arXiv:2411.13111v1 [math.OC] 20 Nov 2024

Optimal investment problem of a renewal risk model with generalized erlang distributed interarrival times *

Linlin Tian^{a^{\dagger}} Bohan Li^{b^{\ddagger}} Guoqing Li^{a§}

a. School of Mathematics and Statistics, Donghua University, Shanghai, P.R. China 201620

b. Center for Financial Engineering, Soochow University, Suzhou, Jiangsu, P.R. China 215006

Abstract This paper explores the optimal investment problem of a renewal risk model with generalized Erlang distributed interarrival times. We assume that the phases of the interarrival time can be observed. The price of the risky asset is driven by the CEV model and the insurer aims to maximize the exponential utility of the terminal wealth by asset allocation. By solving the corresponding Hamilton-Jacobi-Bellman equation, when the interest rate is zero, the concavity of the solution as well as the the explicit expression of the investment policy is shown. When the interest rate is not zero, the explicit expression of the optimal investment strategy is shown, the structure as well as the concavity of the value function is proved.

Keywords: Exponential utility; Renewal process; Stochastic optimal control; Hamilton-Jacobi-Bellman equation

1 Introduction

The optimal investment problem of a general insurer has been studied under various settings since the work of [11]. With the exponential utility, [16] consider the optimal investment strategy for an insurer with jump-diffusion surplus process when the risky asset follow a Geometric Brownian motion in which the closed form of the optimal investment strategy is shown. [14] extends the

^{*}This work was supported by the National Natural Science Foundation of China (No. 12201104, No. 12171257, No. 12201174).

^{*}E :11:1: /:

[†]E-mail:linlin.tian@dhu.edu.cn [‡]Corresponding author, E-mail:bhli@suda.edu.cn

[§]E-mail: 2212267@mail.dhu.edu.cn

results of [16] to the case of multiple risky assets. The optimal investment for an insurer with cointegrated assets with CRRA utility was studied in [6]. [8] studied the excess-of-loss reinsurance and investment strategies under a constant elasticity of variance (CEV) model of the insurer. [15] study the robust optimal portfolio and reinsurance problem under a CEV model.

All above mentioned optimization problems are investigated in the Markovian framework. In the compound Poisson model, the interclaim times are exponential distributed. But exponential distributed interclaim times have no memory about the time elapsed since the last claim. To overcome such drawback, probabilist brought out the renewal process to characterize the surplus of the insurance company. For example, for the Erlang(n) interclaim times, [1] and [2] calculate the moment-generating function of the discounted dividends of horizontal barrier strategies and show that in general horizontal barrier strategies is not the optimal dividend strategy. [10] show that the optimal dividend strategy is of phase-wise barrier strategy for the Erlang(n) interclaim times. Later, [3, 4] investigate the optimal investment and dividend problem of the Sparre Andersen model in the framework of viscosity solution. One can refer [5, 13] for more control studies about renewal surplus process. As far as we know, the explicit solution of optimal utility of renewal process is not easy to find, thus, the paper explores the optimal investment problem of Erlang(n) distributed interclaim renewal process.

Now we describe the renewal claim process formally. Let $\{J_t\}$ be a homogenous Markov chain on the state space $\{1, 2, \dots, n\}$ with an intensity matrix of the form

$$\left(\begin{array}{ccccc} -\lambda_1 & \lambda_1 & 0 & \cdots & 0\\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \lambda_n & 0 & 0 & \cdots & -\lambda_n \end{array}\right),$$

i.e., the process moves through $1 \to 2 \to \cdots \to n \to 1 \to \cdots$ and stays in state $i \in \{1, 2, \cdots, n\}$ for an exponential time with parameter λ_i — this is often referred to as an exponential clock. When the exponential clock rings, J_t jumps to the next state. Specifically, when the current state is n, after staying an exponential time with parameter λ_n , J_t will jump to state 1 and a claim occurs. We also assume that the phases of J_t can be observed. It is obvious that if n = 1, then the surplus process degenerates to the compound Poisson process.

As the Markov chain J_t has n states, the Hamilton-Jacobi-Bellman (HJB) equation forms a system of n-dimensional coupled equation. We divide the optimization problem into two cases: the first where the interest rate is zero, and the second where the interest rate is non-zero. The solutions to the Hamilton-Jacobi-Bellman (HJB) equation in these two cases take different mathematical forms. The main challenge lies in demonstrating that the solution to each HJB equation is concave. In the first case, when the interest rate is zero, we prove the concavity of the solution using the Laplace transform of the Markov chain with killing. Additionally, we derive the explicit expressions for both the value function and the optimal investment strategy. In the second case, where the interest rate is non-zero, it is more difficult to express the solution to the HJB equation explicitly. However, we establish the existence of a solution, and rigorously prove the explicit form and concavity of the value function by decoupling the system of simultaneous equations and applying the Banach fixed point theorem.

One might compare the Erlang(n) renewal model with the regime-switching model where various optimization problems have been addressed within the regime-switching framework. For example, [7] explores an optimal investment problem of an insurance company aiming to maximize the minimal expected exponential utility with regime switching. Later, [9] investigates an optimal investment problem of an insurer with risk constraint and regime-switching. Other optimization problems in this area include [12], which explores the optimal dividend problem for an insurance company in the presence of regime shifts, and [18], which studies optimal consumption, investment, and insurance policies under regime switching.

A key distinction between our model and the regime-switching is: in the regime-switching model, external regime influence variables such as the interest rate, claim intensity, return rate, and volatility rate and so on. However, regime changes typically do not lead to a lump sum reduction in wealth. In contrast, in our Erlang(n) model, at the end of phase n, not only the Markov chain enters a new phase (phase 1), but also a claim occurs, resulting in a lump sum decrease in wealth. Given these differences, we focus on the optimal investment problem in the context of Erlang(n) interclaim times.

The main contributions of this work are twofold: 1. The optimal investment problems for the classical compound Poisson claim process are well-established. The Erlang-distributed interclaim model extends the classical compound Poisson model. The explicit solutions derived in our study provide valuable insights for other optimization problems involving renewal processes. 2. To demonstrate the concavity of the solution to the HJB equation, we employ the Laplace transform of the Markov chain, decouple the system of simultaneous equations, and apply the Banach fixed point theorem, showcasing the novelty of the mathematical methodology.

The structure of the paper is organized as follows. Section 2 introduces the surplus process of the insurance company. The goal is to maximize the exponential utility of terminal wealth by investment. The problem is then divided into two cases. Section 3 studies the case where the interest rate is zero, presenting the explicit solution for both the optimal value function and the optimal strategy. In Section 4, we explore the case where the interest rate is non-zero. In this section, we provide the explicit expression for the optimal investment policy and apply the decoupling equation technique, along with the Banach fixed point theorem, to demonstrate the concavity of the solution to the HJB equation. Section 5 analyzes the sensitivity of various parameters on the optimal policy and value function. Finally, Section 6 concludes the paper, while some detailed proofs are provided in the Appendix.

2 Modelling

We work on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ which satisfies the usual condition. Let T > 0be a finite time horizon and \mathscr{F}_t stands the information available before time $t \in [0, T]$. The surplus of the insurer follows

$$dC_t = cdt - d\sum_{i=1}^{N_t} Y_i \tag{2.1}$$

where c > 0 is the premium rate, N_t is a renewal counting process representing the number of claims before time t, $\{Y_i\}_{i=1}^{\infty}$ are independent and identically distributed positive random variables and Y_i represents the size of the *i*-th claim. The interclaim times are independent and follow a generalized Erlang(n) distribution. As we introduced in the last section, the state of the Markov chain can be observed. We also assume that the claim size $\{Y_i\}_{i=1}^{\infty}$ are i.i.d random variables which are independent with the markov chain $\{J_t\}$.

