Mutual Information-oriented ISAC Beamforming Design under Statistical CSI

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Abstract

Existing integrated sensing and communication (ISAC) beamforming design were mostly designed under perfect instantaneous channel state information (CSI), limiting their use in practical dynamic environments. In this paper, we study the beamforming design for multiple-input multiple-output (MIMO) ISAC systems based on statistical CSI, with the weighted mutual information (MI) comprising sensing and communication perspectives adopted as the performance metric. In particular, the operator-valued free probability theory is utilized to derive the closed-form expression for the weighted MI under statistical CSI. Subsequently, an efficient projected gradient ascent (PGA) algorithm is proposed to optimize the transmit beamforming matrix with the aim of maximizing the weighted MI. Numerical results validate that the derived closed-form expression matches well with the Monte Carlo simulation results and the proposed optimization algorithm is able to improve the weighted MI significantly. We also illustrate the trade-off between sensing and communication MI.

Index Terms

Integrated sensing and communication, mutual information, beamforming design, statistical channel state information, free probability theory.

I. INTRODUCTION

The sixth generation (6G) wireless network has stimulated numerous innovative applications, such as smart cities and intelligent transportation. Toward this end, higher requirements of communication and sensing capabilities are raised for numerous nodes within the network. Due to the capability of enabling the dual functionalities of information transmission and target sensing with shared wireless resources, integrated sensing and communication (ISAC) has been regarded as a key enabler for the realization of 6G network [\[1\]](#page-12-0).

As a promising approach to enhance the performances of ISAC systems, beamforming design has been investigated in numerous works adopting various communication and sensing performance metrics. In [\[2\]](#page-12-1), the authors considered transmit beampattern and signal-to-interference-plus-noise (SINR) as sensing and communication performance

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Fig. 1. Considered MIMO ISAC system.

metrics respectively. As a step further, due to its capability of characterizing the lower bound of parameter estimation, the Cramer-Rao Bound (CRB) has also been adopted as a sensing performance metric in ISAC systems [\[3\]](#page-12-2). However, ´ in most existing works, the performance metrics for sensing and communications are diverse, resulting in difficulties in evaluating the trade-offs between sensing and communication performance. To address this issue, some researches have adopted mutual information (MI) as unified performance metric for ISAC systems [\[4\]](#page-12-3), [\[5\]](#page-12-4).

It should be noted that the aforementioned works require perfect channel state information (CSI). However, in practical systems, accurate instantaneous CSI is hard to obtain due to the increasing signaling overhead, especially in a highly dynamic scenario. By contrast, it is easy to obtain the long-term channel statistics, such as the spatial correlation of the antenna array. In [\[6\]](#page-12-5), [\[7\]](#page-12-6), the authors have investigated the beamforming design in communication systems. In [\[6\]](#page-12-5), the authors leveraged the second-order channel statistics to design hybrid analog/digital precoding in mm-wave systems with the aim of enhancing the rate performance. In [\[7\]](#page-12-6), the authors studied the beamforming design for RIS aided MIMO communication systems and maximized the ergodic rate of the system. However, the ISAC beamforming design based on long-term channel statistics remains an open problem.

In this paper, we investigate the MI-oriented transmit beamforming design for a MIMO ISAC system, where an ISAC base station (BS) serves a user equipment (UE) while simultaneously sensing an extended target. Based on the statistical CSIs available at the BS, we formulate a transmit beamforming design problem with the aim of maximizing the weighted MI. Applying the operator-valued free probability theory, we derive the closed-form expression of the weighted MI and reformulate the beamforming design problem. Subsequently, based on the obtained closed-form expression, we propose an efficient projected gradient ascent (PGA) algorithm to solve the problem. Numerical results validate the accuracy of the derived expression, as well as the convergence and effectiveness of the proposed algorithm. In addition, the trade-off between sensing and communication MI is also depicted.

II. SYSTEM MODEL

A. Signal Model

We consider an ISAC system as shown in Fig. [1,](#page-1-0) where a BS is equipped with N_t transmitting antennas and N_r receiving antennas. The BS simultaneously serves a UE equipped with N_u receiving antennas and senses an extended target with L scatters uniformly distributed in the proximity of the center. The transmit beamforming The received signal at UE can be expressed as

$$
\mathbf{Y}_c = \mathbf{H}_c \mathbf{W} \mathbf{S} + \mathbf{N}_C,\tag{1}
$$

where H_c is the channel between UE and BS, $N_C \in \mathbb{C}^{N_u \times N_s}$ is the additive white Gaussian noise (AWGN) at the receiving antennas of UE and $N_C \sim \mathcal{CN}(0, \sigma_c^2 \mathbf{I}_{N_s})$. The received echoes at the BS, $\mathbf{Y}_s \in \mathbb{C}^{N_r}$, can be expressed as

$$
\mathbf{Y}_s = \sum_{l=1}^L \mathbf{G}_l \mathbf{WS} + \mathbf{N}_S,
$$
\n(2)

where $G_l \in \mathbb{C}^{N_r \times N_t}$ is the round-trip channel matrix between the *l*−th scatter and BS, $\mathbf{N}_S \in \mathbb{C}^{N_r \times N_s}$ is the AWGN at the receiving antennas of the BS and $N_S \sim \mathcal{CN}(0, \sigma_s^2 \mathbf{I}_{N_s})$.

B. Channel Model

Due to the limitations of the Kronecker channel model in accurately representing the correlation between transceivers when scattering clusters are uniformly distributed, we employ the Weichselberger MIMO channel model [\[8\]](#page-12-7), which captures the spatial correlation at both ends of the link and their mutual interdependence, for both radar sensing channel and the communication channel. The channels in [\(1\)](#page-2-0) and [\(2\)](#page-2-1) can be expressed as

$$
\mathbf{H}_c = \overline{\mathbf{H}}_c + \widetilde{\mathbf{H}}_c = \overline{\mathbf{H}}_c + \mathbf{U}(\mathbf{M} \odot \mathbf{P})\mathbf{V}^\dagger, \tag{3}
$$

$$
\mathbf{G}_{l} = \overline{\mathbf{G}}_{l} + \widetilde{\mathbf{G}}_{l} = \overline{\mathbf{G}}_{l} + \mathbf{R}_{l}(\mathbf{N}_{l} \odot \mathbf{Q}_{l}) \mathbf{T}_{l}^{\dagger}, 1 \leq l \leq L,
$$
\n(4)

where \overline{H}_c and \overline{G}_l are deterministic matrices which denote the line-of-sight (LoS) component of H_c and G_l . U, V, \mathbf{R}_l and \mathbf{T}_l are deterministic unitary matrices. M and N are deterministic nonnegative matrices which represent the variance profiles of the random components \widetilde{H}_c and \widetilde{G}_l respectively. $P \in \mathbb{C}^{N_u \times N_t}$ and $Q_l \in \mathbb{C}^{N_r \times N_t}$ are complex Gaussian distributed with $P_{i,j} \sim \mathcal{CN}(0,1/T)$ and $[Q_l]_{i,j} \sim \mathcal{CN}(0,1/T)$.

