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Almost invariant subspaces of shift operators and products of Toeplitz and Hankel operators

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Abstract

In this paper we formulate the almost invariant subspaces theorems of backward shift operators in terms of the ranges or kernels of product of Toeplitz and Hankel operators. This approach simplifies and gives more explicit forms of these almost invariant subspaces which are derived from related nearly backward shift invariant subspaces with finite defect. Furthermore, this approach also leads to the surprising result that the almost invariant subspaces of backward shift operators are the same as the almost invariant subspaces of forward shift operators which were treated only briefly in literature.

Keywords: Almost invariant subspaces, shift operators, Toeplitz operators, Hankel operators

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1 Introduction

The recent study of almost invariant subspaces inside Banach spaces attributes its starting point to the paper by Androulakis-Popov-Tcaciuc-Troitsky [1] in 2009, see also [25] [28] [29] [31] [17]. But an equivalent definition on Hilbert spaces occurred in Hoffman [20] in 1978 in connection with essentially invariant subspaces [6] in 1971 (see Definition 5.2 and Definition 5.5). In the sequel, let H be a complex Hilbert space and $B(H)$ be the algebra of bounded linear operators acting on H. For an operator $T \in B(H)$, write $R(T)$ and $N(T)$ for the range and the kernel of T, respectively. If M is a closed subspace of H, then the orthogonal projection from H onto M is denoted by P_M.

Definition 1.1 *[1] Let $T \in B(H)$. A closed subspace H_1 of H is called an almost invariant subspace of T if there is a finite dimensional subspace W such that $TH_1 \subset H_1 + W$. The minimal possible dimension n of W is called the defect of H_1, denoted by $\varsigma(H_1) = \varsigma(T, H_1) = n$, and W is called a minimal defect space of H_1. If W is a minimal defect space of H_1 such that $W \perp H_1$, then W is called a minimal orthogonal defect space of H_1.*

We say H_1 is an almost reducing subspace of T if both H_1 and $H_1^{\perp}$ are almost invariant for T or equivalently H_1 is almost invariant for both T and T^*. To avoid triviality, often H_1 is assumed such that both H_1 and $H_1^{\perp}$ are infinite dimensional, such an H_1 is called a *half-space*. The study of almost invariant subspaces is motivated by the Invariant Subspace Problem which is still open on Hilbert spaces. By a series of papers from several authors [25] [28] [29] [31], Tcaciuc proved the following important theorem.

Theorem 1.2 *Let X be a separable Banach space and T be a bounded linear operator on X. Then T has an almost invariant half-space with defect at most* 1.

On the other hand, the celebrated Beurling-Lax-Halmos (BLH) invariant subspace Theorem [3] [21] [18] for the (forward) shift operator is a cornerstone of modern analysis. It has played an important role in operator theory, function theory and applications. The following question is natural.

Question 1.3 *What are the almost invariant subspaces of the shift operator?*

It turns out the almost invariant subspaces of the backward shift operator are characterized recently, see [9] [10] [8] [23]. By formulating the results of these papers using the ranges of Toeplitz and Hankel operators, we will answer the above motivating question. Now we introduce the function theoretic background of the shift operator. Let L^2 be the space of square integrable functions on the unit circle $\mathbb{T}$ with respect to the normalized Lebesgue measure. Let H^2 be the Hardy space on the open unit disk $\mathbb{D}$. L^∞ and H^∞ are the algebras of bounded functions in L^2 and H^2 respectively. Let E and F be two complex separable Hilbert spaces. Let $B(E,F)$ be the set of bounded linear operators from E into F. L^2_E and H^2_E denote E-valued L^2 and H^2 spaces, respectively. $L^\infty_{B(E)}$ and $H^\infty_{B(E)}$ are operator-valued L^∞ and H^∞ algebras, and $L^\infty_{B(E,F)}$ and $H^\infty_{B(E,F)}$ are operator-valued L^∞ and H^∞ spaces.

Denote by P the projection from L^2_E to H^2_E and $Q := I-P$ the projection from L^2_E to $\overline{zH^2_E} := L^2_E \ominus H^2_E$, where E is any complex separable Hilbert space. Let $\Phi \in L^\infty_{B(E,F)}$. The multiplication operator by Φ from L^2_E into L^2_F is denoted by M_Φ. The (block) Toeplitz operator T_Φ from H^2_E into H^2_F is defined by

$$T_\Phi h = P\left[\Phi h\right], h \in H^2_E.$$

When $E = F$, let $T_z (= T_{zI_E})$ denote the shift operator on H^2_E for some E which the context will make clear. We will also use S_E or just S to denote this shift operator T_z on H^2_E and S^*_E or just S^* for the backward shift. We note that E may be regarded as a subspace H^2_E in the sense that if for each $x \in E$, we define

$$f_x(z) := x \quad \text{for all } z \in \mathbb{T},$$

then $f_x \in H^2_E$, so that $E \cong \{f_x : x \in E\} \subseteq H^2_E$. Thus if there is no confusion in the context of H^2_E, we will still use the notation P_E. Observe that

$$I - S_E S^*_E = P_E.$$

The Toeplitz operator $T_\Phi = A$ is characterized by the operator equation $S^*_F A S_E = A$.

Let J be defined on L^2_E by

$$Jf(z) = \overline{z} f(\overline{z}), \quad f \in L^2_E.$$

J maps $\overline{zH^2_E}$ onto H^2_E, and J maps H^2_E onto $\overline{zH^2_E}$. Furthermore J is a unitary operator,

$$J^* = J, J^2 = I, \ JQ = PJ, \text{ and } JP = QJ.$$

For $\Phi \in L^\infty$, the Hankel operator H_Φ from H^2_E into H^2_F is defined by

$$H_\Phi h = JQ\left[\Phi h\right] = PJ\left[\Phi h\right], \quad h \in H^2_E.$$

The Hankel operator $H_\Phi = A$ is characterized by the operator equation $AS_E = S^*_F A$. Set $\widetilde{\Phi}(z) = \Phi(\overline{z})^*$. It is easy to check that $H^*_\Phi = H_{\widetilde{\Phi}}$. The Toeplitz and Hankel operators are connected in the following basic formula:

$$T_{\Omega\Psi} - T_\Omega T_\Psi = H^*_{\Omega^*} H_\Psi \tag{1}$$

for symbols Ω and Ψ with compatible dimensions, see [4] [24].

We emphasize that again Toeplitz and Hankel operators T_Φ and H_Φ could be between different spaces H^2_E and H^2_F according to whether the symbol Φ acts between different spaces E and F. In this paper, all underlying Hilbert spaces E, F, and E_i for the vector-valued Hardy spaces H^2_E, H^2_F, and $H^2_{E_i}$ will be finite dimensional. We also note that T_Φ and H_Φ could be unbounded operators, where $\Phi \in L^2_{B(E,F)}$, that is, Φ is a matrix-valued function whose entries belong to L^2. In this case an operator equation involving T_Φ and H_Φ is valid when it acts on polynomials or H^∞ functions, see for example [14] where the product of unbounded operator $T_\Phi H_\Psi$ can be interpreted by using the bilinear form $\langle T_\Phi H_\Psi h, g\rangle := \langle H_\Psi h, T_{\Phi^*} g\rangle$ for h, g being polynomials or H^∞ functions.

The S^*_E-almost invariant subspaces (and related nearly S^*_E-invariant subspaces) have been characterized by Chalendar-Chevrot-Partington [9], Chalendar-Gallardo-Partington [10], Chattopadhyay-Das-Pradhan [8], and independently in O'Loughlin [23], see Theorem 3.5 and Corollary 3.6 below for their developments and characterizations.

In this paper, we reformulate their results using operator ranges and prove a number of results on the almost invariant subspaces of S_E^* and S_E. Our approach achieve three goals: first the reformulation simplifies the results of S_E^*-almost invariant subspaces in [9] [10] [8] [23] which were derived as consequences of related nearly S_E^*-invariant subspaces, the reformulation bypasses the use of nearly S_E^*-invariant subspaces; second our approach leads to more explicit examples of S_E^*-invariant subspaces; third surprisingly, we prove that almost invariant subspaces of S_E are the same as the almost invariant subspaces of S_E^* and thus answer our motivating Question 1.3 completely.

Let us briefly outline the plan of the paper. In Section 2, we show that the closures of ranges of finite sums of finite products of Hankel and Toeplitz operators under a mild condition are S_E^*-almost invariant and S_E-almost invariant. In Section 3, we reformulate their results (see Theorem 3.5) as follows.

Theorem 1.4 *A closed subspace M of H_F^2 is S_F^*-almost invariant if and only if one of the following holds:*

(1) $M = R(T_\Theta)$, where $\Theta \in H^\infty_{B(E,F)}$ is inner.

(2) $M = R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$, where $\Theta \in H^\infty_{B(E,E_1)}$ is inner and pure, and $\Phi \in H^2_{B(E_1,F)}$ is such that $T_\Phi (I_{E_1} - T_\Theta T_\Theta^*)$ is a partial isometry.

The idea of using ranges of partial isometries to represent almost invariant subspaces is inspired by [7] [32] [16], where the invariant subspaces of $S_F \oplus S_F^*$ are represented as ranges of partial isometries involving Toeplitz and Hankel operators. Such an approach also enables us to observe the following.

Corollary 1.5 *Let M be a closed subspace of H_F^2. Then M is S_F^*-almost invariant if and only if M is S_F-almost invariant. Consequently, If M is S_F-almost invariant, then M is S_F-almost reducing.*

The above corollary gives explicit forms of S_F-almost invariant subspaces, otherwise, S_F-almost invariant subspaces are represented as $M^\perp$, where M is as in Theorem 1.4. See Remark 3.3 in [10] and Remark 3.7 in [8], where it is observed that $S_E^* M \subset M \oplus W$ if and only if $S_E (M \oplus W)^\perp \subset (M \oplus W)^\perp \oplus W$. The orthogonal complement $M^\perp$ seems difficult to identify, for example, we know M is infinite dimensional if and only if Θ is not a finite Blaschke-Potapov product, but we know $M^\perp$ to be infinite dimensional in some special cases. In Section 4, we study in more details $M = R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$, where Φ is also inner and give more explicit examples of S_E^*-invariant or S_E-invariant subspaces.

