

# Stability of the Inviscid Power-Law Vortex in Self-Similar Coordinates

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## Abstract

We prove that the stationary power-law vortex  $\bar{\omega}(x) = \beta|x|^{-\alpha}$ , which explicitly solves the incompressible Euler equations in  $\mathbb{R}^2$ , is linearly stable in self-similar coordinates with the natural scaling.

## 1 Introduction

Consider the Cauchy problem of the two-dimensional Euler equations in vorticity form:

$$\begin{aligned}\partial_t \omega + (v \cdot \nabla) \omega &= f \\ K_2 * \omega(\cdot, t) &= v(\cdot, t) \\ \omega(\cdot, t = 0) &= \omega_0(\cdot)\end{aligned}\tag{1}$$

Here,  $f(x, t)$  is a real-valued forcing term and the vorticity  $\omega(x, t)$  is a real-valued function defined on  $\mathbb{R}^2 \times [0, T)$ . We consider solutions in the sense of distribution. In particular, we say  $\omega$  is a solution of Equation (1) if the following integral identity holds for every  $\phi \in C_c^\infty(\mathbb{R}^2 \times [0, T))$ :

$$\int_0^T \int_{\mathbb{R}^2} (\omega(\partial_t \phi + (K_2 * \omega) \cdot \nabla \phi) + f \phi) dx dt = - \int_{\mathbb{R}^2} \phi(x, 0) \omega_0(x) dx.\tag{2}$$

Here and in the sequel we denote the standard two-dimensional Biot-Savart kernel by  $K_2$ . Although the operator  $\omega \rightarrow K_2 * \omega$  defined on Schwartz functions cannot be continuously extended to  $L^2$ , as shown in [2], it is well behaved on some closed linear subspaces of  $L^2$ . More generally, we shall show that for every  $2 \leq q < \infty$  there exist some closed linear subspaces of  $L^q$ , which we denote  $L_m^q$ , to which the operator

can be continuously extended. Here  $m \geq 2$  is an integer and  $L_m^q$  will essentially be the space of  $m$ -fold symmetric functions in  $L^q$ . A detailed description appears in subsequent sections.

We shall restrict ourselves to solutions of Equation (1) in a particular class of integrability. First, for any  $1 \leq q \leq \infty$  and  $2 < p \leq \infty$ , we define:

**Definition 1.** *The function  $\omega(x, t)$  is in the class  $\Upsilon_{q,p}^T$  if and only if*

$$\omega \in L_t^q([0, T], (L^1 \cap L^p)_x)$$

$$K_2 * \omega \in L_t^q([0, T], L_x^2)$$

**Definition 2.** *The function  $\omega(x)$  is in the class  $\Upsilon_p^0$  if and only if*

$$\omega \in (L^1 \cap L^p)_x$$

$$K_2 * \omega \in L_x^2$$

**Definition 3.** *The function  $\omega(x, t)$  is in the class  $\Upsilon_{q,p}^\infty$  if  $\omega \in \Upsilon_{q,p}^T$  for all  $T \geq 0$ .*

We also use a slightly different notation if the domain of times does not include the time  $t = 0$ :

**Definition 4.** *Let  $a, b \in \mathbb{R}$ . The function  $\omega(x, t)$  is in the class  $\Upsilon_{q,p}^{[a,b]}$  if and only if*

$$\omega \in L_t^q([a, b], (L^1 \cap L^p)_x)$$

$$K_2 * \omega \in L_t^q([a, b], L_x^2).$$

*We also say that  $\omega \in \Upsilon_{q,p}^{[a,\infty)}$  if and only if  $\omega \in \Upsilon_{q,p}^{[a,b]}$  for every  $b > a$ .*

One generally searches for solutions in  $\Upsilon_{\infty,p}^\infty$  with initial data in  $\Upsilon_p^0$ . The famous theorem of Yudovich, proven in [10], is:

**Theorem 1.1.** *Let  $\omega_0(x) \in \Upsilon_\infty^0$  and let  $f$  be some forcing term such that  $f \in \Upsilon_{1,\infty}^\infty$ . Then there exists a unique solution  $\omega(x, t)$  to Equation (1) in the class  $\Upsilon_{\infty,\infty}^\infty$  with initial data  $\omega_0$ .*

In a remarkable couplet of papers [7] and [8], Vishik provides the first evidence that the so-called ‘‘Yudovich Class’’  $L^1 \cap L^\infty$  is sharp, proving:

**Theorem 1.2.** *For every  $2 < p < \infty$ , there exists  $\omega_0(x) \in \Upsilon_p^0$  and a force  $f \in \Upsilon_{1,p}^\infty$  with the property that there are uncountably many solutions  $\omega(x, t) \in \Upsilon_{\infty,p}^\infty$  to Equation (1) with initial data  $\omega_0$ .*

The monograph [2] provides an alternative proof of Vishik’s theorem while following a similar approach, and we shall generally use the notation and terminology from [2]. Succinctly, one may describe Vishik’s general strategy as constructing an unstable radial vortex in self-similar coordinates that generates non-uniqueness while breaking radial symmetry. We now discuss the interpretation of Vishik’s proof in terms of dynamical systems proposed by the authors of the monograph [2].

We first observe that Equation (1) admits many stationary solutions in the form of “radial vortices” or vortex profiles of the form:

$$\bar{\omega}(x) = g(|x|), \quad \bar{v}(x) = \zeta(|x|x^\perp,$$

where  $x^\perp = (-x_2, x_1)$  and  $K_2 * \bar{\omega} = \bar{v}$ . Suppose one could find a vortex profile  $\bar{\omega}$  that is linearly unstable, for instance that one finds a real and strictly positive eigenvalue  $\lambda$  of the linearized Euler equations and a trajectory on the unstable manifold associated to  $\lambda$  and  $\bar{\omega}$  of the form

$$\omega = \bar{\omega} + \omega_{lin} + o(e^\lambda t).$$

Here  $\omega_{lin} = e^{\lambda t} \eta(x)$  is a solution of the linearized Euler equations. We would then expect “non-uniqueness at time  $t = -\infty$ ” because of the instability of the vortex. The outline proposed in [2] is to instead consider a choice of self-similar coordinates for which self-similar solutions are sufficiently integrable and to find an unstable stationary vortex profile in the self-similar coordinates. Given a positive parameter  $\alpha > 0$ , one such choice of coordinates is given by

$$\xi = xt^{-1/\alpha}, \quad \tau = \log(t)$$

$$v(x, t) = t^{1/\alpha-1} V(\xi, \tau), \quad \omega(x, t) = t^{-1} \Omega(\xi, \tau).$$

The Euler equations in these similarity variables, without force, are given by

$$\partial_\tau \Omega - \left(1 + \frac{\xi}{\alpha} \cdot \nabla_\xi\right) \Omega + V \cdot \nabla_\xi \Omega = 0$$

$$V = K_2 * \Omega.$$

If a self-similar profile  $\Omega$  satisfies  $\|\Omega\|_{L^p} = O(1)$  as  $\tau \rightarrow -\infty$ , then we also have that  $\|\omega\|_{L^p} = O(t^{-1+\frac{2}{p\alpha}})$  as  $t \rightarrow 0^+$ . In particular, choosing  $p = 2/\alpha$  ensures that the Lebesgue norms are  $O(1)$  in both coordinate systems, which is one reason why we consider this a “natural” choice of self-similar coordinates. To prove Vishik’s theorem, one should take  $0 < \alpha \leq 2/p$ , which ensures the desired integrability. If one can find an unstable stationary solution  $\bar{\Omega}$  of the self-similar equations, then

one can hope to prove non-uniqueness at time  $\tau = -\infty$ , which corresponds to non-uniqueness at physical time  $t = 0$ .