C. Problem Formulation

From now on, for convenience of expression, we use $I_s(\sigma^2)$ and $I_c(\sigma^2)$ to denote $I_s(\sigma_s^2)$ and $I_c(\sigma_c^2)$, respectively. According to [\[9\]](#page-13-0), the sensing MI can be expressed as

$$
I_s(\sigma^2) = N_r \mathbb{E}\left[\log\det\left(\mathbf{I}_{N_s} + \frac{1}{\sigma^2} \sum_{l=1}^L (\mathbf{S}^\dagger \mathbf{W}^\dagger \mathbf{G}_l^\dagger \mathbf{G}_l \mathbf{W} \mathbf{S})\right)\right]
$$

= $N_r \mathbb{E}\left[\log\det\left(\mathbf{I}_{LN_r} + \frac{1}{\sigma^2} (\hat{\mathbf{G}} \mathbf{S} \mathbf{S}^\dagger \hat{\mathbf{G}}^\dagger) \right)\right],$ (5)

where the expectation is taken over the random components G_l in [\(4\)](#page-2-2) and random data symbols S , and the matrix \hat{G} is defined as $\hat{G} = [(G_1 W)^{\dagger}, (G_2 W)^{\dagger}, ..., (G_l W)^{\dagger}]^{\dagger}$. Based on the received signal at UE, the communication MI for the considered ISAC system can be expressed as [\[10\]](#page-13-1)

$$
I_c(\sigma^2) = \mathbb{E}\left[\log \det(\mathbf{I}_{N_u} + \frac{1}{\sigma^2} \hat{\mathbf{H}} \hat{\mathbf{H}}^\dagger)\right],\tag{6}
$$

where $\hat{H} = H_c W$, and the expectation is taken over the random components \tilde{H}_c in [\(3\)](#page-2-3).

In order to achieve a balance between communication performance and sensing performance, we define a weighted MI as the performance metric for the proposed ISAC system, which is expressed as

$$
I(\mathbf{W}) = \rho I_s(\sigma^2) + (1 - \rho)I_c(\sigma^2),\tag{7}
$$

where ρ is a weighting factor that determines the weights of communication performance and sensing performance in the ISAC system. Furthermore, the transmit beamforming problem can be formulated as:

$$
\mathbf{P1}\max_{\mathbf{W}} I(\mathbf{W})\tag{8a}
$$

$$
\text{s.t.} \quad \left\| \mathbf{W} \right\|_F \stackrel{2}{\leq} \leq P_t,\tag{8b}
$$

where $\|\cdot\|_F^2$ denotes the Frobenius norm, [\(8b\)](#page-3-0) is the transmit power constraint and P_t is the transmit power budget.

Due to the operation of taking expectation, there is no exact expression for the weighted MI $I(\mathbf{W})$, resulting in difficulties in solving Problem (P1) with conventional methods. To address this issue, we will derive the closed-form expression for the weighted MI $I(\mathbf{W})$ in Section [III.](#page-3-1)

III. PROBLEM REFORMULATION AND PROPOSED ALGORITHM

In this section, we reformulate Problem (P1) and propose an efficient algorithm to solve the reformulated problem. Specially, in Section [III-A,](#page-3-2) we first utilize the free probability theory and the linearization trick to derive the closedform expression of the Cauchy transform. Based on this, we reformulate Problem (P1) by deriving the closed-form expression for the weighted MI of the considered ISAC system. Finally, the PGA algorithm is proposed to solve the reformulated problem in Section [III-B.](#page-7-0)

A. Problem Reformulation

Utilizing the Shannon transform [\[10\]](#page-13-1), $I_s(\sigma^2)$ can be rewritten as

$$
I_s(\sigma^2) = LN_r^2 \mathcal{V}_{\mathbf{B}_1}(\sigma^2) = LN_r^2 \int_0^\infty \log(1 + \frac{1}{\sigma^2} \lambda) f_{\mathbf{B}_1}(\lambda) d\lambda,
$$
\n(9)

where $B_1 = \hat{G} S S^{\dagger} \hat{G}^{\dagger}$, $\mathcal{V}_{B_1}(\sigma^2)$ is the Shannon transform of B_1 , and $f_{B_1}(\lambda)$ is the probability density function (PDF) of B_1 . Furthermore, the relationship between the Shannon transform and the Cauchy transform is applied to obtain the specific form of the Shannon transform, which can be expressed as

$$
\frac{dI_s(\sigma^2)}{dz} = LN_r^2 \frac{dV_{\mathbf{B}_1}(z)}{dz} = -\frac{LN_r^2}{z} - LN_r^2 \mathcal{G}_{\mathbf{B}_1}(-z),\tag{10}
$$

where $z = -\frac{1}{\sigma^2}$, and the Cauchy transform of B_1 is defined as

$$
\mathcal{G}_{\mathbf{B}_1}(z) = \int_0^\infty \frac{1}{z - \lambda} df_{\mathbf{B}_1}(\lambda).
$$
 (11)

Similarly, for the communication MI, we have

$$
\frac{dI_c(\sigma^2)}{dz} = N_u \frac{dV_{\mathbf{B}_2}(z)}{dz} = -\frac{N_u}{z} - N_u \mathcal{G}_{\mathbf{B}_2}(-z),\tag{12}
$$

$$
\mathcal{G}_{\mathbf{B}_2}(z) = \int_0^\infty \frac{1}{z - \lambda} \mathrm{d}f_{\mathbf{B}_2}(\lambda),\tag{13}
$$

where $\mathbf{B}_2 = \hat{\mathbf{H}} \hat{\mathbf{H}}^{\dagger}$, $f_{\mathbf{B_2}}(\lambda)$ is the PDF of \mathbf{B}_2 and $\mathcal{V}_{\mathbf{B}_2}(\sigma^2)$ is the Shannon transform of \mathbf{B}_2 and $\mathcal{G}_{\mathbf{B}_2}(z)$ is the Cauchy transform of B_2 .

To obtain the closed-form expression of $\mathcal{G}_{\mathbf{B}_i}(z)$, free probability theory serves as a powerful analytical tool [\[12\]](#page-13-2). However, in the considered system, \hat{G} and S are not free in the classic free probability aspect, resulting in difficulties in directly obtaining the Cauchy Transform for the product of \hat{G} , S , S^{\dagger} and \hat{G}^{\dagger} . To address this issue, the linearization trick will be adopted to embed the non-free matrices into a lager matrix, in which the deterministic parts and random parts are proved to be asymptotically free. Then the desired Cauchy transform can be obtained from the transformation of the operator-valued Cauchy transform for the embedded matrix. For the convenience of expression, we define $\overline{G} = [(\overline{G}_1 W)^T, (\overline{G}_2 W)^T, ..., (\overline{G}_l W)^T]^T$ and $\overline{H} = \overline{H}_c W$. Note that since I_c and I_s share a similar form, we only provide the detailed derivation of $\mathcal{G}_{\mathbf{B}_1}(z)$.