An observation on Banach spaces [1] says that a closed subspace M is T-almost invariant if and only if M is $(T + T_0)$-invariant for some finite rank operator T_0, see Lemma 5.1 below. In section 5, we first give a more precise result on Hilbert spaces by using the equivalent definition of almost invariant subspaces in [20]. Namely, if M is T-almost invariant, we identify all finite rank operators T_0 such that M is $(T + T_0)$-invariant. We then apply this theorem to S_F^*-almost invariant subspaces. For a given $M = R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$ as in Theorem 1.4, we write down all finite rank operators T_0 in terms of Φ, Θ, M such that M is $(S_F^* + T_0)$-invariant or $(S_F + T_0)$-invariant or $(S_F + T_0)$-reducing.

2 The ranges of products of Toeplitz and Hankel operators are S^*-almost invariant

We begin with:

Lemma 2.1 *Let $\Psi \in L^\infty_{B(E,E_1)}$ and $\Phi \in L^\infty_{B(E_1,F)}$. Then $S_F^* T_\Phi H_\Psi - T_\Phi H_\Psi S_E$ is of finite rank, more precisely,*

$$S_F^* T_\Phi H_\Psi - T_\Phi H_\Psi S_E = S_F^* T_\Phi P_{E_1} H_\Psi. \tag{2}$$

Also, we have

$$S_F H_\Phi T_\Psi - H_\Phi T_\Psi S_E^* = H_\Phi S_{E_1}^* T_\Psi P_E - P_F H_\Phi S_{E_1}^* T_\Psi + S_F H_\Phi P_{E_1} T_\Psi. \tag{3}$$

Proof. Recall $S_F^* T_\Phi S_{E_1} = T_\Phi$ and $S_{E_1}^* H_\Psi = H_\Psi S_E$. Then

$$\begin{aligned} S_F^* T_\Phi S_{E_1} S_{E_1}^* &= T_\Phi S_{E_1}^*, \\ S_F^* T_\Phi \left(I - P_{E_1}\right) &= T_\Phi S_{E_1}^* \quad \text{and hence, } \; S_F^* T_\Phi = T_\Phi S_{E_1}^* + S_F^* T_\Phi P_{E_1} \end{aligned} \tag{4}$$

and

$$\begin{aligned} S_F^* T_\Phi H_\Psi &= \left(T_\Phi S_{E_1}^* + S_F^* T_\Phi P_{E_1}\right) H_\Psi \\ &= T_\Phi S_{E_1}^* H_\Psi + S_F^* T_\Phi P_{E_1} H_\Psi \\ &= T_\Phi H_\Psi S_E + S_F^* T_\Phi P_{E_1} H_\Psi. \end{aligned}$$

This proves (2).

Similarly, it follows from $S_F^* H_\Phi = H_\Phi S_{E_1}$ that

$$\begin{aligned} S_F S_F^* H_\Phi S_{E_1}^* &= S_F H_\Phi S_{E_1} S_{E_1}^*, \\ \left(I - P_F\right) H_\Phi S_{E_1}^* &= S_F H_\Phi \left(I - P_{E_1}\right), \\ S_F H_\Phi &= H_\Phi S_{E_1}^* - P_F H_\Phi S_{E_1}^* + S_F H_\Phi P_{E_1} \end{aligned} \tag{5}$$

and

$$\begin{aligned} S_F H_\Phi T_\Psi &= \left(H_\Phi S_{E_1}^* - P_F H_\Phi S_{E_1}^* + S_F H_\Phi P_{E_1}\right) T_\Psi \\ &= H_\Phi S_{E_1}^* T_\Psi - P_F H_\Phi S_{E_1}^* T_\Psi + S_F H_\Phi P_{E_1} T_\Psi \\ &= H_\Phi T_\Psi S_E^* + H_\Phi S_{E_1}^* T_\Psi P_E - P_F H_\Phi S_{E_1}^* T_\Psi + S_F H_\Phi P_{E_1} T_\Psi. \end{aligned}$$

The proof is complete. ∎

Let H_1 be an almost invariant subspace of T and W is the minimal orthogonal defect space of H_1. So $TH_1 \subset H_1 \oplus W$. Since $TH_1 \subset H_1 \oplus W$ is equivalent to $T^* \left(H_1 \oplus W\right)^\perp \subset H_1^\perp = \left(H_1 \oplus W\right)^\perp \oplus W$. Thus H_1 is T-almost invariant with defect n if and only if $\left(H_1 \oplus W\right)^\perp$ is T^*-almost invariant with defect n. Furthermore, $T^* H_1^\perp = T^* \left[\left(H_1 \oplus W\right)^\perp \oplus W\right] \subset H_1^\perp + T^* W$, so $H_1^\perp$ is T^*-almost invariant with defect less than or equal to n.

Lemma 2.2 *Let $\Psi \in L^\infty_{B(E,E_1)}$ and $\Phi \in L^\infty_{B(E_1,F)}$. The following statements hold.*

(i) $R(T_\Phi H_\Psi)^-$ is S_F^*-almost invariant and $\varsigma(R(T_\Phi H_\Psi)^-) \leq \dim E_1$.

(ii) $R(H_\Psi^* T_\Phi^*)^-$ is S_E^*-almost invariant and $\varsigma(R(H_\Psi^* T_\Phi^*)^-) \leq \dim E_1$.

(iii) $N(T_\Phi H_\Psi)$ is S_E-almost invariant and $\varsigma(S_E, N(T_\Phi H_\Psi)) = \varsigma(S_E^*, R(H_\Psi^* T_\Phi^*)^-) \leq \dim E_1$.

(iv) $N(H_\Psi^* T_\Phi^*)$ is S_F-almost invariant and $\varsigma(S_F, N(T_\Phi H_\Psi)) = \varsigma(S_F^*, R(T_\Phi H_\Psi)^-) \leq \dim E_1$.

Proof. It follows from Lemma 2.1 that $S_F^* T_\Phi H_\Psi = T_\Phi H_\Psi S_E + G$, where G is a finite rank operator with rank less than or equal to $\dim E_1$. Thus $S_F^* R(T_\Phi H_\Psi) \subset R(T_\Phi H_\Psi) + R(G)$. Since G is of finite rank, by taking closure in H_F^2, one see that

$$S_F^* R(T_\Phi H_\Psi)^- \subset R(T_\Phi H_\Psi)^- + R(G),$$

and $R(T_\Phi H_\Psi)^-$ is S_F^*-almost invariant and $\varsigma(R(T_\Phi H_\Psi)^-) \leq rank(G)$. This proves (i).

By taking adjoint, we have

$$S_E^* H_\Psi^* T_\Phi^* = H_\Psi^* T_\Phi^* S_F - G^*.$$

A similar argument shows that $R(H_\Psi^* T_\Phi^*)^-$ is S_E^*-almost invariant and $\varsigma(R(H_\Psi^* T_\Phi^*)^-) \le rank(G^*)$. This proves (ii)

Since $R(H_\Psi^* T_\Phi^*)^- = N(T_\Phi H_\Psi)^\perp$, by the argument just above this lemma, $N(T_\Phi H_\Psi)$ is S_E-almost invariant and

$$\varsigma(S_E, N(T_\Phi H_\Psi)) = \varsigma(S_E^*, R(H_\Psi^* T_\Phi^*)^-) \le \dim E_1.$$

This proves (iii). The proof of (iv) is similar. ■

Remark 2.3 *The above lemma also holds when E and F are infinite dimensional.*

It is natural to ask if $R(T_\Phi H_\Psi)^-$ is S_E-almost invariant, the answer is yes under a more restricted assumption that E_1, E and F are all finite dimensional. But the defect of $R(T_\Phi H_\Psi)^-$ is more complicated.

Lemma 2.4 *Let $\Psi \in L^\infty_{B(E,E_1)}$ and $\Phi \in L^\infty_{B(E_1,F)}$. Then $R(H_\Phi T_\Psi)^-$ is S_F-almost invariant and*

$$\varsigma(S_F,\ R((H_\Phi T_\Psi)^-)) \le \dim E_1 + \dim E + \dim F.$$

Moreover, $N(H_\Phi T_\Psi)$ is S_E^*-almost invariant with $\varsigma(S_E^*, N(H_\Phi T_\Psi)) = \varsigma(S_E, R(T_\Psi^* H_\Phi^*)^-)$. In the case $\Psi^* \in H^\infty_{B(E_1,E)}$, we have $\varsigma(S_F,\ R((H_\Phi T_\Psi)^-) \le \dim E_1 + \dim F$.

Proof. It follows from Lemma 2.1 that $S_F H_\Phi T_\Psi = H_\Phi T_\Psi S_E^* + G$, where G is a finite rank operator with rank less than or equal to $\dim E_1 + \dim E + \dim F$. The rest of the proof is similar to the proof of Lemma 2.2. In the case $\Psi^* \in H^\infty_{B(E_1,E)}$, we note that $H_\Phi S_{E_1}^* T_\Psi P_E = H_\Phi T_\Psi S_E^* P_E = 0$, so G is a finite rank operator with rank less than or equal to $\dim E_1 + \dim F$. ■

To search for better representations of S_E-almost invariant subspaces as ranges of operators, we make another observation.