It is not difficult to see that the only stationary radial vortices solving the self-similar Euler equations are precisely the power-law vortices of the form

$$\bar{\Omega}(\xi) = \beta(2 - \alpha)|\xi|^{-\alpha},$$

and we note that these correspond exactly to radial power-law vortices solving the stationary Euler equations in the original coordinates. The exponent  $\alpha$  of the stationary profile is exactly determined by the choice of scaling for the self-similar coordinates, and the prefactor  $\beta$  is an arbitrary real number. The natural question arises: are the power-law vortices unstable in the self-similar coordinates? This question was also posed by the authors of [2]. An affirmative answer would suggest that non-uniqueness can arise from the (simple and explicit) power-law vortex, while a negative answer shows that a more complex stationary profile that necessarily depends on the angular variable would have to be found if there is any hope to complete the program proposed in [2]. We prove that the power-law vortex is linearly stable, in a way we shall now make more precise.

We can write the linearization of (3) around  $\bar{\Omega}$  as

$$(\partial_\tau - L_{ss})\Omega = 0.$$

The domain of  $L_{ss}$  as an operator into  $L_m^q$  is denoted

$$D_m(L_{ss}) = \{\Omega \in L_m^q : L_{ss}(\Omega) \in L_m^q\}.$$

Thus,  $L_{ss}$  is a closed, densely defined operator. We define the resolvent set of an operator  $L$  on  $L_m^q$  to be the open set of  $z \in \mathbb{C}$  for which  $L - z$  has a bounded inverse from  $L_m^q \rightarrow L_m^q$ . We define the spectrum of  $L$ , which we denote by  $\text{spec}_{m,q}(L)$ , to be the closed set which is the complement of the resolvent set.

**Theorem 1.3.** *Let  $\beta \in \mathbb{R}$  and  $\alpha \in (0, 1)$ . Let  $q$  be such that  $2 \leq q \leq 2/\alpha$  and let  $m \geq 2$  be an integer. If  $\bar{\Omega}(\xi) = \beta(2 - \alpha)|\xi|^{-\alpha}$  is the radial power-law vortex that solves the Euler equations in self-similar coordinates, then for any  $\lambda \in \text{spec}_{m,q}(L_{ss})$ , we have  $\text{Re}(\lambda) \leq a_0 := 1 - \frac{2}{\alpha q} \leq 0$ .*

Using the Hille-Yosida theorem, we have the following related theorem:

**Theorem 1.4.** *Let  $\beta \in \mathbb{R}$  and  $\alpha \in (0, 1)$ . Let  $q$  be such that  $2 \leq q \leq 2/\alpha$  and let  $m \geq 2$  be an integer. The operator  $L_{ss}$  is the generator of a strongly continuous semigroup on  $L_m^q$ , which we denote  $e^{\tau L_{ss}}$ , and the growth bound  $G(L_{ss})$  of  $e^{\tau L_{ss}}$  is*

less than or equal to  $a_0$ . As a consequence, for any  $\epsilon$  there exists a number  $M(\alpha, \epsilon)$  depending only on  $\alpha, \epsilon$  such that

$$\|e^{\tau L_{ss}} \Omega\|_{L^q} \leq M(\alpha, \epsilon) e^{(a_0 + \epsilon)\tau} \|\Omega\|_{L^q} \quad \forall \tau, \forall \Omega \in L_m^q.$$

The choice of Lebesgue space  $L^q$  for  $2 \leq q \leq 2/\alpha$  is justified since one expects non-uniqueness to emerge from an (integrable) singularity at the spatial origin, and the power-law vortex  $\bar{\Omega}$  is in  $L_{loc}^q$  if and only if  $q < 2/\alpha$ . Our theorems prove that the power-law vortex is linearly stable up to the borderline case:  $2 \leq q \leq 2/\alpha$ , and in the case when  $2 \leq q < 2/\alpha$ , we prove that the power-law vortex is exponentially stable. We would also like to remark that our result, and indeed all work done on the non-uniqueness of the incompressible Euler equations with vorticity in the class  $\Upsilon_{\infty, p}^\infty$ , has implications for potential non-uniqueness of Leray-Hopf solutions of the incompressible Navier-Stokes equations. As shown by the authors of [1] in the case with a force, if one can construct an unstable vortex for the Euler equations in self-similar coordinates, one has essentially proven that Leray solutions of the Navier-Stokes equations admit non-unique solutions. Our result confirms the statement in [1] that finding such an unstable vortex is “far from elementary”.

In addition, we may view our result in the context of classical results on hydrodynamics stability. In the comprehensive work [4] by Chandrasekhar, various stability and instability results for fluid motion are described, including the well-known Rayleigh’s criterion for stability of steady inviscid flow between two co-axial cylinders. Rayleigh’s criterion would suggest, but not rigorously prove in the case of our infinite energy power-law vortices on the whole plane, that a vorticity profile  $|x|^{-\alpha}$  is stable whenever  $\alpha < 2$ . We consider our Theorem 1.3 as mathematical proof of the expected stability in this case. In fact, the numerology of our theorem (we get linear stability and local integrability if  $\alpha q \leq 2$  and we need  $q \geq 1$  to get a complete Lebesgue norm), suggests that Rayleigh’s criterion holds exactly true. We also mention the recent work [12] of Zelati and Zillinger, in which the authors consider the linear stability of vorticity profiles with singularities of power-law type. The work in [12], however, only considers the stability at sufficiently high frequency modes.

We now remark the relationship our work has with the stability theorems proven by Arnold in [3] and the related questions proposed by Yudovich in [11]. Let  $\psi$  be the stream function of a stationary solution of the incompressible Euler equations. Arnold proves in [3] that the stationary flow is stable if the velocity profile is convex, or  $\nabla\psi/\nabla\Delta\psi > 0$ . When the velocity profile is concave, or  $\nabla\psi/\nabla\Delta\psi < 0$ , then there are finitely many unstable eigenvalues of the corresponding linear problem. The velocity profile corresponding to the power-law vortex  $\bar{\Omega}$  is concave, so we expect finitely many unstable eigenvalues. Our work improves this to a statement that there

are no unstable eigenvalues for the linearization around  $\bar{\Omega}$ . Yudovich in [11] proposes that understanding the stability (or instability) of ideal fluid flows is an important problem in mathematical hydrodynamics, and we consider our work as progress in that direction.