To obtain the closed-form of $\mathcal{G}_{\mathbf{B}_1}(z)$, we first apply the Anderson's linearization trick [\[12\]](#page-13-2) to construct a block matrix of size $(LN_r + N_s + 2M) \times (LN_r + N_s + 2M)$ as

$$
\mathbf{B}_{L} = \begin{bmatrix} \mathbf{0}_{LN_r \times LN_r} & \mathbf{0}_{LN_r \times M} & \mathbf{0}_{LN_r \times N_s} & \hat{\mathbf{G}} \\ \mathbf{0}_{M \times LN_r} & \mathbf{0}_{M \times M} & \mathbf{S} & -\mathbf{I}_M \\ \mathbf{0}_{N_s \times LN_r} & \mathbf{S}^\dagger & -\mathbf{I}_{N_s} & \mathbf{0}_{N_s \times M} \\ \hat{\mathbf{G}}^\dagger & -\mathbf{I}_M & \mathbf{0}_{M \times N_s} & \mathbf{0}_{M \times M} \end{bmatrix} . \tag{14}
$$

The operator-valued Cauchy transform $\mathcal{G}_{\mathbf{B}_L}^{\mathcal{D}}$ of \mathbf{B}_L [\[12\]](#page-13-2) is given by

$$
\mathcal{G}_{\mathbf{B}_L}^{\mathcal{D}}(\mathbf{\Lambda}(z)) = \mathbb{E}_{\mathcal{D}}\left[\left(\mathbf{\Lambda}(z) - \mathbf{B}_L\right)^{-1} \right],\tag{15}
$$

where $\mathbb{E}_{\mathcal{D}}[\mathbf{X}]$ is defined as

$$
\mathbb{E}_{\mathcal{D}}\left[\mathbf{X}\right] = \begin{bmatrix} \mathbb{E}\left[\mathbf{X}_{1}\right]_{1} & \cdots & \cdots & \cdots & \cdots\\ -\frac{1}{2} & \mathbb{E}\left[\mathbf{X}_{2}\right]_{1} & \cdots & \cdots & \cdots\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \end{bmatrix}, \tag{16}
$$

where $X_1 = \{X\}_{1}^{LN_r}$, $X_2 = \{X\}_{LN_r+1}^{LN_r+M}$, $X_3 = \{X\}_{LN_r+M+1}^{LN_r+M+N_s}$, $X_4 = \{X\}_{LN_r+M+N_s+1}^{LN_r+2M+N_s}$, with the notation ${A}^b_a$ denoting the submatrix of A containing the rows and columns with indices from a to b, i.e., $[{A}^b_a]_{i,j} =$ $[A]_{i+a-1,j+a-1}$ for $1 \le i,j \le b-a+1$, where the notation $[A]_{i,j}$ is the element in the *i*-th row and *j*-th column

of matrix **A**. The matrix function $\Lambda(z)$ is defined as

$$
\mathbf{\Lambda}(z) = \begin{bmatrix} z\mathbf{I}_{LN_r} & \mathbf{0}_{LN_r \times (N_s+2)} \\ \mathbf{0}_{(N_s+2) \times LN_r} & \mathbf{0}_{(N_s+2) \times (N_s+2)} \end{bmatrix} .
$$
 (17)

Then $\mathcal{G}_{\mathbf{B}_1}(z)$ is given by

$$
\mathcal{G}_{\mathbf{B}_1}(z) = \frac{1}{LN_r} \text{Tr}\left(\left\{\mathcal{G}_{\mathbf{B}_L}^{\mathcal{D}}(\mathbf{\Lambda}(z))\right\}^{(1,1)}\right),\tag{18}
$$

where $\{\cdot\}^{(1,1)}$ represents the upper-left $LN_r \times LN_r$ matrix block and the operator Tr(\cdot) represents the trace of the matrix. It can be observed that B_L shares a similar structure with the matrix L proposed in [\[13,](#page-13-3) Prop. 2], which inspires us to use the method in [\[13\]](#page-13-3). Specifically, the Cauchy transform $\mathcal{G}_{\mathbf{B}_1}(z)$ can be obtained with the following proposition.

Proposition 1. The Cauchy transform of B_1 , with $z \in \mathbb{C}^+$, is given by

$$
\mathcal{G}_{\mathbf{B}_1}(z) = \frac{1}{LN_r} \text{Tr} \left[\mathcal{G}_{\widetilde{\mathbf{C}}}(z) \right],\tag{19}
$$

where $\mathcal{G}_{\tilde{C}}(z)$ *satisfies the following equations*

$$
\mathcal{G}_{\widetilde{\mathbf{C}}}(z) = \left(\widetilde{\Psi}(z) - \overline{\mathbf{G}}\Pi^{-1}\overline{\mathbf{G}}^{\dagger}\right)^{-1},\tag{20}
$$

$$
\mathbf{\Pi} = \mathbf{\Psi}(z) - \widetilde{\mathbf{\Phi}}(z)^{-1},\tag{21}
$$

where the matrices $\tilde{\Psi}(z)$, $\Psi(z)$, $\tilde{\Phi}(z)$, $\Phi(z)$ *are respectively denoted as*

$$
\widetilde{\Psi}(z) = z\mathbf{I}_{LN_r} - \text{diag}\left\{\widetilde{\eta}_1(\mathcal{G}_{\mathbf{C}}(z)), \dots, \widetilde{\eta}_L(\mathcal{G}_{\mathbf{C}}(z)\right\},\tag{22}
$$

$$
\Psi(z) = -\sum_{l=1}^{L} \eta_l(\mathcal{G}_{\widetilde{\mathbf{C}}_l}(z)),\tag{23}
$$

$$
\widetilde{\Phi}(z) = -\widetilde{\zeta}(\mathcal{G}_{\widetilde{\mathbf{D}}}(z)),\tag{24}
$$

$$
\Phi(z) = \mathbf{I}_{N_s} - \zeta(\mathcal{G}_{\mathbf{D}}(z)),\tag{25}
$$

where the notation diag(A, \dots, B) *represent the diagonal block matrix constructed by* A, \dots, B *matrices and* $\eta_l(\tilde{C})$ *,* $\tilde{\eta}_l(C)$ *,* $\zeta(D)$ *,* $\tilde{\zeta}(D)$ are the parameterized one-sided correlation matrices, which are shown as [\(46\)](#page-9-0), [\(47\)](#page-9-1), [\(54\)](#page-10-0) and [\(55\)](#page-10-1) in Appendix [A.](#page-9-2) The matrices $\mathcal{G}_{\mathbf{C}}(z)$, $\mathcal{G}_{\mathbf{\widetilde{D}}}(z)$, $\mathcal{G}_{\mathbf{D}}(z)$, and $\mathcal{G}_{\mathbf{\widetilde{C}}_l}(z)$ are defined as