Lemma 2.5 *Let $\Psi \in L^\infty_{B(E,E_1)}$ and $\Phi \in L^\infty_{B(E_1,F)}$. Then $R(T_\Phi T_\Psi)^-$ is S_F-almost invariant and $\varsigma(S_F, R(T_\Phi T_\Psi)^-) \le \dim E_1 + \dim F$. In particular, if $\Phi \in H^\infty_{B(E_1,F)}$, then $\varsigma(S_F,\ R(T_\Phi T_\Psi)^-) \le \dim E_1$.*

Proof. By taking adjoint of (4), we have

$$\begin{aligned} S_{E_1} T_\Psi &= T_\Psi S_E - P_{E_1} T_\Psi S_E, \\ S_F T_\Phi &= T_\Phi S_{E_1} - P_F T_\Phi S_{E_1}. \end{aligned}$$

Hence

$$\begin{aligned} S_F T_\Phi T_\Psi &= (T_\Phi S_{E_1} - P_F T_\Phi S_{E_1})\, T_\Psi \\ &= T_\Phi\, (T_\Psi S_E - P_{E_1} T_\Psi S_E) - P_F T_\Phi S_{E_1} T_\Psi \\ &= T_\Phi T_\Psi S_E - T_\Phi P_{E_1} T_\Psi S_E - P_F T_\Phi S_{E_1} T_\Psi. \end{aligned}$$

Since $rank(T_\Phi P_{E_1} T_\Psi S_E + P_F T_\Phi S_{E_1} T_\Psi) \le \dim E_1 + \dim F$, by a proof similar to the proof of Lemma 2.2, $R(T_\Phi T_\Psi)^-$ is S_F-almost invariant. In the case $\Phi \in H^\infty_{B(E_1,F)}$, $P_F T_\Phi S_{E_1} T_\Psi = P_F S_F T_\Phi T_\Psi = 0$. ■

It turns out the closure of the range of a finite sum of finite products of Toeplitz and Hankel operators is S_E-almost invariant and S_E^*-almost invariant under an appropriate condition.

Theorem 2.6 *Let $T = \sum_{i=1}^{k} A_i \in B(H_E^2, H_F^2)$ with $A_i = \prod_{j=1}^{m_i} C_{j,i}$, where each $C_{j,i}$ is either a (bounded) Hankel operator or a (bounded) Toeplitz operator. Then $R(T)^-$ is S_F-almost invariant and S_F^*-almost invariant, and $N(T)$ is S_E-almost invariant and S_E^*-almost invariant.*

Proof. By using (4) and (5) repeatedly, if for each i, A_i contains an even number of Hankel operators, then

$$S_F^* T = \sum_{i=1}^{k} S_F^* A_i = \sum_{i=1}^{k} (A_i S_E^* + G_i) = T S_E^* + G,$$

where each G_i is of finite rank and $G = \sum_{i=1}^{k} G_i$ is of finite rank. Hence $R(T)^-$ is S_F^*-almost invariant. Similarly, $S_F T = T S_E + G$, where G is of finite rank. Hence $R(T)^-$ is S_F-almost invariant. Note that $T^* = \sum_{i=1}^{k} A_i^*$, where for each i, A_i^* also contains an even number of Hankel operators. Therefore, $R(T^*)^-$ is S_E-almost invariant and S_E^*-almost invariant. By Lemma 2.9, $N(T) = R(T^*)^\perp$ is S_E-almost invariant and S_E^*-almost invariant.

If for each i, A_i contains an odd number of Hankel operators, then $S_F^* T = T S_E + G_1$ and $S_F T = T S_E^* + G_2$, where G_1 and G_2 are of finite rank. The same argument as the above gives the result. ■

In view of the above theorem, the following question is interesting.

Question 2.7 *Let $\Psi \in L^\infty_{B(E,F)}$ and $\Phi \in L^\infty_{B(E,F)}$. When is $R(T_\Psi + H_\Phi)^-$ S_F-almost invariant? When is $R(T_\Psi + H_\Phi)^-$ S_F^*-almost invariant?*

The C^*-algebra generated by all Toeplitz operators is called the Toeplitz algebra and the C^*-algebra generated by all Toeplitz and Hankel operators is called the Toeplitz+Hankel algebra. The operator T in Theorem 2.6 belongs to this algebra.

On H^2, since S is irreducible, a nontrivial S-invariant subspace is not S^*-invariant. It seems surprising that in the examples above S-almost invariant subspace is also S^*-almost invariant, it is interesting to ask if this is the case for other operators.

We include the following two lemmas in this section for future use. Recall $T \in B(H)$.

Lemma 2.8 *If H_1 is an almost invariant subspace of T, then the minimal orthogonal defect space of H_1 is unique.*

Proof. Let W_1 and W_2 be two n-dimensional minimal orthogonal defect spaces of H_1. Then $TH_1 \subset H_1 \oplus W_1$ and $TH_1 \subset H_1 \oplus W_2$. By the minimality of W_1, $TH_1 + H_1 \supset W_1$. Hence $H_1 \oplus W_2 \supset TH_1 + H_1 \supset W_1$. This implies $W_2 \supset W_1$. Similarly, $W_1 \supset W_2$. So $W_1 = W_2$. ■

We here record the argument just above Lemma 2.2 as a lemma for future use.

Lemma 2.9 *The following statements hold.*

(i) H_1 is T-almost invariant with defect n if and only if $H_1^\perp$ is T^*-almost invariant with defect n.

(ii) If $TH_1 \subset H_1 + W$ and $\varsigma(T, H_1) = \dim W$, then $T[H_1 + W] \subset [H_1 + W] + TW$ and $\varsigma(T, H_1 + W) = \dim(TW) - \dim(TW \cap (W + H_1))$.

Proof. We already argued that if H_1 is T-almost invariant with defect n, then $H_1^\perp$ is T^*-almost invariant with defect less than or equal to n. If $\varsigma(T^*, H_1^\perp) = k < n$, then by applying what we just proved to $H_1^\perp$, we will get $H_1 = \left(H_1^\perp\right)^\perp$ is T-almost invariant with defect less than or equal to k, which is a contradiction. This proves (i).

Now we prove (ii). Let W be a minimal defect space of H_1. Observe $T[H_1 + W] = TH_1 + TW \subset [H_1 + W] + TW = [H_1 + W] + [TW \ominus (TW \cap (W + H_1))]$. Thus $\varsigma(T, H_1 + W) \le \dim(TW) - \dim(TW \cap (W + H_1))$. On the other hand, assume $T[H_1 + W] \subset [H_1 + W] \oplus G$ for some finite dimensional subspace G. Since W is a minimal defect space of H_1, we know $TH_1 + H_1 \supset W$. Thus $H_1 + T[H_1 + W] \supset W + TW$ and

$$[H_1 + W] \oplus G \supset H_1 + T[H_1 + W] \supset H_1 + W + TW.$$

Hence $\dim G \ge \dim(TW) - \dim(TW \cap (W + H_1))$. That is, $\varsigma(T, H_1 + W) \ge \dim(TW) - \dim(TW \cap (W + H_1))$. ■

It is interesting to note that by iteration,

$$\varsigma(T,H_1) \geq \varsigma(T,H_1+W) \geq \varsigma(T,H_1+W+TW) \geq \cdots \geq \varsigma(T,H_1+W+TW+\cdots+T^kW).$$

So it is possible for $H_1+W+TW+\cdots+T^kW$ to become an invariant subspace of T.

3 Representations of S-almost invariant subspaces.

The closely related nearly S^*-invariant subspaces actually predates S^*-almost invariant subspaces.

Definition 3.1 *A closed subspace M of H^2_E is said to be nearly S^*_E-invariant if $h \in M$ and $h(0)=0$ implies that $S^*_E h \in M$.*

Nearly S^*-invariant subspaces of H^2 are useful in describing kernels of Toeplitz operators and finding inverses of Toeplitz operators [13] [11]. The following characterization (due to Hitt [19] and Sarason [26]) will be useful for us, see Theorem 30.15 in [13]. For $x, y \in H^\infty$, an admissible pair (x,y) of an outer function G is a Pythagorean pair (x,y) such that $|x|^2+|y|^2=1$ on $\mathbb{T}$, x is outer and $G=x/(1-y)$.

Theorem 3.2 *[19] [26] Let M be a nearly S^*-invariant subspace of H^2. Then M has one of the following forms:*

(i) There exists an inner function θ such that $\theta(0)\neq 0$ and $M=\theta H^2$.

(ii) There exists a function g of unit norm in M such that $g(0)>0$ and an inner function θ such that $\theta(0)=0$ and θ divides the function y, where $(x,y)$is admissible pair for the outer factor G of g, such that $M=T_gK_\theta$, where $K_\theta=H^2\ominus\theta H^2$.

The requirement that θ divides the function y is equivalent to T_g acts isometrically on K_θ so that indeed T_gK_θ is a closed subspace of H^2, see Theorem 30.14 in [13].

The case (ii) should contain the S^*-invariant subspace K_ϕ. Indeed, in this case if

$$g=(1-|\phi(0)|^2)^{-1/2}(1-\overline{\phi(0)}\phi) \;\text{ and }\; \theta=\frac{\phi(0)-\phi}{1-\overline{\phi(0)}\phi}, \tag{6}$$

then $T_gK_\theta=K_\phi$ (cf. [27]).

The vector-valued version of nearly S^*-invariant subspaces of H^2_E, where E is of finite dimension was given by Chalendar-Chevrot-Partington [9]. Recently the concept of nearly S^*-invariant subspace with finite defect was introduced, and a description of nearly S^*-invariant subspace with finite defect inside H^2 was obtained in Chalendar-Gallardo-Partington [10]. The vector-valued version of nearly S^*-invariant subspace with finite defect inside H^2_E was described first in Chattopadhyay-Das-Pradhan [8], then independently in O'Loughlin [23].

Definition 3.3 *A closed subspace M of H^2_E is said to be nearly S^*_E-invariant with defect p if there is a p-dimensional subspace G (called a defect space of M that may be taken to be orthogonal to M) such that if $h\in M$ and $h(0)=0$, then $S^*_Eh\in M+G$. The smallest possible p is said to be the defect of M, denoted by $\eta(S^*_E,M)=p$.*

It is clear that if M is S^*_E-almost invariant with $\varsigma(S^*_E,M)=p$, then M is nearly S^*_E-invariant with $\eta(S^*_E,M)\leq p$. But if M is S^*_E-almost invariant with $\varsigma(S^*_E,M)=p$, it is possible that M is not nearly S^*_E-invariant, see for example Proposition 2.6 in [8]. We will see below when a S^*_E-almost invariant subspace is nearly S^*-invariant. It has been shown in Proposition 2.2 in [10] and Proposition 2.2 in [8] that a nearly S^*_E-invariant subspace is S^*_E-almost invariant.

Next we generalize this result to a nearly S^*_E-invariant subspace with finite defect.