Besides the application to fluid dynamics, we consider an intriguing aspect of our work to be the novel techniques we use in our analysis of the Euler equations linearized around the singular power-law vortex. The singularity of the background vortex leads to an unbounded operator  $L_{ss}$  with unbounded coefficients, which, to our knowledge, cannot be handled by any previously known method. For example,  $L_{ss}$  cannot be thought of as the compact perturbation of a skew-adjoint operator, which would be case if the background vortex were smooth.

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## 2 Exponential Self-Similar Coordinates

We now more precisely discuss the change of coordinates used by Vishik in [7] and [8] as well as by the authors of [2]. Given a solution  $\omega(x, t)$  of Equation (1) on  $\mathbb{R}^2 \times [T_0, T_1]$ , we introduce a function  $\Omega$  on  $\mathbb{R}^2 \times [\log T_0, \log T_1]$  given by the following transformation. We set  $\tau = \log t$ ,  $\xi = xt^{-1/\alpha}$ , and let

$$\Omega(\xi, \tau) = e^\tau \omega(e^{\tau/\alpha} \xi, e^\tau).$$

The reverse transformation is given by:

$$\omega(x, t) = t^{-1} \Omega(t^{-1/\alpha} x, \log t).$$

If the vector field  $v(x, t)$  is given by  $(K_2 * \omega)(x, t)$  and the vector field  $V(\xi, \tau)$  is given by  $(K_2 * \Omega)(\xi, \tau)$ , then we have the following transformation rules:

$$V(\xi, \tau) = e^{\tau(1-1/\alpha)} v(e^{\tau/\alpha} \xi, e^\tau),$$

$$v(x, t) = t^{-1+1/\alpha} V(t^{-1/\alpha} x, \log t).$$

Vishik in [7] and [8] as well as the authors of [2] show that if  $\omega(x, t)$  satisfies Equation (1), then the function  $\Omega(\xi, \tau)$  satisfies

$$\begin{aligned} \partial_\tau \Omega - \left(1 + \frac{\xi}{\alpha} \cdot \nabla\right) \Omega + (V \cdot \nabla) \Omega &= 0 \\ K_2 * \Omega(\cdot, \tau) &= V(\cdot, \tau) \end{aligned} \tag{3}$$

We usually refer to this system of coordinates as the exponential self-similar coordinates or simply the self-similar coordinates.

Now we introduce exponential self-similar polar coordinates. In particular, if  $(r, \theta, t)$  are the usual polar coordinates on  $\mathbb{R}^2 \times [T_1, T_2]$ , we let  $\rho = rt^{-1/\alpha}$ ,  $\theta$  unchanged, and  $\tau = \log t$  to get a new  $(\rho, \theta, \tau)$  system of coordinates. We consider the velocity form of the Euler Equations:

$$\begin{aligned}\partial_t v + (v \cdot \nabla)v &= -\nabla p \\ \operatorname{div} v &= 0\end{aligned}\tag{4}$$

which we write in terms of exponential self-similar polar coordinates as:

$$\begin{aligned}\partial_\tau V_\rho + \left(\frac{1}{\alpha} - 1\right)V_\rho - \frac{\rho}{\alpha}\partial_\rho V_\rho + V_\rho\partial_\rho V_\rho + \frac{V_\theta}{\rho}\partial_\theta V_\rho - \frac{V_\theta^2}{\rho} &= -\partial_\rho P \\ \partial_\tau V_\theta + \left(\frac{1}{\alpha} - 1\right)V_\theta - \frac{\rho}{\alpha}\partial_\rho V_\theta + V_\rho\partial_\rho V_\theta + \frac{V_\theta}{\rho}\partial_\theta V_\theta + \frac{V_\rho V_\theta}{\rho} &= -\frac{1}{\rho}\partial_\theta P \\ \partial_\rho(\rho V_\rho) + \partial_\theta V_\theta &= 0\end{aligned}\tag{5}$$

One can see that, up to a constant prefactor, the only stationary radial vortex  $V = V_\theta(\rho)e_\theta$  satisfying the exponential self-similar equations is precisely the power-law vortex with  $V_\theta(\rho) = \beta\rho^{1-\alpha}$ , where  $\beta$  is any real number.

**Proposition 2.1.** *The unique solution of Equation (5) of the form  $V = V_\theta(\rho)e_\theta$  is given by the profile  $V_\theta(\rho) = \beta\rho^{1-\alpha}$ , where  $\beta$  is any real number.*

*Proof.* The divergence-free condition is clearly satisfied. The second listed equation in Equation (5) simplifies to  $(1/\alpha - 1)V_\theta(\rho) - (\rho/\alpha)V_\theta'(\rho) = 0$ , a first-order differential equation whose unique solution is  $V_\theta(\rho) = \beta\rho^{1-\alpha}$ . For an appropriate choice of the pressure  $P$ , the equation listed first in Equation (5) will also be satisfied.  $\square$

We are thus led to define the velocity profile:

$$\bar{V}(\rho) = \beta\rho^{1-\alpha}e_\theta\tag{6}$$

and its associated vorticity profile:

$$\bar{\Omega}(\rho) = \beta(2 - \alpha)\rho^{-\alpha}.\tag{7}$$

Note that  $\bar{\Omega}$  corresponds exactly by the transformation back to physical coordinates to the stationary radial vortex with vorticity profile  $\beta(2 - \alpha)r^{-\alpha}$ .

### 3 Stability of the Power-Law Vortex

The goal of this section is to prove Theorem 1.3 (which consequently will also prove Theorem 1.4). Theorem 1.3 is the final result of a technical analysis of the linearization of the Euler equations around a singular power-law vortex. We write the linearization as

$$(\partial_\tau - L_{ss})\Omega = 0.$$

The main technical issue, for which no standard method can be used, is the highly singular behavior of the coefficients of  $L_{ss}$ . For example, we cannot simply say that  $L_{ss}$  is the compact perturbation of a skew-adjoint operator, which, on the contrary, would be the case if the background vortex profile  $\bar{\Omega}$  were smooth.

We now introduce some useful notation. We denote  $L_m^q$  to be the closed linear subspace of  $L^q$  of elements that are  $m$ -fold symmetric. In other words, if  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the counterclockwise rotation of angle  $\theta$  around the origin, then a function  $f \in L_m^q$  satisfies

$$f = f \circ R_{2\pi/m}.$$

It will be convenient to define the following closed linear subspaces of  $L_m^q$ :

$$U_k := \{f(r)e^{ik\theta} : f \in L^q(\mathbb{R}^+, r dr)\}$$

so we have the direct sum:

$$L_m^q = \bigoplus_{k \in \mathbb{Z}} U_{km}.$$

We denote the Schwartz space by  $\mathcal{S}$  and its dual by  $\mathcal{S}^*$ . We begin by stating a slight improvement of a lemma from [2]. The improvement allows for a continuous extension of  $K_2*$  to  $L_m^q$  whenever  $2 \leq q < \infty$ .

