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# EISENSTEIN SERIES ON METAPLECTIC COVERS AND MULTIPLE DIRICHLET SERIES

*par*

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**Résumé.** — We computed the first Whittaker coefficient of an Eisenstein series on a global metaplectic group induced from the torus and related the result with a Weyl group multiple Dirichlet series attached to the (dual) root system of the group under a mild assumption on the root system and the degree of the metaplectic cover. This confirms a conjecture of Brubaker-Bump-Friedberg.

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## 1. Introduction

**1.1. Statement of main result.** — Let  $k$  be a global field with ring of adèles  $\mathbb{A}$ ,  $\mathfrak{D} = (\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$  be a semisimple simply-connected based root datum,  $n$  be a positive integer such that  $k$  contains all  $2n$ -th roots of unity, let  $S$  be a finite set of places of  $k$  containing all the archimedean places and satisfies some additional conditions (see §2.1.5). There are two objects that are naturally associated to the above data  $(k, \mathfrak{D}, n, S)$ :

- (1) the *metaplectic Eisenstein series*  $E(g, \Phi_{\lambda, S}, \lambda)$ , which is an Eisenstein series on the global metaplectic group  $\tilde{G}_{\mathbb{A}}$  with root datum  $\mathfrak{D}$  induced from the torus, defined in §5.2.
- (2) the *Weyl group multiple Dirichlet series* (WMDS)  $Z_{\Psi}(s_1, \dots, s_r)$ , which is a Dirichlet series in multiple variables with twisted multiplicative coefficients, defined in §3.3.

The goal of this article is to prove the following

**Theorem.** — *Suppose the metaplectic dual root datum  $\mathfrak{D}_{(\mathbb{Q}, n)}^\vee$  is of adjoint type (see §3.1). Let  $\psi$  be an additive character of  $\mathbb{A}/k$  that is unramified over every place  $v \notin S$ . For  $\lambda \in \text{Gode}$  in the Godement region (5.8), the first Whittaker coefficient of  $E(g, \lambda)$*

$$\int_{U_{\mathbb{A}}^-/U_k^-} E(\lambda, \Phi_{\lambda, S}, u^-) \psi(u^-)^{-1} du$$

is equal to

$$[T_{o_S} : T_{0, o_S}] Z_{\Psi}(s_1, \dots, s_r)$$

where  $s_i = -\langle \lambda, \alpha_i^\vee \rangle$ ,  $\alpha_1^\vee, \dots, \alpha_r^\vee$  are the simple coroots.

See Theorem 5.3.3 for a precise statement. This confirmed the so called *Eisenstein conjecture* by Brubaker-Bump-Friedberg.

**1.1.1. Eisenstein conjecture.** — In [7, 11], Brubaker-Bump-Friedberg conjectured that the Whittaker coefficients of a metaplectic Eisenstein series are Weyl group multiple Dirichlet series. The main result Theorem 5.3.3 of this article confirms this conjecture under the mild assumption that the metaplectic dual root datum is of adjoint type.

The Eisenstein conjecture was proved for the root system  $A_2$  by Brubaker-Bump-Friedberg-Hoffstein in [10] by direct computations, then proved by Brubaker-Bump-Friedberg in [9] for type  $A$  root systems, and subsequently by Friedberg-Zhang in [18] for type  $B$  root systems. In these works the key ingredient of the proof is a combinatorial description of the  $p$ -part in terms of crystal graphs or Gelfand-Tsetlin patterns. Our approach to the Eisenstein conjecture works uniformly for all types of root systems and used a different description of the  $p$ -part as metaplectic Casselman-Shalika formula due to McNamara [33] and Chinta-Gunnells [13].

**1.1.2. Eisenstein series on metaplectic groups.** — Let  $G$  be a split semisimple simply-connected group over a global field  $k$  containing all  $n$ -th roots of unity. For each place  $v$ , let  $G_v = G(k_v)$ , the *metaplectic cover*  $\tilde{G}_v$  of  $G_v$  is a certain central extension of  $G_v$  by  $\mu_n(k)$ , the group of  $n$ -th roots of unity in  $k$ . See §4.2 for this construction. There is a *global metaplectic group*  $\tilde{G}_{\mathbb{A}}$  which is a central extension of  $G_{\mathbb{A}}$  by  $\mu_n(k)$ . See §5.1.3.

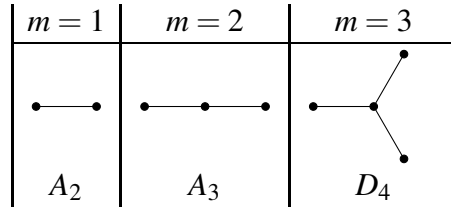
In particular, the group  $G_k$  of  $k$ -rational points splits canonically in  $\widetilde{G}_{\mathbb{A}}$ , so it makes sense to talk about automorphic forms on  $\widetilde{G}_{\mathbb{A}}$ , which are functions on  $\widetilde{G}_{\mathbb{A}}/G_k$  satisfying some analytic properties. For example, classical modular forms of half-integral weights can be viewed as automorphic forms on the two-fold metaplectic cover of  $\mathrm{SL}_2(\mathbb{A})$  for the field of rational numbers  $\mathbb{Q}$ . In the context of modular forms of half integral weights, the Whittaker coefficients of metaplectic Eisenstein series were first studied by Hecke, which results in the Dirichlet series of a quadratic Dirichlet character.

Kubota first considered coverings groups of  $\mathrm{SL}_2$  of degree larger than 2 in [27]. In [28, 29] Kubota computed the Fourier coefficients of an Eisenstein series on the 3-fold metaplectic cover of  $\mathrm{SL}_2(\mathbb{A})$  for the number field  $k = \mathbb{Q}(\sqrt{-3})$ , and results in a Dirichlet series whose coefficients are cubic Gauss sums. The Gauss sum coefficients are not multiplicative, so this Dirichlet series is not an Euler product as in the 2-fold cover case.

In the pioneering work [26] of Kazhdan-Patterson they defined the metaplectic  $n$ -fold covers for  $\mathrm{GL}_r$  both locally and globally. They computed the Whittaker coefficients of the Borel Eisenstein series on the  $n$ -fold cover of  $\mathrm{GL}_2(\mathbb{A})$  they constructed, and the result is essentially a Gauss sum Dirichlet series as in the computation of Kubota.

There are also some work on the Whittaker coefficients of Eisenstein series of *maximal parabolic* type on metaplectic groups. For example, see [1, 12].

**1.1.3. Weyl group multiple Dirichlet series.** — The idea of Weyl group multiple Dirichlet series originates in the study of moments of quadratic  $L$ -functions. This idea goes back to the work of Goldfeld-Hoffstein [25] on the first moment, and was summarized in the paper [15] of Diaconu-Goldfeld-Hoffstein. Roughly speaking, to estimate the  $m$ -th moment, they are lead to study a certain multiple Dirichlet series whose group of functional equation is a Coxeter group for the Coxeter diagram with one central node and  $m$  nodes connected to the central node. The following image shows the dynkin diagrams for  $m = 1, 2, 3$ , which are Dynkin diagrams for the root systems of type  $A_2, A_3, D_4$  respectively.



Weyl group multiple Dirichlet series (WMDS) are multiple Dirichlet series  $Z_{\Psi}(s_1, \dots, s_r)$  attached to a root system with nice analytic properties: they have meromorphic continuations to the whole complex space and has group of functional equations isomorphic to the Weyl group of the root system. They were first defined in the series of papers [7, 10, 11]. Concretely,

$$Z_{\Psi}(s_1, \dots, s_r) = \sum_{C_1, \dots, C_r \in (\mathfrak{o}_S \setminus \{0\}) / \mathfrak{o}_S^{\times}} H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}$$

where the  $H$ -coefficients are twisted multiplicative, namely

$$\begin{aligned} & \frac{H(C_1 C'_1, \dots, C_r C'_r)}{H(C_1, \dots, C_r) H(C'_1, \dots, C'_r)} \\ &= \prod_{i=1}^r \left( \frac{C_i}{C'_i} \right)_S^{Q(\alpha_i^{\vee})} \left( \frac{C'_i}{C_i} \right)_S^{Q(\alpha_i^{\vee})} \prod_{1 \leq i < j \leq r} \left( \frac{C_i}{C'_j} \right)_S^{B(\alpha_i^{\vee}, \alpha_j^{\vee})} \left( \frac{C'_i}{C_j} \right)_S^{B(\alpha_i^{\vee}, \alpha_j^{\vee})} \end{aligned} \quad (1.1)$$

where  $Q : \Lambda^\vee \rightarrow \mathbb{Z}$  is the unique Weyl group invariant quadratic form on the coweight lattice  $\Lambda^\vee$  normalized such that its value on short coroots is equal to 1, and  $B$  is the associated bilinear form by (3.1).  $\Psi$  is another function in  $C_1, \dots, C_r$  that plays a less important role.

Because of twisted multiplicativity, the  $H$ -coefficients are determined by the values of  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r})$  for every  $v \notin S$  and  $r$ -tuple of non-negative integers  $k_1, \dots, k_r$ , where  $\pi_v$  is a prime element of  $\mathfrak{o}_S$  corresponding to the place  $v$ . It is convenient to form the generating series

$$N(v) = \sum_{k_1, \dots, k_r \geq 0} H(\pi_v^{k_1}, \dots, \pi_v^{k_r}) e^{-k_1 \alpha_1^\vee - \dots - k_r \alpha_r^\vee} \in \mathbb{C}[[\Lambda^\vee]] \quad (1.2)$$

which is called the  $v$ -part of the series. Correctly chosen  $v$ -parts will result in the desired analytic continuation and functional equations of the series  $Z_\Psi$ .

There are several equivalent ways to construct the  $v$ -part:

- The original construction in [7, 11] by directly define  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r})$  as a certain product of Gauss sums. This only works in the *stable case*, namely when  $n$  is sufficiently large with respect to the root system.
- The combinatorial construction by Brubaker-Bump-Friedberg in [8, 9] by crystal graphs. This construction goes beyond the stable case and leads to a prove of the Eisenstein conjecture.
- The construction by Chinta-Gunnells in [13] by averaging the constant polynomial over a non-standard Weyl group action on  $\mathbb{C}(\Lambda^\vee)$ . It was then proved by McNamara that the  $v$ -parts constructed in this way are equal to the values of certain values of a metaplectic Whittaker function on  $\widehat{G}_v$ , and these values are given by a metaplectic version of the Casselman-Shalika formula. See also [37].
- The construction by a statistical mechanical model by Brubaker-Bump-Chinta-Friedberg-Gunnells in [6] and [2]. This model was then intensively studied by Brubaker, Buciumas, Bump, Friedberg, Gray, Gustafsson, and other people in a series of papers. See [4] and reference therein.

**1.1.4. Consequences.** — The work of Mœglin-Waldspurger [34] on Eisenstein series are applicable for metaplectic covers, so the metaplectic Eisenstein series have analytic continuations and functional equations similar to non-metaplectic Eisenstein series. As a result, the Whittaker coefficient also have analytic continuation and functional equations, so logically the proof of the Eisenstein conjecture leads to a new proof of the analytic properties of Weyl group multiple Dirichlet series.

Nevertheless, we hope that our computation shed some light in the definition of Weyl group multiple Dirichlet series for *affine* root systems. In the affine case the Weyl group is an infinite group, and in general it is still unknown how to construct the  $p$ -parts so that the Weyl group multiple Dirichlet series still have analytic continuations and functional equations. Some work in this direction were done by Diaconu, Whitehead, Pasol, Popa, Sawin in [16, 17, 40, 44, 45].

In an ongoing work, we compute the first Whittaker coefficient of a Borel Eisenstein series on an *affine metaplectic group* defined in [38] for  $k = \mathbb{F}_q(t)$  the rational function field over a finite field. We hope that the analytic properties of Eisenstein series on affine Kac-Moody groups developed by Garland in [20–23] can be generalized to affine metaplectic groups and help us to establish the analytic properties of the affine Weyl group multiple Dirichlet series.

**1.2. Local and global Whittaker functionals for metaplectic groups.** — In this section we introduce the features of the Whittaker coefficients of metaplectic Eisenstein series. A substantial difference with the non-metaplectic case is that the Whittaker coefficients are not factorizable as Euler products, essentially due to failure of local uniqueness of Whittaker models for metaplectic groups.

**1.2.1. Non-metaplectic case.** — To compare with the metaplectic case, we recall that in the non-metaplectic case, the Whittaker coefficients of a Borel Eisenstein series is factorizable as an Euler product. More concretely, let  $G$  be a split simple simply-connected group over  $k$  with Borel subgroup  $B = TU$  where  $T$  is a split torus and  $U$  the unipotent radical. Let  $\lambda \in X^*(T) \otimes \mathbb{C}$  be a complex weight, which induces a character  $\chi_\lambda : T_\mathbb{A} \rightarrow \mathbb{C}^*$ . The adelic group  $G_\mathbb{A}$  has Iwasawa decomposition  $G_\mathbb{A} = K_\mathbb{A} T_\mathbb{A} U_\mathbb{A}$ , and we can define a function  $\Phi_\lambda^b : G_\mathbb{A} \rightarrow \mathbb{C}$  by

$$\Phi_\lambda^b(ktu) = \chi_{\lambda+\rho}(t)$$

The function  $\Phi_\lambda^b = \prod_v \Phi_{\lambda,v}^b$  is factorizable and for  $v \nmid \infty$   $\Phi_{\lambda,v}^b$  is the normalized spherical vector in the local principal series representations  $I_v^b(\lambda)$ . The Eisenstein series induced from  $\Phi_\lambda^b$  is

$$E^b(\lambda, g) = \sum_{\gamma \in G_k/B_k} \Phi_\lambda^b(g\gamma)$$

By a standard unfolding process, the Whittaker coefficients of  $E(\lambda, g)$  for a generic character  $\psi$  are given by

$$W^b(\lambda, a) = \int_{U_\mathbb{A}^-/U_k^-} E^b(\lambda, au^-) \psi(u^-) du^- = (\text{archimedean contributions}) \cdot \prod_{v \nmid \infty} W_{\lambda,v}^b(a_v)$$

where  $W_{\lambda,v}^b$  is a spherical Whittaker function on  $G_v$ , and  $W_{\lambda,v}^b(a_v)$  are given by the Casselman-Shalika formula.

**1.2.2. Metaplectic case: non-factorizable.** — In the metaplectic case, we still want to consider the Eisenstein series induced from a factorizable section of principal series representations, but several complications arise:

- We have to exclude a finite number of "bad" non-archimedean places where the notion of "spherical vector" does not make sense. This corresponds to the choice of the finite set  $S$  of places.
- For "good" places  $v \notin S$ , the metaplectic torus  $\tilde{T}_v$  is not commutative, so the principal series representations  $I_v(\lambda)$  are induced not from  $\tilde{T}$  but from a maximal abelian subgroup  $\tilde{T}_0$  of  $\tilde{T}$ . We take the factorizable section  $\Phi_\lambda$  such that  $\Phi_{\lambda,v}$  is the normalized spherical vector in  $I_v(\lambda)$ .
- After a similar unfolding computation, the first Whittaker coefficient is not factorizable:

$$W(\lambda, 1) = \int_{U_\mathbb{A}^-/U_k^-} E(\lambda, u^-) \psi(u^-) du^- = \sum_{\eta \in T_k/T_{0,k}} (\text{contributions from } S) \cdot \prod_{v \notin S} W_{\lambda,v}(\eta) \tag{1.3}$$

where  $T_0$  is the subgroup of  $T$  corresponding to the metaplectic lattice  $\Lambda_0^\vee \subseteq \Lambda^\vee = X_*(T)$  defined by (3.2), and  $W_{\lambda,v}(\eta)$  is given by

$$q^{\langle \rho, \lambda^\vee \rangle} \int_{U_v^-} \Phi_{\lambda,v}(u^- \eta) \psi_v(u_v^-) du^-$$

where  $\Phi_{\lambda, \nu}$  is a spherical vector.

**1.2.3. Local Whittaker functions and  $\nu$ -parts of WMDS.** — It turns out that those  $W_{\lambda, \nu}(\eta)$  appeared in (1.3) are computable. To see this, let  $\pi_\nu$  be a uniformizer of  $k_\nu$ . Essentially by the relation between metaplectic Whittaker functions and the  $\nu$ -parts of WMDS in [33], for any  $\lambda^\nu \in \Lambda^\nu$  we have

$$q^{\langle \rho, \lambda^\nu \rangle} \int_{U_\nu^-} \Phi_{\lambda, \nu}(u^- \pi_\nu^{\lambda^\nu}) \psi_\nu(u_\nu^-) du^- = \sum_{\substack{k_1, \dots, k_r \geq 0 \\ k_1 \alpha_1^\nu + \dots + k_r \alpha_r^\nu - \lambda^\nu \in \Lambda_0^\nu}} H(\pi_\nu^{k_1}, \dots, \pi_\nu^{k_r}) q_\nu^{-k_1 s_1 - \dots - k_r s_r} \quad (1.4)$$

where  $s_i = -\langle \lambda, \alpha_i^\nu \rangle$  and  $q_\nu$  is the residue characteristic of  $k_\nu$ . This is a sub-summation of the  $\nu$ -part (1.2) of the WMDS.

For every  $\eta \in T_k$  we can factorize  $\eta$  in  $T_\nu$  as  $\pi_\nu^{\lambda^\nu} \eta^\nu$  for  $\eta^\nu \in T_{\mathcal{O}_\nu}$ , then we have

$$W_{\lambda, \nu}(\eta) = D(\eta; \nu) \sum_{\substack{k_1, \dots, k_r \geq 0 \\ k_1 \alpha_1^\nu + \dots + k_r \alpha_r^\nu - \lambda^\nu \in \Lambda_0^\nu}} H(\pi_\nu^{k_1}, \dots, \pi_\nu^{k_r}) q_\nu^{-k_1 s_1 - \dots - k_r s_r}$$

where  $D(\eta; \nu)$  is a root of unity obtained when lifting the factorization  $\eta = \pi_\nu^{\lambda^\nu} \eta^\nu$  to the central extension  $\tilde{T}_\nu$  of  $T_\nu$ .

**1.2.4. Two local-to-global "gluing" processes.** — As a result,  $\prod_{\nu \notin S} W_{\lambda, \nu}(\eta)$  is equal to  $D(\eta) := \prod_{\nu \notin S} D(\eta; \nu)$  times a product of some sub-summations of the  $\nu$ -parts. In other words, it "glues" the different  $\nu$ -parts by a factor  $D(\eta)$ .

Correspondingly, there is another "gluing" process for the WMDS: suppose we already know all the  $\nu$ -parts for a WMDS, the coefficients  $H(C_1, \dots, C_r)$  then can be "glued" from the coefficients  $H(\pi_\nu^{k_1}, \dots, \pi_\nu^{k_r})$  in the  $\nu$ -parts by twisted multiplicativity. So  $H(C_1, \dots, C_r)$  is a product of  $H(\pi_\nu^{k_1^\nu}, \dots, \pi_\nu^{k_r^\nu})$  for  $\nu \notin S$  together with a root of unity  $D(C_1, \dots, C_r)$  coming from twisted multiplicativity, where  $k_i^\nu = \text{ord}_\nu C_i$ . This is done in Lemma 3.3.1.

The crucial point in the proof is that these two gluing processes are *the same*. Concretely, let  $\eta_k(C_1, \dots, C_r) \in T_k$  be the element defined by (4.1), then we have  $D(\eta_k(C_1, \dots, C_r)) = D(C_1, \dots, C_r)$ . See Lemma 5.3.7. This compatibility of gluing essentially allows us reduce the Eisenstein conjecture — the global statement that

– Whittaker coefficients of metaplectic Eisenstein series equals to WMDS

to the local statement (1.4) relating local Whittaker integrals with the  $\nu$ -parts of WMDS, which are local counterparts of the Whittaker coefficient of metaplectic Eisenstein series and the WMDS respectively.

Finally, the Eisenstein conjecture follows the fact that the "contribution from  $S$ " in (1.3) can be matched with the function  $\Psi$  in the definition of WMDS.

**1.2.5. Total Whittaker functionals.** — There is one more local-global relation that can be seen from the proof of the Eisenstein conjecture. Namely, in the right hand side (1.3) we can think of each term of the summation for  $\eta \in T_k/T_{0,k}$  as applying a global Whittaker functional  $L_\eta$  to the factorizable section  $\Phi_\lambda$  for the principal series representations, namely the Whittaker coefficient can be viewed as a sum over all the different Whittaker functionals applied to  $\Phi_\lambda$ . We call this sum of all the Whittaker functional the *total Whittaker functional*, and the Eisenstein conjecture can be stated as

$$\text{total Whittaker functional applied to } \Phi_\lambda = \text{WMDS} \quad (1.5)$$

There is a local counterpart of the "total Whittaker functional". In fact, a basis of the space of Whittaker functionals of  $I_v(\lambda)$  can be given by

$$L_{\lambda, \mu^\vee} : I_v(\lambda) \rightarrow \mathbb{C}, \varphi \mapsto q^{\langle \lambda + \rho, \mu^\vee \rangle} \int_{U_v^-} \varphi(u^- \pi_v^{\mu^\vee}) \psi_v(u_v^-) du^-$$

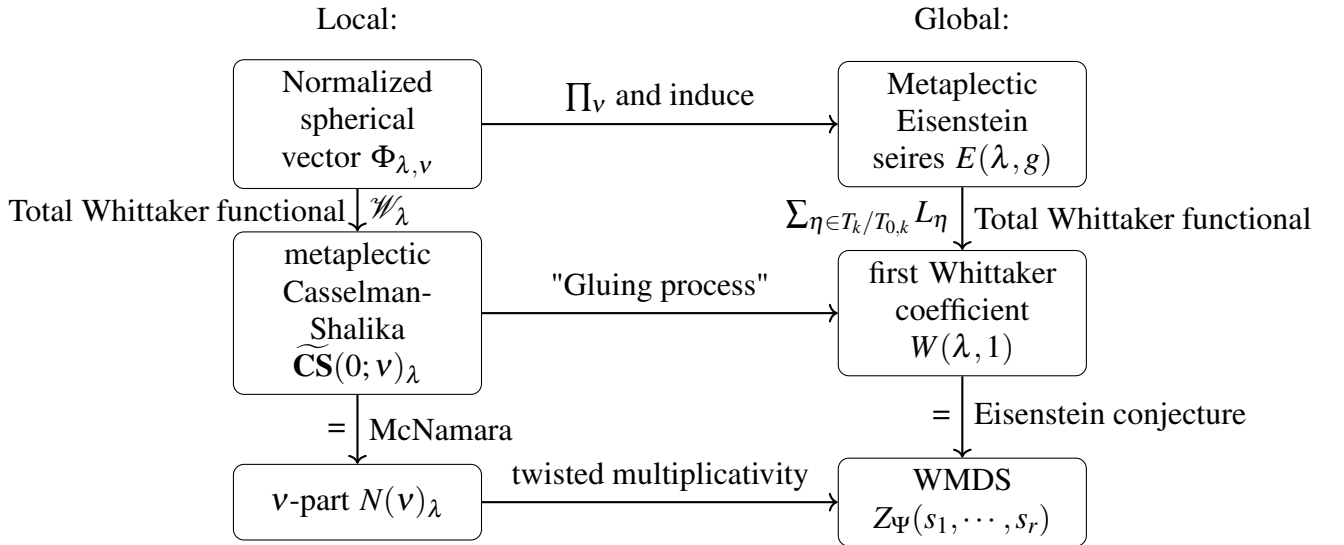
for  $\mu^\vee \in \Lambda^\vee / \Lambda_0^\vee$ . We form a *total Whittaker functional*  $\mathscr{W}_\lambda = \sum_{\mu^\vee \in \Lambda^\vee / \Lambda_0^\vee} L_{\lambda, \mu^\vee}$ . In [33] it was proved that

$$\mathscr{W}_\lambda(\Phi_{\lambda, \nu}) = \widetilde{\mathbf{CS}}(0; \nu)_\lambda = N(\nu)_\lambda$$

where  $\widetilde{\mathbf{CS}}(0; \nu) \in \mathbb{C}[\Lambda^\vee]$  is a metaplectic version of the Casselman-Shalika formula and the subscript  $\lambda$  means the evaluation of the polynomial at  $\lambda$  by  $e^{\mu^\vee} \mapsto q_v^{-\langle \lambda, \mu^\vee \rangle}$ ,  $N(\nu)$  is the  $\nu$ -part of the WMDS. See (3.4) and §4.4, §4.5 for more details. This is the local counterpart of (1.5), namely

$$\text{total Whittaker functional applied to } \Phi_{\lambda, \nu} = \nu - \text{part of WMDS} \quad (1.6)$$

The local-global correspondence (1.5) (1.6) generalizes the Euler product factorization of Whittaker coefficients of Eisenstein series on non-metaplectic groups into local Casselman-Shalika formulas. Informally speaking, the Eisenstein conjecture (1.5) essentially follows from the compatibility of the two gluing processes in §1.2.4 and the purely local result (1.6). The following diagram summarizes all these local-global relations.



