

Strong XOR Lemma for Information Complexity

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Abstract

For any $\{0, 1\}$ -valued function f , its n -folded XOR is the function $f^{\oplus n}$ where $f^{\oplus n}(X_1, \dots, X_n) = f(X_1) \oplus \dots \oplus f(X_n)$. Given a procedure for computing the function f , one can apply a “naive” approach to compute $f^{\oplus n}$ by computing each $f(X_i)$ independently, followed by XORing the outputs. This approach uses n times the resources required for computing f .

In this paper, we prove a strong XOR lemma for *information complexity* in the two-player randomized communication model: if computing f with an error probability of $O(n^{-1})$ requires revealing I bits of information about the players’ inputs, then computing $f^{\oplus n}$ with a constant error requires revealing $\Omega(n) \cdot (I - 1 - o_n(1))$ bits of information about the players’ inputs. Our result demonstrates that the naive protocol for computing $f^{\oplus n}$ is both information-theoretically optimal and asymptotically tight in error trade-offs.

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1 Introduction

For a function $f : \mathcal{X} \rightarrow \{0, 1\}$ and any natural number n , let $f^{\oplus n} : \mathcal{X}^n \rightarrow \{0, 1\}$ denote the function defined by $f^{\oplus n}(X_1, \dots, X_n) = f(X_1) \oplus \dots \oplus f(X_n)$. The questions surrounding *XOR lemmas* focus on the relationship between the *resources* needed to compute f and those required for computing $f^{\oplus n}$. Given a procedure \mathcal{P} for computing f , one could naively compute $f^{\oplus n}(X_1, \dots, X_n)$ by evaluating each $f(X_i)$ independently via \mathcal{P} and then taking the XOR of the n output bits. This strategy uses n times the resources required to compute f . The central question of XOR lemmas asks whether this naive protocol is resource-optimal.

Question 1. *For which regimes of parameters (ρ, ρ') and which notions of *resources* does computing $f^{\oplus n}$ with probability ρ' require $\Omega(n)$ times the resources needed for computing f with probability ρ ?*

XOR lemmas are closely connected to the *Direct Sum* problem, where we seek to compute $f(X_i, Y_i)$ for all $i \in [n]$, and have been extensively studied under various resource models, including circuit size [Yao82; Lev87; Imp95; IW97; GNW11], query complexity [Sha03; She11; Dru12; BKLS20; BKST24], and decision-tree complexity [Hoz24]. Another related question is the *Direct Product* problem: does the success probability of $f^{\oplus n}$ or the *advantage* of $f^{\oplus n}$ decay exponentially as n increases under limited resources? This question has been studied in contexts of game values [Raz98; Hol07; Raz08; Rao08].

In this paper, we consider the computation of a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ in a *two-player randomized communication* setting. In this model, Alice receives an input $X \in \mathcal{X}$, and Bob receives $Y \in \mathcal{Y}$. The goal is for the players to compute $f(X, Y)$ by exchanging a sequence of messages (M^1, M^2, \dots, M^r) . For odd rounds i , Alice generates M^i based on her input X and the preceding messages $M^{<i}$; for even i , Bob generates M^i based on Y and the preceding messages $M^{<i}$. Both players also have access to private randomness and shared public randomness. The randomized messaging schemes (M^1, M^2, \dots, M^r) are called (*randomized*) *protocols*. Notably, the computation of $f^{\oplus n}$ can also be modeled in this two-player setting: Alice receives (X_1, \dots, X_n) , Bob receives (Y_1, \dots, Y_n) , and their goal is to compute $f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n)$ through running a protocol. Many results exist when the resource of interest is the total length of messages, namely the *communication complexity*, such as Direct Sum results [CSWY01; Sha03; JRS03; HJMR07; BCCR10], Direct Product results [Kla10; BRWY13b; BRWY13a; Jai15; IR24b], and XOR lemmas [BCCR10; Yu22; IR24a; IR24b].

Beyond suggesting trade-offs in the required resources, the naive protocol also provides insight into the optimal trade-offs between error rates, which occur when $(\rho, \rho') = (\frac{1}{2} + \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha^n}{2})$ for some *advantage* $\alpha \in (0, 1)$. To see this, suppose p and q are independent $\{0, 1\}$ -valued random variables with advantages α_p and α_q , meaning they can be predicted by \hat{p} and \hat{q} with probabilities $\frac{1}{2} + \frac{\alpha_p}{2}$ and $\frac{1}{2} + \frac{\alpha_q}{2}$, respectively. Then, the XOR of $p \oplus q$ can be predicted by $\hat{p} \oplus \hat{q}$ with probability $\frac{1}{2} + \frac{\alpha_p \alpha_q}{2}$, since

$$\begin{aligned} \Pr(\hat{p} \oplus \hat{q} = p \oplus q) &= \Pr(\hat{p} = p \wedge \hat{q} = q) + \Pr(\hat{p} \neq p \wedge \hat{q} \neq q) \\ &= \Pr(\hat{p} = p) \cdot \Pr(\hat{q} = q) + \Pr(\hat{p} \neq p) \cdot \Pr(\hat{q} \neq q) && (p \perp q) \\ &= \left(\frac{1}{2} + \frac{\alpha_p}{2}\right) \cdot \left(\frac{1}{2} + \frac{\alpha_q}{2}\right) + \left(\frac{1}{2} - \frac{\alpha_p}{2}\right) \cdot \left(\frac{1}{2} - \frac{\alpha_q}{2}\right) \\ &= \frac{1}{2} + \frac{\alpha_p \alpha_q}{2}. \end{aligned}$$

In the naive protocol, each $f(X_i, Y_i)$ is computed with probability $\rho = \frac{1}{2} + \frac{\alpha}{2}$, achieving advantage α . Thus, computing the XOR of these n bits yields an advantage of α^n , corresponding to $\rho' = \frac{1}{2} + \frac{\alpha^n}{2}$.¹ Thus, Question 1 is worth investigating in two parameter regimes:

1. $(\rho, \rho') = (\frac{9}{10}, \frac{1}{2} + 2^{-n})$, corresponding to the optimal trade-off where $\alpha = \Theta(1)$, and
2. $(\rho, \rho') = (1 - \frac{1}{n}, \frac{9}{10})$, corresponding to the optimal trade-off where $\alpha = 1 - \Theta(1/n)$.

¹It is reasonable to consider regimes in terms of advantage (ρ, ρ') or error probabilities $(1 - \rho, 1 - \rho')$ only *asymptotically*, as it is possible to *boost* the success probability by making multiple independent runs of the protocol, taking the majority answer. For example, $T = O(1)$ runs suffice to boost the success probability from 0.501 to any constant below 1, or from error $n^{-0.01}$ to $n^{-O(1)}$.

1.1 Our Results

In this paper, we provide an affirmative answer to Question 1 (up to vanishing additive losses) in the regime where $(\rho, \rho') = (1 - \frac{1}{n}, \frac{9}{10})$ when the resource of interest is *information*. For a function g and error parameter $\varepsilon \in (0, 1)$, let the *information complexity* of g with error ε , denoted $\mathbf{I}(g, \varepsilon)$, be the maximum amount of information each player learns about the other's input by the end of a protocol that computes g with probability at least $1 - \varepsilon$. This “resource” represents the (worst-case) amount of information players must learn to compute f accurately. With this notion, we prove a strong XOR lemma for information complexity.

Theorem 2 (Strong XOR Lemma for Information Complexity). *There exists a universal constant $\lambda \in (0, 1)$ and $c_1 > 0$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ and any positive integer n , we have*

$$\mathbf{I}(f^{\oplus n}, 1/10) \geq c_1 n \cdot \left(\mathbf{I}(f, n^{-1}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right).$$

Our result is asymptotically tight (up to vanishing additive losses), as demonstrated by the following result. Its proof whose proof will be deferred to Appendix A.

Theorem 3. *There exists a universal constant $c_2 > 0$ such that for any $\{0, 1\}$ -valued function f and positive integer n , we have*

$$\mathbf{I}(f^{\oplus n}, 1/10) \leq c_2 n \cdot \mathbf{I}(f, n^{-1}).$$

Our proof of Theorem 2 relies on a distributional version of the XOR lemma for *information cost*. For a function g , error parameter $\varepsilon \in (0, 1)$, and input distribution μ , let $\text{IC}_\mu(g, \varepsilon)$ denote the *information cost* of g with error ε , defined as the minimum information learned by each player about the other's input while using a protocol that computes g with probability at least $1 - \varepsilon$ over inputs drawn from μ . We establish a strong XOR lemma for distributional information cost, using it to prove Theorem 2.

Theorem 4 (Strong XOR Lemma for Distributional Information Cost). *There exists a universal constant $\lambda \in (0, 1)$ and $c_3 > 0$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$, any positive integer n , and any input distribution μ over $\mathcal{X} \times \mathcal{Y}$, we have*

$$\text{IC}_{\mu^n}(f^{\oplus n}, 1/10) \geq c_3 n \cdot \left(\text{IC}_\mu(f, n^{-\lambda}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right).$$

1.2 XOR Lemmas for Exponentially Small Advantage

Another significant regime is $(\rho, \rho') = (\frac{2}{3}, \frac{1}{2} + 2^{-n})$, representing the tight upper bound for $\alpha = \Theta(1)$. However, in this setting, an XOR lemma for information complexity does not hold. Consider the following protocol that computes $f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n)$ with probability $\frac{1}{2} + 2^{-n}$ while revealing only an exponentially small amount of information: for a 2^{-n+1} fraction of inputs, Alice sends her entire input to Bob, allowing him to compute $f^{\oplus n}$ exactly; otherwise, Alice sends nothing, and Bob outputs a random bit which is correct with probability $1/2$. This protocol achieves success probability $\frac{1}{2} + 2^{-n}$, but Alice reveals only $2^{-n+1} \cdot n \cdot \log|\mathcal{X}|$ bits of information, meaning that an XOR lemma cannot hold in this regime.

On the contrary, a recent result by [Yu22] provides a positive answer to Question 1 when the resource of interest is *communication*, showing that if any r -round protocol computing f with probability $2/3$ requires C bits of communication, then any protocol computing $f^{\oplus n}$ with probability $1/2 + 2^{-n}$ requires $n \cdot (r^{-O(r)} \cdot C - 1)$ bits. [IR24b] extends this to the *Direct Product* setting, as well as eliminating the exponent “ $-O(r)$ ” from Yu's result.

2 Related Work

We restrict our attention to the following question, as it immediately implies our Distributional XOR Lemma.

Question 5. *Given a communication protocol π for computing $f^{\oplus n}$ with error $1/10$ over μ^n with information cost \mathcal{I} , can we construct a new protocol η for computing f with error $n^{-O(1)}$ over μ with information cost $\approx \mathcal{I}/n$?*

An easier variant of this question is implied by known results, where we allow the same constant error $1/10$ for computing f .

2.1 The ‘‘Folklore’’ Input Embedding Procedure

The XOR Lemma for distributional information cost is known to be true in the setting where $\rho = \rho'$ due to the work by [BJKS04] which was made explicit by [BBCR10].

Theorem 6 (Theorem 2.4 of [BBCR10]; informal). *For any function f , error parameter $\varepsilon \in (0, 1)$, and an input distribution μ , the following holds.*

*Given a protocol π for computing $f^{\oplus n}$ with error ε over μ^n with information cost at most \mathcal{I} ,
then there exists a protocol η for computing f with error ε over μ with information cost $\mathcal{I}/n + O(1)$.*

For completeness, we roughly sketch the proof of the theorem by constructing the protocol η via by *embedding* an input (x, y) into one of the n coordinates, and then execute π .

Protocols η for computing $f(x, y)$ where (x, y) are drawn from μ
<ol style="list-style-type: none"> 1. Players use public randomness to sample a uniform index $J \in [n]$ and partial inputs $X_{<J}$ and $Y_{>J}$ 2. Alice embeds $X_J = x$ and privately samples $X_{>J}$ conditioned on $Y_{>J}$. 3. Bob embeds $Y_J = y$ and privately samples $Y_{<J}$ conditioned on $X_{<J}$. 4. Players execute π to compute $f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n)$. 5. Alice sends Bob an extra bit indicating $f^{\oplus n-J}(X_{>J}, Y_{>J})$, and Bob sends Alice an extra bit indicating $f^{\oplus J-1}(X_{<J}, Y_{<J})$. 6. Players recover $f(x, y)$ by computing $f(x, y) := f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n) \oplus f^{\oplus n-J}(X_{>J}, Y_{>J}) \oplus f^{\oplus J-1}(X_{<J}, Y_{<J}).$

Figure 1: A protocol η for computing f on inputs $(x, y) \sim \mu$ via the embedding method.

The protocol η can also be interpreted as follows: we list n protocols (π_1, \dots, π_n) for which π_j corresponds to $\eta \mid J = j$, and execute a π_j for a random $j \in [n]$. It can be shown by calculation that the information costs of these n protocols sum up to at most $\mathcal{I} + O(n)$. Since η picks a uniform random coordinate j and runs π_j . Thus, its information cost is at most

$$\frac{1}{n} \cdot (\mathcal{I} + O(n)) = \frac{\mathcal{I}}{n} + O(1).$$

Nevertheless, η is ineffective in boosting correctness as it only succeeds with probability $\rho' = \rho$. To see this, observe that both players always compute $f^{\oplus n-J}(X_{>J}, Y_{>J})$ and $f^{\oplus J-1}(X_{<J}, Y_{<J})$ correctly. Thus, the correctness of $f(x, y)$ inherits from that of $f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n)$. Since $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ is distributed exactly like μ^n , the probability that η is correct remains ρ . This roughly proves Theorem 6.

By plugging $\rho = 9/10$ (i.e. the error is $1/10$) into Theorem 6, and take the supremum over protocol π , we have shown a distributional xor lemma with preserving error. As a corollary, a similar bound can be shown for the information complexity.²

Theorem 7. *There exists a universal constant c_4 such that the following holds. For any $\{0, 1\}$ -valued function f , any positive integer n , and any input distribution μ , it holds that*

$$\text{IC}_{\mu^n}(f^{\oplus n}, 1/10) \geq n \cdot (\text{IC}_{\mu}(f, 1/10) - O(1))$$

and consequently

$$\mathbf{I}(f^{\oplus n}, 1/10) \geq c_4 n \cdot (\mathbf{I}(f, 1/10) - O(1)).$$

Our main results in Theorem 2 and Theorem 4 can be interpreted as improving the asymptotic of errors of Theorem 7 from $(\frac{1}{10}, \frac{1}{10})$ to $(\frac{1}{10}, n^{-O(1)})$ where it is asymptotically-tight. Attempting to obtain polynomially-small error for f poses as the main technical challenge of our work.

2.2 Driving Down the Errors

To bring the error probability of f down from $\frac{1}{10}$ to n^{-1} in the XOR Lemmas, [Yu22] presents an alternative view of the input embedding procedure. On a high level, Yu proposes a *decomposition* procedure that split the protocol π for computing $f^{\oplus n}$ into two protocols: a protocol $\pi^{(n)}$ for computing f over μ , and a protocol $\pi^{(<n)}$ for computing $f^{\oplus n-1}$ over μ^{n-1} , for which their information costs add up to $\mathcal{I} + O(1)$. More importantly, it holds “pointwisely” that the advantage of π is equal to the product of the advantage of $\pi^{(n)}$ and the advantage of $\pi^{(<n)}$. These observations motivate the reasoning that at least one of the following cases should occur:

- (1) $\pi^{(n)}$ is a “good” protocol for computing f , as it has low information cost and errs with small probability. In this case, we have found the desired protocol $\eta := \pi^{(n)}$.
- (2) $\pi^{(<n)}$ is “better than average” for computing $f^{\oplus n-1}$. In this case, we recursively decompose $\pi^{(<n)}$ into $\pi^{(n-1)}$ and $\pi^{(<n-1)}$, until we land in case (1).

This preliminary idea of *protocol decomposition* led [Yu22] to the Strong XOR Lemma for *communication complexity* in the regime where $(\rho, \rho') = (\frac{1}{2} + 2^{-n}, \frac{2}{3})$. However, to the best of our knowledge, the majority of the techniques used in Yu’s proof do *not* transfer to our regime, where $(\rho, \rho') = (\frac{9}{10}, 1 - \frac{1}{n})$. While Yu’s work serves as a foundational building block, our approach eventually diverges. We address these distinctions in Section 3.3.

3 Technical Overview

In this section, we present our main lemma and outline its proof. The complete proof will be detailed in a sequence of Sections 7 and Section 8. Theorem 2 and Theorem 4 follows directly from Lemma 8. For clarity, their proofs are deferred to Section 9.

Lemma 8 (Main Technical Lemma). *There exists a universal constant $C > 0$ and $\lambda \in (0, 1)$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$:*

If there exists a (standard) communication protocol π for computing $f^{\oplus n}$ over an input distribution μ^n such that it errs with probability $\frac{1}{10}$ and has information cost \mathcal{I} ,

then there exists a (standard) communication protocol η for computing f over an input distribution μ such that it errs with probability $n^{-\lambda}$ and has information cost at most $C \cdot \left(\frac{\mathcal{I}}{n} + \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} + 1\right)$.

To prove Lemma 8, we assume its setup. Let π be a protocol that computes $f^{\oplus n}$ on the input distribution μ^n with information cost \mathcal{I} and advantage $\frac{4}{5}$.³ Below, we sketch the approach to obtaining the protocol η that satisfies the requirements of Lemma 8.

²We shall not show the reduction explicitly; however, it is remarkably similar to the proof of Theorem 3 appeared in Appendix A.

³The advantage $\frac{4}{5}$ corresponds to the error probability $\frac{1}{10}$.

3.1 Binary Protocol Decomposition from [Yu22]

Our starting point is a slight modification of the protocol decomposition procedure introduced in [Yu22]. It is worth noting that in the work of [Yu22], their decomposition yields two *unbalanced* protocols: a protocol $\pi^{(n)}$ for computing f , and a protocol $\pi^{(<n)}$ for computing $f^{\oplus n-1}$. In our work, we split the protocol equally so the two smaller protocols both compute $f^{\oplus n/2}$. This turns out to be an important aspect of our decomposition, as it will later yields a very clean analysis. Specifically, given a protocol π for computing $f^{\oplus n}$, we construct two protocols, namely π_0 and π_1 , as follows.

Algorithm 1 : Protocol π_0 for computing $f^{\oplus n/2}$

Input: Alice receives input x on $n/2$ coordinates and Bob receives input y on $n/2$ coordinates

- 1: Alice sets $X_{<n/2}$ to x and Bob sets $Y_{<n/2}$ to y
 - 2: Alice and Bob publicly samples $Y_{>n/2}$
 - 3: Alice privately samples $X_{>n/2}$
 - 4: Players run π pretending that their inputs are (X, Y) on n instances to compute $f^{\oplus n}(X, Y)$
 - 5: Alice sends Bob an extra bit b_0 indicating $f^{\oplus n/2}(X_{>n/2}, Y_{>n/2})$
 - 6: Both players recover $f^{\oplus n/2}(x, y) = f^{\oplus n/2}(X_{<n/2}, Y_{<n/2}) = f^{\oplus n}(X, Y) \oplus b_0$.
-

Algorithm 2 : Protocol π_1 for computing $f^{\oplus n/2}$

Input: Alice receives input x on $n/2$ coordinates and Bob receives input y on $n/2$ coordinates

- 1: Alice sets $X_{>n/2}$ to x and Bob sets $Y_{>n/2}$ to y
 - 2: Alice and Bob publicly samples $X_{<n/2}$
 - 3: Bob privately samples $Y_{<n/2}$
 - 4: Players run π pretending that their inputs are (X, Y) on n instances to compute $f^{\oplus n}(X, Y)$
 - 5: Bob sends Alice an extra bit b_1 indicating $f^{\oplus n/2}(X_{<n/2}, Y_{<n/2})$
 - 6: Both players recover $f^{\oplus n/2}(x, y) = f^{\oplus n/2}(X_{>n/2}, Y_{>n/2}) = f^{\oplus n}(X, Y) \oplus b_1$.
-

One might notice that the protocol π_0 could still achieve the same success probability even if the *final bit* b_0 were omitted. This is because, by computing b_0 herself, Alice can answer $f^{\oplus n/2}(X_0, Y_0)$ on Bob's behalf. However, the final bit remains essential in our decomposition due to technical reasons that we will later require that *both players* know the value of b_0 . As a result, b_0 could add up to one bit to the information cost of π_0 . In what follows, we make one simplification: when analyzing the information costs, we account only for the cost incurred by the messages in the original protocol, neglecting the cost of the final bits. Eventually, we will address how to lift this assumption.

