

Capacity of loop-erased random walk

Maarten Markering

University of Cambridge

Abstract

We study the capacity of loop-erased random walk (LERW) on \mathbb{Z}^d . For $d \geq 4$, we prove a strong law of large numbers and give explicit expressions for the limit in terms of the non-intersection probabilities of a simple random walk and a two-sided LERW. Along the way, we show that four-dimensional LERW is ergodic. For $d = 3$, we show that the scaling limit of the capacity of LERW is random. We show that the capacity of the first n steps of LERW is of order $n^{1/\beta}$, with β the growth exponent of three-dimensional LERW. We express the scaling limit of the capacity of LERW in terms of the capacity of Kozma's scaling limit of LERW.

1 Introduction

In this paper, we study the *capacity* of the *loop-erased random walk* (LERW) on \mathbb{Z}^d for $d \geq 3$. LERW was introduced by Lawler in [17]. It has since become a central object in modern probability theory. This is partly due to its many connections to other models, chief among them being the uniform spanning tree. In that context, the capacity of LERW has become an important object of study. The capacity of a set $A \subset \mathbb{Z}^d$ can be considered as a rescaled hitting probability of A by a simple random walk (SRW) started far away. Through Wilson's algorithm, the capacity of a loop-erased random walk is related to the length and density of branches in the uniform spanning tree/forest. Using Hutchcroft's interlacement Aldous-Broder algorithm developed in [13], the capacity of loop-erased random walk played an important role in determining the scaling of several exponents for the uniform spanning tree on \mathbb{Z}^d for $d \geq 5$ by Hutchcroft [13] and $d = 4$ by Hutchcroft and Sousi [14] and Hutchcroft and Halberstam [12]. In particular, a weak law of large numbers for the capacity of LERW was established in [14]. A finite analogue of the capacity of LERW was also instrumental in establishing the scaling limit of the uniform spanning tree on the torus in dimensions $d \geq 4$ [4, 5, 22, 24, 26].

Before stating the results in this paper, we first give a formal definition of the LERW and capacity. Let Ω be the set of finite, one-sided infinite and two-sided infinite nearest-neighbour paths in \mathbb{Z}^d . Let \mathcal{F} be the sigma-algebra on Ω generated by events depending only on a finite number of indices (we call those events cylinder events). Throughout the paper, all events and random variables will be measurable with respect to \mathcal{F} . We will also need the definition of the left-shift operator $T: \Omega \rightarrow \Omega$, given by $(T\omega)(k) = \omega(k+1) - \omega(1)$.

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Definition 1.1 (Loop-erased random walk). Let $\omega \in \Omega$ be a one-sided nearest-neighbour, transient path in \mathbb{Z}^d . The loop-erasure $\text{LE}(\omega)$ of ω is the path obtained by erasing loops from ω chronologically. To be precise, $\text{LE}(\omega)_i = \omega_{\ell_i}$, where the times $(\ell_i)_{i \geq 0}$ are defined inductively as $\ell_0 = 0$ and $\ell_{i+1} = 1 + \max\{k: \omega_k = \omega_{\ell_i}\}$. When S is a simple random walk on \mathbb{Z}^d for $d \geq 3$, we write

$$\eta := \text{LE}(S[0, \infty)). \quad (1.1)$$

This is well-defined, since simple random walk is almost surely transient for $d \geq 3$.

Definition 1.2 (Capacity). Let W be a simple random walk on \mathbb{Z}^d for $d \geq 3$ and denote by \mathbb{P}_y the law of W with $W(0) = y \in \mathbb{Z}^d$. For a set $A \subset \mathbb{Z}^d$, the capacity of A is defined as

$$\text{Cap}(A) := \lim_{\|y\| \rightarrow \infty} \frac{\mathbb{P}_y(W[0, \infty) \cap A \neq \emptyset)}{G(0, y)} = \sum_{a \in A} \mathbb{P}_a(W[1, \infty) \cap A = \emptyset), \quad (1.2)$$

where G is the SRW Green's function on \mathbb{Z}^d and $\|y\|$ denotes the Euclidean norm of y .

We refer to [19] for more background on the capacity, including a proof that the two expressions above are equal. Throughout the paper, we let W, S denote simple random walks and denote their laws by \mathbb{P}^W and \mathbb{P}^S . We drop the superscripts if the law is clear from the context. As in the previous two definitions, we will use W in the context of a random walk hitting another set and we will use S in the context of a random walk being loop-erased.

Although motivated by applications to other models, we study the capacity of LERW in its own right in this paper. We establish a strong law of large numbers (SLLN) for $\text{Cap}(\eta[0, n])$ in dimensions $d \geq 4$. In three dimensions, we show that $\text{Cap}(\eta[0, n])$ has a non-deterministic scaling limit. These complement earlier results on the capacity of SRW, which has a rich history. Already in 1968, Jain and Orey established a SLLN for the capacity of the range of SRW in dimensions $d \geq 5$ [15]. This was then extended to a central limit theorem for $d \geq 6$ by Asselah, Schapira and Sousi in [7]. The same authors also proved a SLLN and central limit theorem for $d = 4$ in [8]. Furthermore, in [9], Chang showed that the scaling limit in $d = 3$ is random.