**1.3. Organization of this paper.** — In Section 2 we introduce the number theoretic concepts that are needed in this paper, including  $S$ -integers, Hilbert and power residue symbols, and Gauss sums.

In Section 3 we recall the construction of Weyl group multiple Dirichlet series and its  $p$ -parts. We use the Chinta-Gunnells averaging method to construct the  $p$ -part, but we follow more closely to the presentation in [37].

Section 4 contains the construction of metaplectic covers over non-archimedean local fields, and the basic properties of the metaplectic group and the unramified principal series representations, in particular we investigate the Whittaker functionals carefully. We adopt the viewpoint of universal principal series as in [24].

In Section 5 we recall the construction of global metaplectic groups and define the Eisenstein series we study. Usually the Borel Eisenstein series is induced from a section of the principal series representations, but for bad primes we don't have a good description of the unramified principal series representation for the local metaplectic group, so instead we treat all the bad places in  $S$  together. This idea comes from [9]. Then we computed the first Whittaker coefficient of this Eisenstein series and thereby proved the Eisenstein conjecture.

In Appendix A we proved a relation between the local Whittaker functionals and intertwiners on the universal principal series representation, which is equivalent to the computation of "Kazhdan-Patterson scattering matrix" in the literature.

In Appendix B we formulate a relation between twisted multiplicativity and factorizable genuine functions on the torus.

**1.4. Acknowledgement.** — This work started as a joint one with Manish Patnaik. We thank him for the many discussions we have had about these topics, for his constant help and guidance throughout this work, and for generously sharing his thoughts on related matters. This article would not exist but for them.

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## 2. Number theoretic notations

### 2.1. Local and global fields, $S$ -integers. —

**2.1.1. Local fields.** — Let  $\mathcal{F}$  be a non-archimedean local field with discrete valuation  $v : \mathcal{F}^* \rightarrow \mathbb{Z}$  and corresponding norm  $|\cdot|_v$ . Define

$$\mathcal{O}_v := \{x \in \mathcal{F}^* : |x|_v \leq 1\} \cup \{0\} \quad \text{and} \quad \mathfrak{p}_v = \{x \in \mathcal{F}^* : |x|_v < 1\} \cup \{0\} \quad (2.1)$$

as the ring of integers of  $\mathcal{F}$  along with its maximal ideal respectively. Write  $\kappa(v) = \mathcal{O}_v/\mathfrak{p}_v$  for the corresponding residue field, a finite field whose cardinality will be denoted as  $q_v$ . Pick a uniformizer  $\pi_v \in \mathcal{F}$ , i.e.  $|\pi_v|_v = 1$ . If there is no danger of confusion, we drop  $v$  from our notation and write  $\mathcal{O}$  for  $\mathcal{O}_v$ ,  $\pi$  for  $\pi_v$ , etc.

By an archimedean local field  $\mathcal{F}$  we mean  $\mathbb{R}$  or  $\mathbb{C}$ . In this case we define the norm  $|\cdot|$  on  $\mathcal{F}$  to be the usual absolute value on  $\mathbb{R}$  or norm on  $\mathbb{C}$ .

**2.1.2. Global fields.** — Throughout this work, we let  $k$  denote a global field, so either a number field or a function field of a curve  $C$  over a finite field. Write  $\mathcal{V}_k$ , or just  $\mathcal{V}$  for the set of (equivalence classes of) valuations of  $k$ . In the case that  $k$  is a number field, we write  $\mathcal{V}_\infty$  for the set of archimedean places and  $\mathcal{V}_{\text{fin}}$  for the set of finite places. Often we write  $v \mid \infty$  or  $v \nmid \infty$  to denote that  $v \in \mathcal{V}_\infty$  or  $v \notin \mathcal{V}_\infty$ .

**2.1.3. Completions of global fields.** — For  $v \in \mathcal{V}_k$ , we write  $k_v$  for the corresponding completion. If  $v \in \mathcal{V}_{\text{fin}}$ , then  $k_v$  is a non-archimedean local field with ring of integers denoted as  $\mathcal{O}_v$ . Let  $|\cdot|_v$  denote the corresponding norm on  $k_v^*$  and  $\pi_v \in \mathcal{O}_v$  be a uniformizer. If  $v \in \mathcal{V}_\infty$ , then  $k_v$  is either isomorphic to  $\mathbb{R}$  or  $\mathbb{C}$ , and we say that  $k$  is totally real or totally imaginary if all  $k_v$  are isomorphic to  $\mathbb{R}$  or  $\mathbb{C}$  respectively.

Fix  $k$  a global field. We maintain the same notations as in §2.1.2-§2.1.3.



**2.1.4.  $S$ -integers.** — Let  $S \subset \mathcal{V}_k$  be a finite set of places containing  $\mathcal{V}_\infty$ . Let

$$\mathfrak{o}_S = \{x \in k : |x|_v \leq 1 \text{ for all } v \notin S\} \quad (2.2)$$

be the ring of  $S$ -integers. We recall that  $\mathfrak{o}_S$  is a Dedekind domain and that for some possibly larger (finite) set of places  $S' \supset S$ ,  $\mathfrak{o}_{S'}$  will be a principal ideal domain. We shall also write  $\mathfrak{o}_S^\times \subset \mathfrak{o}_S$  for the group of  $S$ -units, *i.e.*

$$\mathfrak{o}_S^\times := \{x \in k \setminus \{0\} \mid |x|_v = 1 \text{ for } v \notin S\} \quad (2.3)$$

**2.1.5. Conditions on  $S$ .** — Fix a positive integer  $n$ , and let  $k$  be a global field. Assume that  $k$  contains all  $2n$ -th roots of unity, let  $S \subset \mathcal{V}_k$  be a finite set of places that satisfies

- $S \supset \mathcal{V}_\infty$ ;
- if  $k$  is a number field and the prime ideal of  $v \in \mathcal{V}_k$  is above the (rational) prime factors of  $n$ , then  $v \in S$ ;
- if  $v \in \mathcal{V}_k$  is ramified over  $\mathbb{Q}$  (in the number field case) or  $\mathbb{F}(T)$  (in the function field case), then  $v \in S$ ;
- $\mathfrak{o}_S := \{x \in k \mid x_v \in \mathcal{O}_v \text{ for } v \notin S\}$  is a principal ideal domain.

**2.1.6. Choices of prime elements.** — As we have assumed  $S$  is sufficiently large so that  $\mathfrak{o}_S$  is a principal ideal domain, and since the prime ideals in  $\mathfrak{o}_S$  are in bijection with places  $v \in \mathcal{V}_k \setminus S$ , for every place  $v \notin S$ , the corresponding prime ideal can be given by  $\mathfrak{p}_v = (\pi_v)$  for a single generator  $\pi_v \in \mathfrak{o}_S$ . We fix once and for all the generators  $\pi_v$  for all places  $v \notin S$ . Under the map  $k \rightarrow k_v$ , the element  $\pi_v \in \mathfrak{o}_S$  is sent to a uniformizer in  $\mathcal{O}_v$  of the same name, and in this way we fixed a uniformizer of  $\mathcal{O}_v$  for every  $v \notin S$  at the same time.

Note that if  $f(x)$  is any polynomial in one variable, then  $f(\pi_v)$  can be regarded as an element of  $\mathfrak{o}_S$  or  $\mathcal{O}_v$ . If however  $f(x)$  is a power series in  $x$  with infinitely many non-zero coefficients, then  $f(\pi_v)$  can only be regarded as an element in  $\mathcal{O}_v$ . We hope these comments may clarify any confusion in the notation which may arise.

**2.1.7. A commonly used factorization.** — We shall often use the following description. Assume  $S$  satisfies the conditions of §2.1.5, and let  $C \in \mathfrak{o}_S \setminus \{0\}$ . If  $v \notin S$ , may write  $C = \pi_v^m C^v$  with  $C^v \in \mathfrak{o}_S$  satisfying  $\gcd(C^v, \pi_v) = 1$ . The image of  $C^v \in k_v$  lies in  $\mathcal{O}_v^\times$ , so that the image of  $C$  in  $k_v$  is equal to

$$C_v := \pi_v^m u \text{ with } u \in \mathcal{O}_v^* \text{ the image of } C^v \text{ in } k_v. \quad (2.4)$$

So the number  $m$  is equal to  $\text{ord}_v(C)$ . We shall sometimes write  $m := n_v(C)$  and write  $C_v := \pi_v^{n_v(C)}$ .

**2.2. Hilbert and Power Residue Symbols.** — In the rest of this section, we fix a positive integer  $n$ . For any field  $e$ , let  $\mu_n(e)$  be the subgroup of  $e^*$  of  $n$ -th roots of unity in  $e$ . We say that  $e$  contains all  $n$ -th roots of unity if the set  $\mu_n(e)$  has cardinality  $n$ .

**2.2.1.** — Let  $\mathcal{F}$  be a local field such which contains all  $n$ -th roots of unity. Let  $(\cdot, \cdot) : \mathcal{F}^* \times \mathcal{F}^* \rightarrow \mu_n(\mathcal{F})$  be the Hilbert symbol as defined in [5], namely they are the *inverse* of the Hilbert symbols defined in [36]. The main properties of this symbol which we need are summarized as follows (see [5, Prop. 1, p. 164]). First of all  $(\cdot, \cdot)$  is a *symbol*, *i.e.*

1. (Bi-multiplicativity)  $(aa', b) = (a, b)(a', b)$  and  $(a, bb') = (a, b)(a, b')$  for  $a, b \in \mathcal{F}^*$ .
2. (Inverse)  $(a, b)^{-1} = (b, a)$
3.  $(a, -a) = 1$  and if  $a \neq 1$ ,  $(a, 1 - a) = 1$

Moreover, we have

**Proposition.** — (i) If  $\mathcal{F}$  is complex we have  $(a, b) = 1$  for all  $a, b \in \mathbb{C}^*$ .

(ii) If  $\mathcal{F}$  is non-archimedean with residue characteristic  $q$  and  $n$  does not divide the residue characteristic of  $\mathcal{F}$  (namely  $n \nmid (q-1)$ ), then  $(a, b) = 1$  for  $a, b \in \mathcal{O}^\times$ .

(iii) if  $\mathcal{F}$  is non-archimedean and  $2n \mid (q-1)$ , then we have  $(\pi, \pi) = 1$  for the uniformizer  $\pi$  of  $\mathcal{F}$ .

Note that in general  $(a, b) = 1$  if and only if  $a$  is a norm from  $\mathcal{F}(b^{1/n})$ .

**2.2.2.  $S$ -power residue symbols.** — Let  $k$  be a global field which contains all  $n$ -th roots of unity. We fix, once and for all,  $\varepsilon : \mu_n(k) \hookrightarrow \mathbb{C}^*$  an embedding into the complex numbers. For every  $v \in \mathcal{V}_k$  the embedding  $k \hookrightarrow k_v$  induces an isomorphism  $\mu_n(k) \xrightarrow{\sim} \mu_n(k_v)$ , and using this isomorphism we identify  $\mu_n(k_v)$  with  $\mu_n(k)$  for every  $v$ , so the Hilbert symbols on  $k_v$  takes value in  $\mu_n(k)$ .

For  $x, a \in \mathfrak{o}_S$  we define the  $S$ -power residue symbol following [14] by

$$\left(\frac{x}{a}\right)_S = \begin{cases} \prod_{v \notin S} (x, a)_v & \text{if } \gcd(x, a) = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

where  $(x, a)_v$  is the local Hilbert symbol on  $k_v$  introduced in §2.2.1.

**2.2.3. The symbol  $(\cdot, \cdot)_S$ .** — Fix the conditions on  $k, n$ , and  $S \subset \mathcal{V}_k$  as in §2.1.5. Let  $\mathbb{A}$  be the ring of adèles of  $k$ . Define

$$k_S = \prod_{v \in S} k_v, \quad \mathbb{A}_S = \left(\prod_{v \notin S} \mathcal{O}_v\right) \times \left(\prod_{v \in S} k_v\right), \quad \text{and} \quad \mathbb{A}^S = \prod_{v \notin S} k_v. \quad (2.6)$$

Note that  $\mathfrak{o}_S = k \cap \mathbb{A}_S, \mathbb{A} = \mathbb{A}^S \mathbb{A}_S$ .

The  $n$ -th order  $S$ -Hilbert symbol  $(-, -)_S : k_S^\times \times k_S^\times \rightarrow \mu_n$  is defined by

$$(x, y)_S := \prod_{v \in S} (x, y)_v = \prod_{v \notin S} (x, y)_v^{-1}. \quad (2.7)$$

The following lemma follows from [5, Lemma 2].

**Lemma.** — (i)  $(x, y)_S = 1$  for  $x, y \in \mathfrak{o}_S^\times$ .

(ii) Let  $\Omega = \mathfrak{o}_S^\times k_S^{\times, n}$ , then  $(x, y)_S = 1$  for  $x, y \in \Omega$ .

### 2.3. Gauss Sums. —

**2.3.1. Additive and multiplicative characters.** — Let  $\mathcal{F}$  be a non-archimedean local field. Let  $\psi : \mathcal{F} \rightarrow \mathbb{C}^*$  be an additive character. The *conductor* of  $\psi$  is the maximal integer  $k$  such that  $\psi|_{\pi^{-k}\mathcal{O}}$  is trivial.  $\psi$  is called *unramified* if it has conductor 0.

Note that for any  $x \in \mathcal{F}^*$ , the map  $y \mapsto (x, y)$  is a multiplicative character of  $\mathcal{F}^*$ .

**2.3.2. Gauss sums.** — Let  $\psi : \mathcal{F} \rightarrow \mathbb{C}^*$  be an unramified character. Define

$$G(k, b) = q \int_{\mathcal{O}^\times} (r, \pi)^k \psi(\pi^b r^{-1}) dr$$

In particular, for  $b = -1$ , we denote  $\mathbf{g}_k := G(k, -1)$ . We summarize the properties of Gauss sums as follows:

**Proposition ([37]).** — Suppose  $2n \mid (q-1)$ .

(i) If  $b \leq -2$ , then  $G(k, b) = 0$ .

(ii) If  $b \geq 0$ , then

$$G(k, b) = q \int_{\mathcal{O}^\times} (r, \pi)^{-k} dr = \begin{cases} q-1, & n \mid k \\ 0, & \text{otherwise.} \end{cases}$$

- (iii)  $\mathfrak{g}_k = \mathfrak{g}_l$  if  $n|(k-l)$ ;
- (iv)  $\mathfrak{g}_0 = -1$ ;
- (v)  $\mathfrak{g}_k \mathfrak{g}_{-k} = q$  if  $n \nmid k$ .

### 3. Weyl group multiple Dirichlet series

#### 3.1. Root and Metaplectic datum. —

**3.1.1. Root data.** — For a more detailed exposition of this part see [3]. A *root datum* is a quadruple  $\mathfrak{D} = (\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$  where

- $\Lambda$  and  $\Lambda^\vee$  are free abelian groups of finite rank, in duality by a pairing  $\langle -, - \rangle : \Lambda \times \Lambda^\vee \rightarrow \mathbb{Z}$ .
- $\Phi$  and  $\Phi^\vee$  are finite subsets of  $\Lambda$  and  $\Lambda^\vee$  respectively, there is a bijection  $\Phi \rightarrow \Phi^\vee$  denoted  $\alpha \mapsto \alpha^\vee$ .

They are supposed to satisfy the following axioms:

(RD1)  $\langle \alpha, \alpha^\vee \rangle = 2$  for every  $\alpha \in \Phi$ .

(RD2)  $s_\alpha(\Phi) \subseteq \Phi$  and  $s_{\alpha^\vee}(\Phi^\vee) \subseteq \Phi^\vee$ , where  $s_\alpha : \Lambda \rightarrow \Lambda$  is defined by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha.$$

and similarly  $s_{\alpha^\vee}(\Phi)$  is defined by

$$s_{\alpha^\vee}(y) = y - \langle y, \alpha \rangle \alpha^\vee.$$

Let  $W$  be the Weyl group of the root datum. It is identified with the subgroup of  $\text{Aut}(\Lambda)$  generated by  $\{s_\alpha : \alpha \in \Phi\}$  and the subgroup of  $\text{Aut}(\Lambda^\vee)$  generated by  $\{s_{\alpha^\vee} : \alpha^\vee \in \Phi^\vee\}$ .

**3.1.2. Based root data.** — It follows that  $(\Phi, \Lambda \otimes \mathbb{Q})$  is a root system. A *based root datum* is a sextuple  $\mathfrak{D} = (\Lambda, \Phi, \Delta, \Lambda^\vee, \Phi^\vee, \Delta^\vee)$  such that  $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$  is a root datum and  $\Delta \subseteq \Phi$  is an ordered basis of the root system  $(\Phi, \Lambda \otimes \mathbb{Q})$ . Since  $\Delta \subseteq \Lambda$  and  $\Delta^\vee \subseteq \Lambda^\vee$  determines  $\Phi$  and  $\Phi^\vee$  uniquely, usually a based root datum is denoted as the quadruple  $\mathfrak{D} = (\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$ . Usually we denote the ordered set of simple roots  $\Delta$  by  $\Delta = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$  and similarly denote  $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ .

The based root datum is called

- *semisimple* if  $\text{rank} \Lambda = \text{rank} \Lambda^\vee = |\Delta|$ ,
- *adjoint* if  $\Delta$  is a basis of  $\Lambda$ ,
- *simply-connected* if  $\Delta^\vee$  is a basis of  $\Lambda^\vee$ .

**3.1.3. Weyl group.** — For  $i = 1, \dots, r$  let  $s_i \in W$  be the simple reflection corresponding to  $\alpha_i$  or  $\alpha_i^\vee$ .  $(W, \{s_i : i = 1, \dots, r\})$  is a Coxeter system.

**3.1.4. Positive roots and coroots.** — let  $\Phi_+$  (resp.  $\Phi_+^\vee$ ) be the set of positive roots in  $\Phi$  with respect to  $\Delta$  (resp. positive coroots in  $\Phi^\vee$  with respect to  $\Delta^\vee$ ). Let  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \in \Lambda$  be the Weyl vector. It has the property that  $\langle \rho, \alpha_i^\vee \rangle = 1$  for every simple coroot  $\alpha_i^\vee \in \Delta^\vee$ .

**3.1.5. Metaplectic structures.** — In the rest of this section we fix a positive integer  $n$ . Let  $Q : \Lambda^\vee \rightarrow \mathbb{Z}$  be the unique  $\mathbb{Z}$ -valued  $W$ -invariant quadratic form on  $\Lambda^\vee$  such that  $Q$  takes value 1 on short coroots. Let  $B : \Lambda^\vee \times \Lambda^\vee \rightarrow \mathbb{Z}$  be the associated bilinear form defined by

$$B(\lambda^\vee, \mu^\vee) = Q(\lambda^\vee + \mu^\vee) - Q(\lambda^\vee) - Q(\mu^\vee). \quad (3.1)$$

We define

$$\Lambda_0^\vee = \{\lambda^\vee \in \Lambda^\vee : B(\lambda^\vee, \alpha_i^\vee) \equiv 0 \pmod{n} \text{ for all } i \in I\}. \quad (3.2)$$

We note that by  $W$ -invariance of  $Q$  and  $B$  we actually have

$$B(\lambda^\vee, \alpha_i^\vee) = \langle \lambda^\vee, \alpha_i \rangle Q(\alpha_i^\vee).$$

For every  $\alpha^\vee \in \Phi^\vee$ , let  $n(\alpha^\vee)$  be the smallest integer such that  $n(\alpha^\vee)Q(\alpha^\vee)$  is a multiple of  $n$ , and let  $\tilde{\alpha} = n(\alpha^\vee)\alpha^\vee \in \Lambda^\vee$ .

**Lemma** ([32]). — (i)  $\tilde{\alpha}^\vee \in \Lambda_0^\vee$  for every  $\alpha^\vee \in \Phi^\vee$  and  $\{\tilde{\alpha}^\vee : \alpha^\vee \in \Phi^\vee\}$  is a root system in  $\Lambda_0^\vee \otimes \mathbb{Q}$ .

(ii) Moreover, let  $\Lambda_0 \in \Lambda \otimes \mathbb{Q}$  be the dual lattice of  $\Lambda_0^\vee$ , namely

$$\Lambda_0 = \{\lambda \in \Lambda \otimes \mathbb{Q} : \langle \lambda, \tilde{\alpha}_i^\vee \rangle = 1 \text{ for } i = 1, 2, \dots, r.\}$$

Then  $(\Lambda_0^\vee, \{n_i \alpha_i^\vee\}_{i=1}^r, \Lambda_0, \{n_i^{-1} \alpha_i\}_{i=1}^r)$  is a based root datum, called the metaplectic dual root datum of  $(\mathcal{D}, Q, n)$ , denoted  $\mathcal{D}_{(Q,n)}^\vee$ .

**3.1.6.** — In the sequel we will use the following notations: we let  $Q_i := Q(\alpha_i^\vee)$ ,  $B_{ij} = B(\alpha_i^\vee, \alpha_j^\vee)$ , and  $n_i := n(\alpha_i^\vee)$ .

## 3.2. The Chinta-Gunnells action. —

**3.2.1.** *The generic ring and specializations.* — Let  $\mathbb{C}_v = \mathbb{C}[v, v^{-1}]$ ,  $\mathbb{C}_{v,\mathfrak{g}}$  be the ring  $\mathbb{C}_v[\mathfrak{g}_k : k \in \mathbb{Z}]$  where  $\mathfrak{g}_k$  are formal parameters satisfying the conditions

- $\mathfrak{g}_k = \mathfrak{g}_l$  if  $n|(k-l)$ .
- $\mathfrak{g}_0 = -1$ .
- if  $n \nmid k$  then  $\mathfrak{g}_{-k}\mathfrak{g}_k = v^{-1}$ .

For every non-archimedean local field  $\mathcal{F}$ , let  $s_{\mathcal{F}} : \mathbb{C}_{v,\mathfrak{g}} \rightarrow \mathbb{C}$  be the ring homomorphism defined by

$$v \mapsto q^{-1}, \mathfrak{g}_k \mapsto \mathbf{g}_k$$

where  $\mathbf{g}_k$  are the Gauss sums defined in §2.3.2. It induces a map

$$s_{\mathcal{F}} : \mathbb{C}_{v,\mathfrak{g}}[\Lambda^\vee] \rightarrow \mathbb{C}[\Lambda^\vee].$$

For  $p \in \mathbb{C}[\Lambda^\vee]$ ,  $s_{\mathcal{F}}(p)$  is called the *specialization* of  $p$  in  $\mathcal{F}$ .