The insurer is allowed to invest in a financial market consisting two assets, one risk-free asset and one risky asset. The price process $S_0(t)$ of the risk-free asset follows

$$dS_0(t) = rS_0(t)dt,$$

where r > 0 is the risk-free interest rate. We assume that the price process of the risky asset is driven by the CEV model

$$dS(t) = S(t)(\mu dt + \sigma S(t)^{\beta} dW(t)),$$

where $\mu > r$ is the expected instantaneous return rate of the risky asset; k > 0 is a constant; $\beta \ge 0$ represents the elasticity parameter; W(t) is a standard Brownian motion which is independent of the compound renewal process. When $\beta = 0$, a CEV model degenerates to a geometric Brownian motion. The investment strategy is denoted by $\{a_t\}_{0 \le t \le T}$, where $a_t \in \mathbb{R}$ denotes the total amount of money invested in the risky asset. Under the strategy a, the surplus process of the insurance company follows

$$dX_t^a = (rX_t + (\mu - r)a_t + c)dt + \sigma S_t^\beta a_t dW_t - d\sum_{i=1}^{N_t} Y_i$$
(2.2)

We call a strategy $\{a_t\}_{0 \le t \le T}$ admissible if for any $t \in [0, T]$, a_t is \mathscr{F}_t progressively measurable, $\mathbb{E}[\int_0^{+\infty} a_t^2 S(t)^{2\beta} dt] < +\infty$ and the equation (2.2) admits a unique strong solution. Denote \mathscr{U}_{ad} the set of all admissible strategies.

We consider the optimal investment problem which aims to maximize exponential utility of the terminal wealth, mathematically speaking, the utility function of the insurer is defined as

$$U(x) = -\frac{1}{m}e^{-mx},$$

where m > 0 is a constant. Such an utility function plays an important role in mathematical finance and actuarial science. The constant m is called the absolute risk aversion parameter. For any investment strategy $\{a_t\}_{0 \le t \le T} \in \mathscr{U}_{ad}$ and any initial state (s, x, i), define the utility of the strategy a as

$$J(t, x, s, i; a) = \mathbb{E}\left[U(X_T^a)|X_t = x, S_t = s, J_t = i\right].$$
(2.3)

The value function is defined as

$$V(t, x, s, i) = \sup_{\{a\} \in \mathscr{U}_{ad}} J(t, x, s, i; a),$$
(2.4)

The aim of our paper is to find the optimal policy $a^* \in \mathscr{U}_{ad}$ so that $J(t, x, s, i; a^*) = V(t, x, s, i)$. By dynamic programming principle, we derive the HJB equation of the optimization problem (2.4) which is an *n*-dimensional coupled equation:

$$\begin{aligned} v_t(t,x,s,i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t,x,s,i)(c+a(\mu-r)+rx) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t,x,s,i) \right. \\ \left. + \sigma^2 a s^{2\beta+1} v_{xs}(t,x,s,i) \right\} + \mu s v_s(t,x,s,i) + \frac{1}{2}\sigma^2 s^{2\beta+2} v_{ss}(t,x,s,i) \\ \left. + \lambda_i (v(t,x,s,i+1) - v(t,x,s,i)) = 0, \quad i = 1, 2, \cdots, n-1; \\ v_t(t,x,s,i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t,x,s,i)(c+a(\mu-r)+rx) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t,x,s,i) \right. \\ \left. + \sigma^2 a s^{2\beta+1} v_{xs}(t,x,s,i) \right\} + \mu s v_s(t,x,s,n) + \frac{1}{2}\sigma^2 s^{2\beta+2} v_{ss}(t,x,s,i) \\ \left. + \lambda_n (\mathbb{E}[v(t,x-Y,s,1)] - v(t,x,s,i)) = 0, \quad i = n, \end{aligned}$$

$$(2.5)$$

with boundary condition

$$v(T, x, s, i) = -\frac{1}{m}e^{-mx}, \quad i = 1, 2, 3, \dots, n.$$
 (2.6)

We treat the case of r = 0 and $r \neq 0$ differently since there is an explicit solution for the case of r = 0. There is no explicit expression for the case of $r \neq 0$ but the structure of the solution can be expressed. The next section deals with the case of r = 0 first.

3 When the interest rate is 0

If the interest rate is 0, then equation (2.5) degenerates to the following equations:

$$\begin{aligned} v_t(t, x, s, i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t, x, s, i)(c + a\mu) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t, x, s, i) \right. \\ &+ \sigma^2 a s^{2\beta + 1} v_{xs}(t, x, s, i) \right\} + \mu s v_s(t, x, s, i) + \frac{1}{2}\sigma^2 s^{2\beta + 2} v_{ss}(t, x, s, i) \\ &+ \lambda_i (v(t, x, s, i + 1) - v(t, x, s, i)) = 0, \quad i = 1, 2, \cdots, n - 1; \\ v_t(t, x, s, i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t, x, s, i)(c + a\mu) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t, x, s, i) \right. \\ &+ \sigma^2 a s^{2\beta + 1} v_{xs}(t, x, s, i) \right\} + \mu s v_s(t, x, s, i) + \frac{1}{2}\sigma^2 s^{2\beta + 2} v_{ss}(t, x, s, i) \\ &+ \lambda_n (\mathbb{E}[v(t, x - Y, s, 1)] - v(t, x, s, i)) = 0, \quad i = n. \end{aligned}$$

Notice that if there exists a concave solution for (3.1), for each *i*, the maximizer of (3.1) $a^*(t, x, s, i) = -\frac{\mu v_x(t, x, s, i) + \sigma^2 s^{2\beta+1} v_{xs}(t, x, s, i)}{\sigma^2 s^{2\beta} v_{xx}(t, x, s, i)}$. In what follows, we look for a continuously differentiable concave solution for (3.1). We conjecture that the solution of (3.1) takes the form of

$$v(t, x, s, i) = -\frac{1}{m} \exp\left\{-mx + \frac{\mu^2}{2\sigma^2}(t-T)s^{-2\beta}\right\} \psi_i(t), i = 1, 2, \dots, n,$$
(3.2)

where $\psi_i(t), i = 1, 2, \dots, n$ are some unknown functions which will be determined later. After direct calculations,

$$v_{t} = v(t, x, s, i) \frac{\mu^{2}}{2\sigma^{2}} s^{-2\beta} - \frac{1}{m} \exp\left\{-mx + \frac{\mu^{2}}{2\sigma^{2}}(t-T)s^{-2\beta}\right\} \psi'(t),$$

$$v_{x} = -mv(t, x, s, i), v_{xx} = m^{2}v(t, x, s, i)$$

$$v_{s} = \beta \frac{\mu^{2}(T-t)}{\sigma^{2}} s^{-2\beta-1}v(t, x, s, i),$$

$$v_{ss} = \left[\frac{\beta^{2}\mu^{4}(T-t)^{2}}{\sigma^{4}} s^{-4\beta-2} - \frac{\beta\mu^{2}(T-t)}{\sigma^{2}}(2\beta+1)s^{-2\beta-2}\right]v(t, x, s, i),$$

$$v_{xs} = -m\beta \frac{\mu^{2}(T-t)}{\sigma^{2}} s^{-2\beta-1}v(t, x, s, i).$$
(3.3)

Substituting

$$a^{*}(t, x, s, i) = -\frac{\mu v_{x}(t, x, s, i) + \sigma^{2} s^{2\beta+1} v_{xs}(t, x, s, i)}{\sigma^{2} s^{2\beta} v_{xx}(t, x, s, i)}$$
(3.4)

and (3.3) into (3.1) and eliminating same terms show that $\{\psi_i\}_{i=1}^n$ satisfies

$$\begin{cases} \psi_1'(t) - (cm + \frac{(2\beta+1)\mu^2\beta(T-t)}{2} + \lambda_1)\psi_1(t) + \lambda_1\psi_2(t) = 0, \\ \psi_2'(t) - (cm + \frac{(2\beta+1)\mu^2\beta(T-t)}{2} + \lambda_2)\psi_2(t) + \lambda_2\psi_3(t) = 0, \\ \cdots \\ \psi_{n-1}'(t) - (cm + \frac{(2\beta+1)\mu^2\beta(T-t)}{2} + \lambda_{n-1})\psi_{n-1}(t) + \lambda_{n-1}\psi_n(t) = 0, \\ \psi_n'(t) - (cm + \frac{(2\beta+1)\mu^2\beta(T-t)}{2} + \lambda_n)\psi_n(t) + \lambda_n\psi_1(t)\mathbb{E}e^{mY} = 0. \end{cases}$$
(3.5)

