot{\omega}} \otimes \tilde{\text{U}}(1)_{T_3} \otimes \tilde{\text{U}}(1)_{\text{PQ}_3} \right], \quad (2)$$

with the $\tilde{\text{U}}(1)_{\text{PQ}_i}$ being the anomalous global Peccei-Quinn symmetries [11]. The most general gauge-invariant Yukawa couplings at least include the following renormalizable and non-renormalizable terms²

$$-\mathcal{L}_Y = Y_{\mathcal{B}} \overline{\mathbf{8}}_{\mathbf{F}}^\omega \mathbf{28}_{\mathbf{F}} \overline{\mathbf{8}}_{\mathbf{H},\omega} + Y_{\mathcal{T}} \mathbf{28}_{\mathbf{F}} \mathbf{28}_{\mathbf{F}} \mathbf{70}_{\mathbf{H}} \\ + Y_{\mathcal{D}} \overline{\mathbf{8}}_{\mathbf{F}}^{\dot{\omega}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}} + \frac{c_4}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}}^\dagger \mathbf{63}_{\mathbf{H}} + H.c.. \quad (3)$$

All renormalizable Yukawa couplings and the Wilson coefficient of the non-renormalizable term are expected to be $(Y_{\mathcal{B}}, Y_{\mathcal{T}}, Y_{\mathcal{D}}, c_4) \sim \mathcal{O}(1)$. Accordingly, the non-anomalous global $\tilde{\text{U}}(1)_T$ charges and

¹The rank-2 and the rank-3 chiral IRAFFSs are named after the SU(8) rank-2 and rank-3 anti-symmetric fermions of $A_2 = \mathbf{28}_{\mathbf{F}}$ and $A_3 = \mathbf{56}_{\mathbf{F}}$, respectively.

²The term of $\mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \mathbf{28}_{\mathbf{H}} + H.c.$ vanishes due to the anti-symmetric property [9]. Instead, only a $d = 5$ non-renormalizable term of $\frac{1}{M_{\text{pl}}} \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}}^\dagger \mathbf{63}_{\mathbf{H}}$ is possible to generate masses for vectorlike fermions in the $\mathbf{56}_{\mathbf{F}}$. Since it transforms as an $\widetilde{\text{SU}}(5)_{\dot{\omega}}$ vector and carries non-vanishing $\tilde{\text{U}}(1)_{\text{PQ}}$ charge of $p + 3q_3 \neq 0$ from Eq. (4), it is only possible due to the gravitational effect.

the anomalous global $\widetilde{U}(1)_{\text{PQ}}$ charges for fermions and Higgs fields are assigned in Tab. 1, where the anomalous global $\widetilde{U}(1)_{\text{PQ}}$ charges are assigned such that

$$p : q_2 \neq -3 : +2, \quad p : q_3 \neq -3 : +1. \quad (4)$$

The adjoint Higgs field of $\mathbf{63}_{\mathbf{H}}$ in Eq. (3) will maximally break the symmetry as follows [12]

$$\begin{aligned} \text{SU}(8) &\xrightarrow{\langle \mathbf{63}_{\mathbf{H}} \rangle} \mathcal{G}_{441}, \quad \mathcal{G}_{441} \equiv \text{SU}(4)_s \otimes \text{SU}(4)_W \otimes \text{U}(1)_{X_0} \\ \langle \mathbf{63}_{\mathbf{H}} \rangle &= \frac{1}{4} \text{diag}(-\mathbb{I}_{4 \times 4}, +\mathbb{I}_{4 \times 4}) v_U. \end{aligned} \quad (5)$$

In the physical basis, the maximally symmetry breaking pattern defines the $\text{U}(1)_{X_0}$ charges for the $\text{SU}(8)$ fundamental representation as follows

$$\mathcal{X}_{0,\text{phys}}(\mathbf{8}) \equiv \text{diag}\left(\underbrace{-\frac{1}{4}\mathbb{I}_{4 \times 4}}_{4_s}, \underbrace{+\frac{1}{4}\mathbb{I}_{4 \times 4}}_{4_w}\right). \quad (6)$$

With the maximally $\text{SU}(8)$ symmetry breaking pattern in Eq. (5), the gauge coupling of α_{X_0} associated with the $\text{U}(1)_{X_0}$ charges is identical to the $\text{U}(1)_1$ gauge coupling of α_1 obtained from the Cartan subalgebra [13]. Accordingly, several relevant $\text{SU}(8)$ irreps are decomposed as follows ³

$$\mathbf{8} \hookrightarrow (\mathbf{4}, \mathbf{1}, -\frac{1}{4}) \oplus (\mathbf{1}, \mathbf{4}, +\frac{1}{4}), \quad (7a)$$

$$\mathbf{28} \hookrightarrow (\mathbf{6}, \mathbf{1}, -\frac{1}{2}) \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2}) \oplus (\mathbf{4}, \mathbf{4}, 0), \quad (7b)$$

$$\mathbf{56} \hookrightarrow (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, -\frac{3}{4}) \oplus (\mathbf{4}, \mathbf{6}, +\frac{1}{4}) \oplus (\mathbf{6}, \mathbf{4}, -\frac{1}{4}), \quad (7c)$$

$$\mathbf{70} \hookrightarrow (\mathbf{1}, \mathbf{1}, -1) \oplus (\mathbf{1}, \mathbf{1}, +1) \oplus (\mathbf{4}, \bar{\mathbf{4}}, +\frac{1}{2}) \oplus (\bar{\mathbf{4}}, \mathbf{4}, -\frac{1}{2}) \oplus (\mathbf{6}, \mathbf{6}, 0), \quad (7d)$$

in the physical basis. Separately, we will also define the *algebraic basis* for the conformal embedding, where the $\text{U}(1)_{X_0}$ charges for the $\text{SU}(8)$ fundamental representation are integers

$$\mathcal{X}_{0,\text{alg}}(\mathbf{8}) \equiv \text{diag}\left(\underbrace{-\mathbb{I}_{4 \times 4}}_{4_s}, \underbrace{+\mathbb{I}_{4 \times 4}}_{4_w}\right). \quad (8)$$

All $\text{SU}(8)$ irreps in Tab. 1 also carry integer-valued $\text{U}(1)_{X_0}$ charges as follows

$$\mathbf{8} \hookrightarrow (\mathbf{4}, \mathbf{1}, -1) \oplus (\mathbf{1}, \mathbf{4}, +1), \quad (9a)$$

$$\mathbf{28} \hookrightarrow (\mathbf{6}, \mathbf{1}, -2) \oplus (\mathbf{1}, \mathbf{6}, +2) \oplus (\mathbf{4}, \mathbf{4}, 0), \quad (9b)$$

$$\mathbf{56} \hookrightarrow (\mathbf{1}, \bar{\mathbf{4}}, +3) \oplus (\bar{\mathbf{4}}, \mathbf{1}, -3) \oplus (\mathbf{4}, \mathbf{6}, +1) \oplus (\mathbf{6}, \mathbf{4}, -1), \quad (9c)$$

$$\mathbf{70} \hookrightarrow (\mathbf{1}, \mathbf{1}, -4) \oplus (\mathbf{1}, \mathbf{1}, +4) \oplus (\mathbf{4}, \bar{\mathbf{4}}, +2) \oplus (\bar{\mathbf{4}}, \mathbf{4}, -2) \oplus (\mathbf{6}, \mathbf{6}, 0). \quad (9d)$$

³The decompositions of the conjugated irreps are neglected, since their conformal dimensions are identical to the original irreps.

There can be three possible sequential symmetry breaking patterns ⁴, namely,

$$\begin{aligned}
\text{SWW} & : \text{SU}(8) \hookrightarrow \mathcal{G}_{441} \hookrightarrow \mathcal{G}_{341} \hookrightarrow \mathcal{G}_{331} \hookrightarrow \mathcal{G}_{\text{SM}}, \\
& \mathcal{G}_{341} \equiv \text{SU}(3)_c \otimes \text{SU}(4)_W \otimes \text{U}(1)_{X_1}, \quad \mathcal{G}_{331} \equiv \text{SU}(3)_c \otimes \text{SU}(3)_W \otimes \text{U}(1)_{X_2}, \\
& v_{441} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{341} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{331} \simeq 4.8 \times 10^{13} \text{ GeV}, \quad (10a)
\end{aligned}$$

$$\begin{aligned}
\text{WSW} & : \text{SU}(8) \hookrightarrow \mathcal{G}_{441} \hookrightarrow \mathcal{G}_{431} \hookrightarrow \mathcal{G}_{331} \hookrightarrow \mathcal{G}_{\text{SM}}, \\
& \mathcal{G}_{431} \equiv \text{SU}(4)_s \otimes \text{SU}(3)_W \otimes \text{U}(1)_{X_1}, \\
& v_{441} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{431} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{331} \simeq 4.8 \times 10^{13} \text{ GeV}, \quad (10b)
\end{aligned}$$

$$\begin{aligned}
\text{WWS} & : \text{SU}(8) \hookrightarrow \mathcal{G}_{441} \hookrightarrow \mathcal{G}_{431} \hookrightarrow \mathcal{G}_{421} \hookrightarrow \mathcal{G}_{\text{SM}}, \\
& \mathcal{G}_{421} \equiv \text{SU}(4)_s \otimes \text{SU}(2)_W \otimes \text{U}(1)_{X_2}, \quad \mathcal{G}_{\text{SM}} \equiv \text{SU}(3)_c \otimes \text{SU}(2)_W \otimes \text{U}(1)_Y, \\
& v_{441} \simeq 1.4 \times 10^{17} \text{ GeV}, \quad v_{431} \simeq 4.8 \times 10^{15} \text{ GeV}, \quad v_{421} \simeq 1.1 \times 10^{15} \text{ GeV}, \quad (10c)
\end{aligned}$$

which have been analyzed in details in Refs. [10, 13, 14]. Three intermediate symmetry breaking stages exist above the EW scale by counting the rank of the SU(8) group. Through the analysis of the emergent non-anomalous global $\tilde{\text{U}}(1)_T$ symmetry in Eq. (2), we derived the non-anomalous global $B-L$ symmetry in the SM, and one unique SM Higgs doublet of $(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})''_{\mathbf{H}} \subset \mathbf{70}_{\mathbf{H}}$ was conjectured [8] based on (i) its vanishing global $B-L$ charge, and (ii) its renormalizable Yukawa coupling to the top quark at the tree level. Due to the distinctive symmetry properties of three-generational SM fermions, and with the unique SM Higgs doublet of $(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})''_{\mathbf{H}} \subset \mathbf{70}_{\mathbf{H}}$ as the guidance, we further found in Ref. [10] that all SM quark/lepton masses as well as the CKM mixing pattern can be explained with (i) both $d=5$ direct Yukawa coupling terms and indirect Yukawa coupling terms to the $d=5$ irreducible Higgs mixing operators induced by the inevitable gravitational effects that break the global flavor symmetries in Eq. (2), (ii) three reasonable intermediate symmetry breaking scales suggested in Eqs. (10), and (iii) precise identifications of the SM flavors in their UV irreps [10, 14]. Specifically, the $d=5$ direct Yukawa coupling terms include

$$\begin{aligned}
c_4 \mathcal{O}_{\mathcal{F}}^{(4,1)} & \equiv c_4 \mathbf{56}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{28}}_{\mathbf{H}, \dot{\omega}} \cdot \mathbf{70}_{\mathbf{H}}, \\
c_5 \mathcal{O}_{\mathcal{F}}^{(5,1)} & \equiv c_5 \mathbf{28}_{\mathbf{F}} \mathbf{56}_{\mathbf{F}} \cdot \overline{\mathbf{8}}_{\mathbf{H}, \omega} \cdot \mathbf{70}_{\mathbf{H}}. \quad (11)
\end{aligned}$$

For the later convenience, we shall focus on the SWW symmetry breaking pattern in Eq. (10a) throughout this paper. Accordingly, we tabulate the fermion representations at various stages of the SU(8) model in Tabs. 2, 3, and 4.