$$
\mathcal{G}_{\mathbf{C}}(z) = \left(\Psi(z) - \overline{\mathbf{G}}^{\dagger} \widetilde{\Psi}(z)^{-1} \overline{\mathbf{G}}^{\dagger} - \widetilde{\Phi}(z)^{-1}\right)^{-1},\tag{26}
$$

$$
\mathcal{G}_{\widetilde{\mathbf{D}}}(z) = \Phi(z)^{-1},\tag{27}
$$

$$
\mathcal{G}_{\mathbf{D}}(z) = \left(\widetilde{\Phi}(z) - \left(\Psi(z) - \overline{\mathbf{G}}^{\dagger}\widetilde{\Psi}(z)\overline{\mathbf{G}}\right)^{-1}\right)^{-1},\tag{28}
$$

$$
\mathcal{G}_{\widetilde{\mathbf{C}}_l}(z) = \{ \mathcal{G}_{\widetilde{\mathbf{C}}}(z) \}_{1+(l-1)N_r}^{lN_r}.
$$
\n(29)

Proof. The proof of this Proposition 1 is similar to the method presented in [\[13,](#page-13-3) Prop. 2]. Therefore we provide a brief outline of the proof. First, we prove that the deterministic and random components of the linearized matrix are free. Then, by applying the subordination formula, we derive the equation for the operator-valued Cauchy transform. Finally, using the matrix inversion formula, we decompose the operator-valued Cauchy transform, thereby obtaining the above expressions. \Box

Based on the closed-form expression of the Cauchy transformation $\mathcal{G}_{\mathbf{B}_1}(z)$, we can derive the closed-form expression of the Shannon transform $\mathcal{V}_{\mathbf{B}_1}(z)$ in the following proposition.

Proposition 2. The Shannon transform $V_{\mathbf{B}_1}(z)$, with $z \in \mathbb{C}^+$, is given by [\(44\)](#page-7-1), where $\widetilde{\Psi}(-z)$, $\Psi(-z)$, $\widetilde{\Phi}(-z)$, $\mathcal{G}_{\widetilde{\mathbf{D}}}(-z)$, $\mathcal{G}_{\mathbf{D}}(-z)$, and $\mathcal{G}_{\widetilde{\mathbf{C}}}(-z)$ are given by [\(20\)](#page-5-0)-[\(25\)](#page-5-1) in Proposition [1.](#page-5-2)

Proof. The proof of Propsition [2](#page-6-0) is given in Appendix [B](#page-10-2) .

 \Box

Similarly, the Cauchy transform $\mathcal{G}_{\mathbf{B}_2}(z)$ of \mathbf{B}_2 and the Shannon transform $\mathcal{V}_{\mathbf{B}_2}(z)$, with $z \in \mathbb{C}^+$, are given by

$$
\mathcal{G}_{\mathbf{B}_2}(z) = \frac{1}{N_u} \text{Tr} \left[\mathcal{G}_{\widetilde{\mathbf{E}}}(z) \right],\tag{30}
$$

$$
\mathcal{V}_{\mathbf{B}_2}(z) = \frac{1}{N_u} \log \det \left(\frac{\widetilde{\mathbf{\Omega}}(-z)}{-z} \right) + \frac{1}{N_u} \text{Tr} \left(\mathcal{G}_{\widetilde{\mathbf{E}}}(-z)\widetilde{\tau} \right)
$$

$$
- \frac{1}{N_u} \log \det \left(\mathcal{G}_{\mathbf{E}}(-z) \right), \tag{31}
$$

where $\eta_l(\widetilde{\mathbf{C}})$, $\widetilde{\eta}_l(\mathbf{C})$, $\tau(\mathbf{E})$, $\widetilde{\tau}(\widetilde{\mathbf{E}})$ are the parameterized one-sided correlation matrices, which are shown as [\(46\)](#page-9-0), [\(47\)](#page-9-1), [\(50\)](#page-9-3) and [\(51\)](#page-9-4) in [A.](#page-9-2) The matrices $\mathcal{G}_{\widetilde{\mathbf{E}}}(z)$, $\mathcal{G}_{\mathbf{E}}(z)$, $\widetilde{\mathbf{\Omega}}(z)$ and $\mathbf{\Omega}(z)$ satisfy the following equations

$$
\mathcal{G}_{\widetilde{\mathbf{E}}}(z) = \left(\widetilde{\mathbf{\Omega}}(z) - \overline{\mathbf{H}}\mathbf{\Omega}(z)^{-1}\overline{\mathbf{H}}^{\dagger}\right)^{-1},\tag{32}
$$

$$
\mathcal{G}_{\mathbf{E}}(z) = \left(\Omega(z) - \overline{\mathbf{H}}^{\dagger} \widetilde{\Omega}(z)^{-1} \overline{\mathbf{H}}\right)^{-1},\tag{33}
$$

$$
\widetilde{\mathbf{\Omega}}(z) = z\mathbf{I}_{N_u} - \widetilde{\tau}(\mathcal{G}_{\mathbf{E}}(z)),\tag{34}
$$

$$
\Omega(z) = 1 - \tau(\mathcal{G}_{\widetilde{\mathbf{E}}}(z)).
$$
\n(35)

Then the sensing MI , the communication MI and the weighted MI can be rewritten as

$$
I_s(z, \mathbf{W}) = LN_r^2 \mathcal{V}_{\mathbf{B}_1}(\sigma^2),\tag{36}
$$

$$
I_c(z, \mathbf{W}) = N_u \mathcal{V}_{\mathbf{B}_2}(\sigma^2),\tag{37}
$$

$$
I(z, \mathbf{W}) = \rho I_s(z, \mathbf{W}) + (1 - \rho)I_c(z, \mathbf{W}).
$$
\n(38)

Consequently, we reformulate the problem (P1) as

$$
\mathbf{(P2)}\max_{\mathbf{W}} I(z, \mathbf{W})\tag{39a}
$$

$$
s.t. (8b). \t(39b)
$$

$$
\mathcal{V}_{\mathbf{B}_1}(z) = \frac{1}{LN_r} \log \det \left(\frac{\tilde{\Psi}(-z)}{-z} \right) + \frac{1}{LN_r} \log \det \left(\Psi(-z) - \overline{\mathbf{G}}^{\dagger} \tilde{\Psi}(-z)^{-1} \overline{\mathbf{G}}^{\dagger} - \tilde{\Phi}(-z)^{-1} \right) + \frac{1}{LN_r} \log \det \left(\tilde{\Phi}(-z) \right)
$$