**3.2.2.** *The Chinta-Gunnells action.* — Let  $J$  be the smallest multiplicative-closed subset of  $\mathbb{C}_{v,\mathfrak{g}}[\Lambda^\vee]$  containing  $1 - e^{-\tilde{\alpha}_i^\vee}$  and  $1 - ve^{-\tilde{\alpha}_i^\vee}$  for every  $\alpha^\vee \in \Phi^\vee$ . Let  $\mathbb{C}_{v,\mathfrak{g}}[\Lambda^\vee]_J$  be the localization of  $\mathbb{C}_{v,\mathfrak{g}}[\Lambda^\vee]$  by  $J$ .

For each  $\lambda^\vee \in \Lambda^\vee$  and  $\alpha_i \in \Delta$ , following [37] we define

$$s_i \star e^{\lambda^\vee} = \frac{e^{s_i \lambda^\vee}}{1 - ve^{-\tilde{\alpha}_i^\vee}} \left( (1-v)e^{\text{res}_{n_i} \left( \frac{B(\lambda^\vee, \alpha_i^\vee)}{Q(\alpha_i^\vee)} \right) \alpha_i^\vee} - v^{\mathfrak{g}_{Q(\alpha_i^\vee) + B(\lambda^\vee, \alpha_i^\vee)}} e^{-\alpha_i^\vee} (e^{\tilde{\alpha}_i^\vee} - 1) \right) \quad (3.3)$$

where  $\text{res}_{n_i} : \mathbb{Z} \rightarrow \{0, 1, \dots, n_i - 1\}$  is the residue map for division by  $n(\alpha^\vee)$ . It is proved in [37] that this formula could be extended to a  $W$ -action on  $\mathbb{C}_{v,\mathfrak{g}}[\Lambda^\vee]_J$ . We also note that  $\frac{B(\lambda^\vee, \alpha_i^\vee)}{Q(\alpha_i^\vee)} = \langle \lambda^\vee, \alpha_i \rangle$ .

We define the following element in  $\mathbb{C}_{v,\mathfrak{g}}$ :

$$\widetilde{\text{CS}}(\lambda^\vee) = q^{-\langle \rho, \lambda^\vee \rangle} \prod_{\alpha \in R_+} \frac{1 - ve^{-n(\alpha^\vee)\alpha^\vee}}{1 - e^{-n(\alpha^\vee)\alpha^\vee}} \sum_{w \in W} (-1)^{\ell(w)} \left( \prod_{\beta^\vee \in R^\vee(w^{-1})} e^{-n(\beta^\vee)\beta^\vee} \right) w \star e^{\lambda^\vee} \quad (3.4)$$

This is the metaplectic version of the Casselman-Shalika formula [33, 37].

**3.3. Definition of Weyl group multiple Dirichlet series.** — Let  $k$  be a global field containing all  $2n$ -th root of unity and  $S \subset \mathcal{V}_k$  a finite set of places satisfying the conditions of §2.1.5. Recall also the  $S$ -power residue symbols  $(\cdot)_S$  defined in §2.2.3. We also fix a based root datum  $\mathfrak{D} = (\Lambda, \{\alpha_i\}_{i \in I}, \Lambda^\vee, \{\alpha_i^\vee\}_{i \in I})$  which is semisimple and simply-connected. Let  $Q$  and  $B$  be defined as in §3.1.5. Write  $r := |I|$ . Fix an embedding  $\varepsilon : \mu_n(k) \hookrightarrow \mathbb{C}^*$ .

Throughout this article, we will use the following notation for  $r$ -tuples of objects in a set  $X$ : by  $\underline{C} \in X^r$  we mean an  $r$ -tuple of objects  $\underline{C} = (C_1, \dots, C_r)$  with  $C_i \in X$  for  $i = 1, \dots, r$ .

**3.3.1. Twisted multiplicativity.** — A function  $H : (\mathfrak{o}_S \setminus \{0\})^r \rightarrow \mathbb{C}$  is said to be *twisted multiplicative* if for  $C_1, \dots, C_r, C'_1, \dots, C'_r \in \mathfrak{o}_S \setminus \{0\}$  with  $\gcd(C_1 \cdots C_r, C'_1 \cdots C'_r) = 1$ , we have

$$\begin{aligned} & \frac{H(C_1 C'_1, \dots, C_r C'_r)}{H(C_1, \dots, C_r) H(C'_1, \dots, C'_r)} \\ &= \prod_{i=1}^r \varepsilon \left( \frac{C_i}{C'_i} \right)_S^{Q(\alpha_i^\vee)} \varepsilon \left( \frac{C'_i}{C_i} \right)_S^{Q(\alpha_i^\vee)} \prod_{1 \leq i < j \leq r} \varepsilon \left( \frac{C_i}{C_j} \right)_S^{B(\alpha_i^\vee, \alpha_j^\vee)} \varepsilon \left( \frac{C'_i}{C'_j} \right)_S^{B(\alpha_i^\vee, \alpha_j^\vee)} \end{aligned} \quad (3.5)$$

Note that there are  $\varepsilon$ 's because we defined the power residue symbol to take value in  $\mu_n(k)$ .

Recall the conventions from §2.1.7. In particular, for each  $C \in \mathfrak{o}_S \setminus \{0\}$  the integers  $n_\nu(C)$  are defined for all  $\nu \notin S$ , and almost all of them are equal to 0.

Since  $\mathfrak{o}_S$  is a unique factorization domain by our assumption on  $S$ , every element  $C \in \mathfrak{o}_S \setminus \{0\}$  factorizes uniquely as

$$C = a \cdot \prod_{\nu \notin S} \pi_\nu^{n_\nu(C)}$$

where  $a \in \mathfrak{o}_S^\times$  is an  $S$ -unit. Such a factorization depends on the choice of  $\pi_\nu$  for each  $\nu \notin S$ , for which we fixed as in §2.1.6. Let  $\mathfrak{o}_S^+$  be the subset of  $\mathfrak{o}_S$  consisting of elements with  $a = 1$  in this unique factorization, namely

$$\mathfrak{o}_S^+ = \left\{ \prod_{\nu \notin S} \pi_\nu^{k_\nu} : k_\nu = 0 \text{ for almost all } \nu \notin S \right\} \subseteq \mathfrak{o}_S \setminus \{0\}. \quad (3.6)$$

Clearly  $\mathfrak{o}_S^+$  is a set of representatives of  $(\mathfrak{o}_S \setminus \{0\})/\mathfrak{o}_S^\times$ . Also note that in the notation of §2.1.7, for every  $C \in \mathfrak{o}_S^+$  we have

$$C = \prod_{\nu \notin S, \pi_\nu | C} C_\nu \quad (3.7)$$

**Lemma.** — Let  $C_1, \dots, C_r \in \mathfrak{o}_S^+$ . Then we have

$$H(C_1, \dots, C_r) = \varepsilon(D(C_1, \dots, C_r)) \prod_{\nu \notin S} H\left(\pi_\nu^{n_\nu(C_1)}, \dots, \pi_\nu^{n_\nu(C_r)}\right) \quad (3.8)$$

where we define

$$D(C_1, \dots, C_r) = \prod_{\substack{\omega, \nu \notin S \\ \omega \neq \nu}} \left( \prod_{i=1}^r (C_{i,\nu}, C_{i,\omega})_\nu^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (C_{i,\omega}, C_{j,\nu})_\nu^{B_{ij}} \right) \quad (3.9)$$

*Démonstration.* — For every  $\underline{C} = (C_1, \dots, C_r) \in (\mathfrak{o}_S^+)^r$  we define

$$E(\underline{C}) = \frac{H(C_1, \dots, C_r)}{\prod_{\nu \notin S} H(\pi_\nu^{n_\nu(C_1)}, \dots, \pi_\nu^{n_\nu(C_r)})} \quad (3.10)$$

So our goal is to prove that  $\varepsilon(D(\underline{C})) = E(\underline{C})$  for every  $\underline{C} = (C_1, \dots, C_r) \in (\mathfrak{o}_S^+)^r$ . We do this by induction on the size of the set  $\Sigma(\underline{C})$  of all the prime factors of  $C_1, \dots, C_r$ . If  $|\Sigma(\underline{C})| = 1$ , clearly by definition we have  $\varepsilon(D(\underline{C})) = E(\underline{C}) = 1$ .

To finish the induction, we let  $\underline{C}' = (C_1\pi^{l_1}, \dots, C_r\pi^{l_r})$  for  $\pi = \pi_{v_0}$  with  $v_0 \notin \Sigma(\underline{C})$ , so that  $\Sigma(\underline{C}') = \Sigma(\underline{C}) \sqcup \{v_0\}$ . Then by (3.5) we have

$$\begin{aligned} & H(C_1\pi^{l_1}, \dots, C_r\pi^{l_r}) \\ &= H(C_1, \dots, C_r)H(\pi^{l_1}, \dots, \pi^{l_r}) \prod_{i=1}^r \varepsilon\left(\frac{C_i}{\pi^{l_i}}\right)_S^{Q_i} \varepsilon\left(\frac{\pi^{l_i}}{C_i}\right)_S^{Q_i} \prod_{1 \leq i < j \leq r} \varepsilon\left(\frac{C_i}{\pi^{l_j}}\right)_S^{B_{ij}} \varepsilon\left(\frac{\pi^{l_i}}{C_j}\right)_S^{B_{ij}} \end{aligned}$$

so by (2.5) we have

$$\begin{aligned} \frac{E(\underline{C}')}{E(\underline{C})} &= \left( \prod_{i=1}^r \varepsilon(C_i, \pi^{l_i})_{v_0}^{Q_i} \right) \left( \prod_{i=1}^r \prod_{v|C_i} \varepsilon(\pi^{l_i}, C_i)_v^{Q_i} \right) \\ &\quad \cdot \left( \prod_{1 \leq i < j \leq r} \varepsilon(C_i, \pi^{l_j})_{v_0}^{B_{ij}} \right) \left( \prod_{1 \leq i < j \leq r} \prod_{v|C_i} \varepsilon(\pi^{l_i}, C_j)_v^{B_{ij}} \right) \\ &= \prod_{v \in \Sigma(\underline{C})} \left( \prod_{i=1}^r \varepsilon(C_{i,v}, \pi^{l_i})_{v_0}^{Q_i} \right) \left( \prod_{i=1}^r \varepsilon(\pi^{l_i}, C_{i,v})_v^{Q_i} \right) \\ &\quad \cdot \left( \prod_{1 \leq i < j \leq r} \varepsilon(C_{i,v}, \pi^{l_j})_{v_0}^{B_{ij}} \right) \left( \prod_{1 \leq i < j \leq r} \varepsilon(\pi^{l_i}, C_{j,v})_v^{B_{ij}} \right) \end{aligned}$$

where we used (3.7) and the fact that for  $v \notin S$  we have  $(\mathcal{O}_v^\times, \mathcal{O}_v^\times)_v = 1$ .

On the other hand, by (3.9) we have

$$\begin{aligned} \frac{D(\underline{C}')}{D(\underline{C})} &= \left( \prod_{\substack{\omega=v_0 \\ v \in \Sigma(\underline{C})}} \left( \prod_{i=1}^r (C_{i,v}, \pi^{l_i})_v^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (\pi^{l_i}, C_{j,v})_v^{B_{ij}} \right) \right) \\ &\quad \cdot \left( \prod_{\substack{v=v_0 \\ \omega \in \Sigma(\underline{C})}} \left( \prod_i (\pi^{l_i}, C_{i,\omega})_{v_0}^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (C_{i,\omega}, \pi^{l_j})_{v_0}^{B_{ij}} \right) \right) \\ &= \prod_{v \in \Sigma(\underline{C})} \left( \prod_{i=1}^r (C_{i,v}, \pi^{l_i})_v^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (\pi^{l_i}, C_{j,v})_v^{B_{ij}} \right) \\ &\quad \cdot \left( \prod_{i=1}^r (\pi^{l_i}, C_{i,v})_{v_0}^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (C_{i,v}, \pi^{l_j})_{v_0}^{B_{ij}} \right) \end{aligned}$$

So we conclude that whenever  $|\Sigma(\underline{C}')| = |\Sigma(\underline{C})| + 1$ , we have

$$\frac{\varepsilon(D(\underline{C}'))}{\varepsilon(D(\underline{C}))} = \frac{E(\underline{C}')}{E(\underline{C})}$$

and thus by induction, we proved that  $\varepsilon(D(\underline{C})) = E(\underline{C})$  for every  $\underline{C} = (C_1, \dots, C_r) \in (\mathfrak{o}_S^+)^r$ .  $\square$

**3.3.2. Defining the Weyl Group Multiple Dirichlet Series (WMDS).** — Following [7, 11], Weyl group multiple Dirichlet series associated to  $(\mathfrak{D}, n, S, \mathbb{Q})$  is a function in complex variables  $s_1, \dots, s_r$  of the form

$$Z_\Psi(s_1, \dots, s_r) = \sum_{0 \neq C_1, \dots, C_r \in \mathfrak{o}_S / \mathfrak{o}_S^\times} H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}. \quad (3.11)$$

where  $H : (\mathfrak{o}_S - \{0\})^r \rightarrow \mathbb{C}$  is some twisted multiplicative function and  $\Psi : (k_S^\times)^r \rightarrow \mathbb{C}$  satisfies

$$\frac{\Psi(\gamma_1 C_1, \dots, \gamma_r C_r)}{\Psi(C_1, \dots, C_r)} = \prod_{i=1}^r \varepsilon(\gamma_i, C_i)_S^{\mathbb{Q}_i} \prod_{1 \leq i < j \leq r} \varepsilon(\gamma_i, C_j)_S^{\mathbb{B}_{ij}} \quad (3.12)$$

for  $C_1, \dots, C_r \in k_S^\times$  and  $\gamma_1, \dots, \gamma_r \in \Omega$  (we recall that  $\Omega$  was defined in Lemma 2.2.3). In practice, we have a specific choice of  $H$  in mind, and allow ourselves some freedom in specifying  $\Psi$ .

Note that in the definition (3.11), the summation is over the set  $((\mathfrak{o}_S \setminus \{0\}) / \mathfrak{o}_S^\times)^r$  because the function  $H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}$  is invariant under  $(\mathfrak{o}_S^\times)^r$ . In practice we find it more convenient to sum over the set  $(\mathfrak{o}_S^+)^r$  of representatives of  $((\mathfrak{o}_S \setminus \{0\}) / \mathfrak{o}_S^\times)^r$ , namely we actually use

$$Z_\Psi(s_1, \dots, s_r) = \sum_{C_1, \dots, C_r \in \mathfrak{o}_S^+} H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}. \quad (3.13)$$

When  $\Psi$  is unchanged throughout the discussion, we drop it from the notation.

**3.3.3. Specifying  $H$ .** — By Lemma 3.3.1, the values of the coefficients  $H(C_1, \dots, C_r)$  for  $C_1, \dots, C_r \in \mathfrak{o}_S^+$  are determined by the values  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r})$  for all places  $v \notin S$  and  $k_1, \dots, k_r \geq 0$ . As mentioned above, we have a specific choice in mind for these, and they are defined as follows: for non-negative integers  $k_1, \dots, k_r$ , first define the elements  $\mathcal{H}(k_1, \dots, k_r) \in \mathbb{C}_{v, \mathfrak{g}}$  using

$$\begin{aligned} \widetilde{\mathbf{CS}}(0) &:= q^{-\langle \rho, \lambda^\vee \rangle} \prod_{\alpha \in R_+} \frac{1 - v e^{-n(\alpha^\vee)} \alpha^\vee}{1 - e^{-n(\alpha^\vee)} \alpha^\vee} \sum_{w \in W} (-1)^{\ell(w)} \left( \prod_{\beta^\vee \in R^\vee(w^{-1})} e^{-n(\beta^\vee)} \beta^\vee \right) w \star e^0 \\ &= \sum_{k_1, \dots, k_r \geq 0} \mathcal{H}(k_1, \dots, k_r) v^{k_1 + \dots + k_r} e^{-k_1 \alpha_1^\vee - \dots - k_r \alpha_r^\vee} \end{aligned}$$

Then, for  $v \notin S$ , and  $k_1, \dots, k_r \geq 0$  integers, define  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r})$  as the specialization of  $\mathcal{H}(k_1, \dots, k_r)$  by setting  $v \mapsto q_v^{-1}$  and  $\mathfrak{g}_i \mapsto \mathfrak{g}_i$  for  $i \in \mathbb{Z}$ . For those  $k_1, \dots, k_r \in \mathbb{Z}$  such that  $\mathcal{H}(k_1, \dots, k_r)$  is not defined, we then take  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r}) = 0$ . For future reference, we define the support of  $\widetilde{\mathbf{CS}}(0)$  to be the following collection of  $\Lambda^\vee$ ,

$$\text{supp} \widetilde{\mathbf{CS}}(0) = \{k_1 \alpha_1^\vee + \dots + k_r \alpha_r^\vee \mid \mathcal{H}(k_1, \dots, k_r) \neq 0\}. \quad (3.14)$$

Note that this is actually the "inverse" of the support of  $\widetilde{\mathbf{CS}}(0)$ .

**Example.** — (i) Suppose  $n \geq 3$ . For the root system of type  $A_1$ , let  $\alpha^\vee$  be the unique positive root, then we have

$$\widetilde{\mathbf{CS}}(0) = 1 + v \mathfrak{g}_1 e^{-\alpha^\vee}.$$

(ii) Suppose  $n \geq 4$ . For the root system of type  $A_2$ , let  $\alpha_1^\vee, \alpha_2^\vee$  be the two positive simple roots, then we have

$$\widetilde{\mathbf{CS}}(0) = 1 + v\mathfrak{g}_1 e^{-\alpha_1^\vee} + v\mathfrak{g}_1 e^{-\alpha_2^\vee} + v^3 \mathfrak{g}_1 \mathfrak{g}_2 e^{-2\alpha_1^\vee - \alpha_2^\vee} + v^3 \mathfrak{g}_1 \mathfrak{g}_2 e^{-\alpha_1^\vee - 2\alpha_2^\vee} + v^4 \mathfrak{g}_1^2 \mathfrak{g}_2 e^{-2\alpha_1^\vee - 2\alpha_2^\vee}.$$

**Remarks.** — There are different ways to define the coefficients  $H(\pi_v^{k_1}, \dots, \pi_v^{k_r})$  for  $v \notin S$  as mentioned in §1.1.3. The description given here is essentially using the averaging method of Chinta-Gunnells [13].

**3.3.4.** *The maps  $\log^S$  and  $\log_v$ .* — Recall that for each  $v \notin S$ , let  $n_v(C_i)$  be defined as in §2.1.7. Define two maps

$$\log_v : (\mathfrak{o}_S^+)^r \rightarrow \Lambda^\vee, \quad \underline{C} \mapsto \log_v \underline{C} := \sum_{i=1}^r n_v(C_i) \alpha_i^\vee \quad (3.15)$$

$$\log^S : (\mathfrak{o}_S^+)^r \rightarrow \bigoplus_{v \notin S} \Lambda^\vee, \quad \underline{C} \mapsto \log^S \underline{C} := (\log_v \underline{C})_{v \notin S}. \quad (3.16)$$

**3.3.5.** *Regrouping the sum  $Z$ .* — By (3.9) we see that  $H(\underline{C}) := H(C_1, \dots, C_r) = 0$  if for some  $v \notin S$  we have  $\log_v \underline{C} \notin \text{supp } \widetilde{\mathbf{CS}}_v(0)$ . Hence, if we define

$$\text{supp}(Z) = \{ \underline{C} = (C_1, \dots, C_r) \in (\mathfrak{o}_S^+)^r : \log_v \underline{C} \in \text{supp } \widetilde{\mathbf{CS}}_v(0) \text{ for all } v \notin S \}. \quad (3.17)$$

Then we have

$$Z(s_1, \dots, s_r) = \sum_{\underline{C} \in \text{supp}(Z)} H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N} C_1^{-s_1} \dots \mathbb{N} C_r^{-s_r}. \quad (3.18)$$

We shall actually need a slight refinement of this description in the proof of the main result Theorem 5.3.3. To state it, consider

$$p_Z : \text{supp}(Z) \xrightarrow{\log^S} \bigoplus_{v \notin S} \Lambda^\vee \longrightarrow \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee, \quad (3.19)$$

where the latter map is the natural projection (recall the sublattice  $\Lambda_0^\vee \subset \Lambda^\vee$  was defined in (3.2)). For any  $\underline{\lambda}^\vee = (\lambda_v^\vee) \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ , we may describe its fiber under  $p$  as

$$\text{supp}(Z; \underline{\lambda}^\vee) := p_Z^{-1}(\underline{\lambda}^\vee) = \{ \underline{C} \in \text{supp}(Z) : \log_v \underline{C} \equiv -\lambda_v^\vee \pmod{\Lambda_0^\vee} \} \quad (3.20)$$

Then we have a decomposition of  $\text{supp}(Z)$  into a disjoint union

$$\text{supp}(Z) = \bigsqcup_{\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee} \text{supp}(Z; \underline{\lambda}^\vee)$$

For  $\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ , let

$$Z_{\underline{\lambda}^\vee}(s_1, \dots, s_r) = \sum_{\underline{C} \in \text{supp}(Z; \underline{\lambda}^\vee)} H(C_1, \dots, C_r) \Psi(C_1, \dots, C_r) \mathbb{N} C_1^{-s_1} \dots \mathbb{N} C_r^{-s_r}. \quad (3.21)$$

Then we have

$$Z(s_1, \dots, s_r) = \sum_{\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee} Z_{\underline{\lambda}^\vee}(s_1, \dots, s_r). \quad (3.22)$$

**3.3.6.** — For every place  $v \notin S$  and  $\underline{C} \in (\mathfrak{o}_S^+)^r$  let

$$\begin{aligned} D(\underline{C}; v) &= \left( \prod_i \varepsilon(C_{i,v}, C_i^\vee)_v^{-Q_i} \right) \left( \prod_{i < j} \varepsilon(C_i^\vee, C_{j,v})_v^{B_{ij}} \right) \\ &= \left( \prod_i \prod_{\omega \neq v} \varepsilon(C_{i,v}, C_{i,\omega})_v^{-Q_i} \right) \left( \prod_{i < j} \prod_{\omega \neq v} \varepsilon(C_{i,\omega}, C_{j,v})_v^{B_{ij}} \right) \end{aligned} \quad (3.23)$$



They are elements of  $\mu_n(k)$  that only depend on the prime factorization of  $\underline{C}$ , and it is not hard to see that for  $\underline{C}, \underline{C}' \in \text{supp}(Z; \underline{\lambda}^\vee)$ , we have  $D(\underline{C}; \nu) = D(\underline{C}'; \nu)$  for every  $\nu \notin S$ , so  $D(\underline{C}; \nu)$  depends only on  $\underline{\lambda}^\vee = \log^S \underline{C} \in \bigoplus_{\nu \notin S} \Lambda^\vee / \Lambda_0^\vee$ , and we also denote it by  $D(\underline{\lambda}^\vee; \nu)$ ,

Note that by (3.9) we have

$$D(\underline{C}) = \prod_{\nu \notin S} D(\underline{C}; \nu) \quad (3.24)$$

for  $\underline{C} \in (\mathfrak{o}_S^+)^r$ . So for  $\underline{C}, \underline{C}' \in \text{supp}(Z; \underline{\lambda}^\vee)$  we also have  $D(\underline{C}) = D(\underline{C}')$  and we similarly denote  $D(\underline{C})$  by  $D(\underline{\lambda}^\vee)$ . These will be used in the proof of the main result Theorem 5.3.3 of this paper.

#### 4. Local metaplectic groups

Let  $\mathcal{F}$  be a non-archimedean local field of characteristic 0,  $\mathbf{G}$  be a simply-connected Chevalley group with root datum  $\mathfrak{D}$ ,  $n$  be a positive integer such that  $\mathcal{F}$  contains all  $n$ -th roots of unity. In this section we will construct the metaplectic central extension of  $\mathbf{G}(\mathcal{F})$  by  $\mu_n(\mathcal{F})$ , denoted  $\tilde{\mathbf{G}}$ , and study the unramified principal series representations of  $\tilde{\mathbf{G}}$  under a stronger assumption that  $2n \mid (q-1)$ , where we recall that  $q$  is the residue characteristic of  $\mathcal{F}$ .