1.2 The main results

The GUT can only be valid when the gauge coupling unification is achieved in terms of their renormalization group equations (RGEs) ⁵. Based on the extended gauge sectors and the suggested intermediate symmetry breaking scales in Eqs. (10), we perform the analyses of the gauge coupling RGEs in Refs. [13, 14]. It turns out that the gauge couplings cannot achieve the unification in the minimal SU(8) setup, with effects of (i) different hypotheses of the Higgs fields' masses in the spectrum, (ii) the reasonable threshold effects [16, 18], and (iii) the possible $d=5$ gravity-induced operators of

$$\mathcal{O}_{\text{HSW}} \equiv -\frac{1}{2} \frac{c_{\text{HSW}}}{M_{\text{pl}}} \text{Tr}[\mathbf{63}_{\mathbf{H}} U^{\mu\nu} U_{\mu\nu}], \quad (12)$$

⁴The acronyms stand for the strong-weak-weak (SWW), weak-strong-weak (WSW), and weak-weak-strong (WWS) symmetry breaking patterns.

⁵Some early studies of the RGEs based on the SU(5) and SO(10) groups include Refs. [15–17].

SU(8)	\mathcal{G}_{441}	\mathcal{G}_{341}	\mathcal{G}_{331}	\mathcal{G}_{SM}
$\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega}$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega}$	$(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} : \mathcal{D}_R^{\Omega c}$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} : \tilde{\mathcal{N}}_L^{\Omega}$ $(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})_{\mathbf{F}}^{\Omega} : \mathcal{L}_L^{\Omega} = (\mathcal{E}_L^{\Omega}, -\mathcal{N}_L^{\Omega})^T$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega'} : \tilde{\mathcal{N}}_L^{\Omega'}$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} : \tilde{\mathcal{N}}_L^{\Omega''}$
	$(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{4}}, -\frac{1}{4})_{\mathbf{F}}^{\Omega}$	$(\mathbf{1}, \overline{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega}$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''}$	

Table 2: The SU(8) fermion representation of $\overline{\mathbf{8}}_{\mathbf{F}}^{\Omega}$ under the $\mathcal{G}_{441}, \mathcal{G}_{341}, \mathcal{G}_{331}, \mathcal{G}_{\text{SM}}$ subgroups for the three-generational SU(8) theory, with $\Omega \equiv (\omega, \dot{\omega})$. Here, we denote $\underline{\mathcal{D}}_R^{\Omega c} = \underline{d}_R^{\Omega c}$ for the SM right-handed down-type quarks, and $\mathcal{D}_R^{\Omega c} = \mathfrak{D}_R^{\Omega c}$ for the right-handed down-type heavy partner quarks. Similarly, we denote $\underline{\mathcal{L}}_L^{\Omega} = (\underline{\ell}_L^{\Omega}, -\underline{\nu}_L^{\Omega})^T$ for the left-handed SM lepton doublets, and $\mathcal{L}_L^{\Omega} = (\mathfrak{e}_L^{\Omega}, -\mathfrak{n}_L^{\Omega})^T$ for the left-handed heavy lepton doublets. All left-handed neutrinos of $\tilde{\mathcal{N}}_L$ are sterile neutrinos, which are \mathcal{G}_{SM} -singlets and do not couple to the EW gauge bosons.

SU(8)	\mathcal{G}_{441}	\mathcal{G}_{341}	\mathcal{G}_{331}	\mathcal{G}_{SM}
$\mathbf{28}_{\mathbf{F}}$	$(\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}} : \underline{\mathfrak{D}}_L$
		$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} : \underline{t}_R^c$
		$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}} : (\underline{\mathfrak{e}}_R^c, \underline{\mathfrak{n}}_R^c)^T$
	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}} : \underline{\mathfrak{n}}_R^c$
			$(\mathbf{1}, \overline{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}$	$(\mathbf{1}, \overline{\mathbf{2}}, +\frac{1}{2})_{\mathbf{F}}' : (\underline{\mathfrak{n}}_R^c, -\underline{\mathfrak{e}}_R^c)^T$
		$(\mathbf{4}, \mathbf{4}, 0)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)_{\mathbf{F}} : \underline{\tau}_R^c$
			$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})_{\mathbf{F}} : (\underline{t}_L, \underline{b}_L)^T$
	$(\mathbf{4}, \mathbf{4}, 0)_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{4}, -\frac{1}{12})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}' : \underline{\mathfrak{D}}_L'$
			$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})_{\mathbf{F}}'' : \underline{\mathfrak{D}}_L''$
		$(\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})_{\mathbf{F}}''$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})_{\mathbf{F}}'' : (\underline{\mathfrak{e}}_R^c, \underline{\mathfrak{n}}_R^c)^T$
$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}''$			$(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}' : \underline{\mathfrak{n}}_R^c$ $(\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}'' : \underline{\mathfrak{n}}_R^c$	

Table 3: The SU(8) fermion representation of $\mathbf{28}_{\mathbf{F}}$ under the $\mathcal{G}_{441}, \mathcal{G}_{341}, \mathcal{G}_{331}, \mathcal{G}_{\text{SM}}$ subgroups for the three-generational SU(8) theory. All SM fermions are marked with underlines.

SU(8)	\mathcal{G}_{441}	\mathcal{G}_{341}	\mathcal{G}_{331}	\mathcal{G}_{SM}
56_F	$(\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})'''_{\mathbf{F}} : (\mathbf{n}'''_R, -\mathbf{e}'''_R)^T$
			$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'_{\mathbf{F}} : \mu_R^c$
			$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)''_{\mathbf{F}} : \mathfrak{E}_R^c$
	$(\bar{\mathbf{4}}, \mathbf{1}, -\frac{3}{4})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_{\mathbf{F}} : u_R^c$
			$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, -1)_{\mathbf{F}} : \mathfrak{E}_L$
			$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'_{\mathbf{F}} : (c_L, s_L)^T$
	$(\mathbf{4}, \mathbf{6}, +\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{6}, +\frac{1}{6})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{3}, 0)'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})'''_{\mathbf{F}} : \mathfrak{D}''_L$
			$(\mathbf{3}, \bar{\mathbf{3}}, +\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \bar{\mathbf{2}}, +\frac{1}{6})''_{\mathbf{F}} : (\mathfrak{d}_L, -u_L)^T$
			$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, +\frac{2}{3})_{\mathbf{F}} : \mathfrak{U}_L$
	$(\mathbf{6}, \mathbf{4}, -\frac{1}{4})_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{6}, +\frac{1}{2})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{3}, +\frac{1}{3})'_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{2}, +\frac{1}{2})''''_{\mathbf{F}} : (\mathbf{e}''''_R, \mathbf{n}''''_R)^T$
			$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, 0)'''_{\mathbf{F}} : \check{\mathbf{n}}'''_R^c$
			$(\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})''_{\mathbf{F}}$	$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})''''_{\mathbf{F}} : (\mathbf{n}''''_R, -\mathbf{e}''''_R)^T$
$(\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_{\mathbf{F}}$		$(\mathbf{3}, \mathbf{3}, 0)''_{\mathbf{F}}$	$(\mathbf{1}, \mathbf{1}, +1)'''_{\mathbf{F}} : e_R^c$	
		$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{2}, +\frac{1}{6})'''_{\mathbf{F}} : (u_L, d_L)^T$	
		$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}} : \mathfrak{D}''''_L$	
$(\bar{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{3}, -\frac{1}{3})_{\mathbf{F}}$	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})''''_{\mathbf{F}} : \mathfrak{D}''''_L$		
	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6})_{\mathbf{F}} : (\mathfrak{d}_R^c, u_R^c)^T$		
	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}}$	$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})''_{\mathbf{F}} : \mathfrak{U}_R^c$		
			$(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'''_{\mathbf{F}} : c_R^c$	

Table 4: The SU(8) fermion representation of **56_F** under the $\mathcal{G}_{441}, \mathcal{G}_{341}, \mathcal{G}_{331}, \mathcal{G}_{\text{SM}}$ subgroups for the three-generational SU(8) theory. All SM fermions are marked with underlines.

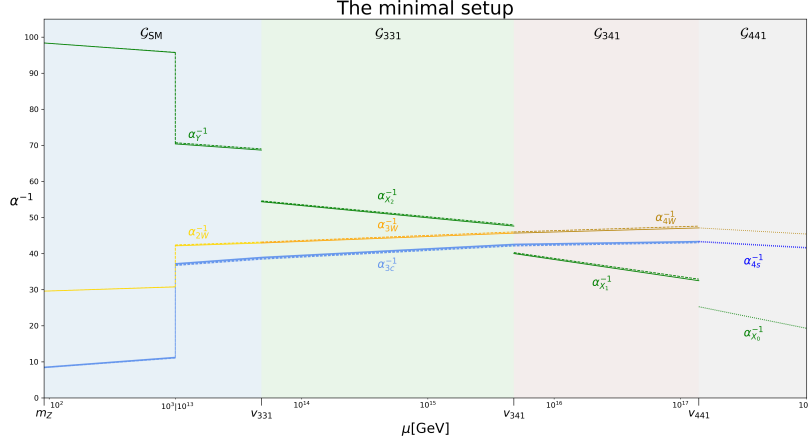


Figure 1: The RGEs of the minimal non-SUSY SU(8) setup through the SWW symmetry breaking pattern in Eq. (10a). The dashed lines and the solid lines represent the one-loop and two-loop RGEs, respectively. The interval of $10^3 \text{ GeV} \lesssim \mu \lesssim 10^{13} \text{ GeV}$ are zoomed out in order to highlight the behaviors in three intermediate symmetry breaking scales.

suggested by Hill-Shafi-Wetterich [19, 20] taken into account. Here, the $U_{\mu\nu}$ represents the SU(8) field strength tensor. After the GUT-scale symmetry breaking with the VEV in Eq. (5), this HSW operator shifts two SU(4)_{s/W} field strengths of $G_{\mu\nu} = G_{\mu\nu}^{\bar{A}} T_{\text{SU}(4)_s}^{\bar{A}}$ and $W_{\mu\nu} = W_{\mu\nu}^{\bar{I}} T_{\text{SU}(4)_W}^{\bar{I}}$ as follows

$$-\frac{1}{2}\left(1 - \frac{1}{4}c_{\text{HSW}}\zeta_0\right) \text{Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{2}\left(1 + \frac{1}{4}c_{\text{HSW}}\zeta_0\right) \text{Tr}(W_{\mu\nu}W^{\mu\nu}), \quad (13)$$

with $\zeta_0 \equiv v_U/M_{\text{pl}}$. The non-Abelian gauge coupling unification is modified into

$$\left(1 - \frac{1}{4}c_{\text{HSW}}\zeta_0\right)\alpha_{4s}(v_U) = \left(1 + \frac{1}{4}c_{\text{HSW}}\zeta_0\right)\alpha_{4W}(v_U) = \alpha_U(v_U), \quad (14)$$

while the Abelian gauge coupling is untouched with the maximally symmetry breaking pattern in Eq. (5). A typical example of the RGEs in the minimal SU(8) setup is given in Fig. 1, with the survival hypothesis [5, 21–25] to the Higgs spectrum⁶. In other words, the difficulty of achieving the gauge coupling unification mainly comes from: (i) the defined *Cartan discontinuities* of U(1) gauge couplings in Eq. (116), and (ii) the suggested intermediate symmetry breaking scales, which were previously determined according to the observed SM quark/lepton masses and the CKM mixing patterns [10].

Furthermore, the previous analyses of different symmetry breaking patterns [14] seem to suggest a following gauge coupling relation of

$$\alpha_{4s}^{-1}(v_U) \approx \alpha_{4W}^{-1}(v_U) \approx 2\alpha_{X_0}^{-1}(v_U) = 40, \quad v_U \approx 5.0 \times 10^{17} \text{ GeV}. \quad (15)$$

With the reasonable threshold effects taken into account, we have found that such a large discrepancy between the non-Abelian and the Abelian gauge couplings cannot be eliminated. It was first shown by Ginsparg [26] that the gauge coupling unification in the string theory can be achieved as

$$k_s\alpha_{4s}(v_U) = k_W\alpha_{4W}(v_U) = k_1\alpha_1(v_U) = \alpha_U(v_U), \quad (16)$$

⁶An alternative assumption to the minimal SU(8) Higgs spectrum was also considered in Ref. [13] and lead to a very similar behavior of the RGEs.

with (k_s, k_W, k_1) being the affine levels of the corresponding gauge couplings in the physical basis ⁷. In this paper, we prove that the reasonable conformal embedding is based on the affine Lie algebra of

$$\widehat{\mathfrak{su}}(4)_{s, k_s=1} \oplus \widehat{\mathfrak{su}}(4)_{W, k_W=1} \oplus \widehat{\mathfrak{u}}(1)_{1, k_1, \text{phys}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}, \quad (17)$$

in the *physical basis*, and the gauge coupling unification can be achieved with the $\mathcal{N} = 1$ supersymmetric (SUSY) extension between the scale of $v_{441} \leq \mu \leq v_U$.