**Lemma 3.1.** *For every  $m \geq 2$  there exists a unique continuous operator  $T : L_m^q \rightarrow \mathcal{S}^*$  satisfying:*

1. *If  $\varphi \in \mathcal{S}$ , then  $T\varphi = K_2 * \varphi$  in the sense of distribution.*
2. *There exists a constant  $C > 0$  such that for every  $\varphi \in L_m^q$  there exists  $v(\varphi) := v \in W_{loc}^{1,q}$  such that*

- $R^{-1}\|v\|_{L^q(B_R)} + \|Dv\|_{L^q(B_R)} \leq C\|\varphi\|_{L^q(\mathbb{R}^2)}$  for all  $R > 0$
- $\operatorname{div} v = 0$  and  $T(\varphi) = v(\varphi)$  in the sense of distribution.



We shall denote  $T$  interchangeably with  $\nabla^\perp \Delta^{-1}$  or  $K_{2*}$ , although we shall mostly use the latter notation. Recall that we wrote the linearization of (3) around  $\bar{\Omega}$  as

$$(\partial_\tau - L_{ss})\Omega = 0$$

where the linear operator  $L_{ss}$  is given by:

$$L_{ss}\Omega = \Omega + \left(\frac{\xi}{\alpha} - \bar{V}\right) \cdot \nabla \Omega - \nabla^\perp \Delta^{-1} \Omega \cdot \nabla \bar{\Omega}.$$

The domain of  $L_{ss}$  as an operator into  $L_m^q$  is denoted

$$D_m(L_{ss}) = \{\Omega \in L_m^q : L_{ss}(\Omega) \in L_m^q\}.$$

Thus,  $L_{ss}$  is a closed, densely defined operator. For a general operator  $L$ , we define the resolvent set of  $L$  on  $L_m^q$  to be the open set of  $z \in \mathbb{C}$  for which  $L - z$  has a bounded inverse from  $L_m^q \rightarrow L_m^q$ . We define the spectrum of  $L$ , which we denote by  $\text{spec}_{m,q}(L)$ , to be the closed set which is the complement of the resolvent set.

We take the opportunity to restate the theorems that we shall prove. Our first theorem states that the real part of any element of the spectrum of  $L_{ss}$  is nonpositive: a linear stability result.

**Theorem 3.2.** *Let  $\beta \in \mathbb{R}$  and  $\alpha \in (0, 1)$ . Let  $q$  be such that  $2 \leq q \leq 2/\alpha$  and let  $m \geq 2$  be an integer. If  $\bar{\Omega}(\xi) = \beta(2 - \alpha)|\xi|^{-\alpha}$  is a radial power-law vortex that solves the Euler equations in self-similar coordinates, then for any  $\lambda \in \text{spec}_{m,q}(L_{ss})$ , we have  $\text{Re}(\lambda) \leq a_0 := 1 - \frac{2}{\alpha q} \leq 0$ .*

The value of the parameter  $m$  will not change the proof or the statement of the theorem in any way. Functionally, in the proof that follows, one may change any appearance of “ $2k$ ” with “ $mk$ ” and get an identical proof for the general case. For this reason, and to simplify our presentation, we may assume that  $m = 2$ .

Recall the Hille-Yosida Theorem, stated, for example, as Corollary 3.6 of Chapter 2 in [5]:

**Theorem 3.3** (Hille-Yosida). *Let  $w \in \mathbb{R}$ . Consider the linear operator  $A : D(A) \subset X \rightarrow X$  on the Banach space  $X$ . Then  $A$  generates a strongly continuous semigroup  $T(t)$  satisfying*

$$\|T(t)\| \leq Ce^{wt} \quad \forall t \geq 0$$

*if and only if  $A$  is a closed operator,  $D(A)$  is dense in  $X$ , and for any  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda > w$ ,  $\lambda$  is in the resolvent of  $A$  and for a constant  $C > 0$  independent of  $\lambda$ :*

$$\|(A - \lambda I)^{-1}\| \leq \frac{C}{\text{Re } \lambda - w}.$$

We would like to use the Hille-Yosida theorem to also conclude our Theorem 1.4, which we restate:

**Theorem 3.4.** *Let  $\beta \in \mathbb{R}$  and  $\alpha \in (0, 1)$ . Let  $q$  be such that  $2 \leq q \leq 2/\alpha$  and let  $m \geq 2$  be an integer. The operator  $L_{ss}$  is the generator of a strongly continuous semigroup on  $L_m^q$ , which we denote  $e^{\tau L_{ss}}$ , and the growth bound  $G(L_{ss})$  of  $e^{\tau L_{ss}}$  is less than or equal to  $a_0$ . As a consequence, for any  $\epsilon$  there exists a number  $M(\alpha, \epsilon)$  depending only on  $\alpha, \epsilon$  such that*

$$\|e^{\tau L_{ss}}\Omega\|_{L^q} \leq M(\alpha, \epsilon)e^{(a_0+\epsilon)\tau}\|\Omega\|_{L^q} \quad \forall \tau, \forall \Omega \in L_m^q.$$

By the Hille-Yosida theorem, Theorem 3.4 will follow as a consequence of Theorem 3.2 and the following proposition:

**Proposition 3.5.** *Let  $\beta \in \mathbb{R}$  and  $\alpha \in (0, 1)$ . For any  $m \geq 2$ , for any  $2 \leq q \leq 2/\alpha$ , and for any  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda > 1 - \frac{2}{q\alpha}$ ,*

$$\|(L_{ss} - \lambda I)^{-1}\| \leq \frac{K_\alpha}{\text{Re } \lambda - (1 - \frac{2}{q\alpha})}$$

for a constant  $K_\alpha > 0$  depending only on  $\alpha$ .

We now proceed to prove these claims.