**4.0.1. Notation conventions.** — In this section we will use the notations in §2.1.1 and §3.1. In addition, we will use boldface letters  $\mathbf{A}, \mathbf{B}, \dots$  to denote group schemes over  $\mathbb{Z}$ , and use the corresponding letters  $A, B, \dots$  for the corresponding groups of  $\mathcal{F}$ -points  $\mathbf{A}(\mathcal{F}), \mathbf{B}(\mathcal{F}), \dots$ . The groups of  $\mathcal{O}$ -points will be denoted  $A_{\mathcal{O}}, B_{\mathcal{O}}, \dots$ .

**4.1. Chevalley groups over  $p$ -adic fields.** — For this part we follow [41].

**4.1.1. Chevalley group scheme and subgroups.** — Let  $\mathbf{G}$  be the Chevalley group scheme associated to a semisimple simply-connected based root datum  $\mathfrak{D} = (\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$ . It is a group scheme over  $\mathbb{Z}$ , and for every field  $e$ , the group scheme  $\mathbf{G} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(e)$  obtained from  $\mathbf{G}$  by base change is a split semisimple simply-connected group scheme over  $k$ . Let  $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$  be the maximal split torus and the Borel subgroup that are constructed along with the Chevalley group  $\mathbf{G}$ .

**4.1.2. Root subgroups.** — For each root  $\alpha \in \Phi$  there is a root subgroup  $\mathbf{U}_\alpha \subseteq \mathbf{G}$ , and an isomorphism  $\times_\alpha : \mathbf{G}_m \xrightarrow{\sim} \mathbf{U}_\alpha$  that are constructed along with  $\mathbf{G}$ . Let  $\mathbf{U}$  be the unipotent radical generated by  $\mathbf{U}_\alpha$  for  $\alpha \in \Phi_+$ , let  $\mathbf{U}^-$  be the unipotent radical generated by  $\mathbf{U}_\alpha$  for  $\alpha \in \Phi_-$ . We have semidirect products  $\mathbf{B} = \mathbf{U}\mathbf{T}$ ,  $\mathbf{B}^- = \mathbf{U}^-\mathbf{T}$ .

we will denote  $\times_{\alpha_i}$  (resp.  $\times_{-\alpha_i}$ ) by  $\times_i$  (resp.  $\times_{-i}$ ), and denote  $U_{\alpha_i}$  (resp.  $U_{-\alpha_i}$ ) by  $U_i$  (resp.  $U_{-i}$ ).

**4.1.3. Torus.** — The torus  $\mathbf{T}$  is split, and  $\Lambda^\vee$  is the cocharacter lattice of  $\mathbf{T}$ . For every  $\lambda^\vee \in \Lambda^\vee$ , the corresponding cocharacter of  $\mathbf{T}$  is denoted  $(-)^{\lambda^\vee} : \mathbf{G}_m \rightarrow \mathbf{T}$ . Also the group functor  $\mathbf{T}$  can be identified with the functor that sends every (unital commutative) ring  $R$  into the group  $\Lambda^\vee \otimes_{\mathbb{Z}} R^\times$ .

If  $\alpha_i^\vee \in \Lambda^\vee$  is a simple coroot, the cocharacter  $(-)^{\alpha_i^\vee}$  is also denoted  $h_i : \mathbf{G}_m \rightarrow \mathbf{T}$ .

Let  $e$  be any field. Since  $\mathfrak{D}$  is simply-connected,  $\Lambda^\vee$  is spanned by the simple coroots  $\Delta^\vee$ , thus every element in  $\mathbf{T}(e)$  can be uniquely written as  $h_1(F_1) \cdots h_r(F_r)$  for some  $F_1, \dots, F_r \in e^*$ . We introduce the following notation: for an  $r$ -tuple  $\underline{F} = (F_1, \dots, F_r) \in (e^*)^r$ ,

let

$$\eta_e(\underline{F}) = h_1(F_1) \cdots h_r(F_r) \in \mathbf{T}(e). \quad (4.1)$$

**4.1.4.  $p$ -adic groups.** — Let  $G = \mathbf{G}(\mathcal{F})$  be the group of  $\mathcal{F}$ -points of  $\mathbf{G}$ . Then  $B, B^-, U, U^-, T$  are all subgroups of  $G$ . Let  $K = G_{\mathcal{O}}$ . It is a maximal compact subgroup of  $G$ .

**Lemma.** — *The map  $\Lambda^\vee \rightarrow T_{\mathcal{O}} \backslash T$ ,  $\lambda^\vee \mapsto T_{\mathcal{O}} \pi^{\lambda^\vee}$  is an isomorphism of abelian groups.*

Let  $\log_\vee : T/T_{\mathcal{O}} \rightarrow \Lambda^\vee$  be the inverse of this isomorphism.

## 4.2. Metaplectic covers. —

**4.2.1. Steinberg's universal central extension.** — Let  $\mathcal{E}$  be the universal central extension of  $G$  in [41]. Recall that  $\mathcal{E}$  is generated by the symbols  $\mathbf{x}_\alpha(s)$  for  $\alpha \in \Phi$ ,  $s \in \mathcal{F}$  subject to the relations in *loc. cit.* There is a natural homomorphism  $p : \mathcal{E} \rightarrow G$  sending  $\mathbf{x}_\alpha(s)$  to  $x_\alpha(s)$ , whose kernel is denoted  $C$ . Then we have a short exact sequence

$$0 \rightarrow C \rightarrow \mathcal{E} \xrightarrow{p} G \rightarrow 1 \quad (4.2)$$

which makes  $\mathcal{E}$  a central extension of  $G$ . We recall the description of  $C$  by Steinberg and Matsumoto. For each  $\alpha \in \Phi$  and  $t \in \mathcal{F}^*$  we introduce the following elements in  $\mathcal{E}$ :

$$\mathbf{w}_\alpha(t) = \mathbf{x}_\alpha(t) \mathbf{x}_{-\alpha}(-t^{-1}) \mathbf{x}_\alpha(t), \quad \mathbf{h}_\alpha(s) = \mathbf{w}_\alpha(s) \mathbf{w}_\alpha(1)^{-1}.$$

Then for each  $\alpha \in \Phi$  and  $s, t \in \mathcal{F}^*$  there is a central element  $\mathbf{c}_{\alpha^\vee}(s, t) \in C$  such that

$$\mathbf{h}_\alpha(s) \mathbf{h}_\alpha(t) \mathbf{h}_\alpha(st)^{-1} = \mathbf{c}_{\alpha^\vee}(s, t). \quad (4.3)$$

Steinberg proved that

- For every short (or long) coroot  $\alpha^\vee$ , the elements  $\mathbf{c}_{\alpha^\vee}(s, t)$  are equal. Let  $\mathbf{c}(s, t)$  be the element for short coroots.
- There is a  $W$ -invariant quadratic form  $Q : \Lambda^\vee \rightarrow \mathbb{Z}$  such that

$$\mathbf{c}_{\alpha^\vee}(s, t) = \mathbf{c}(s, t)^{Q(\alpha^\vee)}.$$

In fact  $Q$  is exactly the quadratic form we defined in §3.1.5 under the same notation.

Let  $\mathcal{T}$  be the subgroup of  $\mathcal{E}$  generated by  $\mathbf{h}_\alpha(s)$  for  $\alpha \in \Phi$ ,  $s \in \mathcal{F}^*$ . Let  $B$  be the quadratic form associated to  $Q$  by (3.1). Then one can show that the following relation holds in  $\mathcal{T}$ :

$$[\mathbf{h}_\alpha(s), \mathbf{h}_\beta(t)] = \mathbf{c}(s, t)^{B(\alpha^\vee, \beta^\vee)}. \quad (4.4)$$

**Theorem** ([30]). —  *$C$  is generated by  $\mathbf{c}(s, t)$  for  $s, t \in \mathcal{F}^*$  subject to the relations*

$$\mathbf{c}(s, -s) = (1 - s, s) = 1, \quad \mathbf{c}(s, t) = \mathbf{c}(t, s)^{-1}, \quad \mathbf{c}(s, t) \mathbf{c}(s, u) = \mathbf{c}(s, tu).$$

**4.2.2. Metaplectic extension of  $G$ .** — By pushing out the short exact sequence (4.2) via the map

$$m : C \rightarrow \mu_n(\mathcal{F}), \quad \mathbf{c}_{\alpha^\vee}(s, t) \mapsto (s, t)^{Q(\alpha^\vee)}$$

given by  $n$ -th Hilbert symbol  $(-, -)$ , we obtain a central extension  $\tilde{G}$  of  $G$  by  $\mu_n(\mathcal{F})$  which fits into the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \longrightarrow & \mathcal{E} & \xrightarrow{p} & G \longrightarrow 1 \\ & & \downarrow m & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mu_n(\mathcal{F}) & \xrightarrow{i} & \tilde{G} & \xrightarrow{p} & G \longrightarrow 1 \end{array}$$

$\tilde{G}$  is called the *metaplectic cover* of  $G$ , and the induced map  $\tilde{G} \rightarrow G$  is also denoted by  $p$ . Let  $\tilde{x}_\alpha(s)$  be the image of  $\mathbf{x}_\alpha(s)$  in  $\tilde{G}$ , similar for  $\tilde{w}_\alpha(s)$  and  $\tilde{h}_\alpha(s)$ .

For every subgroup  $H \subseteq G$ , let  $\tilde{H} = p^{-1}(H)$  be the preimage of  $H$  in  $\tilde{G}$ . The subgroup  $E$  is said to *split* in  $\tilde{G}$  if there exists an injective group homomorphism  $s : H \rightarrow \tilde{G}$  which is a section of  $p$ , namely  $p \circ s = \text{id}_H$ . In this case we have  $\tilde{H} \cong \mu_n(\mathcal{F}) \times H$ .

**4.2.3. Splitting of unipotent radicals.** — The unipotent radicals  $U$  and  $U^-$  splits canonically in  $\tilde{G}$ :

**Proposition.** — (i) For each  $\alpha \in \Phi$ , the map  $s : U_\alpha \rightarrow \tilde{G}$ ,  $\mathbf{x}_\alpha(s) \mapsto \tilde{x}_\alpha(s)$  is a injective group homomorphism, and is a section of  $p$ .

(ii) The map  $s : U \rightarrow \tilde{G}$  obtained by putting the maps in (i) together for  $\alpha \in \Phi_+$  is a injective group homomorphism, and is a section of  $p$ . So  $s$  is a canonical splitting of  $U$  in  $\tilde{G}$ . The same holds for  $U^-$ .

We identify the images of the canonical splitting for  $U$  with  $U$ , and similar for  $U^-$ .

**4.2.4. Splitting of maximal compact subgroup.** —

**Proposition** ([35, 39]). — If  $(q, n) = 1$ , then there is a splitting of  $K$  in  $\tilde{G}$  whose image in  $\tilde{G}$  is generated by  $\{\tilde{x}_\alpha(s) : \alpha \in \Phi, s \in \mathcal{O}\}$ .

Note that the splitting of  $K$  is not unique in general. We fix the above choice of splitting and identify  $K$  with the image of the splitting, namely we view  $K$  as a subgroup of  $\tilde{G}$  via this splitting.

**4.2.5. Metaplectic torus.** — Let  $\tilde{T} = p^{-1}(T)$  be the *metaplectic torus*. It is a central extension of  $T$  by  $\mu_n(\mathcal{F})$ , and the restriction of  $p$  to  $\tilde{T}$  is given by  $\tilde{h}_\alpha(s) \mapsto h_\alpha(s)$ . By (4.3), (4.4) we have the following relations in  $\tilde{T}$ : for  $\alpha, \beta \in \Phi$  and  $s, t \in \mathcal{F}^*$ ,

$$[h_\alpha(s), h_\beta(t)] = (s, t)^{B(\alpha^\vee, \beta^\vee)}.$$

$$\tilde{h}_\alpha(s)\tilde{h}_\alpha(t)\tilde{h}_\alpha(st)^{-1} = (s, t)^{Q(\alpha^\vee)}. \quad (4.5)$$

Since  $\mathbf{G}$  is simply connected, every element in  $T$  can be expressed as  $\eta(\underline{F}) = h_1(F_1) \cdots h_r(F_r)$  for  $\underline{F} = (F_1, \dots, F_r) \in \mathcal{F}^*$  (in this section we always work over the local field  $\mathcal{F}$ , so we simply denote  $\eta_{\mathcal{F}}$  by  $\eta$ ). We define a map  $i : T \rightarrow \tilde{T}$  by

$$i(h_1(F_1) \cdots h_r(F_r)) = \tilde{h}_1(F_1) \cdots \tilde{h}_r(F_r) \text{ for } F_1, \dots, F_r \in \mathcal{F}^*. \quad (4.6)$$

The map  $i$  is a section of  $p : \tilde{T} \rightarrow T$ , but it is not a group homomorphism in general.

We will often use the following simple observation without mentioning:

**Lemma.** — Every element  $\tilde{t} \in \tilde{T}$  can be uniquely factorized as  $\tilde{t} = i(t)\zeta$  for  $\zeta \in \mu_n(\mathcal{F})$ ,  $t \in T$ .

*Démonstration.* — Take  $t = p(\tilde{t})$ , then we have  $p(i(t)^{-1}\tilde{t}) = t^{-1}t = 1$ , so  $i(t)^{-1}\tilde{t} \in \ker p = \mu_n(\mathcal{F})$ . Let  $\zeta = i(t)^{-1}\tilde{t}$ , then  $\tilde{t} = i(t)\zeta$ .  $\square$

**4.2.6. Center of  $\tilde{T}$ .** — Let  $\mathbf{T}_0$  be the group functor given by sending a ring  $R$  to  $\Lambda_0^\vee \otimes_{\mathbb{Z}} R^\times$ . It is a split torus with cocharacter lattice  $\Lambda_0^\vee$ . The inclusion  $\Lambda_0^\vee \hookrightarrow \Lambda^\vee$  induces an isogeny  $\mathbf{T}_0 \rightarrow \mathbf{T}$ . By abuse of notation, we denote by  $T_0$  the image of the induced map  $\mathbf{T}_0(\mathcal{F}) \rightarrow \mathbf{T}(\mathcal{F})$ . It is subgroup of  $T$  generated by elements of the form  $s^{\lambda^\vee}$  for  $s \in \mathcal{F}^*$  and  $\lambda^\vee \in \Lambda_0^\vee$ . Similarly, we define  $T_{0, \mathcal{O}}$  to be the subgroup of  $T$  generated by  $s^{\lambda^\vee}$  for  $s \in \mathcal{O}^*$  and  $\lambda^\vee \in \Lambda_0^\vee$ .

**Proposition** ([43]). —  $Z(\tilde{T}) = \tilde{T}_0$ .

**4.2.7.** When  $(q, n) = 1$ . — In this subsection we suppose  $(q, n) = 1$ , so that the splitting of  $K$  in Proposition 4.2.4 induces a splitting of  $T_\theta$ . Since we also assumed that  $\mathcal{F}$  contains all  $n$ -th roots of unity, we actually have  $n|(q-1)$ , and thus the Hilbert symbol  $(-, -)$  is unramified, namely  $(a, b) = 1$  for  $a, b \in \mathcal{O}^\times$  (see §2.2.1). This implies that  $i|_{T_\theta} : T_\theta \rightarrow \tilde{T}$  is a group homomorphism, and coincides with the splitting of  $T_\theta$  induced by Proposition 4.2.4.

By abuse of notation, for every  $\lambda^\vee \in \Lambda^\vee$ , the element  $i(\pi^{\lambda^\vee})$  in  $\tilde{T}$  is also denoted by  $\pi^{\lambda^\vee}$ . Then each element  $\tilde{a} \in \tilde{T}$  has a unique decomposition

$$\tilde{a} = \pi^{\lambda^\vee} a' \zeta \text{ where } a' \in T_\theta, \lambda^\vee \in \Lambda^\vee, \zeta \in \mu_n(\mathcal{F}). \quad (4.7)$$

Let  $T_* = T_0 T_\theta$ . In the decomposition (4.7), we have  $T_0 = \pi^{\Lambda_0^\vee} T_{0, \theta}$  and  $T_* = \pi^{\Lambda_0^\vee} T_\theta$ .

**Lemma.** —  $\tilde{T}_*$  is an abelian subgroup of  $\tilde{T}$ .

*Démonstration.* — By Proposition 4.2.6,  $\tilde{T}_0$  is central in  $\tilde{T}$ . Also  $T_\theta$  is abelian because the Hilbert symbol is unramified. So  $\tilde{T}_* = \tilde{T}_0 T_\theta$  is abelian.  $\square$

Actually by [32], if  $2n|(q-1)$ , then  $\tilde{T}_*$  is a maximal abelian subgroup of  $\tilde{T}$ . Logically we don't need this result in this article.

**4.2.8.** *The section i.* — We record the following basic computation, which will be used repeatedly in the sequel:

**Proposition.** — For  $\underline{E}, \underline{E}' \in (\mathcal{F}^*)^r$ , we have

$$i(\eta(\underline{E}\underline{E}')) = d(\underline{E}, \underline{E}') i(\eta(\underline{E})) i(\eta(\underline{E}')) \quad (4.8)$$

where  $d(\underline{E}, \underline{E}') \in \mu_n(\mathcal{F})$  is given by

$$d(\underline{E}, \underline{E}') = \left( \prod_{i=1}^r (F_i, F'_i)^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (F'_i, F_j)^{B_{ij}} \right) \quad (4.9)$$

*Démonstration.* — By (4.5) we have

$$\begin{aligned} i(\eta(\underline{E}\underline{E}')) &= \tilde{h}_1(F_1 F'_1) \cdots \tilde{h}_r(F_r F'_r) \\ &= \left( \prod_{i=1}^r (F_i, F'_i)^{-Q_i} \right) \tilde{h}_1(F_1) \tilde{h}_1(F'_1) \cdots \tilde{h}_r(F_r) \tilde{h}_r(F'_r) \\ &= \left( \prod_{i=1}^r (F_i, F'_i)^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (F'_i, F_j)^{B_{ij}} \right) \tilde{h}_1(F_1) \cdots \tilde{h}_r(F_r) \tilde{h}_1(F'_1) \cdots \tilde{h}_r(F'_r) \\ &= d(\underline{E}, \underline{E}') i(\eta(\underline{E})) i(\eta(\underline{E}')). \end{aligned}$$

$\square$

**Remarks.** —  $d(\underline{E}, \underline{E}')$  is a cocycle of the central extension  $\tilde{T}$  of  $T$  by  $\mu_n(\mathcal{F})$  after we identify  $T \cong (\mathcal{F}^*)^r$  by the map  $\eta$ .

**Corollary.** — Suppose the metaplectic dual root datum  $\mathfrak{D}_{(\mathbb{Q}, n)}^\vee$  defined by Lemma 3.1.5 (ii) is of adjoint type, namely the lattice  $\Lambda_0^\vee$  is spanned by  $n_i \alpha_i^\vee$  for  $i = 1, 2, \dots, r$ . For  $t \in T$  and  $t' \in T_0$  we have  $i(tt') = i(t)i(t') = i(t')i(t)$ .

*Démonstration.* — It suffices to prove this for  $t' = s^{\lambda^\vee}$  with  $s \in \mathcal{F}^*$ ,  $\lambda^\vee \in \Lambda_0^\vee$ . Suppose  $\lambda^\vee = k_1 \alpha_1^\vee + \cdots + k_r \alpha_r^\vee$ , since  $\Lambda_0^\vee$  is spanned by  $n_i \alpha_i^\vee$  for  $i = 1, 2, \dots, r$ , we have  $n_i | k_i$  for

$i = 1, \dots, r$ . Suppose  $t = \eta(\underline{F})$ ,  $t' = \eta(\underline{F}')$ , then we have  $F'_i = s^{k_i}$  for  $i = 1, \dots, r$ , so

$$d(\underline{F}, \underline{F}') = \left( \prod_{i=1}^r (F_i, s)^{-k_i Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (F'_i, s)^{k_i B_{ij}} \right)$$

Note that  $n|k_i Q_i$  because  $n_i|k_i$ . Also  $n|k_i B_{ij}$  because  $k_i B_{ij} = \langle \alpha_j^\vee, \alpha_i \rangle k_i Q_i$ . Thus  $d(\underline{F}, \underline{F}') = 1$  and we are done.  $\square$

**4.2.9. Iwasawa decomposition.** — We recall the following analogue of Iwasawa decomposition for metaplectic groups:

**Proposition.** — *Every element  $g \in \tilde{G}$  can be written as  $g = ktu$  for  $k \in K$ ,  $t \in \tilde{T}$ ,  $u \in U$ . The decomposition is not unique in general, but the class of  $t$  in  $\tilde{T}_\mathcal{O} \backslash \tilde{T}$  is uniquely determined by  $g$ .*

In fact, by (4.7), every element  $g \in \tilde{G}$  can be written as  $g = k\pi^{\lambda^\vee} \zeta u$  for  $k \in K$ ,  $\lambda^\vee \in \Lambda^\vee$ ,  $\zeta \in \mu_n(\mathcal{F})$ ,  $u \in U$ , and  $\lambda^\vee$  is uniquely determined by  $g$ .

**4.3. Universal unramified principal series representations.** — In this subsection we assume that  $2n|(q-1)$ . We develop the theory of unramified principal series in the same way as in [24] for reductive groups, namely we adopt the formalism of "universal principal series".

**4.3.1. Genuine functions.** — We fix an embedding  $\varepsilon : \mu_n(\mathcal{F}) \rightarrow \mathbb{C}^*$ . For any central extension of groups

$$1 \rightarrow \mu_n(\mathcal{F}) \rightarrow \tilde{E} \rightarrow E \rightarrow 0$$

we say a function  $f : \tilde{E} \rightarrow \mathbb{C}$  on  $\tilde{E}$  is  $\varepsilon$ -genuine if it satisfies  $f(\zeta x) = \varepsilon(\zeta)f(x)$  for any  $\zeta \in \mu_n(\mathcal{F})$ ,  $x \in \tilde{E}$ . Since  $\varepsilon$  is fixed, such functions are usually called genuine for short.

**4.3.2. Universal character.** — We consider a "universal" unramified character of  $T_0$  defined by

$$\chi_{\text{univ}} : T_0/T_{0,\mathcal{O}} \rightarrow \mathbb{C}[\Lambda_0^\vee]^\times, \pi^{\mu^\vee} \mapsto e^{\mu^\vee}. \quad (4.10)$$

It induces a genuine character  $\tilde{\chi}_{\text{univ}} : \tilde{T}_* \rightarrow \mathbb{C}[\Lambda_0^\vee]^\times$  by

$$\tilde{\chi}_{\text{univ}}(\pi^{\mu^\vee} a' \zeta) = e^{\mu^\vee} \varepsilon(\zeta) \text{ for } \mu^\vee \in \Lambda_0^\vee, a' \in T_\mathcal{O}, \zeta \in \mu_n(\mathcal{F}). \quad (4.11)$$

It is right  $T_\mathcal{O}$ -invariant by definition.

Let  $i_{\text{univ}} = \text{Ind}_{\tilde{T}_*}^{\tilde{T}}(\tilde{\chi}_{\text{univ}})$ . It is the space of genuine functions  $f : \tilde{T} \rightarrow \mathbb{C}[\Lambda_0^\vee]$  such that

$$f(a\pi^{\mu^\vee} a' \xi) = \varepsilon(\xi)f(a)e^{\mu^\vee}, \zeta \in \mu_n(\mathcal{F}), a \in \tilde{T}, \mu^\vee \in \Lambda_0^\vee, a' \in T_\mathcal{O}.$$

on which  $\tilde{T}$  acts by left translations. In particular, these functions are right  $T_\mathcal{O}$ -invariant.