The rest of the paper is organized as follows. The conformal embedding of the affine Lie algebra in Eq. (17) will be derived in Sec. 2. In Sec. 3, we propose an $\mathcal{N} = 1$ SUSY extension to the SU(8) theory, and study the corresponding RGEs between the scale of $v_{441} \leq \mu \leq v_U$. Such type of SUSY extension achieves the gauge coupling unification according to the corresponding affine Lie algebras, with the reasonable assumptions of the mass spectra. We summarize the results and discuss the underlying issues in Sec. 4. In App. A, we provide the necessary results of affine Lie algebra, with the emphasis on the root and weight system defined on the Cartan-Weyl basis. In App. B, we provide the general results of the U(1) Cartan discontinuities at different symmetry breaking stages.

2 The conformal embedding of the $\widehat{\mathfrak{su}}(8)_{k_U}$ affine Lie algebra

Throughout this section, we shall discuss the affine embeddings of the form $\oplus_i \widehat{\mathfrak{h}}_{i, k_i} \subset \widehat{\mathfrak{g}}_k$, where $\widehat{\mathfrak{h}}_{i, k_i}$ and $\widehat{\mathfrak{g}}_k$ represent some subalgebras and parent algebra, respectively. Mostly, we will use the group invariants defined with the normalization of $T(\square)_{\mathfrak{alg}} = 1$ in what we called the algebraic basis, and the corresponding quantities are denoted with a subscript of \mathfrak{alg} . The central issue is to look for the conformal embedding of the $\widehat{\mathfrak{su}}(8)_{k_U}$ affine Lie algebra according to the maximally symmetry breaking pattern in Eq (5). Some well-known constraints to the $\widehat{\mathfrak{su}}(8)_{k_U}$ affine Lie algebra are

1. $k_U \in \mathbb{Z}^+$,
2. the central charge should satisfy $c(\widehat{\mathfrak{su}}(8)_{k_U}) \leq 22$, with the central charge defined by

$$c(\widehat{\mathfrak{g}}_k) \equiv \frac{k \dim(\mathfrak{g})}{k + g^\vee}, \quad (18)$$

for an affine Lie algebra of $\widehat{\mathfrak{g}}_k$. With the dual Coxeter number of $g^\vee = N$ for $\mathfrak{g} = \mathfrak{su}(N)$ according to Eq. (72b), there is an upper bound to the $\widehat{\mathfrak{su}}(8)_{k_U}$ level as

$$\widehat{\mathfrak{su}}(8)_{k_U} \quad : \quad k_U \leq \frac{176}{41}, \quad \text{or } k_U \leq 4. \quad (19)$$

The conformal embedding is a subclass of the affine embeddings of $\oplus_i \widehat{\mathfrak{h}}_{i, k_i} \subset \widehat{\mathfrak{g}}_k$ that preserves the conformal invariance, which can be satisfied when the corresponding central charges are equal

$$\sum_i c(\widehat{\mathfrak{h}}_{i, k_i}) = c(\widehat{\mathfrak{g}}_k) \Rightarrow \sum_i \frac{k_i \dim \mathfrak{h}_i}{k_i + h_i^\vee} = \frac{k \dim \mathfrak{g}}{k + g^\vee}, \quad (20)$$

with (h_i^\vee, g^\vee) being the dual Coxeter numbers of \mathfrak{h}_i and \mathfrak{g} . For the Abelian $\widehat{\mathfrak{u}}(1)_{k_1}$, the central charge is reduced to

$$c(\widehat{\mathfrak{u}}(1)_{k_1}) = \frac{k_1 \dim \mathfrak{u}(1)}{k_1} = 1, \quad (21)$$

⁷Some early studies of the higher affine levels in the string-inspired unified theories include Refs. [27, 28].

since the $\mathfrak{u}(1)$ has no dual Coxeter number. It is known that the conformal embeddings are only possible for $k = 1$. For example, the $\widehat{\mathfrak{su}}(2)_1 \subset \widehat{\mathfrak{su}}(3)_1$ cannot be a conformal embedding, since the central charges are not equal

$$c(\widehat{\mathfrak{su}}(2)_1) = \frac{1 \times 3}{1 + 2} = 1, \quad c(\widehat{\mathfrak{su}}(3)_1) = \frac{1 \times 8}{1 + 3} = 2 \Rightarrow c(\widehat{\mathfrak{su}}(2)_1) \neq c(\widehat{\mathfrak{su}}(3)_1). \quad (22)$$

Instead, the $\widehat{\mathfrak{su}}(2)_4 \subset \widehat{\mathfrak{su}}(3)_1$ can be a conformal embedding, since

$$c(\widehat{\mathfrak{su}}(2)_4) = \frac{4 \times 3}{4 + 2} = 2, \quad c(\widehat{\mathfrak{su}}(3)_1) = \frac{1 \times 8}{1 + 3} = 2 \Rightarrow c(\widehat{\mathfrak{su}}(2)_4) = c(\widehat{\mathfrak{su}}(3)_1). \quad (23)$$

With these relations, let us turn to our flavor-unified $SU(8)$ theory with the maximally breaking pattern in Eq. (5). If one has to achieve the conformal embedding of $\widehat{\mathfrak{su}}(4)_{s, k_s} \oplus \widehat{\mathfrak{su}}(4)_{W, k_W} \oplus \hat{\mathfrak{u}}(1)_{1, k_1} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}$, the conformal invariance in Eq. (20) fixes the affine levels of both $\widehat{\mathfrak{su}}(4)_s \oplus \widehat{\mathfrak{su}}(4)_W$ subalgebras to be

$$\frac{15 k_s}{k_s + 4} + \frac{15 k_W}{k_W + 4} + 1 = 7 \Rightarrow (k_s, k_W) = (1, 1). \quad (24)$$

According to the relation in Eq. (59), the corresponding affine levels of two non-Abelian Lie algebras are consistent with the relation of $\alpha_{4s}^{-1}(v_U) \approx \alpha_{4W}^{-1}(v_U)$ from the RGEs in Fig. 1. If one considers the generalized conformal embedding of the form $\widehat{\mathfrak{su}}(n_s)_{s, k_s} \oplus \widehat{\mathfrak{su}}(n_W)_{W, k_W} \oplus \hat{\mathfrak{u}}(1)_{1, k_1} \subset \widehat{\mathfrak{su}}(N)_1$ with $N = n_s + n_W$, one finds that the following equality

$$\frac{(n_s^2 - 1) k_s}{k_s + n_s} + \frac{(n_W^2 - 1) k_W}{k_W + n_W} + 1 = N - 1 \quad (25)$$

between the central charges always holds when $(k_s, k_W) = (1, 1)$. Meanwhile, the level of $\hat{\mathfrak{u}}(1)_{k_1}$ is undetermined by the conformal invariance. Alternatively, if one proposes a different conformal embedding of $\widehat{\mathfrak{su}}(4)_2 \otimes \widehat{\mathfrak{su}}(4)_2 \otimes \hat{\mathfrak{u}}(1)_{k_1} \subset \widehat{\mathfrak{su}}(N)_1$, the relation in Eq. (20) leads to

$$2 \times \frac{15 \times 2}{2 + 4} + 1 = N - 1 \Rightarrow N = 12. \quad (26)$$

This means an embedding with higher affine levels can only be achieved through $\widehat{\mathfrak{su}}(4)_2 \oplus \widehat{\mathfrak{su}}(4)_2 \oplus \hat{\mathfrak{u}}(1)_{k_1} \subset \widehat{\mathfrak{su}}(12)_1$, which is inconsistent with our minimal setup described in Sec. 1.

The unitarity constraint [29] further requires that the conformal dimension of the massless and highest-weight states should satisfy

$$h(\mathcal{R}) \leq 1, \quad (27)$$

where the conformal dimensions of the highest-weight states are given by

$$h(\mathcal{R}) \equiv \frac{C_2(\mathcal{R})_{\mathfrak{alg}} / (\boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h)}{k_U + g_U^\vee}, \quad (28)$$

with $g_U^\vee = 8$ for $\mathfrak{g} = \mathfrak{su}(8)$, and $(\boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h) = 2$ according to the normalization in the algebraic basis⁸. More generically, the conformal dimensions of the rank- k anti-symmetric irreps in the $\widehat{\mathfrak{su}}(N)_{k_U=1}$ theories are

$$h(A_k) = \frac{k(N - k)}{2N}, \quad (29)$$

⁸Here, the $\boldsymbol{\alpha}_h$ represents the longest root of the Lie algebra \mathfrak{g} . For the $\mathfrak{su}(N)$ Lie algebra, the length of all equally-long roots is normalized as $(\boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h) = 2$ in the algebraic basis.

according to Eq. (83). Specifically with $C_2(\mathbf{8})_{\text{alg}} = C_2(\overline{\mathbf{8}})_{\text{alg}} = 63/8$, $C_2(\mathbf{28})_{\text{alg}} = C_2(\overline{\mathbf{28}})_{\text{alg}} = 27/2$, $C_2(\mathbf{56})_{\text{alg}} = 135/8$, and $C_2(\mathbf{70})_{\text{alg}} = 18$ according to Eqs. (83), we find the following conformal dimensions

$$h(\mathbf{8}) = h(\overline{\mathbf{8}}) = \frac{7}{16}, \quad h(\mathbf{28}) = h(\overline{\mathbf{28}}) = \frac{3}{4}, \quad h(\mathbf{56}) = \frac{15}{16}, \quad h(\mathbf{70}) = 1, \quad (30)$$

in the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory. This means all massless states with the anti-symmetric irreps satisfy the unitarity constraint.

Next, we also need to match the conformal dimensions of the fields through the affine branching rules of

$$\hat{\lambda} \mapsto \bigoplus_{\hat{\mu}} b_{\hat{\lambda}, \hat{\mu}} \hat{\mu}, \quad (31)$$

with $\hat{\lambda}$ being the irreps of the $\hat{\mathfrak{g}}_k$, and $b_{\hat{\lambda}, \hat{\mu}}$ being the multiplicity of the irreps $\hat{\mu}$ of the $\bigoplus_i \hat{\mathfrak{h}}_{i, k_i}$. This leads to the equality between the conformal dimensions as follows

$$h_{\hat{\lambda}} + n = h_{\hat{\mu}}, \quad h_{\hat{\lambda}} = \frac{C_2(\lambda)_{\text{alg}}}{2(1 + g^\vee)}, \quad h_{\hat{\mu}} = \frac{C_2(\mu)_{\text{alg}}}{2(k_i + h_i^\vee)}, \quad (32)$$

for the decompositions in Eqs. (7) with some grade n . For the Abelian $\hat{\mathfrak{u}}(1)_{k_0}$, there is no dual Coxeter number g^\vee or the Weyl vector ρ , hence the corresponding conformal dimension becomes

$$h(\hat{\mathfrak{u}}(1)) = \frac{(\lambda, \lambda)}{2k_0} = \frac{\mathcal{X}_0^2}{2k_0}(\omega_j, \omega_j) \Rightarrow \frac{\mathcal{X}_{0, \text{alg}}^2}{4k_{0, \text{alg}}}, \quad (33)$$

with the normalization of the fundamental weights in Eq. (73). At the grade = 0 and the level of $k_{0, \text{alg}} = 4$, we find that

$$\begin{aligned} \mathbf{8} &\hookrightarrow (\mathbf{4}, \mathbf{1}, -1) \oplus (\mathbf{1}, \mathbf{4}, +1) : h(\mathbf{8}) = \frac{7}{16}, \\ h(\mathbf{4}, \mathbf{1}, -1) &= h(\mathbf{1}, \mathbf{4}, +1) = \frac{15/8}{1+4} + \frac{(\pm 1)^2}{4 \times 4} = \frac{7}{16}, \end{aligned} \quad (34a)$$

$$\begin{aligned} \mathbf{28} &\hookrightarrow (\mathbf{6}, \mathbf{1}, -2) \oplus (\mathbf{1}, \mathbf{6}, +2) \oplus (\mathbf{4}, \mathbf{4}, 0) : h(\mathbf{28}) = \frac{3}{4}, \\ h(\mathbf{6}, \mathbf{1}, -2) &= h(\mathbf{1}, \mathbf{6}, +2) = \frac{5/2}{1+4} + \frac{(\pm 2)^2}{4 \times 4} = \frac{3}{4}, \\ h(\mathbf{4}, \mathbf{4}, 0) &= 2 \times \frac{15/8}{1+4} = \frac{3}{4}, \end{aligned} \quad (34b)$$