Denote $L(t) := \int_0^t (cm + \frac{(2\beta+1)\mu^2\beta(T-u)}{2}) du$. Multiplying $e^{-L(t)}$ on both sides of (3.5) gives

$$\begin{cases} \varphi_1'(t) - \lambda_1 \varphi_1(t) + \lambda_1 \varphi_2(t) = 0, \\ \varphi_2'(t) - \lambda_2 \varphi_2(t) + \lambda_2 C_3(t) = 0, \\ \cdots \\ \varphi_{n-1}'(t) - \lambda_{n-1} \varphi_{n-1}(t) + \lambda_{n-1} \varphi_n(t) = 0, \\ \varphi_n'(t) - \lambda_n \varphi_n(t) + \lambda_n \mathbb{E}(e^{mY}) \varphi_1(t) = 0, \end{cases}$$

or equivalently,

$$\begin{pmatrix} \varphi_1'(t) \\ \varphi_2'(t) \\ \vdots \\ \varphi_{n-1}'(t) \\ \varphi_n'(t) \end{pmatrix} = \hat{Q} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_{n-1}(t) \\ \varphi_n(t) \end{pmatrix}, \qquad (3.6)$$

where

$$\hat{Q} = \begin{pmatrix} \lambda_1 & -\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & -\lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & -\lambda_{n-1} \\ -\lambda_n \mathbb{E}(e^{mY}) & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

The boundary condition of $\varphi_i(t)$ is

$$\varphi_i(T) = e^{-L(T)} > 0.$$
 (3.7)

As we can see, every element of the matrix \hat{Q} is constant. Combining the boundary condition (3.7), the explicit solution of $\varphi_i(t)$ can be derived.

$$\begin{pmatrix} \varphi_{1}(t) \\ \varphi_{2}(t) \\ \vdots \\ \varphi_{n-1}(t) \\ \varphi_{n}(t) \end{pmatrix} = e^{\hat{Q}(t-T)} \begin{pmatrix} \varphi_{1}(T) \\ \varphi_{2}(T) \\ \vdots \\ \varphi_{n-1}(T) \\ \varphi_{n}(T) \end{pmatrix} = e^{\hat{Q}(t-T)} \begin{pmatrix} e^{-L(T)} \\ e^{-L(T)} \\ \vdots \\ e^{-L(T)} \\ e^{-L(T)} \\ e^{-L(T)} \end{pmatrix},$$
(3.8)

Now we show that the solution $\varphi_i(t), i = 1, 2 \cdots, n$, are non-negative. To see this, we only need to show that every element of the matrix $e^{\hat{Q}(t-T)}$ is non-negative.

Lemma 3.1. Every element of the matrix $e^{\hat{Q}(t-T)}$ is non-negative.

Actually, the matrix $e^{\hat{Q}(t-T)}$ can be seen as the transition matrix of a Markov chain with killing. Thus, the non-negativity of every element is undoubtable. We leave the detailed proof in the Appendix for the readability of the whole context of the paper.

Lemma 3.2. The function v(t, x, s, i) is concave about x.

Proof. From above calculations, we can notice that the concavity of v(t, x, s, i) is equivalent to the non-negativity of $\{\varphi_i(t)\}_{i=1}^n$. By Lemma 3.1, we get that every element of the matrix $e^{\hat{Q}(T-t)}$ is nonnegative. The proof is complete.

Until now, we solve a continuously differentiable concave solution for the HJB equation, i.e.,

$$v(t, x, s, i) = -\frac{1}{m} \exp\left\{-mx + \frac{\mu^2}{2\sigma^2}(t-T)s^{-2\beta}\right\} \psi_i(t), i = 1, 2, \dots, n,$$
(3.9)

where $\psi_i(t) = \varphi_i(t)e^{L(t)}$ and $\varphi_i(t)$ is given by (3.8).

Now we formulate a verification theorem to show that under suitable conditions, the solution of the HJB equation (2.6)-(3.1) is indeed the optimal value function when the interest rate is 0.

Theorem 3.3. (verification theorem) Denote $H_1(u) = \frac{\mu + \mu^2 \beta(T-t)}{\sigma^2 s^{2\beta} m}$, if $-8H_1(u) + 16\frac{\sigma^2}{\mu^2}H_1^2(u) < \frac{\mu^2}{\sigma^2}$ for all $u \in [0,T]$, then the value function V(t, x, s, i) = v(t, x, s, i), where v is shown in (3.9). The optimal investment policy is

$$a^*(t) = \frac{\mu + \mu^2 \beta(T-t)}{\sigma^2 s^{2\beta} m}.$$

The detailed proof is in the Appendix.

4 When the interest rate is not 0

When the interest rate is not 0, the HJB equation is

$$\begin{aligned} v_t(t, x, s, i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t, x, s, i)(c + a(\mu - r) + rx) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t, x, s, i) \right. \\ &+ \sigma^2 a s^{2\beta + 1} v_{xs}(t, x, s, i) \right\} + \mu s v_s(t, x, s, i) + \frac{1}{2}\sigma^2 s^{2\beta + 2} v_{ss}(t, x, s, i) \\ &+ \lambda_i (v(t, x, s, i + 1) - v(t, x, s, i)) = 0, \quad i = 1, 2, \cdots, n - 1; \\ v_t(t, x, s, i) + \sup_{a \in \mathbb{R}} \left\{ v_x(t, x, s, i)(c + a(\mu - r) + rx) + \frac{1}{2}\sigma^2 a^2 s^{2\beta} v_{xx}(t, x, s, i) \right. \\ &+ \sigma^2 a s^{2\beta + 1} v_{xs}(t, x, s, i) \right\} + \mu s v_s(t, x, s, n) + \frac{1}{2}\sigma^2 s^{2\beta + 2} v_{ss}(t, x, s, i) \\ &+ \lambda_n (\mathbb{E}[v(t, x - Y, s, 1)] - v(t, x, s, i)) = 0, \quad i = n, \end{aligned}$$

with boundary condition

$$v(T, x, s, i) = -\frac{1}{m}e^{-mx}, \quad i = 1, 2, 3, \dots, n.$$
 (4.2)

In this section, if there exists a concave solution for (4.1), for each *i*, the maximizer of (4.1)

$$a^{*}(t, x, s, i) = -\frac{(\mu - r)v_{x}(t, x, s, i) + \sigma^{2}s^{2\beta + 1}v_{xs}(t, x, s, i)}{\sigma^{2}s^{2\beta}v_{xx}(t, x, s, i)}.$$
(4.3)