**Lemma.** — *Let  $f$  be a left  $T_\mathcal{O}$ -invariant function in  $i_{\text{univ}}$ . Then  $f$  is uniquely determined by its value at 1, and  $f(\pi^{\mu^\vee} a' \xi) = 0$  if  $\mu^\vee \notin \Lambda_0^\vee$ .*

*Démonstration.* — By (4.7), every element in  $\tilde{T}$  can be decomposed uniquely as  $\pi^{\mu^\vee} a' \zeta$  for  $\mu^\vee \in \Lambda^\vee$ ,  $a' \in T_\mathcal{O}$ , so  $f$  is completely determined by its values at  $\pi^{\mu^\vee}$  for  $\mu^\vee \in \Lambda^\vee$ . Let  $\gamma \in \mathcal{O}^*$  be an element such that  $\varepsilon(\gamma, \pi)$  is a primitive  $n$ -th root of unity. Then for each simple coroot  $\beta^\vee \in \Pi^\vee$ , we have

$$f(\pi^{\mu^\vee}) = f(\pi^{\mu^\vee} h_\beta(\gamma)) = \varepsilon(\pi, \gamma)^{B(\mu^\vee, \beta^\vee)} f(h_b(\gamma)\pi^{\mu^\vee}) = \varepsilon(\pi, \gamma)^{B(\mu^\vee, \beta^\vee)} f(\pi^{\mu^\vee}).$$

So for  $\mu^\vee \notin \Lambda_0^\vee$ , we can take some  $\beta^\vee \in \Pi^\vee$  such that  $n \nmid B(\mu^\vee, \beta^\vee)$ . This forces  $f(\mu^\vee) = 0$  for  $\mu^\vee \notin \Lambda_0^\vee$ . For  $\mu^\vee \in \Lambda_0^\vee$ , we have  $f(\pi^{\mu^\vee}) = e^{\mu^\vee} f(1)$ , so  $f$  is totally determined by  $f(1) \in \mathbb{C}[\Lambda_0^\vee]$ .  $\square$

**4.3.3. Universal unramified principal series.** — We define the universal unramified principal series representation of  $\tilde{G}$  by  $M_{\text{univ}} := \text{Ind}_{\tilde{T}U}^{\tilde{G}}(i_{\text{univ}})$  (parabolic induction). By transitivity of induction, we can equivalently define  $M_{\text{univ}} := \text{Ind}_{\tilde{T}_*U}^{\tilde{G}}(\tilde{\chi}_{\text{univ}})$ . We will use the second model, so  $M_{\text{univ}}$  is the space of functions  $\varphi : \tilde{G} \rightarrow \mathbb{C}[\Lambda_0^\vee]$  such that

$$\varphi(g\pi^{\mu^\vee} a' \zeta u) = \varepsilon(\zeta) q^{\langle \rho, \mu^\vee \rangle} e^{\mu^\vee} \varphi(g), \zeta \in \mu_n(\mathcal{F}), g \in \tilde{G}, \mu^\vee \in \Lambda_0^\vee, a' \in T_\theta, u \in U. \quad (4.12)$$

on which  $\tilde{G}$  acts by left translations. It is also a  $\mathbb{C}[\Lambda_0^\vee]$ -module.

**Proposition.** — *Let  $\varphi$  be a left  $K$ -invariant function in  $M_{\text{univ}}$ . Then  $\varphi|_{\tilde{T}}$  is a left  $T_\theta$ -invariant function in  $i_{\text{univ}}$ . In particular,  $\varphi$  is determined by its value at 1.*

Let  $\Phi$  be the unique  $K$ -invariant element in  $M_{\text{univ}}$  such that  $\Phi(1) = 1$ . Then  $\Phi$  is a genuine left  $K$ -invariant right  $T_\theta U$ -invariant function on  $\tilde{G}$ . It follows from the proof of Lemma 4.3.2 that  $\Phi$  is supported on  $p^{-1}(K\pi^{\lambda^\vee} U)$  if and only if  $\lambda^\vee \in \Lambda_0^\vee$ , and the value of  $\Phi$  on  $K\pi^{\lambda^\vee} U\xi$  is  $q^{\langle \lambda^\vee, \rho \rangle} e^{\lambda^\vee} \varepsilon(\xi)$  for  $\lambda^\vee \in \Lambda_0^\vee$ .

**4.3.4. Back to the unramified principal series representations.** — For  $\lambda \in \Lambda \otimes \mathbb{C}$ , we define an unramified character  $\chi_\lambda$  of  $T_0$  by

$$\chi_\lambda(\pi^{\mu^\vee} a') = q^{-\langle \lambda, \mu^\vee \rangle} \text{ for } \mu^\vee \in \Lambda_0^\vee, a' \in T_{0,\theta}.$$

$\chi_\lambda$  extends to an unramified genuine character  $\tilde{\chi}_\lambda$  of  $\tilde{T}_*$  by extending trivially on  $T_\theta$ :

$$\tilde{\chi}_\lambda(\pi^{\mu^\vee} a' \zeta) = \varepsilon(\zeta) q^{-\langle \lambda, \mu^\vee \rangle} \text{ for } \mu^\vee \in \Lambda_0^\vee, a' \in T_\theta, \varepsilon(\zeta) \in \mu_n(\mathcal{F}). \quad (4.13)$$

Note that  $\tilde{\chi}_\lambda$  is right  $T_\theta$ -invariant by definition.

Let  $i(\lambda) = \text{Ind}_{\tilde{T}_*}^{\tilde{T}}(\tilde{\chi}_\lambda)$  be the representation of  $\tilde{T}$  induced from the character  $\tilde{\chi}_\lambda$  of  $\tilde{T}_*$ . Concretely  $i(\lambda)$  is the space of functions  $f : \tilde{T} \rightarrow \mathbb{C}$  such that

$$f(a\pi^{\mu^\vee} a' \xi) = \varepsilon(\xi) f(a) q^{-\langle \lambda, \mu^\vee \rangle}, \zeta \in \mu_n(\mathcal{F}), a \in \tilde{T}, \mu^\vee \in \Lambda_0^\vee, a' \in T_\theta.$$

The "usual" unramified principal series representations associated to  $\lambda \in \Lambda \otimes \mathbb{C}$  is defined by  $I(\lambda) := \text{Ind}_{\tilde{T}U}^{\tilde{G}}(i(\lambda)) \cong \text{Ind}_{\tilde{T}_*U}^{\tilde{G}}(\tilde{\chi}_\lambda)$ . In the second model, it is the space of functions  $\varphi : \tilde{G} \rightarrow \mathbb{C}$  such that

$$\varphi(g\pi^{\mu^\vee} a' \zeta u) = \varepsilon(\zeta) q^{\langle \rho - \lambda, \mu^\vee \rangle} \varphi(g), \zeta \in \mu_n(\mathcal{F}), g \in \tilde{G}, \mu^\vee \in \Lambda_0^\vee, a' \in T_\theta, u \in U. \quad (4.14)$$

on which  $\tilde{G}$  acts by left translations. We have  $I(\lambda) = M_{\text{univ}} \otimes_{\mathbb{C}[\Lambda_0^\vee]} \mathbb{C}_\lambda$ , where  $\mathbb{C}_\lambda$  is the one-dimensional  $\mathbb{C}[\Lambda_0^\vee]$ -module on which  $e^{\mu^\vee}$  acts by  $q^{-\langle \mu^\vee, \lambda \rangle}$ . Let  $\Phi_\lambda$  be the image of  $\Phi$  in  $I(\lambda)$ . It is the unique  $K$ -invariant function in  $I(\lambda)$  such that  $\Phi_\lambda(1) = 1$ .

**4.3.5. Intertwiners.** — Let  $J$  be the smallest multiplicative subset of  $\mathbb{C}[\Lambda_0^\vee]$  containing the elements  $(1 - q^{-1}e^{-\tilde{\alpha}^\vee})$  and  $(1 - e^{-\tilde{\alpha}^\vee})$  for every  $\alpha^\vee \in \Phi^\vee$ , let  $\mathbb{C}[\Lambda_0^\vee]_J$  be the completion at  $J$ . For any  $w \in W$ , define the intertwiner  $I_w : M_{\text{univ}} \otimes \mathbb{C}[\Lambda_0^\vee]_J \rightarrow M_{\text{univ}} \otimes \mathbb{C}[\Lambda_0^\vee]_J$  by

$$I_w(\varphi)(g) = w \int_{U_w} \varphi(guw) du \quad (4.15)$$

where  $U_w = U \cap wU^{-}w^{-1}$  and the  $w$  in the front means the  $W$ -action on  $\mathbb{C}[\Lambda_0^\vee]$ . Then we have

- $I_w$  is  $\tilde{G}$ -equivariant.
- $I_w(e^{\lambda^\vee} \varphi) = e^{w\lambda^\vee} \varphi$  for  $\lambda^\vee \in \Lambda_0^\vee$ .
- $I_{w_1 w_2} = I_{w_1} I_{w_2}$  if  $\ell(w) = \ell(w_1) + \ell(w_2)$ .

We have the following metaplectic version of the Gindikin-Karpelevich formula:

**Proposition** ([31]). — *Let  $s_i$  be a simple reflection. The effect of  $I_{s_i}$  on the spherical vector  $\Phi$  is given by*

$$I_{s_i} \Phi = \frac{1 - q^{-1} e^{\tilde{\alpha}_i^\vee}}{1 - e^{\tilde{\alpha}_i^\vee}} \Phi$$

Also for every  $\lambda \in \Lambda \otimes \mathbb{C}$ ,  $I_w$  induces an intertwiner  $I_w(\lambda) : I(\lambda) \rightarrow I(w\lambda)$  defined by

$$I_w(\varphi)(g) = \int_{U_w} \varphi(guw) du.$$

which is the usual intertwiner on unramified principal series representations.

**4.4. Whittaker functionals.** — We fix an additive character  $\psi : \mathcal{F} \rightarrow \mathbb{C}^*$  of conductor  $\mathcal{O}$ .

**4.4.1. Generic unramified character of  $U^-$ .** — For every simple root  $\alpha_i \in \Delta$ ,  $\psi$  induces a character of  $U_{-i}$  via the isomorphism  $\times_{-i} : \mathcal{F} \xrightarrow{\sim} U_{-i}$ , denoted  $\psi_{-i}$ . Then we define a character of  $U^-$  by

$$\psi : U^- \rightarrow U^-/[U^-, U^-] \cong \prod_{i \in I} U_{-i} \xrightarrow{\prod_{i \in I} \psi_{-i}} \mathbb{C}^*$$

which we still denote by  $\psi$  by abuse of notations.

**4.4.2. Universal Whittaker functionals.** — For every  $\mathbb{C}[\Lambda_0^\vee]$ -valued smooth representation  $(r, V)$  of  $\tilde{G}$ , let  $\text{Wh}(V)$  be the space of linear functionals  $L : V \rightarrow \mathbb{C}[\Lambda_0^\vee]$  such that

$$L(r(u^-)v) = \psi(u^-)L(v) \text{ for all } v \in V, u^- \in U^-. \quad (4.16)$$

**Proposition.** —  *$\text{Wh}(M_{\text{univ}})$  is a free  $\mathbb{C}[\Lambda_0^\vee]$ -module of rank  $|\Lambda^\vee/\Lambda_0^\vee|$ . For every  $\lambda^\vee \in \Lambda^\vee/\Lambda_0^\vee$  define a linear functional  $L_{\lambda^\vee}$  on  $I_u$  by*

$$L_{\lambda^\vee}(f) = q^{-\langle \lambda^\vee, \rho \rangle} \int_{U^-} f(u^- \pi^{\lambda^\vee}) \psi(u^-)^{-1} du^- \quad (4.17)$$

Let  $\Gamma$  be a set of representatives of  $\Lambda^\vee/\Lambda_0^\vee$ . Then  $\{L_{\lambda^\vee} : \lambda^\vee \in \Gamma\}$  is a set of  $\mathbb{C}[\Lambda_0^\vee]$ -module generators of  $\text{Wh}(M_{\text{univ}})$ .

**4.4.3. The total Whittaker functional  $\mathcal{W}$ .** — Note that

$$L_{\lambda^\vee + \mu^\vee}(f) = e^{\mu^\vee} L_{\lambda^\vee}(f)$$

for  $\mu^\vee \in \Lambda_0^\vee$ , so  $L_{\lambda^\vee}(f)e^{-\lambda^\vee} \in \mathbb{C}[\Lambda^\vee]$  only depends on the residue class of  $\lambda^\vee$  in  $\Lambda_0^\vee$ . We also define the "total" Whittaker functional  $\mathcal{W} : M_{\text{univ}} \rightarrow \mathbb{C}[\Lambda^\vee]$  by

$$\mathcal{W}(f) = \sum_{\lambda^\vee \in \Lambda^\vee/\Lambda_0^\vee} L_{\lambda^\vee}(f) e^{-\lambda^\vee} \quad (4.18)$$

**Proposition.** —  *$\mathcal{W} : M_{\text{univ}} \rightarrow \mathbb{C}[\Lambda^\vee]$  is surjective.*

*Démonstration.* — The proof is similar to the proof of [9, Proposition 3]. For any  $\xi^\vee \in \Lambda^\vee$ , it suffices to find a preimage for  $e^{\xi^\vee}$ . Let  $\varphi_{\xi^\vee}$  be the function supported on  $U_{\mathcal{O}}^- \tilde{T}U$  whose value at  $k\pi^{\mu^\vee} a \zeta u$  is equal to

$$\begin{cases} \varepsilon(\zeta) q^{\langle \rho, \mu^\vee \rangle} e^{\xi^\vee + \mu^\vee}, & \text{if } \xi^\vee + \mu^\vee \in \Lambda_0^\vee, \\ 0 & \text{otherwise.} \end{cases}$$

for  $k \in U_{\mathcal{O}}^-$ ,  $\mu^\vee \in \Lambda^\vee$ ,  $a \in T_{\mathcal{O}}$ ,  $\zeta \in \mu_n$ ,  $u \in U$ . By Iwasawa decomposition, the function  $\varphi_{\xi^\vee}$  is well-defined and is a member of  $M_{\text{univ}}$ . Now we consider

$$L_{\lambda^\vee}(\varphi_{\xi^\vee}) = q^{-\langle \lambda^\vee, \rho \rangle} \int_{U^-} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee}) \psi(u^-)^{-1} du^-.$$

Since the support of  $\varphi_{\xi^\vee}$  is in  $U_{\mathcal{O}}^- \tilde{T}U$ ,  $\varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee}) = 0$  unless  $u^- \in U_{\mathcal{O}}^-$ , so the integration is actually taken over  $U_{\mathcal{O}}^-$ , on which the character  $\psi$  is trivial. Thus

$$\begin{aligned} L_{\lambda^\vee}(\varphi_{\xi^\vee}) &= q^{-\langle \lambda^\vee, \rho \rangle} \int_{U_{\mathcal{O}}^-} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee}) du^- \\ &= q^{-\langle \lambda^\vee, \rho \rangle} \varphi_{\xi^\vee}(\pi^{\lambda^\vee}) = \begin{cases} e^{\lambda^\vee + \xi^\vee} & \text{if } \xi^\vee + \lambda^\vee \in \Lambda_0^\vee, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As a result,  $\mathscr{W}(\varphi_{\xi^\vee}) = L_{-\xi^\vee}(\varphi_{\xi^\vee}) e^{\xi^\vee} = e^{\xi^\vee}$ .  $\square$

#### 4.5. Unramified Whittaker functions. —

**4.5.1.** *The unramified Whittaker functions.* — The unramified metaplectic Whittaker functions are the functions

$$W_{\lambda^\vee}(g) = q^{-\langle \lambda^\vee, \rho \rangle} \int_{U^-} \Phi(gu^- \pi^{\lambda^\vee}) \psi(u^-)^{-1} du^-. \quad (4.19)$$

for  $\lambda^\vee \in \Lambda^\vee$ . They satisfy

$$W_{\lambda^\vee}(kgu^- \zeta) = \psi(u^-) \varepsilon(\zeta) W_{\lambda^\vee}(g) \text{ for } k \in K, g \in \tilde{G}, u^- \in U^-, \zeta \in \mu_n(\mathcal{F}). \quad (4.20)$$

and

$$W_{\lambda^\vee + \mu^\vee}(g) = e^{\mu^\vee} W_{\lambda^\vee}(g) \quad (4.21)$$

thus  $W_{\lambda^\vee}(g) e^{-\lambda^\vee}$  depends only on the residue class of  $\lambda^\vee$  in  $\Lambda^\vee / \Lambda_0^\vee$ . Also note that

$$W_{\lambda^\vee}(g) = L_{\lambda^\vee}(g \cdot \Phi)$$

where  $g \cdot \Phi$  is the  $\tilde{G}$ -action on the universal principal series  $M_{\text{univ}}$  by left translations.

**4.5.2.** *The "total" Whittaker function.* — In [37] Patnaik-Puskas also defined a "total" Whittaker function  $W : \tilde{G} \rightarrow \mathbb{C}[\Lambda^\vee]$  by

$$W(g) = \int_{U^-} \tilde{\Phi}(gu^-) \psi(u^-) du^- \quad (4.22)$$

where  $\tilde{\Phi} : \tilde{G} \rightarrow \mathbb{C}[\Lambda^\vee]$  is the function whose value on  $K\pi^{\lambda^\vee} U\xi$  is  $q^{\langle \lambda^\vee, \rho \rangle} e^{\lambda^\vee} \varepsilon(\xi)$ . The values of  $\tilde{\Phi}$  and  $\Phi$  coincide on  $K\pi^{\lambda^\vee} U\xi$  for  $\lambda^\vee \in \Lambda_0^\vee$ . The main result of loc. cit. is

**Theorem.** — For every  $\lambda^\vee \in \Lambda_+^\vee$  we have  $W(\pi^{\lambda^\vee}) = \widetilde{\text{CS}}(\lambda^\vee)$ .

The relation between the Whittaker functions  $W_{\lambda^\vee}^\vee$  for  $\lambda^\vee \in \Lambda^\vee$  and Patnaik-Puskas' Whittaker function  $W(g)$  is



**Proposition.** — We have

$$W(g) = \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} e^{-\lambda^\vee} W_{\lambda^\vee}(g).$$

In particular,

$$W(1) = \mathcal{W}(\Phi) = \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} e^{-\lambda^\vee} L_{\lambda^\vee}(\Phi).$$

*Démonstration.* — The functions  $W$  and  $W_{\lambda^\vee}$  all satisfy (4.20), so by Iwasawa decomposition, it suffices to prove this theorem for  $g = \pi^\eta$ . For simplicity we prove it for  $g = 1$  (and in this paper we only need this result for  $g = 1$ ), the proof for a general  $\pi^\eta$  is similar.

By Iwasawa decomposition, we have

$$\begin{aligned} W(1) &= \int_{U^-} \tilde{\Phi}(u^-) \psi(u^-) du^- = \sum_{\mu^\vee \in \Lambda^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \varepsilon(\xi) q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee} \psi(u^-) du^- \\ &= \sum_{\mu^\vee \in \Lambda^\vee} q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \varepsilon(\xi) \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \psi(u^-) du^-. \end{aligned}$$

And similarly,

$$\begin{aligned} W_{\lambda^\vee}(1) &= q^{-\langle \lambda^\vee, \rho \rangle} \int_{U^-} \Phi(u^- \pi^{\lambda^\vee}) \psi(u^-) du^- \\ &= q^{-\langle \lambda^\vee, \rho \rangle} \sum_{\mu^\vee \in \Lambda^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \Phi(u^- \pi^{\lambda^\vee}) \psi(u^-) du^- \\ &= \sum_{\mu^\vee \in -\lambda^\vee + \Lambda_0^\vee} q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee + \lambda^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \varepsilon(\xi) \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \psi(u^-) du^- \end{aligned}$$

since  $\Phi$  is supported on  $\Lambda_0^\vee$ . Thus

$$\begin{aligned} &\sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} e^{-\lambda^\vee} W_{\lambda^\vee}(1) \\ &= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \sum_{\mu^\vee \in -\lambda^\vee + \Lambda_0^\vee} q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \varepsilon(\xi) \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \psi(u^-) du^- \\ &= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \sum_{\mu^\vee \in -\lambda^\vee + \Lambda_0^\vee} q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \varepsilon(\xi) \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \psi(u^-) du^- \\ &= \sum_{\mu^\vee \in \Lambda^\vee} q^{\langle \mu^\vee, \rho \rangle} e^{\mu^\vee} \sum_{\xi \in \mu_n(\mathcal{K})} \varepsilon(\xi) \int_{K\pi^{\mu^\vee} U \xi \cap U^-} \psi(u^-) du^- \\ &= W(1). \end{aligned}$$

□

In particular,  $e^{-\lambda^\vee} W_{\lambda^\vee}(1)$  is the part of  $W(1)$  supported on  $-\lambda^\vee + \Lambda_0^\vee$ .

**Corollary.** — We have

$$e^{-\lambda^\vee} L_{\lambda^\vee}(\Phi) = e^{-\lambda^\vee} W_{\lambda^\vee}(1) = \sum_{\substack{k_1, \dots, k_r \geq 0 \\ k_1 \alpha_1^\vee + \dots + k_r \alpha_r^\vee - \lambda^\vee \in \Lambda_0^\vee}} H(\pi^{k_1}, \dots, \pi^{k_r}) q^{-k_1 - \dots - k_r} e^{-k_1 \alpha_1^\vee - \dots - k_r \alpha_r^\vee}$$

*Démonstration.* — This follows from Theorem 4.5.2, Theorem 4.5.3 and the definition of  $H(\pi^{k_1}, \dots, \pi^{k_r})$  in §3.3.3. □

**4.6. Kazhdan-Patterson's scattering matrix.** — We have the following result on the relation between the total Whittaker functional  $\mathscr{W}$  and the intertwiner  $I_{s_i}$  for a simple reflection  $s_i$ . It is equivalent to the computation of Kazhdan-Patterson's scattering matrix [19, 26].

**Proposition.** — For a simple reflection  $s_i$ , for any  $\Phi_\infty \in M_{\text{univ}, \infty}$  we have

$$\mathscr{W}(I_{s_i}\Phi_\infty) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - e^{-\tilde{\alpha}_i^\vee}} s_i \star \mathscr{W}(\Phi_\infty)$$

where  $s_i \star -$  is the Chinta-Gunnells action defined by (3.3).

The proof of this proposition will be given in the appendix.

**4.7. Invariance of  $\widetilde{\mathbf{CS}}(0)$  under the Chinta-Gunnells action.** —

**Theorem** ([13]). — Let

$$D = \prod_{\alpha \in \Phi_+} (1 - q^{-1}e^{-\tilde{\alpha}^\vee})$$

Then  $D^{-1}\widetilde{\mathbf{CS}}(0)$  is invariant under the Chinta-Gunnells action (3.3), namely  $s_i \star (D^{-1}\widetilde{\mathbf{CS}}(0)) = D^{-1}\widetilde{\mathbf{CS}}(0)$  for every simple reflection  $s_i$ .