$$\begin{aligned} \mathbf{56} &\hookrightarrow (\mathbf{1}, \overline{\mathbf{4}}, +3) \oplus (\overline{\mathbf{4}}, \mathbf{1}, -3) \oplus (\mathbf{4}, \mathbf{6}, +1) \oplus (\mathbf{6}, \mathbf{4}, -1) : h(\mathbf{56}) = \frac{15}{16}, \\ h(\mathbf{1}, \overline{\mathbf{4}}, +3) &= h(\overline{\mathbf{4}}, \mathbf{1}, -3) = \frac{15/8}{1+4} + \frac{(\pm 3/4)^2}{2 \times (1/2)} = \frac{15}{16}, \\ h(\mathbf{4}, \mathbf{6}, +1) &= h(\mathbf{6}, \mathbf{4}, -1) = \frac{15/8}{1+4} + \frac{5/2}{1+4} + \frac{(\pm 1/4)^2}{2 \times (1/2)} = \frac{15}{16}, \end{aligned} \quad (34c)$$

$$\begin{aligned} \mathbf{70} &\hookrightarrow (\mathbf{1}, \mathbf{1}, -4) \oplus (\mathbf{1}, \mathbf{1}, +4) \oplus (\mathbf{4}, \overline{\mathbf{4}}, +2) \oplus (\overline{\mathbf{4}}, \mathbf{4}, -2) \oplus (\mathbf{6}, \mathbf{6}, 0) : h(\mathbf{70}) = 1, \\ h(\mathbf{1}, \mathbf{1}, -4) &= h(\mathbf{1}, \mathbf{1}, +4) = \frac{(\pm 4)^2}{4 \times 4} = 1, \quad h(\mathbf{6}, \mathbf{6}, 0) = 2 \times \frac{5/2}{1+4} = 1, \\ h(\mathbf{4}, \overline{\mathbf{4}}, +2) &= h(\overline{\mathbf{4}}, \mathbf{4}, -2) = 2 \times \frac{15/8}{1+4} + \frac{(\pm 2)^2}{4 \times 4} = 1, \end{aligned} \quad (34d)$$

with $C_2(\mathbf{4})_{\text{alg}} = C_2(\overline{\mathbf{4}})_{\text{alg}} = 15/4$ and $C_2(\mathbf{6})_{\text{alg}} = 5$. With the conformal dimension defined in Eq. (33), all conformal dimensions of the fields in Tab. 1 are found to match through their decompositions during the maximal symmetry breaking pattern.

With the above results, we find a conformal embedding of $\widehat{\mathfrak{su}}(4)_1 \oplus \widehat{\mathfrak{su}}(4)_1 \oplus \widehat{\mathfrak{u}}(1)_{4,\text{alg}} \subset \widehat{\mathfrak{su}}(8)_1$ in the algebraic basis. The conformal dimension in Eq. (33) in the physical basis becomes

$$h(\widehat{\mathfrak{u}}(1)) = \frac{\mathcal{X}_{0,\text{phys}}^2}{4k_{0,\text{phys}}}. \quad (35)$$

With the relation of $\mathcal{X}_{0,\text{phys}} = \frac{1}{4}\mathcal{X}_{0,\text{alg}}$, we find that

$$k_{0,\text{phys}} = \frac{1}{16}k_{0,\text{alg}} = \frac{1}{4}, \quad (36)$$

since the conformal dimension should be invariant under the basis conversion from Eq. (33) to Eq. (35). Thus, we conclude a conformal embedding of the affine Lie algebra of $\widehat{\mathfrak{su}}(4)_{k_s=1} \oplus \widehat{\mathfrak{su}}(4)_{k_W=1} \oplus \widehat{\mathfrak{u}}(1)_{k_{0,\text{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}$ in the physical basis. To achieve the gauge coupling unification, the following relation at the scale of v_U must be satisfied

$$\alpha_U(v_U) = \alpha_{4s}(v_U) = \alpha_{4W}(v_U) = \frac{1}{4}\alpha_{X_0}(v_U). \quad (37)$$

The RGE behaviors displayed in Fig. 1 are inconsistent with the above relation. In particular, the gauge coupling of the α_{X_0} in the minimal SU(8) theory was too small to reach the condition in Eq. (37).

3 The RGEs of the SUSY $\widehat{\mathfrak{su}}(8)_{k_U=1}$

Historically, an $\mathcal{N} = 1$ SUSY extension to the SU(5) theory was first considered in Ref. [17], where a minimal extension of a chiral superfield of $\overline{\mathbf{5}}_{\mathbf{H}}$ is necessary to cancel the anomaly of the $\mathbf{5}_{\mathbf{H}}$ in the minimal SU(5) theory when it was promoted to be supersymmetric. Moreover, two chiral superfields of $(\mathbf{5}_{\mathbf{H}}, \overline{\mathbf{5}}_{\mathbf{H}})$ formulate the holomorphic Yukawa coupling terms of

$$W_Y = \overline{\mathbf{5}}_{\mathbf{F}}\mathbf{10}_{\mathbf{F}}\overline{\mathbf{5}}_{\mathbf{H}} + \mathbf{10}_{\mathbf{F}}\mathbf{10}_{\mathbf{F}}\mathbf{5}_{\mathbf{H}}, \quad (38)$$

in the SU(5) superpotential.

When promoting all SU(8) fields in Tab. 1 to be chiral superfields, the super partners of the $\overline{\mathbf{8}}_{\mathbf{H},\omega} \oplus \overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}}$ bring the total anomaly of

$$\sum_{\omega} \text{Anom}(\overline{\mathbf{8}}_{\mathbf{H},\omega}) + \sum_{\dot{\omega}} \text{Anom}(\overline{\mathbf{28}}_{\mathbf{H},\dot{\omega}}) = -24. \quad (39)$$

Naively, there may be two possible SUSY extensions [13] with the additional chiral superfields in the spectrum to cancel the anomaly

$$\{H\}_{\text{I}} = \mathbf{8}_{\mathbf{H}}^{\omega} \oplus \mathbf{28}_{\mathbf{H}}^{\dot{\omega}}, \quad \omega = (3, \text{IV}, \text{V}, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VII}, \text{VIII}, \text{IX}), \quad (40a)$$

$$\{H\}_{\text{II}} = \mathbf{36}_{\mathbf{H}}^{\rho}, \quad \rho = (\odot, \ominus). \quad (40b)$$

Unfortunately, the SUSY extension in Eq. (40b) involve two rank-2 symmetric chiral superfields, which can only appear as the descendant states in the spectrum. It turns out the corresponding conformal

dimension violates the unitarity constraint in Eq. (27), and this rules the extension in Eq. (40b). One also has to guarantee that all Yukawa coupling terms should still arise from the holomorphic superpotential. Three renormalizable Yukawa couplings in Eq. (3) are obviously holomorphic terms in the superpotential, while the non-renormalizable term therein can be modified into the following holomorphic term of

$$W_Y \supset \frac{c_4}{M_{\text{pl}}} \mathbf{56}_F \mathbf{56}_F \mathbf{28}_H \hat{\omega} \mathbf{63}_H. \quad (41)$$

Two $d = 5$ direct Yukawa coupling terms in Eq. (11) are also holomorphic terms in the superpotential.

Below, we derive the corresponding RGEs according to the SWW symmetry breaking pattern of Eq. (10a) for the SUSY extension in Eq. (40a). Generically, the two-loop RGE of a gauge coupling of α_Υ is given by [30]

$$\frac{d\alpha_\Upsilon(\mu)}{d \log \mu} = \frac{b_\Upsilon^{(1)}}{2\pi} \alpha_\Upsilon^2(\mu) + \left(\sum_{\Upsilon'} \frac{b_{\Upsilon\Upsilon'}^{(2)}}{8\pi^2} \alpha_{\Upsilon'}(\mu) \right) \cdot \alpha_\Upsilon^2(\mu). \quad (42)$$

We will always assume a set of SUSY RGEs between the $v_{441} \leq \mu \leq v_U$, and the corresponding one- and two-loop β coefficients are

$$\begin{aligned} \text{non - Abelian} \quad : \quad b_\Upsilon^{(1)} &= -3C_2(\mathcal{G}_\Upsilon) + \sum_\Phi T(\mathcal{R}_\Upsilon^\Phi), \\ b_{\Upsilon\Upsilon'}^{(2)} &= -\frac{34}{3}C_2(\mathcal{G}_\Upsilon)^2 + \sum_\Phi [6C_2(\mathcal{R}_{\Upsilon'}^\Phi) + 4C_2(\mathcal{G}_\Upsilon)] T(\mathcal{R}_\Upsilon^\Phi), \end{aligned} \quad (43a)$$

$$\begin{aligned} \text{Abelian} \quad : \quad b_\Upsilon^{(1)} &= \sum_\Phi (\mathcal{X}_\Upsilon^\Phi)^2, \\ b_{\Upsilon\Upsilon'}^{(2)} &= 6 \sum_{\Phi, \Upsilon'} (\mathcal{X}_{\Upsilon'}^\Phi)^2 \cdot (\mathcal{X}_\Upsilon^\Phi)^2. \end{aligned} \quad (43b)$$

Here, $C_2(\mathcal{G}_\Upsilon)$ is the quadratic Casimir of the group \mathcal{G}_Υ , $T(\mathcal{R}_\Upsilon^\Phi)$ are trace invariants of the chiral superfield Φ with $\mathcal{R}_\Upsilon^\Phi \in \mathcal{G}_\Upsilon$ and the physical normalization of $T(\square)_{\text{phys}} = 1/2$, and $\mathcal{X}_\Upsilon^\Phi$ are the U(1) charges at different stages. The non-SUSY RGEs between the scales of $v_{\text{EW}} \leq \mu \leq v_{441}$ are contributed by the one- and two-loop β coefficients of

$$\begin{aligned} \text{non - Abelian} \quad : \quad b_\Upsilon^{(1)} &= -\frac{11}{3}C_2(\mathcal{G}_\Upsilon) + \frac{2}{3} \sum_F T(\mathcal{R}_\Upsilon^F) + \frac{1}{3} \sum_S T(\mathcal{R}_\Upsilon^S), \\ b_{\Upsilon\Upsilon'}^{(2)} &= -\frac{34}{3}C_2(\mathcal{G}_\Upsilon)^2 + \sum_F \left[2 \sum_{\Upsilon'} C_2(\mathcal{R}_{\Upsilon'}^F) + \frac{10}{3}C_2(\mathcal{G}_\Upsilon) \right] T(\mathcal{R}_\Upsilon^F) \\ &+ \sum_S \left[4 \sum_{\Upsilon'} C_2(\mathcal{R}_{\Upsilon'}^S) + \frac{2}{3}C_2(\mathcal{G}_\Upsilon) \right] T(\mathcal{R}_\Upsilon^S), \end{aligned} \quad (44a)$$

$$\begin{aligned} \text{Abelian} \quad : \quad b_\Upsilon^{(1)} &= \frac{2}{3} \sum_F (\mathcal{X}_\Upsilon^F)^2 + \frac{1}{3} \sum_S (\mathcal{X}_\Upsilon^S)^2, \\ b_{\Upsilon\Upsilon'}^{(2)} &= 2 \sum_{F, \Upsilon'} (\mathcal{X}_{\Upsilon'}^F)^2 \cdot (\mathcal{X}_\Upsilon^F)^2 + 4 \sum_{S, \Upsilon'} (\mathcal{X}_{\Upsilon'}^S)^2 \cdot (\mathcal{X}_\Upsilon^S)^2. \end{aligned} \quad (44b)$$

Between the $v_{441} \leq \mu \leq v_U$, all chiral superfields besides of $\mathbf{63}_H$ in Tab. 1 are assumed to be massless. Specifically, all chiral superfields in the second columns of Tabs. 2, 3, and 4 are massless, together with the following massless fields

$$\begin{aligned}
& (\bar{\mathbf{4}}, \mathbf{1}, +\frac{1}{4})_{H,\omega} \oplus (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{H,\omega} \subset \bar{\mathbf{8}}_{H,\omega}, \\
& (\mathbf{6}, \mathbf{1}, +\frac{1}{2})_{H,\dot{\omega}} \oplus (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{H,\dot{\omega}} \oplus (\bar{\mathbf{4}}, \bar{\mathbf{4}}, 0)_{H,\dot{\omega}} \subset \bar{\mathbf{28}}_{H,\dot{\omega}}, \\
& (\mathbf{4}, \bar{\mathbf{4}}, +\frac{1}{2})_H \oplus (\bar{\mathbf{4}}, \mathbf{4}, -\frac{1}{2})_H \oplus (\mathbf{6}, \mathbf{6}, 0)_H \oplus (\mathbf{1}, \mathbf{1}, +1)_H \oplus (\mathbf{1}, \mathbf{1}, -1)_H \subset \mathbf{70}_H, \\
& (\mathbf{4}, \mathbf{1}, -\frac{1}{4})_{H,\omega} \oplus (\mathbf{1}, \mathbf{4}, +\frac{1}{4})_{H,\omega} \subset \mathbf{8}_{H,\omega}, \\
& (\mathbf{6}, \mathbf{1}, -\frac{1}{2})_{H,\dot{\omega}} \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_{H,\dot{\omega}} \oplus (\mathbf{4}, \mathbf{4}, 0)_{H,\dot{\omega}} \subset \mathbf{28}_{H,\dot{\omega}}.
\end{aligned} \tag{45}$$

Correspondingly, we find that

$$\begin{aligned}
& (b_{4s}^{(1)}, b_{4W}^{(1)}, b_{X_0}^{(1)}) = (47, 47, 59), \\
& b_{G_{441}}^{(2)} = \begin{pmatrix} 3181/2 & 945/2 & 30 \\ 945/2 & 3181/2 & 30 \\ 450 & 450 & 93 \end{pmatrix},
\end{aligned} \tag{46}$$

according to Eqs. (43).