Similar with Section 3, we look for a continuously differentiable concave solution for the HJB equation (4.1)-(4.2). We conjecture that the solution takes the form of

$$v(t,x,s,i) = -\frac{1}{m} \exp\left\{-mxe^{r(T-t)} - \frac{(\mu-r)^2}{4\sigma^2\beta r} [1 - e^{2\beta r(t-T)}]s^{-2\beta}\right\} \psi_i(t), i = 1, 2, \dots, n.$$
(4.4)

where $\psi_i(t), i = 1, 2, \dots, n$, are some deterministic function which will be determined later. After simple calculations, we can get that

$$v_t(t, x, s, i) = v(t, x, s, 1) \left(mrxe^{r(T-t)} + \frac{(\mu - r)^2}{2\sigma^2} e^{2\beta r(t-T)} s^{-2\beta} \right) - \frac{1}{m} \exp\left\{ -mxe^{r(T-t)} - \frac{(\mu - r)^2}{4\sigma^2\beta r} [1 - e^{2\beta r(t-T)}] s^{-2\beta} \right\} \psi'_i(t),$$
(4.5)

$$v_x(t, x, s, i) = v(t, x, s, i) \left\{ -me^{r(T-t)} \right\},$$
(4.6)

$$v_{xx}(t, x, s, i) = v(t, x, s, i) \left\{ -me^{r(T-t)} \right\}^2,$$
(4.7)

$$v_s(t, x, s, i) = v(t, x, s, i) \left\{ \frac{(\mu - r)^2}{2\sigma^2 r} [1 - e^{2\beta r(t - T)}] s^{-2\beta - 1} \right\},$$
(4.8)

$$v_{ss}(t,x,s,i) = v(t,x,s,i) \left\{ \frac{(\mu-r)^4}{4\sigma^4 r^2} [1 - e^{2\beta r(t-T)}]^2 s^{-4\beta-2} \right\} + v(t,x,s,i) \left\{ -(2\beta+1) \frac{(\mu-r)^2}{2\sigma^2 r} [1 - e^{2\beta r(t-T)}] s^{-2\beta-2} \right\},$$
(4.9)

$$v_{xs}(t,x,s,i) = v(t,x,s,i) \left\{ -me^{r(T-t)} \frac{(\mu-r)^2}{2\sigma^2 r} [1 - e^{2\beta r(t-T)}] s^{-2\beta-1} \right\}.$$
(4.10)

Substituting (4.3) and (4.5)-(4.10) into (4.4) shows that $\{\psi_i\}_{i=1}^n$ should satisfy

$$\begin{cases} \psi_1'(t) - (cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(t-T)}] + \lambda_1)\psi_1(t) + \lambda_1\psi_2(t) = 0, \\ \psi_2'(t) - (cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(t-T)}] + \lambda_2)\psi_2(t) + \lambda_2\psi_3(t) = 0, \\ \cdots \\ \psi_{n-1}'(t) - (cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(t-T)}] + \lambda_{n-1})\psi_{n-1}(t) + \lambda_{n-1}\psi_n(t) = 0, \\ \psi_n'(t) - (cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(t-T)}] + \lambda_n)\psi_n(t) + \lambda_n\psi_1(t)\mathbb{E}e^{mYe^{r(T-t)}} = 0. \end{cases}$$

$$(4.11)$$

Together with (2.6) we get that the boundary condition of $\{\psi_i\}_{i=1}^n$ is

$$\psi_i(T) = 1, i = 1, 2, 3, \cdots, n.$$
 (4.12)

We need to solve a continuously differentiable and non-negative solution for (4.11)-(4.12). The non-negativity of the solution is to make sure that the solution of (4.4) is concave. We use the technique of changing variables can be used to simplify (4.11).

Denote $F_i(t)$ one primitive function of $cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(t-T)}], i = 1, 2, \cdots, n,$ or in other words,

$$F_i(t) = \int_0^t \left\{ cme^{r(T-u)} + \frac{(2\beta+1)(\mu-r)^2}{4r} [1 - e^{2\beta r(u-T)}] \right\} du.$$
(4.13)

Multiplying $e^{-F_i(t)}$ on both sides of (4.11) gives

$$\begin{split} e^{-F_{1}(t)}\psi_{1}'(t) - e^{-F_{1}(t)} \left(cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^{2}}{4r}[1 - e^{2\beta r(t-T)}] + \lambda_{1}\right)\psi_{1}(t) \\ & +\lambda_{1}\psi_{2}(t)e^{-F_{1}(t)} = 0, \\ e^{-F_{2}(t)}\psi_{2}'(t) - e^{-F_{2}(t)} \left(cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^{2}}{4r}[1 - e^{2\beta r(t-T)}] + \lambda_{2}\right)\psi_{2}(t) \\ & +\lambda_{2}\psi_{3}(t)e^{-F_{2}(t)} = 0, \\ \cdots, \\ e^{-F_{n-1}(t)}\psi_{n-1}'(t) - e^{-F_{n-1}(t)} \left(cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^{2}}{4r}[1 - e^{2\beta r(t-T)}] + \lambda_{n-1}\right)\psi_{n-1}(t) \\ & +\lambda_{n-1}\psi_{n}(t)e^{-F_{n-1}(t)} = 0, \\ e^{-F_{n}(t)}\psi_{n}'(t) - e^{-F_{n}(t)}(cme^{r(T-t)} + \frac{(2\beta+1)(\mu-r)^{2}}{4r}[1 - e^{2\beta r(t-T)}] + \lambda_{n})\psi_{n}(t) \\ & +\lambda_{n}\psi_{1}(t)\mathbb{E}e^{mYe^{r(T-t)}}e^{-F_{n}(t)} = 0. \end{split}$$

Denote

$$\varphi_i(t) := e^{-F_i(t)} \psi_i(t), \quad i = 1, 2, 3, \dots, n.$$
(4.14)

It turns out that

$$\varphi_1'(t) - \lambda_1 \varphi_1(t) + \lambda_1 \varphi_2(t) = 0,$$

$$\varphi_2'(t) - \lambda_2 \varphi_2(t) + \lambda_2 \varphi_3(t) = 0,$$

$$\dots \qquad (4.15)$$

$$\varphi_{n-1}'(t) - \lambda_{n-1} \varphi_{n-1}(t) + \lambda_{n-1} \varphi_n(t) = 0,$$

$$\varphi_n'(t) - \lambda_n \varphi_n(t) + \lambda_n \mathbb{E}(e^{mY e^{r(T-t)}}) \varphi_1(t) = 0,$$

or equivalently,

$$\begin{pmatrix} \varphi_1'(t) \\ \varphi_2'(t) \\ \vdots \\ \varphi_{n-1}'(t) \\ \varphi_n'(t) \end{pmatrix} = Q(t) \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \vdots \\ \varphi_{n-1}(t) \\ \varphi_n(t) \end{pmatrix}, \qquad (4.16)$$

where

$$Q(t) = \begin{pmatrix} \lambda_1 & -\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & -\lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & -\lambda_{n-1} \\ -\lambda_n z(t) & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

is a $n \times n$ matrix and $z(t) := \mathbb{E}(e^{mYe^{r(T-t)}})$. The boundary condition is

$$\varphi_i(T) = e^{-F_i(T)} > 0.$$
 (4.17)

The ordinary differential equations (4.15) looks quite simple and the existence as well as uniqueness of the solution is undoubtable. But showing an explicit expression of the solution is not easy since Cayley–Hamilton theorem is not applicable due to the fact that for some $u, t \in [0, T]$,

$$Q(t)Q(u) \neq Q(u)Q(t).$$

We show that the solution of (4.15) $\{\varphi_i(t)\}_{i=1}^n$ is non-negative. The non-negativity is to verify the concavity of the solution of the HJB equation.

Denote $z(t) := \mathbb{E}(e^{mYe^{r(T-t)}})$ for simplicity. Denote $\bar{z} := \sup_{s \in [0,T]} |z(s)|$ and $\bar{\lambda} := \max\{\lambda_1, \dots, \lambda_n\}$. For a given $\delta \leq \frac{1}{2\bar{z}\lambda}$, we divide the time interval [0,T] into the N subintervals $[0,T-(N-1)\delta], \dots, [T-k\delta, T-(k-1)\delta]$, such that $T \leq N\delta$. Now consider the system (4.15) in the time interval $[T-k\delta, T-(k-1)\delta]$:

$$\begin{aligned} \varphi_1'(t) - \lambda_1 \varphi_1(t) + \lambda_1 \varphi_2(t) &= 0, \\ \varphi_2'(t) - \lambda_2 \varphi_2(t) + \lambda_2 C_3(t) &= 0, \\ \cdots & t \in [T - k\delta, T - (k - 1)\delta], \\ \varphi_{n-1}'(t) - \lambda_{n-1} \varphi_{n-1}(t) + \lambda_{n-1} \varphi_n(t) &= 0, \\ \varphi_n'(t) - \lambda_n \varphi_n(t) + \lambda_n z(t) \varphi_1(t) &= 0. \end{aligned}$$
(4.18)

Consider the non-negative valued continuous function space $C([T-k\delta, T-(k-1)\delta]; \mathbb{R}_+)$ equipped with the supremum norm $||v||_{\infty} := \sup_{s \in [T-k\delta, T-(k-1)\delta]} |v(s)|$ which is a Banach space. If k = 1, then the boundary condition is $\varphi_i(T-(k-1)\delta) = e^{-F_i(T)} > 0$. If $k \neq 1$, we assume that the same system defined in the former time interval $[T-(k-1)\delta, T-(k-2)\delta]$ admits a unique non-negative solution $\bar{\varphi}_i(t) \in C([T-k\delta, T-(k-1)\delta]; \mathbb{R}_{\geq 0})$, then let $\varphi_i(T-(k-1)\delta) := \bar{\varphi}_i(T-(k-1)\delta)$.