We also give explicit expressions for the limiting values. In dimensions $d \geq 5$, we show that the SLLN limit equals the non-intersection probability of a SRW and a *two-sided loop-erased random walk*. Two-sided LERW for $d \geq 5$ was first constructed by Lawler in [17]. Much later, Lawler, Sun and Wu constructed two-sided LERW for $d = 4$ in [20]. We show that (two-sided) LERW is ergodic in four dimensions. We then use an alternative formula for the capacity to express the limit in terms of non-intersection probabilities of a SRW and two-sided LERW. Both the ergodicity of LERW for $d = 4$ as well as the alternative capacity formula are of independent interest. In three dimensions, we do not consider the two-sided LERW. Rather, we express the scaling limit of $\text{Cap}(\eta[0, n])$ as a function of the scaling limit of LERW, which was first constructed by Kozma in [16]. Furthermore, we show that the correct scaling for the capacity is $n^{1/\beta}$, with β being the growth exponent of three-dimensional LERW.

We state the laws of large numbers for dimensions $d \geq 5$, $d = 4$ and $d = 3$ in Sections 1.1, 1.2 and 1.3 respectively. In Section 1.4, we give an overview of the proofs. Finally, in Section 1.5, we give an outline of the rest of the paper.

1.1 Dimensions $d \geq 5$

Before stating the SLLN in high dimensions, we first need to introduce the *two-sided* loop-erased random walk, which we denote by $\hat{\eta}$. The random variable $\hat{\eta}: \mathbb{Z} \rightarrow \mathbb{Z}^d$ takes values in the space

of two-sided infinite simple paths. Intuitively speaking, $\hat{\eta}$ can be viewed as an infinite loop-erased random walk ‘seen from the middle’, i.e., $\hat{\eta}$ is the weak limit of $T^k\eta$ as $k \rightarrow \infty$. The two-sided LERW was first introduced by Lawler in [17, Section 5] and is defined as follows.

Definition 1.3. Let $d \geq 5$. Let S_1, S_2 be independent simple random walks on \mathbb{Z}^d started from 0, conditioned such that $\text{LE}(S_1)[1, \infty) \cap S_2[1, \infty) = \emptyset$. The two-sided loop-erased random walk $\hat{\eta}$ is defined as the union of $\text{LE}(S_1)$ and $\text{LE}(S_2)$, i.e., for $n \geq 0$, $\hat{\eta}(n) = \text{LE}(S_1)(n)$ and $\hat{\eta}(-n) = \text{LE}(S_2)(n)$.

We are now ready to state the strong law of large numbers for the capacity of loop-erased random walk in dimensions $d \geq 5$.

Theorem 1.4. *Let $d \geq 5$. Then almost surely,*

$$\lim_{n \rightarrow \infty} \frac{\text{Cap}(\eta[0, n])}{n} = \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0). \quad (1.3)$$

1.2 Dimension $d = 4$

In \mathbb{Z}^4 , we have $\mathbb{P}(W[1, \infty) \cap \text{LE}(S[0, \infty)) = \emptyset) = 0$. So we cannot define two-sided LERW as above. Note that in high dimensions, $\hat{\eta}[0, \infty)$ has the law of a one-sided LERW weighted by the probability that a SRW started from the same point does not intersect it. Although this probability is 0 for an infinite SRW, we can still consider the n -step escape probabilities and show that they converge. Let

$$X_n := (\log n)^{\frac{1}{3}} \mathbb{P}_0^W(W[1, n] \cap \eta[0, \infty) = \emptyset \mid \eta). \quad (1.4)$$

Theorem 1.5 ([20]). *There exists a nontrivial random variable X_∞ such that*

$$X_\infty = \lim_{n \rightarrow \infty} X_n \quad (1.5)$$

almost surely and in L^p for all $p > 0$.

We can now define two-sided LERW in the same way as in high dimensions, but instead using the random variable X_∞ instead of $\mathbb{P}^W(W[1, \infty) \cap \eta[0, \infty) = \emptyset \mid \eta)$.

Definition 1.6. We define the two-sided loop-erased random walk $\hat{\eta}$ as the random variable on two-sided infinite paths in Ω such that $\hat{\eta}[0, \infty)$ is absolutely continuous with respect to η with Radon-Nikodym derivative given by $\frac{X_\infty}{\mathbb{E}[X_\infty]}$. It was shown in [20] that $(T\hat{\eta})[0, \infty) \stackrel{d}{=} \hat{\eta}[0, \infty)$. Thus, we can extend $\hat{\eta}$ to the space of two-sided infinite paths by defining $\hat{\eta}[-a, b] \stackrel{d}{=} \hat{\eta}[0, a+b] - \hat{\eta}(a)$.

In order to formulate the SLLN for $d = 4$, we need the following extension of Theorem 1.5.

Theorem 1.7. *Let $d = 4$. Then there exist random variables \hat{X}_∞ and \hat{X}_∞^+ depending on $\hat{\eta}$ such that*

$$\begin{aligned} \hat{X}_\infty^+ &= \lim_{n \rightarrow \infty} \hat{X}_n^+ := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[1, n] = \emptyset \mid \hat{\eta}), \\ \hat{X}_\infty &= \lim_{n \rightarrow \infty} \hat{X}_n := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[0, n] = \emptyset \mid \hat{\eta}) \end{aligned} \quad (1.6)$$

almost surely and in L^p for every $p < 3$.

We prove a SLLN for $d = 4$ in terms of \hat{X}_∞ and \hat{X}_∞^+ .