Proposition 3.4 *If a closed subspace M of H^2_E is nearly S^*_E-invariant with $\eta(S^*_E, M) = p$, then M is S^*_E-almost invariant with $\varsigma(S^*_E, M) \leq p + \dim E$.*

Proof. Assume M is nearly S^*_E-invariant with defect p. That is, there is a p-dimensional subspace W such that if $h \in M$ and $h(0) = 0$, then $S^*_E h \in M + W$. Set $M_1 = \{h \in M : h(0) = 0\} = M \cap zH^2_E$. Thus $S^*_E M_1 \in M + W$. Set

$$W_1 := M \ominus M_1 = M \ominus M \cap zH^2_E.$$

By a lemma from [9], $\dim W_1 \leq \dim E$. Then

$$S^*_E M = S^*_E \left(M_1 \oplus W_1\right) \subset S^*_E M_1 + S^*_E W_1 \subset M + W + S^*_E W_1.$$

Hence $\varsigma(S^*_E, M) \leq \dim(W + S^*_E W_1) \leq p + \dim E$. ■

So the set of nearly S^*_E-invariant subspaces with finite defect is the same as the set of S^*_E-almost invariant subspaces.

Theorem 3.5 *[9] [10] [8] [23] Let M be a closed subspace of $H^2_{\mathbb{C}^m}$ that is nearly S^*-invariant with defect p. Then:*

(i) In the case where there are functions in M that do not vanish at 0,

$$M = \{G : G(z) = G_0(z)k_0(z) + z\sum_{i=1}^{p} g_i(z)k_i(z) : (k_0, \cdots, k_p) \in K\}, \tag{7}$$

where G_0 is the matrix of size $m \times r$ whose columns consist of any orthonormal basis of $M \ominus \left(M \cap zH^2_{\mathbb{C}^m}\right)$, $\{g_1, \cdots, g_p\}$ is any orthonormal basis of the defect space W, and $K \subset H^2_{\mathbb{C}^{r+p}}$ is a S^*-invariant subspace. Furthermore, $\|G\|^2 = \sum_{i=0}^{p} \|k_i\|^2$.

(ii) In the case all functions in M vanish at 0,

$$M = \{G : G(z) = z\sum_{i=1}^{p} g_i(z)k_i(z) : (k_0, \cdots, k_p) \in K\},$$

with the same notation as in (i) except that $K \subset H^2_{\mathbb{C}^p}$ is a S^*-invariant subspace. Furthermore, $\|G\|^2 = \sum_{i=1}^{p} \|k_i\|^2$.

Conversely if a closed subspace M of $H^2_{\mathbb{C}^m}$ has a representation as in (i) or (ii), then it is nearly S^*-invariant with defect p.

The proof of the above theorem generalizes Hitt's algorithm as in proving Theorem 3.2 and uses a lemma of [2] about $C_{\cdot 0}$ contractions.

Corollary 3.6 *[10] [8] [23] A closed subspace M of $H^2_{\mathbb{C}^m}$ is S^*-almost invariant with defect p if and only if it satisfies the conditions of the above theorem together with an extra condition that the column space of S^*G_0 is contained in $M + W$ in case (i), while case (ii) is unchanged.*

Using Lemma 2.2, we reformulate the above theorem and corollary in terms of ranges of Toeplitz and Hankel operators. (We have to use unbounded Toeplitz operators, but the justification should be easy).

Recall $\Theta \in H^\infty_{B(E,E_1)}$ is (left) inner if $\Theta(z)^*\Theta(z) = I_E$ for almost all $z \in \mathbb{T}$ for some $E \subset E_1$. Similarly, $\Theta \in H^\infty_{B(E,E_1)}$ is right inner if $\Theta(z)\Theta(z)^* = I_{E_1}$ for almost all $z \in \mathbb{T}$ for some $E_1 \subset E$. When $E = E_1$ and Θ is both left and right inner, we say Θ is a two-sided inner function or a square inner function. In short, if Θ is left inner, then we just say Θ is inner. If $\Theta \in H^\infty_{B(E,E_1)}$ is inner, let $K_\Theta = H^2_{E_1} \ominus \Theta H^2_E$ denote the model space. The celebrated Beurling-Lax-Halmos (BLH) Theorem [3] [21] [18] says an invariant subspace of S_{E_1} is of the form ΘH^2_E, and consequently, an invariant subspace of $S^*_{E_1}$ is of the form K_Θ. It follows that T_Θ is an isometry from H^2_E into $H^2_{E_1}$. When Θ is two-sided inner, by (1), $H^*_{\Theta^*}H^*_\Theta = I - T_\Theta T^*_\Theta$, so $H^*_{\Theta^*}$ is a partial isometry and $R(H^*_{\Theta^*}) = K_\Theta$. It is known that the kernel of a Hankel operator is S-invariant. However, when Θ is just left inner, $N(H_{\Theta^*}) \supset \Theta H^2_E$ and $R(H^*_{\Theta^*}) \subset K_\Theta$ in general. Thus representing ΘH^2_E as the kernel of a Hankel operator is studied in [15] when $\dim E_1 < \infty$, see also [12] for a study of this problem when $\dim E_1 = \infty$. See also [22] where the connection of Hankel operators and shift invariant subspaces on Dirichlet spaces are studied. Connected to this, we have the following result.

Lemma 3.7 *[12] Let $\Theta \in H^\infty_{B(E,E_1)}$ be inner. Then there exists $\Phi \in L^\infty_{B(E_1,E_1)}$ such that $N(H_\Phi) = \Theta H^2_E$. Consequently, $K_\Theta = R(H_{\widetilde{\Phi}})^-$.*

K_Θ is unchanged if we replace Θ by $\Theta \oplus U$, where U is a constant unitary operator. To clarify this situation, we recall a classical concept. Let $B \in H^\infty_{B(E,E_1)}$ and B is a contractive operator-valued function. The function B is called purely contractive (or just pure) if $\|B(0)v\| < \|v\|$ for all $v \in E\backslash\{0\}$. Any contractive $B \in H^\infty_{B(E,E_1)}$ admits a decomposition such that

$$B(z) = B_1(z) \oplus U : E = E_1 \oplus E_2 \to F = F_1 \oplus F_2, \tag{8}$$

where B_1 is $B(E_1, F_1)$-valued and pure in the sense that $\|B_1(0)v\| < \|v\|$ for $v \in E_1\backslash\{0\}$ and U is a unitary constant from E_2 onto F_2. The B_1 is referred to be the purely contractive part and U is the unitary part of B, see [30, Page 194]. It is easy to see that with respect to the decomposition (8),

$$I_F - T_B T^*_B = \left(I_{F_1} - T_{B_1}T^*_{B_1}\right) \oplus 0_{F_2}.$$

Therefore, to represent K_Θ, we can assume Θ is inner and pure.

Theorem 3.8 *A closed subspace M of H^2_F is S^*_F-almost invariant if and only if one of the following holds:*

(1) $M = R(T_\Theta)$, where $\Theta \in H^\infty_{B(E,F)}$ is inner. In this case, $\varsigma(S^*_F, M) = \dim E - rank(U)$, where U is the unitary part of Θ.

(2) $M = R(T_\Phi\,(I_{E_1} - T_\Theta T^*_\Theta))$, where $\Theta \in H^\infty_{B(E,E_1)}$ is inner and pure, $\Phi \in H^2_{B(E_1,F)}$, $\dim E_1 < \infty$, and $T_\Phi\,(I_{E_1} - T_\Theta T^*_\Theta)$ is a partial isometry. In this case, a minimal defect space is $W := R(S^*_F T_\Phi P_{E_1}) \ominus [R(S^*_F T_\Phi P_{E_1}) \cap M]$ and $\varsigma(S^*_F, M) = \dim W$.

Proof. The "if" direction follows from Theorem 2.6. For the "only if" direction, set $F = \mathbb{C}^m$. We will deal with case (i) in Theorem 3.5 since the proof of case (ii) is similar. Then in case (i) $E_1 = \mathbb{C}^{r+p}$, $\Phi = \begin{bmatrix} G_0 & zg_1 & \cdots & zg_p \end{bmatrix}$ is an inner matrix function of size $m \times (r+p)$. Since $K \subset H^2_{\mathbb{C}^{r+p}}$ is S^*-invariant, either $K = H^2_{\mathbb{C}^{r+p}}$ and hence, $M = R(T_\Phi)$ and (1) holds, or by BLH Theorem, $K = K_\Theta = H^2_{\mathbb{C}^{r+p}} \ominus \Theta H^2_{\mathbb{C}^n}$, where $\Theta \in H^\infty_{B(E,E_1,)}$ for $E = \mathbb{C}^n$ with $n \leq r+p$. Note that $I - T_\Theta T^*_\Theta$ is the projection from $H^2_{\mathbb{C}^{r+p}}$ onto K_Θ. By (7), $M = T_\Phi K_\Theta = R(T_\Phi\,(I_{E_1} - T_\Theta T^*_\Theta))$. Now the equation $\|G\|^2 = \sum_{i=0}^p \|k_i\|^2$ is equivalent to that T_Φ acts on K_Θ as an isometry. Namely, for all $h \in H^2_{E_1}$,

$$\langle T_\Phi\,(I - T_\Theta T^*_\Theta)\,h, T_\Phi\,(I - T_\Theta T^*_\Theta)\,h\rangle = \langle (I - T_\Theta T^*_\Theta)\,h, h\rangle\,.$$

Equivalently,

$$[T_\Phi\,(I - T_\Theta T^*_\Theta)]^*\,[T_\Phi\,(I - T_\Theta T^*_\Theta)] = (I - T_\Theta T^*_\Theta)\,. \tag{9}$$

Since $(I - T_\Theta T^*_\Theta)$ is a projection, we know $[T_\Phi\,(I - T_\Theta T^*_\Theta)]$ is a partial isometry. The proof of the "only if" direction is complete.

If $M = R(T_\Theta)$, then by (4), we have

$$S^*_F T_\Theta = T_\Theta S^*_E + S^*_F T_\Theta P_E.$$

Thus the defect space W of $M = R(T_\Theta)$ is a subspace $R(S_F^* T_\Theta P_E)$, where

$$R(S_F^* T_\Theta P_E) = Span\left\{\overline{z}\left[\Theta(z) - \Theta(0)\right] e : e \in E\right\}.$$

Note that $\overline{z}\left[\Theta(z) - \Theta(0)\right] e \in R(T_\Theta)^\perp$, and hence the minimal orthogonal defect space W is $R(S_F^* T_\Theta P_E)$. It is easy to see that $\dim R(S_F^* T_\Theta P_E) = \dim E - rank(U)$, where U is the unitary part of $\Theta(z)$.