Between the $v_{341} \leq \mu \leq v_{441}$, the spectrum contains massless fermions of

$$\begin{aligned}
& [(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_F^\Omega \oplus (\mathbf{1}, \mathbf{1}, 0)_F^\Omega] \oplus (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_F^\Omega \subset \bar{\mathbf{8}}_F^\Omega, \\
& \Omega = (\omega, \dot{\omega}), \quad \omega = (3, V, VI), \quad \dot{\omega} = (\dot{1}, \dot{2}, \dot{VII}, \dot{VIII}, \dot{IX}), \\
& (\mathbf{1}, \mathbf{1}, 0)_F^{IV} \subset \bar{\mathbf{8}}_F^{IV}, \\
& (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_F \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2})_F \oplus (\mathbf{3}, \mathbf{4}, -\frac{1}{12})_F \subset \mathbf{28}_F, \\
& (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})_F \oplus [(\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})'_F \oplus (\mathbf{1}, \mathbf{1}, -1)_F] \oplus [(\mathbf{3}, \mathbf{6}, +\frac{1}{6})_F \oplus (\mathbf{1}, \mathbf{6}, +\frac{1}{2})'_F] \\
& \oplus [(\mathbf{3}, \mathbf{4}, -\frac{1}{12})'_F \oplus (\bar{\mathbf{3}}, \mathbf{4}, -\frac{5}{12})_F] \subset \mathbf{56}_F,
\end{aligned} \tag{47}$$

together with the massless Higgs fields of

$$\begin{aligned}
& (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{H,3,V,VI} \subset \bar{\mathbf{8}}_{H,3,V,VI}, \\
& (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{H,i,\dot{2},\dot{VIII}} \subset \bar{\mathbf{28}}_{H,i,\dot{2},\dot{VIII}}, \\
& (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{H,i,\dot{2},\dot{VII},\dot{IX}} \subset (\bar{\mathbf{4}}, \bar{\mathbf{4}}, 0)_{H,i,\dot{2},\dot{VII},\dot{IX}} \subset \bar{\mathbf{28}}_{H,i,\dot{2},\dot{VII},\dot{IX}}, \\
& (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})'_H \subset (\mathbf{4}, \bar{\mathbf{4}}, +\frac{1}{2})_H \subset \mathbf{70}_H, \\
& (\mathbf{1}, \mathbf{4}, +\frac{1}{4})^i_H \subset (\mathbf{4}, \mathbf{4}, 0)^i_H \subset \mathbf{28}_H^i.
\end{aligned} \tag{48}$$

Correspondingly, we find that

$$\begin{aligned} (b_{3c}^{(1)}, b_{4W}^{(1)}, b_{X_1}^{(1)}) &= \left(-\frac{5}{3}, -\frac{17}{6}, +\frac{497}{36}\right), \\ b_{\mathcal{G}_{341}}^{(2)} &= \begin{pmatrix} 226/3 & 75/2 & 97/36 \\ 20 & 887/6 & 95/12 \\ 194/9 & 475/4 & 4333/216 \end{pmatrix}, \end{aligned} \quad (49)$$

according to Eqs. (44).

Between the $v_{331} \leq \mu \leq v_{341}$, the spectrum contains massless fermions of

$$\begin{aligned} & \left[(\bar{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega} \right] \oplus \left[(\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{F}}^{\Omega} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\Omega''} \right] \subset \bar{\mathbf{8}}_{\mathbf{F}}^{\Omega}, \\ & \Omega = (\omega, \dot{\omega}), \quad \omega = (3, \text{VI}), \quad \dot{\omega} = (\dot{1}, \dot{2}, \text{VIII}, \text{IX}), \\ & (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{IV}} \subset \bar{\mathbf{8}}_{\mathbf{F}}^{\text{IV}}, \quad (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{V}, \text{V}''} \subset \bar{\mathbf{8}}_{\mathbf{F}}^{\text{V}}, \\ & (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}} \oplus (\mathbf{1}, \mathbf{1}, 0)_{\mathbf{F}}^{\text{VII}''} \subset \bar{\mathbf{8}}_{\mathbf{F}}^{\text{VII}}, \\ & (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}} \oplus (\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}} \subset \mathbf{28}_{\mathbf{F}}, \\ & (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}' \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}' \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}' \oplus (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{F}}'' \right] \\ & \oplus \left[(\mathbf{3}, \mathbf{3}, 0)_{\mathbf{F}}'' \oplus (\bar{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})_{\mathbf{F}}''' \right] \subset \mathbf{56}_{\mathbf{F}}. \end{aligned} \quad (50)$$

together with the massless Higgs fields of

$$\begin{aligned} & (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, 3, \text{VI}} \subset (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, 3, \text{VI}} \subset \bar{\mathbf{8}}_{\mathbf{H}, 3, \text{VI}}, \\ & (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})'_{\mathbf{H}, \dot{2}, \text{VIII}} \subset (\mathbf{1}, \mathbf{6}, -\frac{1}{2})_{\mathbf{H}, \dot{2}, \text{VIII}} \subset \bar{\mathbf{28}}_{\mathbf{H}, \dot{2}, \text{VIII}}, \\ & (\mathbf{1}, \bar{\mathbf{3}}, -\frac{1}{3})_{\mathbf{H}, \dot{2}, \text{IX}} \subset (\mathbf{1}, \bar{\mathbf{4}}, -\frac{1}{4})_{\mathbf{H}, \dot{2}, \text{IX}} \subset (\bar{\mathbf{4}}, \bar{\mathbf{4}}, 0)_{\mathbf{H}, \dot{2}, \text{IX}} \subset \bar{\mathbf{28}}_{\mathbf{H}, \dot{2}, \text{IX}}, \\ & (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}''' \subset (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})'_{\mathbf{H}} \subset (\mathbf{4}, \bar{\mathbf{4}}, +\frac{1}{2})_{\mathbf{H}} \subset \mathbf{70}_{\mathbf{H}}. \end{aligned} \quad (51)$$

Correspondingly, we find that

$$\begin{aligned} (b_{3c}^{(1)}, b_{3W}^{(1)}, b_{X_2}^{(1)}) &= \left(-5, -\frac{23}{6}, +\frac{82}{9}\right), \\ b_{\mathcal{G}_{331}}^{(2)} &= \begin{pmatrix} 12 & 12 & 2 \\ 12 & 113/3 & 38/9 \\ 16 & 304/9 & 304/27 \end{pmatrix}, \end{aligned} \quad (52)$$

according to Eqs. (44).

Between the $v_{\text{EW}} \leq \mu \leq v_{331}$, the massless fields include three-generational SM fermions together with one SM Higgs doublet of

$$(\mathbf{1}, \bar{\mathbf{2}}, +\frac{1}{2})_{\mathbf{H}}''' \subset (\mathbf{1}, \bar{\mathbf{3}}, +\frac{2}{3})_{\mathbf{H}}''' \subset (\mathbf{1}, \bar{\mathbf{4}}, +\frac{3}{4})'_{\mathbf{H}} \subset (\mathbf{4}, \bar{\mathbf{4}}, +\frac{1}{2})_{\mathbf{H}} \subset \mathbf{70}_{\mathbf{H}}, \quad (53)$$

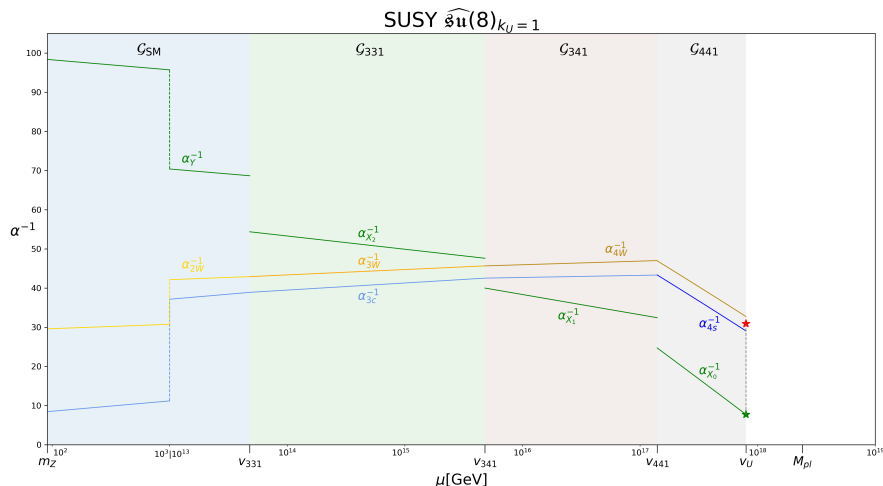


Figure 2: The RGEs of the SUSY $\widehat{\mathfrak{su}}(8)_{k_U=1}$ setup. The gauge couplings between $10^3 \text{ GeV} \lesssim \mu \lesssim 10^{13} \text{ GeV}$ evolve according to the SM β coefficients in Eq. (54) and are zoomed out in order to highlight the behaviors in three intermediate symmetry breaking scales.

and we have the usual SM β coefficients of

$$\begin{aligned}
 (b_{3c}^{(1)}, b_{2W}^{(1)}, b_Y^{(1)}) &= \left(-7, -\frac{19}{6}, +\frac{41}{6}\right), \\
 b_{\mathcal{G}_{SM}}^{(2)} &= \begin{pmatrix} -26 & 9/2 & 11/6 \\ 12 & 35/6 & 3/2 \\ 44/3 & 9/2 & 199/18 \end{pmatrix}.
 \end{aligned} \tag{54}$$

The corresponding RGEs of the SUSY $\widehat{\mathfrak{su}}(8)_{k_U=1}$ extension are plotted in Fig. 2, with three intermediate scales given in Eq. (10a). The benchmark point is marked by \star and reads

$$\begin{aligned}
 c_{\text{HSW}} &\approx 0.72, \quad v_U \approx 8.0 \times 10^{17} \text{ GeV}, \\
 \alpha_{4s}^{-1}(v_U) &= \alpha_{4W}^{-1}(v_U) \approx 30.9, \quad \alpha_{X_0}^{-1}(v_U) \approx 7.71.
 \end{aligned} \tag{55}$$

A natural $\mathcal{O}(1)$ Wilsonian coefficient of c_{HSW} in Eq. (12) is found to compensate for the small discrepancy between two non-Abelian gauge couplings.

4 Summary

Though the previous results in Ref. [10] suggests encouraging outcome in addressing the SM quark/lepton masses and the CKM mixing patterns, the minimal setup based on the intermediate scales in Eq. (10a) cannot achieve the gauge coupling unification [13] in the context of the field theory. In this paper, we describe the gauge coupling unification in the SUSY extensions of the affine $\widehat{\mathfrak{su}}(8)_{k_U=1}$ Lie algebra. The SUSY version is obtained from the field contents of the minimal setup in Tab. 1, with the additional chiral superfields in Eq. (40a) that (i) cancel the anomalies, and (ii) satisfy the unitarity constraint. According to the RGEs plotted in Fig. 2, the SUSY extension achieves the gauge coupling unification in Eq. (37), and the benchmark point was numerically presented in Eq. (55). The unification scales of $\sim \mathcal{O}(10^{18}) \text{ GeV}$ are precious close to the Planck scale, which further suggests the intrinsic connections between the GUT and the underlying quantum gravity theory, such as the string theory.