Theorem 1.8. *Let $d = 4$. Then almost surely,*

$$\lim_{n \rightarrow \infty} \frac{(\log n)^{2/3} \text{Cap}(\eta[0, n])}{n} = \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+]. \quad (1.7)$$

1.3 Dimension $d = 3$

Lastly, we show a non-deterministic law of large numbers for $d = 3$. We relate the rescaled capacity of LERW in \mathbb{Z}^d to the capacity of its scaling limit in \mathbb{R}^d . Existence of the scaling limit of η as the mesh size tends to 0 was first shown by Kozma in [16] along a dyadic subsequence. This was later strengthened to the following statement in [1], which is what we will need for the law of large numbers for the capacity. Let \mathcal{C} be the collection of parametrized curves $\lambda: [0, \infty) \rightarrow \mathbb{R}^3$ with $\lambda(0) = 0$ and $\lim_{t \rightarrow \infty} \lambda(t) = \infty$. We also consider the LERW η to be an element of \mathcal{C} , with the natural parametrization. Consider the metric χ on \mathcal{C} given by

$$\chi(\lambda_1, \lambda_2) := \sum_{k=1}^{\infty} 2^{-k} \sup_{0 \leq t \leq k} \min\{|\lambda_1(t) - \lambda_2(t)|, 1\} \quad (1.8)$$

for $\lambda_1, \lambda_2 \in \mathcal{C}$. Let

$$\beta := \lim_{n \rightarrow \infty} \frac{\log n}{\log \mathbb{E} \|\eta(n)\|} \in \left(1, \frac{5}{3}\right] \quad (1.9)$$

be the *growth exponent* for LERW in \mathbb{Z}^3 . A priori, it is not clear that β is well-defined. In [27], Shiraishi proved existence of the limit. Furthermore, Hernández-Torres, Li and Shiraishi proved convergence of LERW to Kozma's scaling limit for the full sequence $(n^{1/\beta} \eta(n \cdot))_{n \in \mathbb{N}}$ in [1].

Theorem 1.9 ([27], [1]). *The limit in (1.9) exists. Furthermore, there exists a random infinite parametrized curve $\gamma \in \mathcal{C}$ such that $n^{-1/\beta} \eta(n \cdot)$ converges weakly to γ in (\mathcal{C}, χ) as $n \rightarrow \infty$.*

Our last main theorem expresses the scaling limit of $\text{Cap}(\eta[0, n])$ in \mathbb{Z}^3 in terms of the capacity of $\gamma[0, 1]$. First, we need to define the capacity of a subset of \mathbb{R}^3 .

Definition 1.10. Let $A \subset \mathbb{R}^3$. Let M be a standard Brownian motion and denote by \mathbb{P}_y^M the law of M with $M(0) = y$. The *capacity* of A is defined¹ as

$$\text{Cap}_{\mathbb{R}^3}(A) := \lim_{\|y\| \rightarrow \infty} \frac{\mathbb{P}_y^M(M[0, \infty) \cap A \neq \emptyset)}{G_{\mathbb{R}^3}(0, y)}, \quad (1.10)$$

with $G_{\mathbb{R}^3}$ the Green's function of Brownian motion on \mathbb{R}^3 .

From now on, when discussing three-dimensional capacity, we will use the subscripts $\text{Cap}_{\mathbb{Z}^3}$ and $\text{Cap}_{\mathbb{R}^3}$ to avoid confusion. In higher dimensions, we only consider discrete capacity and drop the subscript. We are now ready to state the law of large numbers for the capacity of three-dimensional LERW.

Theorem 1.11. *Let $d = 3$. Then*

$$\lim_{n \rightarrow \infty} \frac{\text{Cap}_{\mathbb{Z}^3}(\eta[0, n])}{3n^{1/\beta}} = \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \quad (1.11)$$

in distribution.

¹As in the discrete case, there are several different equivalent definitions of capacity, but for our purposes we only need the one stated here. We refer to [23] for more background.

1.4 Proof sketch

1.4.1 Dimensions $d \geq 5$

The proof of the law of large numbers in high dimensions is the most straightforward one. Recall that the capacity can be written as

$$\frac{1}{n} \text{Cap}(\eta[0, n]) = \frac{1}{n} \sum_{k=0}^n \mathbb{P}_{\eta(k)}^W(W[1, \infty) \cap \eta[0, n] = \emptyset \mid \eta). \quad (1.12)$$

If the probabilities in the sum above were i.i.d., a strong law of large numbers would follow immediately. This is of course not the case, but we can still apply an ergodic theorem to deduce the law of large numbers as follows. As $k \rightarrow \infty$, the part of η near $\eta(k)$ is distributed as $\hat{\eta}$. So $\frac{1}{n} \text{Cap}(\eta[0, n])$ and $\frac{1}{n} \text{Cap}(\hat{\eta}[0, n])$ should have the same limit, which we show in Proposition 2.2. Furthermore, $\hat{\eta}$ is stationary and ergodic. So by Birkhoff's ergodic theorem,

$$\begin{aligned} \frac{1}{n} \text{Cap}(\hat{\eta}[0, n]) &= \frac{1}{n} \sum_{k=0}^n \mathbb{P}_0^W(W[1, \infty) \cap (T^k \hat{\eta})[-k, n-k] \mid \hat{\eta}) \\ &\approx \frac{1}{n} \sum_{k=0}^n \mathbb{P}_0^W(W[1, \infty) \cap (T^k \hat{\eta})(-\infty, \infty) \mid \hat{\eta}) \\ &\rightarrow \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0), \quad n \rightarrow \infty, \end{aligned} \quad (1.13)$$

which was what we wanted to show. The approximate equality comes from the fact that in dimensions $d \geq 5$, a LERW and a SRW only intersect each other near their starting points.

1.4.2 Dimension $d = 4$

The overall picture in four dimensions is the same as in higher dimensions, but there are several subtleties. Dimension 4 is critical for the intersection of LERW and SRW. This means that an infinite SRW and an infinite LERW started from the same point intersect almost surely, and the probability that the first n steps intersect decays polylogarithmically in n . So while $\frac{\text{Cap}(\eta[0, n])}{n} \rightarrow 0$ almost surely as $n \rightarrow \infty$, we obtain a non-trivial limit if we multiply by a polylogarithmic correction. It turns out that the correct scaling is $\frac{(\log n)^{2/3}}{n}$, which we will now explain.