Observe that in both protocols π_0 and π_1 , the input pair (x, y) is drawn from the distribution $\mu^{n/2}$. Moreover, to be able to execute π , the players pretend that their inputs are consisting of n instances by filling up the missing coordinates so that their "artificial" inputs (X, Y) distribute exactly like μ^n . Notice further that in π_0 , Alice knows both $X_{>n/2}$ and $Y_{>n/2}$; thus, she computes b_0 correctly with probability 1. Similarly, in π_1 Bob computes b_1 correctly with probability 1.

Let I_0 and I_1 denote the information costs of π_0 and π_1 respectively. [Yu22] made an elegant observation that when decomposing the protocols as above, their information costs and advantages also admits algebraic decomposition.

Decomposition of Information Costs. For π_0 , the information cost from Alice's side is $I(M : Y_{<n/2} | X_{<n/2} Y_{>n/2})$, and from Bob's side is $I(M : X_{<n/2} | Y)$. For π_1 , the information cost from Alice's side is $I(M : Y_{>n/2} | X)$, and from Bob's side is $I(M : X_{>n/2} | X_{<n/2} Y_{>n/2})$. Observe that

$$\begin{aligned}
& I(M : Y_{<n/2} | X_{<n/2} Y_{>n/2}) + I(M : Y_{>n/2} | X) \\
&= I(M : Y_{<n/2} | X Y_{>n/2}) + I(M : Y_{>n/2} | X) && \text{(rectangle property)} \\
&= I(M : Y | X) && \text{(chain rule)}
\end{aligned}$$

and similarly we also have $I(M : X_{<n/2} | Y_{<n/2}X_{>n/2}) + I(M : X_{>n/2} | Y) = I(M : Y | X)$. Therefore, we have $I_0 + I_1 = \mathcal{I}$. This suggests that the information costs of the protocols decompose additively.

Decomposition of Advantage. Denote the following set of random variables which depends on the randomness of $MX_{<n/2}Y_{>n/2}$:

$$\begin{aligned} A_0 &= \text{adv}(f^{\oplus n/2}(X_{<n/2}, Y_{<n/2}) | MX_{<n/2}Y_{>n/2}) \\ A_1 &= \text{adv}(f^{\oplus n/2}(X_{>n/2}, Y_{>n/2}) | MX_{<n/2}Y_{>n/2}) \\ Z &= \text{adv}(f^{\oplus n/2}(X, Y) | MX_{<n/2}Y_{>n/2}) \end{aligned}$$

where $\text{adv}(a | W)$ denotes the advantage of the bit a conditioned on event W .

Conditioned on $MX_{<n/2}Y_{>n/2}$, A_0 is the advantage of π_0 (from Alice's perspective) and A_1 is the advantage of π_1 (from Bob's perspective). Moreover, we have $f^{\oplus n/2}(X_{<n/2}, Y_{<n/2}) \perp f^{\oplus n/2}(X_{>n/2}, Y_{>n/2}) | MX_{<n/2}Y_{>n/2}$ due to the *rectangle property* of communication protocols. This implies $Z = A_0A_1$ ⁴. This suggests that advantages of the protocols decompose multiplicatively.

However, a caveat arises: the equality $Z = A_0A_1$ only holds pointwise. Yet we know that the advantage of π_0 is $\mathbb{E}(A_0)$, the advantage of π_1 is $\mathbb{E}(A_1)$, and the advantage of π is at most $\mathbb{E}(Z)$. Only did we have an assumption that $\mathbb{E}(A_0)\mathbb{E}(A_1) \geq \mathbb{E}(A_0A_1) = \mathbb{E}(Z)$, we would have been able to conclude that advantages decompose multiplicatively. However, the reality is that it is not always the case that the “ \geq ” holds.

As a thought experiment, let us consider the (incorrect) implications of the information costs and advantages decompositions if they had held. If we apply the decomposition on π_0 and π_1 again, we could obtain four protocols for computing $f^{\oplus n/4}$. Repeating this recursively, we would eventually end up with n protocols for computing f . By the additive decomposition of information costs and the (incorrect) multiplicative decomposition of advantages, those n protocols must have their information costs summing to \mathcal{I} and their advantages multiplying to at least $4/5$. By the averaging argument, there must exist a protocol η computing f with information cost $O(\mathcal{I}/n)$ and advantage at least $(4/5)^{1/n} = 1 - \Theta(1/n)$ which is equivalent to success probability $1 - \Theta(1/n)$.

While this approach fails due to the false premise, the idea of recursively breaking down the protocols remains useful.

3.2 Our Approach: “Conditional” Protocol Decomposition

To address the aforementioned issues, we propose a modified protocol decomposition approach. Specifically, let W denote the event that $Z \geq 0.01$. Then, we obtain protocols π_0 and π_1 by applying binary decomposition to the “protocol” $\pi | W$ (thus the name *conditional* decomposition.)⁵

We also introduce a new parameter of interest, called *disadvantage*, denoted ε , which is defined as one minus the protocol's advantage. Generally, this quantity is twice of the error probability when outputting the more-likely bit. Thus, by having players output the more-likely bit, the disadvantage provide a measure of error probability, accurate within a factor of 2. Let $\varepsilon = 1 - \frac{4}{5} = \frac{1}{5}$ denote the disadvantage of π , and let ε_0 and ε_1 represent the disadvantages of π_0 and π_1 , respectively.

⁴If p, q are independent $\{0, 1\}$ -valued random variables, then $\text{adv}(p \oplus q) = \text{adv}(p) \cdot \text{adv}(q)$.

⁵Careful readers might flag that $\pi | W$ no longer aligns with the conventional description of communication protocols. We will address this point shortly.

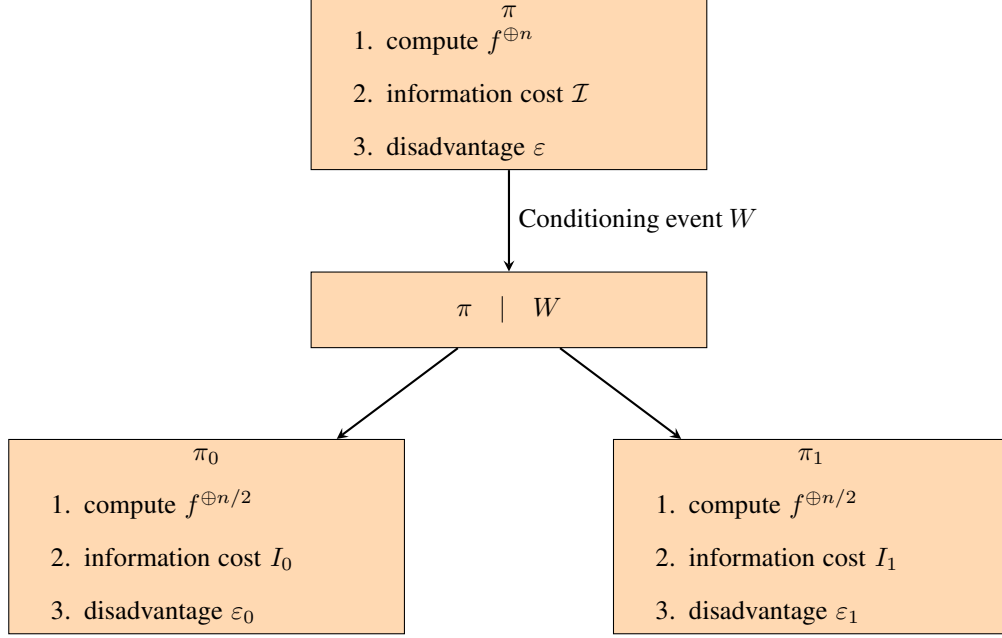


Figure 2: A single-level “conditional” decomposition of a protocol π for $f^{\oplus n}$ into two protocols π_0 and π_1 for $f^{\oplus n/2}$.

With the new way of decomposition, we can show that ⁶

$$\varepsilon_0 + \varepsilon_1 \leq 1.98\varepsilon \quad \text{and} \quad I_0 + I_1 \leq \mathcal{I} \cdot e^{O(\varepsilon)}. \quad (1)$$

In other words, the conditional decomposition yields a geometric decay in disadvantages and a near-linear decay in information costs, adjusted by a small multiplicative factor of $e^{O(\varepsilon)}$.

Next, consider applying this procedure recursively. For each protocol π_S , we introduce a conditioning event W_S and derive two smaller protocols, π_{S0} and π_{S1} , via a binary decomposition of $\pi_S | W_S$. Let the collection of protocols π_S with $|S| = k$ be referred to as the *level k*. Upon reaching level $m = \log_2 n$, we have obtained n protocols $\{\pi_S\}_{|S|=m}$, each of which computes f .

Decomposition of Disadvantages. By the first inequality of (1), we see that the average of ε ’s across level $k + 1$ decreases from that of level k to a multiplicative factor of 0.99. As a consequence, the average ε ’s for the leaf level $m = \log_2 n$ (i.e. where the protocols are for single-instance of f) is at most $(0.99)^{\log_2 n} \varepsilon < n^{-0.01}$.

Decomposition of Information Costs. The breakdown of information costs is much more intricate. To illustrate potential outcomes following Equation (1), let us assume that the ε values are well-balanced: for S , we consider $\varepsilon_{S0} \approx \varepsilon_{S1} \approx 0.99\varepsilon_S$. This implies that $\varepsilon_S \approx (0.99)^{|S|}\varepsilon$. Under this assumption, the *total* information costs of the protocols at level $k + 1$ increase from those at level k by a factor of $\exp(0.99^k \varepsilon)$. Overall, this accumulation leads to a blow-up factor of $O(1)$ due to the geometric sum. Therefore, we can expect the sum of information costs at level $m = \log_2 n$ to be $O(\mathcal{I})$.

By averaging arguments, there must exist a protocol η at the leaf level $m = \log_2 n$ with an information cost of $O(\mathcal{I}/n)$ and an error probability of $n^{-0.01}$.

Nonetheless, the argument outlined above encounters several technical difficulties.

⁶Proof omitted. A stronger statement will be discussed in Section 7.

- (1) The calculation assumes a balanced split of information costs across all decompositions; however, this cannot be guaranteed.⁷

Our earlier argument can be interpreted as a probabilistic proof: we sample an index S uniformly from $[n]$, and in expectation, π_S has small disadvantage and low information cost. To address issue (1), we can sample S from a more carefully-constructed distribution \mathcal{D} . We will show, *without assuming a balanced decomposition of information costs*, that in expectation over $S \sim \mathcal{D}$, a protocol π_S exhibits the desired properties: it has small disadvantage of $n^{-0.01}$ and an information cost of approximately $O(\mathcal{I}/n)$.

Recall that under our assumption, the information cost of π_S does not account for the missing bits, denoted B_S , which include all the “final bits” along the recursive decomposition. Let us now address this assumption by computing the information cost incurred by B_S . Since each level of the decomposition adds one extra bit to the protocol, and π_S is at level $m = \log_2 n$, the length of B_S must be of $\log_2 n$ bits. Naively, this means B_S could contain up to $\log_2 n$ bits of information, which is too costly. To correct this, it can be shown that B_S contains, in expectation, only $O(1)$ information about the inputs (X_S, Y_S) at coordinate S . Therefore, the “true” information cost of the protocol π_S is bounded by $O(\frac{\mathcal{I}}{n} + 1)$.

However, we are not finished yet; we still encounter additional issues:

- (2a) π_S is not a *standard* communication protocol.
(2b) The input distribution of π_S is no longer μ .

Let us first elaborate on issue (2a). In *standard* communication protocols, each player is restricted to generating the next message based solely on their input and the past messages exchanged (as well as their private and public randomness.) For instance, the protocol π is standard due to the set-up of Lemma 8. In contrast, $\pi \mid W$ does not meet this criterion because the conditioning event W can introduce arbitrary correlations between Alice’s messages and Bob’s input, and vice versa. Protocols that allow such correlations are referred to as *generalized* communication protocols. Consequently, π_S is no longer a standard protocol, while it remains a generalized protocol.

Issue (2b) stems from a similar source: the sequence of conditioning events, denoted by E , that are recursively applied throughout the decomposition. This conditioning can distort the input distributions of π_S from μ to $\mu \mid E$ in unpredictable ways.

Fortunately, both issues can be resolved simultaneously. The underlying intuition is that on average, each conditioning event occurs with a probability very close to 1, suggesting that the overall distortion induced by E is minimal in expectation. As a result, we can expect the protocol π_S to be “close” to a standard protocol while operating on an input distribution that is “close” to μ .

Somewhat-more formally, we can augment the protocol π_S with another desirable set of properties: it is statistically-close to some standard protocol η (in terms of KL-Divergence) whose input distribution aligns precisely with μ . Importantly, the protocol η maintains an error probability of $n^{-0.001}$ over input distribution μ and achieves information cost of at most $C \cdot (\frac{\mathcal{I}}{n} + o_n(1) + 1)$ for some absolute constant C . In summary, the *standard* protocol η exhibits all the required properties, thereby proving Lemma 8.

Notably, due to technical reasons, the final description of our the conditional decomposition procedure must deviate from the overview provided here. Nevertheless, the overall flow and main ideas of the proof remain largely the same.

3.3 Key Differences from [Yu22]

As briefly mentioned, to the best of our understanding, the techniques from Yu’s work do not extend the distributional XOR lemma to our regime, where $(\rho, \rho') = (\frac{9}{10}, 1 - \frac{1}{n})$. At a glance, both our approach and Yu’s adopt a similar strategy: recursively applying conditional decomposition until we obtain a protocol η for computing f with low “cost” and small distributional error. However, the sequence of conditioning events E inevitably affects the distribution, distorting it in an unpredictable way. The key distinction lies in the specification of the conditioning events used in each paper.

⁷One might be tempted to use the average of ε ’s across level k as a proxy to the exponent of the blowups from level k to level $k + 1$, but to the best of our attempt, this approach fails due to the convexity of e^x .

To expand on this, at each level of decomposition, Yu’s approach involves a conditioning event that occurs with constant probability $O(1)$. Accumulating across all levels, E occurs with probability $O(1)$ on average. This poses a fatal challenge in our regime, where we can tolerate only polynomially-small distributional error for f . To understand why, we recognize that the guarantee of “small distributional error” of η is evaluated against its own the input distribution $\mu \mid E$, rather than the “true” input distribution μ . Consequently, the error of η with respect to μ is:

$$\begin{aligned} & \Pr_{(x,y) \sim \mu} (\eta \text{ errs on } (x, y)) \\ &= \Pr(E) \cdot \Pr_{(x,y) \sim \mu \mid E} (\eta \text{ errs on } (x, y)) + \Pr(\overline{E}) \cdot \Pr_{(x,y) \sim \mu \mid \overline{E}} (\eta \text{ errs on } (x, y) \mid E) \\ &\geq \Pr(\overline{E}) \cdot \Pr_{(x,y) \sim \mu \mid \overline{E}} (\eta \text{ errs on } (x, y)) \end{aligned}$$

Since we have no guarantees over $\Pr_{(x,y) \sim \mu \mid \overline{E}} (\eta \text{ errs on } (x, y))$, this probability can be as large as 1, causing such error to be as large as $\Pr(\overline{E}) = \Omega(1)$.

In contrast, in our work, we propose a set of “simple” conditioning events, each of which occurs with probability $1 - o(1)$. These events result in $\Pr(E) = 1 - \frac{1}{\text{poly}(n)}$ on average. In this scenario, the error of η with respect to the input distribution μ is polynomially-bounded:

$$\begin{aligned} & \Pr_{(x,y) \sim \mu} (\eta \text{ errs on } (x, y)) \\ &= \Pr(E) \cdot \Pr_{(x,y) \sim \mu \mid E} (\eta \text{ errs on } (x, y)) + \Pr(\overline{E}) \cdot \Pr_{(x,y) \sim \mu \mid \overline{E}} (\eta \text{ errs on } (x, y) \mid E) \\ &\leq \Pr_{(x,y) \sim \mu \mid E} (\eta \text{ errs on } (x, y)) + \Pr(\overline{E}) \\ &\leq 1/\text{poly}(n) \qquad (\eta \text{ errs w.p. } 1/\text{poly}(n) \text{ on } \mu \mid E, \text{ and } \Pr(\overline{E}) = 1/\text{poly}(n)) \end{aligned}$$

which within the desired range of distributional error. This rough calculation plays an important role in addressing issue (2b) in Section 8.

In short, the amount of distortion in Yu’s decomposition is too large to manage in the regime where we allow only polynomially small error for computing f , whereas our decomposition incurs only low distortion that remains manageable.

Another distinction between our approach and Yu’s is that, at each level of the recursive procedure, we split a protocol for $f^{\oplus k}$ into two protocols for computing $f^{\oplus k/2}$, whereas Yu’s approach splits it into two protocols: one for computing f and one for computing $f^{\oplus(k-1)}$. Such a “binary” decomposition is necessary to ensure the intersection of all events we condition on across the levels has high probability.

3.4 Paper Organization

In Section 4, we establish the conventions used throughout this paper and review basic information theory concepts. Section 5 introduces key notations and properties of communication protocols, along with various cost types relevant to our proofs. In Section 6, we prove essential lemmas. Section 7 presents our recursive decomposition and addresses issue (1). In Section 8, we tackle issues (2a) and (2b) simultaneously, thereby completing the proof of Lemma 8. Finally, in Section 9, we use Lemma 8 to prove our main theorems. Note that some algebraically intensive proofs are deferred to Appendix A.

4 Preliminaries

For variables X , Y , and Z , we write $X \perp Y \mid Z$ to indicate that X and Y are independent when conditioned on Z . Similarly, for an event E , we write $X \perp Y \mid E$ to indicate that X and Y are independent given that E occurs. We may write the joint distributions interchangeably by X, Y or XY .

Following standard notation, an uppercase letter represents a variable, while the corresponding lowercase letter denotes its value. For a distribution π supported over multiple variables, let $\pi(X)$ represent the marginal distribution

of X . For a value x , let $\pi(x)$ denote the probability that $X = x$ under π . For an event W , let $\pi(W)$ denote the probability of W occurring under the distribution π . We also define the conditional analogues: let $\pi(X | Y)$ be the distribution of $X | Y$, let $\pi(X | y)$ represent the distribution of X conditioned on $Y = y$, and let $\pi(X | W)$ denote the distribution of X conditioned on event W occurring. We write $X \sim \pi$ and $X \sim \pi | W$ to indicate sampling X from the distribution $\pi(X)$ and the conditional distribution $\pi(X | W)$, respectively.

For a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ and any natural number n , we denote by $f^{\oplus n} : \mathcal{X}^n \times \mathcal{Y}^n \rightarrow \{0, 1\}$ the function such that

$$f^{\oplus n}(X_1, \dots, X_n, Y_1, \dots, Y_n) = f(X_1, Y_1) \oplus \dots \oplus f(X_n, Y_n).$$

Occasionally, we may refer to $f^{\oplus n}$ simply as f^{\oplus} , omitting the number of instances.

We also define *advantage* and *disadvantage* of $\{0, 1\}$ -valued random variables.

Definition 9 (Advantage and Disadvantage.). *Let π be a distribution and b be a $\{0, 1\}$ -valued random variable. Denote the advantage of b with respect to π to be*

$$\text{adv}^\pi(b) := |2 \cdot \pi(b = 0) - 1| = |2 \cdot \pi(b = 1) - 1|.$$

Moreover, for any event W , we might write $\text{adv}^\pi(b | W)$ and $\text{adv}^{\pi|W}(b)$ interchangeably. Conversely, denote the disadvantage of b with respect to π to be $1 - \text{adv}^\pi(b)$.

Note that disadvantage of b is in fact twice the error probability of predicting b with its more-likely value among $\{0, 1\}$. To see this, suppose that b takes on the value 0 with probability $\frac{1}{2} + \frac{\alpha}{2}$. Then, by predicting b with 0 (i.e. its more-likely bit), we err with probability $\frac{1}{2} - \frac{\alpha}{2}$, which is half of b 's disadvantage.