If $M = R(T_\Phi\left(I_{E_1} - T_\Theta T_\Theta^*\right))$, then by using (4) twice, we have

$$\begin{aligned} & S_F^* T_\Phi\left(I - T_\Theta T_\Theta^*\right) \\ & = \left(T_\Phi S_{E_1}^* + S_F^* T_\Phi P_{E_1}\right)\left(I - T_\Theta T_\Theta^*\right) \\ & = T_\Phi S_{E_1}^*\left(I - T_\Theta T_\Theta^*\right) + S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right) \\ & = T_\Phi\left(I - T_\Theta T_\Theta^*\right) S_{E_1}^* + G, \end{aligned} \tag{10}$$

where

$$G = -T_\Phi S_{E_1}^* T_\Theta P_E T_\Theta^* + S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right). \tag{11}$$

Note $R(T_\Phi S_{E_1}^* T_\Theta P_E T_\Theta^*) \subset R(T_\Phi S_{E_1}^* T_\Theta P_E)$ and

$$R(T_\Phi S_{E_1}^* T_\Theta P_E) = \left\{\Phi(z)\overline{z}\left[\Theta(z) - \Theta(0)\right] e : e \in E\right\}.$$

Since $\Phi(z)\overline{z}\left[\Theta(z) - \Theta(0)\right] e \in M$, the defect space W is a subspace of $R(S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right))$. We claim that if Θ is pure, then

$$R(P_{E_1}\left(I - T_\Theta T_\Theta^*\right)) = R(P_{E_1}). \tag{12}$$

Thus a minimal defect space W is $R(S_F^* T_\Phi P_{E_1}) \ominus \left[R(S_F^* T_\Phi P_{E_1}) \cap M\right]$. To prove the claim, assume there exists $e_0 \in E_1$ such that $e_0 \perp R(P_{E_1}\left(I - T_\Theta T_\Theta^*\right))$. Then for any $e_1 \in E_1$

$$\begin{aligned} 0 & = \left\langle e_0, P_{E_1}\left(I - T_\Theta T_\Theta^*\right) e_1\right\rangle_{E_1} \\ & = \left\langle e_0, e_1 - P_{E_1}\left(T_\Theta \Theta(0)^* e_1\right)\right\rangle_{E_1} \\ & = \left\langle e_0, \left(I_{E_1} - \Theta(0)\Theta(0)^*\right) e_1\right\rangle_{E_1}. \end{aligned}$$

Since Θ is pure, $\Theta(0)$ is a strict contraction. Hence $e_0 = 0$ and the claim is proved. ∎

For a scalar Toeplitz operator T_φ, where $\varphi \in L^\infty$, it is known that T_φ is a partial isometry if and only if φ is inner [5]. But it is difficult to characterize when $T_\Phi\left(I_{E_1} - T_\Theta T_\Theta^*\right)$ is a partial isometry, see Theorem 3.2 above for the case when both $\Phi \equiv g$ and $\Theta \equiv \theta$ are scalar-valued functions.

Corollary 3.9 *A closed subspace M of H_F^2 is S_F^*-almost invariant if and only if either $M = R(T_\Theta)$, where $\Theta \in H_{B(E,F)}^\infty$ is inner or $M = R(T_\Phi H_\Psi)^-$, where $\Phi \in H_{B(E,F)}^\infty$ and $\Psi \in L_{B(F,E)}^\infty$.*

Proof. By the above theorem, either $M = R(T_\Theta)$ or $M = T_\Phi K_\Theta$. By Lemma 3.7, $K_\Theta = R(H_\Psi)^-$ for some $\Psi \in L_{B(F,E)}^\infty$. Thus $M = T_\Phi K_\Theta = R(T_\Phi H_\Psi)^-$. ∎

The above theorem seems to suggest that "an extra condition that the column space of $S_F^* G_0$ is contained in $M + G$ in case (i)" in Corollary 3.6 is not needed which is not the case, what the above theorem says is that without this condition, M is still S_F^*-almost invariant, but the defect of M is more than p. Corollary 3.6 shows how to get a S_F^*-almost invariant subspace with defect p from a nearly S_F^*-invariant subspace with defect p by this extra condition. Our theorem combines these two concepts using Proposition 3.4. Theorem 3.8 captures the essential part of Theorem 3.5 and Corollary 3.6 and leaves out some details. We think such a reformulation and simplification is useful. The approach to view M as the range of an operator involving Toeplitz and Hankel operators is fruitful. Of course we can also add back more details (a more precise information of Φ) if we need to. We can pick out nearly S_F^*-invariant subspaces from S_F^*-almost invariant subspaces in the following way.

Corollary 3.10 *Let $M := R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$ be as in Theorem 3.8 (2). Then M is nearly S_F^*-invariant if and only if rank $[\Phi(0)] = \dim E_1$. In particular, $\dim E_1 \le \dim F$. If $M = R(T_\Theta)$, where $\Theta \in H^\infty_{B(E,F)}$ is inner, then M is nearly S_F^*-invariant if and only if rank $[\Theta(0)] = \dim F$. In particular, Θ is two-sided inner.*

Proof. Assume *rank* $[\Phi(0)] = \dim E_1$. Let $h \in M$ be such that $h(0) = 0$. Write $h = \Phi k$, where $k \in K_\Theta$. Then $0 = h(0) = \Phi(0)k(0)$ implies $k(0) = 0$ since $\Phi(0)$ is full column rank. Hence $S_F^* h = \Phi S_{E_1}^* k \in M$ and M is nearly S_F^*-invariant.

On the other hand, assume M is nearly S_F^*-invariant. Let $h \in M$ be such that $h(0) = 0$. Write $h = \Phi k$, where $k \in K_\Theta$. Then

$$\begin{aligned} S_F^* h(z) &= \Phi(z) S_{E_1}^* k(z) + \overline{z}\,[\Phi(z) - \Phi(0)]\,k(0) \\ &= \Phi(z) S_{E_1}^* k(z) + \overline{z}\Phi(z)k(0) \in M \end{aligned}$$

implies that $\overline{z}\Phi(z)k(0) = \Phi(z)k_1(z)$ for some $k_1 \in K_\Theta$. Thus $\Phi(z)k(0) = \Phi(z)zk_1(z)$ and $k(0) = zk_1(z)$. This can only happen if $k(0) = 0$. In conclusion, $\Phi(0)k(0) = 0$ implies $k(0) = 0$. If we show the set of all the possible $k(0)$ is E_1, then the above implication proves that *rank* $[\Phi(0)] = \dim E_1$. Indeed, the set of all possible $k(0) = P_{E_1} k_1(z)$, where $k_1 \in K_\Theta$, is $R(P_{E_1}(I - T_\Theta T_\Theta^*))$, which is equal to $R(P_{E_1})$ by (12) since Θ is pure.

In the case $M = R(T_\Theta)$, similarly, we can prove that M is nearly S_F^*-invariant if and only if *rank* $[\Theta(0)] = \dim F$. ■

The above corollary contains Lemma 2.4 in [8], where Φ is assumed to be a diagonal inner function. By Theorem 3.8 and Lemma 2.4, a S_F^*-almost invariant subspace is S_F-almost invariant, special examples in the scalar-valued case ($\dim F = 1$) have been observed in Proposition 2.2 in [10] and in the vector-valued case in Proposition 2.2 in [8]. We also have the converse.

Corollary 3.11 *Let M be a closed subspace of H_F^2. Then M is S_F^*-almost invariant if and only if M is S_F-almost invariant.*

Proof. If M is S_F^*-almost invariant, then by the above theorem, $M = R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$. It follows from Theorem 2.6 that M is S_F-almost invariant.

If M is S_F-almost invariant, then by Lemma 2.9, $M^\perp$ is S_F^*-almost invariant. By what we just proved, $M^\perp$ is S_F-almost invariant. By Lemma 2.9 again, $M = (M^\perp)^\perp$ is S_F^*-almost invariant. ■

Quite amazingly, we obtain the same characterization of S_F-almost invariant subspaces. We also find the defect space.

Theorem 3.12 *Under the same notation as in Theorem 3.8, a closed subspace M of H_F^2 is S_F-almost invariant if and only if one of the following holds:*

(1) $M = R(T_\Theta)$, where $\Theta \in H^\infty_{B(E,F)}$ is inner. In this case, $\varsigma(S_F,\, M) = 0$.

(2) $M = R(T_\Phi (I_{E_1} - T_\Theta T_\Theta^*))$, where $\Theta \in H^\infty_{B(E,E_1)}$ is inner and pure, $\Phi \in H^2_{B(E_1,F)}$, $\dim E_1 < \infty$, and $T_\Phi (I_{E_1} - T_\Theta T_\Theta^*)$ is a partial isometry. In this case, a minimal defect space W is $R(T_\Phi T_\Theta P_E) \ominus [R(T_\Phi T_\Theta P_E) \cap M]$ and $\varsigma(S_F,\, M) = \dim W$.