Like in high dimensions, the strong law of large numbers for $\hat{\eta}$ is the same as for η , so we only consider two-sided LERW. We first show in Lemma 3.6 that the limits

$$\begin{aligned} \hat{X}_\infty &= \lim_{n \rightarrow \infty} \hat{X}_n := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[0, n] = \emptyset \mid \hat{\eta}) \\ \hat{X}_\infty^+ &= \lim_{n \rightarrow \infty} \hat{X}_n^+ := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[1, n] = \emptyset \mid \hat{\eta}) \end{aligned} \quad (1.14)$$

exist, building on a very similar result from [20]. In other words, the intersection probability of $W[1, \infty)$ and $\hat{\eta}[0, n]$ decays like $(\log n)^{-1/3}$. So the intersection probability of $W[1, \infty)$ and $\hat{\eta}[-n, n]$ should decay like $(\log n)^{-2/3}$. So

$$\text{Cap}(\eta[0, n]) = \sum_{k=0}^n \mathbb{P}_{\hat{\eta}(k)}^W(W[1, \infty) \cap \hat{\eta} = \emptyset \mid \hat{\eta}) \asymp (\log n)^{-2/3} n. \quad (1.15)$$

Note that the escape probabilities in the expression above are *two-sided* escape probabilities: W started from $\eta(k)$ has to escape both $\eta[0, k]$ and $\eta[k + 1, n]$. However, we only know the scaling limits of the one-sided escape probabilities from (1.14). So in order to obtain a precise expression for the limit, we use a different formula for the capacity. In Lemma 3.3, we show that²

$$\text{Cap}(\hat{\eta}[0, n]) = \sum_{k=0}^n \mathbb{P}_{\hat{\eta}(k)}^W(W[1, \infty) \cap \hat{\eta}[\hat{\eta}(k), n] = \emptyset \mid \hat{\eta}) \mathbb{P}_{\hat{\eta}(k)}^W(W[1, \infty) \cap \hat{\eta}[\hat{\eta}(k+1), n] = \emptyset \mid \hat{\eta}) \quad (1.16)$$

In other words, instead of writing the capacity as the sum of two-sided escape probabilities, we write the capacity as the sum of the product of two one-sided escape probabilities. This can be rewritten further in terms of \hat{X} and \hat{X}^+ :

$$\begin{aligned} \text{Cap}(\hat{\eta}[0, n]) &= \sum_{k=0}^n (\log k)^{-2/3} \hat{X}_{n-k}(T^k \hat{\eta}) \hat{X}_{n-k}^+(T^k \hat{\eta}) \\ &\approx \sum_{k=0}^n (\log n)^{-2/3} \hat{X}_{\infty}(T^k \hat{\eta}) \hat{X}_{\infty}^+(T^k \hat{\eta}). \end{aligned} \quad (1.17)$$

The approximate equality is due to the fact that $W[1, \infty)$ is most likely to hit $\hat{\eta}[0, n]$ close to the origin. The desired strong law of large numbers now follows from ergodicity and stationarity of $\hat{\eta}$ with respect to T , which we prove in Proposition 3.1.

1.4.3 Dimension $d = 3$

The picture in three dimensions is entirely different. The capacity of LERW no longer scales (almost) linearly and the limit law is no longer deterministic. In three dimensions, the capacity of a set that has dimension strictly greater than 1 is of the same order as its diameter. Since the LERW is roughly β -dimensional, the correct scaling for $\text{Cap}(\eta[0, n])$ is $n^{1/\beta}$, where β is the growth exponent of LERW as defined in Section 1.3. This also means that the global behaviour of the LERW influences the capacity, which implies that the scaling limit must be non-deterministic. The proof goes as follows. We use the same strategy as in [9].

Recall that γ denotes the scaling limit of LERW in three dimensions. For a set A in \mathbb{Z}^3 or \mathbb{R}^3 , let $B(A, \varepsilon) := \{x : \inf_{a \in A} \|x - a\| \leq \varepsilon\}$ denote the ε -sausage of A . We first show that a SRW started from distance $\delta n^{1/\beta}$ from $\eta[0, n]$ is very likely to hit $\eta[0, n]$ for δ small enough. So the ‘hittability’ of $\eta[0, n]$ is close to the ‘hittability’ of $B(\eta[0, n], \delta n^{1/\beta})$. This implies that

$$\text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \approx \text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], \delta n^{1/\beta})). \quad (1.18)$$

Furthermore, $n^{-1/\beta} \eta[0, n]$ converges in distribution to $\gamma[0, 1]$. So we can couple η and γ in such a way that $B(n^{-1/\beta} \eta[0, n], \delta) \approx B(\gamma[0, 1], \delta)$, which implies

$$\text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], \delta n^{1/\beta})) \approx \text{Cap}_{\mathbb{Z}^3}(B(n^{1/\beta} \gamma[0, 1], \delta n^{1/\beta})). \quad (1.19)$$

Using a strong coupling of SRW and Brownian motion and the fact that $G_{\mathbb{R}^3}(0, y) \sim 3G_{\mathbb{Z}^3}(0, y)$, we then show

$$\text{Cap}_{\mathbb{Z}^3}(B(n^{1/\beta} \gamma[0, 1], \delta n^{1/\beta})) \approx 3 \text{Cap}_{\mathbb{R}^3}(B(n^{1/\beta} \gamma[0, 1], \delta n^{1/\beta})) = 3n^{1/\beta} \text{Cap}_{\mathbb{R}^3}(B(\gamma[0, 1], \delta)). \quad (1.20)$$

²This formula for the capacity was known to some people in the field (e.g. Asselah [6]), but the author has not seen this precise expression anywhere in the existing literature. A related expression appears in [21].