Fact 10. *Let π be a distribution, and let b_1 and b_2 be independent $\{0, 1\}$ -valued random variables in the same probability space π . Then, we have*

$$\text{adv}^\pi(b_1 \oplus b_2) = \text{adv}^\pi(b_1) \cdot \text{adv}^\pi(b_2).$$

The proof of Fact 10 was informally given in the Introduction; thus, shall be omitted.

4.1 Basic Information Theory

For the following set of definitions, let X, Y, Z be arbitrary discrete variables in the probability space π .

Definition 11 (Entropy and Conditional Entropy). *We denote:*

- *The entropy of X is defined as:*

$$H(X) = \mathbb{E}_{x \sim \pi} \log \frac{1}{\pi(x)} = \sum_x \pi(x) \cdot \log \frac{1}{\pi(x)}.$$

We may abuse the notion that $0 \cdot \log \frac{1}{0} = 0$, or equivalently only consider the summation over $x \in \text{supp}(X)$.

- *The conditional entropy of $X | Y$ is defined as:*

$$H(X | Y) = \mathbb{E}_{y \sim \pi} H(X | Y = y)$$

Theorem 12 (Chain Rule for Entropy). $H(X, Y) = H(X) + H(Y | X) = H(Y) + H(X | Y)$.

Rearrange it, we have the definition of mutual information.

Definition 13 (Mutual Information). *The mutual information between X and Y is defined as:*

$$I(X : Y) = H(X) - H(X | Y) = H(Y) - H(Y | X) = H(X) + H(Y) - H(X, Y).$$

As a by-product, the entropy is subadditive.

Theorem 14 (Subadditivity of Entropy). *It holds that*

$$H(X, Y) \leq H(X) + H(Y).$$

The equality is achieved when $X \perp Y$.

Definition 15 (Conditional Mutual Information and Chain Rule). *The conditional mutual information is defined as:*

$$I(Y : Z | X) = I(XY : Z) - I(X : Z).$$

Rearranging it yields a chain rule for mutual information:

$$I(XY : Z) = I(X : Z) + I(Y : Z | X).$$

The following definition measures the closeness of distributions.

Definition 16 (KL-Divergence and Total Variation Distance). *Let $\pi(X)$ and $\eta(X)$ distributions over the variable X . Denote the following distances between the two distributions.*

1. KL-Divergence: $D\left(\frac{\pi(X)}{\eta(X)}\right) = \mathbb{E}_{x \sim \pi(X)} \log \frac{\pi(x)}{\eta(x)}$
2. Total Variation Distance: $\|\pi(X) - \eta(X)\| = \sum_x |\pi(x) - \eta(x)|$.

Remarks that for the KL-Divergence, we shall write $D\left(\frac{\pi(X)}{\eta(X)}\right)$ and $D(\pi(X) \parallel \eta(X))$ interchangeably. For both distance functions, we might drop their variables when the context is clear.

Lemma 17 (Pinsker's Inequality). *For any distributions π and η , we have:*

$$\|\pi - \eta\| = O\left(\sqrt{D(\pi \parallel \eta)}\right).$$

5 Formalizing Communication Protocols

In this section, we define communication protocol and introduce various definitions which will be used throughout the paper. It is important to emphasize that we do not claim novelty regarding these definitions, propositions, theorems, or their proofs. However, they serve as the foundations for our work.

5.1 Distributional View of Communication Protocols

Recall that in the two-player communication model, the players' task is to compute a function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ using a (randomized) protocol π that dictates a sequence of messages M . Suppose that the input pair (X, Y) is drawn from some distribution μ . We can examine π as a distribution over a set of random variables (X, Y, \mathbf{M}) , where $\pi(X, Y)$ represents the input distribution μ , and \mathbf{M} governs the public randomness M^0 , and the sequence of messages $M^+ = (M^1, M^2, \dots, M^r)$. More precisely, we interpret the *standard* communication protocol as a distribution over these variables.

Definition 18 (Standard Communication Protocols). *Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$, μ be an input distribution to f , and $\rho \in (0, 1)$ be the error parameter. We say $\pi = (X, Y, \mathbf{M})$ is a standard protocol for computing f on input distribution μ with probability ρ iff*

- $\pi(x, y) = \mu(x, y)$
- $\mathbf{M} = (M^0, M^1, M^2, \dots, M^r)$ consists of public randomness M^0 and a sequence of messages $M^+ = (M^1, M^2, \dots, M^r)$ such that each M^i only depends on $M^{<i}$ and the sender's input.

- Towards the end of the protocol, Alice and Bob output M^r which correctly computes $f(x, y)$ with probability ρ .

To expand on the last bullet, we assume that the last message sent by one of the players is their answer to $f(x, y)$. Any protocol can be converted to this form by having the first player who knows the answer send it to the other, adding only one additional bit to the message. We also assume that Alice sends the odd messages (M^1, M^3, \dots) and Bob sends the even messages (M^2, M^4, \dots) .

A key characteristic of standard protocols is the restriction that each player generates a message based solely on their own inputs and the previously exchanged messages. Specifically, assuming Alice always speaks first, a standard protocol π must satisfy the conditions $M^i \perp Y \mid X, M^{<i}$ for all odd i , and $M^i \perp X \mid Y, M^{<i}$ for all even i . In this work, we will explore an extended notion of standard protocols in which each message can be correlated with the receiver's inputs, referred to as a *generalized protocol*.

Definition 19 (Generalized Communication Protocols). *Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$, μ be an input distribution to f , and $\rho \in (0, 1)$ be the error parameter. We say $\pi = (X, Y, M)$ is a generalized protocol for computing f on input distribution μ with probability ρ iff*

- $\pi(x, y) = \mu(x, y)$
- $M = (M^0, M^1, M^2, \dots, M^r)$ consists of public randomness M^0 and a sequence of messages $M^+ = (M^1, M^2, \dots, M^r)$ such that each M^i only depends on $M^{<i}$ and the both players' input.
- Towards the end of the protocol, Alice and Bob output M^r which correctly computes $f(x, y)$ with probability ρ .

5.2 Information Costs and Information Complexity

We now define our resource of interest: *information*.

Definition 20 (Information Cost of a Protocol). *For a (standard or generalized) protocol $\pi = (X, Y, M)$, we denote the (internal) information cost of π to be:*

$$\begin{aligned} \text{IC}(\pi) &= I(M : X \mid Y M_0) + I(M : Y \mid X M_0) \\ &= I(M^+ : X \mid Y M_0) + I(M^+ : Y \mid X M_0). \end{aligned}$$

To reason about the information cost, let us examine the first term. In Bob's view, at the end of the protocol he learns the message M^+ , while already knows his own input Y and the public randomness M_0 . Hence, the amount of information that he gains of Alice's input X is exactly is $I(M : X \mid Y M_0)$. We also have the symmetric term for Alice's gain. In other words, the *information cost* captures the amount of information that both parties learns from executing a protocol, or equivalently the amount of information that the protocol reveals to the players.

The following set of notions are borrowed from [Bra15]

Definition 21 (Distributional Information Complexity). *Let f be a $\{0, 1\}$ -valued function and $\varepsilon > 0$. Let μ be an input distribution. Then, the distributional information complexity of π of a function f with error ε and distribution μ is*

$$\text{IC}_\mu(f, \varepsilon) = \inf_{\pi: \Pr_{(x,y) \sim \mu}(\pi(x,y) \neq f(x,y)) \leq \varepsilon} \text{IC}(\pi).$$

Definition 22 (Max-Distributional Information Complexity). *The max-distributional information complexity of a function f with error ε is*

$$\text{IC}_D(f, \varepsilon) = \max_{\mu} \text{IC}_\mu(f, \varepsilon).$$

Definition 23 (Information Complexity). *The information complexity of a function f with error ε is*

$$\text{IC}(f, \varepsilon) = \inf_{\pi \text{ that errs w.p. at most } \varepsilon \text{ on any inputs}} \max_{\mu} \text{IC}_\mu(\pi).$$

In other words, the *information complexity* of f is defined as the information cost of the best protocol that solves f with a probability of failure at most ε , evaluated against its worst-case input distribution. By definition, it is trivial to see that $\text{IC}(f, \varepsilon) \geq \text{IC}_D(f, \varepsilon)$. The following theorem, by setting $\alpha = \frac{1}{2}$, implies that they are asymptotically equivalent.

Theorem 24 (Theorem 3.5 of [Bra15]). *Let f be any function, and let $\varepsilon \geq 0$ be an error parameter. For each value of the parameter $0 < \alpha < 1$ we have*

$$\text{IC}(f, \frac{\varepsilon}{\alpha}) \leq \frac{\text{IC}_D(f, \varepsilon)}{1 - \alpha}.$$

5.3 Operations on Protocols

Here, we define a set of operations that can be applied on a protocol, turning it into another protocol(s) with some desirable properties. The first operation decomposes a protocol π into two π_0 and π_1 , each running on a smaller set of inputs.

Definition 25 ((Binary) Protocol Decomposition). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol where \mathbf{M} consists of public randomness M^0 and a sequence of messages $M^+ = (M^1, M^2, \dots, M^r)$. Let $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ be the partition of input coordinates. The binary protocol decomposition of π yields two protocols π_0 and π_1 with the following distributions:*

- $\pi_0 = (X_0, Y_0, \mathbf{M}^{\pi_0})$ where $\mathbf{M}^{\pi_0} = (M^0, Y_1 \circ M^1, M^2, \dots, M^r)$
- $\pi_1 = (X_1, Y_1, \mathbf{M}^{\pi_1})$ where $\mathbf{M}^{\pi_1} = (M^0 \circ X_0, M^1, M^2, \dots, M^r)$

It is worth noting that we will eventually consider a slight variant of decomposition which, roughly speaking, applies the decomposition to a *conditional* distribution $\pi \mid W$ for some event W .

The next operation captures sending one additional message on top of a protocol.

Definition 26 (Appending Messages). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol, and let B be an arbitrary distribution representing the additional message. Denote $\pi \odot B$ to be a generalized protocol where the players execute π , followed by sending B . Distribution-wise, this can be written as*

$$\pi \odot B = (X, Y, (\mathbf{M}, B)).$$

It is worth remarking that the distributions of π and $\pi \odot B$ over the variables (X, Y, \mathbf{M}, B) are in fact identical. Their only distinction is that $\pi \odot B$ includes B as a part of the messages, while π does not. Hence, we can also write the distribution of $\pi \odot B$ as:

$$(\pi \odot B)(X, Y, \mathbf{M}, B) := \pi(X, Y, \mathbf{M}) \cdot \pi(B \mid X, Y, \mathbf{M}).$$

Last but not least, given a generalized protocol, we can convert it into a new standard protocol via the following procedure.

Definition 27 (Standardization). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol, and let μ be an arbitrary input distribution. Say $\pi' = \text{standardize}(\pi, \mu)$ is the standardization of π with respect to μ iff π' admits the following distribution:*

$$\pi'(X, Y, \mathbf{M}) = \pi(M^0) \cdot \mu(X, Y) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i \mid X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i \mid Y M^{<i}).$$

A key observation is that $\pi' = \text{standardize}(\pi, \mu)$ is a standard protocol. This follows from the definition: in each odd round i , Alice generates her message M^i from the distribution $\pi(M^i \mid X M^{<i})$, which depends solely on her input and the previous messages. In other words, each of Alice's message in π' disregards any correlation with Bob's inputs. A similar argument applies to Bob's messages.

5.4 Rectangle Properties

The *rectangle property* is a fundamental aspect of communication protocols. The following set of notions were introduced in [Yu22].

Definition 28 (Rectangle Property). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol. We say π has the rectangle property with respect to μ iff there exists nonnegative functions g_1 and g_2 such that*

$$\pi(X, Y, \mathbf{M}) = \mu(X, Y) \cdot g_1(X, \mathbf{M}) \cdot g_2(Y, \mathbf{M}).$$

Fact 29. *Standard communication protocols admit the rectangle properties.*

Recall that by Definition 25, the protocol decomposition procedure splits inputs into two parts. To facilitate such partition, [Yu22] proposed the *partial rectangle property*.

Definition 30 (Partial Rectangle Property). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol such that $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$. We say π has the partial rectangle property with respect to μ iff there exists nonnegative functions g_1, g_2, g_3 such that*

$$\pi(X, Y, \mathbf{M}) = \mu(X, Y) \cdot g_1(X, \mathbf{M}) \cdot g_2(Y, \mathbf{M}) \cdot g_3(X_0, Y_1, \mathbf{M}).$$

It turns out that partial rectangle property suffices to ensure the independence between inputs after the decomposition.

Proposition 31. *If a generalized protocol $\pi = (X, Y, \mathbf{M})$ has the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$ where $\mu_0 \perp \mu_1$, then $X_1 \perp Y_0 \mid X_0 Y_1 \mathbf{M}$ in the distribution π .*

Proof. It follows from the partial rectangular property that

$$\begin{aligned} & \pi(X_1 Y_0 \mid X_0 Y_1 \mathbf{M}) \\ &= \frac{\pi(X, Y, \mathbf{M})}{\pi(X_0, Y_1, \mathbf{M})} \\ &= \frac{\mu(X, Y) \cdot g_1(X, \mathbf{M}) \cdot g_2(Y, \mathbf{M}) \cdot g_3(X_0, Y_1, \mathbf{M})}{\pi(X_0, Y_1, \mathbf{M})} \\ &= (\mu_0(X_0, Y_0) \cdot g_2(Y, \mathbf{M})) \cdot \left(\mu_1(X_1, Y_1) \cdot g_1(X, \mathbf{M}) \cdot \frac{g_3(X_0, Y_1, \mathbf{M})}{\pi(X_0, Y_1, \mathbf{M})} \right). \end{aligned}$$

Now, conditioned on $X_0 Y_1 \mathbf{M}$, the first term only depends on Y_0 and the second term only depends on X_1 . This proves the conditional independence. \square

More importantly, the partial rectangle property of π implies the rectangle property of its decomposed protocols if μ is a product distribution. The following lemma will be used recursively throughout our proofs.

Lemma 32. *Suppose that a generalized protocol $\pi = (X, Y, \mathbf{M})$ has the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$ where $\mu_0 \perp \mu_1$. Let $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ and $\pi_1 = (X_0, Y_0, \mathbf{M}^{(\pi_1)})$ be generalized protocols obtained via decomposing π (recall Definition 25.) Then, π_0 has a rectangle property with respect to μ_0 , and π_1 has a rectangle property with respect to μ_1 .*

Proof. Consider

$$\begin{aligned} \pi_0(X_0, Y_0, \mathbf{M}^{(\pi_0)}) &= \pi(X_0, Y, \mathbf{M}) \\ &= \sum_{X_1} \pi(X = (X_0, X_1), Y, \mathbf{M}) \\ &= \sum_{X_1} \mu(X, Y) \cdot g_1(X, \mathbf{M}) \cdot g_2(Y, \mathbf{M}) \cdot g_3(X_0, Y_1, \mathbf{M}) \\ &= \mu_0(X_0, Y_0) \cdot g_2(Y, \mathbf{M}) \cdot \left(\sum_{X_1} \mu_1(X_1, Y_1) \cdot g_1(X, \mathbf{M}) \cdot g_3(X_0, Y_1, \mathbf{M}) \right). \end{aligned}$$

Notice that the second term is a function of $Y_0, \mathbf{M}^{(\pi_0)}$. The third term is a function of $X_1, \mathbf{M}^{(\pi_0)}$. This concludes rectangle property of π_0 . The proof of the rectangle property of π_1 also follows closely. \square

5.5 θ -cost and γ -cost

The θ -cost of a generalized protocol π (with respect to μ) roughly measure a combination of two distances: the closeness of π to $\text{standardize}(\pi, \mu)$, and the closeness of its input distribution of $\pi(X, Y)$ to μ .⁸

Definition 33 (Pointwise- θ -cost). *For a generalized protocol $\pi = (X, Y, \mathbf{M})$, input distribution μ , and points (X, Y, \mathbf{M}) , denote the pointwise- θ -cost of π with respect to μ at (X, Y, \mathbf{M}) by the following quantity:*

$$\begin{aligned} \theta_\mu(\pi @ X, Y, \mathbf{M}) &= \log \frac{\pi(X, Y | M^0)}{\mu(X, Y)} + \sum_{\text{odd } i \geq 1} \log \frac{\pi(M^i | X, Y, M^{<i})}{\pi(M^i | X, M^{<i})} + \sum_{\text{even } i \geq 2} \log \frac{\pi(M^i | X, Y, M^{<i})}{\pi(M^i | Y, M^{<i})} \\ &= \log \left(\frac{\pi(X, Y, \mathbf{M})}{\pi(M^0) \cdot \mu(X, Y) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | X, M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y, M^{<i})} \right) \\ &= \log \frac{\pi(X, Y, \mathbf{M})}{\eta(X, Y, \mathbf{M})} \end{aligned}$$

where $\eta = \text{standardize}(\pi, \mu)$ is the standardization of π with respect to μ .

We can also define the θ -cost of a protocol as follows.

Definition 34 (θ -cost). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol. The θ -cost of π with respect to μ is defined as*

$$\theta_\mu(\pi) = \mathbb{E}_{(X, Y, \mathbf{M}) \sim \pi} \theta_\mu(\pi @ X, Y, \mathbf{M}).$$

By expanding the pointwise- θ -cost, the following fact is immediate.

Fact 35. *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol, and let μ be an input distribution. Let $\eta = \text{standardize}(\pi, \mu)$ be the standardization of π with respect to μ . Then, we have*

$$\theta_\mu(\pi) = D \left(\frac{\pi(X, Y, \mathbf{M})}{\eta(X, Y, \mathbf{M})} \right).$$

Observation 36. *If π is a standard protocol with input distribution $\pi(x, y) = \mu$, then $\theta_\mu(\pi) = 0$.*

It turns out that the pointwise- θ -cost admit linearity when undergoing a decomposition.

Lemma 37 (Linearity of pointwise- θ -cost). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol with the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$. Let $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ and $\pi_1 = (X_1, Y_1, \mathbf{M}^{(\pi_1)})$ be generalized protocols obtained via decomposing π (recall Definition 25.) Then, for any point (X, Y, \mathbf{M}) , we have*

$$\theta_\mu(\pi @ X, Y, \mathbf{M}) = \theta_{\mu_0}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \theta_{\mu_1}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}).$$

Next, we discuss the γ -cost which roughly measures the amount of information that each parties reveals about their inputs via the messages.

⁸Our version of θ -cost and γ -cost are equivalent to the logarithmic version of the θ -cost and χ^2 -cost from [Yu22], respectively.

Definition 38 (Pointwise- γ -cost). For a generalized protocol $\pi = (X, Y, \mathbf{M})$, input distribution μ , and points (X, Y, \mathbf{M}) , denote the pointwise- γ -cost of π with respect to μ at (X, Y, \mathbf{M}) by the following quantities:

$$\begin{aligned}\gamma_{\mu,A}(\pi @ X, Y, \mathbf{M}) &= \log \frac{\pi(X | Y\mathbf{M})}{\mu(X | Y)} \\ \gamma_{\mu,B}(\pi @ X, Y, \mathbf{M}) &= \log \frac{\pi(Y | X\mathbf{M})}{\mu(Y | X)}.\end{aligned}$$

Similar to the θ -cost, we can define the γ -cost of a protocol. Additionally, the pointwise- γ -cost also admit linearity when undergoing a decomposition.

Definition 39 (γ -cost). Let π be a generalized protocol. The γ -cost of π with respect to μ is defined as

$$\gamma_{\mu}(\pi) = \mathbb{E}_{(X,Y,\mathbf{M}) \sim \pi} \gamma_{\mu,A}(\pi @ X, Y, \mathbf{M}) + \mathbb{E}_{(X,Y,\mathbf{M}) \sim \pi} \gamma_{\mu,B}(\pi @ X, Y, \mathbf{M}).$$

Lemma 40 (Linearity of pointwise- γ -cost). Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol with the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$. Let $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ and $\pi_1 = (X_1, Y_1, \mathbf{M}^{(\pi_1)})$ be generalized protocols obtained via decomposing π (recall Definition 25.) Then, for any point (X, Y, \mathbf{M}) , we have

$$\begin{aligned}\gamma_{\mu,A}(\pi @ X, Y, \mathbf{M}) &= \gamma_{\mu_0,A}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \gamma_{\mu_1,A}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}) \\ \gamma_{\mu,B}(\pi @ X, Y, \mathbf{M}) &= \gamma_{\mu_0,B}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \gamma_{\mu_1,B}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}).\end{aligned}$$

The proofs of Lemma 37 and 40 are deferred to the appendix.