More generally, we prove the following conformal embedding of

$$\widehat{\mathfrak{su}}(n_s)_{k_s=1} \oplus \widehat{\mathfrak{su}}(n_W)_{k_W=1} \oplus \widehat{\mathfrak{u}}(1)_{k_{1,\text{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(N)_{k_U=1}, \quad n_s + n_W = N, \quad (56)$$

in the physical basis. The $U(1)_1$ charges of the fundamental representation in the physical basis are normalized as ⁹

$$\mathbf{N} = (\mathbf{n}_s, \mathbf{1}, -\frac{n_W}{\sqrt{2n_s n_W N}}) \oplus (\mathbf{1}, \mathbf{n}_W, +\frac{n_s}{\sqrt{2n_s n_W N}}). \quad (57)$$

Thus, the equality between the conformal dimensions leads to

$$\begin{aligned} \frac{N-1}{2N} &= \frac{n_s-1}{2n_s} + \frac{1}{4k_{1,\text{phys}}} \left(-\frac{n_W}{\sqrt{2n_s n_W N}} \right)^2 = \frac{n_W-1}{2n_W} + \frac{1}{4k_{1,\text{phys}}} \left(+\frac{n_s}{\sqrt{2n_s n_W N}} \right)^2 \\ \Rightarrow k_{1,\text{phys}} &= \frac{1}{4}. \end{aligned} \quad (58)$$

This means the conformal embedding in Eq. (56) always leads to an affine level of $k_{1,\text{phys}} = \frac{1}{4}$ in the physical basis, regardless of the specific symmetry breaking pattern. This means the gauge coupling unification is generalized from the relation in Eq. (37) to the following

$$\alpha_U(v_U) = \alpha_s(v_U) = \alpha_W(v_U) = \frac{1}{4}\alpha_1(v_U), \quad (59)$$

regardless of the specific symmetry breaking patterns in the theory based any $\widehat{\mathfrak{su}}(N)_{k_U=1}$ affine Lie algebra.

For the SUSY $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory, there may be two non-maximally symmetry breaking patterns of

$$\begin{aligned} \text{SU}(8) &\xrightarrow{\langle 63_H \rangle} \mathcal{G}_{531}/\mathcal{G}_{351}, \\ \mathcal{G}_{531} &\equiv \text{SU}(5)_s \otimes \text{SU}(3)_W \otimes \text{U}(1)_{X_0}, \quad \mathcal{G}_{351} \equiv \text{SU}(3)_s \otimes \text{SU}(5)_W \otimes \text{U}(1)_{X_0}, \\ \text{embedding} &: \widehat{\mathfrak{su}}(5)_{k_{s/W}=1} \oplus \widehat{\mathfrak{su}}(3)_{k_{W/s}=1} \oplus \widehat{\mathfrak{u}}(1)_{k_{1,\text{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}, \end{aligned} \quad (60a)$$

$$\begin{aligned} \text{SU}(8) &\xrightarrow{\langle 63_H \rangle} \mathcal{G}_{621}, \quad \mathcal{G}_{621} \equiv \text{SU}(6)_s \otimes \text{SU}(2)_W \otimes \text{U}(1)_{X_0}, \\ \text{embedding} &: \widehat{\mathfrak{su}}(6)_{k_s} \oplus \widehat{\mathfrak{su}}(2)_{k_W} \oplus \widehat{\mathfrak{u}}(1)_{k_{1,\text{phys}}=\frac{1}{4}} \subset \widehat{\mathfrak{su}}(8)_{k_U=1}, \end{aligned} \quad (60b)$$

according to Witten [31]. Along both symmetry breaking patterns, the conformal invariance in Eq. (20) fixes the levels of both subalgebras to be

$$\begin{aligned} \text{SU}(8) \hookrightarrow \mathcal{G}_{531}/\mathcal{G}_{351} &: \frac{24k_s}{k_s+5} + \frac{8k_W}{k_W+3} + 1 = 7 \Rightarrow (k_s, k_W) = (1, 1), \\ \text{SU}(8) \hookrightarrow \mathcal{G}_{621} &: \frac{35k_s}{k_s+6} + \frac{3k_W}{k_W+2} + 1 = 7 \Rightarrow (k_s, k_W) = (1, 1). \end{aligned} \quad (61)$$

Therefore, these two non-maximally symmetry breaking patterns can only achieve the gauge coupling unification in the context of the conformal embedding when the relation in Eq. (59) holds.

⁹Notice that, one has to evaluate the RGE of the $U(1)_1$ gauge coupling for the unification in any unified theory. For example, the $U(1)_1$ gauge coupling in the Georgi-Glashow $\text{SU}(5) \hookrightarrow \mathcal{G}_{\text{SM}}$ theory is normalized as $g_1 = \sqrt{\frac{5}{3}}g_Y$. In the symmetry breaking patterns of $\text{SU}(8) \hookrightarrow \text{SU}(4) \otimes \text{SU}(4) \otimes \text{U}(1)$ or $\text{SU}(9) \hookrightarrow \text{SU}(6) \otimes \text{SU}(3) \otimes \text{U}(1)$, no additional normalizations are needed.

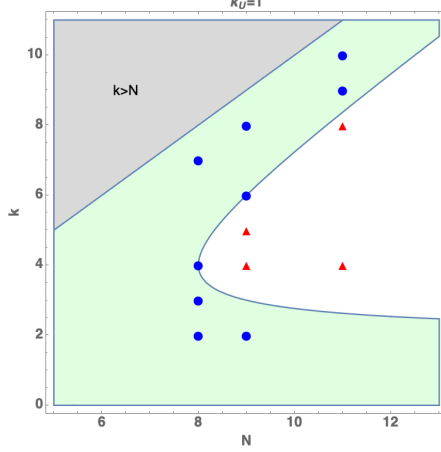


Figure 3: The unitarity allowed region (green shaded) to the rank- k anti-symmetric irreps of the $\widehat{\mathfrak{su}}(N)_{k_U=1}$ theories. The blue circles and red triangles represent the allowed and the excluded irreps, respectively. The gray shaded region with $k > N$ is not considered.

Finally, we comment on the conformal embedding for the non-minimal flavor-unified theories beyond the $\widehat{\mathfrak{su}}(8)_{k_U=1}$. Given the RGE behaviors in the minimal SU(8) theory, it is reasonable to conjecture that the similar behaviors can also exist when one extends the flavor-unified theories beyond the affine Lie algebra of $\widehat{\mathfrak{su}}(8)$. Here, we list the matter contents in two possible non-minimal extensions of

$$\begin{aligned} \text{SU}(9) & : \quad \{f_L\}_{\text{SU}(9)}^{n_g=3} = \left[\overline{\mathbf{9}}_{\mathbf{F}}^\lambda \oplus \mathbf{36}_{\mathbf{F}} \right] \oplus \left[\overline{\mathbf{9}}_{\mathbf{F}}^\lambda \oplus \mathbf{126}_{\mathbf{F}} \right], \\ & \quad \{H\}_{\text{SU}(9)}^{n_g=3} = \overline{\mathbf{9}}_{\mathbf{H},\lambda} \oplus \overline{\mathbf{84}}_{\mathbf{H},\lambda} \oplus \overline{\mathbf{126}}_{\mathbf{H}} \oplus \mathbf{80}_{\mathbf{H}}, \\ & \quad \lambda = (3, \text{IV}, \text{V}, \text{VI}, \text{VII}), \quad \dot{\lambda} = (\dot{1}, \dot{2}, \text{II}\dot{\text{X}}, \text{I}\dot{\text{X}}, \dot{\text{X}}), \end{aligned} \quad (62a)$$

$$\text{SU}(11) : \quad \{f_L\}_{\text{SU}(11)}^{n_g=3} = \mathbf{330}_{\mathbf{F}} \oplus \overline{\mathbf{165}}_{\mathbf{F}} \oplus \overline{\mathbf{55}}_{\mathbf{F}} \oplus \overline{\mathbf{11}}_{\mathbf{F}}, \quad (62b)$$

which were previously proposed in Refs. [8] and [5]. Both of them involve higher-rank anti-symmetric irreps and lead to three-generational SM fermions mathematically. In Fig. 3, we plot the unitarity constrained regions (28) to the rank- k anti-symmetric irreps A_k from the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ (1) (3), $\widehat{\mathfrak{su}}(9)_{k_U=1}$, and $\widehat{\mathfrak{su}}(11)_{k_U=1}$ theories (62). Apparently, both the $\widehat{\mathfrak{su}}(9)_{k_U=1}$ and the $\widehat{\mathfrak{su}}(11)_{k_U=1}$ extensions contain the irreps excluded by the unitarity constraints, as compared to the allowed $\widehat{\mathfrak{su}}(8)_{k_U=1}$ theory. In other words, the gauge coupling unifications in such non-minimal extensions cannot be achieved with the relation of (59) through the conformal embedding.

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A Some results of the affine Lie algebra

In this appendix, we review some necessary results of the Lie algebra and the affine Lie algebra. We follow the conventions in the textbook of the conformal field theory [32] closely. All results described in this section are given in the algebraic basis.

A.1 The Lie algebra: Cartan-Weyl basis, simple roots, and the highest weight

The Lie algebra of \mathfrak{g} is defined according to the commutation relations among $\dim(\mathfrak{g})$ generators as follows

$$[T^A, T^B] = if^{ABC}T^C, \quad A, B, C = 1, \dots, \dim(\mathfrak{g}), \quad (63)$$

where f^{ABC} are known as the structure constants. \mathfrak{g} has a maximal set of commuting generators to form the Cartan subalgebra, and we denote them by

$$\{\mathcal{H}^i\}, \quad i = 1, \dots, r, \quad r = \text{rank}(\mathfrak{g}), \quad [\mathcal{H}^i, \mathcal{H}^j] = 0. \quad (64)$$

All other $\dim(\mathfrak{g}) - r$ generators of E^α are the ladder operators, and their commutators with the Cartan generators lead to the roots α of the \mathfrak{g} as follows

$$[\mathcal{H}^i, E^\alpha] = \alpha^i E^\alpha, \quad \alpha \equiv (\alpha^1, \dots, \alpha^r). \quad (65)$$

Since for any root α , the $-\alpha$ is also a root, one thus partitions the set of all roots $\{\alpha\}$ in Eq. (65) into the set of positive root and negative roots as

$$\Delta \equiv \Delta_+ \oplus \Delta_-, \quad \dim\Delta_+ = \dim\Delta_- = \frac{1}{2}(\dim(\mathfrak{g}) - r). \quad (66)$$

The mutual commutators between the raising operators of E^α and lowering operators of $E^{-\alpha}$ are

$$[E^\alpha, E^{-\alpha}] = \frac{2}{(\alpha, \alpha)} \alpha \cdot \mathcal{H}. \quad (67)$$

All roots can be generally written as linear combinations of other roots, and the r linearly independent set of roots are known as the simple roots of α_i (with $i = 1, \dots, r$). The simple coroots are defined according to the simple roots as

$$\alpha_i^\vee \equiv \frac{2\alpha_i}{(\alpha_i, \alpha_i)}. \quad (68)$$

For the $\mathfrak{su}(N)$ Lie algebras, the scalar products between the simple roots are normalized as $(\alpha_j, \alpha_j) = 2$, hence one has $\alpha_i^\vee = \alpha_i$ and $(\alpha_j^\vee, \alpha_j^\vee) = 2$ as well. The scalar products between the simple roots and the coroots define the $\mathfrak{su}(N)$ Cartan matrix

$$A_{ij} \equiv (\alpha_i, \alpha_j^\vee), \quad A_{ii} = 2, \quad A_{i, i\pm 1} (1 < i < N - 1) = A_{12} = A_{N-1, N-2} = -1. \quad (69)$$

Among all roots in the Δ , there is a unique *highest root* θ , which can be expressed as the summation of all simple roots or simple coroots

$$\theta = \sum_{i=1}^{N-1} a_i \alpha_i = \sum_{i=1}^{N-1} a_i^\vee \alpha_i^\vee, \quad a_i, a_i^\vee \in \mathbb{Z}. \quad (70)$$

The a_i and a_i^\vee are the *marks* and the *comarks* and one has $a_i = a_i^\vee = 1$ for the $\mathfrak{su}(N)$ Lie algebra. The highest root $\boldsymbol{\theta}$ for the $\mathfrak{su}(N)$ is given by

$$\boldsymbol{\theta} = (1, \underbrace{0, \dots, 0}_{N-3}, 1). \quad (71)$$

The *Coxeter number* g and the *dual Coxeter number* g^\vee can be defined as

$$g \equiv \sum_{i=1}^{N-1} a_i + 1, \quad (72a)$$

$$g^\vee \equiv \sum_{i=1}^{N-1} a_i^\vee + 1. \quad (72b)$$

The dual Coxeter number reads $g^\vee = N$ for the $\mathfrak{su}(N)$ Lie algebra, while the Abelian $\mathfrak{u}(1)$ Lie algebra has no dual Coxeter number.