+
$$
\frac{1}{LN_r} \text{Tr} \left(\mathcal{G}_{\tilde{\mathbf{C}}}(-z) \left(-z \mathbf{I}_{LNr} - \tilde{\Psi}(-z) \right) \right) + \frac{1}{LN_r} \text{Tr} \left(\mathcal{G}_{\tilde{\mathbf{D}}}(-z) \zeta \right) + \frac{1}{LN_r} \log \det \left(\Phi(-z) \right), \qquad (44)
$$

$$
[\nabla g_{\mathbf{W}}]_{i,j} = \rho N_r \text{Tr} \left(\left(-\Psi + \mathbf{W}^{\dagger} \overline{\mathbf{G}} \tilde{\Psi}^{-1} \overline{\mathbf{G}} \mathbf{W} + \tilde{\Phi}^{-1} \right)^{-1} \mathbf{E}_{i,j}^{\dagger} \overline{\mathbf{G}}^{\dagger} \tilde{\Psi}^{-1} \overline{\mathbf{G}} \mathbf{W} \right)
$$

+
$$
(\rho - 1) \text{Tr} \left(\left(\mathbf{\Omega} - \mathbf{W}^{\dagger} \overline{\mathbf{H}}^{\dagger} \tilde{\mathbf{\Omega}}^{-1} \overline{\mathbf{H}} \mathbf{W} \right)^{-1} \mathbf{E}_{i,j}^{\dagger} \overline{\mathbf{H}}^{\dagger} \tilde{\mathbf{\Omega}}^{-1} \overline{\mathbf{H}} \mathbf{W} \right), \qquad (45)
$$

B. Proposed PGA Algorithm

In order to solve the problem (P2), we propose the PGA algorithm. Firstly, we calculate the gradient of the weighted MI in [\(38\)](#page-6-1). Then the updated beamforming matrix at the $(i + 1)$ -th iteration is updated as

$$
\widetilde{\mathbf{W}}^{[i+1]} = \mathbf{W}^{[i]} + \lambda \nabla g_{\mathbf{W}} \left(\mathbf{W}^{[i]} \right), \tag{40}
$$

where $W^{[i]}$ denotes the beamforming matrix at the *i*-th iteration, λ denotes the step size, and the element of gradient $\nabla g_{\mathbf{W}}$ of $I(z, \mathbf{W})$ is given by [\(45\)](#page-7-2), where $\mathbf{E}_{i,j}$ is a matrix whose elements satisfy

$$
[\mathbf{E}_{i,j}]_{s,t} = \begin{cases} 1, & if \ s = i \ and \ t = j, \\ 0, & otherwise. \end{cases}
$$
 (41)

To satisfy the transmit power constraint [\(8b\)](#page-3-0), the solution $\widetilde{W}^{[i+1]}$ is then projected onto the feasible region. The obtained solution at the $(i + 1)$ -th iteration is given by

$$
\mathbf{W}^{[i+1]} = \operatorname{Proj}_{W} \left(\widetilde{\mathbf{W}}^{[i+1]} \right), \tag{42}
$$

where the projection operator is given as

$$
\text{Proj}_{W} = \begin{cases} \mathbf{W}, & if \|\mathbf{W}\|_{F}^{2} \le P_{t}, \\ \sqrt{P_{t}} \frac{\mathbf{W}}{\|\mathbf{W}\|_{F}}, & otherwise. \end{cases}
$$
(43)

The detailed algorithm is presented in Algorithm 1.

Algorithm 1 PGA algorithm

initialize: Set the convergence criterion ε , $i = 0$, and randomly generate the beamforming matrix $\mathbf{W}^{[0]}$. Calculate $I^{[0]}$ based on [\(38\)](#page-6-1). repeat Update $W^{[i+1]}$ based on [\(42\)](#page-7-3).

Calculate $I^{[i+1]}$ based on [\(38\)](#page-6-1). $i = i + 1.$ until $|I^{[i]} - I^{[i-1]}| \leq \varepsilon$. output Optimal beamforming matrix W.

Fig. 2. (a) Communication MI and sensing MI versus SNR. (b) Weighted MI versus the number of iterations, where $\rho = 0.5$.

IV. NUMERICAL RESULTS

In this section, we provide numerical results to verify the accuracy of the derived closed-form weighted MI expression and the effectiveness of the proposed PGA algorithm. For the channel settings, the deterministic components are modeled as the direct links between uniform planar arrays (UPA) equipped at both the transmitter and the receiver. In addition, statistical characteristics parameters of the channel including the deterministic unitary matrices U, V, \mathbf{R}_l and \mathbf{T}_l , as well as the variance matrices M and \mathbf{N}_l , are generated randomly but fixed in each Monte Carlo simulation. Unless otherwise stated, the number of scatters, data streams and signal samples are set as $L = 2$, $M = N_u$, and $N_s = M$, respectively. In addition, the transmit power budget is set as $P_t = N_t$. It should be noted that since the radar detection link distance is typically longer than the communication link distance to UE, without loss of generality, we assume that the signal-noise-ratio (SNR) at UE is 20 dB higher than the SNR at BS. Besides,the SNR mentioned in this section refers to the SNR at UE. Furthermore, the beamforming matrix is set as $W = \sqrt{P_t/M} I_M$ without optimization. All simulation results are generated by averaging over 10^6 channel realizations.

Subsequently, the convergence of the proposed PGA algorithm under different numbers of antennas is plotted in Fig. [2\(](#page-8-0)b). The weighting factor and the SNR are set as $\rho = 0.5$ and 10 dB respectively. It can be observed that the weighted MI increases with the number of antennas. This is because a larger number of antennas provides higher diversity gain and greater design DOFs. In addition, the weighted MI increases with the iterations and converges within 3 iterations, demonstrating the fast convergence of the proposed PGA algorithm.

To demonstrate the effectiveness of the proposed optimization algorithm, we show the optimized weighted MI with different numbers of antennas for the considered system in Fig. [3\(](#page-9-5)a). The weighting factor is set as $\rho = 0.4$. It can be observed that with the same SNR, the proposed PGA algorithm can improve the weighted MI.

To further depict the trade-off between sensing and communication MI, we show the optimized weighted MI with weighting factors ρ ranging from 0 to 1 in Fig. [3\(](#page-9-5)b). The number of antennas and the SNR are set as $N_t = N_r = N_u = 8$ and 10 dB respectively. As the weighting factor increases, the ISAC system places greater emphasis on sensing performance, and conversely, prioritizes communication performance when the weighting factor

Fig. 3. (a) Weighted MI versus SNR with different schemes, where $\rho = 0.4$. (b) Sensing MI versus Communication MI. decreases.

V. CONCLUSIONS

In this paper, we investigated the beamforming design for a practical MIMO ISAC system. To derive the closedform expression of the weighted MI for the considered system, we resorted to the operator-valued free probability. Based on the closed-form expression, we proposed the PGA algorithm to design the transmit beamforming matrix for maximizing the weighted MI. Simulation results verified the accuracy of the derived closed-form expression and the effectiveness of the proposed algorithm.