6 Useful Facts and Lemmas

In this section, we present key facts and lemmas which will be needed later in the paper.

Fact 41. Denote $\mathcal{H}(p) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}$ for any $p \in (0, 1)$. Then, we have:

1. \mathcal{H} is concave.
2. For any $p \in (0, \frac{1}{2}]$, we have $\mathcal{H}(p) \leq 2p \log \frac{1}{p}$.

Lemma 42. Let A, B, C be variables and E be an event in the same probability space with p . Then,

$$I(A : B | CE) \leq \frac{1}{p} \cdot [I(A : B | C) + \mathcal{H}(p)].$$

Proof. Recall that $I(A : B | C\mathbb{1}_E) \geq p \cdot I(A : B | CE)$. Moreover, we have

$$\begin{aligned}I(A : B | C\mathbb{1}_E) &= H(A | C\mathbb{1}_E) - H(A | BC\mathbb{1}_E) \\ &\leq H(A | C) - H(A | BC\mathbb{1}_E) \\ &= H(A | C) - [H(A\mathbb{1}_E | BC) - H(\mathbb{1}_E | BC)] \\ &\leq H(A | C) - [H(A | BC) - H(\mathbb{1}_E)] \\ &= I(A : B | C) + H(\mathbb{1}_E) \\ &= I(A : B | C) + \mathcal{H}(p).\end{aligned}$$

Combining two inequalities conclude the proof. □

Lemma 43. Let π and η be probability distributions and E be an event in the same probability space. Then, we have

$$\mathbf{D}\left(\frac{\pi(X|E)}{\eta(X)}\right) \leq \frac{1}{\pi(E)} \cdot \left[\mathbf{D}\left(\frac{\pi(X)}{\eta(X)}\right) + \mathcal{H}(\pi(E)) \right].$$

Proof. Consider

$$\begin{aligned} \mathbf{D}\left(\frac{\pi(X)}{\eta(X)}\right) &= \sum_x \pi(x) \cdot \log \frac{\pi(x)}{\eta(x)} \\ &= \sum_x \pi(x|E) \cdot \pi(E) \log \frac{\pi(x)}{\eta(x)} + \sum_x \pi(x\bar{E}) \cdot \log \frac{\pi(x)}{\eta(x)} \end{aligned}$$

Note that

$$\begin{aligned} \sum_x \pi(x|E) \log \frac{\pi(x)}{\eta(x)} &\geq \sum_x \pi(x|E) \log \frac{\pi(xE)}{\eta(x)} \\ &= \sum_x \pi(x|E) \log \frac{\pi(x|E) \cdot \pi(E)}{\eta(x)} \\ &= \log \pi(E) \cdot \sum_x \pi(x|E) + \sum_x \pi(x|E) \log \frac{\pi(x|E)}{\eta(x)} \\ &= \log \pi(E) + \mathbf{D}\left(\frac{\pi(X|E)}{\eta(X)}\right) \end{aligned}$$

Also via Jensen,

$$\begin{aligned} \sum_x \pi(x\bar{E}) \cdot \log \frac{\pi(x)}{\eta(x)} &\geq \sum_x \pi(x\bar{E}) \cdot \log \frac{\pi(x\bar{E})}{\eta(x)} \\ &\geq \pi(\bar{E}) \cdot \log(\pi(\bar{E})) \end{aligned} \quad (\text{Jensen's})$$

Combining the two inequalities yield the result. \square

The next lemma offers its extension.

Lemma 44. Let π and η be probability distributions and E be an event in the same probability space. Then, we have

$$\mathbb{E}_{y \sim \pi|E} \mathbf{D}\left(\frac{\pi(X|yE)}{\eta(X)}\right) \leq \frac{1}{\pi(E)} \cdot \left(\mathbb{E}_{y \sim \pi} \mathbf{D}\left(\frac{\pi(X|y)}{\eta(X)}\right) + \mathcal{H}(\pi(E)) \right)$$

Proof. For any value of y , we have

$$\begin{aligned} \pi(y|E) \cdot \mathbf{D}\left(\frac{\pi(X|yE)}{\eta(X)}\right) &\leq \pi(y|E) \cdot \left[\frac{1}{\pi(E|y)} \cdot \mathbf{D}\left(\frac{\pi(X|y)}{\eta(X)}\right) + \mathcal{H}(\pi(E|y)) \right] \\ &= \frac{\pi(y)}{\pi(E)} \cdot \mathbf{D}\left(\frac{\pi(X|y)}{\eta(X)}\right) + \pi(y|E) \cdot \mathcal{H}(\pi(E|y)) \end{aligned} \quad (\text{Lemma 43})$$

Summing over y , we have

$$\begin{aligned}
\mathbb{E}_{y \sim \pi|E} \mathbf{D} \left(\frac{\pi(X|yE)}{\eta(X)} \right) &\leq \frac{1}{\pi(E)} \cdot \mathbb{E}_{y \sim \pi} \mathbf{D} \left(\frac{\pi(X|y)}{\eta(X)} \right) + \sum_y \pi(y|E) \cdot \mathcal{H}(\pi(E|y)) \\
&\leq \frac{1}{\pi(E)} \cdot \mathbb{E}_{y \sim \pi} \mathbf{D} \left(\frac{\pi(X|y)}{\eta(X)} \right) + \frac{1}{\pi(E)} \sum_y \pi(y) \cdot \mathcal{H}(\pi(E|y)) \quad (\pi(y|E) \leq \frac{\pi(y)}{\pi(E)}) \\
&\leq \frac{1}{\pi(E)} \cdot \mathbb{E}_{y \sim \pi} \mathbf{D} \left(\frac{\pi(X|y)}{\eta(X)} \right) + \frac{1}{\pi(E)} \cdot \mathcal{H} \left(\sum_y \pi(y) \cdot \pi(E|y) \right) \quad (\text{Jensen's}) \\
&= \frac{1}{\pi(E)} \cdot \mathbb{E}_{y \sim \pi} \mathbf{D} \left(\frac{\pi(X|y)}{\eta(X)} \right) + \frac{1}{\pi(E)} \cdot \mathcal{H}(\pi(E))
\end{aligned}$$

Rearranging it yields the result. \square

The following lemma will be used for analyzing the “losses” incurred by protocol standardization.

Lemma 45 (Coupling Lemma). *Let μ and μ' be distributions over supports \mathcal{A} . There exists a random process such that at the end of the process, we obtain a and a' such that a distributes according to μ , a' distributes according to μ' , and the probability that $a \neq a'$ is at most $O(\|\mu - \mu'\|)$.*

Lemma 45 appears as the “correlated sampling” in Lemma 7.5 of [RY20]. We present our proof in the Appendix.

7 Obtaining a “Nice” Generalized Protocol for f

For the next two sections, we will prove Lemma 8. We assume the setup of the lemma as follows. Let $\pi = (X, Y, \mathbf{M})$ where $\mathbf{M} = (M^0, M^1, \dots, M^r)$ be a standard protocol that computes $f^{\oplus n}$ over the input distribution μ^n with disadvantage $\varepsilon = 1/5$.⁹ Denote the input $(X, Y) \sim \mu^n$ by $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ such that each (X_i, Y_i) is independently drawn from μ . Also denote the information cost of π by \mathcal{I} .

We will eventually show that there exists a standard protocol η that computes f over μ , errs with probability $1/\text{poly}(n)$, and has information cost approximately $O(\mathcal{I}/n + 1)$. In this section, we address a relaxed version of this statement: showing that there exists a “generalized” protocol that “almost” achieves these properties while remaining “close” to being standard.

7.1 Refinements of Conditional Decomposition

We begin by discussing a revision of the *conditional decomposition* described in Section 3, which we will eventually apply recursively. For simplicity, in this subsection, we demonstrate only the top level of the recursion.

Conventions. We assume that $n = 2^m$ is a power of two, and label the n coordinates with $\{0, 1\}^m$. Additionally, for any string $S \in \{0, 1\}^{\leq m}$, we denote μ_S , X_S , and Y_S as the input distributions, Alice’s input coordinates, and Bob’s input coordinates for the $2^{m-|S|}$ instances prefixed with S , respectively. Let $\alpha \in (0, 1)$ be a constant such that $\alpha = 1 - 2\varepsilon$.¹⁰ Let $\tau \in (0, 1)$ be a constant such that $\frac{1+\sqrt{\alpha}}{2} = 2^{-\tau}$.

Recall the binary protocol decomposition procedures.

Definition 25 ((Binary) Protocol Decomposition). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol where \mathbf{M} consists of public randomness M^0 and a sequence of messages $M^+ = (M^1, M^2, \dots, M^r)$. Let $X = (X_0, X_1)$ and $Y = (Y_0, Y_1)$ be the partition of input coordinates. The binary protocol decomposition of π yields two protocols π_0 and π_1 with the following distributions:*

⁹This disadvantage $\varepsilon = 1/5$ corresponds to a success probability of $1/10$ for π over μ^n .

¹⁰As long as π errs on μ^n with probability at most $1/4 - O(1)$, we have ε is twice the error, which is at most $1/2 - O(1)$. Then, we are able to set $\alpha = 1 - 2\varepsilon = \Omega(1)$.

- $\pi_0 = (X_0, Y_0, \mathbf{M}^{\pi_0})$ where $\mathbf{M}^{\pi_0} = (M^0, Y_1 \circ M^1, M^2, \dots, M^r)$
- $\pi_1 = (X_1, Y_1, \mathbf{M}^{\pi_1})$ where $\mathbf{M}^{\pi_1} = (M^0 \circ X_0, M^1, M^2, \dots, M^r)$

Remark. In the protocol π_0 , we can interpret the first two messages, $(M^0, Y_1 \circ M^1)$, as Alice drawing Y_1 and prepending it to her first message to Bob, allowing him to recover Y_1 . A crucial observation is that, distribution-wise, the distribution of \mathbf{M}^{π_0} remains unchanged if the first two terms are replaced by $(M^0 \circ Y_1, M^1, \dots, M^r)$. This perspective, in fact, corresponds to the protocol decomposition discussed earlier in the overview. These interpretations are equivalent in terms of *information* cost; however, their *communication* costs differ, as having Alice send Y_1 to Bob significantly increases the amount of communication.

Conditional Decomposition of π . We begin by discussing the first level of our decomposition, where we use the protocol π to derive two *generalized* protocols, π_0 and π_1 , each of which computes the function $f^{\oplus n/2}$. We highlight that the process can be viewed as two steps: (1) apply the binary decomposition to $\pi \mid W$, for some event W , to obtain $\tilde{\pi}_0$ and $\tilde{\pi}_1$, and (2) complete the each protocol by appending the “final bit”.

Conditional Decomposition of $\pi = (X, Y, \mathbf{M})$ into π_0 and π_1 .

1. Let \mathcal{G} be a set of (X_0, Y_1, \mathbf{M}) such that $\text{adv}^\pi(f^{\oplus n}(X, Y) \mid X_0, Y_1, \mathbf{M}) \geq \alpha$.
2. Let $W := [(X_0, Y_1, \mathbf{M}) \in \mathcal{G}]$ be an event that (X_0, Y_1, \mathbf{M}) yields $\text{adv}^\pi(f^{\oplus n}(X, Y) \mid X_0, Y_1, \mathbf{M}) \geq \alpha$.
3. Apply the Binary Decomposition to $\pi \mid W$. This yields two protocols, each of which computes $f^{\oplus n/2}$.
 - $\tilde{\pi}_0 = (X_0, Y_0, \tilde{\mathbf{M}}_0)$ where $\tilde{\mathbf{M}}_0^0 = M^0$ and $\tilde{\mathbf{M}}_0^+ = (Y_1 \circ M^1, M^2, \dots, M^r)$
 - $\tilde{\pi}_1 = (X_1, Y_1, \tilde{\mathbf{M}}_1)$ where $\tilde{\mathbf{M}}_1^0 = M^0 \circ X_0$ and $\tilde{\mathbf{M}}_1^+ = (Y_1 \circ M^1, M^2, \dots, M^r)$
- 4a. The protocol $\pi_0 = (X_0, Y_0, \mathbf{M}_0)$ is obtained as follows.
 - (i) Upon finishing $\tilde{\pi}_0$, Alice determines the more-likely bit B_0 among the posterior distribution

$$\tilde{\pi}_0(f^{\oplus n/2}(X_0, Y_0) \mid X_0, \tilde{\mathbf{M}}_0).$$
 - (ii) Alice sends B_0 to Bob, and both players declare B_0 as the answer.

Equivalently, distribution-wise we can write $\pi_0 := \tilde{\pi}_0 \odot B_0$.
- 4b. The protocol $\pi_1 = (X_1, Y_1, \mathbf{M}_1)$ is obtained as follows.
 - (i) Upon finishing $\tilde{\pi}_1$, Bob determines the more-likely bit B_1 among the posterior distribution

$$\tilde{\pi}_1(f^{\oplus n/2}(X_1, Y_1) \mid Y_1, \tilde{\mathbf{M}}_1).$$
 - (ii) Bob sends B_1 to Alice, and both players declare B_1 as the answer.

Equivalently, distribution-wise we can write $\pi_1 := \tilde{\pi}_1 \odot B_1$.

Figure 3: The “conditional” decomposition of π into π_0 and π_1 .

As an important remark, we observe the distribution of $\pi_0(X_0, Y_0, \mathbf{M}_0)$ and $\tilde{\pi}_0(X_0, Y_0, \mathbf{M}_0, B_0)$ are in fact identical. However, their differences lies within the language of communication protocol: that π_0 contains the final message

B_0 while $\tilde{\pi}_0$ does not.

Notice further that in the procedure of $\tilde{\pi}_0$, Alice knows X_0 and \tilde{M}_0 . Therefore, she can determine B_0 by herself. In other words, we have $B_0 \perp Y_0 \mid X_0 \tilde{M}_0$ in the the distribution of $\tilde{\pi}_0$. With the same reasoning, we have $B_1 \perp X_1 \mid Y_1 \tilde{M}_1$ in the the distribution of $\tilde{\pi}_1$.

Following our decomposition, it can be shown that π_0 and π_1 has the rectangle property with respect to μ_0 and μ_1 respectively.

Claim 46. *If π has a rectangle property with respect to μ_\emptyset , then π_0 has a rectangle property with respect to μ_0 , and π_1 has a rectangle property with respect to μ_1 .*

Proof. First we will show that $\pi \mid W$ has a partial rectangle property with respect to (μ_0, μ_1) . Due to the rectangle property of π , let the functions g_1, g_2 be such that at any point (X, Y, M) , we have

$$\pi(X, Y, M) = \mu(X, Y) \cdot g_1(X, M) \cdot g_2(Y, M).$$

Then, we have

$$\begin{aligned} \pi(X, Y, M \mid W) &= \pi(X, Y, M) \cdot \frac{\pi(W \mid X, Y, M)}{\pi(W)} \\ &= \mu(X, Y) \cdot g_1(X, M) \cdot g_2(Y, M) \cdot \frac{\mathbb{1}[(X_0, Y_1, M) \in \mathcal{G}]}{\pi(W)}. \end{aligned}$$

Note that the last term can be interpreted as a function of (X_0, Y_1, M) . Therefore, $\pi \mid W$ has the partial rectangle property.

Lemma 32 then implies that $\tilde{\pi}_0$ has a rectangle property with respect to μ_0 . Let the functions g_3, g_4 be such that for any points (X_0, Y_0, \tilde{M}_0) , we have

$$\tilde{\pi}_0(X_0, Y_0, \tilde{M}_0) = \mu(X_0, Y_0) \cdot g_3(X_0, \tilde{M}_0) \cdot g_4(Y_0, \tilde{M}_0).$$

Then, we have

$$\begin{aligned} \pi_0(X_0, Y_0, M_0) &= \tilde{\pi}_0(X_0, Y_0, \tilde{M}_0, B_0) \\ &= \tilde{\pi}_0(X_0, Y_0, \tilde{M}_0) \cdot \tilde{\pi}_0(B_0 \mid X_0, Y_0, \tilde{M}_0) \\ &= \mu(X_0, Y_0) \cdot g_3(X_0, \tilde{M}_0) \cdot g_4(Y_0, \tilde{M}_0) \cdot \tilde{\pi}_0(B_0 \mid X_0, Y_0, \tilde{M}_0) \\ &= \mu(X_0, Y_0) \cdot \left(g_3(X_0, \tilde{M}_0) \cdot \tilde{\pi}_0(B_0 \mid X_0, \tilde{M}_0) \right) \cdot g_4(Y_0, \tilde{M}_0) \quad (B_0 \perp Y_0 \mid X_0 M_0 \text{ in } \tilde{\pi}_0) \end{aligned}$$

Recall that M_0 consists of \tilde{M}_0 and B_0 . Therefore, the second term is a function of (X_0, M_0) , and the third term is a function of (Y_0, M_0) . This proves that π_0 has a rectangle property with respect to μ_0 . The same proof applies for π_1 . \square

Now, we discuss the decomposition of advantages when a protocol undergoes the conditional decomposition. As a starting point, [Yu22] observed the following.

Claim 47 (Multiplicity of Advantages.). *Suppose that π has a rectangle property with respect to μ_\emptyset . For any value of (X_0, Y_1, M) , we have*

$$\text{adv}^\pi(f^{\oplus n/2}(X_0, Y_0) \mid X_0, Y_1, M) \cdot \text{adv}^\pi(f^{\oplus n/2}(X_1, Y_1) \mid X_0, Y_1, M) = \text{adv}^\pi(f^{\oplus n}(X, Y) \mid X_0, Y_1, M).$$

Proof. The rectangle property of π implies $X_1 \perp Y_0 \mid X_0 Y_1 M$. Thus, we have $f^{\oplus n/2}(X_0, Y_0) \perp f^{\oplus n/2}(X_1, Y_1) \mid X_0 Y_1 M$ (because first term only depends on Y_0 and the second term only depends on X_1 . By Fact 10, the proof is concluded. \square

For brevity, we denote random variables over the randomness of (X_0, Y_1, \mathbf{M}) :

$$\begin{aligned} Z &:= \text{adv}^\pi (f^{\oplus n}(X, Y) \mid X_0, Y_1, \mathbf{M}) \\ A_0 &:= \text{adv}^\pi (f^{\oplus n/2}(X_0, Y_0) \mid X_0, Y_1, \mathbf{M}) \\ A_1 &:= \text{adv}^\pi (f^{\oplus n/2}(X_1, Y_1) \mid X_0, Y_1, \mathbf{M}) \end{aligned}$$

With these notions, we realize that W is, in fact, simply the event that $(Z \geq \alpha)$. Furthermore, Claim 47 can be expressed as $A_0 A_1 = Z$. According to the decomposition outlined in Figure 3, the advantage of π_0 is $\mathbb{E}(A_0 \mid W)$. To see this, let us try to understand the advantage of π_0 from Alice's side. Let (X_0, Y_1, \mathbf{M}) be arbitrary point which occurs with probability $\pi(X_0, Y_1, \mathbf{M} \mid W)$. Since Alice knows (X_0, Y_1, \mathbf{M}) and outputs the more-likely answer B_0 , she can compute $f^{\oplus n/2}(X_0, Y_0)$ correctly with probability

$$\frac{1}{2} + \frac{\text{adv}^{\tilde{\pi}_0}(f^{\oplus n/2}(X_0, Y_0) \mid X_0, \tilde{\mathbf{M}}_0)}{2} = \frac{1}{2} + \frac{\text{adv}^\pi(f^{\oplus n/2}(X_0, Y_0) \mid X_0, Y_1, \mathbf{M}, W)}{2}.$$

Therefore, the advantage of π_0 from Alice's perspective is

$$\sum_{(X_0, Y_1, \mathbf{M})} \pi(X_0, Y_1, \mathbf{M} \mid W) \cdot \text{adv}^\pi(f^{\oplus n/2}(X_0, Y_0) \mid X_0, Y_1, \mathbf{M}, W) = \mathbb{E}(A_0 \mid W).$$

Finally, by sending Bob an additional bit indicating her answer to $f^{\oplus n/2}(X_0, Y_0)$, Bob can achieve the same advantage. A similar argument applies to π_1 , where its advantage is $\mathbb{E}(A_1 \mid W)$.