Finally, for δ small, a Brownian motion started from the boundary of $B(\gamma[0, 1], \delta)$ will hit $\gamma[0, 1]$ with high probability. Thus,

$$3n^{1/\beta} \text{Cap}_{\mathbb{R}^3}(B(\gamma[0, 1], \delta)) \approx 3n^{1/\beta} \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]). \quad (1.21)$$

The desired law of large numbers then follows by combining (1.18)–(1.21).

1.5 Outline

In Section 2, we give the proof of Theorem 1.4. We prove Theorem 1.8 in Section 3. Finally, we give the proof of Theorem 1.11 in Section 4.

2 Capacity of LERW in high dimensions

In this section, we consider loop-erased random walk in dimensions $d \geq 5$ and prove Theorem 1.4. Recall the definition of two-sided LERW in high dimensions and the left-shift operator T from Section 1. We first state some facts about two-sided LERW. We then prove Theorem 1.4 by showing a lower and upper bound.

Proposition 2.1 ([17]). *Two-sided LERW $\hat{\eta}$ satisfies the following properties:*

- (i) *the law $\hat{\eta}$ is absolutely continuous with respect to the law of η ;*
- (ii) *$T^k \eta$ converges weakly to $\hat{\eta}$ as $k \rightarrow \infty$;*
- (iii) *$\hat{\eta}$ is stationary with respect to T , i.e., $T\hat{\eta} \stackrel{d}{=} \hat{\eta}$;*
- (iv) *η and $\hat{\eta}$ are ergodic with respect to T , i.e., for all events E such that $\mathbb{P}(\eta \in E \Delta T(E)) = 0$, we have $\mathbb{P}(\eta \in E) \in \{0, 1\}$.*

As a corollary, we show that it suffices to prove a strong law of large numbers for $\text{Cap}(\hat{\eta}[0, n])$.

Proposition 2.2. *Assume that $\frac{1}{n} \text{Cap}(\hat{\eta}[0, n])$ converges almost surely. Then $\frac{1}{n} \text{Cap}(\eta[0, n])$ converges almost surely to the same limit.*

Proof. Note that the law of $\hat{\eta}$ is absolutely continuous with respect to the law of η . So if $\frac{1}{n} \text{Cap}(\hat{\eta}[0, n])$ converges almost surely, $\frac{1}{n} \text{Cap}(\hat{\eta}[0, n])$ converges with positive probability. Since η is ergodic with respect to T and $\frac{1}{n} \text{Cap}(\hat{\eta}[0, n])$ converges if and only if $\frac{1}{n} \text{Cap}((T\hat{\eta})[0, n])$ converges, the probability must equal 1. \square

Proof of Theorem 1.4. By Proposition 2.2, it suffices to prove a strong law of large numbers for the two-sided LERW. We first prove a lower bound. By the definition of capacity and recalling that \mathbb{P}^W is the law of a SRW W , we have

$$\begin{aligned} \text{Cap}(\hat{\eta}[0, n]) &= \sum_{k=0}^n \mathbb{P}_{\hat{\eta}(k)}^W(W[1, \infty) \cap \hat{\eta}[0, n] = \emptyset \mid \hat{\eta}) \\ &\geq \sum_{k=0}^n \mathbb{P}_{\hat{\eta}(k)}^W(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid \hat{\eta}) \\ &= \sum_{k=0}^n \mathbb{P}_0^W(W[1, \infty) \cap (T^k \hat{\eta})(-\infty, \infty) = \emptyset \mid \hat{\eta}). \end{aligned} \quad (2.1)$$

Since $\hat{\eta}$ is ergodic and stationary with respect to T , it follows by Birkhoff's ergodic's theorem that almost surely,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \text{Cap}(\hat{\eta}[0, n]) &= \mathbb{E}^{\hat{\eta}}[\mathbb{P}_0^W(W[1, \infty) \cap (T^k \hat{\eta})(-\infty, \infty) = \emptyset \mid \hat{\eta})] \\ &= \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0), \end{aligned} \quad (2.2)$$

which settles the lower bound.

We now turn to the upper bound. Let $m \in \mathbb{N}$. Then by subadditivity of capacity,

$$\begin{aligned} \frac{1}{n} \text{Cap}(\hat{\eta}[0, n]) &\leq \frac{1}{n} \sum_{k=0}^{\lceil \frac{n}{m} \rceil - 1} \text{Cap}(\hat{\eta}[mk, m(k+1)]) \\ &= \frac{1}{m} \frac{1}{\frac{n}{m}} \sum_{k=0}^{\lceil \frac{n}{m} \rceil - 1} \text{Cap}(((T^m)^k \hat{\eta})[0, m]) \end{aligned} \quad (2.3)$$

From the proofs in [17], it follows that $\hat{\eta}$ is also ergodic with respect to any power of T , so also with respect to T^m . Therefore, again by Birkhoff's ergodic theorem,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \text{Cap}(\hat{\eta}[0, n]) \leq \frac{1}{m} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])]. \quad (2.4)$$

Therefore, it suffices to show

$$\lim_{m \rightarrow \infty} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])] = \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0). \quad (2.5)$$