*Démonstration.* — We have  $\widetilde{\mathbf{CS}}(0) = W(1) = \mathscr{W}(\Phi)$ . Consider  $\mathscr{W}(I_{s_i}\Phi)$ . On the one hand, by the Gindikin-Karpelevich formula (4.3.5) we have

$$\mathscr{W}(I_{s_i}\Phi) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - e^{-\tilde{\alpha}_i^\vee}} \mathscr{W}(\Phi) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - e^{-\tilde{\alpha}_i^\vee}} \widetilde{\mathbf{CS}}(0).$$

On the other hand, by Proposition 4.6 we also have

$$\mathscr{W}(I_{s_i}\Phi) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - e^{-\tilde{\alpha}_i^\vee}} s_i \star \mathscr{W}(\Phi) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - e^{-\tilde{\alpha}_i^\vee}} s_i \star \widetilde{\mathbf{CS}}(0)$$

Thus we have

$$s_i \star \widetilde{\mathbf{CS}}(0) = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}} \widetilde{\mathbf{CS}}(0)$$

The result follows from the fact that  $s_i \star D = s_i D$  and

$$s_i D / D = \frac{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}{1 - q^{-1}e^{-\tilde{\alpha}_i^\vee}}.$$

□

## 5. Whittaker coefficients of metaplectic Eisenstein series

Let  $k$  be a number field containing all  $2n$ -th roots of unity (in particular,  $k$  is totally imaginary). Let  $\mathbb{A}$  be the ring of adèles of  $k$ . Let  $\mathbf{G}$  be a split semisimple simply-connected reductive group over  $k$  with root datum  $\mathcal{D}$ . In this section we define the global metaplectic group  $\widetilde{G}_{\mathbb{A}}$ , which is a central extension of the adelic group  $G_{\mathbb{A}}$  by  $\mu_n(k)$ . The group  $G_k$  of  $k$ -rational points in  $G_{\mathbb{A}}$  splits canonically in  $\widetilde{G}_{\mathbb{A}}$ , so we can talk about automorphic forms on  $G_k \backslash \widetilde{G}_{\mathbb{A}}$ . In particular, we define certain Eisenstein series on  $\widetilde{G}_{\mathbb{A}}$  induced from the Borel subgroup. The main result of this paper is the computation of the first Whittaker coefficient of this Eisenstein series, which results in a Weyl group multiple Dirichlet series as defined in (3.11), as conjectured by Brubaker-Bump-Friedberg [7].

### 5.1. Global metaplectic covers. —

**5.1.1.** — For any place  $v$  of  $k$ , the inclusion  $k \hookrightarrow k_v$  induces an isomorphism  $\mu_n(k) \cong \mu_n(k_v)$ . Let  $\mu_n(\mathbb{A}) = \bigoplus_v \mu_n(k_v)$ . There is a canonical map  $m : \mu_n(\mathbb{A}) \rightarrow \mu_n(k)$  obtained by identifying all the components with  $\mu_n(k)$ , and take the product of all the components.

**5.1.2.** — For each finite place  $v \nmid \infty$ , let  $\tilde{G}_v$  be the metaplectic central extension of  $G_v = \mathbf{G}(k_v)$  defined in §4.2. For  $v \nmid n$ , let  $K_v \subseteq \tilde{G}_v$  be a splitting of the maximal compact subgroup  $\mathbf{G}(\mathcal{O}_v)$  of  $G_v$ . For  $v | \infty$ , let  $\tilde{G}_v = G_v \times \mu_n(k)$  be the trivial central extension of  $\tilde{G}_v$  by  $\mu_n(k_v)$ . (Note that by our assumption,  $k$  is totally imaginary, so for each  $v | \infty$ ,  $k_v \cong \mathbb{C}$  and  $G(\mathbb{C})$  has no nontrivial central extension by  $\mu_n(k)$ .)

Let  $G_{\mathbb{A}} = \prod'_v G_v$  be the restricted direct product of  $\{G_v : v \in \mathcal{V}_k\}$  with respect to  $\{K_v = \mathbf{G}(\mathcal{O}_v) : v \nmid \infty\}$ . Let  $\prod'_v \tilde{G}_v$  be the restricted direct product of  $\{\tilde{G}_v : v \in \mathcal{V}_k\}$  with respect to  $\{K_v : v \nmid \infty, v \nmid n\}$ . The maps  $p_v : \tilde{G}_v \rightarrow G_v$  induces a map  $p'_{\mathbb{A}} = \prod_v p_v : \prod'_v \tilde{G}_v \rightarrow G_{\mathbb{A}}$ , which makes  $\prod'_v \tilde{G}_v$  a central extension of  $G_{\mathbb{A}}$ . The kernel of  $p'_{\mathbb{A}}$  is isomorphic the restricted direct product of  $\{\mu_n(k_v) : v \in \mathcal{V}_k\}$  with respect to the trivial subgroups  $\{1\}$  over the finite places not dividing  $n$ , so  $\ker p'_{\mathbb{A}} \cong \mu_n(\mathbb{A})$ , and we will always make this identification.

**5.1.3.** — Let  $\tilde{G}_{\mathbb{A}}$  be the pushout of the central extension  $p'_{\mathbb{A}} : \prod'_v \tilde{G}_v \rightarrow G_{\mathbb{A}}$  via  $m : \mu_n(\mathbb{A}) \rightarrow \mu_n(k)$ , which fits into the following exact commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_n(\mathbb{A}) & \longrightarrow & \prod'_v \tilde{G}_v & \xrightarrow{p'_{\mathbb{A}}} & G_{\mathbb{A}} \longrightarrow 1 \\ & & \downarrow m & & \downarrow m' & & \parallel \\ 1 & \longrightarrow & \mu_n(k) & \longrightarrow & \tilde{G}_{\mathbb{A}} & \xrightarrow{p_{\mathbb{A}}} & G_{\mathbb{A}} \longrightarrow 1 \end{array}$$

So the natural map  $p_{\mathbb{A}} : \tilde{G}_{\mathbb{A}} \rightarrow G_{\mathbb{A}}$  makes  $\tilde{G}_{\mathbb{A}}$  a central extension of  $G_{\mathbb{A}}$  by  $\mu_n(k)$ .

**Proposition.** — [35] *The group of  $k$ -rational points  $G_k := \mathbf{G}(k)$  splits canonically in  $\tilde{G}_{\mathbb{A}}$ . We view  $G_k$  as a subgroup of  $\tilde{G}_{\mathbb{A}}$  via this splitting.*

**5.1.4. Local metaplectic groups as subgroups of  $\tilde{G}_{\mathbb{A}}$ .** — The global metaplectic group  $\tilde{G}_{\mathbb{A}}$  is not a restricted direct product of  $\tilde{G}_v$  for  $v \in \mathcal{V}_k$ , but we still have the following:

**Proposition.** — *The composition of the natural embedding  $\tilde{G}_v \hookrightarrow \prod'_v \tilde{G}_v$  with  $m' : \prod'_v \tilde{G}_v \rightarrow \tilde{G}_{\mathbb{A}}$  is an injection. We view  $\tilde{G}_v$  as a subgroup of  $\tilde{G}_{\mathbb{A}}$  with respect to this injection.*

By abuse of notations, given  $g_v \in \tilde{G}_v$  for every  $v \in \mathcal{V}_k$ , the element  $m'(\prod_v g_v)$  in  $\tilde{G}_{\mathbb{A}}$  is also denoted by  $\prod_v g_v$  or  $(g_v)_v$ .

**5.1.5. Unipotent radicals.** — Let  $U_{\mathbb{A}}$  be the subgroup of  $G_{\mathbb{A}}$  defined by

$$U_{\mathbb{A}} = \{(u_v)_v \in G_{\mathbb{A}} : u_v \in U_v \text{ for all } v \in \mathcal{V}_k\}$$

and similarly define  $U_{\mathbb{A}}^-$ . Then  $U_{\mathbb{A}} = \prod'_v U_v$  and  $U_{\mathbb{A}}^- = \prod_v U_v'$  are restricted direct products of the local unipotent radicals. Both  $U_{\mathbb{A}}$  and  $U_{\mathbb{A}}^-$  split canonically in  $\tilde{G}_{\mathbb{A}}$ , and we will view them also as subgroups of  $\tilde{G}_{\mathbb{A}}$  via the canonical splitting.

**5.1.6. Torus.** — Let  $\tilde{T}_{\mathbb{A}} = p_{\mathbb{A}}^{-1}(T_{\mathbb{A}})$ . It is also a central extension of  $T_{\mathbb{A}}$  by  $\mu_n(k)$ . The canonical splitting of  $G_k$  in  $\tilde{G}_{\mathbb{A}}$  induces a splitting  $i_k : T_k \rightarrow \tilde{T}_{\mathbb{A}}$  given by

$$i_k(\eta) = \prod_v i_v(\eta) \text{ for } \eta \in T_k. \quad (5.1)$$

**5.1.7. Genuine functions.** — We fix an injection  $\varepsilon : \mu_n(k) \rightarrow \mathbb{C}^*$ , then we can define  $\varepsilon$ -genuine functions on  $\tilde{G}_\mathbb{A}$  or  $\tilde{T}_\mathbb{A}$  in the same way as in §4.3.1, namely a function  $\varphi : \tilde{G}_\mathbb{A} \rightarrow \mathbb{C}$  is called  $\varepsilon$ -genuine if  $\varphi(\zeta g) = \varepsilon(\zeta)\varphi(g)$  for  $\zeta \in \mu_n(k)$ ,  $g \in \tilde{G}_\mathbb{A}$ , and similar for  $\tilde{T}_\mathbb{A}$ . Since  $\varepsilon$  is fixed, such functions are simply called genuine.

**5.1.8. Factorizable functions.** — Let  $\tilde{G}_{\mathbb{A}^S} = \prod_{v \notin S} \tilde{G}_v$ . Note that  $\varepsilon$  induced injections  $\varepsilon : \mu_n(k_v) \rightarrow \mathbb{C}$  for every  $v \in \mathcal{V}_k$ , so it makes sense to talk about genuine functions on each  $\tilde{G}_v$ . Clearly the restriction of a genuine function on  $\tilde{G}_{\mathbb{A}^S}$  to  $\tilde{G}_v$  is genuine. Conversely we have

**Proposition.** — *For every  $v \notin S$ , let  $\varphi_v : \tilde{G}_v \rightarrow \mathbb{C}$  be a genuine function on  $\tilde{G}_v$  such that the product  $\prod_{v \notin S} \varphi_v : \prod_{v \notin S} \tilde{G}_v \rightarrow \mathbb{C}$  is a well-defined function on  $\prod_{v \notin S} \tilde{G}_v$ , namely for every  $(g_v)_{v \notin S} \in \prod_{v \notin S} \tilde{G}_v$ , the infinite product  $\prod_{v \notin S} \varphi_v(g_v)$  is convergent. Then the function  $\prod_{v \notin S} \varphi_v : \prod_{v \notin S} \tilde{G}_v \rightarrow \mathbb{C}$  descends to a genuine function  $\varphi$  on  $\tilde{G}_{\mathbb{A}^S}$ . By abuse of notion, the function  $\varphi$  is also denoted  $\prod_{v \notin S} \varphi_v$ .*

And a similar result holds for  $\tilde{T}_{\mathbb{A}^S}$ . A genuine function on  $\tilde{G}_{\mathbb{A}^S}$  or  $\tilde{T}_{\mathbb{A}^S}$  is called *factorizable* if it can be obtained from a family  $(\varphi_v)_{v \notin S}$  of local genuine functions in this way.

**5.1.9. Genuine functions on  $\tilde{G}_\mathbb{A}$ .** — Similarly, let  $\varphi^S$  be a genuine function on  $\tilde{G}_{\mathbb{A}^S}$ , let  $\varphi_S$  be a genuine function on  $\tilde{G}_S$ . Then we can define a genuine function  $\varphi$  on  $\tilde{G}_\mathbb{A} = \tilde{G}_{\mathbb{A}^S} \times_{\mu_n(k)} \tilde{G}_S$  by descending the function  $\varphi^S \cdot \varphi_S$  on  $\tilde{G}_{\mathbb{A}^S} \times \tilde{G}_S$ . By abuse of notation, the genuine function  $\varphi$  on  $\tilde{G}_\mathbb{A}$  is also denoted  $\varphi^S \cdot \varphi_S$ .

The following simple lemma will be used without mentioning:

**Lemma.** — *Let  $\varphi^S = \prod_{v \notin S} \varphi_v$  be a factorizable function on  $\tilde{G}_{\mathbb{A}^S}$ , let  $\varphi_S$  be a factorizable function on  $\tilde{G}_S$ , let  $\varphi = \varphi^S \cdot \varphi_S$ . Then for  $g = (g_v)_v \in \tilde{G}_\mathbb{A}$  and  $\eta \in T_k$  we have*

$$\varphi(g\eta) = \left( \prod_{v \notin S} \varphi_v(g_v i_v(\eta)) \right) \varphi_S(g_S i_S(\eta))$$

where  $g_S = (g_v)_{v \in S} \in \tilde{G}_S$ . The same is true if  $\tilde{G}_\mathbb{A}$  is replaced by  $\tilde{T}_\mathbb{A}$ .

The lemma is a simple consequence of the fact that (5.1) is a splitting of  $T_k$  in  $\tilde{G}_\mathbb{A}$ .

**5.2. Metaplectic Eisenstein series.** — Still let  $k$  be a totally imaginary number field containing and pick  $n$  and  $S$  to satisfy the conditions in §2.1.5.

**5.2.1.  $\mathbf{G}$  over Bad places.** — We deal with the places in  $S$  together. Heuristically, in this section,  $k_S$  and  $\mathfrak{o}_S$  plays the same role as  $\mathcal{F}$  and  $\mathcal{O}$  in the local case. This idea comes from [9]. Let  $G_S = \mathbf{G}(k_S)$ , let  $U_S = \prod_{v \in S} U_v$ ,  $U_S^- = \prod_{v \in S} U_v^-$ ,  $T_S = \prod_{v \in S} T_v$ ,  $T_{0,S} = \prod_{v \in S} T_{0,v}$  be the corresponding subgroups of  $G_S$ . Let  $p_S : \tilde{G}_S \rightarrow G_S$  be the central extension of  $G_S$  by  $\mu_n(k)$  defined by a similar pushout as in §5.1.3. Note that the unipotent radicals  $U_S$ ,  $U_S^-$  still splits canonically in  $\tilde{G}_S$ , and the torus  $\tilde{T}_S = p_S^{-1}(T_S)$  is a central extension of  $T_S$  by  $\mu_n(k)$ .

**5.2.2. The torus  $\tilde{T}_S$ .** — To distinguish between different places, for  $v \in \mathcal{V}_k$ ,  $s \in k_v$  and  $i = 1, \dots, r$  let  $h_{i,v}(s)$  be the corresponding element in  $T_v$ , where we recall that the notation  $h_i$  was defined in §4.1.3.

For every  $t = (t_v)_{v \in S} \in k_S$ , let  $h_{i,S}(t) = \prod_{v \in S} h_{i,v}(t_v)$  be the corresponding element in  $T_S$ , let  $\tilde{h}_{i,S}(t_v) = \prod_{v \in S} \tilde{h}_{i,v}(t_v)$  be the corresponding element in  $\tilde{T}_S$ . For  $\underline{c} \in (\mathfrak{o}_S \setminus \{0\})^r$ , we

define

$$\eta_S(\underline{C}) = \tilde{h}_{1,S}(C_1) \cdots \tilde{h}_{r,S}(C_r) \in \tilde{T}_S.$$

Let  $i_S : T_S \rightarrow \tilde{T}_S$  be the map given by

$$i_S(h_{1,S}(s_{1,v}) \cdots h_{r,S}(s_{r,v})) = \tilde{h}_{1,S}(s_{1,v}) \cdots \tilde{h}_{r,S}(s_{r,v}) \text{ for } s_1, \dots, s_r \in k_S^*. \quad (5.2)$$

This is a section of  $p_S : \tilde{T}_S \rightarrow T_S$  but not a group homomorphism.

Note that  $T_{0_S}$  is an abelian group by Lemma 2.2.3 (i), and also by this lemma,  $i_S|_{T_{0_S}} : T_{0_S} \rightarrow \tilde{T}_S$  is a group homomorphism, so it a splitting of  $T_{0_S} \in \tilde{T}_S$ . We view  $T_{0_S}$  as a subgroup of  $\tilde{T}_S$  via this splitting. Moreover, we have

**Proposition.** —  $\tilde{T}_{*,S} = \tilde{T}_{0,S}T_{0_S}$  is an abelian subgroup of  $\tilde{T}_S$ .

*Démonstration.* —  $\tilde{T}_{0,S}$  is central in  $\tilde{T}_{*,S}$  by Lemma 4.2.6, and  $T_{0_S}$  is abelian by Lemma 2.2.3.  $\square$

We expect  $\tilde{T}_S$  to be a maximal abelian subgroup of  $\tilde{T}_S$  so that it plays the role of the maximal abelian subgroup  $\tilde{T}_{*,v}$  of  $\tilde{T}_v$  for  $v \notin S$ , but we don't need this.

**5.2.3. Induced representation of  $\tilde{G}_S$ .** — The following mimics §4.3.4. For an element  $a = h_{1,S}(C_1) \cdots h_{n,S}(C_n)$  of  $T_{0,S}$  and  $\mu \in \Lambda \otimes \mathbb{C}$ , we define

$$|a|_S^\mu := |C_1|_S^{\langle \mu, \alpha_1^\vee \rangle} \cdots |C_r|_S^{\langle \mu, \alpha_r^\vee \rangle} \quad (5.3)$$

Now every  $\lambda \in \Lambda \otimes \mathbb{C}$  induces a character  $\chi_{\lambda,S} : T_{0,S} \rightarrow \mathbb{C}^*$  by  $\chi_{\lambda,S}(a) = |a|_S^{\lambda - \rho}$  which is  $T_{0_S}$ -invariant. It extends to a character of  $T_{*,S}$  by the composition

$$T_{*,S} \twoheadrightarrow T_{*,S}/T_{0_S} \cong T_{0,S}/T_{0,S} \xrightarrow{\chi_{\lambda,S}} \mathbb{C}$$

This character of  $T_{*,S}$  is still denoted by  $\chi_{\lambda,S}$ . Let  $\tilde{\chi}_{\lambda,S}$  be the genuine character of  $\tilde{T}_{*,S}$  defined by

$$\tilde{\chi}_{\lambda,S}(i_S(a)\zeta) = \varepsilon(\zeta)\chi_{\lambda,S}(a) \text{ for } \zeta \in \mu_n(k), a \in T_{*,S}.$$

It is right  $T_{0_S}$ -invariant.

Let  $i_S(\lambda) = \text{Ind}_{\tilde{T}_{*,S}}^{\tilde{T}_S}(\tilde{\chi}_{\lambda,S})$  be the induced representation. Similar to the previous section, it consists of functions  $f : \tilde{T}_S \rightarrow \mathbb{C}$  such that  $f(aa') = f(a)\tilde{\chi}_{\lambda,S}(a')$  for  $a \in \tilde{T}_S, a' \in \tilde{T}_{*,S}$ .

Let  $I_S(\lambda)$  be the induced representation  $I_S(\lambda) = \text{Ind}_{\tilde{T}_S U_S}^{\tilde{G}_S} i_S(\lambda)$  or  $\text{Ind}_{\tilde{T}_{*,S} U_S}^{\tilde{G}_S}(\tilde{\chi}_{\lambda,S})$ . The second model consists of functions  $\Phi : \tilde{G}_S \rightarrow \mathbb{C}$  such that

$$\Phi(gau) = \Phi(g)\tilde{\chi}_{\lambda,S}(a)$$

for  $g \in \tilde{G}_S, a \in \tilde{T}_{*,S}, u \in U_S$ .

Let  $\Phi_{\lambda,S}$  be a function in  $\text{Ind}_{\tilde{T}_{*,S} U_S}^{\tilde{G}_S}(\tilde{\chi}_{\lambda,S})$ . It satisfies

$$\Phi_{\lambda,S}(gi_S(a)a'\zeta u) = \varepsilon(\zeta)\Phi_{\lambda,S}(g)|a|_S^{\lambda - \rho} \text{ for } g \in \tilde{G}_S, \zeta \in \mu_n(k), a \in T_{0,S}, a' \in T_{0_S}, u \in U_S. \quad (5.4)$$

**5.2.4.** — Now fix  $\lambda \in \Lambda \otimes \mathbb{C}$ . Let  $\Phi_{\lambda, \nu}$  be the function  $\Phi_\lambda$  defined in §4.3.4 for  $\nu \notin S$ . By Proposition 5.1.8 we can define a genuine function  $\prod_{\nu \notin S} \Phi_{\lambda, \nu}$  on  $\tilde{G}_{\mathbb{A}^S}$  because  $\Phi_{\lambda, \nu}|_{G_{\mathcal{O}_\nu}} = 1$  for all  $\nu \notin S$ . We define a genuine function  $\Phi_{\lambda, *}: \tilde{G}_{\mathbb{A}} \rightarrow \mathbb{C}$  by

$$\Phi_{\lambda, *}(g) = \left( \prod_{\nu \notin S} \Phi_{\lambda, \nu} \right) \cdot \Phi_{\lambda, S} \quad (5.5)$$

in the notation of §5.1.9.

**Lemma.** — *The function  $\Phi_{\lambda, *}$  is right  $T_{0, k}$ -invariant.*

*Démonstration.* — Let  $g = (g_\nu)_\nu \in \tilde{G}_{\mathbb{A}}$ ,  $a \in T_{0, k}$ . We have

$$\Phi_{\lambda, *}(ga) = \left( \prod_{\nu \notin S} \Phi_{\lambda, *}(g_\nu i_\nu(a)) \right) \cdot \Phi_{\lambda, S}(g_S i_S(a)) = \left( \prod_{\nu \notin S} \Phi_{\lambda, *}(g_\nu) |a|_\nu^{\lambda - \rho} \right) \cdot \Phi_{\lambda, S}(g_S) |a|_S^{\lambda - \rho} = \Phi_{\lambda, *}(g).$$

□

**5.2.5.** — The function  $\Phi_{\lambda, *}$  is only invariant under  $T_{0, k}$  but not invariant under  $T_k$ , so following [26] we now introduce the function  $\Phi_{\lambda, 0}(x): \tilde{G}_{\mathbb{A}} \rightarrow \mathbb{C}$ ,

$$\Phi_{\lambda, 0}(x) = \sum_{\eta \in T_k / T_{0, k}} \Phi_{\lambda, *}(x\eta) \quad (5.6)$$

This function is left  $K^S = \prod_{\nu \notin S} K_\nu$ -invariant and right  $U_{\mathbb{A}} T_k$  invariant.

**5.2.6. Metaplectic Eisenstein series.** — Using the function  $\Phi_{0, \lambda}$ , we form the *metaplectic Eisenstein series*

$$E(\lambda, \Phi_{\lambda, S}, g) = \sum_{\gamma \in G_k / B_k} \Phi_{\lambda, 0}(g\gamma) \quad (5.7)$$

**Proposition** ([34]). — (i) *Let*

$$\text{Gode} = \{\lambda \in \Lambda \otimes \mathbb{C} : \langle \text{Re}(\lambda), \alpha_i^\vee \rangle \leq -1 \text{ for every simple coroot } \alpha_i^\vee\}. \quad (5.8)$$

*be the Godement region.  $E(\lambda, \Phi_{\lambda, S}, g)$  is absolutely convergent for  $(g, \lambda) \in \tilde{G}(\mathbb{A}) \times \text{Gode}$ . Moreover, the convergence is uniform on compact subsets of  $(g, \lambda) \in \tilde{G}(\mathbb{A}) \times \text{Gode}$ , in particular, for a fixed  $g \in \tilde{G}(\mathbb{A})$ , the function  $E(g, \Phi_{\lambda, S}, \lambda)$  is a holomorphic function in  $\lambda \in \text{Gode}$ .*

(ii) *For a fixed  $g \in \tilde{G}(\mathbb{A})$ , the function  $E(g, \Phi_{\lambda, S}, \lambda)$  has a meromorphic continuation to  $\lambda \in \Lambda \otimes \mathbb{C}$ .*

**5.2.7. Functional equations.** — For  $\lambda \in \text{Gode}$ , let  $M(w, \lambda): \text{Ind}_{\tilde{T}_{*, S} U_S}^{\tilde{G}_S}(\chi_{\lambda, S}) \rightarrow \text{Ind}_{\tilde{T}_{*, S} U_S}^{\tilde{G}_S}(\chi_{w\lambda, S})$  be the intertwiner

$$M(w, \lambda) \Phi_{\lambda, S}(g) = \int_{U_{w, S}} \Phi_{\lambda, S}(guw) du$$

It is known that the intertwiners  $M(w, \lambda)$  also have meromorphic continuations to  $\lambda \in \Lambda \otimes \mathbb{C}$ .