Denote $\varepsilon_0 = 1 - \text{adv}^{\pi_0}(f^{\oplus n/2}(X_0, Y_0)) = 1 - \mathbb{E}(A_0 \mid W)$ and $\varepsilon_1 = 1 - \text{adv}^{\pi_1}(f^{\oplus n/2}(X_0, Y_0)) = 1 - \mathbb{E}(A_1 \mid W)$ to be the disadvantages of π_0 and π_1 respectively. Again, by letting Alice output the more-likely, the protocol π_0 errs with probability

$$\frac{1}{2} - \frac{\text{adv}^{\pi_0}(f^{\oplus n/2}(X_0, Y_0))}{2} = \frac{\varepsilon_0}{2}$$

and vice-versa for Bob in π_1 . Thus, it is intuitive to use ε_0 and ε_1 as a proxy to the error of π_0 and π_1 respectively.

The following claim shows that the sum of ε_0 and ε_1 cannot be too large.

Claim 48. $\varepsilon_0 + \varepsilon_1 \leq \frac{2}{1 + \sqrt{\alpha}} \cdot (1 - \mathbb{E}(Z \mid W))$.

Proof. Conditioned on W (meaning $Z \geq \alpha$) we derive:

$$A_0 + A_1 \geq 2\sqrt{A_0 A_1} = 2\sqrt{Z} \geq 2 \cdot \frac{Z + \sqrt{\alpha}}{1 + \sqrt{\alpha}}$$

where the last inequality is equivalent to $(\sqrt{Z} - 1)(\sqrt{Z} - \sqrt{\alpha}) \leq 0$. Taking an expectation conditioned on W , the inequality becomes:

$$\mathbb{E}(A_0 \mid W) + \mathbb{E}(A_1 \mid W) \geq 2 \cdot \frac{\mathbb{E}(Z \mid W) + \sqrt{\alpha}}{1 + \sqrt{\alpha}}.$$

Recall that $\varepsilon_0 = 1 - \mathbb{E}(A_0 \mid W)$ and $\varepsilon_1 = 1 - \mathbb{E}(A_1 \mid W)$. Rearranging it concludes the proof. \square

The following claim shows that the conditioning event $W = (Z \geq \alpha)$ occurs with substantial probability.

Claim 49. $\pi(W) \geq \frac{1 - \varepsilon - \alpha}{\mathbb{E}(Z \mid W) - \alpha}$.

Proof. By definition, \overline{W} implies $Z \leq \alpha$. We also have $\mathbb{E}(Z) \geq \text{adv}^\pi(f^{\oplus n}(X, Y)) = 1 - \varepsilon$ via the triangle inequality. Then, we have

$$\begin{aligned} 1 - \varepsilon &\leq \mathbb{E}(Z) = \pi(W) \cdot \mathbb{E}(Z \mid W) + \pi(\overline{W}) \cdot \mathbb{E}(Z \mid \overline{W}) \\ &\leq \pi(W) \cdot \mathbb{E}(Z \mid W) + (1 - \pi(W)) \cdot \alpha \end{aligned}$$

Rearranging it concludes the proof. \square

7.2 Obtaining Generalized Protocols for f

We now describe the *conditional decomposition* that is applied to π_S for an arbitrary $S \in \{0, 1\}^{\leq m-1}$ (Figure 4), which is simply a succinct generalization of Figure 3. Precisely, the conditional decomposition of π_S yields two protocols, π_{S0} and π_{S1} , each of which computes f^\oplus on half of the inputs that π_S handles. Specifically, for an event W_S (to be defined shortly), we decompose $\pi_S \mid W_S$ according to Definition 25, yielding two protocols, $\tilde{\pi}_{S0}$ and $\tilde{\pi}_{S1}$. To derive the protocol π_{S0} , we further let Alice compute the more likely answer of $f^\oplus(X_{S0}, Y_{S0})$ based on her knowledge, denoted B_{S0} , and send it to Bob. This bit B_{S0} serves as the players' answer to $f^\oplus(X_{S0}, Y_{S0})$ in π_{S0} . Equivalently, we can interpret the protocol π_{S0} as appending B_{S0} to $\tilde{\pi}_{S0}$, i.e., $\pi_{S0} := \tilde{\pi}_{S0} \odot B_{S0}$. The protocol π_{S1} is obtained analogously.

Conditional Decomposition of π_S into π_{S0} and π_{S1}.
<ol style="list-style-type: none"> 1. Let $\tilde{\pi}_{S0} = (X_{S0}, Y_{S0}, \tilde{M}_{S0})$ and $\tilde{\pi}_{S1} = (X_{S1}, Y_{S1}, \tilde{M}_{S1})$ be the protocols obtained from decomposing $\pi_S \mid W_S$ via the procedure given in Definition 25. 2. Let B_{S0} be the more-likely value of $f^\oplus(X_{S0}, Y_{S0})$ in the distribution $\tilde{\pi}_{S0}(f^\oplus(X_{S0}, Y_{S0}) \mid X_{S0}\tilde{M}_{S0})$ as computed by Alice. The protocol π_{S0} is defined to be $\pi_{S0} := \tilde{\pi}_{S0} \odot B_{S0}$. 3. Let B_{S1} be the more-likely value of $f^\oplus(X_{S1}, Y_{S1})$ in the distribution $\tilde{\pi}_{S1}(f^\oplus(X_{S1}, Y_{S1}) \mid Y_{S1}\tilde{M}_{S1})$ as computed by Bob. The protocol π_{S1} is defined to be $\pi_{S1} := \tilde{\pi}_{S1} \odot B_{S1}$.

Figure 4: The conditional decomposition procedure of π_S into π_{S0} and π_{S1}

We remark that the event W_S is to be defined shortly via Table 1. The following observation is immediate, as by the end of $\tilde{\pi}_{S0}$, Alice knows X_{S0} and \tilde{M}_{S0} , and vice versa for Bob.

Observation 50. For any S , we have $B_{S0} \perp Y_{S0} \mid X_{S0}\tilde{M}_{S0}$ in the distribution of $\tilde{\pi}_{S0}$, and $B_{S1} \perp X_{S1} \mid Y_{S1}\tilde{M}_{S1}$ in the distribution of $\tilde{\pi}_{S1}$.

Next, we describe our *recursive procedure* (Figure 5) which applies the conditional decomposition in lexicographical order: for $k = 0, 1, \dots, m-1$, we split 2^k protocols $\{\pi_S\}_{|S|=k}$ into 2^{k+1} protocols $\{\pi_{S'}\}_{|S'|=k+1}$. We begin with $k = 0$, which corresponds to the single protocol $\pi_\emptyset = \pi$ for $f^{\oplus n}$. By the end of this process, after completing round $k = m-1$, we will have n protocols $\{\pi_S\}_{|S|=m}$ for f . We collectively refer to the protocols $\{\pi_S\}_{|S|=k}$ as the *level k* .

Recursive Procedure \mathcal{P}
<ol style="list-style-type: none"> 1. set π_\emptyset to π 2. for each $k = 0, 1, \dots, m-1$, 3. for each $S \in \{0, 1\}^k$, 4. apply the conditional decomposition from Figure 4 on π_S to obtain π_{S0} and π_{S1}

Figure 5: A recursive procedure begins with a standard protocol π for computing $f^{\oplus n}$ and ultimately yields n generalized protocols $\{\pi_S\}_{|S|=m}$, each of which computes f .

For any S , we define the following set of parameters related to the protocol π_S . Note that the case of $S = \emptyset$ corresponds to the first level of conditional decomposition discussed in the previous subsection. We also note that when $S = \emptyset$, we might drop the subscript S . By doing that, the notations become consistent with our earlier discussions (e.g. π_\emptyset becomes π , X_\emptyset becomes X , or ε_\emptyset becomes ε , etc.)

Parameters related to a generalized protocol $\pi_S = (X_S, Y_S, \mathbf{M}_S)$
<ul style="list-style-type: none"> • (X_S, Y_S) denotes the input of $2^{m- S }$ coordinates associated with the protocol π_S. • \mathbf{M}_S collectively denotes: <ul style="list-style-type: none"> – M_S^0 denotes public randomness of π_S – $M_S^+ = (M_S^1, M_S^2, \dots)$ denotes the transcript of communication, where M_S^i indicates the message sent in round i. • μ_S is the desired input distributions $\mu^{2^{m- S }}$ associated with the protocol π_S. • Z_S, A_{S0}, A_{S1} to be the random variable over the randomness of $(X_{S0}, Y_{S1}, \mathbf{M}_S)$ for which $Z_S = \text{adv}^{\pi_S}(f^\oplus(X_S, Y_S) \mid X_{S0}, Y_{S1}, \mathbf{M}_S)$ $A_{S0} = \text{adv}^{\pi_S}(f^\oplus(X_{S0}, Y_{S0}) \mid X_{S0}, Y_{S1}, \mathbf{M}_S)$ $A_{S1} = \text{adv}^{\pi_S}(f^\oplus(X_{S1}, Y_{S1}) \mid X_{S0}, Y_{S1}, \mathbf{M}_S)$ • W_S is an event that $Z_S \geq \alpha$. • ε_S is the disadvantage of π_S, denoted as $1 - \text{adv}^{\pi_S}(f^\oplus(X_S, Y_S))$. • $I_S =$ information cost of π_S. • $p_S = \pi_S(W_S)$ • χ_S to be defined recursively such that $\chi_\emptyset = 1$, and $\chi_{S0} = \chi_{S1} = p_S \cdot \chi_S$ for any S.

Table 1: A set of parameters and variables related to $\pi_S = (X_S, Y_S, \mathbf{M}_S)$.

Let us now rationalize the sequence of χ_S . Intuitively, χ_S is the probability of all conditioning events that are attached with π_S . To see this, we notice that $\pi_\emptyset = \pi$ is unconditioned; i.e. is conditioned by an event with probability $1 =: \chi_\emptyset$. Next, π_0 and π_1 is a result of decomposition of $\pi \mid W$. In other words, they are attached with the conditioning event W which occurs with probability $\pi(W)$ which happens to be equal to χ_{S0} and χ_{S1} . We can then apply this reasoning inductively.

It turns out that the analogues of Claims 46, 47, 48, and 49 hold at any level of the iterative process. These results are summarized in the following lemma.

Lemma 51. *For any S , we have:*

- (i) π_S has a rectangle property with respect to μ_S .
- (ii) $\pi_S \mid W_S$ has a partial rectangle property with respect to (μ_{S0}, μ_{S1}) .
- (iii) Pointwise $Z_S = A_{S0}A_{S1}$.
- (iv) $\varepsilon_{S0} + \varepsilon_{S1} \leq \frac{2}{1+\sqrt{\alpha}} \cdot (1 - \mathbb{E}(Z_S \mid W_S))$.

$$(v) \ p_S \geq \frac{1 - \varepsilon_S - \alpha}{\mathbb{E}(Z_S | W_S) - \alpha}.$$

$$(vi) \ \varepsilon_{S_0} = 1 - \mathbb{E}(A_{S_0} | W_S) \text{ and } \varepsilon_{S_1} = 1 - \mathbb{E}(A_{S_1} | W_S)$$

For brevity, we only sketch its proof, as it highly resembles the contents of Section 7.1.

Proof Sketch. We first prove statements (i) by induction. The base case, where $S = \emptyset$, is trivial due to the fact that π is a standard protocol. For the induction step, suppose that the statement is true for $|S| = k$. Then, for any S with $|S| = k$, π_S has the rectangle property with respect to μ_S . Following the approach in the proof of Claim 46, we can show that π_{S_0} has the rectangle property with respect to π_{S_0} and π_{S_1} has the rectangle property with respect to π_{S_1} . Apply this argument to all $|S| = k$ concludes the induction step.

For (ii), given (i) the statement is equivalent to the first half of the proof of Claim 46.

Having established (i) and (ii), the proofs for statements (iii), (iv), and (v) closely follow those of Claims 47, 48, and 49, respectively. For the sake of succinctness, we omit their proofs.

Finally, (vi) follows from the fact that $\text{adv}^{\pi_{S_0}}(f^\oplus(X_{S_0}, Y_{S_0})) = \mathbb{E}(A_{S_0} | W_S)$ by the same reasoning we argued in the earlier subsection. \square

7.3 Related Bounds

In this subsection, we present several useful inequality bounds regarding the parameters we have set. These bounds shall be used repeatedly throughout the section.

Claim 52. For any S , we have $\mathbb{E}(Z_S | W_S) \geq 1 - \varepsilon_S$.

Proof. Consider the following calculation:

$$\begin{aligned} \mathbb{E}(Z_S) &= \sum_{(X_{S_0}, Y_{S_1}, \mathbf{M}_S)} \pi_S(X_{S_0}, Y_{S_1}, \mathbf{M}_S) \cdot \text{adv}^{\pi_S}(f^\oplus(X_S, Y_S) | X_{S_0} Y_{S_1} \mathbf{M}_S) \\ &= \sum_{(X_{S_0}, Y_{S_1}, \mathbf{M}_S)} \pi_S(X_{S_0}, Y_{S_1}, \mathbf{M}_S) \cdot |2 \cdot \pi_S(f^\oplus(X_S, Y_S) = 0 | X_{S_0} Y_{S_1} \mathbf{M}_S) - 1| \\ &\geq \left| 2 \left(\sum_{(X_{S_0}, Y_{S_1}, \mathbf{M}_S)} \pi_S(X_{S_0}, Y_{S_1}, \mathbf{M}_S) \cdot \pi_S(f^\oplus(X_S, Y_S) = 0 | X_{S_0} Y_{S_1} \mathbf{M}_S) \right) - 1 \right| \\ &= |2 \cdot \pi_S(f^\oplus(X_S, Y_S) = 0) - 1| \\ &= \text{adv}^{\pi_S}(f^\oplus(X_S, Y_S)) \\ &= 1 - \varepsilon_S \end{aligned}$$

where the only inequality uses the triangle inequality. Then, we can derive

$$\begin{aligned} 1 - \varepsilon_S &= \mathbb{E}(Z_S) \\ &= \Pr(W_S) \cdot \mathbb{E}(Z_S | W_S) + \Pr(\overline{W_S}) \cdot \mathbb{E}(Z_S | \overline{W_S}) \\ &\leq \Pr(W_S) \cdot \mathbb{E}(Z_S | W_S) + \Pr(\overline{W_S}) \cdot \mathbb{E}(Z_S | W_S) \\ &= \mathbb{E}(Z_S | W_S). \end{aligned}$$

where the inequality follows from the notation of $W_S = (Z_S \geq \alpha)$; thus, $\mathbb{E}(Z_S | \overline{W_S}) \leq \alpha \leq \mathbb{E}(Z_S | W_S)$. This concludes the proof. \square

Corollary 53. For any k , we have $\sum_{|S|=k} \varepsilon_S \leq 2^{\tau k} \varepsilon$.

Proof. Combining Lemma 51 and 52, we have $\varepsilon_{S_0} + \varepsilon_{S_1} \leq \frac{2}{1 + \sqrt{\alpha}} \cdot (1 - \mathbb{E}(Z_S | W_S)) \leq 2^{-\tau} \varepsilon_S$. Induction on $|S|$ finishes the proof. \square

Corollary 54. For any S , we have $\varepsilon_S \leq \varepsilon$.

Proof. It suffices to show that $\varepsilon_{S_0}, \varepsilon_{S_1} \leq \varepsilon_S$ for any S , as the fact follows inductively. By Lemma 51, we have $Z_S = A_{S_0}A_{S_1} \leq A_S$ holds pointwisely. Moreover, by Lemma 51 and Claim 52, the disadvantage of π_{S_0} is $\mathbb{E}(A_{S_0} | W_S) \geq \mathbb{E}(Z_S | W_S) \geq 1 - \varepsilon_S$. Plugging in $\varepsilon_{S_0} = 1 - \mathbb{E}(A_{S_0} | W_S)$ yields $\varepsilon_{S_0} \leq \varepsilon_S$. Similarly, we also have $\varepsilon_{S_1} \leq \varepsilon_S$. \square

The following claim is critical to several of our proofs in the next subsection.

Claim 55. $\sum_{|S|=m} \chi_S = \Omega(n)$.

Proof. For any string S with $|S| \leq m$, denote *potential* of π_S to be

$$\Lambda_S = \underbrace{\left(2^{-|S|} \log \frac{1}{\chi_S}\right)}_{\phi_S} + \frac{1}{\varepsilon} \cdot \underbrace{\left(\frac{1 + \sqrt{\alpha}}{2}\right)^{|S|}}_{\psi_S} \varepsilon_S$$

We claim that a conditional decomposition which splits π_S into π_{S_0} and π_{S_1} never decreases total potentials; that is for any S we must have $\Lambda_{S_0} + \Lambda_{S_1} \leq \Lambda_S$. To see this, consider

$$\begin{aligned} \phi_{S_0} + \phi_{S_1} - \phi_S &= 2^{-|S|} \cdot \log \frac{\chi_S}{\sqrt{\chi_{S_0}\chi_{S_1}}} \\ &= 2^{-|S|} \log \frac{1}{p_S} \\ &\leq 2^{-|S|} \log \left(\frac{\mathbb{E}(Z_S | W_S) - \alpha}{1 - \varepsilon_S - \alpha} \right) && \text{(Lemma 51)} \\ &\leq 2^{-|S|} \cdot \frac{\mathbb{E}(Z_S | W_S) - (1 - \varepsilon_S)}{1 - \varepsilon_S - \alpha} && (\log x \leq x - 1) \\ &= 2^{-|S|} \cdot \frac{\mathbb{E}(Z_S | W_S) - (1 - \varepsilon_S)}{\varepsilon} \end{aligned}$$

where the last equality follows Claim 52 that $\mathbb{E}(Z_S | W_S) \geq 1 - \varepsilon_S$, and Corollary 54 that $1 - \varepsilon_S - \alpha \geq 1 - \varepsilon - \alpha = \varepsilon$. Moreover,

$$\begin{aligned} \psi_{S_0} + \psi_{S_1} - \psi_S &= \left(\frac{1 + \sqrt{\alpha}}{2}\right)^{|S|} \cdot \left(\frac{1 + \sqrt{\alpha}}{2}\right)^{|S|} \cdot (\varepsilon_{S_0} + \varepsilon_{S_1}) - \varepsilon_S \\ &\leq \left(\frac{1 + \sqrt{\alpha}}{2}\right)^{|S|} \cdot (1 - \varepsilon_S - \mathbb{E}(Z_S | W_S)) && \text{(Lemma 51)} \\ &\leq 2^{-|S|} \cdot (1 - \varepsilon_S - \mathbb{E}(Z_S | W_S)) && \text{(Claim 52)} \end{aligned}$$

Combining the two inequalities, we have $\Lambda_{S_0} + \Lambda_{S_1} \leq \Lambda_S$ for any S . Applying it recursively, we have $\Lambda_\emptyset \geq \sum_{|S|=m} \Lambda_S$ which leads to:

$$1 = \Lambda_\emptyset \geq \sum_{|S|=m} \Lambda_S = \left(\sum_{|S|=m} \frac{1}{n} \log \frac{1}{\chi_S} \right) + \frac{1}{\varepsilon} \cdot \left(\sum_{|S|=m} n^{-\tau} \varepsilon_S \right) \geq \sum_{|S|=m} \frac{1}{n} \log \frac{1}{\chi_S}.$$

In other words, we have

$$\frac{1}{n} \sum_S \log \chi_S \geq -1.$$

Then, by the AM-GM inequality, we have

$$\sum_S \chi_S \geq n \cdot \left(\prod_S \chi_S \right)^{1/n} \geq ne^{-1}$$

as wished. □

The following patterns will be prevalent throughout this section.

Lemma 56. *Let $c > 0$ be an arbitrary constant. For each $S \in \{0, 1\}^{\leq m}$, let $q_S \in \mathbb{R}_{\geq 0}$ satisfying the following inequality:*

$$q_{S0} + q_{S1} \leq \frac{1}{p_S} \cdot (q_S + c \cdot \mathcal{H}(p_S)). \quad (2)$$

Then, we have

$$\sum_{|S|=m} \chi_S q_S \leq q_\emptyset + O(n^\tau \log n).$$

Lemma 57. *Let $c, c' > 0$ be arbitrary constants. For each $S \in \{0, 1\}^{\leq m}$, let $q_S \in \mathbb{R}_{\geq 0}$ satisfying the following inequality:*

$$q_{S0} + q_{S1} \leq \frac{1}{p_S} \cdot (q_S + c \cdot \mathcal{H}(p_S)) + c' \quad (3)$$

Then, we have

$$\sum_{|S|=m} \chi_S q_S \leq q_\emptyset + O(n).$$

The proof of Lemma 56 and Lemma 57 involves algebraic computations and will be deferred to the Appendix.