APPENDIX A

PARAMETERIZED ONE-SIDED CORRELATION MATRICES

We define the parameterized one-sided correlation matrices of G_l as

$$
\eta_l(\widetilde{\mathbf{C}}) = \mathbb{E}[\widetilde{\mathbf{G}}_l^\dagger \widetilde{\mathbf{C}} \widetilde{\mathbf{G}}_l] = \frac{1}{T} \mathbf{T}_l \mathbf{\Pi}_l(\widetilde{\mathbf{C}}) \mathbf{T}_l^\dagger, 1 \le l \le L,\tag{46}
$$

$$
\widetilde{\eta}_l(\mathbf{C}) = \mathbb{E}[\widetilde{\mathbf{G}}_l \mathbf{C} \widetilde{\mathbf{G}}_l^\dagger] = \frac{1}{T} \mathbf{R}_l \widetilde{\mathbf{\Pi}}_l(\mathbf{C}) \mathbf{R}_l^\dagger, 1 \le l \le L,\tag{47}
$$

where $\widetilde{C} \in \mathbb{C}^{N_t \times N_t}$ and $C_l \in \mathbb{C}^{N_r \times N_r}$ are any Hermitian matrices, $\Pi_l(\widetilde{C})$ and $\widetilde{\Pi}_l(C)$ are diagonal matrices with the entries given by

$$
\left[\mathbf{\Pi}_{l}(\widetilde{\mathbf{C}})\right]_{i,i} = \sum_{j=1}^{N_r} \left([\mathbf{N}_l]_{j,i} \right)^2 \left[\mathbf{R}_l^\dagger \widetilde{\mathbf{C}} \mathbf{R}_l \right]_{j,j}, 1 \le i \le N_t,
$$
\n(48)

$$
\left[\widetilde{\mathbf{H}}_{l}(\mathbf{C})\right]_{i,i} = \sum_{j=1}^{N_t} \left([\mathbf{N}_l]_{i,j} \right)^2 \left[\mathbf{T}_l^{\dagger} \mathbf{C} \mathbf{T}_l \right]_{j,j}, 1 \le i \le N_r. \tag{49}
$$

Similarly, the parameterized one-sided correlation matrices of H_c , are defined as

$$
\tau(\mathbf{E}) = \mathbb{E}[\widetilde{\mathbf{H}}_c^{\dagger} \mathbf{E} \widetilde{\mathbf{H}}_c] = \frac{1}{T} \mathbf{V} \Sigma(\mathbf{E}) \mathbf{V}^{\dagger},\tag{50}
$$

$$
\widetilde{\tau}(\widetilde{\mathbf{E}}) = \mathbb{E}[\widetilde{\mathbf{H}}_c \widetilde{\mathbf{E}} \widetilde{\mathbf{H}}_c^{\dagger}] = \frac{1}{T} \mathbf{U} \widetilde{\mathbf{\Sigma}} (\widetilde{\mathbf{E}}) \mathbf{U}^{\dagger},\tag{51}
$$

where $\widetilde{\mathbf{E}} \in \mathbb{E}^{N_t \times N_t}$ and $\mathbf{E} \in \mathbb{C}^{N_u \times N_u}$ are arbitrary Hermitian matrices. $\Sigma(\mathbf{E})$ and $\widetilde{\Sigma}(\widetilde{\mathbf{E}})$ are diagonal matrices with the entries given by

$$
\left[\mathbf{\Sigma}(\mathbf{E})\right]_{i,i} = \sum_{j=1}^{N_u} \left([\mathbf{M}]_{j,i} \right)^2 \left[\mathbf{U}^\dagger \mathbf{D} \mathbf{U} \right]_{j,j},\tag{52}
$$

$$
\left[\widetilde{\mathbf{\Sigma}}(\widetilde{\mathbf{E}})\right]_{i,i} = \sum_{j=1}^{N_t} \left([\mathbf{M}]_{i,j} \right)^2 \left[\mathbf{V}^\dagger \widetilde{\mathbf{E}} \mathbf{V} \right]_{j,j}.
$$
\n(53)

The parameterized one-sided correlation matrices of S are

$$
\zeta(\mathbf{D}) = \mathbb{E}[\mathbf{S}^{\dagger} \mathbf{D} \mathbf{S}] = \frac{1}{N_s} \text{Tr}(\mathbf{D}) \mathbf{I}_{N_s},\tag{54}
$$

$$
\widetilde{\zeta}(\widetilde{\mathbf{D}}) = \mathbb{E}[\mathbf{S}\widetilde{\mathbf{D}}\mathbf{S}^{\dagger}] = \frac{1}{N_s} \text{Tr}(\widetilde{\mathbf{D}}) \mathbf{I}_M,\tag{55}
$$

where $\widetilde{\mathbf{D}} \in \mathbb{C}^{N_s \times N_s}$ and $\mathbf{D} \in \mathbb{C}^{M \times M}$ are any Hermitian matrices.

APPENDIX B

PROOF OF PROPOSITION [1](#page-5-2)

To prove the Proposition [2,](#page-6-0) we need to prove that [\(10\)](#page-3-3) holds with the $\mathcal{V}_{\mathbf{B}_1}(z)$ given in [\(44\)](#page-7-1). The partial derivative of $V_{\mathbf{B}}(z)$ with respect to z is given by

$$
\frac{d}{dz}\mathcal{V}_{\mathbf{B}_1}(z) = \frac{1}{LN_r}\frac{d}{dz}\log\det\left(\frac{\widetilde{\Psi}}{-z}\right) + \frac{1}{LN_r}\frac{d}{dz}\log\det\left(\Psi - \overline{\mathbf{G}}^{\dagger}\widetilde{\Psi}^{-1}\overline{\mathbf{G}}^{\dagger} - \widetilde{\Phi}^{-1}\right) + \frac{1}{LN_r}\frac{d}{dz}\log\det\left(\widetilde{\Phi}\right),
$$

$$
+ \frac{1}{LN_r}\frac{d}{dz}\text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{C}}}\left(-z\mathbf{I}_{LNr} - \widetilde{\Psi}\right)\right) + \frac{1}{LN_r}\frac{d}{dz}\text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{D}}}\zeta\right) + \frac{1}{LN_r}\frac{d}{dz}\log\det\left(\Phi\right),\tag{56}
$$

where notation ($-z$) are omitted for convenience. For a matrix function $K(z)$, the following equations hold:

$$
\frac{d}{dz}\log\det \mathbf{K}(z) = \text{Tr}\left(\mathbf{K}(z)^{-1}\frac{d\mathbf{K}(z)}{dz}\right),\tag{57}
$$

$$
\operatorname{Tr}\left(\frac{d\mathbf{K}(z)^{-1}}{dz}\right) = -\operatorname{Tr}\left(\mathbf{K}(z)^{-1}\frac{d\mathbf{K}(z)}{dz}\mathbf{K}(z)^{-1}\right). \tag{58}
$$

According to [\[14,](#page-13-4) Lemma1], the following equations hold:

$$
\operatorname{Tr}(\mathbf{A}_1 \eta_l(\mathbf{A}_2)) = \operatorname{Tr}(\mathbf{A}_2 \tilde{\eta}_l(\mathbf{A}_1)), 1 \le l \le L,\tag{59}
$$

$$
\operatorname{Tr}(\mathbf{B}_1\zeta(\mathbf{B}_2)) = \operatorname{Tr}(\mathbf{B}_2\tilde{\zeta}(\mathbf{B}_1)).\tag{60}
$$

Then we will simplify the terms on the right-hand side of equation [\(56\)](#page-10-3) based on [\(57\)](#page-10-4),[\(58\)](#page-10-5),[\(59\)](#page-10-6) and [\(60\)](#page-10-7).