Note that as $k, m-k \rightarrow \infty$, we have $\mathbb{P}(W[1, \infty) \cap (\hat{\eta}(-\infty, -k) \cup \hat{\eta}(m-k, \infty)) \neq \emptyset) \rightarrow 0$. So indeed, by stationarity of $\hat{\eta}$,

$$\begin{aligned} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])] &= \frac{1}{m} \sum_{k=0}^m \mathbb{P}(W[1, \infty) \cap \hat{\eta}[0, m] = \emptyset \mid W(0) = \hat{\eta}(k)) \\ &= \frac{1}{m} \sum_{k=0}^m \mathbb{P}(W[1, \infty) \cap \hat{\eta}[-k, m-k] = \emptyset \mid W(0) = \hat{\eta}(0) = 0) \\ &= \frac{1}{m} \sum_{k=0}^m \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0) + o(1) \\ &= \mathbb{P}(W[1, \infty) \cap \hat{\eta}(-\infty, \infty) = \emptyset \mid W(0) = \hat{\eta}(0) = 0) + o(1), \quad m \rightarrow \infty, \end{aligned} \quad (2.6)$$

which concludes the proof of Theorem 1.4. \square

3 Capacity of LERW in four dimensions

In this section, we consider LERW in four dimensions. We first show that two-sided LERW is ergodic in four dimensions in Section 3.1. We then prove Theorem 1.8 in Section 3.2.

3.1 Ergodicity of two-sided LERW

Proposition 3.1. *The random variable $\hat{\eta}$ is ergodic with respect to T .*

Throughout the paper, we mostly consider LERW η as its own process, without reference to the underlying SRW S . However, for this proof, we will need several facts about S and the loop-erasing procedure. Recall the definition of the loop-erasure times ℓ_i from Definition 1.1. Also define $\rho_j = \max\{i \geq 0: \ell_i \leq j\}$, which is the number of points up until time j that are kept in the loop-erasure. We say n is a *cut time* of the walk S if $S[0, n] \cap S[n+1, \infty) = \emptyset$. The behaviour of ℓ_i , ρ_j and the cut times of S in four dimensions are well understood. In particular, ℓ_i and ρ_j are well concentrated around $i(\log i)^{1/3}$ and $j(\log j)^{-1/3}$ and S has a large amount of cut times. We summarize the results we need in the following lemma, whose bounds are implicitly stated in the proofs of [18, Lemma 7.7.4 and Theorem 7.7.5].

Lemma 3.2. *We have*

$$\lim_{n \rightarrow \infty} \mathbb{E} [|n - \rho_{n(\log n)^{-1/3}}|] = \lim_{n \rightarrow \infty} \mathbb{E} [|(\log n)^{1/3}n - \ell_n|] = 0. \quad (3.1)$$

Furthermore, there exists $C > 0$ such that for all n and all m such that $|n - m| \geq (\log m)^{-6}m$,

$$\mathbb{P}(\text{there are no cut times between } n \text{ and } m) \leq C \frac{\log \log m}{\log m}. \quad (3.2)$$

Proof of Proposition 3.1. The proof is an adaptation of [17, Proposition 5.9]. Recall that $\hat{\eta}$ is ergodic if for all events E with $\mathbb{P}(\hat{\eta} \in T(E) \Delta E) = 0$, we have $\mathbb{P}(\hat{\eta} \in E) \in \{0, 1\}$. Let $\zeta = (\zeta_0, \dots, \zeta_m)$, $\xi = (\xi_0, \dots, \xi_m)$ be finite self-avoiding paths. Since $\hat{\eta}$ is absolutely continuous with respect to η , it suffices to prove that $\hat{\eta}$ is ergodic with respect to T . Since \mathcal{F} is generated by cylinder sets, it suffices to show that for all finite self-avoiding paths $\zeta = (\zeta_0, \dots, \zeta_m)$ and $\xi = (\xi_0, \dots, \xi_m)$ [11, Lemma 6.7.4],

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{P}(\eta[0, m] = \zeta, \eta[k, k+m] = \xi) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{P}(\eta[k, k+m] = \xi) \mathbb{P}(\eta[0, m] = \zeta). \quad (3.3)$$

By the domain Markov property for loop-erased random walk,

$$\mathbb{P}(\eta[k, k+m] = \xi \mid \eta[0, m] = \zeta) = \mathbb{P}(\eta[k-m, k] = \xi \mid S[1, \infty) \cap \zeta = \emptyset, S_0 = \zeta_m), \quad (3.4)$$

where S is the SRW such that $\eta = \text{LE}(S)$. Let F be the event that $S[1, \infty) \cap \zeta = \emptyset$. Then it suffices to prove that for every ξ and ζ such that F has strictly positive probability,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{P}(\eta[k, k+m] = \xi \mid F) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{P}(\eta[k, k+m] = \xi). \quad (3.5)$$

Let

$$H^n(\xi) = \frac{1}{n} \sum_{k=0}^n \mathbf{1}_{\{\eta_k^{k+m} = \xi\}}. \quad (3.6)$$

We divide the interval $[0, n(\log n)^{1/3}]$ into $(\log n)^{1/2}$ intervals I_j of size $n(\log n)^{-1/6}$, separated by buffers L_j and R_j of size $n(\log n)^{-6}$ as follows. Let $a_j = \lfloor jn(\log n)^{-1/6} \rfloor$, $a_j^- = a_j - \lfloor n(\log n)^{-6} \rfloor$, and $a_j^+ = a_j + \lfloor n(\log n)^{-6} \rfloor$. Define $L_j = [a_j, a_j^+]$, $I_j = [a_j^+, a_{j+1}^-]$ and $R_j = [a_{j+1}^-, a_{j+1}]$.³ Let

$$H_j^n(\xi) = \sum_{k=\rho_{a_j}}^{\rho_{a_{j+1}}-1} \mathbf{1}_{\eta_k^{m+k}=\xi}, \quad \bar{H}_j^n(\xi) = \sum_{k=0}^{\lfloor a_1(\log a_1)^{-1/3} \rfloor} \mathbf{1}_{\{\text{LE}(S[a_j, a_{j+1}])_k^{k+m}=\xi\}}. \quad (3.7)$$