7.4 Certifying a “Nice” Generalized Protocol for f

We will show that following the recursive procedure (Figure 5), there exists an index $S \in \{0, 1\}^m$ such that the protocol π_S has small error (i.e. disadvantage), small information cost, and small θ -cost with respect to μ . The proof will rely on a probabilistic argument: we will show that if the index S is sampled from the “proportional” distribution \mathcal{D} over $\{0, 1\}^m$, then π_S has these properties in expectation. Specifically, we define the distribution \mathcal{D} as follows.

A distribution \mathcal{D} sampling an index

- Let \mathcal{D} be a distribution over $\{0, 1\}^m$ where $\mathcal{D}(S) = \frac{\chi_S}{\sum_{|S'|=m} \chi_{S'}}$ (i.e. is proportional to χ_S .)

Figure 6: The “proportional” distribution \mathcal{D} for sampling an index $S \in \{0, 1\}^m$.

We wish for these properties for the protocol π_S .

Lemma 58 (Decomposition Lemma; informal). *Over the distribution $S \sim \mathcal{D}$, the generalized protocol π_S has the following properties in expectation.*

- (1) π_S errs with small probability $\frac{1}{\text{poly}(n)}$.
- (2) π_S has small information cost $\approx O(\mathcal{I}/n + 1)$.
- (3) π_S has small θ -cost $\frac{1}{\text{poly}(n)}$ with respect to μ .

For the remaining of this section, we will prove a formal version of Lemma 58.

π_S **Has Small Error.** We first prove property (1). Recall that the disadvantage of π_S is ε_S , which is exactly twice the error of π_S . Thus, it suffices to bound the expectation of ε_S .

Claim 59. $\mathbb{E}(\varepsilon_S) \leq O(n^{-(1-\tau)})$.

Proof. It follows by the calculation:

$$\begin{aligned} \mathbb{E}(\varepsilon_S) &= \frac{\sum_{|S|=m} \varepsilon_S \chi_S}{\sum_{|S|=m} \chi_S} \leq \frac{\sum_{|S|=m} \varepsilon_S}{\sum_{|S|=m} \chi_S} \leq \frac{\varepsilon n^\tau}{\Omega(n)} && \text{(Corollary 53 and Claim 55)} \\ &= O(n^{-(1-\tau)}). \end{aligned}$$

□

π_S **Has Small Information Cost.** Next, we will prove property (2). Directly upper-bounding the information cost of π_S can be quite complicated. Instead, we will use the γ -cost (Definition 39) as an intermediate. For any subset $S \in \{0, 1\}^m$, denote:

$$\Gamma_S = \gamma_{\mu_S}(\pi_S).$$

Observe that Γ_\emptyset is precisely the information cost of π , which is \mathcal{I} . This is due to:

$$\begin{aligned} \Gamma_\emptyset &= \gamma_{\mu^n}(\pi) \\ &= \mathbb{E}_{(X,Y,M) \sim \pi} \gamma_{\mu^n, A}(\pi @ X, Y, M) + \mathbb{E}_{(X,Y,M) \sim \pi} \gamma_{\mu^n, B}(\pi @ X, Y, M) \\ &= \mathbb{E}_{(X,Y,M) \sim \pi} \log \frac{\pi(X | YM)}{\mu^n(X | Y)} + \mathbb{E}_{(X,Y,M) \sim \pi} \log \frac{\pi(Y | XM)}{\mu^n(Y | X)} \\ &= \mathbb{E}_{(X,Y,M) \sim \pi} \log \frac{\pi(X | YM)}{\pi(X | Y)} + \mathbb{E}_{(X,Y,M) \sim \pi} \log \frac{\pi(Y | XM)}{\pi(Y | X)} \\ &= \mathcal{I}. \end{aligned}$$

The following claim argues that the γ -cost are low in expectation.

Claim 60. $\mathbb{E}(\Gamma_S) = O(\mathcal{I}/n + 1)$.

Proof. We will first show that $\Gamma_{S_0} + \Gamma_{S_1} \leq \frac{1}{p_S} \cdot [\Gamma_S + 2\mathcal{H}(p_S)] + 4$. To do so, we compute $\gamma_{\mu_S}(\pi_S | W_S)$ in two different ways.

1. Denote the protocol π_S by (X_S, Y_S, M_S) . Recall that $\pi_S | W_S$ undergoes a binary decomposition into $\tilde{\pi}_{S_0} = (X_{S_0}, Y_{S_0}, \tilde{M}_{S_0})$ and $\tilde{\pi}_{S_1} = (X_{S_1}, Y_{S_1}, \tilde{M}_{S_1})$. Moreover, distribution-wise we have $\pi_{S_0} = (X_{S_0}, Y_{S_0}, M_{S_0})$ where $M_{S_0} = (\tilde{M}_{S_0}, B_{S_0})$ and $\pi_{S_1} = (X_{S_1}, Y_{S_1}, M_{S_1})$ where $M_{S_1} = (\tilde{M}_{S_1}, B_{S_1})$.

By definition, we have:

$$\begin{aligned} \gamma_{\mu_S}(\pi_S | W_S) &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \gamma_{\mu_S, A}(\pi_S | W_S @ X_S, Y_S, M_S) \\ &\quad + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \gamma_{\mu_S, B}(\pi_S | W_S @ X_S, Y_S, M_S). \end{aligned}$$

Following Lemma 51, we know that $\pi_S | W_S$ has the partial rectangle property with respect to (μ_{S_0}, μ_{S_1}) . The first term, via the linearity of pointwise- γ -cost (Lemma 40), has become:

$$\begin{aligned} &\mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \gamma_{\mu_S, A}(\pi_S | W_S @ X_S, Y_S, M_S) \\ &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \gamma_{\mu_{S_0}, A}(\tilde{\pi}_{S_0} @ X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \gamma_{\mu_{S_1}, A}(\tilde{\pi}_{S_1} @ X_{S_1}, Y_{S_1}, \tilde{M}_{S_1}) \\ &= \mathbb{E}_{(X_{S_0}, Y_{S_0}, M_{S_0}) \sim \tilde{\pi}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} + \mathbb{E}_{(X_{S_1}, Y_{S_1}, M_{S_1}) \sim \tilde{\pi}_{S_1}} \log \frac{\tilde{\pi}_{S_1}(X_{S_1} | Y_{S_1} \tilde{M}_{S_1})}{\mu_{S_1}(X_{S_1} | Y_{S_1})}. \end{aligned}$$

Next, consider the following calculation.

$$\begin{aligned}
\Gamma_{S_0} &= \gamma_{\mu_{S_0}}(\pi_{S_0}) \\
&= \gamma_{\mu_{S_0}}(\tilde{\pi}_{S_0} \odot B_{S_0}) \\
&= \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}, B_{S_0}) \sim \tilde{\pi}_{S_0} \odot B_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0} B_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \\
&\quad + \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}, B_{S_0}) \sim \tilde{\pi}_{S_0} \odot B_{S_0}} \log \frac{\tilde{\pi}_{S_0}(Y_{S_0} | X_{S_0} \tilde{M}_{S_0} B_{S_0})}{\mu_{S_0}(Y_{S_0} | X_{S_0})}.
\end{aligned}$$

Let us expand on the first term:

$$\begin{aligned}
&\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}, B_{S_0}) \sim \tilde{\pi}_{S_0} \odot B_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0} B_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \\
&= \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \mathbb{E}_{B_{S_0} \sim \tilde{\pi}_{S_0} | X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0} B_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \\
&= \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \mathbb{E}_{B_{S_0} \sim \tilde{\pi}_{S_0} | X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}} \left(\log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} + \log \frac{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} Y_{S_0} \tilde{M}_{S_0})}{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} Y_{S_0})} \right) \\
&= \left(\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \right) \\
&\quad + \left(\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \mathbb{E}_{B_{S_0} \sim \tilde{\pi}_{S_0} | X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} Y_{S_0} \tilde{M}_{S_0})}{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} Y_{S_0})} \right) \\
&= \left(\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \right) + \left(\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} I(B_{S_0} : \tilde{M}_{S_0} | X_{S_0} Y_{S_0}) \right) \\
&\leq \left(\mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \log \frac{\tilde{\pi}_{S_0}(X_{S_0} | Y_{S_0} \tilde{M}_{S_0})}{\mu_{S_0}(X_{S_0} | Y_{S_0})} \right) + 1
\end{aligned}$$

where the inequality follows the fact that $I(B_{S_0} : M_{S_0} | X_{S_0} Y_{S_0}) \leq |B_{S_0}| = 1$.

Combining with the other symmetric terms, we have

$$\Gamma_{S_0} + \Gamma_{S_1} \leq \gamma_{\mu_S}(\pi_S | W_S) + 4.$$

2. Recall that

$$\gamma_{\mu_S}(\pi_S | W_S) = \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(X_S | Y_S M_S W_S)}{\mu_S(X_S | Y_S)} + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(Y_S | X_S M_S W_S)}{\mu_S(Y_S | X_S)}.$$

For the first term, we have

$$\begin{aligned}
&\mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(X_S | Y_S M_S W_S)}{\mu_S(X_S | Y_S)} \\
&= \mathbb{E}_{(Y_S, M_S) \sim \pi_S | W_S} \mathbb{D} \left(\frac{\pi_S(X_S | Y_S M_S W_S)}{\mu_S(X_S | Y_S)} \right) \\
&\leq \frac{1}{p_S} \cdot \mathbb{E}_{(Y_S, M_S) \sim \pi_S} \left[\mathbb{D} \left(\frac{\pi_S(X_S | Y_S M_S)}{\mu_S(X_S | Y_S)} \right) + \mathcal{H}(p_S) \right] \quad (\text{Lemma 44}) \\
&= \frac{1}{p_S} \cdot \left[\mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(X_S | Y_S M_S)}{\mu_S(X_S | Y_S)} + \mathcal{H}(p_S) \right].
\end{aligned}$$

Combining with its symmetric term, we have

$$\gamma_{\mu_S}(\pi_S | W_S) \leq \frac{1}{p_S} \cdot [\Gamma_S + 2\mathcal{H}(p_S)].$$

Putting together the two calculations, we have $\Gamma_{S_0} + \Gamma_{S_1} \leq \frac{1}{p_S} \cdot [\Gamma_S + 2\mathcal{H}(p_S)] + 4$ for any S . By Lemma 57, we have

$$\sum_{|S|=m} \chi_S \Gamma_S \leq \Gamma_\emptyset + O(n) = \mathcal{I} + O(n).$$

Recall via Claim 55 that $\sum_{|S|=m} \chi_S = \Omega(n)$. Therefore, we have

$$\mathbb{E}(\Gamma_S) = \frac{\sum_{|S|=m} \chi_S \Gamma_S}{\sum_{|S|=m} \chi_S} = \frac{\mathcal{I} + O(n)}{\Omega(n)} = O(\mathcal{I}/n + 1)$$

as wished. \square

As a corollary, we derive the same upper bound for the expectation of information cost.

Claim 61. $\mathbb{E}(I_S) = O(\mathcal{I}/n + 1)$.

Proof. It suffices to show that for any S , the information cost of π_S is upper-bounded by Γ_S (hence the reason that we use Γ_S as a proxy.) This is due to the following calculation.

$$\begin{aligned} \Gamma_S &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(X_S | Y_S M_S)}{\mu_S(X_S | Y_S)} + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(Y_S | X_S M_S)}{\mu_S(Y_S | X_S)} \\ &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(X_S | Y_S M_S)}{\pi_S(X_S | Y_S M_S^0)} + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(Y_S | X_S M_S)}{\pi_S(Y_S | X_S M_S^0)} \\ &\quad + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(X_S | Y_S M_S^0)}{\mu_S(X_S | Y_S)} + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(Y_S | X_S M_S^0)}{\mu_S(Y_S | X_S)} \\ &= I(M_S : X_S | Y_S M_S^0) + I(M_S : Y_S | X_S M_S^0) \\ &\quad + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(X_S | Y_S M_S^0)}{\mu_S(X_S | Y_S)} + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S} \log \frac{\pi_S(Y_S | X_S M_S^0)}{\mu_S(Y_S | X_S)} \\ &= I_S + \mathbb{E}_{(Y_S, M_S^0) \sim \pi_S} \mathbb{D} \left(\frac{\pi_S(X_S | Y_S M_S^0)}{\mu_S(X_S | Y_S)} \right) + \mathbb{E}_{(X_S, M_S^0) \sim \pi_S} \mathbb{D} \left(\frac{\pi_S(Y_S | X_S M_S^0)}{\mu_S(Y_S | X_S)} \right) \\ &\geq I_S \end{aligned}$$

where the last inequality is due to the fact that KL-divergences are always non-negative. \square

π_S **Has Small θ -Cost.** Finally, we prove property (3). Recall the definition of the θ -cost via Definition 34. For any S such that $|S| \leq m$, denote

$$\Theta_S = \theta_{\mu_S}(\pi_S).$$

With this notions, we have $\Theta_\emptyset = 0$ because $\pi_\emptyset = \pi$ is a standard protocol whose input distribution is exactly μ^n (via Observation 36.)

We will soon need the following lemma.

Lemma 62. For any $S \in \{0, 1\}^{\leq m}$ and $i \geq 1$, we have

$$\mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{\leq i} W_S)}{\pi_S(M_S^i | X_S M_S^{\leq i} W_S)} \leq \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{\leq i} W_S)}{\pi_S(M_S^i | X_S M_S^{\leq i})}.$$

Proof. Observe that the upper term of both sides are identical. Using linearity of expectation, it is equivalent to showing that

$$\mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \geq 0.$$

Consider:

$$\begin{aligned} & \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \\ &= \sum_{X_S, M_S^{<i}} \pi_S(M_S^{<i} X_S | W_S) \log \frac{\pi_S(M_S^i | X_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \\ &= \sum_{X_S, M_S^{<i}} \pi_S(M_S^{<i} X_S | W_S) \cdot \pi_S(M_S^i | X_S M_S^{<i} W_S) \cdot \log \frac{\pi_S(M_S^i | X_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \\ &= \sum_{X_S, M_S^{<i}} \pi_S(M_S^{<i} X_S | W_S) \cdot \sum_{M_S^i} \pi_S(M_S^i | X_S M_S^{<i} W_S) \cdot \log \frac{\pi_S(M_S^i | X_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \\ &= \sum_{X_S, M_S^{<i}} \pi_S(M_S^{<i} X_S | W_S) \cdot \mathbb{D} \left(\frac{\pi_S(M_S^i | X_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} \right) \\ &\geq 0. \end{aligned}$$

This concludes the proof. \square

The following claim argues that the θ -cost are low in expectation.

Claim 63. $\mathbb{E}(\Theta_S) = O(n^{-(1-\tau)} \log n)$.

Proof. We will first show that $\Theta_{S_0} + \Theta_{S_1} \leq \frac{1}{p_S} \cdot [\Theta_S + \mathcal{H}(p_S)]$. To do so, we calculate $\theta_{\mu_S}(\pi_S | W_S)$ in two different ways.

1. Denote the protocol π_S by (X_S, Y_S, M_S) . Recall that $\pi_S | W_S$ undergoes a binary decomposition into $\tilde{\pi}_{S_0} = (X_{S_0}, Y_{S_0}, \tilde{M}_{S_0})$ and $\tilde{\pi}_{S_1} = (X_{S_1}, Y_{S_1}, \tilde{M}_{S_1})$. Moreover, distribution-wise we have $\pi_{S_0} = (X_{S_0}, Y_{S_0}, M_{S_0})$ where $M_{S_0} = (\tilde{M}_{S_0}, B_{S_0})$ and $\pi_{S_1} = (X_{S_1}, Y_{S_1}, M_{S_1})$ where $M_{S_1} = (\tilde{M}_{S_1}, B_{S_1})$.

By Lemma 51, we know that $\pi_S | W_S$ has the partial rectangle property with respect to (μ_{S_0}, μ_{S_1}) . By the linearity of pointwise- θ -cost (Lemma 37), we have

$$\begin{aligned} & \theta_{\mu_S}(\pi_S | W_S) \\ &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \theta_{\mu_S}(\pi_S | W_S @ X, Y, M) \\ &= \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \theta_{\mu_{S_0}}(\tilde{\pi}_{S_0} @ X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) + \mathbb{E}_{(X_S, Y_S, M_S) \sim \pi_S | W_S} \theta_{\mu_{S_1}}(\tilde{\pi}_{S_1} @ X_{S_1}, Y_{S_1}, \tilde{M}_{S_1}) \\ &= \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) \sim \tilde{\pi}_{S_0}} \theta_{\mu_{S_0}}(\tilde{\pi}_{S_0} @ X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}) + \mathbb{E}_{(X_{S_1}, Y_{S_1}, \tilde{M}_{S_1}) \sim \tilde{\pi}_{S_1}} \theta_{\mu_{S_1}}(\tilde{\pi}_{S_1} @ X_{S_1}, Y_{S_1}, \tilde{M}_{S_1}) \end{aligned}$$

On the other hand, we can write:

$$\begin{aligned} \Theta_{S_0} &= \theta_{\mu_{S_0}}(\pi_{S_0}) = \theta_{\mu_{S_0}}(\tilde{\pi}_{S_0} \odot B_{S_0}) \\ &= \mathbb{E}_{(X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}, B_{S_0}) \sim \tilde{\pi}_{S_0} \odot B_{S_0}} \theta_{\mu_{S_0}}(\pi_{S_0} @ X_{S_0}, Y_{S_0}, \tilde{M}_{S_0}, B_{S_0}) \end{aligned}$$

By $\pi_{S_0} = \tilde{\pi}_{S_0} \odot B_{S_0}$, we know that the protocol π_{S_0} and $\tilde{\pi}_{S_0}$ are identical, up to the last bit B_{S_0} of π_{S_0} . Therefore, we have

$$\theta_{\mu_{S_0}}(\pi_{S_0} @ X, Y, \mathbf{M}) - \theta_{\mu_{S_0}}(\tilde{\pi}_{S_0} @ X, Y, \mathbf{M}) = \log \frac{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} Y_{S_0} \mathbf{M}_{S_0})}{\tilde{\pi}_{S_0}(B_{S_0} | X_{S_0} \mathbf{M}_{S_0})} = 0$$

where the last equality follows Fact 50 that $B_{S_0} \perp Y_{S_0} | X_{S_0} \tilde{\mathbf{M}}_{S_0}$ in the distribution $\tilde{\pi}_{S_0}$.