### 3.1 Proof of Lemma 3.1

Let  $f \in L_m^q \cap \mathcal{S}$  and let  $v = K_2 * f$ . We first claim that

$$\int_{B_R} v = 0 \quad \text{for every } R > 0. \quad (8)$$

Indeed, we have  $v = \nabla^\perp h$ , where  $h$  is the unique classical solution of  $\Delta h = f$  given by  $K * f$  where  $K(x) = \frac{1}{2\pi} \log |x|$ . Since the kernel  $K$  is invariant under all rotations  $R_\theta$  and since  $f$  is  $m$ -fold symmetric, we conclude that  $h$  is  $m$ -fold symmetric. Therefore,

$$R_{-2\pi/m} \nabla h(R_{2\pi/m}x) = \nabla h(x).$$

Thus, integrating in  $x$  we conclude

$$\int_{B_R} \nabla h = R_{2\pi/m} \int_{B_R} \nabla h,$$

so

$$\int_{B_R} \nabla h = \frac{1}{m} \sum_{k=0}^{m-1} R_{2k\pi/m} \int_{B_R} \nabla h.$$

However, since  $m \geq 2$ , the sum is zero, which shows that  $\int_{B_R} \nabla h = 0$ . Finally, with the property just shown, we may use the Poincaré inequality to conclude:

$$R^{-1} \|v\|_{L^q(B_R)} + \|Dv\|_{L^q(B_R)} \leq C \|f\|_{L^q(B_R)}$$

since  $\|Dv\|_{L^q} \leq \|f\|_{L^q}$  by the Calderón-Zygmund theorem.

### 3.2 Fourier Expansion of the Linearized Equations

In what follows, we shall abuse notation and denote the (exponential self-similar) radial coordinate by  $r$  instead of  $\rho$ . As previously mentioned, we may without loss of generality examine only the case when the parameter  $m = 2$ .

We recall that the inverse Laplacian  $\Delta^{-1}$  preserves rotational symmetries. Thus, for any  $\Omega \in L_2^q \cap \mathcal{S}$ ,  $\Delta^{-1}\Omega \in L_2^q \cap C^\infty$ . Now we decompose

$$\Delta^{-1}\Omega = \sum_{k \in \mathbb{Z}} f_k(r) e^{2ik\theta} \tag{9}$$

where  $f_k(r) \in L^q(\mathbb{R}^+, r dr)$  for all  $k \in \mathbb{Z}$ . Let  $\lambda = \lambda_1 + i\lambda_2$  be our putative element of the spectrum, where  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $\lambda_1 > 1 - \frac{2}{q\alpha}$ . We consider the eigenvalue equation  $L_{ss}\Omega - \lambda\Omega = 0$ , or equivalently

$$(1 - (\lambda_1 + i\lambda_2))\Omega + \left(\frac{\xi}{\alpha} - \bar{V}\right) \cdot \nabla \Omega - \nabla^\perp \Delta^{-1}\Omega \cdot \nabla \bar{\Omega} = 0.$$

Using the decomposition (9), the above linear partial differential equation becomes equivalent to an infinite family of ordinary differential equations. The ordinary differential equation corresponding to the parameter  $k$  is:

$$\begin{aligned} & (1 - (\lambda_1 + i\lambda_2))\left(\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}\right)(f_k(r)e^{2ik\theta}) + \\ & + \left(\frac{r}{\alpha}e_r - \beta r^{1-\alpha}e_\theta\right) \cdot \left(\partial_r\left(\left(\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}\right)(f_k(r)e^{2ik\theta})\right)e_r + \frac{\partial_\theta}{r}\left(\left(\partial_r^2 + \frac{\partial_r}{r} + \frac{\partial_\theta^2}{r^2}\right)(f_k(r)e^{2ik\theta})\right)e_\theta\right) + \\ & + \left(\frac{\partial_\theta}{r}(f_k(r)e^{2ik\theta})e_r - \partial_r(f_k(r)e^{2ik\theta})e_\theta\right) \cdot \left(-\alpha(2-\alpha)\beta r^{-1-\alpha}e_r\right) = 0. \end{aligned}$$

Simplifying, we get

$$\begin{aligned} & \frac{r}{\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right)' + (1 - (\lambda_1 + i\lambda_2)) \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) \\ & - 2ik\beta r^{-\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) - 2ikr^{-2-\alpha} \alpha(2-\alpha)\beta f_k(r) = 0 \end{aligned} \quad (10)$$

### 3.3 Characterization of the Spectrum

We want to show that the intersection of the spectrum of  $L_{ss}$  with  $\{z : \operatorname{Re} z > 1 - \frac{2}{q\alpha}\}$  is in fact empty. In particular, for any  $\operatorname{Re} \lambda > 1 - \frac{2}{q\alpha}$ ,  $\lambda$  is in the resolvent of  $L_{ss}$ . Our proof will also directly show the bound required on the resolvent to apply the Hille-Yosida theorem, so this section will finish the proof of both Theorem 3.2 and Proposition 3.5.

Proving that  $L_{ss} - \lambda I$  is surjective with bounded resolvent is equivalent to proving that the operator  $R_\lambda(g) := (L_{ss} - \lambda I)^{-1}(g)$  exists and is bounded as an operator into  $D_2(L_{ss})$ . We thus assume that  $g$  is an arbitrary function in  $L_2^q$ . We shall now reduce our problem to showing that the unique solution of an ordinary differential equation boundary value problem is in some Lebesgue space with quantified  $L^q$  bound.

In particular, that  $L_{ss} - \lambda I$  is surjective is equivalent to stating that the inhomogeneous ordinary differential equation

$$\begin{aligned} & \frac{r}{\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right)' + (1 - (\lambda_1 + i\lambda_2)) \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) \\ & - 2ik\beta r^{-\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) - 2ikr^{-2-\alpha} \alpha(2-\alpha)\beta f_k(r) = g_k(r) \end{aligned} \quad (11)$$

has a solution  $f_k(r)$  with  $f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \in L^q(\mathbb{R}, r dr)$  for any given function  $g_k(r) \in L^q(\mathbb{R}, r dr)$ . That  $L_{ss} - \lambda I$  is injective is equivalent to stating that the homogeneous ordinary differential equation Equation (10) has no non-trivial solution or, equivalently, that  $L_{ss} - \lambda I = g$  has a unique solution for every choice of the function  $g$ .

Henceforth, we drop the subscript notation showing dependence on  $k$  and simply write  $f(r)$  and  $g(r)$ . Moreover, we define:

$$u(r) := f''(r) + \frac{f'(r)}{r} - \frac{4k^2 f(r)}{r^2},$$

which is the quantity we wish to control in  $L^q(r dr)$ . We now perform the change of variables  $e^t = r$  and define:

$$\psi(t) := f(e^t) e^{(2/q-2)t} \quad G(t) := g(e^t) e^{2t/q} \quad U(t) := u(e^t) e^{2t/q}.$$

These functions are chosen so that

$$\begin{aligned} g(r) \in L^q(rdr) &\iff G(t) \in L^q(dt) \\ u(r) \in L^q(rdr) &\iff U(t) \in L^q(dt) \\ f(r)/r^2 \in L^q(rdr) &\iff \psi(t) \in L^q(dt). \end{aligned}$$