Lemma 4.1. For any $k = 1, \dots, N$, the equation system (4.18) admits a unique solution $(\varphi_1, \dots, \varphi_n)$ such that $\varphi_i(t) \in C^1([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$ for all $i = 1, \dots, n$.

Proof. We decouple the system (4.18) by constructing a map $\Phi : C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+) \ni \hat{\varphi}_1 \longmapsto \varphi_1 \in C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$ as follows

$$\begin{cases} \varphi_1'(t) - \lambda_1 \varphi_1(t) + \lambda_1 \varphi_2(t) = 0, \\ \varphi_2'(t) - \lambda_2 \varphi_2(t) + \lambda_2 \varphi_3(t) = 0, \\ \cdots \\ \varphi_{n-1}'(t) - \lambda_{n-1} \varphi_{n-1}(t) + \lambda_{n-1} \varphi_n(t) = 0, \\ \varphi_n'(t) - \lambda_n \varphi_n(t) + \lambda_n z(t) \hat{\varphi}_1(t) = 0, \end{cases}$$

$$(4.19)$$

Since $\hat{\varphi}_1(t)$ and z(t) are non-negative functions, it is easy to show that

$$\varphi_n(t) = e^{-\lambda_n (T - (k-1)\delta - t)} \bar{\varphi}_n (T - (k-1)\delta) + \lambda_n \int_t^{T - (k-1)\delta} e^{-\lambda_n (s-t)} z(s) \hat{\varphi}_1(s) ds > 0.$$
(4.20)

Hence $\varphi_n(t)$ belongs to $C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$. We conduct this procedure to the equation of $\varphi_{n-1}, \dots, \varphi_1$ one-by-one. It can be shown that the system of (4.19) admits a unique solution $(\varphi_1, \dots, \varphi_n)$, and all the entries belong to $C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$. Therefore Φ is a self-map. If we consider another input $\hat{\varphi}_1^{\varepsilon}(t)$ to the system,

1

$$\begin{pmatrix} (\varphi_1^{\varepsilon})'(t) - \lambda_1 \varphi_1^{\varepsilon}(t) + \lambda_1 \varphi_2^{\varepsilon}(t) = 0, \\ (\varphi_2^{\varepsilon})'(t) - \lambda_2 \varphi_2^{\varepsilon}(t) + \lambda_2 \varphi_3^{\varepsilon}(t) = 0, \\ \dots \\ (\varphi_{n-1}^{\varepsilon})'(t) - \lambda_{n-1} \varphi_{n-1}^{\varepsilon}(t) + \lambda_{n-1} \varphi_n^{\varepsilon}(t) = 0, \\ (\varphi_n^{\varepsilon})'(t) - \lambda_n \varphi_n^{\varepsilon}(t) + \lambda_n z(t) \hat{\varphi}_1^{\varepsilon}(t) = 0, \\ \end{pmatrix}$$
(4.21)

The last equation can be written as

$$\varphi_n^{\varepsilon}(t) = e^{-\lambda_n(T-t)}\bar{\varphi}_i(T-(k-1)\delta) + \lambda_n \int_t^{T-(k-1)\delta} e^{-\lambda_n(s-t)} z(s)\hat{\varphi}_1^{\varepsilon}(s)ds.$$
(4.22)

We take the difference of (4.20) and (4.22),

$$\varphi_n^{\varepsilon}(t) - \varphi_n(t) = \lambda_n \int_t^{T - (k-1)\delta} e^{-\lambda_n(s-t)} z(s) \left(\hat{\varphi}_1^{\varepsilon}(s) - \hat{\varphi}_1(s)\right) ds \le \delta \bar{\lambda} \bar{z} \|\hat{\varphi}_1^{\varepsilon} - \hat{\varphi}_1\|_{\infty}.$$

For the second-to-last equation, we have

$$\varphi_{n-1}^{\varepsilon}(t) - \varphi_{n-1}(t) = \lambda_{n-1} \int_{t}^{T-(k-1)\delta} e^{-\lambda_{n-1}(s-t)} z(s) \left(\varphi_{n}^{\varepsilon}(t) - \varphi_{n}(t)\right) ds$$
$$\leq \delta \bar{\lambda} \bar{z} \|\varphi_{n}^{\varepsilon} - \varphi_{n}\|_{\infty} \leq \delta^{2} \bar{\lambda}^{2} \bar{z}^{2} \|\hat{\varphi}_{1}^{\varepsilon} - \hat{\varphi}_{1}\|_{\infty}.$$

We take this procedure to other equations in systems (4.19) and (4.21) to deduce that

$$\|\varphi_1^{\varepsilon} - \varphi_1\|_{\infty} \le \delta^n \bar{\lambda}^n \bar{z}^n \|\hat{\varphi}_1^{\varepsilon} - \hat{\varphi}_1\|_{\infty} \le \frac{1}{2^n} \|\hat{\varphi}_1^{\varepsilon} - \hat{\varphi}_1\|_{\infty}.$$
(4.23)

Hence the map Φ is contractive map. By the Banach fixed point theorem, there exists a unique fixed point $\varphi_1 \in C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$ such that $\Phi(\varphi_1) = \varphi_1$. Until now, we proved that the equation system (4.18) admits a unique solution $(\varphi_1, \dots, \varphi_n)$ such that, for any $i = 1, \dots, n$, $\varphi_i(t) \in C([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$. By the continuity of z(t), the system (4.18) further shows that each $\varphi_i(t) \in C^1([T - k\delta, T - (k - 1)\delta]; \mathbb{R}_+)$.

Theorem 4.2. The equation system (4.15) admits a unique solution $(\varphi_1, \dots, \varphi_n)$ such that $\varphi_i(t) \in C^1([0,T]; \mathbb{R}_+)$ for all $i = 1, \dots, n$.

Proof. We past all the solutions in the N subintervals $[T - k\delta, T - (k - 1)\delta]$ together to obtain the solution to system (4.15).

The non-negativity of φ_i implies the concavity of v(t, x, s, i) about x. In summary, we construct a continuously differentiable concave solution v(t, x, s, i) for the HJB equation (4.1)-(4.2). In what follows, a verification theorem is provided to show that under suitable conditions, a continuously differentiable solution of the HJB equation is indeed the optimal value function defined in (2.4).

Theorem 4.3. (verification theorem) For the case of interest rate not being 0, let v(t, x, s, i)defined in (4.4) be the solution of HJB equation (4.1)-(4.2), and the parameters satisfy that for all $u \in [0,T]$, $-16(\mu - r)\hat{Q}(u) + 64\hat{Q}(u)^2 \leq \mu^2$, where $\hat{Q}(u) = (\mu - r) + (1 - e^{2\beta r(T-u)})\frac{(\mu - r)^2}{2r}$, the optimal value function V(t, x, s, i) = v(t, x, s, i). The optimal investment policy is

$$a^* = \frac{(\mu - r) + (1 - e^{2\beta r(T-t)})\frac{(\mu - r)^2}{2r}}{\sigma^2 s^{2\beta} m e^{r(T-t)}}.$$
(4.24)

Detailed proof is shown in the Appendix.