Combining with the symmetric terms, we have

$$\theta_{\mu_S}(\pi_S | W_S) = \Theta_{S_0} + \Theta_{S_1}.$$

2. As a result of Lemma 62, we have

$$\begin{aligned} & \sum_{\text{odd } i} \mathbb{E}_{(X_S, Y_S, \mathbf{M}_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i} W_S)} + \sum_{\text{even } i} \mathbb{E}_{(X_S, Y_S, \mathbf{M}_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{<i} W_S)}{\pi_S(M_S^i | Y_S M_S^{<i} W_S)} \\ & \leq \sum_{\text{odd } i} \mathbb{E}_{(X_S, Y_S, \mathbf{M}_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{<i} W_S)}{\pi_S(M_S^i | X_S M_S^{<i})} + \sum_{\text{even } i} \mathbb{E}_{(X_S, Y_S, \mathbf{M}_S) \sim \pi_S | W_S} \log \frac{\pi_S(M_S^i | X_S Y_S M_S^{<i} W_S)}{\pi_S(M_S^i | Y_S M_S^{<i})} \end{aligned}$$

By patching $\mathbb{E}_{(X_S, Y_S, \mathbf{M}_S) \sim \pi_S | W_S} \log \frac{\pi_S(X_S, Y_S | W_S M_S^0)}{\mu_S(X_S, Y_S)}$ into both sides, we get:

$$\theta_{\mu_S}(\pi_S | W_S) \leq \mathbf{D} \left(\frac{\pi_S(X_S, Y_S, \mathbf{M}_S | W_S)}{\gamma(X_S, Y_S, \mathbf{M}_S)} \right)$$

where $\gamma = \text{standardize}(\pi_S, \mu_S)$ is the standardization of π_S with respect to μ_S . Using Lemma 43, we can further bound

$$\begin{aligned} \theta_{\mu_S}(\pi_S | W_S) & \leq \mathbf{D} \left(\frac{\pi_S(X_S, Y_S, \mathbf{M}_S | W_S)}{\gamma(X_S, Y_S, \mathbf{M}_S)} \right) \\ & \leq \frac{1}{p_S} \cdot \left[\mathbf{D} \left(\frac{\pi_S(X_S, Y_S, \mathbf{M}_S)}{\gamma(X_S, Y_S, \mathbf{M}_S)} \right) + \mathcal{H}(p_S) \right] \quad (\text{Lemma 43}) \\ & = \frac{1}{p_S} \cdot [\theta_{\mu_S}(\pi_S) + \mathcal{H}(p_S)] \quad (\text{Fact 35}) \\ & = \frac{1}{p_S} \cdot [\Theta_S + \mathcal{H}(p_S)]. \end{aligned}$$

Putting together the two calculations, we have $\Theta_{S_0} + \Theta_{S_1} \leq \frac{1}{p_S} \cdot [\Theta_S + \mathcal{H}(p_S)]$ for any S . By Lemma 56, we have

$$\sum_{|S|=m} \chi_S \Theta_S \leq \Theta_\emptyset + O(n^\tau \log n) = O(n^\tau \log n).$$

Recall via Claim 55 that $\sum_{|S|=m} \chi_S = \Omega(n)$. Therefore, we have

$$\mathbb{E}(\Theta_S) = \frac{\sum_{|S|=m} \chi_S \Theta_S}{\sum_{|S|=m} \chi_S} = \frac{O(n^\tau \log n)}{\Omega(n)} = O(n^{-(1-\tau)} \log n)$$

as wished. \square

As a summary of this section, we reformulate our decomposition lemma.

Lemma 64 (Decomposition Lemma; formal). *There exists $S \in \{0, 1\}^m$ such that the protocol π_S for solving f has the following properties:*

- (1) π_S errs with small probability: $\varepsilon_S = O(n^{-(1-\tau)})$.
- (2) π_S has small information cost: $I_S = O(\mathcal{I}/n + 1)$.
- (3) π_S has small θ -cost with respect to μ : $\theta_\mu(\pi_S) = O(n^{-(1-\tau)} \log n)$.

Proof. Consider sampling $S \sim \mathcal{D}$. Applying Markov's Inequality to Claim 59, Claim 61, and Claim 63, each the following events occurs with probability at least 0.99: $\varepsilon_S = O(n^{-(1-\tau)})$, $I_S = O(\mathcal{I}/n + 1)$, and $\theta_\mu(\pi_S) = O(n^{-(1-\tau)} \log n)$. Via the union bound, the three event occurs simultaneously with positive probability. Therefore, there must exists $S \in \{0, 1\}^m$ for which all three events holds. \square

8 Obtaining a Standard Protocol

Let π_S be a protocol that satisfies Lemma 64. We identify two remaining issues arising from the conditioning events that associate with the protocol π_S . First, π_S is not a standard protocol. Second, the input distribution of π_S is no longer μ ; therefore, the low distributional error of π_S is evaluated against a different input distribution μ' . We address both issues simultaneously. Notably, the low θ -cost of π_S implies that π_S is “close” to being a standard protocol, and that μ' is “close” to μ . Thus, there are hopes that we can transform π_S into a standard protocol with the correct input distribution μ , while incurring only small losses in both information cost and distributional error. It turns out that this task can be achieved via the *standardization* as restated below.

Definition 27 (Standardization). *Let $\pi = (X, Y, M)$ be a generalized protocol, and let μ be an arbitrary input distribution. Say $\pi' = \text{standardize}(\pi, \mu)$ is the standardization of π with respect to μ iff π' admits the following distribution:*

$$\pi'(X, Y, M) = \pi(M^0) \cdot \mu(X, Y) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | XM^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | YM^{<i}).$$

The following lemma guarantees that standardizing a generalized protocol results in only small losses in information cost and distributional error, provided that the θ -cost of the protocol is small.

Lemma 65. *Suppose that a generalized protocol $\pi = (X, Y, M)$ has information cost $\text{IC}(\pi) = I^\pi(M : X | YM_0) + I^\pi(M : Y | XM_0)$, and errs with probability ρ over the input distribution $\pi(x, y)$. Let μ be another input distribution, and let $\eta = \text{standardize}(\pi, \mu)$ be the standardization of π with respect to μ . Denote $\ell = D(\pi \| \eta)$. Then, we have:*

- (1) *the information cost of η is at most $\text{IC}(\pi) + O\left(\mathcal{H}(\sqrt{\ell}) + \sqrt{\ell} \cdot \log(|\mathcal{X}| \cdot |\mathcal{Y}|)\right)$.*
- (2) *η errs with probability at most $\rho + O(\sqrt{\ell})$ over the input distribution μ .*

Proof. By Pinsker's Inequality, we have $\|\pi(X, Y, M) - \eta(X, Y, M)\| \leq O(\sqrt{\ell})$. By Lemma 45, there exists a random process that generates $(X^\pi, Y^\pi, M^\pi) \sim \pi(X, Y, M)$, $(X^\eta, Y^\eta, M^\eta) \sim \eta(X, Y, M)$, and crucially the probability that $(X^\pi, Y^\pi, M^\pi) \neq (X^\eta, Y^\eta, M^\eta)$ is at most $\|\pi(X, Y, M) - \eta(X, Y, M)\| \leq O(\sqrt{\ell})$. Let F be such event.

To prove (1), consider:

$$\begin{aligned} & I^\eta(M : X | YM_0) \\ &= \Pr(\overline{F}) \cdot I^\eta(M : X | YM_0 \overline{F}) + \Pr(F) \cdot I^\eta(M : X | YM_0 F) \\ &= \Pr(\overline{F}) \cdot I^\pi(M : X | YM_0 \overline{F}) + \Pr(F) \cdot I^\eta(M : X | YM_0 F) && \text{(Conditioned on } \overline{F}, \text{ we have } \eta = \pi) \\ &\leq I^\pi(M : X | YM_0) + \mathcal{H}(\Pr(\overline{F})) + \Pr(F) \cdot I^\eta(M : X | YM_0 F) && \text{(Lemma 42)} \\ &\leq I^\pi(M : X | YM_0) + \mathcal{H}(O(\sqrt{\ell})) + O(\sqrt{\ell}) \cdot \log |\mathcal{X}| && (I^\eta(M : X | Y\overline{E}) \leq \mathcal{H}(X) \leq \log |\mathcal{X}|) \end{aligned}$$

Combining with a symmetric term, we will have

$$\text{IC}(\eta) \leq \text{IC}(\pi) + 2 \cdot \mathcal{H}(O(\sqrt{\ell})) + O(\sqrt{\ell}) \cdot \log(|\mathcal{X}| \cdot |\mathcal{Y}|).$$

To prove (2), observe that

$$\begin{aligned}\Pr_{(x,y)\sim\mu}(\eta \text{ errs on } (x,y)) &= \Pr(\overline{F}) \cdot \Pr_{(x,y)\sim\mu|\overline{F}}(\eta \text{ errs on } (x,y)) + \Pr(F) \cdot \Pr_{(x,y)\sim\mu|F}(\eta \text{ errs on } (x,y)) \\ \Pr_{(x,y)\sim\pi}(\pi \text{ errs on } (x,y)) &= \Pr(\overline{F}) \cdot \Pr_{(x,y)\sim\pi|\overline{F}}(\pi \text{ errs on } (x,y)) + \Pr(F) \cdot \Pr_{(x,y)\sim\pi|F}(\pi \text{ errs on } (x,y))\end{aligned}$$

Recall again that that conditioned on \overline{F} , we have $\eta = \pi$. Therefore,

$$\begin{aligned}&\Pr_{(x,y)\sim\mu}(\eta \text{ errs on } (x,y)) \\ &= \Pr_{(x,y)\sim\pi}(\pi \text{ errs on } (x,y)) + \Pr(F) \cdot \left[\Pr_{(x,y)\sim\mu|F}(\eta \text{ errs on } (x,y)) - \Pr_{(x,y)\sim\pi|F}(\pi \text{ errs on } (x,y)) \right] \\ &\leq \rho + O(\sqrt{\ell}).\end{aligned}$$

This concludes the proof. \square

Completing the Proof of Lemma 8. Let $\eta = \text{standardize}(\pi_S, \mu)$ be the standardization of π_S with respect to μ . Note that by the promise of Lemma 64, we have $\text{IC}(\pi_S) = I_S = O(\mathcal{I}/n + 1)$, and the its distributional error is $\rho = \frac{\varepsilon_S}{2} = O(n^{-(1-\tau)})$. Plus, we can upper-bound ℓ using Fact 35:

$$\ell := D(\pi_S \parallel \eta) = \theta_\mu(\pi_S) = O(n^{-(1-\tau)} \log n).$$

Recall $\tau \in (0, 1)$ is an absolute constant. We apply Lemma 65 to the generalized protocol π_S . As a result, the standard protocol $\eta = \text{standardize}(\pi, \mu)$ has the following properties:

1. η computes f and errs over input distribution with probability $\rho + O(\sqrt{\ell}) = O(\sqrt{n^{-(1-\tau)} \log n}) \leq n^{-\lambda}$
2. η has information cost at most $O(\frac{\mathcal{I}}{n} + 1) + O(\mathcal{H}(\sqrt{\ell}) + \sqrt{\ell} \cdot \log(|\mathcal{X}| \cdot |\mathcal{Y}|)) \leq C \cdot \left(\frac{\mathcal{I}}{n} + \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} + 1\right)$

for some absolute constants $\lambda \in (0, 1)$ and $C > 0$. This completes the proof of Lemma 8.

9 Proof of XOR Lemmas

Recall our main lemma which we have proved

Lemma 8 (Main Technical Lemma). *There exists a universal constant $C > 0$ and $\lambda \in (0, 1)$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$:*

If there exists a (standard) communication protocol π for computing $f^{\oplus n}$ over an input distribution μ^n such that it errs with probability $\frac{1}{10}$ and has information cost \mathcal{I} ,

then there exists a (standard) communication protocol η for computing f over an input distribution μ such that it errs with probability $n^{-\lambda}$ and has information cost at most $C \cdot \left(\frac{\mathcal{I}}{n} + \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} + 1\right)$.

Now we use it to derive our main results of Theorem 2 and Theorem 4. We note that throughout our proofs (including those in the Appendix), we assume that the infima in Definition 21 and Definition 23 are attained by some protocol π . This assumption can be relaxed by instead considering a sequence of protocols $\{\pi_i\}_{i \geq 1}$ with corresponding costs that converge to the infimum.

9.1 XOR Lemma for Distributional Information Cost

Theorem 4 (Strong XOR Lemma for Distributional Information Cost). *There exists a universal constant $\lambda \in (0, 1)$ and $c_3 > 0$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$, any positive integer n , and any input distribution μ over $\mathcal{X} \times \mathcal{Y}$, we have*

$$\text{IC}_{\mu^n}(f^{\oplus n}, 1/10) \geq c_3 n \cdot \left(\text{IC}_{\mu}(f, n^{-\lambda}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right).$$

Proof. By definition of distributional information cost, let π be a protocol for solving $f^{\oplus n}$ over μ^n that errs with probability $1/10$ and has information cost $\mathcal{I} = \text{IC}_{\mu^n}(f^{\oplus n}, 1/10)$. Let η be the protocol for solving f obtained from Lemma 8. We then derive:

$$\begin{aligned} \text{IC}_{\mu}(f, n^{-\lambda}) &= \inf_{\pi; \Pr_{(x,y) \sim \mu}(\pi(x,y) \neq f(x,y)) \leq n^{-\lambda}} \text{IC}(\pi) \\ &\leq \text{IC}(\eta) \quad (\eta \text{ is a protocol satisfying the infimum conditions}) \\ &\leq C \cdot \left(\frac{\mathcal{I}}{n} + \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} + 1 \right). \end{aligned}$$

Rearranging the inequality completes the proofs. \square

9.2 XOR Lemma for Information Complexity

We will need the following lemma to boost the success probability. Its proof will be shown in the appendix.

Lemma 66. *Let g be a $\{0, 1\}$ -valued function, and $\varepsilon' < \varepsilon < (2e)^{-1} - 0.01$. Then, we have $\text{IC}(g, \varepsilon') \leq O\left(\frac{\log(1/\varepsilon')}{\log(1/\varepsilon)}\right) \cdot \text{IC}(g, \varepsilon)$.*

Now we are ready to prove Theorem 2.

Theorem 2 (Strong XOR Lemma for Information Complexity). *There exists a universal constant $\lambda \in (0, 1)$ and $c_1 > 0$ such that for any function $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ and any positive integer n , we have*

$$\mathbf{I}(f^{\oplus n}, 1/10) \geq c_1 n \cdot \left(\mathbf{I}(f, n^{-1}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right).$$

Proof. Let μ be the maximizer of $\text{IC}_{\mu}(f, n^{-\lambda})$ so that $\text{IC}_{\mathcal{D}}(f, n^{-\lambda}) = \text{IC}_{\mu}(f, n^{-\lambda})$. Consider

$$\begin{aligned} \text{IC}(f^{\oplus n}, 1/10) &\geq \text{IC}_{\mathcal{D}}(f^{\oplus n}, 1/10) \geq \text{IC}_{\mu^n}(f^{\oplus n}, 1/10) \\ &\geq C n \cdot \left(\text{IC}_{\mu}(f, n^{-\lambda}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right) \\ &\geq C n \cdot \left(\text{IC}_{\mathcal{D}}(f, n^{-\lambda}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right) \end{aligned}$$

Let us now focus on the quantity $\text{IC}_{\mathcal{D}}(f, n^{-\lambda})$. By Theorem 24, we have $\text{IC}_{\mathcal{D}}(f, n^{-\lambda}) \geq \frac{\text{IC}(f, 2n^{-\lambda})}{2}$. Moreover, we know that $\text{IC}(f, n^{-1}) \leq O(\text{IC}(f, 2n^{-\lambda}))$ via Lemma 66. Combining everything, we have:

$$\text{IC}(f^{\oplus n}, 1/10) \geq c_1 n \cdot \left(\text{IC}(f, n^{-1}) - \frac{\log(|\mathcal{X}| \cdot |\mathcal{Y}|)}{n^\lambda} - 1 \right)$$

for some absolute constant $c_1 > 0$. \square

The following Theorem will be proved in the Appendix.

Theorem 3. *There exists a universal constant $c_2 > 0$ such that for any $\{0, 1\}$ -valued function f and positive integer n , we have*

$$\mathbf{I}(f^{\oplus n}, 1/10) \leq c_2 n \cdot \mathbf{I}(f, n^{-1}).$$

Theorem 2 and Theorem 3 together establish an asymptotically tight relationship (up to vanishing additive losses) between the information complexities of f and $f^{\oplus n}$.

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A Missing Proofs

We will restate and provide the missing proofs from the earlier sections.

Lemma 45 (Coupling Lemma). *Let μ and μ' be distributions over supports \mathcal{A} . There exists a random process such that at the end of the process, we obtain a and a' such that a distributes according to μ , a' distributes according to μ' , and the probability that $a \neq a'$ is at most $O(\|\mu - \mu'\|)$.*

Proof. The random process operates as follows: we interpret the randomness as a sequence of pairs $(a_1, \rho_1), (a_2, \rho_2), \dots$, where each pair (a_i, ρ_i) is drawn uniformly from $\mathcal{A} \times [0, 1]$. We then set a to be a_i for the smallest i such that $\rho_i < \mu(a_i)$, and we set a' to be $a_{i'}$ for the smallest i' such that $\rho_{i'} < \mu'(a_{i'})$. It is straightforward to see that a follows the distribution μ and a' follows the distribution μ' . Thus, the next step is to bound the probability that $a \neq a'$, which is also upper-bounded by the probability that $i \neq i'$.

For any $a \in \text{supp}(\mathcal{A})$ denote an interval $U_a := [0, \max\{\mu(a), \mu'(a)\}]$ and $D_a := [\min\{\mu(a), \mu'(a)\}, \max\{\mu(a), \mu'(a)\}]$. Also denote

$$\mu \cup \mu' := \{(a, U_a) ; a \in \text{supp}(\mathcal{A})\} \quad \text{and} \quad \mu \triangle \mu' := \{(a, D_a) ; a \in \text{supp}(\mathcal{A})\}$$

and their volumes

$$\text{vol}(\mu \cup \mu') := \sum_{a \in \text{supp}(\mathcal{A})} |U_a| \quad \text{and} \quad \text{vol}(\mu \triangle \mu') := \sum_{a \in \text{supp}(\mathcal{A})} |D_a|$$

We say that $(a, \rho) \in \mu \cup \mu'$ if and only if $\rho \in U_a$. Similarly, we say that $(a, \rho) \in \mu \triangle \mu'$ if and only if $\rho \in D_a$. Let j be the smallest index such that $(a_j, \rho_j) \in \mu \cup \mu'$. Then, $a_1 \neq a_2$ occurs if and only if $(a_j, \rho_j) \in \mu \triangle \mu'$. Thus, we have:

$$\begin{aligned} \Pr(a_1 \neq a_2) &= \Pr[(a_j, \rho_j) \in \mu \triangle \mu' \mid (a_j, \rho_j) \in \mu \cup \mu'] \\ &= \frac{\text{vol}(\mu \triangle \mu')}{\text{vol}(\mu \cup \mu')} \\ &= \frac{\sum_{a \in \text{supp}(\mathcal{A})} |D_a|}{\sum_{a \in \text{supp}(\mathcal{A})} |U_a|} \\ &= \frac{\sum_{a \in \text{supp}(\mathcal{A})} |\mu(a) - \mu'(a)|}{\sum_{a \in \text{supp}(\mathcal{A})} \max(\mu(a), \mu'(a))} \\ &\leq 2 \cdot \frac{\|\mu - \mu'\|}{\sum_{a \in \text{supp}(\mathcal{A})} \mu(a) + \mu'(a)} \quad (p + q \leq 2 \cdot \max(p, q)) \\ &= \|\mu - \mu'\|. \end{aligned}$$

This concludes the proof. □

Lemma 37 (Linearity of pointwise- θ -cost). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol with the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$. Let $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ and $\pi_1 = (X_1, Y_1, \mathbf{M}^{(\pi_1)})$ be generalized protocols obtained via decomposing π (recall Definition 25.) Then, for any point (X, Y, \mathbf{M}) , we have*

$$\theta_\mu(\pi @ X, Y, \mathbf{M}) = \theta_{\mu_0}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \theta_{\mu_1}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}).$$

Proof. Throughout this proof, convenience, we will write $\mathbf{M}_{(\pi_0)}$ instead of $\mathbf{M}^{(\pi_0)}$ and $\mathbf{M}_{(\pi_1)}$ instead of $\mathbf{M}^{(\pi_1)}$. Recall that $\pi_0 = (X_0, Y_0, \mathbf{M}_{(\pi_0)})$ where $\mathbf{M}_{(\pi_0)} = (M^0, Y_1 \circ M^1, M^2, \dots, M^r)$. We then can write:

$$\begin{aligned}
& \theta_{\mu_0}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) \\
&= \log \left(\frac{\pi_0(X_0, Y_0, \mathbf{M}_{(\pi_0)})}{\pi_0(\mathbf{M}_{(\pi_0)}^0) \cdot \mu_0(X_0, Y_0) \cdot \prod_{\text{odd } i \geq 1} \pi_0(\mathbf{M}_{(\pi_0)}^i | X_0, \mathbf{M}_{(\pi_0)}^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi_0(\mathbf{M}_{(\pi_0)}^i | Y_0, \mathbf{M}_{(\pi_0)}^{<i})} \right) \\
&= \log \left(\frac{\pi_0(X_0, Y_0, \mathbf{M}_{(\pi_0)}^+ | \mathbf{M}_{(\pi_0)}^0)}{\mu_0(X_0, Y_0) \cdot \pi_0(\mathbf{M}_{(\pi_0)}^1 | X_0, \mathbf{M}_{(\pi_0)}^0) \cdot \prod_{\text{odd } i \geq 3} \pi(M^i | X_0, Y_1, M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right) \\
&= \log \left(\frac{\pi(X_0, Y, M^+ | M^0)}{\mu_0(X_0, Y_0) \cdot \pi(Y_1 M^1 | X_0 M^0) \cdot \prod_{\text{odd } i \geq 3} \pi(M^i | X_0 Y_1 M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right) \\
&= \log \left(\frac{\pi(X_0, Y, M^+ | M^0)}{\mu_0(X_0, Y_0) \cdot \pi(Y_1 | X_0 M^0) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | X_0 Y_1, M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right)
\end{aligned}$$