Consider the event that L_j and R_j contains a cut time. Then $\eta[\rho_{a_j}, \rho_{a_{j+1}}]$ and $\text{LE}(S[a_j, a_{j+1}])$ differ only differ on the sections contributed by L_j and R_j . The number of indices of L_j contributes to η is $\rho_{a_j^+} - \rho_{a_j} \leq a_j^+ - a_j = n(\log n)^{-6}$. The contribution of L_j to $\text{LE}(S[a_j, a_{j+1}])$ is also at most $a_j^+ - a_j$. The same estimates hold for R_j . Furthermore, the number of indices counted by $H_j^n(\xi)$ and $\bar{H}_j^n(\xi)$ differs by at most $|\rho_{a_{j+1}} - \rho_{a_j} - \lfloor a_1(\log a_1)^{-1/3} \rfloor|$. By Lemma 3.2, the probability that L_j or R_j does not contain a cut time is bounded by $\lesssim \frac{\log \log n}{\log n}$ and $\mathbb{E}|n - \rho_{n(\log n)^{-1/3}}| = o(1)$. We show that we can approximate $H^n(\xi)$ by the sum over $\bar{H}_j^n(\xi)$. Combining the above, we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[H^n(\xi) - \frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \mid F \right] \right| \\ & \leq \left| \mathbb{E} \left[H^n(\xi) - \frac{1}{n} \sum_{k=0}^{\rho_{n(\log n)^{1/3}}} \mathbf{1}_{\{\eta_k^{k+m}=\xi\}} \mid F \right] \right| \\ & \quad + \left| \mathbb{E} \left[\frac{1}{n} \sum_{k=0}^{\rho_{n(\log n)^{1/3}}} \mathbf{1}_{\{\eta_k^{k+m}=\xi\}} - \frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \mid F \right] \right| \\ & \leq \frac{1}{n} \mathbb{E}[|n - \rho_{n(\log n)^{1/3}}|] \mathbb{P}(F)^{-1} \\ & \quad + \mathbb{E} \left[\frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} |H_j^n(\xi) - \bar{H}_j^n(\xi)| \mathbf{1}_{\{\text{Each } L_j \text{ and } R_j \text{ contains a cut time}\}} \right] \mathbb{P}(F)^{-1} \\ & \quad + \mathbb{P}(\text{There is a } j \text{ such that } L_j \text{ or } R_j \text{ does not contain a cut time}) \mathbb{P}(F)^{-1} \\ & \lesssim o(1) + \frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} \left(2n(\log n)^{-6} + \mathbb{E}[|\rho_{a_{j+1}} - \rho_{a_j} - \lfloor a_1(\log a_1)^{-1/3} \rfloor|] \right) + \frac{\log \log n (\log n)^{1/2}}{\log n} \\ & \lesssim o(1) + (\log n)^{-5} + \frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} a_1(\log a_1)^{-1/3} o(a_1) + \frac{\log \log n}{(\log n)^{1/2}} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.8)$$

So it suffices to show that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \mid F \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{n} \sum_{j=0}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \right]. \quad (3.9)$$

³The precise size and amount of intervals do not really matter. The important thing is that with high probability, all buffers contains a cut time.

First note that we can ignore the term for $j = 0$. Furthermore, as $m \rightarrow \infty$, the law of $S[m, \infty) - S_m$ conditioned to avoid a finite set A converges in total variation norm to the law of $S[0, \infty)$. So

$$\begin{aligned} & \left| \mathbb{E} \left[\frac{1}{n} \sum_{j=1}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \mid F \right] - \mathbb{E} \left[\frac{1}{n} \sum_{j=1}^{\lfloor (\log n)^{1/2} \rfloor} \bar{H}_j^n(\xi) \right] \right| \\ & \leq \frac{1}{n} \sum_{j=1}^{\lfloor (\log n)^{1/2} \rfloor} \sum_{k=0}^{a_1 (\log a_1)^{-1/3}} |\mathbb{P}(\text{LE}(S[[a_j], [a_{j+1}]])_k^{k+m} = \xi \mid F) - \mathbb{P}(\text{LE}(S[[a_j], [a_{j+1}]])_k^{k+m} = \xi)| \\ & \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \tag{3.10}$$

We conclude that $|\mathbb{E}[H(n, \xi) \mid F] - \mathbb{E}[H(n, \xi)]| \rightarrow 0$ as $n \rightarrow \infty$, and so that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{k=0}^n \mathbb{P}(\eta[k, k+m] = \xi \mid \eta[0, m] = \zeta) - \frac{1}{n} \sum_{k=0}^n \mathbb{P}(\eta[k, k+m] = \xi) \right| = 0 \tag{3.11}$$

for all self-avoiding paths ζ, ξ . It only remains to show that, $\frac{1}{n} \sum_{k=0}^n \mathbb{P}(\eta[k, k+m] = \xi)$ converges. Let f denote the Radon-Nikodym derivative of $\hat{\eta}$ with respect to η and let \mathbb{E}^η denote expectation with respect to η . Let $\varepsilon > 0$. Since f is integrable, there exists a simple function $\sum_i \alpha_i \mathbf{1}_{A_i}$ such that $\mathbb{E}^\eta[|f - \sum_i \alpha_i \mathbf{1}_{A_i}|] < \varepsilon$ and A_i are cylinder sets. By the above, there exists N such that for all $n \geq N$, we have for all i , $|\frac{1}{n} \sum_{k=0}^n \mathbb{E}[\mathbf{1}_{\eta[k, k+m] = \xi} \mathbf{1}_{A_i}] - \frac{1}{n} \sum_{k=0}^n \mathbb{P}(\eta[k, k+m] = \xi) \mathbb{E}[\mathbf{1}_{A_i}]| < \varepsilon$. So for all $n \geq N$, by stationarity of $\hat{\eta}$,