**Proposition** ([34]). — For every  $w \in W$ , we have the following functional equation

$$E(g, M(w, \lambda) \Phi_{\lambda, S}, w\lambda) = \left( \prod_{\substack{\alpha^\vee \in \Phi_+^\vee \\ w^{-1}\alpha^\vee < 0}} \frac{\zeta_S(-\langle \lambda, \alpha^\vee \rangle)}{\zeta_S(-\langle \lambda, \alpha^\vee \rangle + 1)} \right) E(g, \Phi_{\lambda, S}, \lambda). \quad (5.9)$$

where  $\zeta_S(z) = \prod_{v \notin S} (1 - q_v^{-z})^{-1}$  is the partial zeta function of  $k$  with local factors in  $S$  removed.

### 5.3. Whittaker coefficients. —

**5.3.1. Remark on Haar measures.** — In this article we will only do integration on the unipotent radical  $U^-$  and its subgroups over local and global fields, so we simply take Tamagawa measures (see [42]). In particular, for any unipotent group  $\mathbf{N}$  over  $k$ , the Tamagawa measure on  $N_\mathbb{A}$  induces an invariant measure on the quotient  $N_\mathbb{A}/N_k$ , under which the volume of  $N_\mathbb{A}/N_k$  is equal to 1.

**5.3.2. First Whittaker coefficient.** — From now on we will fix the function  $\Phi_{\lambda, S}$  and simply denote the Eisenstein series by  $E(g, \lambda)$ . We take an additive character  $\psi : \mathbb{A}/k \rightarrow \mathbb{C}^*$ , which factorizes as  $\psi = \prod_v \psi_v$  for  $v \in \mathcal{V}(k)$ , where  $\psi_v : k_v \rightarrow \mathbb{C}^*$  is an additive character of the local field  $k_v$ . We suppose that  $\psi_v$  is non-trivial for every place  $v$  and unramified for every non-archimedean place  $v \notin S$ .

For  $\lambda \in \text{Gode}$  and  $g \in \tilde{G}_\mathbb{A}$ , the first *Whittaker coefficient* of the metaplectic Eisenstein series  $E(\lambda, g)$  introduced in (5.7) is defined to be

$$W(\lambda, g) = \int_{U_\mathbb{A}^-/U_k^-} E(\lambda, gu^-) \psi(u^-)^{-1} du^-, \quad (5.10)$$

where  $du^-$  is defined to be the (quotient) Haar measure on  $U_\mathbb{A}^-/U_k^-$  such that  $U_\mathbb{A}^-/U_k^-$  has total volume 1. Note that since  $U_\mathbb{A}^-/U_k^-$  is compact, the Whittaker coefficient is absolutely convergent for  $\lambda \in \text{Gode}$ . Our goal in this section is to relate this first Whittaker coefficient to the Weyl group multiple Dirichlet series introduced in §3.3.2.

**5.3.3. Statement of main result.** — We can now state the main result of this section, the so-called *Eisenstein conjecture* of [7, 11]. The proof will occupy the remainder of this section.

**Theorem.** — Let  $k$  be a number field containing all  $2n$ -th roots of unity. Choose the finite set of places  $S \subset \mathcal{V}_k$  to satisfy the conditions of §2.1.5. Let  $\mathfrak{D}$  be a semisimple simply-connected root datum such that the metaplectic dual root datum  $\mathfrak{D}_{(S, Q, n)}^\vee$  defined in Lemma 3.1.5 (ii) is of adjoint type. Let  $\lambda \in \text{Gode}$ , let  $s_i := \langle \rho - \lambda, \alpha_i^\vee \rangle = 1 - \langle \lambda, \alpha_i^\vee \rangle$  for  $i \in I$ . Then the coefficient  $W(\lambda, 1)$  from (5.10) is well-defined and we have

$$W(\lambda, 1) = [T_{\mathfrak{o}_S} : T_{0, \mathfrak{o}_S}] Z_\Psi(s_1, \dots, s_r), \quad (5.11)$$

where  $Z_\Psi(s_1, \dots, s_r)$  is the Weyl group multiple Dirichlet series attached to  $(\mathfrak{D}, n, S, Q)$  (see §3.3.2) with  $H(C_1, \dots, C_r)$  as in 3.3.3 and

$$\Psi(C_1, \dots, C_r) := \mathbb{N}C_1^{s_1} \cdots \mathbb{N}C_r^{s_r} \int_{U_S^-} \Phi_{\lambda, S}(u_S^- \tilde{h}_{1, S}(C_1) \cdots \tilde{h}_{r, S}(C_r)) \psi_v(u_S^-)^{-1} du_S^-, \quad (5.12)$$

where  $\psi_S = \prod_{v \in S} \psi_v$ .

The proof will be broken down into three parts: in the rest of this subsection we use a standard unfolding argument to rewrite the Whittaker coefficient as a sum over  $\eta \in T_k/T_{0,k}$  of two functions  $I^S(\eta)$  and  $I_S(\eta)$  that can actually be defined on a larger domain, and we establish the invariance properties of these two functions.

Next we rewrite this summation into a summation over  $\bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$  and compare with the regrouped summation of the WMDS in (3.22), so it suffices to prove the equality for the sub-summations for both the Whittaker coefficient and the WMDS for each  $\underline{\lambda} = (\lambda_v^\vee)^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ . In these sub-summations, the  $\Psi$  function in the WMDS is equal to the  $I_S$ -function on the Whittaker side, so it boils down to prove the equality

$$I^S(\underline{\lambda}^\vee) = \sum_{\underline{C} \in \text{supp}(Z; \underline{\lambda}^\vee)} H(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}.$$

Then note that  $I^S$  factorizes into a product of local integrals  $I_v$  for  $v \notin S$ , and each  $I_v$  is equal to the product of the part of the  $v$ -part supported on  $-\lambda_v^\vee$  times the factor  $D(\underline{\lambda}^\vee; v)$  defined in (3.23). Finally we span the product local integrations, the  $v$ -parts glue into the right hand side of the above equality because of Lemma 3.3.1.

**5.3.4. Unfolding the metaplectic Eisenstein series.** — The first step in our argument will be a standard unfolding argument similar to the one used when computing Whittaker coefficients of Borel Eisenstein series on a reductive group. Note that, due to the local non-uniqueness of Whittaker functionals, this does not result in an Euler product as in the linear case.

**Claim.** — For  $a \in \tilde{T}_\mathbb{A}$  and  $\lambda$  as above, we have

$$W(\lambda, a) := \sum_{\eta \in T_k/T_{0,k}} \int_{U_\mathbb{A}^-} \Phi_{\lambda, *}(a u^- \eta) \psi(u^-)^{-1} du^-. \quad (5.13)$$

*Démonstration.* — By definition, for  $a \in \tilde{T}_\mathbb{A}$ , we have

$$W(\lambda, a) = \int_{U_\mathbb{A}^-/U_k^-} \sum_{\gamma \in G_k/B_k} \Phi_{\lambda, 0}(a u^- \gamma) \psi(u^-)^{-1} du^-. \quad (5.14)$$

Decomposing, using the Bruhat decomposition,  $G_k/B_k := \sqcup_{w \in W} U_{w,k} \dot{w} B_k/B_k$ , we find that the above can be decomposed, using the right  $B_k$ -invariance of  $\Phi_{\lambda, 0}$ , as

$$\int_{U_\mathbb{A}^-/U_k^-} \sum_{w \in W} \sum_{u_w \in U_w} \Phi_{\lambda, 0}(a u^- u_w \dot{w}) \psi(u^-)^{-1} du^- = \int_{U_\mathbb{A}^-/U_k^-} \sum_{w \in W} \sum_{\gamma \in U_{w,k}^-} \Phi_{\lambda, 0}(a u^- \dot{w} u_{-w,k}) \psi(u^-)^{-1} d\mathbf{u} \quad (5.15)$$

where  $U_{w,k}^- = \dot{w}^{-1} U_{w,k} \dot{w}$  is a subgroup of  $U_k^-$ . Writing also  $U_k^{-,w} := U_k^- \cap \dot{w} U_k^- \dot{w}^{-1}$ , we have a factorization  $U_k^- = U_k^{-,w} U_{w,k}^-$ . A standard Fubini type argument then shows the above can be rewritten as

$$\sum_{w \in W} \int_{U_\mathbb{A}^-/U_k^{-,w}} \Phi_{\lambda, 0}(a u^- w) \psi(u^-)^{-1} du^-. \quad (5.16)$$

Similarly decomposing  $U_\mathbb{A}^- = U_{w,\mathbb{A}}^- U_{\mathbb{A}}^{-,w}$  we find that the above is equal to



$$\sum_{w \in W} \int_{U_{w, \mathbb{A}}^-} \int_{U_{\mathbb{A}}^{w, -} / U_k^{w, -}} \Phi_{\lambda, 0}(au_1^- u_2^- w) \psi(u_1^- u_2^-)^{-1} du_1^- du_2^- \quad (5.17)$$

$$= \sum_{w \in W} \int_{U_{w, \mathbb{A}}^-} \left( \int_{U_{\mathbb{A}}^{w, -} / U_k^{w, -}} \psi(u_2^-)^{-1} du_2^- \right) \Phi_{\lambda, 0}(au_1^- w) \psi(u_1^-)^{-1} du_1^-. \quad (5.18)$$

If  $w \neq e$ , the inner integral over  $U_{\mathbb{A}}^{w, -} / U_k^{w, -}$  vanishes, so that we are just left with

$$\int_{U_{\mathbb{A}}^-} \Phi_{\lambda, 0}(au^-) \psi(u^-)^{-1} du^-. \quad (5.19)$$

Now, if we substitute in the definition (5.6), the claim follows.  $\square$

**5.3.5.** *The functions  $I^S(\eta), I_S(\eta)$ .* — We now focus on the case when  $a = 1$ . Using the previous Claim, we may now write  $W(\lambda, 1)$  as

$$\sum_{\eta \in T_k / T_{0, k}} \underbrace{\left( |\eta|_v^{\rho - \lambda} \prod_{v \notin S} \int_{U_v^-} \Phi_{\lambda, v}(u_v^- i_v(\eta)) \psi_v(u_v^-)^{-1} du_v^- \right)}_{I^S(\eta)} \cdot \underbrace{\left( |\eta|_S^{\rho - \lambda} \int_{U_S^-(k_S)} \Phi_{\lambda, S}(u_S^- i_S(\eta)) \psi_S(u_S^-)^{-1} du_S^- \right)}_{I_S(\eta)}. \quad (5.20)$$

Note that both  $I^S(\eta)$  and  $I_S(\eta)$  are now defined on  $T_k$  and right invariant under  $T_{0, k}$ . In fact, we may extend these functions by

$$I^S(\eta) := \prod_{v \notin S} |\eta_v|_v^{\rho - \lambda} \int_{U_v^-} \Phi_{\lambda, v}(u_v^- \eta_v) \psi_v(u_v^-)^{-1} du_v^- \quad (5.21)$$

for  $\eta = \prod_{v \notin S} \eta_v \in \tilde{T}_{\mathbb{A}^S}$  with  $\eta_v \in \tilde{T}_v$ , and

$$I_S(\eta) := |\eta|_S^{\rho - \lambda} \int_{U_S^-} \Phi_{\lambda, S}(u_S^- \eta) \psi_S(u_S^-)^{-1} du_S^- \text{ for } \eta \in \tilde{T}_S. \quad (5.22)$$

For every  $v \notin S$  we also define  $I_v : \tilde{T}_v \rightarrow \mathbb{C}$  by

$$I_v(\eta_v) = |\eta_v|_v^{\rho - \lambda} \int_{U_v^-} \Phi_{\lambda, v}(u_v^- \eta_v) \psi_v(u_v^-)^{-1} du_v^- \quad (5.23)$$

Then in the notation introduced after Proposition 5.1.8, we have  $I^S = \prod_{v \notin S} I_v$ . Note that here by abuse of notation, we extend the absolute value functions  $|\cdot|_v : T_v \rightarrow \mathbb{C}$  to  $\tilde{T}_v$  via composition with  $p_v : \tilde{T}_v \rightarrow T_v$ , and similar for  $|\cdot|_S$ .

**Proposition.** — *The functions  $I^S, I_S, I_v$  are all genuine. The function  $I_v : \tilde{T}_v \rightarrow \mathbb{C}$  is right  $T_{*, v}$ -invariant, and the function  $I_S : \tilde{T}_S \rightarrow \mathbb{C}$  is right  $T_{*, S}$ -invariant. (Note that by Corollary 4.2.8, for every  $v \notin S$ ,  $i_v|_{T_{*, v}}$  is a splitting of  $T_{*, v}$ , and we view  $T_{*, v}$  as a subgroup of  $\tilde{T}_v$  via this splitting.)*

*Démonstration.* — The functions  $I_v$  and  $I_S$  are clearly genuine. By Proposition 5.1.8, the function  $I^S = \prod_{v \notin S} I_v$  is also genuine.

The invariance of  $I_v$  follows from the fact that  $\Phi_{\lambda, v}$  satisfies the property (4.14). Similarly, the invariance of  $I_S$  follows from the fact that  $\Phi_{\lambda, S}$  satisfies the property (5.4).  $\square$

**5.3.6.** — In the end of this subsection we will prove a key property of the functions  $I$  and  $J$  which links the summation (5.20) to the summation of Weyl group multiple Dirichlet series. Before stating and proving this key property, we need some preparations.

For any  $\underline{C} \in (\mathfrak{o}_S \setminus \{0\})^r$ , say  $\underline{C} = (C_1, \dots, C_r)$  with  $C_i \in (\mathfrak{o}_S \setminus \{0\})$ , let

$$\eta_v(\underline{C}) := \eta_{k_v}(\underline{C}) = h_{1,v}(C_1) \cdots h_{r,v}(C_r) \in T_v \text{ for } v \in \mathcal{V}_k, \quad (5.24)$$

Where we recall that the notation  $\eta_{k_v}$  was defined in (4.1). Let  $i_v : T_v \rightarrow \tilde{T}_v$  be the map defined by (5.2) for the local field  $k_v$ .

**Lemma.** — For  $\underline{C}, \underline{C}' \in (\mathfrak{o}_S^+)^r$  with  $\log_\omega(\underline{C}') \in \Lambda_0^\vee$  for every place  $\omega \notin S$ . Then for every place  $v$  we have

$$i_v(\eta_v(\underline{C}\underline{C}')) = i_v(\eta_v(\underline{C}))i_v(\eta_v(\underline{C}')) \quad (5.25)$$

*Démonstration.* — If  $v$  is archimedean there is nothing to prove. If  $v$  is non-archimedean, suppose  $\underline{C}' = (C'_1, \dots, C'_r)$  and  $C'_i = \prod_{\omega \notin S} \pi_\omega^{k_i^\omega}$ . Then for every  $\omega \notin S$  we have  $\log_\omega \underline{C}' = \sum_{i=1}^r k_i^\omega \alpha_i^\vee$  and thus  $n_i | k_i^\omega$  for every  $\omega \notin S$  and  $i = 1, \dots, r$ . Thus

$$\eta_v(\underline{C}') = \left( \prod_{\omega \notin S} \pi_\omega^{k_1^\omega \alpha_1^\vee} \right) \cdots \left( \prod_{\omega \notin S} \pi_\omega^{k_r^\omega \alpha_r^\vee} \right) \in T_{0,v}$$

because  $k_i^\omega \alpha_i^\vee \in \Lambda_0^\vee$  for every  $\omega \notin S$  and  $i = 1, \dots, r$ . The lemma follows from Corollary 4.2.8.  $\square$

**5.3.7.** — Let  $C \in \mathfrak{o}_S^+$ . Recall that we can factor

$$C = \prod_{v \notin S} \pi_v^{m_v} \text{ with } \text{and almost all of the } m_v = 0. \quad (5.26)$$

For ease of notation, we sometimes just write  $C_v \in \mathfrak{o}_S$  for  $\pi_v^{m_v}$  and

$$C^\vee := \prod_{\omega \neq v, \omega \notin S} C_\omega. \quad (5.27)$$

With this notation, we may write

$$C = C_v \cdot C^\vee \quad (5.28)$$

A similar factorization holds for  $\underline{C}$  which we then write as

$$\underline{C} = \left( \prod_{v \notin S} C_{1,v}, \dots, \prod_{v \notin S} C_{r,v} \right) \text{ with } a_i \in \mathfrak{o}_S^\times \quad (5.29)$$

$$= (C_{1,v} C_1^\vee, \dots, C_{r,v} C_r^\vee). \quad (5.30)$$

Writing  $\underline{C}_v := (C_{1,v}, \dots, C_{r,v})$ ,  $\underline{C}^\vee = (C_1^\vee, \dots, C_r^\vee)$ , we may also write

$$\underline{C} = \underline{C}_v \underline{C}^\vee, \quad (5.31)$$

where the multiplication is componentwise in the above expression.

**Lemma.** — Let  $\underline{C} \in (\mathfrak{o}_S^+)^r$ . For any  $v \notin S$ , we have in  $\tilde{T}_v$

$$i_v(\eta_v(\underline{C})) = D(\underline{C}; v) i_v(\eta_v(\underline{C}_v)) \cdot i_v(\eta_v(\underline{C}^\vee)), \quad (5.32)$$

where  $D(\underline{C}; v)$  is defined by (3.23).

*Démonstration.* — By Proposition 4.2.8 we have

$$i_v(\eta_v(\underline{C})) = d(\underline{C}_v, \underline{C}^v) i_v(\eta_v(\underline{C}_v)) \cdot i_v(\eta_v(\underline{C}^v))$$

The lemma follows from the fact that  $D(\underline{C}; v) = d(\underline{C}_v, \underline{C}^v)$ , which directly follows from the definitions (3.23) and (4.9).  $\square$

**5.3.8.** *The key property of the functions  $I_S$  and  $I^S$ .* —

**Lemma.** — *The functions  $I_S(\cdot)$  and  $I_v(\cdot)$  are constant on the fibers of the map  $p_Z$  defined in (3.19), namely if  $\underline{C}_1, \underline{C}_2 \in p_Z^{-1}(\underline{\lambda}^\vee)$  for some  $\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ , then  $I_S(i_S(\eta_S(\underline{C}_1))) = I_S(i_S(\eta_S(\underline{C}_2)))$  and  $I_v(i_v(\eta_v(\underline{C}_1))) = I_v(i_v(\eta_v(\underline{C}_2)))$ . We call these common values  $I_S(\underline{\lambda}^\vee)$  and  $I_v(\underline{\lambda}^\vee)$  respectively.*

*Démonstration.* — Let us first treat the case of  $I_S$ : it suffices to prove that for  $\underline{C}' = (C'_1, \dots, C'_r) \in (\mathfrak{o}_S^+)^r$  with  $\log_v \underline{C}' \in \Lambda_0^\vee$  for every  $v \notin S$ , we have  $I_S(i_S(\eta(\underline{C}\underline{C}')))) = I_S(i_S(\eta(\underline{C})))$ .

Note that the condition  $\log_v \underline{C}' \in \Lambda_0^\vee$  for every  $v \notin S$  implies that  $\eta_S(\underline{C}') \in T_{0,S}$  (see the proof of Lemma 5.3.6). Now we have  $i_S(\eta_S(\underline{C}\underline{C}')) = i_S(\eta_S(\underline{C})) i_S(\eta_S(\underline{C}'))$  by Lemma 5.3.6, so  $I_S(i_S(\eta_S(\underline{C}\underline{C}')))) = I_S(i_S(\eta_S(\underline{C})) i_S(\eta_S(\underline{C}')))) = I_S(i_S(\eta_S(\underline{C})))$  by right  $T_{*,S}$ -invariance of  $I_S$  from Proposition 5.3.5.

As for the second claim, it suffices to prove that for  $\underline{C}' = (C'_1, \dots, C'_r) \in (\mathfrak{o}_S^+)^r$  with  $\log_\omega \underline{C}' \in \Lambda_0^\vee$  for every  $\omega \notin S$ , we have  $I_v(i_v(\eta_v(\underline{C}\underline{C}')))) = I_v(i_v(\eta_v(\underline{C})))$ . Note that the condition  $\log_\omega \underline{C}' \in \Lambda_0^\vee$  for every  $\omega \notin S$  implies that  $\eta(\underline{C}') \in T_{0,v}$  (see the proof of Lemma 5.3.6). Now we have  $i_v(\eta_v(\underline{C}\underline{C}')) = i_v(\eta_v(\underline{C})) i_v(\eta_v(\underline{C}'))$  by Lemma 5.3.6, so  $I_v(i_v(\eta_v(\underline{C}\underline{C}')))) = I_v(i_v(\eta_v(\underline{C})) i_v(\eta_v(\underline{C}')))) = I_v(i_v(\eta_v(\underline{C})))$  by right  $T_{*,v}$ -invariance of  $I_v$  from Proposition 5.3.5.  $\square$

## 5.4. Proof of Theorem 5.3.3. —

**5.4.1.** *Step 1: Investigating  $W$ .* — For every  $\underline{\lambda}^\vee = (\lambda_v^\vee)_v \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$  recall that we let  $\pi^{\underline{\lambda}}$  be the equivalence class of  $\prod_v \pi_v^{\lambda_v^\vee}$  in  $T_k / T_{0,k}$ . Since  $\mathfrak{o}_S$  is a unique factorization domain, a set of representatives of  $T_k / T_{0,k}$  can be given by

$$\{\pi^{\underline{\lambda}^\vee} u : \underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee, u \in T_{\mathfrak{o}_S} / T_{0,\mathfrak{o}_S}\}.$$

So from (5.20) we have

$$\begin{aligned} W(\lambda, 1) &= \sum_{\eta \in T_k / T_{0,k}} \left( \prod_{v \notin S} |\eta|_v^{\rho-\lambda} \int_{U_v^-} \Phi_{\lambda,v}(u_v^- i_v(\eta)) \psi_v(u_v^-)^{-1} du_v^- \right) \\ &\quad \left( |\eta|_S^{\rho-\lambda} \int_{U_S^-} \Phi_{\lambda,S}(u_S^- i_S(\eta)) \psi_S(u_S^-)^{-1} du_S^- \right) \\ &= \sum_{\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee} \sum_{u \in T_{\mathfrak{o}_S} / T_{0,\mathfrak{o}_S}} \left( \prod_{v \notin S} |\pi^{\underline{\lambda}^\vee}|_v^{\rho-\lambda} \int_{U_v^-} \Phi_{\lambda,v}(u_v^- i_v(\pi^{\underline{\lambda}^\vee})) \psi_v(u_v^-)^{-1} du_v^- \right) \\ &\quad \left( |\pi^{\underline{\lambda}^\vee}|_S^{\rho-\lambda} \int_{U_S^-} \Phi_{\lambda,S}(u_S^- i_S(\pi^{\underline{\lambda}^\vee})) \psi_S(u_S^-)^{-1} du_S^- \right) \\ &= [T_{\mathfrak{o}_S} : T_{0,\mathfrak{o}_S}] \sum_{\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee} I^S(i^S(\pi^{\underline{\lambda}^\vee})) I_S(i_S(\pi^{\underline{\lambda}^\vee})) \end{aligned} \quad (5.33)$$

where  $I^S$  and  $I_S$  are the functions defined in (5.21) and (5.22) respectively.

**5.4.2. Step 2: Matching the summations.** — Comparing the summation in (5.33) with the regrouping of the summation of the Weyl group multiple Dirichlet series given in (3.22), it suffices to show that

$$I^S(i^S(\pi^{\underline{\lambda}^\vee}))I_S(i_S(\pi^{\underline{\lambda}^\vee})) = Z_{\underline{\lambda}^\vee}(s_1, \dots, s_r) \quad (5.34)$$

for every  $\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ .

**Proposition.** — For every  $\underline{C} = (C_1, \dots, C_r) \in \text{supp}(Z; \underline{\lambda}^\vee)$ , we have

$$\Psi(C_1, \dots, C_r) = I_S(i_S(\pi^{\underline{\lambda}^\vee}))$$

*Démonstration.* — Clearly for  $\underline{C} \in \text{supp}(Z; \underline{\lambda}^\vee)$  we have  $\eta_S(\underline{C}) = \pi^{\underline{\lambda}^\vee}$ . Note that we can rewrite the definition of  $\Psi$  as

$$\Psi(\underline{C}) = |i_S(\eta_S(\underline{C}))|_S^{\rho-\lambda} \int_{U_S^-} \Phi_{\lambda, S}(u_S^- i_S(\eta_S(\underline{C}))) \psi_S(u_S)^{-1} du_S^- = I_S(i_S(\eta_S(\underline{C}))).$$

The proposition follows from Lemma 5.3.8. □

**Lemma.** — The function  $\Psi(C_1, \dots, C_r)$  satisfies (3.12).

*Démonstration.* — Since  $\Psi(\underline{C}) = I_S(i_S(\eta_S(\underline{C})))$ , we only need to prove that for  $\underline{C} \in (k_S^\times)^r$  and  $\underline{C}' \in \Omega^r = \mathfrak{o}_S^\times k_S^{\times, n}$ , we have

$$I_S(i_S(\eta_S(\underline{C}\underline{C}')))) = \left( \prod_{i=1}^r \varepsilon(C'_i, C_i)_S^{Q_i} \right) \left( \prod_{i < j} \varepsilon(C'_i, C_j)_S^{B_{ij}} \right) I_S(i_S(\eta_S(\underline{C}))).$$

By Proposition 4.2.8 we have

$$i_S(\eta_S(\underline{C}\underline{C}')) = \left( \prod_{v \in S} d_v(\underline{C}, \underline{C}') \right) i_S(\eta_S(\underline{C})) i_S(\eta_S(\underline{C}')).$$

where

$$d_v(\underline{C}, \underline{C}') = \left( \prod_{i=1}^r (C_i, C'_i)_v^{-Q_i} \right) \left( \prod_{1 \leq i < j \leq r} (C'_i, C_j)_v^{B_{ij}} \right) \quad (5.35)$$

Thus we have

$$\begin{aligned} I_S(i_S(\eta_S(\underline{C}\underline{C}')))) &= \left( \prod_{v \in S} \varepsilon(d_v(\underline{C}, \underline{C}')) \right) I_S(i_S(\eta_S(\underline{C})) i_S(\eta_S(\underline{C}')))) \\ &= \left( \prod_{i=1}^r \varepsilon(C'_i, C_i)_S^{Q_i} \right) \left( \prod_{i < j} \varepsilon(C'_i, C_j)_S^{B_{ij}} \right) I_S(i_S(\eta_S(\underline{C})) i_S(\eta_S(\underline{C}')))) \end{aligned}$$

because  $I_S$  is a genuine function. Finally, by  $\underline{C}' \in \Omega^r$  we have  $\eta_S(\underline{C}') \in T_{0,S} T_{\mathfrak{o}_S}$ , so the lemma follows from the right invariance of  $I_S$  under  $T_{*,S} = T_{0,S} T_{\mathfrak{o}_S}$ . □

**5.4.3. Step 3: gluing local integrations.** — Comparing (5.34) to the  $Z_{\underline{\lambda}^\vee}$  summation (3.21) and use Proposition 5.4.2, it boils down to prove the following

**Lemma.** — For  $\underline{\lambda}^\vee \in \bigoplus_{v \notin S} \Lambda^\vee / \Lambda_0^\vee$ , we have

$$I^S(\underline{\lambda}^\vee) = \sum_{\underline{C} \in \text{supp}(Z; \underline{\lambda}^\vee)} H(C_1, \dots, C_r) \mathbb{N}C_1^{-s_1} \dots \mathbb{N}C_r^{-s_r}. \quad (5.36)$$

*Démonstration.* — For  $\underline{C} = (C_1, \dots, C_r) \in \text{supp}(Z; \underline{\lambda}^\vee)$ , first we have  $I^S(\pi^{\underline{\lambda}^\vee}) = \prod_{v \notin S} I_v(i_v \eta_v(\underline{C}))$ . From Lemma 5.3.7 we have

$$i_v(\eta_v(\underline{C})) = D(\underline{C}; v) i_v(\eta_v(\underline{C}_v)) \cdot i_v(\eta_v(\underline{C}^v)). \quad (5.37)$$

Hence

$$I_v(i_v \eta_v(\underline{C})) = I_v(D(\underline{C}; v) i_v(\eta_v(\underline{C})) \cdot i_v(\eta_v(\underline{C}^v))) = \varepsilon(D(\underline{C}; v)) I_v(i_v(\eta_v(\underline{C}))). \quad (5.38)$$

Thus we have

$$I(i_v \eta_v(\underline{C})) = \underbrace{\prod_{v \notin S} \varepsilon(D(\underline{C}, v))}_{\varepsilon(D(\underline{C}))} \cdot \prod_{v \notin S} I_v(i_v \eta_v(\underline{C})). \quad (5.39)$$

Write  $\lambda_v^\vee := \log_v \eta_v(\underline{C})$ , so that  $\underline{\lambda}^\vee = (\lambda_v^\vee)_v$ . By Corollary 4.5.2,

$$I_v(i_v \eta_v(\underline{C})) = \sum_{\substack{k_1, \dots, k_r \geq 0 \\ k_1 \alpha_1^\vee + \dots + k_r \alpha_r^\vee - \lambda_v^\vee \in \Lambda_0^\vee}} H(\pi_v^{k_1}, \dots, \pi_v^{k_r}) q_v^{-k_1 s_1 - \dots - k_r s_r}. \quad (5.40)$$

We are done using Lemma 3.3.1. □

## Appendice A. Intertwiners and Chinta-Gunnells actions

In this appendix we prove the following formula:

**Theorem.** — Let  $s_i$  be a simple reflection, let  $\varphi_{\xi^\vee} \in M_{\text{univ}}$  be a function such that  $\mathscr{W}(\varphi_{\xi^\vee}) = e^{\xi^\vee}$  (such a function exists due to Proposition 4.4.3) Then

$$W(I_{s_i} \varphi_{\xi^\vee}) = \frac{1 - q^{-1}}{1 - e^{n(\alpha_i^\vee)} \alpha_i^\vee} e^{s_i \xi^\vee + \text{res}_{n(\alpha_i^\vee)}(\langle \xi^\vee, \alpha_i \rangle) \alpha_i^\vee} + q^{-1} \mathbf{g}_{(1 + \langle \xi^\vee, \alpha_i \rangle) Q(\alpha_i^\vee)} e^{s_i \bullet \xi^\vee}$$

where we note that the intertwiner  $I_{s_i}$  is defined in Theorem 4.15.

*Démonstration.* — The condition that  $\mathscr{W}(\varphi_{\xi^\vee}) = e^{\xi^\vee}$  is equivalent to the condition that

$$\int_{U^-} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee}) \psi(u^-)^{-1} du^- = \begin{cases} q^{\langle \rho, \lambda^\vee \rangle} e^{\xi^\vee + \lambda^\vee}, & \xi^\vee + \lambda^\vee \in \Lambda_0^\vee \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
\mathcal{W}(I_s, \varphi_{\xi^\vee}) &= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} \int_{U^-} (I_s, \varphi_{\xi^\vee})(u^- \pi^{\lambda^\vee}) \psi(u^-)^{-1} du^- \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \int_F \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee} x_i(s) s_i) \psi(u^-)^{-1} du^- ds \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \int_F \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee} x_{-i}(s^{-1})) (-s)^{\alpha_i^\vee} x_i(-s^{-1}) \psi(u^-)^{-1} du^- ds \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \int_F \varphi_{\xi^\vee}(u^- x_{-i}(\pi^{-\langle \lambda^\vee, \alpha_i \rangle} s^{-1}) \pi^{\lambda^\vee} s^{\alpha_i^\vee}) \psi(u^-)^{-1} du^- ds \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \int_F \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee} s^{\alpha_i^\vee}) \psi(\pi^{-\langle \lambda^\vee, \alpha_i \rangle} s^{-1}) \psi(u^-)^{-1} du^- ds \right) e^{-\lambda^\vee}
\end{aligned}$$

Next we apply  $F - \{0\} = \bigsqcup_{k \in \mathbb{Z}} \pi^k \mathfrak{o}^\times$  and write  $s = \pi^k r$ . Note that  $s^{\alpha_i^\vee} = h_i(s) = h_i(\pi^k r) = h_i(\pi^k) h_i(r) (\pi^k, r)^{-Q(\alpha_i^\vee)} = \pi^{k\alpha_i^\vee} r^{\alpha_i^\vee} (r, \pi)^{kQ_i}$ . So the above is equal to

$$\begin{aligned}
&\sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \sum_{k \in \mathbb{Z}} q^{-k} \int_{\mathfrak{o}^\times} (r, \pi)^{kQ_i} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee} \pi^{k\alpha_i^\vee} r^{\alpha_i^\vee}) \psi(\pi^{-\langle \lambda^\vee, \alpha_i \rangle - k} r^{-1}) \psi(u^-)^{-1} du^- dr \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( q^{-\langle \rho, \lambda^\vee \rangle} s_i \int_{U^-} \sum_{k \in \mathbb{Z}} q^{-k} \int_{\mathfrak{o}^\times} (r, \pi)^{kQ_i} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee + k\alpha_i^\vee}) \psi(\pi^{-\langle \lambda^\vee, \alpha_i \rangle - k} r^{-1}) \psi(u^-)^{-1} du^- dr \right) e^{-\lambda^\vee} \\
&= \sum_{\lambda^\vee \in \Lambda^\vee / \Lambda_0^\vee} \left( \sum_{k \in \mathbb{Z}} q^{-\langle \rho, \lambda^\vee \rangle - k} \left( \int_{\mathfrak{o}^\times} (r, \pi)^{kQ_i} \psi(\pi^{-\langle \lambda^\vee, \alpha_i \rangle - k} r^{-1}) dr \right) \left( s_i \int_{U^-} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee + k\alpha_i^\vee}) \psi(u^-)^{-1} du^- \right) \right) e^{-\lambda^\vee}
\end{aligned}$$

We first look at the integration

$$\int_{U^-} \varphi_{\xi^\vee}(u^- \pi^{\lambda^\vee + k\alpha_i^\vee}) \psi(u^-)^{-1} du^-.$$

It vanishes if  $\lambda^\vee + k\alpha_i^\vee + \xi^\vee$  is not in  $\Lambda_0^\vee$ . If there exists  $k$  such that  $\lambda^\vee + k\alpha_i^\vee + \xi^\vee \in \Lambda_0^\vee$ , since the summation over  $\lambda^\vee$  is over a set of representatives for the quotient  $\Lambda^\vee / \Lambda_0^\vee$ , we have the flexibility to adjust  $\lambda^\vee$  in the same  $\Lambda_0^\vee$ -coset, so we can simply take  $\lambda^\vee = -\xi^\vee - k\alpha_i^\vee$ . This means that we can manage the summation over  $\lambda^\vee$  to be a summation over  $-\xi^\vee - m\alpha_i^\vee$  for  $m$  running through a complete residue system modulo  $n(\alpha_i^\vee)$ . Here we still have the flexibility to adjust each  $m$  by a multiple of  $n(\alpha_i^\vee)$ .

So from now on we set  $\lambda^\vee = -\xi^\vee - m\alpha_i^\vee$  and sum over  $m \bmod n(\alpha_i^\vee)$ . Then the summation over  $k$  is actually a summation over  $k \in \mathbb{Z}$  with the same residue modulo  $n(\alpha_i^\vee)$  as  $m$  (such that  $\lambda^\vee + k\alpha_i^\vee + \xi^\vee = (k - m)\alpha_i^\vee \in \Lambda_0^\vee$ ).

Next we look at the integration

$$\int_{\mathfrak{o}^\times} (r, \pi)^{kQ_i} \psi(\pi^{-\langle \lambda^\vee, \alpha_i \rangle - k} r^{-1}) dr$$

which is essentially a Gauss sum defined in §2.3.2. In particular by the discussion in §2.3.2, it vanishes if  $-\langle \lambda^\vee, \alpha_i \rangle - k < -1$ , so for  $k$  it suffices to sum over  $k \leq -\langle \lambda^\vee, \alpha_i \rangle + 1$  and  $k \equiv m \pmod{n(\alpha_i^\vee)}$ . The boundary point  $k = -\langle \lambda^\vee, \alpha_i \rangle + 1$  is included in the summation only if  $-\langle \lambda^\vee, \alpha_i \rangle + 1 \equiv \langle \xi^\vee, \alpha_i \rangle + 2m + 1 \equiv m \pmod{n(\alpha_i^\vee)}$ , namely  $m \equiv -\langle \xi^\vee, \alpha_i \rangle - 1 \pmod{n(\alpha_i^\vee)}$ . Since we have the flexibility to modify  $m$  by a multiple of  $n(\alpha_i^\vee)$  we can

simply take  $m = -\langle \xi^\vee, \alpha_i \rangle - 1$ , then  $k = -\langle \lambda^\vee, \alpha_i \rangle + 1 = \langle \xi^\vee, \alpha_i \rangle + 2m + 1 = -\langle \xi^\vee, \alpha_i \rangle - 1 = m$ . The contribution to the total sum is then equal to

$$\begin{aligned} & q^{-\langle \rho, -\xi^\vee - m\alpha_i^\vee \rangle - k} q^{-1} \mathbf{g}_{(\langle \xi^\vee, \alpha_i \rangle + 1)Q(\alpha_i^\vee)}(s_i q^{\langle \rho, -\xi^\vee \rangle}) e^{\xi^\vee - (\langle \xi^\vee, \alpha_i \rangle + 1)\alpha_i^\vee} \\ &= q^{-1} \mathbf{g}_{(\langle \xi^\vee, \alpha_i \rangle + 1)Q(\alpha_i^\vee)} e^{s_i \bullet \xi^\vee} \end{aligned}$$

For other  $m$ , the boundary point is not included and the summation on  $k$  is over  $k \leq -\langle \lambda^\vee, \alpha_i \rangle$ . In this case, also as discussed in §2.3.2, the Gauss sum does not vanish only if  $n|kQ(\alpha_i^\vee)$ , namely  $n(\alpha_i^\vee)|k$ , and in this case the Gauss sum integration is equal to  $(1 - q^{-1})$ . However, our summation of  $k$  is also restricted to those with the same residue as  $m$  modulo  $n(\alpha_i^\vee)$ . This means that the only other congruence class with a nonzero contribution to the sum is the congruence class of 0, and by our flexibility on choosing  $m$  we simply take  $m = 0$ . Then the summation of  $k$  is over  $k \leq \langle \xi^\vee, \alpha_i \rangle$  and  $n(\alpha_i^\vee)|k$ , namely the summation is over

$$\{k = \langle \xi^\vee, \alpha_i \rangle - \text{res}_{n(\alpha_i^\vee)}(\langle \xi^\vee, \alpha_i \rangle) - jn(\alpha_i^\vee) : j \geq 0\}$$

and the contribution is equal to

$$\begin{aligned} & \sum_{j \geq 0} q^{\langle \rho, \xi^\vee \rangle - k} (1 - q^{-1}) (s_i q^{\langle \rho, -\xi^\vee + k\alpha_i^\vee \rangle} e^{k\alpha_i^\vee}) e^{\xi^\vee} \\ &= \sum_{j \geq 0} (1 - q^{-1}) e^{\xi^\vee + (-\langle \xi^\vee, \alpha_i \rangle + \text{res}_{n(\alpha_i^\vee)}(\langle \xi^\vee, \alpha_i \rangle) + jn(\alpha_i^\vee))\alpha_i^\vee} \\ &= \frac{1 - q^{-1}}{1 - e^{n(\alpha_i^\vee)\alpha_i^\vee}} e^{s_i \xi^\vee + \text{res}_{n(\alpha_i^\vee)}(\langle \xi^\vee, \alpha_i \rangle)\alpha_i^\vee} \end{aligned}$$

Hence  $W(I_{s_i} \varphi_{\xi^\vee})$  is equal to the sum of the two contributions above (note that this is true even when  $-\langle \xi^\vee, \alpha_i \rangle - 1 \equiv 0 \pmod{n(\alpha_i^\vee)}$ ): in this case the two contributions are contributions over different  $k$ 's).  $\square$

## Appendix B. Factorizable functions on the torus and twisted multiplicativity

**B.0.1. Twisted multiplicativity as a kind of factorizability.** — In this appendix we record a simple result relating factorizable functions on the torus and twisted multiplicative coefficients. We keep the same notations as in Section 5.

**Proposition.** — For every  $v \notin S$ , let  $f_v : \tilde{T}_v/T_{\mathcal{O}_v} \rightarrow \mathbb{C}$  be a right  $T_{\mathcal{O}_v}$ -invariant genuine function such that  $f_v(1) = 1$ . Let  $f^S = \prod_{v \notin S} f_v$  be the genuine function on  $\tilde{T}_{\Delta^S}$  as in Proposition 5.1.8. For  $\underline{C} \in (\mathfrak{o}_S^+)^r$  we define a function

$$H_f(\underline{C}) = f^S(i^S(\eta^S(\underline{C}))) = \prod_{v \notin S} f_v(i_v(\eta_v(\underline{C}))).$$

Then the function is twisted multiplicative, namely it satisfies (3.5).

*Démonstration.* — For  $\underline{C}$  and  $\underline{C}'$  coprime, let  $\Sigma \subseteq \mathcal{V}_k$  be the set of prime factors of  $\underline{C}$  and let  $\Sigma' \subseteq \mathcal{V}_k$  be the set of prime factors of  $\underline{C}'$ , then  $\Sigma \cap \Sigma' = \emptyset$ . For  $\omega \notin S$  such that  $\omega \notin \Sigma \sqcup \Sigma'$

we have  $\eta_\omega(\underline{CC}') \in T_{\mathcal{O}_\omega}$ , so  $f_\omega(i_\omega(\eta_\omega(\underline{CC}))) = 1$ . This implies that

$$\begin{aligned}
H_f(\underline{CC}') &= \prod_{v \in \Sigma \cup \Sigma'} f_v(i_v(\eta_v(\underline{CC}))) \\
&= \left( \prod_{v \in \Sigma} f_v(i_v(\eta_v(\underline{CC}))) \right) \left( \prod_{v \in \Sigma'} f_v(i_v(\eta_v(\underline{C}'\underline{C}))) \right) \\
&= \left( \prod_{v \in \Sigma} f_v(i_v(\eta_v(\underline{C}))i_v(\eta_v(\underline{C}')))\varepsilon(d_v(\underline{C}, \underline{C}')) \right) \left( \prod_{v \in \Sigma'} f_v(i_v(\eta_v(\underline{C}'))i_v(\eta_v(\underline{C})))\varepsilon(d_v(\underline{C}', \underline{C})) \right) \\
&= \left( \prod_{v \in \Sigma} f_v(i_v(\eta_v(\underline{C}')))\varepsilon(d_v(\underline{C}, \underline{C}')) \right) \left( \prod_{v \in \Sigma'} f_v(i_v(\eta_v(\underline{C})))\varepsilon(d_v(\underline{C}', \underline{C})) \right) \\
&= H_f(\underline{C})H_f(\underline{C}')\varepsilon \left( \prod_{v \in \Sigma} d_v(\underline{C}, \underline{C}') \right) \varepsilon \left( \prod_{v \in \Sigma'} d_v(\underline{C}', \underline{C}) \right)
\end{aligned}$$

where we used Lemma 4.2.8, (5.35) and the properties of  $f_v$ . Finally we note that by (5.35) and (2.5) we have

$$\begin{aligned}
\prod_{v \in \Sigma} d_v(\underline{C}, \underline{C}') &= \prod_{v \in \Sigma} \left( \prod_{i=1}^r (C_i, C'_i)_{v^{-Q_i}} \right) \left( \prod_{1 \leq i < j \leq r} (C'_i, C_j)_{v^{B_{ij}}} \right) \\
&= \prod_{i=1}^r \left( \frac{C'_i}{C_i} \right)_S^{Q_i} \prod_{1 \leq i < j \leq r} \left( \frac{C'_i}{C_j} \right)_S^{B_{ij}}.
\end{aligned}$$

and similarly

$$\prod_{v \in \Sigma'} d_v(\underline{C}', \underline{C}) = \prod_{i=1}^r \left( \frac{C_i}{C'_i} \right)_S^{Q_i} \prod_{1 \leq i < j \leq r} \left( \frac{C_i}{C'_j} \right)_S^{B_{ij}}$$

Thus the function  $H_f$  satisfies (3.5) on  $(\mathfrak{o}_S^+)^r$ .  $\square$

In view of this Proposition, twisted multiplicativity should be thought as a metaplectic version of factorizability. Namely,  $\widetilde{T}_{\mathbb{A}^S} = \prod_{v \notin S} \widetilde{T}_v$  is a central extension of  $T_{\mathbb{A}^S}$  by  $\mu_n(k)$ , and there is a natural notion of factorizable genuine functions on  $\widetilde{T}_{\mathbb{A}^S}$ . Now we have a injective natural group homomorphism  $(\mathfrak{o}_S^+)^r \rightarrow T_{\mathbb{A}^S}$  by  $\underline{C} \mapsto (\eta_v(\underline{C}))_{v \notin S}$ . We can thus pullback the central extension  $\widetilde{T}_{\mathbb{A}^S} \rightarrow T_{\mathbb{A}^S}$  to  $(\mathfrak{o}_S^+)^r$ , which yields a central extension of the group  $(\mathfrak{o}_S^+)^r$  denoted  $\widetilde{(\mathfrak{o}_S^+)^r}$ , and the section  $i^S = \prod_{v \notin S} i_v$  induces a section  $i$  of  $\widetilde{(\mathfrak{o}_S^+)^r} \rightarrow (\mathfrak{o}_S^+)^r$ .

$$\begin{array}{ccc}
\widetilde{(\mathfrak{o}_S^+)^r} & \longrightarrow & \widetilde{T}_{\mathbb{A}^S} \\
i \uparrow \downarrow p & & p \downarrow \uparrow i^S \\
(\mathfrak{o}_S^+)^r & \hookrightarrow & T_{\mathbb{A}^S}
\end{array}$$

By the section  $i$ , a genuine function on  $\widetilde{(\mathfrak{o}_S^+)^r}$  pulls back to a function on  $(\mathfrak{o}_S^+)^r$ . The above Proposition then asserts that if we pullback a *factorizable* function on  $\widetilde{T}_{\mathbb{A}^S}$  to  $\widetilde{(\mathfrak{o}_S^+)^r}$  and then to  $(\mathfrak{o}_S^+)^r$ , we results in a twisted multiplicative function on  $(\mathfrak{o}_S^+)^r$ .



**B.0.2. A question.** — Unfortunately, Proposition B.0.1 does not give directly a proof of the Eisenstein conjecture, because the functions  $I_\nu$  does not satisfy  $I_\nu(1) = 1$ . Also, the value of  $I^S(\eta^S(\underline{C}))$  is not equal to  $H(\underline{C})$  unless we make an additional assumption that  $n$  is larger than *twice* of the dual Coxeter number of  $\mathbf{G}$ . So what we did in the proof of the Eisenstein conjecture is more complicated.

However, in view of Proposition B.0.1, it is tempting to ask the following question:

**Question.** — *Is there a genuine factorizable function  $f^S$  on  $\tilde{T}_{\Delta^S}$  such that  $H(\underline{C}) = H_f(\underline{C})$ ?*

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