Given  $G(t) \in L^q(dt)$ , our new goal is to solve for  $U(t) \in L^q(dt)$  (or alternatively  $\psi(t) \in W^{2,2}(dt)$ ). The ordinary differential equations in terms of the new variable and functions are

$$U(t) = \psi''(t) + \left(4 - \frac{4}{q}\right)\psi'(t) + \left(4 - 4k^2 + \frac{4}{q^2} - \frac{8}{q}\right)\psi(t)$$

and

$$\frac{1}{\alpha}U'(t) + \left(1 - \frac{2}{\alpha q} - \lambda_1 - i\lambda_2\right)U(t) - 2ik\beta e^{-\alpha t}U(t) - 2ik\alpha(2 - \alpha)\beta e^{-\alpha t}\psi(t) = G(t).$$

In the case when  $k = 0$ , we can integrate the equation of first order and get:

$$U(t) = ce^{t(2/q + \alpha(\lambda - 1))} + \alpha e^{t(2/q + \alpha(\lambda - 1))} \int_0^t e^{-s(2/q + \alpha(\lambda - 1))} G(s) ds.$$

The unique choice of constant that will make  $U$  integrable is precisely

$$c = -\alpha \int_0^\infty e^{-s(1 + \alpha(\lambda - 1))} G(s) ds,$$

with which we have

$$U(t) = -\alpha \int_{\mathbb{R}} \chi_{(-\infty, 0)}(t - s) e^{(t-s)(2/q + \alpha(\lambda - 1))} G(s) ds.$$

By Young's convolution inequality, it follows that  $U(t) \in L^q(dt)$  is a solution of the ordinary differential equation with

$$\|U\|_q \leq \frac{\alpha \|G\|_q}{2/q + \alpha(\lambda - 1)}.$$

This is the unique solution, since in the case when  $k = 0$ , the homogeneous problem reduces to :

$$\frac{1}{\alpha}U'(t) + \left(1 - \frac{2}{\alpha q} - \lambda_1 - i\lambda_2\right)U(t) = 0,$$

whose only solution is given by  $U(t) = c_1 e^{t(2/q + \alpha(\lambda - 1))}$ , which is not in any  $L^q$  space, unless identically zero.

We may henceforth without loss of generality assume that  $k \geq 1$ . Now, integrating the differential equation of first order yields

$$\begin{aligned}
U(t) &= c_1 \exp \left( -2ik\beta e^{-\alpha t} + t(2/q + \alpha(\lambda - 1)) \right) + \\
&+ \alpha \exp \left( -2ik\beta e^{-\alpha t} + t(2/q + \alpha(\lambda - 1)) \right) \int_0^t \exp \left( 2ik\beta e^{-\alpha s} - s(2/q + \alpha(\lambda - 1)) \right) G(s) ds + \\
&+ 2ik\beta \alpha^2 (2 - \alpha) \exp \left( -2ik\beta e^{-\alpha t} + t(2/q + \alpha(\lambda - 1)) \right) \times \\
&\quad \times \int_0^t \exp \left( 2ik\beta e^{-\alpha s} - s(2/q + \alpha\lambda) \right) \psi(s) ds. \tag{12}
\end{aligned}$$

The constant  $c_1$  depends on the choice of initial condition. Also, integrating the differential equation of second order gets us:

$$\begin{aligned}
\psi(t) &= c_2 e^{-(2k+2-2/q)t} - \frac{e^{-(2k+2-2/q)t}}{4k} \int_0^t e^{(2k+2-2/q)s} U(s) ds + \\
&+ c_3 e^{(2k-2+2/q)t} + \frac{e^{(2k-2+2/q)t}}{4k} \int_0^t e^{-(2k-2+2/q)s} U(s) ds. \tag{13}
\end{aligned}$$

Finally to have  $\psi(t) \in L^q(dt)$  we use Equation (13), with the unique choice of constants:

$$c_2 = -\frac{1}{4k} \int_{-\infty}^0 e^{(2k+2-2/q)s} U(s) ds$$

and

$$c_3 = -\frac{1}{4k} \int_0^{\infty} e^{-(2k-2+2/q)s} U(s) ds.$$

These bounded linear functionals of  $U$  are uniquely chosen so that  $\psi \in L^q(dt)$  if  $U \in L^q(dt)$ . In addition, for smooth functions  $u, f$ , Equation (13) implies that  $\Delta^{-1}u = f$  in the classical sense, while the unique choice of  $c_2, c_3$  above guarantee the desired integrability of  $f$ . We may thus write  $\psi$  in terms of the convolution operator:

$$\psi(t) = -\frac{1}{4k} \int_{\mathbb{R}} \left( e^{-(2k+2-2/q)(t-s)} \chi_{(0,\infty)}(t-s) + e^{(2k-2+2/q)(t-s)} \chi_{(-\infty,0)}(t-s) \right) U(s) ds$$

We can denote the kernel above by

$$K_1(t, s) := \left( e^{-(2k+2-2/q)(t-s)} \chi_{(0,\infty)}(t-s) + e^{(2k-2+2/q)(t-s)} \chi_{(-\infty,0)}(t-s) \right).$$

We remark that given a function  $g$ , there is also at most one unique choice of the parameter  $c_1$  such that the function  $U(t)$  lies in  $L^q(dt)$  and satisfies Equation (12). In fact, the bounded functional  $c_1(U)$  will be given by

$$c_1 = -\alpha \int_0^\infty \exp(2ik\beta e^{-\alpha s} - s(2/q + \alpha(\lambda - 1)))G(s)ds - \\ 2ik\beta\alpha^2(2 - \alpha) \int_0^\infty \exp(2ik\beta e^{-\alpha s} - s(2/q + \alpha\lambda))\psi(s)ds$$

and Equation (12) becomes

$$U(t) = -\alpha \int_{\mathbb{R}} \exp(-2ik\beta e^{-\alpha t} + 2ik\beta e^{-\alpha s} + (t-s)(2/q + \alpha(\lambda - 1)))\chi_{(-\infty, 0)}(t-s)G(s)ds - \\ 2ik\beta\alpha^2(2 - \alpha) \times \\ \times \int_{\mathbb{R}} e^{-\alpha s} \exp(-2ik\beta e^{-\alpha t} + 2ik\beta e^{-\alpha s} + (t-s)(2/q + \alpha(\lambda - 1)))\chi_{(-\infty, 0)}(t-s)\psi(s)ds.$$

We denote the kernel:

$$K_2(t, s) := \exp(-2ik\beta e^{-\alpha t} + 2ik\beta e^{-\alpha s} + (t-s)(2/q + \alpha(\lambda - 1)))\chi_{(-\infty, 0)}(t-s).$$

We thus have the following Fredholm integral equation for  $U$ :

$$U(t) = -\alpha \int_{\mathbb{R}} K_2(t, s)G(s)ds + \frac{i\alpha^2(2 - \alpha)\beta}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_2(t, s)e^{-\alpha s}K_1(s, r)U(r)drds.$$