5 Sensitivity Analysis

This section explores sensitivity of the optimal policy and the optimal value function about different parameters. We assume that the interclaim times follow Erlang (2) distribution and claim size follows the uniform distribution on [0, 1]. Unless otherwise stated, the parameters are shown in the following table.



Figure 5.1: The optimal strategy a^* about stock price s at time t = 1.

Figure 5.2: The optimal strategy a^* about stock price s and time t.

From the explicit expression we can see that the optimal policy is irrelevant with the current surplus x, the current phase i. The optimal policy is correlated with current time t, the stock

price s, the interest rate r, the drift μ and the volatility σ of the stock price. Figure 5.1 shows the effect of stock process s on the optimal investment policy a^* when t = 1. We can see that the investment amount on stocks is decreasing with the increasing of stock price. This phenomenon is in line with intuition since as stock price increasing, the cost of holding stocks will be higher. The volatility decreases when the stock price increases, thus, holding a large amount of stocks can not produce high profits. Eventually, when the stock price is high, the optimal policy should be decreasing the amount of money invested in stocks. Figure 5.2 is a 3-dimensional picture showing the effect of time t and stock price s on the optimal investment policy a^* from which we can see that the amount of investment increases as the time t increases which means as t approaches to the terminal time, the insurer tends to take more risk in order to get higher returns.



Figure 5.3: The value function V about time t at x = 2 and s = 1.

Figure 5.4: The value function V about stock price s at time t = 1, x = 2.

Figure 5.3 shows the value function about variable t when x = 2, s = 1. It is obvious the value function is negative-valued and decreasing with respect to t. From the picture we can also see that $V(t, x, s, 1) \ge V(t, x, s, 2)$. This cames from the fact that the state 2 is more "close" to the claim time, i.e., when the insurer is in state 2, the insurance company need to undertake upcoming claims. Figure 5.4 shows the picture of V about the variable s when t = 1 and x = 2 from which we can see that the value function is decreasing with respect to s. This is because we adopt the exponential utility which means the decision-maker's attitude toward risk does not change with wealth levels. Thus, when the stock prices is increasing, the volatility risk of stocks is also increasing leading to that the utility of the insurer decreases.

6 Conclusion

This paper copes the optimal investment of the renewal surplus process in which the interclaim times are Erlang(n) distributed. As there are n state in the Markov chain, Laplace transform of the Markov chain, decoupling the n-dimensional coupled equation and Banach fixed point theorem are used to prove the concavity of the value function. Our results show that the optimal policy is irrelevant with the wealth and the current phase of Erlang(n) distribution. The utility of the insurer decreases when the current time is close to the next claim. We further focus on the optimal investment of renewal process when the phase of Erlang(n) distribution can not be observed.

7 Appendix

Proof of Lemma 3.1 Denote $\zeta := \mathbb{E}(e^{mY})$. Without loss of generality, we consider the matrix \mathbb{O} has the form of

(λ_1	λ_1	0		0	0	
() -	$-\lambda_2$	λ_2		0	0	
()	0	$-\lambda_3$		0	0	
:		÷	÷	÷	÷	÷	
()	0	0		$-\lambda_{n-1}$	λ_{n-1}	
λ_r	$_{n}\zeta$	0	0		0	$-\lambda_n$)

and show that for any t > 0, every element of the matrix $e^{\mathbb{O}t}$ is non-negative. Consider a Markov chain \tilde{J}_t with the transition matrix

$-\lambda_1$	λ_1	0		0	0)	١
0	$-\lambda_2$	λ_2		0	0	
0	0	$-\lambda_3$		0	0	
	÷	:	÷	:	•	
0	0	0		$-\lambda_{n-1}$	λ_{n-1}	
$\lambda_n \zeta$	0	0		0	$-\lambda_n \zeta$	

From the transition matrix, we can see that

$$\mathbb{P}(\tilde{J}_t = n, t \in [0, s] | \tilde{J}_0 = n) = e^{-\lambda_n \zeta s} = 1 - \lambda_n \zeta s + o(s),$$
$$\mathbb{P}(\tilde{J}_{[0,s]} = i | \tilde{J}_0 = i) = 1 - \lambda_i s + o(s), i \neq n.$$

If the initial state $i \neq n$, define $f_{ij}(t) := \mathbb{E} \left[\mathbf{1}_{\{X_t=j\}} | X_0 = i \right]$, then

$$f_{ij}(t) = (1 - \lambda_i s) f_{ij}(t - s) + \lambda_i s f_{i+1,j}(t - s) + o(s),$$

Thus,

$$\frac{f_{ij}(t) - f_{ij}(t-s)}{s} = \frac{-\lambda_i s f_{ij}(t-s) + \lambda_i s f_{i+1,j}(t-s) + o(s)}{s},$$

Letting $s \downarrow 0$ gives

$$f'_{ij}(t) = -\lambda_i f_{ij}(t) + \lambda_i f_{i+1,j}.$$
(7.1)

If the initial state i = n, define $f_{ij}(t) = \mathbb{E}\left[e^{\lambda_i(\zeta-1)\int_0^t \mathbf{1}_{\{\bar{J}_u=i\}}du}\mathbf{1}_{\{X_t=j\}}|X_0=i\right]$. We can obtain that

$$f_{nj}(t) = e^{\lambda_n(\zeta - 1)s} (1 - \lambda_n \zeta s) f_{nj}(t - s) + f_{1j}(t) \lambda_n \zeta s + o(s).$$
(7.2)

Notice that

$$e^{\lambda_n(\zeta-1)s} = 1 + \lambda_n(\zeta-1)s + o(s).$$
(7.3)

Substituting (7.3) into (7.2) gives

$$f_{nj}(t) = (1 + \lambda_n(\zeta - 1)s)(1 - \lambda_n\zeta s)f_{nj}(t - s) + f_{1j}(t)\lambda_n\zeta s + o(s)$$
$$= (1 - \lambda_n s)f_{nj}(t - s) + f_{1j}(t)\lambda_n\zeta s + o(s),$$

which gives

$$\frac{f_{nj}(t) - f_{nj}(t-s)}{s} = \frac{-\lambda_n s f_{nj}(t-s) + f_{1j}(t)\lambda_n \zeta s + o(s)}{s}.$$
(7.4)

Letting $s \downarrow 0$ in (7.4) gives

$$f'_{nj}(t) = -\lambda_n f_{nj}(t) + f_{1j}\lambda_n\zeta.$$
(7.5)

Combining (7.1) and (7.5), we can see that the matrix $\{f_{ij}\}_{n \times n}$ is the solution of $f'(t) = \mathbb{O}f(t)$ with boundary condition $f(0) = \mathbf{E}$, where \mathbf{E} is the identity matrix, or in other words, $f(t) = e^{\mathbb{O}t}$. By the definition of $f_{ij}(t)$, we can see that $f_{ij}(t)$ is non-negative, which means that every element of $e^{\mathbb{O}t}$ is non-negative. The proof is complete.