The fundamental weights of $\boldsymbol{\omega}_i$ are defined from the simple coroots as follows

$$(\boldsymbol{\omega}_i, \boldsymbol{\alpha}_j^\vee) = \delta_{ij}. \quad (73)$$

The highest-weight states, denoted as $\boldsymbol{\lambda}$, are defined by

$$E^\alpha |\boldsymbol{\lambda}\rangle = 0, \quad \forall \alpha \in \Delta_+, \quad (74)$$

and are expressed in terms of the Dynkin labels as follows

$$\boldsymbol{\lambda} = \sum_{i=1}^r \lambda_i \boldsymbol{\omega}_i, \quad \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{N-1}) = \{\ell_1, \dots, \ell_{N-1}\}, \quad \lambda_i, \ell_i \in \mathbb{Z}, \quad (75)$$

with

$$\ell_i = \lambda_i + \dots + \lambda_{N-1} \quad (76)$$

representing the number of boxes in the i^{th} row of a corresponding Young tableau. For instance, the Dynkin labels for the rank- k anti-symmetric, the rank- k symmetric, and the adjoint representations are expressed as follows

$$\boldsymbol{\lambda}(A_k) = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{N-k-1}), \quad (77a)$$

$$\boldsymbol{\lambda}(S_k) = (k, \underbrace{0, \dots, 0}_{N-2}), \quad (77b)$$

$$\boldsymbol{\lambda}(\text{adj}) = (1, \underbrace{0, \dots, 0}_{N-3}, 1). \quad (77c)$$

For any irrep, all weights can be constructed by subtracting the simple roots from the corresponding highest-weight state, and the total numbers of the weights through this procedure correspond to the

dimension of the irrep. The scalar products between the $\mathfrak{su}(N)$ fundamental weights are expressed as the quadratic form matrix of

$$\begin{aligned}
F_{ij} &= (\boldsymbol{\omega}_i, \boldsymbol{\omega}_j) = \frac{1}{2}(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j)(A^{-1})_{ij} = \frac{1}{N}\min(i, j) \cdot (N - \max(i, j)) \\
&= \frac{1}{N} \begin{pmatrix} N-1 & N-2 & N-3 & \dots & 2 & 1 \\ N-2 & 2(N-2) & 2(N-3) & \dots & 4 & 2 \\ N-3 & 2(N-3) & 3(N-3) & \dots & 6 & 3 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 2 & 4 & 6 & \dots & 2(N-2) & N-2 \\ 1 & 2 & 3 & \dots & N-2 & N-1 \end{pmatrix}, \tag{78}
\end{aligned}$$

where $(A^{-1})_{ij}$ is the inverse of the Cartan matrix (69), and we have used the normalization of $(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k) = 2$. The scalar products between two highest-weight states are thus given by

$$(\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}) = \sum_{i,j} \lambda_i^{(1)} \lambda_j^{(2)} (\boldsymbol{\omega}_i, \boldsymbol{\omega}_j) = \sum_{i,j} \lambda_i^{(1)} \lambda_j^{(2)} F_{ij}. \tag{79}$$

A special type of the weight is the Weyl vector, which is given by the summation over all fundamental weights, or half of the summation over all positive roots as follows

$$\rho \equiv \sum_{i=1}^{N-1} \boldsymbol{\omega}_i = \frac{1}{2} \sum_{\boldsymbol{\alpha} \in \Delta_+} \boldsymbol{\alpha} = \underbrace{(1, \dots, 1)}_{N-1}. \tag{80}$$

The Abelian $\mathfrak{u}(1)$ Lie algebra has no Weyl vector.

The quadratic Casimir operator of \mathcal{Q} is defined by

$$\mathcal{Q} \equiv \sum_i \mathcal{H}^i \mathcal{H}^i + \sum_{\boldsymbol{\alpha} \in \Delta_+} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha). \tag{81}$$

in the Cartan-Weyl basis. Since it commutes with all generators, it is mostly convenient to evaluate the eigenvalue of $C_2(\boldsymbol{\lambda})_{\text{alg}}$ on the highest-weight state of $|\boldsymbol{\lambda}\rangle$, which reads¹⁰

$$\begin{aligned}
\sum_{i=1}^r \mathcal{H}^i \mathcal{H}^i |\boldsymbol{\lambda}\rangle &= (\boldsymbol{\lambda}, \boldsymbol{\lambda}) |\boldsymbol{\lambda}\rangle, \\
\sum_{\boldsymbol{\alpha} \in \Delta_+} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha) |\boldsymbol{\lambda}\rangle &= \sum_{\boldsymbol{\alpha} \in \Delta_+} [E^\alpha, E^{-\alpha}] |\boldsymbol{\lambda}\rangle = \sum_{\boldsymbol{\alpha} \in \Delta_+} (\boldsymbol{\alpha}, \boldsymbol{\lambda}) |\boldsymbol{\lambda}\rangle = (2\rho, \boldsymbol{\lambda}) |\boldsymbol{\lambda}\rangle, \\
\Rightarrow C_2(\boldsymbol{\lambda})_{\text{alg}} &= (\boldsymbol{\lambda} + 2\rho, \boldsymbol{\lambda}), \tag{82}
\end{aligned}$$

where we have used the commutator in Eq. (67) and the definition of the Weyl vector in Eq. (80). For the rank- k anti-symmetric representation in Eq. (77a), one finds that

$$\begin{aligned}
C_2(A_k)_{\text{alg}} &= \underbrace{(0, \dots, 0)}_{k-1}, 1, \underbrace{(0, \dots, 0)}_{r-k} \cdot \underbrace{(2, \dots, 2)}_{k-1}, 3, \underbrace{(2, \dots, 2)}_{r-k} \\
&= 2 \frac{N-k}{N} \sum_{j=1}^{k-1} j + 2 \frac{k}{N} \sum_{j=k+1}^{N-1} (N-j) + 3 \frac{k(N-k)}{N} = \frac{N+1}{N} k(N-k). \tag{83}
\end{aligned}$$

¹⁰The eigenvalue of the quadratic Casimir operator differ by a factor 2 from the conventions in the particle physics.

For the rank- k symmetric representation in Eq. (77b), one finds that

$$\begin{aligned}
C_2(S_k)_{\text{alg}} &= \underbrace{(k, 0, \dots, 0)}_{N-2} \cdot \underbrace{(k+2, 2, \dots, 2)}_{N-2} \\
&= \frac{N-1}{N} k(k+2) + 2k \sum_{j=2}^{N-1} \frac{N-j}{N} \\
&= \frac{N-1}{N} k(N+k). \tag{84}
\end{aligned}$$

For the adjoint representation with the highest-weight state being the highest root of $\boldsymbol{\lambda} = \boldsymbol{\theta}$, one finds the well-known result of

$$\begin{aligned}
C_2(\boldsymbol{\theta})_{\text{alg}} &= (\boldsymbol{\theta} + 2\rho, \boldsymbol{\theta}) = 2 + 2(\rho, \boldsymbol{\theta}) \\
&= 2 + 2 \sum_{i,j=1}^{N-1} a_i^\vee(\omega_j, \boldsymbol{\alpha}_i^\vee) = 2 + 2 \sum_{i=1}^{N-1} a_i^\vee = 2g^\vee, \tag{85}
\end{aligned}$$

by using the definition (72b).

A.2 The affine Lie algebra: Cartan-Weyl basis, simple roots, and the highest weight

Next, we generalize the Lie algebra \mathfrak{g} into the *loop algebra* of $\tilde{\mathfrak{g}}$ as

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \tag{86}$$

such that the commutation relations in Eq. (63) become

$$[T^A_m, T^B_n] = if^{ABC} T^C_{m+n}, \tag{87}$$

with $T^A_m \equiv T^A \otimes t^m$. A central extension to the commutator is given by

$$[T^A_m, T^B_n] = if^{ABC} T^C_{m+n} + \hat{k} n \delta^{AB} \delta_{n+m,0}, \tag{88}$$

with

$$[T^A_m, \hat{k}] = 0. \tag{89}$$

In other words, the central extension only exists when $n+m=0$. The affine Lie algebra is defined by

$$\hat{\mathfrak{g}} \equiv \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0. \tag{90}$$

The L_0 is known as the grading operator of

$$L_0 \equiv -t \frac{d}{dt}, \tag{91}$$

and its action on the generators gives arise to the *grade* n as below

$$[L_0, T^A_n] = -n T^A_n. \tag{92}$$

The maximal Cartan subalgebra of the affine Lie algebra $\hat{\mathfrak{g}}$ is given by

$$\{\mathcal{H}_0^i, \hat{k}, L_0\}, \quad i = 1, \dots, r, \tag{93}$$

while all other generators of $\{E_n^\alpha\}$ and $\{\mathcal{H}_n^i\}$ (with $n \neq 0$) form the ladder operators.

The affine roots include those associated with the ladder operators of E^{α_n}

$$\hat{\alpha} \equiv (\alpha; 0; n) = \alpha + n\delta, \quad \alpha \in \Delta, \quad \delta = (\mathbf{0}; 0; 1), \quad n \in \mathbb{Z}, \quad (94)$$

as well as an extra simple root of

$$\alpha_0 \equiv (-\theta; 0; 1) = -\theta + \delta. \quad (95)$$

Here, the $(\alpha; 0; 0)$ represents the finite part of the affine roots $\hat{\alpha}$, θ is the highest root of \mathfrak{g} in Eq. (70), and the δ is known as the imaginary root due to its zero length of $(\delta, \delta) = 0$.

The affine weights are generally denoted as

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda), \quad (96)$$

with the scalar products defined by

$$(\hat{\lambda}, \hat{\sigma}) \equiv (\lambda, \sigma) + k_\lambda n_\sigma + k_\sigma n_\lambda. \quad (97)$$

The fundamental weights of the $\hat{\mathfrak{g}}$ is given by

$$\hat{\omega}_i \equiv a_i^\vee \hat{\omega}_0 + (\omega_i; 0; 0), \quad \hat{\omega}_0 = (\mathbf{0}; 1; 0), \quad (98)$$

with the $\mathbf{0}$ representing the $N - 1$ zeros in the *basic fundamental weight* of $\hat{\omega}_0$. The corresponding scalar products are given by

$$(\hat{\omega}_i, \hat{\omega}_0) = (\hat{\omega}_0, \hat{\omega}_0) = 0, \quad (\hat{\omega}_i, \hat{\omega}_j) = (\omega_i, \omega_j) = F_{ij}, \quad (i, j \neq 0). \quad (99)$$

For the $\mathfrak{g} = \mathfrak{su}(N)$, the quadratic form matrix of F_{ij} was previously given in Eq. (78). Any affine weight can be expanded in terms of the fundamental weights $\hat{\omega}_i$ and the imaginary root δ as

$$\hat{\lambda} = \sum_{i=0}^r \lambda_i \hat{\omega}_i + \ell \delta, \quad \ell \in \mathbb{R}, \quad (100)$$

with ℓ being the *grade*. For the affine Lie algebra of $\widehat{\mathfrak{su}}(N)_k$, the affine level is given by

$$k \equiv \hat{\lambda}(\hat{k}) = (\hat{\lambda}, \delta) = \sum_{i=0}^r \lambda_i. \quad (101)$$

with $a_i^\vee = 1$ for the $\mathfrak{su}(N)$. The affine weights are also expressed in terms of the Dynkin labels as follows

$$\hat{\lambda} = [\lambda_0; \lambda_1, \dots, \lambda_r]. \quad (102)$$

For all $\widehat{\mathfrak{su}}(8)_{k_U=1}$ rank- k anti-symmetric irreps, their affine weights can still be expressed as the following highest-weight states

$$\hat{\lambda}(A_{k \text{ HW}}) = [0; \underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{7-k}]. \quad (103)$$