However, we can integrate the kernels in the  $s$  variable first:

$$\int_{\mathbb{R}} K_2(t, s)e^{-\alpha s}K_1(s, r)ds = \\ = F_1(t, r) \int_{\mathbb{R}} \exp(2ik\beta e^{-\alpha s} - \alpha s - (2/q + \alpha(\lambda - 1) + 2k + 2 - 2/q)s)\chi_{(\max(t, r), \infty)}(s)ds + \\ + F_2(t, r) \int_{\mathbb{R}} \exp(2ik\beta e^{-\alpha s} - \alpha s - (2/q + \alpha(\lambda - 1) - (2k - 2 + 2/q))s)\chi_{(t, r)}(s)ds.$$

Where

$$F_1(t, r) = \exp(-2ik\beta e^{-\alpha t} + (2/q + \alpha(\lambda - 1))t + r(2k + 2 - 2/q))$$

and

$$F_2(t, r) = \exp(-2ik\beta e^{-\alpha t} + (2/q + \alpha(\lambda - 1))t - r(2k - 2 + 2/q)).$$

We remark that

$$\frac{d}{ds} \left( \frac{i}{2\alpha k\beta} e^{2ik\beta e^{-\alpha s}} \right) = e^{2ik\beta e^{-\alpha s} - \alpha s}.$$

Thus integrating by parts yields (for some generic  $m > 0$ ):

$$\begin{aligned} \int_t^\infty e^{2ik\beta e^{-\alpha s} - \alpha s} e^{-ms} ds &= -\frac{i}{2\alpha k\beta} e^{2ik\beta e^{-\alpha t}} e^{-mt} - \frac{i}{2\alpha k\beta} \int_t^\infty e^{2ik\beta e^{-\alpha s}} e^{-ms} ds = \\ &= \frac{-i}{2\alpha k\beta} \left( 1 + \frac{1}{m} \right) e^{2ik\beta e^{-\alpha t}} e^{-mt} - \frac{1}{m} \int_t^\infty e^{2ik\beta e^{-\alpha s} - \alpha s} e^{-ms} ds. \end{aligned}$$

Simplifying we get (for  $m > 0$ ):

$$\int_t^\infty e^{2ik\beta e^{-\alpha s} - \alpha s} e^{-ms} ds = \frac{1}{2ik\beta\alpha} e^{2ik\beta e^{-\alpha t}} e^{-mt}. \quad (14)$$

Likewise, integrating by parts yields (for some generic  $m \in \mathbb{R}$  not necessarily positive):

$$\int_t^r e^{2ik\beta e^{-\alpha s} - \alpha s} e^{-ms} ds = \frac{1}{2ik\beta\alpha} \left( e^{2ik\beta e^{-\alpha t}} e^{-mt} - e^{2ik\beta e^{-\alpha r}} e^{-mr} \right). \quad (15)$$

Applying Equation (14) and Equation (15) to the original integral under consideration (and using that  $\lambda > 1 - \frac{2}{q\alpha}$ ) gets us

$$\begin{aligned} &\int_{\mathbb{R}} K_2(t, s) e^{-\alpha s} K_1(s, r) ds = \\ &= \frac{1}{2ik\beta\alpha} F_1(t, r) \chi_{(0, \infty)}(t - r) e^{2ik\beta e^{-\alpha t}} e^{-(2/q + \alpha(\lambda - 1) + 2k + 2 - 2/q)t} + \\ &+ \frac{1}{2ik\beta\alpha} F_1(t, r) \chi_{(-\infty, 0)}(t - r) e^{2ik\beta e^{-\alpha r}} e^{-(2/q + \alpha(\lambda - 1) + 2k + 2 - 2/q)r} + \\ &+ \frac{1}{2ik\beta\alpha} F_2(t, r) \chi_{(-\infty, 0)}(t - r) \left( e^{2ik\beta e^{-\alpha t}} e^{-(2/q + \alpha(\lambda - 1) - (2k - 2 + 2/q))t} - \right. \\ &\quad \left. e^{2ik\beta e^{-\alpha r}} e^{-(2/q + \alpha(\lambda - 1) - (2k - 2 + 2/q))r} \right). \end{aligned}$$

Simplifying further (there is a remarkable cancellation that occurs) we achieve:

$$\begin{aligned} &2ik\beta\alpha \int_{\mathbb{R}} K_2(t, s) e^{-\alpha s} K_1(s, r) ds = \\ &e^{-(2k + 2 - 2/q)(t - r)} \chi_{(0, \infty)}(t - r) + \chi_{(-\infty, 0)}(t - r) e^{(2k - 2 + 2/q)(t - r)} = K_1(t, r). \end{aligned}$$

We conclude that

$$U(t) = -\alpha \int_{\mathbb{R}} K_2(t, s) G(s) ds + \frac{\alpha(2 - \alpha)}{4k} \int_{\mathbb{R}} K_1(t, r) U(r) dr.$$



### 3.3.1 The Integral Transforms

We have a convolution operator:

$$\Phi_1(U)(t) := \int_{\mathbb{R}} K_1(t, s)U(s)ds.$$

Now, by Young's convolution inequality, we have

$$\|\Phi_1(U)(t)\|_q \leq \|U\|_q \left( \int_0^\infty e^{-(2k+2-2/q)y} dy + \int_{-\infty}^0 e^{(2k-2+2/q)y} dy \right) \leq \frac{2}{(2k-2+2/q)} \|U\|_q.$$

In a similar manner, we denote

$$\Phi_2(U) := \int_{\mathbb{R}} K_2(t, s)U(s)ds$$

and observe that

$$|\Phi_2(U)| \leq \int_{\mathbb{R}} \hat{K}_2(t, s)U(s)ds$$

where

$$\hat{K}_2(t, s) := e^{(t-s)(2/q+\alpha(\lambda-1))} \chi_{(-\infty, 0)}(t-s)$$

is the kernel of a convolution operator. Thus, arguing as before, we apply the Young convolution inequality and get

$$\|\Phi_2(U)(t)\|_q \leq \|U\|_q \left( \int_0^\infty e^{-(2/q+\alpha(\lambda-1))y} dy \right) \leq \frac{1}{2/q + \alpha(\lambda-1)} \|U\|_q.$$

### 3.3.2 Conclusion

We recall what we have heretofore shown. We found two bounded integral transforms from  $L^q \rightarrow L^q$ , given by  $\Phi_1$  and  $\Phi_2$ . Showing that the operator we are working on is surjective then becomes equivalent to showing (for every  $G \in L^q$ ) the existence of a fixed point of the map

$$U \rightarrow -\alpha\Phi_2(G) + \frac{\alpha(2-\alpha)}{4k}\Phi_1(U).$$

This, by the Banach fixed point theorem, is equivalent to showing that the map  $\frac{\alpha(2-\alpha)}{4k}\Phi_1$  is contractive. However, by our work in the previous subsection, we know that this is true, since:

$$\left\| \frac{\alpha(2-\alpha)}{4k}\Phi_1(U) \right\|_{L^q} \leq \frac{2\alpha(2-\alpha)}{4k(2k-2+2/q)} \leq \frac{\alpha(2-\alpha)}{4/q} < 1$$

since  $q\alpha \leq 2$ . Thus the unique solubility of the ordinary differential equation, hence the surjectivity and injectivity of the map, is proven.