Proof. By using (4), we have

$$\begin{aligned} &S_F T_\Phi (I - T_\Theta T_\Theta^*) \\ &= T_\Phi S_{E_1} - T_\Phi T_\Theta S_E T_\Theta^* \\ &= T_\Phi S_{E_1} - T_\Phi T_\Theta (T_\Theta^* S_{E_1} - P_E T_\Theta^* S_{E_1}) \\ &= T_\Phi (I - T_\Theta T_\Theta^*) S_{E_1} + T_\Phi T_\Theta P_E T_\Theta^* S_{E_1}. \end{aligned} \tag{13}$$

Therefore, the defect space W is a subspace of $R(T_\Phi T_\Theta P_E) = \{\Phi(z)\Theta(z)e : e \in E\}$. We now claim $R(P_E T_\Theta^* S_{E_1}) = E$. To prove the claim, assume there exists $e_0 \in E$ such that $e_0 \perp R(P_E T_\Theta^* S_{E_1})$. Then for any $h \in H^2_{E_1}$,

$$\begin{aligned} 0 &= \langle e_0, P_E T_\Theta^* S_{E_1} h\rangle_E = \langle e_0, P_E T_\Theta^* S_{E_1} h\rangle_{H^2_E} \\ &= \langle e_0, T_\Theta^* S_{E_1} h\rangle_{H^2_E} = \langle \Theta e_0, zh\rangle_{H^2_E} \\ &= \langle \overline{z}\left[\Theta(z) - \Theta(0)\right] e_0, h\rangle_{H^2_E} . \end{aligned}$$

Thus $\overline{z}\left[\Theta(z) - \Theta(0)\right] e_0 = 0$ and $\|\Theta(z)e_0\|_{H^2_E} = \|\Theta(0)e_0\|_{H^2_E}$. Since T_Θ is an isometry, $\|\Theta(z)e_0\|_{H^2_E} = \|e_0\|_E$. Since Θ is pure, $\|\Theta(0)e_0\|_{H^2_E} < \|e_0\|_E$ if $e_0 \neq 0$. Hence $e_0 = 0$ and the claim is proved. Thus a minimal defect space W is $R(T_\Phi T_\Theta P_E) \ominus \left[R(T_\Phi T_\Theta P_E) \cap M\right]$. ■

4 Examples of S_F-almost invariant subspaces

In spite of a complete characterization of S_F^*-almost invariant subspaces and S_F-almost invariant subspaces by [9] [10] [8] [23] and by Theorem 3.8 and Theorem 3.12, a good understanding of those subspaces requires to clarify the relation between Φ and Θ which seems difficult, as this can be already seen in Theorem 3.2 which is the simplest case when both $\Phi \equiv g$ and $\Theta \equiv \theta$. are scalar-valued functions. In this section, we give some examples of such Φ and Θ.

For $a \in \mathbb{D}$, let

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z} \tag{14}$$

be the automorphism of $\mathbb{D}$. Then $b(z) = \lambda \prod_{i=1}^n \varphi_{a_i}(z)$ is a finite Blaschke product, where $a_i \in \mathbb{D}$ and $|\lambda| = 1$. Let F be a subspace of E and P_F be the projection from E onto F. The inner function

$$Q(z, a, F, E) = \varphi_a(z)\left(I_E - P_F\right) + P_F$$

is called a Blaschke-Potapov factor. To avoid triviality, we assume $F \neq E$ whenever we write down a Blaschke-Potapov factor. Let $B(z) := U \prod_{i=1}^n Q(z, a_i, F_i, E)$, where $U \in B(E)$ is unitary, $a_i \in \mathbb{D}$ and F_i is a subspace of E for $1 \le i \le n$. Such a $B \in H^\infty_{B(E)}$ is called a finite Blaschke-Potapov product. It is known that K_Θ is of finite dimension if and only if Θ is a finite Blaschke-Potapov product [24].

Proposition 4.1 *Let $\Phi \in H^2_{B(E_1,F)}$ be inner and $\Theta \in H^\infty_{B(E,E_1)}$ be inner and pure. Let*

$$M := R(T_\Phi \left(I_{E_1} - T_\Theta T_\Theta^*\right)).$$

The following statements hold.

(i) M is S_F-almost invariant and S_F^*-almost invariant.

(ii) $\varsigma(S_F, M) = \varsigma(S_F^*, M^\perp) = \dim E$ and $\varsigma(S_F, M^\perp) = \varsigma(S_F^*, M) = \dim E_1 - rank(U)$, where U is the unitary part of Φ.

(iii) $M^\perp = R(T_{\Phi\Theta}) \oplus K_\Phi = R(T_{\Phi_1}\left(I - T_{\Theta_1} T_{\Theta_1}^*\right))$, where Φ_1 and Θ_1 are defined by

$$\Phi_1 = \begin{bmatrix} \Phi\Theta & I_F \end{bmatrix} \in H^\infty_{B(E\oplus F, F)} \quad \text{and} \quad \Theta_1 = \begin{bmatrix} 0 \\ \Phi \end{bmatrix} \in H^\infty_{B(E, E\oplus F)}. \tag{15}$$

(iv) M is a half-space if and only if Θ is not a finite Blaschke-Potapov product.

Proof. By assumption, we have $E \subset E_1 \subset F$. Since Φ is inner, by Theorem 3.12, $\varsigma(S_F,\, M) = \dim R(T_\Phi T_\Theta P_E)$. But $T_\Phi T_\Theta P_E = T_{\Phi\Theta} P_E$ and $\Phi\Theta$ is inner, so $\dim R(T_\Phi T_\Theta P_E) = \dim E$. By Lemma 2.9, $\varsigma(S_F^*,\, M^\perp) = \varsigma(S_F,\, M) = \dim E$.

By Theorem 3.8, $\varsigma(S_F^*,\, M) = \dim W$, where $W := R(S_F^* T_\Phi P_{E_1}) \ominus [R(S_F^* T_\Phi P_{E_1}) \cap M]$ is a minimal defect space. We claim $R(S_F^* T_\Phi P_{E_1}) \cap M = \{0\}$. Assume $e_1 \in E_1$ be such that for some $h \in H^2_{E_1}$

$$S_F^* T_\Phi P_{E_1} e_1 = T_\Phi \left(I_{E_1} - T_\Theta T_\Theta^*\right) h.$$

Set $g := \left(I_{E_1} - T_\Theta T_\Theta^*\right) h$. It follows that

$$[\Phi(z) - \Phi(0)]\, e_1 = z\Phi(z) g \;\text{ and }\; \Phi(z)\,(e_1 - zg) = \Phi(0) e_1.$$

Since Φ is inner,

$$\|\Phi(0)e_1\|^2 = \|\Phi(z)\,(e_1 - zg)\|^2 = \|(e_1 - zg)\|^2 = \|e_1\|^2 + \|zg\|^2\,.$$

Thus $g = 0$. This proves the claim $R(S_F^* T_\Phi P_{E_1}) \cap M = \{0\}$. So $W = R(S_F^* T_\Phi P_{E_1})$ is a minimal defect space. Write $\Phi = U \oplus \Psi$, where U is the unitary part of Φ and Ψ is the purely contractive part of Φ. Therefore, $W = R(S_F^* T_\Phi P_{E_1}) = R(S_F^* T_{0\oplus\Psi} P_{E_1})$ and $\dim W = \dim E_1 - rank(U)$. This proves (ii).

Next for (iii), we find $M^\perp$. Assume $h \perp M$. Then $T_\Phi^* h \perp K_\Theta$ and there exists $g \in H^2_E$ such that $T_\Phi^* h = \Theta g$. Thus

$$\Phi^* h = \Theta g + \overline{zu(z)} \text{ for some } u \in H^2_{E_1}.$$

Write $h = \Phi h_1 + h_2$, where $h_1 \in H^2_{E_1}$ and $h_2 \in K_\Phi$. Plugging this decomposition of h into the above equation, we have

$$h_1 = \Theta g + \overline{zu(z)} + \Phi^* h_2.$$

Since $\Phi^* h_2 \in \overline{zH^2_{E_1}}$, we have $h_1 = \Theta g$. Hence $h = \Phi h_1 + h_2 = \Phi\Theta g + h_2 \in R(T_{\Phi\Theta}) \oplus K_\Phi$ and $M^\perp \subset R(T_{\Phi\Theta}) \oplus K_\Phi$. The inclusion $R(T_{\Phi\Theta}) \oplus K_\Phi \subset M^\perp$ can be verified. So $M^\perp = R(T_{\Phi\Theta}) \oplus K_\Phi$. According to Theorem 3.12, $R(T_{\Phi\Theta}) \oplus K_\Phi = R(T_{\Phi_1}\left(I - T_{\Theta_1} T_{\Theta_1}^*\right))$ for some Φ_1 and Θ_1. Indeed, if Φ_1 and Θ_1 are defined by (15), then Θ_1 is inner and T_{Φ_1} acts an an isometry on K_{Θ_1}. Since $K_{\Theta_1} = H^2_E \oplus K_\Phi$, we have $M^\perp = R(T_{\Phi_1}\left(I - T_{\Theta_1} T_{\Theta_1}^*\right))$. We can also use this representation $M^\perp$ to compute $\varsigma(S_F^*, M^\perp)$ and $\varsigma(S_F,\, M^\perp)$ directly. This proves (iii).

Since T_Φ is an isometry, M is finite dimensional if and only only if Θ is a finite Blaschke-Potapov product. When Θ is not a finite Blaschke-Potapov product, then M is infinite dimensional. By (iii), $M^\perp$ is always infinite dimensional. Hence M is a half-space if and only if Θ is not a finite Blaschke-Potapov product. This proves (iv). ∎

Example 4.2 *From Proposition 4.1, one can see that if $\Phi \in H^\infty_{B(E)}$ is inner and $\Theta \in H^\infty_{B(E)}$ is inner and pure then*

$$\Phi\big(\operatorname{ran}\,(I_E - T_\Theta T_\Theta^*)\big) = \Phi\big(\operatorname{ran}\,(H_{\Theta^*}^*)\big) = \Phi K_\Theta$$

is S_F-almost invariant. In particular, K_Θ is S_F-almost invariant with $\varsigma(S_F,\, K_\Theta) = \dim E$.

Example 4.2 contains Propositions 1.4 and 1.5 in [10], where $M = \varphi K_\theta$ is considered with φ and θ being two scalar inner functions and it also contains Propositions 2.5 and 2.6 in [8] where $M = (\Psi K_\Theta)^\perp$ is considered with Ψ being a diagonal square inner function and Θ being an inner function.

In view of Proposition 4.1, we make the following conjecture.

Conjecture 4.3 *Let $M := R(T_\Phi\,(I_{E_1} - T_\Theta T_\Theta^*))$ be as in Theorem 3.8. Then M is a half-space if and only if Θ is not a finite Blaschke-Potapov product.*

It follows from Lemma 2.2 (iv) that we can characterize a S_F-almost invariant subspace as the kernel of $H_{\Theta^*} T_\Phi^*$ (when T_Φ^* is unbounded, we simply interpret $\ker H_{\Theta^*} T_\Phi^*$ as $[R(T_\Phi H_{\Theta^*}^*)]^\perp$).