For the first term of [\(56\)](#page-10-3), we can obtain

$$
\frac{d}{dz}\log\det\left(\frac{\widetilde{\Psi}}{-z}\right) = \text{Tr}\left(\left(\frac{\widetilde{\Psi}}{-z}\right)^{-1}\frac{d\left(\frac{\widetilde{\Psi}}{-z}\right)}{dz}\right) = \text{Tr}\left(-\frac{1}{z}\mathbf{I} + \widetilde{\Psi}^{-1}\frac{d\widetilde{\Psi}}{dz}\right) = -\frac{LN_r}{z} + \text{Tr}\left(\widetilde{\Psi}^{-1}\frac{d\widetilde{\Psi}}{dz}\right). \tag{61}
$$

For the second term , the third term and the last term in [\(56\)](#page-10-3), we can get

$$
\frac{d}{dz}\log\det\left(-\widetilde{\Phi}^{-1}+\Delta\right)=\operatorname{Tr}\left(\left(-\widetilde{\Phi}^{-1}+\Delta\right)^{-1}\frac{d\left(-\widetilde{\Phi}^{-1}+\Delta\right)}{dz}\right),\tag{62}
$$

$$
\frac{d}{dz}\log\det\left(\widetilde{\Phi}\right) = \text{Tr}\left(\widetilde{\Phi}^{-1}\frac{d\widetilde{\Phi}}{dz}\right),\tag{63}
$$

$$
\frac{d}{dz}\log\det\left(\mathbf{\Phi}\right) = \text{Tr}\left(\mathbf{\Phi}^{-1}\frac{d\mathbf{\Phi}}{dz}\right),\tag{64}
$$

where Δ is denoted as $\Delta = \Psi - \overline{G}^{\dagger} \tilde{\Psi}^{-1} \overline{G}^{\dagger}$. Therefore \mathcal{G}_{C} and \mathcal{G}_{D} can be expressed as

$$
\mathcal{G}_{\mathbf{C}} = \left(\Delta - \widetilde{\Phi}^{-1}\right)^{-1},\tag{65}
$$

$$
\mathcal{G}_{\mathbf{D}} = \left(\widetilde{\Phi}(z) - \Delta^{-1}\right)^{-1}.\tag{66}
$$

By applying [\(59\)](#page-10-6), [\(60\)](#page-10-7) and the Woodbury identity, the forth and the fifth term can be expressed as

$$
\frac{d}{dz}\text{Tr}\left(\mathcal{G}_{\tilde{C}}\left(-z\mathbf{I}_{LNr}-\tilde{\Psi}\right)\right)
$$
\n
$$
= \text{Tr}\left(\mathcal{G}_{\tilde{C}}\frac{d\left(-z\mathbf{I}_{LNr}-\tilde{\Psi}\right)}{dz}\right) + \text{Tr}\left(\text{diag}\{\tilde{\eta}_{1}\left(\mathcal{G}_{C}\right),...,\tilde{\eta}_{L}\left(\mathcal{G}_{C}\right)\}\frac{d\mathcal{G}_{\tilde{C}}}{dz}\right)
$$
\n
$$
= \text{Tr}\left(\mathcal{G}_{\tilde{C}}\frac{d\left(-z\mathbf{I}_{LNr}-\tilde{\Psi}\right)}{dz}\right) + \text{Tr}\left(\mathcal{G}_{C}\frac{d\left(\sum_{l=1}^{L}\eta_{l}\left(\mathcal{G}_{\tilde{C}_{l}}\right)\right)}{dz}\right)
$$
\n
$$
= -\text{Tr}\left(\mathcal{G}_{\tilde{C}}\right) - \text{Tr}\left(\mathcal{G}_{\tilde{C}}\frac{d\tilde{\Psi}}{dz}\right) - \text{Tr}\left(\mathcal{G}_{C}\frac{d\Psi}{dz}\right)
$$
\n
$$
= -\text{Tr}\left(\mathcal{G}_{\tilde{C}}\right) - \text{Tr}\left(\left(\tilde{\Psi}(z)^{-1} + \tilde{\Psi}(z)^{-1}\overline{\mathbf{G}}\mathcal{G}_{C}\overline{\mathbf{G}}^{\dagger}\tilde{\Psi}(z)^{-1}\right)\frac{d\tilde{\Psi}}{dz}\right) - \text{Tr}\left(\mathcal{G}_{C}\frac{d\Psi}{dz}\right)
$$
\n
$$
= -\text{Tr}\left(\mathcal{G}_{\tilde{C}}\right) - \text{Tr}\left(\tilde{\Psi}^{-1}\frac{d\tilde{\Psi}}{dz}\right) + \text{Tr}\left(\mathcal{G}_{C}\frac{d\left(\overline{\mathbf{G}}^{\dagger}\tilde{\Psi}\overline{\mathbf{G}}\right)}{dz}\right) - \text{Tr}\left(\mathcal{G}_{C}\frac{d\Psi}{dz}\right),\tag{67}
$$

$$
\frac{d\text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{D}}}\zeta\right)}{dz} = \text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{D}}}\frac{d\zeta}{dz}\right) + \text{Tr}\left(\zeta \frac{d\mathcal{G}_{\widetilde{\mathbf{D}}}}{dz}\right)
$$
\n
$$
= \text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{D}}}\frac{d\left(\mathbf{I} - \boldsymbol{\Phi}\right)}{dz}\right) + \text{Tr}\left(\mathcal{G}_{\mathbf{D}}\frac{d\widetilde{\zeta}}{dz}\right)
$$
\n
$$
= -\text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{D}}}\frac{d\boldsymbol{\Phi}}{dz}\right) - \text{Tr}\left(\mathcal{G}_{\mathbf{D}}\frac{d\widetilde{\boldsymbol{\Phi}}}{dz}\right)
$$
\n
$$
= -\text{Tr}\left(\boldsymbol{\Phi}^{-1}\frac{d\boldsymbol{\Phi}}{dz}\right) - \text{Tr}\left(\mathcal{G}_{\mathbf{D}}\frac{d\widetilde{\boldsymbol{\Phi}}}{dz}\right). \tag{68}
$$