Moreover, by the bound we found on  $\Phi_2$ , the solution  $U$  certainly satisfies

$$\|U\|_{L^q} \leq \frac{M_\alpha \|G\|_{L^q}}{\operatorname{Re} \lambda - (1 - \frac{2}{q\alpha})}$$

for all  $\lambda$  with  $\operatorname{Re} \lambda > 1 - \frac{2}{q\alpha}$ , for some choice of constant  $M_\alpha$  large enough (depending only on  $\alpha$ ). This shows that the solution map satisfies the hypotheses of the Hille-Yosida theorem, which finishes our work.

## 4 Appendix

### 4.1 Explicit Solutions of Equation (10)

We present an alternative proof that the point spectrum of  $L_{ss}$  is empty. For simplicity of notation, we assume  $\beta = 1$ , but the proof is entirely the same in the general case. We recall Equation (10) :

$$\begin{aligned} & \frac{r}{\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right)' + (1 - (\lambda_1 + i\lambda_2)) \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) \\ & - 2ikr^{-\alpha} \left( f_k''(r) + \frac{f_k'(r)}{r} - \frac{4k^2 f_k(r)}{r^2} \right) - 2ikr^{-2-\alpha} \alpha(2 - \alpha) f_k(r) = 0. \end{aligned}$$

We consider the above homogeneous ordinary differential equation and perform a change of variables  $z = -2ikr^{-\alpha}$ , since this will put the ordinary differential equation in a form that is already well studied. More precisely, we define a new function  $w_k(z)$  that is given by

$$w_k(z) = f_k((-2ik)^{1/\alpha} z^{-1/\alpha}) \cdot (-2ik)^{-2k/\alpha} z^{2k/\alpha}.$$

The function is defined so that if  $f_k(r) = w(-2ikr^{-\alpha})r^{2k}$  satisfies the homogeneous ordinary differential equation, we get that  $w$  satisfies the following ordinary differential equation:

$$0 = 2kr^{-3\alpha+2k-2} \left( i(\alpha - 4k)r^\alpha (r^\alpha(\alpha\lambda - 2k + 2) + 2i\alpha k) w'(-2ikr^{-\alpha}) + \right.$$

$$+2\alpha k((r^\alpha(\alpha(\lambda+2)-6k+2)+2i\alpha k)w''(-2ikr^{-\alpha})-2i\alpha k w^{(3)}(-2ikr^{-\alpha}))+ \\ +i(\alpha-2)\alpha r^{2\alpha}w(-2ikr^{-\alpha}))$$

Letting  $r = (-2ik)^{1/\alpha}z^{-1/\alpha}$  and simplifying we finally have:

$$0 = z^2 w^{(3)}(z) + z(1-z+\lambda+1+\frac{2-6k}{\alpha})w''(z) + \\ +((\lambda+\frac{2-2k}{\alpha})\alpha^{-1}(\alpha-4k)-z(-1+\alpha^{-1}(\alpha-4k)+1))w'(z) + \alpha^{-1}(\alpha-2)w(z). \quad (16)$$

We put the equation in this form because Equation 07.25.13.0004.01 from [9] states that the general solution of the ordinary differential equation:

$$z^2 w^{(3)}(z) + z(1-z+b_1+b_2)w''(z) + (b_1b_2-z(a_1+a_2+1))w'(z) - a_1a_2w(z) = 0$$

is given exactly by

$$w(z) = c_1 \cdot {}_2\tilde{F}_2(a_1, a_2; b_1, b_2; z) + \\ + c_2 \left( G_{2,3}^{2,2}(z|1-a_1, 1-a_2; 0, 1-b_1, 1-b_2) + G_{2,3}^{2,2}(z|1-a_1, 1-a_2; 0, 1-b_2, 1-b_1) \right) + \\ + c_3 G_{2,3}^{3,2}(-z|1-a_1, 1-a_2; 0, 1-b_1, 1-b_2).$$

Here  ${}_2\tilde{F}_2$  denotes the regularized hypergeometric function and  $G_{m,n}^{p,q}$  denotes the Meijer-G function. The definition of both of these classes of special functions can be found in [6] or [9].

We choose

$$a_1 = \frac{-2k - \sqrt{\alpha^2 - 2\alpha + 4k^2}}{\alpha}, \quad a_2 = \frac{-2k + \sqrt{\alpha^2 - 2\alpha + 4k^2}}{\alpha} \\ b_1 = \frac{\alpha - 4k}{\alpha}, \quad b_2 = \frac{2 - 2k + \alpha\lambda}{\alpha}$$

and denote

$$\mathfrak{q}_{\alpha,k} := \frac{\sqrt{\alpha^2 - 2\alpha + 4k^2}}{\alpha}.$$

Then the solution of Equation (16) is exactly (for any choice of constants  $c_1, c_2, c_3$ ):

$$w(z) = c_1 \cdot {}_2\tilde{F}_2\left(-\frac{2k}{\alpha} - \mathfrak{q}_{\alpha,k}, -\frac{2k}{\alpha} + \mathfrak{q}_{\alpha,k}; 1 - \frac{4k}{\alpha}, \lambda + \frac{2-2k}{\alpha}; z\right) +$$

$$\begin{aligned}
& +c_2 \left( G_{2,3}^{2,2} \left( z \left| 1 + \frac{2k}{\alpha} + \mathfrak{q}_{\alpha,k}, 1 + \frac{2k}{\alpha} - \mathfrak{q}_{\alpha,k}; 0, \frac{4k}{\alpha}, 1 - \lambda + \frac{2k-2}{\alpha} \right) + \right. \\
& \left. + G_{2,3}^{2,2} \left( z \left| 1 + \frac{2k}{\alpha} + \mathfrak{q}_{\alpha,k}, 1 + \frac{2k}{\alpha} - \mathfrak{q}_{\alpha,k}; 0, 1 - \lambda + \frac{2k-2}{\alpha}, \frac{4k}{\alpha} \right) \right) + \\
& +c_3 G_{2,3}^{3,2} \left( -z \left| 1 + \frac{2k}{\alpha} + \mathfrak{q}_{\alpha,k}, 1 + \frac{2k}{\alpha} - \mathfrak{q}_{\alpha,k}; 0, \frac{4k}{\alpha}, 1 - \lambda + \frac{2k-2}{\alpha} \right) \right).
\end{aligned}$$

Using the transformation back to  $f_k(r)$ , we see that we have found the solutions of the homogeneous differential equation (10). Then, by using the asymptotic expansions for the special functions  ${}_2\tilde{F}_2$  and  $G_{m,n}^{p,q}$  described in Chapter 5 of [6], one can see that the solution cannot lie in any Lebesgue space unless it is identically zero. We leave the details to the interested reader.

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