Previously, it was previously argued [28] that the adjoint representation cannot be admitted in the $k = 1$ affine Lie algebra since one would naively express it in terms of the highest-weight state as

$$\hat{\lambda}(\text{adj}_{\text{HW}}) = [0; 1, \underbrace{0, \dots, 0}_5, 1]. \quad (104)$$

Instead, it can be obtained by subtracting a new simple root in Eq. (95) from the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ singlet (and highest-weight) state as follows

$$\hat{\lambda}(\text{adj}_{\text{des}}) = (\mathbf{0}; 1; 0) - \alpha_0 = (\boldsymbol{\theta}; 1; -1). \quad (105)$$

In other words, the adjoint representation should be interpreted as the descendant state from the $\widehat{\mathfrak{su}}(8)_{k_U=1}$ singlet state of $(\mathbf{0}; 1; 0)$. The last entry of -1 in Eq. (105) represents a grade= 1 state, where the minus sign originates from the definition of the grading operator in Eq. (91). Therefore, we find the conformal dimension for the adjoint representation as

$$h(\text{adj}_{\text{des}}) = h(\mathbf{0}; 1; 0) + \text{grade} = 1, \quad (106)$$

which apparently satisfy the unitarity constraint in Eq. (27). One seemingly possible SUSY extension in Eq. (40b) involves two chiral superfields of $\mathbf{36}_{\mathbf{H}^\rho}$. This rank-2 symmetric irrep cannot be understood as the highest-weight state, since

$$\hat{\lambda}(\mathbf{36}_{\text{HW}}) = [0; 2, \underbrace{0, \dots, 0}_6]. \quad (107)$$

Instead, it can only be a $k_U = 1$ and grade= 1 descendant state obtained from the highest-weight state of $\mathbf{28}_{\text{HW}}$ as follows

$$\begin{aligned} \hat{\lambda}(\mathbf{36}_{\text{des}}) &= [-1; 2, \underbrace{0, \dots, 0}_6] = \hat{\lambda}(\mathbf{28}_{\text{HW}}) - \left(\sum_{i=2}^7 \alpha_i\right) - \alpha_0, \\ \text{with } \hat{\lambda}(\mathbf{28}_{\text{HW}}) &= [0; 0, 1, \underbrace{0, \dots, 0}_5]. \end{aligned} \quad (108)$$

Therefore, the conformal dimension for the $\mathbf{36}$ representation as the descendant state violates the unitarity constraint in Eq. (27) as follows

$$h(\mathbf{36}_{\text{des}}) = h(\mathbf{28}_{\text{HW}}) + \text{grade} = \frac{3}{4} + 1 = \frac{7}{4} > 1. \quad (109)$$

It means the naive SUSY extension in Eq. (40b) cannot be realized if one hypothesizes the corresponding affine Lie algebra of $\widehat{\mathfrak{su}}(8)_{k_U=1}$.

B The $U(1)$ Cartan discontinuities

In the class of GUTs beyond the $SU(5)$, one usually has the intermediate symmetry breaking stages of the form

$$SU(N) \otimes U(1)_X \hookrightarrow SU(N-1) \otimes U(1)_{X'}, \quad (110)$$

where the corresponding gauge couplings are denoted as

$$(g_N, g_X), \quad (g_{N-1}, g_{X'}), \quad (111)$$

respectively. Such a symmetry breaking pattern is usually achieved by the Higgs fields of $\Phi \equiv (\mathbf{N}, +\frac{1}{N})_{\mathbf{H}} \in SU(N) \otimes U(1)_X$, whose decomposition and the VEV can be denoted as follows

$$\begin{aligned} (\mathbf{N}, +\frac{1}{N})_{\mathbf{H}} &\hookrightarrow (\mathbf{N}-\mathbf{1}, +\frac{1}{N-1})_{\mathbf{H}} \oplus (\mathbf{1}, 0)_{\mathbf{H}}, \\ \langle \Phi \rangle &= \frac{1}{\sqrt{2}}(\vec{0}_{N-1}, v_N)^T. \end{aligned} \quad (112)$$

To find the relation between two U(1) gauge couplings of g_X and $g_{X'}$, we extrapolate the following gauge boson mass squared terms from the covariant derivative

$$|D_\mu \Phi|^2 \supset \frac{v_N^2}{2} \left(-g_N \sqrt{\frac{N-1}{2N}} W_\mu^{N^2-1} + \frac{1}{N} g_X X_\mu \right)^2, \quad (113)$$

where only the component associated with the last SU(N) Cartan generator of $T_{\text{SU}(N)}^{N^2-1}$ is necessary. Thus, we find the following mixing between gauge bosons of $(W_\mu^{N^2-1}, X_\mu)$

$$\begin{pmatrix} W_\mu^{N^2-1} \\ X_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_N & \sin \theta_N \\ -\sin \theta_N & \cos \theta_N \end{pmatrix} \cdot \begin{pmatrix} Z_\mu^{N-1} \\ X'_\mu \end{pmatrix}, \quad \tan \theta_N = \frac{g_X}{g_N} \sqrt{\frac{2}{N(N-1)}}, \quad (114)$$

where the Z_μ^{N-1} is massive and the X'_μ is massless. By expressing the term in Eq. (113) with the mixing relation in Eq. (114), we have to require the X'_μ gauge coupling to match with the decomposition in Eq. (112)

$$\begin{aligned} & -g_N \sqrt{\frac{1}{2N(N-1)}} W_\mu^{N^2-1} + \frac{g_X}{N} X_\mu \\ \supset & \left(\frac{g_N}{\sqrt{2N(N-1)}} \sin \theta_N + \frac{g_X}{N} \cos \theta_N \right) X'_\mu = \frac{g_{X'}}{N-1} X'_\mu, \end{aligned} \quad (115)$$

which leads to the general U(1) Cartan discontinuities of

$$\alpha_{X'}^{-1} = \frac{2}{N(N-1)} \alpha_N^{-1} + \alpha_X^{-1}. \quad (116)$$

Obviously, this relation originates from the last SU(N) Cartan generator of $T_{\text{SU}(N)}^{N^2-1}$. Specifically, we have the relations of

$$\text{SU}(4) \otimes \text{U}(1)_X \hookrightarrow \text{SU}(3) \otimes \text{U}(1)_{X'} \quad : \quad \alpha_{X'}^{-1} = \frac{1}{6} \alpha_4^{-1} + \alpha_X^{-1}, \quad (117a)$$

$$\text{SU}(3) \otimes \text{U}(1)_X \hookrightarrow \text{SU}(2) \otimes \text{U}(1)_{X'} \quad : \quad \alpha_{X'}^{-1} = \frac{1}{3} \alpha_3^{-1} + \alpha_X^{-1}, \quad (117b)$$

as were previously displayed in Figs. 1 and 2.

References

- [1] H. Georgi and S. L. Glashow, ‘‘Unity of All Elementary Particle Forces,’’ *Phys. Rev. Lett.* **32** (1974) 438–441.
- [2] H. Fritzsch and P. Minkowski, ‘‘Unified Interactions of Leptons and Hadrons,’’ *Annals Phys.* **93** (1975) 193–266.
- [3] N. Cabibbo, ‘‘Unitary Symmetry and Leptonic Decays,’’ *Phys. Rev. Lett.* **10** (1963) 531–533.
- [4] M. Kobayashi and T. Maskawa, ‘‘CP Violation in the Renormalizable Theory of Weak Interaction,’’ *Prog. Theor. Phys.* **49** (1973) 652–657.
- [5] H. Georgi, ‘‘Towards a Grand Unified Theory of Flavor,’’ *Nucl. Phys. B* **156** (1979) 126–134.

- [6] CMS Collaboration, A. Tumasyan *et al.*, “A portrait of the Higgs boson by the CMS experiment ten years after the discovery,” *Nature* **607** no. 7917, (2022) 60–68, [arXiv:2207.00043 \[hep-ex\]](#).
- [7] ATLAS Collaboration, G. Aad *et al.*, “A detailed map of Higgs boson interactions by the ATLAS experiment ten years after the discovery,” *Nature* **607** no. 7917, (2022) 52–59, [arXiv:2207.00092 \[hep-ex\]](#). [Erratum: *Nature* 612, E24 (2022)].
- [8] N. Chen, Y.-n. Mao, and Z. Teng, “The global B – L symmetry in the flavor-unified SU(N) theories,” *JHEP* **04** (2024) 046, [arXiv:2307.07921 \[hep-ph\]](#).
- [9] S. M. Barr, “Doubly Lopsided Mass Matrices from Unitary Unification,” *Phys. Rev. D* **78** (2008) 075001, [arXiv:0804.1356 \[hep-ph\]](#).
- [10] N. Chen, Y.-n. Mao, and Z. Teng, “The Standard Model quark/lepton masses and the Cabibbo-Kobayashi-Maskawa mixing in an SU(8) theory,” [arXiv:2402.10471 \[hep-ph\]](#).
- [11] R. D. Peccei and H. R. Quinn, “CP Conservation in the Presence of Instantons,” *Phys. Rev. Lett.* **38** (1977) 1440–1443.
- [12] L.-F. Li, “Group Theory of the Spontaneously Broken Gauge Symmetries,” *Phys. Rev. D* **9** (1974) 1723–1739.
- [13] N. Chen, Z. Hou, Y.-n. Mao, and Z. Teng, “The gauge coupling evolutions of an SU(8) theory with the maximally symmetry breaking pattern,” *JHEP* **10** (2024) 149, [arXiv:2406.09970 \[hep-ph\]](#).
- [14] N. Chen, Z. Chen, Z. Hou, Z. Teng, and B. Wang, “Further study of the maximally symmetry breaking patterns in an SU(8) theory,” [arXiv:2409.03172 \[hep-ph\]](#).
- [15] H. Georgi, H. R. Quinn, and S. Weinberg, “Hierarchy of Interactions in Unified Gauge Theories,” *Phys. Rev. Lett.* **33** (1974) 451–454.
- [16] L. J. Hall, “Grand Unification of Effective Gauge Theories,” *Nucl. Phys. B* **178** (1981) 75–124.
- [17] S. Dimopoulos and H. Georgi, “Softly Broken Supersymmetry and SU(5),” *Nucl. Phys. B* **193** (1981) 150–162.
- [18] S. Weinberg, “Effective Gauge Theories,” *Phys. Lett. B* **91** (1980) 51–55.
- [19] C. T. Hill, “Are There Significant Gravitational Corrections to the Unification Scale?,” *Phys. Lett. B* **135** (1984) 47–51.
- [20] Q. Shafi and C. Wetterich, “Modification of GUT Predictions in the Presence of Spontaneous Compactification,” *Phys. Rev. Lett.* **52** (1984) 875.
- [21] S. L. Glashow, “The Future of Elementary Particle Physics,” *NATO Sci. Ser. B* **61** (1980) 687.
- [22] R. Barbieri, D. V. Nanopoulos, G. Morchio, and F. Strocchi, “Neutrino Masses in Grand Unified Theories,” *Phys. Lett. B* **90** (1980) 91–97.
- [23] R. Barbieri and D. V. Nanopoulos, “An Exceptional Model for Grand Unification,” *Phys. Lett. B* **91** (1980) 369–375.

- [24] R. Barbieri and D. V. Nanopoulos, “Hierarchical Fermion Masses From Grand Unification,” *Phys. Lett. B* **95** (1980) 43–46.
- [25] F. del Aguila and L. E. Ibanez, “Higgs Bosons in SO(10) and Partial Unification,” *Nucl. Phys. B* **177** (1981) 60–86.
- [26] P. H. Ginsparg, “Gauge and Gravitational Couplings in Four-Dimensional String Theories,” *Phys. Lett. B* **197** (1987) 139–143.
- [27] A. Font, L. E. Ibanez, and F. Quevedo, “Higher Level Kac-Moody String Models and Their Phenomenological Implications,” *Nucl. Phys. B* **345** (1990) 389–430.
- [28] K. R. Dienes, “String theory and the path to unification: A Review of recent developments,” *Phys. Rept.* **287** (1997) 447–525, [arXiv:hep-th/9602045](https://arxiv.org/abs/hep-th/9602045).
- [29] P. Goddard and D. I. Olive, “Kac-Moody and Virasoro Algebras in Relation to Quantum Physics,” *Int. J. Mod. Phys. A* **1** (1986) 303.
- [30] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization,” *Nucl. Phys. B* **222** (1983) 83–103.
- [31] E. Witten, “Dynamical Breaking of Supersymmetry,” *Nucl. Phys. B* **188** (1981) 513.
- [32] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.