Corollary 4.4 *Under the same notation as in Theorem 3.8, a closed subspace M of H^2_F is S_F-almost invariant if and only if $M = N(T^*) = [R(T)]^\perp$, where $T = T_\Phi\left(I_{E_1} - T_\Theta T_\Theta^*\right)$. In this case, $\varsigma(S_F, N(T^*)) = \varsigma(S_F^*, R(T))$.*

Since $T = T_\Phi\left(I_{E_1} - T_\Theta T_\Theta^*\right)$ is a partial isometry,

$$I_F - TT^* = I_F - T_\Phi(I_{E_1} - T_\Theta T_\Theta^*)T_\Phi^* \tag{16}$$

is the projection onto $N(T^*)$.

Corollary 4.5 *Under the same notation as in Theorem 3.8, a closed subspace M of H^2_F is S_F^*-almost invariant if and only if either $M = R(T_\Theta)$ or M is a reproducing kernel space whose kernel $K_M(z,w)$ is of the form*

$$K_M(z,w) = \frac{\Phi(z)(I_{E_1} - \Theta(z)\Theta(w)^*)\Phi(w)^*}{1 - z\overline{w}}. \tag{17}$$

Similarly, a closed subspace M of H^2_F is S_F-almost invariant if and only if either $M = K_\Theta$ or M is a reproducing kernel space whose kernel is of the form

$$K_{M^\perp}(z,w) = \frac{I_F - \Phi(z)(I_{E_1} - \Theta(z)\Theta(w)^*)\Phi(w)^*}{1 - z\overline{w}}. \tag{18}$$

Proof. Set $T := T_\Phi\left(I_{E_1} - T_\Theta T_\Theta^*\right)$. Since T is a partial isometry [13], the reproducing kernel of $R(T)$ is $TT^*\left(k_w(z)I_F\right)$, where $k_w(z) = 1/(1 - z\overline{w})$ is the reproducing kernel of H^2 [13]. It follows from a general fact that

$$\begin{aligned} TT^*\left(k_w(z)I_F\right) &= T_\Phi(I_{E_1} - T_\Theta T_\Theta^*)T_\Phi^* k_w(z)I_F \\ &= \frac{\Phi(z)(I_E - \Theta(z)\Theta(w)^*)\Phi(w)^*}{1 - z\overline{w}}. \end{aligned}$$

Similarly, by (16), the reproducing kernel of $N(T^*)$ is given by (18). ■

We note that the reproducing kernels of $R(T_\Theta)$ and K_Θ in the above corollary are respectively ($\Theta \in H^\infty_{B(E,F)}$ is inner)

$$\frac{\Theta(z)\Theta(w)^*}{1 - z\overline{w}} \text{ and } \frac{I_F - \Theta(z)\Theta(w)^*}{1 - z\overline{w}}.$$

By Corollary 3.11, the kernel in (18) can be represented as a kernel in (17) with different $\Phi(z)$ and $\Theta(z)$, it will be interesting to have a direct proof of this fact.

5 Invariant subspaces of a finite rank perturbation of the shift operator

It has been observed in Proposition 1.3 in [1] that an almost invariant subspace of T on a Banach space is actually an invariant subspace of $T + T_0$, where T_0 is a finite rank operator.

Lemma 5.1 *[1] Let X be a Banach space, $T \in B(X)$ and M be a closed subspace of X. Then M is T-almost invariant if and if M is $(T + T_0)$-invariant for some finite rank operator T_0.*

It turns out that if X is a Hilbert space, for a given M, we can write down all those T_0 such that M is $(T + T_0)$-invariant. Then M is T-invariant for $T \in B(H)$ if and only if $TP_M - P_M TP_M = 0$ and M is T-reducing if and only if $TP_M - TP_M = 0$. This inspires another equivalent notion of almost invariance which appeared in literature much earlier, see for example [20]. To distinguish we temporarily give it a different term.

Definition 5.2 *[20] Let $T \in B(H)$ and M be a closed subspace of H. Then we say that M is T-finite rank invariant if $TP_M - P_MTP_M$ is of finite rank and that M is T-finite rank reducing if $TP_M - P_MT$ is of finite rank.*

It is known that the notion of T-finite rank invariant is equivalent to the notion of T-almost invariant. We write down a proof to illustrate some points.

Lemma 5.3 *Let $T \in B(H)$ and M be a closed subspace of H. Then M is T-finite rank invariant if and only if M is T-almost invariant. Similarly, M is T-finite rank reducing if and only if M is T-almost reducing.*

Proof. Assume $W := TP_M - P_MTP_M = (I - P_M)TP_M$ is of finite rank. Then for any $h \in M$, $Th = P_MTh + Wh \in M + R(W)$. That is, $TM \subset M + R(W)$. So $R(W)$ is a finite dimensional defect space of M and M is T-almost invariant. In fact, $R(W) \perp M$ since $P_MW = P_M(I - P_M)TP_M = 0$. So $R(W)$ is the minimal orthogonal defect space of M.

On the other hand, assume M is T-almost invariant. That is, $TM \subset M \oplus G$, where G is finite dimensional. Set $W := TP_M - P_MTP_M$. It is clear that $W|M^{\perp} = 0$. For $h \in M$, $Wh = Th - P_MTh = (I - P_M)Th \in M \oplus G$. Thus $Wh \in G$ since $Wh \perp M$. This proves $R(W) \subset G$ and W is a finite rank operator. Hence M is T-finite rank invariant.

Similarly, assume $W := TP_M - P_MT$ is of finite rank. Then, $TM \subset M + R(W)$ and $T^*M \subset M + R(W^*)$. Hence M is T-almost reducing.

On the other hand, assume M is T-almost reducing. By what we have just proved, $W_1 := (I - P_M)TP_M$ and $W_2 := (I - P_M)T^*P_M$ are both of finite rank. Then

$$TP_M - P_MT = W_1 - W_2^*$$

is of finite rank. Hence M is T-finite rank reducing. ■

Theorem 5.4 *Let $T \in B(H)$ and M be a closed subspace of H. Assume M is T-almost invariant. Then M is $(T + T_0)$-invariant for some finite rank operator T_0 if and only if*

$$T_0 = -(I - P_M)TP_M + \sum_{i=1}^{k} x_i \otimes y_i + \sum_{j=1}^{m} u_j \otimes v_j, \tag{19}$$

where $x_i \in M$ and $y_i \in H$, $u_j \in H$ and $v_j \in M^{\perp}$ are arbitrary.

Similarly, assume M is T-almost reducing. Then M is $(T+T_0)$-reducing for some finite rank operator T_0 if and only if

$$T_0 = -(I - P_M)TP_M - P_MT(I - P_M) + \sum_{i=1}^{k} x_i \otimes y_i + \sum_{j=1}^{m} u_j \otimes v_j$$

where $x_i, y_i \in M$ and $u_j, v_j \in M^{\perp}$ are arbitrary.

Proof. Set $W := (I-P_M)TP_M$, $W_1 = \sum_{i=1}^{k} x_i \otimes y_i$ and $W_2 = \sum_{j=1}^{m} u_j \otimes v_j$. Assume $T_0 = -W+W_1+W_2$. Then for $h \in M$,

$$\begin{aligned}(T + T_0)\, h &= Th - (TP_M - P_MTP_M)\, h + \textstyle\sum_{i=1}^{k} \langle h, y_i\rangle\, x_i \\ &= P_MTP_Mh + \textstyle\sum_{i=1}^{k} \langle h, y_i\rangle\, x_i \in M.\end{aligned}$$

This proves "if" direction.

Now assume T_0 is a finite rank operator such that M is $(T + T_0)$-invariant. Then for $h \in M$,

$$(T_0 + W)\, h = T_0h + (TP_M - P_MTP_M)\, h = (T + T_0)\, h - P_MTP_Mh \in M.$$

Hence $(T_0 + W)\, P_M$ is a finite rank operator and $R\,[(T_0 + W)\, P_M] \subset M$. Therefore, $(T_0 + W)\, P_M = W_1$ for some W_1 of the form $\sum_{i=1}^{k} x_i \otimes y_i$. Note that

$$(T_0 + W)\, (I - P_M) = T_0(I - P_M) = W_2 = \sum_{j=1}^{m} u_j \otimes v_j$$

for some W_2 of the form $\sum_{j=1}^{m} u_j \otimes v_j$. Thus

$$(T_0 + W) = (T_0 + W)\, P_M + T_0(I - P_M) = W_1 + W_2.$$

This proves (19).

Next we prove the reducing case. The "if" direction comes from a direct verification. We prove the "only if" direction. Assume T_0 is a finite rank operator such that M is $(T + T_0)$-reducing. Set

$$Q := T_0 + (I - P_M)TP_M + P_M T(I - P_M).$$

Since M is T-almost reducing, both $(I - P_M)TP_M$ and $P_M T(I - P_M)$ are of finite rank. Then for $h \in M$,

$$Qh = (T_0 + T)h - P_M Th \in M,$$

and for $h_1 \in M^\perp$

$$Qh_1 = T_0 h_1 + P_M T h_1 = (T_0 + T)h_1 - (I - P_M)Th_1 \in M^\perp.$$

Thus $QM \subset M$ and $QM^\perp \subset M^\perp$. Consequently,

$$Q = P_M Q P_M + (I - P_M)Q(I - P_M) = \sum_{i=1}^{k} x_i \otimes y_i + \sum_{j=1}^{m} u_j \otimes v_j$$

where $x_i, y_i \in M$ and $u_j, v_j \in M^\perp$. The proof is complete. ■

For essentially invariant subspaces, its first appearance was in Brown and Pearcy [6] in 1971 where it was proved that every operator on a complex infinite dimensional Hilbert space admits an essentially invariant subspace. Here is the definition used in [6] [20].

Definition 5.5 *[6] [20] Let $T \in B(H)$ and M be a closed subspace of H. Then we say that M is T-BP essentially invariant if $TP_M - P_M TP_M$ is a compact operator and that M is T-BP essentially reducing if $TP_M - P_M T$ is a compact operator.*

Inspired by Lemma 5.1, Sirotkin and Wallis [29] gave the following definition on a Banach space, but we state it on a Hilbert space.