Proof of Theorem 3.3 Denote J_t the current state of Erlang (n) distributed interclaim time

at time t. For any strategy $\{a\}$, by Itô formula,

$$v(\tau_{n} \wedge T, X_{\tau_{n} \wedge T}, S_{\tau_{n} \wedge T}, J_{\tau_{n} \wedge T}) = v(t, x, s, j) + \int_{t}^{\tau_{n} \wedge T} (v_{t} + (c + \pi_{u}\mu)v_{x} + \mu S_{u}v_{s} + \frac{1}{2}\sigma^{2}S_{u}^{2\beta+1}v_{ss} + \sigma^{2}S_{u}^{2\beta+1}\pi_{u}v_{xs} + \frac{1}{2}v_{xx}\sigma^{2}S_{u}^{2\beta}\pi_{u}^{2})du + \int_{t}^{\tau_{n} \wedge T}\sigma S_{u}^{\beta}\pi_{u}dW_{u} + \sum_{t \leq u \leq (\tau_{n} \wedge T)} (v(u, X_{u}, S_{u}, J_{u}) - v(u, X_{u-}, S_{u}, J_{u-})),$$
(7.6)

where

$$\tau_n = n \wedge \inf\{u > t; |X^{\pi}(s)| \ge n\} \wedge \inf\{u > t; |S(u)| \ge n\},$$
(7.7)

for $n = 1, 2, \cdots$. Since v is a continuously differentiable solution of the HJB equation, taking expectation on both sides of (7.6) leads to

$$\mathbb{E}v(\tau_n \wedge T, X_{\tau_n \wedge T}, S_{\tau_n \wedge T}, J_{\tau_n \wedge T}) \le v(t, x, s, j).$$

By Fatou's lemma, letting $n \to +\infty$ gives

$$\mathbb{E}[U(X_T^{\pi})] \le v(t, x, s, j).$$

On the other hand, we choose the strategy a^* and denote $X^*(t)$ the corresponding surplus which is driven by strategy a^* . To show $\mathbb{E}[U(X_T^*)] = v(t, x, s, j)$, we only need to prove that

$$\lim_{n \to +\infty} \mathbb{E}v(\tau_n \wedge T, X^*_{\tau_n \wedge T}, S_{\tau_n \wedge T}, J_{\tau_n \wedge T}) = v(t, x, s, j).$$
(7.8)

To prove (7.8), we only need to show that $\mathbb{E}[v^2(t, X^*(t), S(t), J(t))] < +\infty$.

Combining (3.9), we get that

$$v^{2}(t, X^{*}(t), S(t), J(t)) = \frac{1}{m^{2}} \exp\left\{-2mX^{*}(t) + \frac{2\mu^{2}}{\sigma^{2}}(t-T)S(t)^{-2\beta}\right\} \psi^{2}_{J(t)}(t)$$

$$\leq M_{1} \exp\left\{-2mX^{*}(t)\right\},$$
(7.9)

where $M_1 > 0$ is a suitable constant. Substituting a^* into (2.2) gives

$$X_t^* = x_0 + \int_0^t (\mu a_u^* + c) du + \int_0^t \sigma S(u)^\beta a_u^* dW_u - \sum_{0 \le u \le t} (X_{u-} - X_u) \chi_{\{ \triangle X_u \ne 0\}},$$
(7.10)

where χ is the indicator function. Substituting (7.10) gives

$$\exp\left\{-2mX^{\pi^{*}}(t)\right\} = \exp\left\{-2m\left(x_{0} + \int_{0}^{t}(\mu a_{u}^{*} + c)du + \int_{0}^{t}\sigma S(u)^{\beta}a_{u}^{*}dW_{u} - \sum_{0\leq u\leq t}(X_{u-} - X_{u})\chi_{\{\Delta X_{u}\neq 0\}}\right)\right\}$$
(7.11)
$$\leq \exp\left\{-2m\left(x_{0} + \int_{0}^{t}(\mu a_{u}^{*} + c)du + \int_{0}^{t}\sigma S(u)^{\beta}a_{u}^{*}dW_{u}\right)\right\}$$
$$\leq M_{2}\exp\left\{-2m\left(\int_{0}^{t}\mu a_{u}^{*}du + \int_{0}^{t}\sigma S(u)^{\beta}a_{u}^{*}dW_{u}\right)\right\},$$

where $M_2 > 0$ is a suitable constant. Substituting $a^*(t) = \frac{\mu + \mu^2 \beta(T-t)}{\sigma^2 S(t)^{2\beta} m}$ into the above inequality, we get that

$$\exp\left\{-2mX^{*}(t)\right\} \leq M_{2}\exp\left\{-2\int_{0}^{t}\frac{\mu^{2}+\mu^{3}\beta(T-u)}{\sigma^{2}}S^{-2\beta}(u)du-2\int_{0}^{t}\frac{\mu+\mu^{2}\beta(T-u)}{\sigma}S(u)^{-\beta}dW_{u}\right\}.$$
(7.12)

Notice that $H_1(u) := \frac{\mu^2 + \mu^3 \beta(T-u)}{\sigma^2}$, then

$$\exp\left\{-2mX^{*}(t)\right\} \leq M_{2} \exp\left\{-2\int_{0}^{t} H_{1}(u)S(u)^{-2\beta}du - 2\int_{0}^{t} \frac{\sigma}{\mu}H_{1}(u)S(u)^{-\beta}dW_{u}\right\}$$
$$=M_{2} \exp\left\{-2\int_{0}^{t} H_{1}(u)S(u)^{-2\beta}du + 4\int_{0}^{t} \frac{\sigma^{2}}{\mu^{2}}H_{1}^{2}(u)S(u)^{-2\beta}du\right\}$$
$$\cdot \exp\left\{-2\int_{0}^{t} \frac{\sigma}{\mu}H_{1}(u)S(u)^{-\beta}dW_{u} - 4\int_{0}^{t} \frac{\sigma^{2}}{\mu^{2}}H_{1}^{2}(u)S(u)^{-2\beta}du\right\}$$
$$:=M_{2} \exp\{\tilde{H}_{1}(t) + \tilde{H}_{2}(t)\},$$
(7.13)

where $\tilde{H}_1(t) = -2 \int_0^t H_1(u) S(u)^{-2\beta} du + 4 \int_0^t \frac{\sigma^2}{\mu^2} H_1^2(u) S(u)^{-2\beta} du$, $\tilde{H}_2(t) = -2 \int_0^t \frac{\sigma}{\mu} H_1(u) S(u)^{-\beta} dW_u - 4 \int_0^t \frac{\sigma^2}{\mu^2} H_1^2(u) S(u)^{-2\beta} du$. By Hölder's inequality, we get that

$$\mathbb{E}\left[\exp\left\{-2mX^{*}(t)\right\}\right] \leq M_{2}\left(\mathbb{E}\left[e^{2\tilde{H}_{1}(t)}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{2\tilde{H}_{2}(t)}\right]\right)^{\frac{1}{2}},\tag{7.14}$$

Applying Theorem 5.1 of [17], we get that $\mathbb{E}\left[e^{2\tilde{H}_1(t)}\right] < +\infty$. By Lemma 4.3 of [17], it is known that $e^{2\tilde{H}_2(t)}$ is a martingale, then $\mathbb{E}\left[e^{2\tilde{H}_2(t)}\right] < +\infty$. The proof is complete.

Proof of Theorem 4.24 Similar with the proof of Theorem 3.3, we only need to show that when applying the optimal strategy a^* defined in (4.24), it holds that

$$\lim_{n \to +\infty} \mathbb{E}v(\tau_n \wedge T, X^*_{\tau_n \wedge T}, S_{\tau_n \wedge T}, J_{\tau_n \wedge T}) = v(t, x, s, j), j = 1, 2, \cdots, n.$$

i.e., we show that under the strategy a^* , $V(\tau_n \wedge T, X^*_{\tau_n \wedge T}, S_{\tau_n \wedge T}, J_{\tau_n \wedge T})$ is uniformly integrable. After direct calculation, we get that there exists a constant $\hat{M}_1 > 0, \hat{M}_2 > 0$,

$$v^{2}(t, X^{*}(t), S(t), J(t)) = \frac{1}{m^{2}} \exp\left\{-2mX^{*}(t)e^{r(T-t)} - \frac{2(\mu-r)^{2}}{4\sigma^{2}\beta r}[1 - e^{2\beta r(t-T)}]S(t)^{-2\beta}\right\}\psi_{Y(t)}^{2}(t)$$

$$\leq \hat{M}_{1} \exp\left\{-2mX^{*}(t)e^{r(T-t)} - \frac{(\mu-r)^{2}}{2\sigma^{2}\beta r}[1 - e^{2\beta r(t-T)}]S(t)^{-2\beta}\right\}$$

$$\leq \hat{M}_{2} \exp\left\{-2me^{r(T-t)}X^{*}(t)\right\}.$$
(7.15)