Recall that $\pi_1 = (X_1, Y_1, \mathbf{M}_{(\pi_1)})$ where $\mathbf{M}_{(\pi_1)} = (M^0 \circ X_0, M^1, M^2, \dots, M^r)$. We then can write:

$$\begin{aligned}
& \theta_{\mu_1}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}) \\
&= \log \left(\frac{\pi_1(X_1, Y_1, \mathbf{M}_{(\pi_1)})}{\pi_1(\mathbf{M}_{(\pi_1)}^0) \cdot \mu_1(X_1, Y_1) \cdot \prod_{\text{odd } i \geq 1} \pi_1(\mathbf{M}_{(\pi_1)}^i | X_1, \mathbf{M}_{(\pi_1)}^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi_1(\mathbf{M}_{(\pi_1)}^i | Y_0, \mathbf{M}_{(\pi_1)}^{<i})} \right) \\
&= \log \left(\frac{\pi_1(X_1, Y_1, \mathbf{M}_{(\pi_1)}^+ | \mathbf{M}_{(\pi_1)}^0)}{\mu_1(X_1, Y_1) \cdot \prod_{\text{odd } i \geq 1} \pi(M_i | X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M_i | X_0 Y_1 M^{<i})} \right) \\
&= \log \left(\frac{\pi(X_1, Y_1, M^+ | X_0 M^0)}{\mu_1(X_1, Y_1) \cdot \prod_{\text{odd } i \geq 1} \pi(M_i | X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M_i | X_0 Y_1 M^{<i})} \right)
\end{aligned}$$

Combining them, we have

$$\begin{aligned}
& \theta_{\mu_0}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \theta_{\mu_1}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}) \\
&= \log \left(\frac{\pi(X_0, Y, M^+ | M^0) \cdot \pi(X_1, Y_1, M^+ | X_0 M^0)}{\mu(X, Y) \cdot \pi(Y_1 | X_0 M^0) \cdot \pi(M^+ | X_0 Y_1 M^0) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right) \\
&= \log \left(\frac{\pi(X_0 Y M^+ | M^0) \cdot \pi(X_1 | X_0 Y_1 M)}{\mu(X, Y) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right)
\end{aligned}$$

while by definition, we have:

$$\theta_{\mu}(\pi @ X, Y, M) = \log \left(\frac{\pi(X, Y, M^+ | M^0)}{\mu(X, Y) \cdot \prod_{\text{odd } i \geq 1} \pi(M^i | X M^{<i}) \cdot \prod_{\text{even } i \geq 2} \pi(M^i | Y M^{<i})} \right).$$

Thus, it suffices to show that

$$\pi(X_0 Y M^+ | M^0) \cdot \pi(X_1 | X_0 Y_1 M) = \pi(X, Y, M^+ | M^0)$$

which is equivalent to

$$X_1 \perp Y_0 | X_0 Y_1 M$$

which is true due to the partial rectangle property of π via Proposition 31. □

Lemma 40 (Linearity of pointwise- γ -cost). *Let $\pi = (X, Y, \mathbf{M})$ be a generalized protocol with the partial rectangle property with respect to $\mu = (\mu_0, \mu_1)$. Let $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ and $\pi_1 = (X_1, Y_1, \mathbf{M}^{(\pi_1)})$ be generalized protocols obtained via decomposing π (recall Definition 25.) Then, for any point (X, Y, \mathbf{M}) , we have*

$$\begin{aligned}\gamma_{\mu, A}(\pi @ X, Y, \mathbf{M}) &= \gamma_{\mu_0, A}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \gamma_{\mu_1, A}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}) \\ \gamma_{\mu, B}(\pi @ X, Y, \mathbf{M}) &= \gamma_{\mu_0, B}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \gamma_{\mu_1, B}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}).\end{aligned}$$

Proof. Recall that $\pi_0 = (X_0, Y_0, \mathbf{M}^{(\pi_0)})$ where $\mathbf{M}^{(\pi_0)} = (M^0, Y_1 \circ M^1, M^2, \dots, M^r)$ and $\pi_1 = (X_1, Y_1, \mathbf{M}^{(\pi_1)})$ where $\mathbf{M}^{(\pi_1)} = (M^0 \circ X_0, M^1, M^2, \dots, M^r)$. We then can write:

$$\begin{aligned}& \delta_{\mu_0, A}(\pi_0 @ X_0, Y_0, \mathbf{M}^{(\pi_0)}) + \delta_{\mu_1, A}(\pi_1 @ X_1, Y_1, \mathbf{M}^{(\pi_1)}) \\ &= \log \frac{\pi_0(X_0 | Y_0 \mathbf{M}^{(\pi_0)})}{\mu_0(X_0 | Y_0)} + \log \frac{\pi_1(X_1 | Y_1 \mathbf{M}^{(\pi_1)})}{\mu_1(X_1 | Y_1)} \\ &= \log \frac{\pi(X_0 | Y \mathbf{M})}{\mu_0(X_0 | Y_0)} + \log \frac{\pi(X_1 | X_0 Y_1 \mathbf{M})}{\mu_1(X_1 | Y_1)} \\ &= \log \frac{\pi(X_0 | Y \mathbf{M})}{\mu_0(X_0 | Y_0)} + \log \frac{\pi(X_1 | X_0 Y \mathbf{M})}{\mu_1(X_1 | Y_1)} \quad (X_1 \perp Y_0 | X_0 Y_1 \mathbf{M} \text{ via Proposition 31}) \\ &= \log \frac{\pi(X | Y \mathbf{M})}{\mu(X, Y)} \\ &= \delta_{\mu, A}(\pi @ X, Y, \mathbf{M}).\end{aligned}$$

The same proof applies for Bob's cost. □

Lemma 56. *Let $c > 0$ be an arbitrary constant. For each $S \in \{0, 1\}^{\leq m}$, let $q_S \in \mathbb{R}_{\geq 0}$ satisfying the following inequality:*

$$q_{S0} + q_{S1} \leq \frac{1}{p_S} \cdot (q_S + c \cdot \mathcal{H}(p_S)). \quad (2)$$

Then, we have

$$\sum_{|S|=m} \chi_S q_S \leq q_\emptyset + O(n^\tau \log n).$$

Proof. For any $k \in \{0, \dots, m\}$, denote $\lambda_k = \sum_{|S|=k} \chi_S q_S$. Trivially, $\lambda_0 = q_\emptyset$. From (2), multiplying both sides by $\chi_{S0} = \chi_{S1} = \chi_S \cdot p_S$, we have:

$$\chi_{S0} q_{S0} + \chi_{S1} q_{S1} \leq \chi_S q_S + c \cdot \chi_S \cdot \mathcal{H}(p_S) \leq \chi_S q_S + c \cdot \mathcal{H}(p_S).$$

Summing it over $|S| = k$ yields:

$$\lambda_{k+1} \leq \lambda_k + c \cdot \sum_{|S|=k} \mathcal{H}(p_S),$$

thus inductively, we will have

$$\sum_{|S|=m} \chi_S q_S = \lambda_m \leq q_\emptyset + c \cdot \sum_{k=0}^{m-1} \sum_{|S|=k} \mathcal{H}(p_S).$$

Therefore, it remains to show that $\sum_{k=0}^{m-1} \sum_{|S|=k} \mathcal{H}(p_S) = O(n^\lambda \log n)$.

Fix any k . We shall upper bound $\sum_{|S|=k} \mathcal{H}(p_S)$. Recall that $\alpha = 1 - 2\varepsilon$. By Lemma 51, we have

$$p_S \geq \frac{1 - \varepsilon_S - \alpha}{\mathbb{E}(Z_S | W_S) - \alpha} \geq \frac{1 - \varepsilon_S - \alpha}{1 - \alpha} = 1 - \frac{\varepsilon_S}{1 - \alpha} = 1 - \frac{\varepsilon_S}{2\varepsilon}.$$

Using Corollary 53, we have:

$$2^{-k} \cdot \sum_{|S|=k} (1 - p_S) \leq 2^{-k} \sum_{|S|=k} \frac{\varepsilon_S}{2\varepsilon} \leq 2^{-(1-\tau)k-1}$$

which is also at most $\frac{1}{2}$. Furthermore, we can derive:

$$\begin{aligned} \sum_{|S|=k} \mathcal{H}(p_S) &= \sum_{|S|=k} \mathcal{H}(1 - p_S) \\ &\leq 2^k \cdot \mathcal{H} \left(2^{-k} \cdot \sum_{|S|=k} (1 - p_S) \right) && (\mathcal{H} \text{ is concave}) \\ &\leq 2^k \cdot \mathcal{H} \left(2^{-(1-\tau)k-1} \right) && (\mathcal{H} \text{ is increasing in } (0, 1/2]) \\ &\leq 2^k \cdot 2 \cdot \mathcal{H} \left(2^{-(1-\tau)k-1} \right) && (\mathcal{H}(x) \leq 2 \cdot p \log \frac{1}{p} \text{ for } p \in (0, 1/2]) \\ &= 2^{\tau k} \cdot ((1 - \tau)k + 1) \\ &\leq 2^{\tau k} \cdot (k + 1). \end{aligned}$$

Summing this over $0 \leq k \leq m - 1 = \log_2 n - 1$ yields:

$$\sum_{k=0}^{m-1} \sum_{|S|=k} \mathcal{H}(p_S) = O(n^\tau \log n)$$

as wished. \square

Lemma 57. *Let $c, c' > 0$ be arbitrary constants. For each $S \in \{0, 1\}^{\leq m}$, let $q_S \in \mathbb{R}_{\geq 0}$ satisfying the following inequality:*

$$q_{S0} + q_{S1} \leq \frac{1}{p_S} \cdot (q_S + c \cdot \mathcal{H}(p_S)) + c' \quad (3)$$

Then, we have

$$\sum_{|S|=m} \chi_S q_S \leq q_\emptyset + O(n).$$

Proof. The proof proceeds similar to that of Lemma 56. For any $k \in \{0, \dots, m\}$, denote $\lambda_k = \sum_{|S|=k} \chi_S q_S$. Trivially, $\lambda_0 = q_\emptyset$. From (3), multiplying both sides by $\chi_{S0} = \chi_{S1} = \chi_S \cdot p_S$, we have:

$$\begin{aligned} \chi_{S0} q_{S0} + \chi_{S1} q_{S1} &\leq \chi_S q_S + c \cdot \chi_S \cdot \mathcal{H}(p_S) + c' \cdot \chi_S p_S \\ &\leq \chi_S q_S + c \cdot \mathcal{H}(p_S) + c'. \end{aligned}$$

Summing it over $|S| = k$ yields:

$$\lambda_{k+1} \leq \lambda_k + c \cdot \sum_{|S|=k} \mathcal{H}(p_S) + c' \cdot 2^k,$$

thus inductively, we will have

$$\sum_{|S|=m} \chi_S q_S = \lambda_m \leq q_\emptyset + c \cdot \sum_{k=0}^{m-1} \sum_{|S|=k} \mathcal{H}(p_S) + c' \cdot \sum_{k=0}^{m-1} 2^k.$$

Recall from the proof of Lemma 56 that $\sum_{k=0}^{m-1} \sum_{|S|=k} \mathcal{H}(p_S) = O(n^\tau \log n)$. Furthermore, we know that $\sum_{k=0}^{m-1} 2^k < 2^m = n$. Therefore, we have

$$\sum_{|S|=m} \chi_S q_S \leq q_\emptyset + O(n)$$

as wished. \square

Lemma 66. *Let g be a $\{0, 1\}$ -valued function, and $\varepsilon' < \varepsilon < (2e)^{-1} - 0.01$. Then, we have $\text{IC}(g, \varepsilon') \leq O\left(\frac{\log(1/\varepsilon')}{\log(1/\varepsilon)}\right) \cdot \text{IC}(g, \varepsilon)$.*

Proof. Recall via Definition 23 that

$$\text{IC}(g, \varepsilon) = \inf_{\pi \text{ that errs w.p. at most } \varepsilon \text{ on any inputs}} \max_{\mu} \text{IC}_{\mu}(\pi).$$

Let π be the minimizer of $\text{IC}(g, \varepsilon)$ for which $\text{IC}(g, \varepsilon) = \max_{\mu} \text{IC}_{\mu}(\pi)$. Consider the following protocol π' for solving g over any input pair (X, Y) :

- (1) Let $T = O\left(\frac{\log(1/\varepsilon')}{\log(1/\varepsilon)}\right)$ be such that $(2e\varepsilon)^{T/2} < \varepsilon'/100$.
- (2) Alice and Bob runs T independent copies of π , and output the majority answer among those copies. Denote the T independent transcripts by (M_1, \dots, M_T)

The probability of π' being incorrect is:

$$\sum_{i=T/2}^T \binom{T}{i} \cdot \varepsilon^i (1-\varepsilon)^{T-i} \leq \sum_{i=T/2}^T \left(\frac{eT}{i}\right)^i \cdot \varepsilon^i \leq \sum_{i=T/2}^T (2e\varepsilon)^i \leq \frac{(2e\varepsilon)^{T/2}}{1-2e\varepsilon} \leq 100 \cdot (2e\varepsilon)^{T/2} < \varepsilon'.$$

Thus, by definition, we have $\text{IC}(g, \varepsilon') \leq \max_{\mu'} \text{IC}_{\mu'}(\pi')$. Let μ' be a maximizer so that

$$\text{IC}(g, \varepsilon') \leq \text{IC}_{\mu'}(\pi'). \tag{4}$$

Consider

$$\text{IC}_{\mu'}(\pi') = I(M_1 \dots M_T : X | Y) + I(M_1 \dots M_T : Y | X)$$

and

$$\begin{aligned} I(M_1 \dots M_T : X | Y) &= H(M_1 \dots M_T | Y) - H(M_1 \dots M_T | XY) \\ &\leq \sum_{i \in [T]} H(M_i | Y) - \sum_{i \in [T]} H(M_i | XY) \\ &= \sum_{i \in [T]} I(M_i : X | Y) \end{aligned}$$

where to obtain the inequality, the first term follows subadditivity of entropy, and the second term follows the fact that the T transcripts $\{M_1, \dots, M_T\}$ are mutually independent conditioned on (X, Y) . Combining with its symmetric term, we have

$$\begin{aligned} \text{IC}_{\mu'}(\pi') &\leq \sum_{i \in [T]} I(M_i : X | Y) + I(M_i : Y | X) = T \cdot \text{IC}_{\mu'}(\pi) \leq T \cdot \max_{\mu} \text{IC}_{\mu}(\pi) \\ &= T \cdot \text{IC}(g, \varepsilon). \end{aligned}$$

Together with (4) and the fact that $T = O\left(\frac{\log(1/\varepsilon')}{\log(1/\varepsilon)}\right)$ completes the proof. \square

Theorem 3. *There exists a universal constant $c_2 > 0$ such that for any $\{0, 1\}$ -valued function f and positive integer n , we have*

$$\mathbf{I}(f^{\oplus n}, 1/10) \leq c_2 n \cdot \mathbf{I}(f, n^{-1}).$$

Proof. It suffices to show that $\text{IC}(f^{\oplus n}, 1/10) \leq n \cdot \text{IC}(f, (10n)^{-1})$ since $\text{IC}(f, (10n)^{-1}) \leq c_2 \cdot \text{IC}(f, n^{-1})$ for some constant $c_2 > 0$ follows from Lemma 66. Recall via Definition 23 that for any function g and $\varepsilon \in (0, 1)$,

$$\text{IC}(g, \varepsilon) = \inf_{\pi \text{ that errs w.p. at most } \varepsilon \text{ on any inputs}} \max_{\mu} \text{IC}_{\mu}(\pi).$$

Let π be a minimizer protocol so that $\text{IC}(f, (10n)^{-1}) = \max_{\mu} \text{IC}_{\mu}(\pi)$. Let π' be the following protocol for solving $f^{\oplus n}$ over inputs $(X_1, \dots, X_n, Y_1, \dots, Y_n)$.

- (1) For each $i \in [n]$, the players run π (using fresh randomness) over an input pair (X_i, Y_i) to compute their belief of $f(X_i, Y_i)$. Denote this bit by b_i and denote the transcript by M_i .
- (2) Players output $b_1 \oplus \dots \oplus b_n$.

We first argue that π' errs with probability at most $1/10$ on any input $(X_1, \dots, X_n, Y_1, \dots, Y_n)$. By the error guarantees of π , each b_i incorrectly computes $f(X_i, Y_i)$ with probability at most $(10n)^{-1}$. Via Union Bounds, all b_i is correct simultaneously with probability at least $9/10$, resulting in their xor being correct. Therefore, the error of π' is at most $1/10$.

Let μ be the distribution over $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ that maximizes $\text{IC}_{\mu}(\pi')$. By definition, we have

$$\text{IC}(f^{\oplus n}, 1/10) \leq \text{IC}_{\mu}(\pi').$$

Now we expand

$$\text{IC}_{\mu}(\pi') = I(M_1 \dots M_n : X_1 \dots X_n \mid Y_1 \dots Y_n) + I(M_1 \dots M_n : Y_1 \dots Y_n \mid X_1 \dots X_n).$$

Furthermore, consider

$$\begin{aligned} & I(M_1 \dots M_n : X_1 \dots X_n \mid Y_1 \dots Y_n) \\ &= H(M_1 \dots M_n \mid Y_1 \dots Y_n) - H(M_1 \dots M_n \mid X_1 \dots X_n Y_1 \dots Y_n) \end{aligned}$$

For the first term, we bound:

$$\begin{aligned} H(M_1 \dots M_n \mid Y_1 \dots Y_n) &= \sum_{i \in [n]} H(M_i \mid Y_1 \dots Y_n M_{<i}) && \text{(Chain Rule)} \\ &\leq \sum_{i \in [n]} H(M_i \mid Y_i). \end{aligned}$$

For the second term, we recall that each M_i only depends on (X_i, Y_i) ; thus, $(X_1, \dots, X_n, Y_1, \dots, Y_n)$, the n transcripts (M_1, \dots, M_n) are mutually independence. Therefore, we can bound:

$$\begin{aligned} H(M_1 \dots M_n \mid X_1 \dots X_n Y_1 \dots Y_n) &= \sum_{i \in [n]} H(M_i \mid X_1 \dots X_n Y_1 \dots Y_n) \\ &= \sum_{i \in [n]} H(M_i \mid X_i Y_i). \end{aligned}$$

Hence, we shall have

$$\begin{aligned} I(M_1 \dots M_n : X_1 \dots X_n \mid Y_1 \dots Y_n) &\leq \sum_{i \in [n]} H(M_i \mid Y_i) - H(M_i \mid X_i Y_i) \\ &= \sum_{i \in [n]} I(M_i : X_i \mid Y_i). \end{aligned}$$

Thus, we now have

$$\mathsf{IC}_\mu(\pi') \leq \sum_{i \in [n]} I(M_i : X_i | Y_i) + I(M_i : Y_i | X_i)$$

Here for each $i \in [n]$, M_i is the transcript of running π over (X_i, Y_i) . Therefore, we have

$$\begin{aligned} I(M_i : X_i | Y_i) + I(M_i : Y_i | X_i) &= \mathsf{IC}_{\mu(X_i, Y_i)}(\pi) \leq \max_{\mu'} \mathsf{IC}_{\mu'}(\pi) \\ &= \mathsf{IC}(f, (10n)^{-1}). \end{aligned}$$

Combining everything, we shall have:

$$\begin{aligned} \mathsf{IC}(f^{\oplus n}, 1/10) &\leq \mathsf{IC}_\mu(\pi') \\ &\leq \sum_{i \in [n]} I(M_i : X_i | Y_i) + I(M_i : Y_i | X_i) \\ &\leq \sum_{i \in [n]} \mathsf{IC}(f, (10n)^{-1}) \\ &= n \cdot \mathsf{IC}(f, (10n)^{-1}) \end{aligned}$$

as wished. □