Definition 5.6 *[29] Let $T \in B(H)$ and M be a closed subspace of H. Then we say that M is T-SW essentially invariant if M is $(T + T_0)$-invariant for some compact operator T_0 and that M is T-SW essentially reducing if M is $(T + T_0)$-reducing for some compact operator T_0.*

Inspired by Definition 1.1, we can make the following definition.

Definition 5.7 *[29] Let $T \in B(H)$ and M be a closed subspace of H. Then we say that M is T-essentially invariant if there exists a compact operator G such that $TM \subset M + R(G)$ and that M is T-essentially reducing if both M and $M^\perp$ are T-essentially invariant.*

As expected, these three definitions are equivalent and we give a proof for clarity.

Lemma 5.8 *Let $T \in B(H)$ and M be a closed subspace of H. Then M is T-BP essentially invariant if and only if M is T-SW essentially invariant if and only if T-essentially invariant. A similar statement holds for essentially reducing subspaces.*

Proof. Assume that M is T-BP essentially invariant. Then $(I - P_M)TP_M$ is compact and M is $[T - (I - P_M)TP_M]$-invariant. Thus M is T-SW essentially invariant.

Assume M is T-SW essentially invariant. That is M is $[T + T_0]$-invariant for some compact operator T_0. Thus $[T + T_0]\, M \subset M$. Then it is easy to see that $TM \subset M + R(T_0)$. Thus M is T-essentially invariant.

Assume M is T-essentially invariant. Let G be a compact operator such that $TM \subset M + R(G)$. Then

$$TM \subset M + R(G) = M + R(P_M G + (I - P_M)G) \subset M \oplus R((I - P_M)G).$$

Set $W := (I - P_M)TP_M$. Then $W|M^\perp = 0$ and $R(W) \subset M^\perp$. For $h \in M$,

$$Wh = Th - P_M Th \subset M \oplus R((I - P_M)G).$$

Therefore, $R(W) \subset R((I - P_M)G)$. Since $(I - P_M)G$ is compact, W is compact. This proves M is T-BP essentially invariant. The proof for the reducing case is similar. ■

Theorem 5.9 *Let $T \in B(H)$ and M be a closed subspace of H. Assume M is T-essentially invariant. Then M is $(T + T_0)$-invariant for some compact operator T_0 if and only if*

$$T_0 = -(I - P_M)TP_M + P_M W_1 + W_2(I - P_M), \tag{20}$$

where $W_1, W_2 \in B(H)$ are two arbitrary compact operators.

Similarly, assume M is T-essentially reducing. Then M is $(T + T_0)$-reducing for compact operator T_0 if and only if

$$T_0 = -(I - P_M)TP_M - P_M T(I - P_M) + P_M W_1 P_M + (I - P_M)W_2(I - P_M),$$

where $W_1, W_2 \in B(H)$ are two arbitrary compact operators.

Proof. Set $W := (I - P_M)TP_M$. By assumption and Lemma 5.8, W is a compact operator. Assume $T_0 = -W + P_M W_1 + W_2(I - P_M)$, where $W_1, W_2 \in B(H)$ are compact. Then for $h \in M$,

$$\begin{aligned}(T + T_0)\,h &= Th - (TP_M - P_M TP_M)\,h + P_M W_1 h \\ &= P_M TP_M h + P_M W_1 h \in M.\end{aligned}$$

This proves "if" direction.

Now assume T_0 is a compact operator such that M is $(T + T_0)$-invariant. Then for $h \in M$,

$$(T_0 + W)\,h = T_0 h + (TP_M - P_M TP_M)\,h = (T + T_0)\,h - P_M TP_M h \in M.$$

Hence, $P_M\,(T_0 + W)\,P_M = (T_0 + W)\,P_M$. Thus

$$\begin{aligned}T_0 + W &= (T_0 + W)\,P_M + (T_0 + W)\,(I - P_M) \\ &= P_M\,(T_0 + W)\,P_M + T_0(I - P_M).\end{aligned}$$

This proves (20) with $W_1 = (T_0 + W)\,P_M$ and $W_2 = T_0$.

The proof for the reducing case is similar. ■

By using Theorem 5.4, for a S^*-almost invariant subspace M, we can write down all finite rank operators T_0 such that M is $(S + T_0)$-invariant.

Theorem 5.10 *Let $M := R(T_\Phi\,(I_{E_1} - T_\Theta T_\Theta^*))$ be as in Theorem 3.8. The following statements hold.*

(i) M is $(S_F + T_0)$-invariant for some finite rank operator T_0 if and only if

$$T_0 = -T_\Phi T_\Theta P_E T_\Theta^* T_\Phi^* + \textstyle\sum_{i=1}^{k} x_i \otimes y_i + \sum_{j=1}^{m} u_j \otimes v_j, \tag{21}$$

where $x_i \in M$ and $y_i \in H$, $u_j \in H$ and $v_j \in M^\perp$ are arbitrary.

(ii) M is $(S_F^* + T_1)$-invariant for some finite rank operator T_1 if and only if

$$T_1 = -S_F^* T_\Phi P_{E_1}\,(I - T_\Theta T_\Theta^*)\,T_\Phi^* + \textstyle\sum_{i=1}^{k} x_i \otimes y_i + \sum_{j=1}^{m} u_j \otimes v_j,$$

where $x_i \in M$ and $y_i \in H$, $u_j \in H$ and $v_j \in M^\perp$ are arbitrary.

(iii) M is (S_F+T_2)-reducing for some finite rank operator T_2 if and only if

$$T_2 = -T_\Phi T_\Theta P_E T_\Theta^* T_\Phi^* - S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right) T_\Phi^* + \textstyle\sum_{i=1}^k x_i \otimes y_i + \sum_{j=1}^m u_j \otimes v_j,$$

where $x_i, y_i \in M$ and $u_j, v_j \in M^\perp$ are arbitrary.

Proof. Set $K := T_\Phi\left(I - T_\Theta T_\Theta^*\right)$ and $W := T_\Phi T_\Theta P_E T_\Theta^* S_{E_1}$. Then (13) become $S_F K = K S_{E_1} + W$. Since K is a partial isometry, $P_M = KK^*$. Thus

$$\begin{aligned}(I-P_M)S_F P_M &= (I-KK^*)S_F KK^* \\ &= (I-KK^*)(KS_{E_1}+W)K^* \\ &= (I-KK^*)WK^* = WK^* - P_M WK^*,\end{aligned}$$

where $(I-KK^*)K = 0$. Since Θ is inner, we have

$$\begin{aligned}WK^* &= T_\Phi T_\Theta P_E T_\Theta^* S_{E_1}\left(I - T_\Theta T_\Theta^*\right) T_\Phi^* \\ &= T_\Phi T_\Theta P_E T_\Theta^*\left(\left(I - T_\Theta T_\Theta^*\right) T_\Phi^* S - T_\Theta P_E T_\Theta^* T_\Phi^*\right) \\ &= T_\Phi T_\Theta P_E T_\Theta^* T_\Phi^*.\end{aligned}$$

Set $W_1 := \sum_{i=1}^k x_i \otimes y_i$ and $W_2 := \sum_{j=1}^m u_j \otimes v_j$, where $x_i \in M$, $y_i, u_j \in H$, $v_j \in M^\perp$. By Theorem 5.4,

$$T_0 = -\left(WK^* - P_M WK^*\right) + W_1 + W_2.$$

Note that $P_M WK^*$ is a finite rank operator of the same form as W_1. Thus $T_0 = -WK^* + W_1 + W_2$.

Next we prove (ii). Set $G := -T_\Phi S_{E_1}^* T_\Theta P_E T_\Theta^* + S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right)$. Then (10) becomes $S_F^* K = K S_{E_1}^* + G$. Note that

$$\begin{aligned}(I-P_M)S_F^* P_M &= (I-KK^*)S_F^* KK^* \\ &= (I-KK^*)\left(KS_{E_1}^* + G\right)K^* \\ &= (I-KK^*)GK^* = GK^* - P_M GK^*. \end{aligned} \tag{22}$$

Since Θ is inner,

$$\begin{aligned}GK^* &= \left[-T_\Phi S_{E_1}^* T_\Theta P_E T_\Theta^* + S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right)\right]\left(I - T_\Theta T_\Theta^*\right) T_\Phi^* \\ &= S_F^* T_\Phi P_{E_1}\left(I - T_\Theta T_\Theta^*\right) T_\Phi^*.\end{aligned}$$

By Theorem 5.4,

$$T_1 = -\left(GK^* - P_M GK^*\right) + W_1 + W_2.$$

Note that $P_M GK^*$ is a finite rank operator of the same form as W_1. Thus $T_1 = -GK^* + W_1 + W_2$.

Now we prove (iii). By taking adjoint, (22) becomes

$$P_M S_F (I - P_M) = KG^* - KG^* P_M.$$

Set $W_1 := \sum_{i=1}^k x_i \otimes y_i$ and $W_2 := \sum_{j=1}^m u_j \otimes v_j$, where $x_i, y_i \in M$, $u_j, v_j \in M^\perp$. By Theorem 5.4,

$$T_2 = -\left(WK^* - P_M WK^*\right) - \left(GK^* - P_M GK^*\right) + W_1 + W_2.$$

Since $M = R(K)$, $P_M K = K$ and $K^* = K^* P_M$. Thus $P_M WK^* = P_M WK^* P_M$ and $P_M GK^* = P_M WK^* P_M$. So $P_M WK^* P_M$ and $P_M WK^* P_M$ are finite rank operators of the same form as W_1. Therefore, $T_2 = -WK^* - GK^* + W_1 + W_2$. ∎

We end up the paper with a question. Let G be a given finite rank operator on H_F^2. If M is $(S_F + G)$-invariant, then M is S_F-almost invariant and a subspace of $R(G)$ is a defect space. Thus $M = R(T_\Phi\left(I - T_\Theta T_\Theta^*\right))$. In this case, the following question naturally arises:

Question 5.11 *How do we find these Φ and Θ in terms of G?*

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