Substituting π^* into (2.2) gives

$$X_{t}^{*} = e^{rt}x_{0} + \int_{0}^{t} e^{r(t-u)} [(\mu - r)a_{u}^{*} + c] du + \int_{0}^{t} e^{r(t-u)} \sigma S(u)^{\beta} a_{u}^{*} dW_{u} - \sum_{0 \le u \le t} e^{r(t-u)} (X_{u-} - X_{u}) \chi_{\{ \bigtriangleup X_{u} \ne 0\}}.$$
(7.16)

Substituting (4.24) and (7.16) into (7.15) gives

$$\begin{split} &\exp\left\{-2me^{r(T-t)}X^*(t)\right\}\\ &=\exp\left\{-2me^{r(T-t)}\left(e^{rt}x_0+\int_0^t e^{r(t-u)}[(\mu-r)a_u^*+c]du\right.\\ &+\int_0^t e^{r(t-u)}\sigma S(u)^\beta a_u^*dW_u-\sum_{0\leq u\leq t}e^{r(t-u)}(X_{u-}-X_u)\chi_{\{\triangle X_u\neq 0\}}\right)\right\}\\ &\leq \hat{M}_3\exp\left\{-2me^{r(T-t)}\int_0^t e^{r(t-u)}(\mu-r)\frac{(\mu-r)+(1-e^{2\beta r(T-u)})\frac{(\mu-r)^2}{2r}}{\sigma^2 S(u)^{2\beta}me^{r(T-u)}}du\\ &-2me^{r(T-t)}\int_0^t e^{r(t-u)}\sigma S(u)^\beta \frac{(\mu-r)+(1-e^{2\beta r(T-u)})\frac{(\mu-r)^2}{2r}}{\sigma^2 S(u)^{2\beta}me^{r(T-u)}}dW_u\right\}\\ &=\hat{M}_3\exp\left\{\int_0^t \left(-2me^{rT}Q_1(u)+4m^2e^{2rT}Q_2^2(u)\right)S(u)^{-2\beta}du\\ &+\int_0^t -2me^{rT}Q_2(u)S(u)^{-\beta}dW_u-\int_0^t 4m^2e^{2rT}Q_2^2(u)S(u)^{-2\beta}du:=M_3\exp\{Q_3(t)+Q_4(t)\}, \end{split}$$

where $\hat{M} > 0$ is a suitable constant and

$$\begin{aligned} Q_1(u) &:= (\mu - r) \frac{(\mu - r) + (1 - e^{2\beta r(T-u)}) \frac{(\mu - r)^2}{2r}}{\sigma^2 m e^{rT}}, \\ Q_2(u) &:= \frac{(\mu - r) + (1 - e^{2\beta r(T-u)}) \frac{(\mu - r)^2}{2r}}{\sigma m e^{rT}}, \\ Q_3(t) &:= \int_0^t \left(-2m e^{rT} Q_1(u) + 4m^2 e^{2rT} Q_2^2(u)\right) S(u)^{-2\beta} du, \\ Q_4(t) &:= \int_0^t -2m e^{rT} Q_2(u) S(u)^{-\beta} dW_u - \int_0^t 4m^2 e^{2rT} Q_2^2(u) S(u)^{-2\beta} du. \end{aligned}$$

By Cauchy-Schwarz inequality,

 $\mathbb{E} \exp\left\{-2me^{r(T-t)}X^*(t)\right\} \le M_3 \mathbb{E}(\exp\{Q_3(t) + Q_4(t)\}) \le M_3 (\mathbb{E} \exp\{2Q_3(t)\})^{\frac{1}{2}} (\mathbb{E} \exp\{2Q_4(t)\})^{\frac{1}{2}}.$ Applying Theorem 5.1 of [17], it holds that

$$\mathbb{E}[e^{2Q_3(t)}] < +\infty.$$

As $-2me^{rT}Q_2(u)$ deterministic and bounded on [0, T], by Lemma 4.3 of [17] we see that $e^{2Q_4(t)}$ is a martingale, then

$$\mathbb{E}[\exp^{2Q_4(t)}] < +\infty.$$

Until now, we show that under suitable conditions, the solution of the HJB equation is indeed the value function of the optimization problem. The optimal policy is deduced by some direct calculations of (3.4).

Acknowledgmen This research is supported by the National Natural Science Foundation of China under grant No.12201104.

Disclosure statement and Data availability statement The authors report that there are no competing interests to declare and there is no data set associated with the paper.

References

- Albrecher, H., Claramunt, M. M., & Mármol, M. (2005). On the distribution of dividend payments in a Sparre Andersen model with generalized Erlang (n) interclaim times. Insurance: Mathematics and Economics, 37(2), 324–334.
- [2] Albrecher, H., & Hartinger, J. (2006). On the non-optimality of horizontal barrier strategies in the Sparre Andersen model. HERMES International Journal of Computer Mathematics and Its Applications, 7, 109–122.
- [3] Bai, L., Ma, J., & Xing, X. (2017). Optimal Dividend and Investment Problems Under Sparre Andersen Model. The Annals of Applied Probability, 27(6), 3588–3632.
- [4] Bai, L., & Ma, J. (2021). Optimal Investment and Dividend Strategy under Renewal Risk Model. SIAM Journal on Control and Optimization, 59(6), 4590–4614.
- [5] Bergel, A. I., & Egidio dos Reis, A. D. (2015). Further developments in the Erlang (n) risk process. Scandinavian Actuarial Journal, 2015(1), 32-48.

- [6] Chiu, M. C., & Wong, H. Y. (2013). Optimal investment for an insurer with cointegrated assets: CRRA utility. Insurance: Mathematics and Economics, 52(1), 52–64.
- [7] Elliott, R. J., & Siu, T. K. (2011). A stochastic differential game for optimal investment of an insurer with regime switching. Quantitative Finance, 11(3), 365–380.
- [8] Gu, A., Guo, X., Li, Z., & Zeng, Y. (2012). Optimal control of excess-of-loss reinsurance and investment for insurers under a CEV model. Insurance: Mathematics and Economics, 51(3), 674–684.
- [9] Liu, J., Yiu, K. F. C., & Siu, T. K. (2014). Optimal investment of an insurer with regimeswitching and risk constraint. Scandinavian Actuarial Journal, 2014(7), 583–601.
- [10] Mishura, Y., & Schmidli, H. (2012). Dividend barrier strategies in a renewal risk model with generalized Erlang interarrival times. North American Actuarial Journal, 16(4), 493–512.
- [11] Hipp, C., Plum, M., 2000. Optimal investment for insurers. Insurance: Mathematics and Economics 27, 215–228.
- [12] Jiang, Z., & Pistorius, M. (2012). Optimal dividend distribution under Markov regime switching. Finance and Stochastics, 16, 449–476.
- [13] Li, S., & Dickson, D. C. (2006). The maximum surplus before ruin in an Erlang (n) risk process and related problems. Insurance: Mathematics and Economics, 38(3), 529–539.
- [14] Wang, N. (2007). Optimal investment for an insurer with exponential utility preference. Insurance: Mathematics and Economics, 40(1), 77–84.
- [15] Zheng, X., Zhou, J., & Sun, Z. (2016). Robust optimal portfolio and proportional reinsurance for an insurer under a CEV model. Insurance: Mathematics and Economics, 67, 77–87.
- [16] Yang, H.L., Zhang, L.H., 2005. Optimal investment for insurer with jump-diffusion risk process. Insurance: Mathematics and Economics 37, 615–634.
- [17] Zeng X D, Taksar M. (2013). A stochastic volatility model and optimal portfolio selection. Quantative Finance, 13, 1547–1558.
- [18] Zou, B., & Cadenillas, A. (2014). Explicit solutions of optimal consumption, investment and insurance problems with regime switching. Insurance: Mathematics and Economics, 58, 159– 167.