Similarly, we can get the following equations

$$
\operatorname{Tr}\left(\left(\Delta - \tilde{\Phi}^{-1}\right)^{-1}\frac{d\left(-\Delta\right)}{dz}\right)
$$
\n
$$
= \operatorname{Tr}\left(\left(\Delta - \tilde{\Phi}^{-1}\right)^{-1}\frac{d\left(\overline{G}^{\dagger}\tilde{\Psi}\overline{G} - \Psi\right)}{dz}\right)
$$
\n
$$
= \operatorname{Tr}\left(\mathcal{G}_{C}\frac{d\left(\overline{G}^{\dagger}\tilde{\Psi}\overline{G}\right)}{dz}\right) - \operatorname{Tr}\left(\mathcal{G}_{C}\frac{d\Psi}{dz}\right) - \operatorname{Tr}\left(\mathcal{G}_{D}\frac{d\tilde{\Phi}}{dz}\right) + \operatorname{Tr}\left(\left(\tilde{\Phi} - \Delta^{-1}\right)^{-1}\frac{d\tilde{\Phi}}{dz}\right),\tag{69}
$$
\n
$$
\operatorname{Tr}\left(\left(\tilde{\Phi} - \Delta^{-1}\right)^{-1}\frac{d\tilde{\Phi}}{dz}\right)
$$
\n
$$
= \operatorname{Tr}\left(\tilde{\Phi}^{-1}\frac{d\tilde{\Phi}}{dz}\right) - \operatorname{Tr}\left(\left(\Delta - \tilde{\Phi}^{-1}\right)^{-1}\frac{d\tilde{\Phi}^{-1}}{dz}\right).
$$
\n
$$
\qquad (70)
$$

Then [\(62\)](#page-11-0) can be expressed as

$$
\frac{d}{dz}\log\det\left(-\tilde{\Phi}^{-1}+\Delta\right)
$$
\n
$$
= -\text{Tr}\left(\left(\Delta - \tilde{\Phi}^{-1}\right)^{-1}\frac{d\left(-\Delta\right)}{dz}\right) - \text{Tr}\left(\left(\Delta - \tilde{\Phi}^{-1}\right)^{-1}\frac{d\tilde{\Phi}^{-1}}{dz}\right),
$$
\n
$$
= -\text{Tr}\left(\mathcal{G}_{\mathbf{C}}\frac{\left(\overline{\mathbf{G}}^{\dagger}\tilde{\mathbf{\Psi}}\overline{\mathbf{G}}\right)}{dz}\right) + \text{Tr}\left(\mathcal{G}_{\mathbf{C}}\frac{d\mathbf{\Psi}}{dz}\right) + \text{Tr}\left(\mathcal{G}_{\mathbf{D}}\frac{d\tilde{\Phi}}{dz}\right) - \text{Tr}\left(\tilde{\Phi}^{-1}\frac{d\tilde{\Phi}}{dz}\right).
$$
\n(71)

Combining [\(61\)](#page-10-8),[\(63\)](#page-11-1),[\(64\)](#page-11-2),[\(67\)](#page-11-3),[\(68\)](#page-11-4),[\(71\)](#page-12-8), we have

$$
\frac{d}{dz}\mathcal{V}_{\mathbf{B}_1}(z) = -\frac{1}{z} - \frac{1}{LN_r}\text{Tr}\left(\mathcal{G}_{\widetilde{\mathbf{C}}}\right) = -\frac{1}{z} - \mathcal{G}_{\mathbf{B}_1}(-z),\tag{72}
$$

and the proof of Proposition [2](#page-6-0) is completed.

REFERENCES

- [1] J. A. Zhang, M. L. Rahman, K. Wu, X. Huang, Y. J. Guo, S. Chen, and J. Yuan, "Enabling joint communication and radar sensing in mobile networks—a survey," *IEEE Commun. Surveys Tuts.*, vol. 24, no. 1, pp. 306–345, 2021.
- [2] F. Liu, C. Masouros, A. Li, H. Sun, and L. Hanzo, "Mu-MIMO communications with MIMO radar: From co-existence to joint transmission," *IEEE Trans. Wireless Commun.*, vol. 17, no. 4, pp. 2755–2770, 2018.
- [3] X. Wang, Z. Fei, J. A. Zhang, and J. Xu, "Partially-connected hybrid beamforming design for integrated sensing and communication systems," *IEEE Trans. Commun.*, vol. 70, no. 10, pp. 6648–6660, 2022.
- [4] R. Xu, L. Peng, W. Zhao, and Z. Mi, "Radar mutual information and communication channel capacity of integrated radar-communication system using MIMO," *ICT Express*, vol. 1, no. 3, pp. 102–105, 2015.
- [5] Y. Liu, G. Liao, and Z. Yang, "Robust OFDM integrated radar and communications waveform design based on information theory," *Signal Process.*, vol. 162, pp. 317–329, 2019.
- [6] M. Dai and B. Clerckx, "Multiuser millimeter wave beamforming strategies with quantized and statistical CSIT," *IEEE Trans. Wireless Commun.*, vol. 16, no. 11, pp. 7025–7038, 2017.
- [7] J. Dang, Z. Zhang, and L. Wu, "Joint beamforming for intelligent reflecting surface aided wireless communication using statistical CSI," *China Commun.*, vol. 17, no. 8, pp. 147–157, 2020.
- [8] W. Weichselberger, M. Herdin, H. Ozcelik, and E. Bonek, "A stochastic MIMO channel model with joint correlation of both link ends," *IEEE Trans. Wireless Commun.*, vol. 5, no. 1, pp. 90–100, 2006.
- [9] Y. Xu, Y. Li, J. A. Zhang, M. Di Renzo, and T. Q. Quek, "Joint beamforming for RIS-assisted integrated sensing and communication systems," *IEEE Trans. Commun.*, 2023.
- [10] A. M. Tulino and S. Verdu, "Random matrix theory and wireless communications," *Found. Trends Commun. Inf. Theory*, vol. 1, pp. 1–182, Jun. 2004, Now Publishers.
- [11] R. R. Muller, "On the asymptotic eigenvalue distribution of concatenated vector-valued fading channels," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 2086–2091, 2002.
- [12] S. T. Belinschi, T. Mai, and R. Speicher, "Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem," *J. Reine Angew. Math. (Crelles J.)*, vol. 2017, no. 732, pp. 21–53, 2017.
- [13] Z. Zheng, S. Wang, Z. Fei, Z. Sun, and J. Yuan, "On the mutual information of multi-RIS assisted MIMO: From operator-valued free probability aspect," *IEEE Trans. Commun.*, vol. 71, no. 12, pp. 6952-6966, 2023.
- [14] S. Wang, Z. Zheng, Z. Fei, J. Guo, J. Yuan and Z. Sun, "Toward Ergodic Sum Rate Maximization of Multiple-RIS-Assisted MIMO Multiple Access Channels Over Generic Rician Fading," *IEEE Trans. Wireless Commun.*, vol. 23, no. 8, pp. 9613-9628, 2024