$$\begin{aligned} \mathbb{P}(\hat{\eta}[0, m] = \xi) &= \frac{1}{n} \sum_{k=1}^n \mathbb{P}(\hat{\eta}[k, k+m] = \xi) \\ &= \frac{1}{n} \sum_{k=1}^n \mathbb{E}^\eta[f \mathbf{1}_{\eta[k, k+m] = \xi}] \\ &= \sum_i \alpha_i \frac{1}{n} \sum_{k=1}^n \mathbb{E}^\eta[\mathbf{1}_{\eta[k, k+m] = \xi} \mathbf{1}_{A_i}] \pm \varepsilon \\ &= \sum_i \alpha_i \frac{1}{n} \sum_{k=1}^n \mathbb{P}(\eta[k, k+m] = \xi) \mathbb{P}(\eta \in A_i) \pm 2\varepsilon \\ &= \frac{1}{n} \sum_{k=1}^n \mathbb{P}(\eta[k, k+m] = \xi) \pm 3\varepsilon. \end{aligned} \tag{3.12}$$

It follows that $\frac{1}{n} \sum_{k=1}^n \mathbb{P}(\eta[k, k+m] = \xi) \rightarrow \mathbb{P}(\hat{\eta}[0, m] = \xi)$ as $n \rightarrow \infty$, which concludes the proof. \square

3.2 Proof of Theorem 1.8

We now prove Theorem 1.8. Recall from Proposition 2.2 that it suffices to prove the strong law of large numbers for $\hat{\eta}$. We first show the following key lemma, which holds in all dimensions. The proof only uses the strong Markov property of SRW.

Lemma 3.3. *Let $A = \{x_1, \dots, x_n\} \subset \mathbb{Z}^d$. Then*

$$\text{Cap}(A) = \sum_{k=1}^n \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_k\} = \emptyset) \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_{k-1}\} = \emptyset). \quad (3.13)$$

Proof. For $B \subset \mathbb{Z}^d$, let $\tau_B := \min\{t \geq 0: W(t) \in B\}$ be the first hitting time of B . The proof is by induction. The statement is clearly true for all sets of size 1. Now assume the statement holds for all sets of size $n-1$. Then

$$\begin{aligned} \text{Cap}(A) &= \sum_{k=1}^n \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_n\} = \emptyset) \\ &= \sum_{k=1}^n [\mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) \\ &\quad - \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset, W[1, \infty) \cap \{x_n\} \neq \emptyset)] \\ &= \text{Cap}(\{x_1, \dots, x_{n-1}\}) + \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) \\ &\quad - \sum_{k=1}^n \mathbb{P}_{x_k}(\tau_{x_n}^+ = \tau_{\{x_1, \dots, x_n\}}^+ < \infty) \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) \\ &= \text{Cap}(\{x_1, \dots, x_{n-1}\}) \\ &\quad + \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) \left(1 - \sum_{k=1}^n \mathbb{P}_{x_n}(\tau_{x_k}^+ = \tau_{\{x_1, \dots, x_n\}}^+ < \infty)\right) \\ &= \text{Cap}(\{x_1, \dots, x_{n-1}\}) \\ &\quad + \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) (1 - \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_n\} \neq \emptyset)) \\ &= \text{Cap}(\{x_1, \dots, x_{n-1}\}) \\ &\quad + \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_{n-1}\} = \emptyset) \mathbb{P}_{x_n}(W[1, \infty) \cap \{x_1, \dots, x_n\} = \emptyset) \\ &= \sum_{k=1}^n \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_k\} = \emptyset) \mathbb{P}_{x_k}(W[1, \infty) \cap \{x_1, \dots, x_{k-1}\} = \emptyset), \end{aligned} \quad (3.14)$$

which completes the proof. \square

The above lemma is very useful, since it rewrites the capacity as the sum of one-sided escape probabilities. This allows us to use Theorem 1.5. First of all, we define

$$X_\infty^+ := \lim_{n \rightarrow \infty} X_n^+ := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, n) \cap \eta[1, \infty) = \emptyset \mid \eta), \quad (3.15)$$

whose existence was also proved in [20]. Note that X_n and X_n^+ are defined as the non-intersection probability of an n -step SRW and an infinite LERW. However, for our purposes, we need to study the non-intersection probability of an infinite SRW and an n -step LERW. We first show that these have the same scaling limit.

Lemma 3.4. *Let*

$$\begin{aligned} \tilde{X}_n^+ &= (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \eta[1, n) = \emptyset \mid \eta), \\ \tilde{X}_n &= (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \eta[0, n) = \emptyset \mid \eta), \end{aligned} \quad (3.16)$$

Then $X_\infty = \lim_{n \rightarrow \infty} \tilde{X}_n$ and $X_\infty^+ = \lim_{n \rightarrow \infty} \tilde{X}_n^+$ almost surely and in L^p for every $p < 3$.⁴

In the proof of this lemma, we will need certain hitting estimates on four-dimensional SRWs. Because they are only used in this proof, we will simply refer the reader to the relevant literature when needed, and not restate them in this paper. We will also need the following analysis lemma, which will be used multiple times in this paper. The lemma allows us to strengthen subsequential convergence to full convergence under certain conditions.

Lemma 3.5. *Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be strictly increasing sequences and let f be a decreasing function such that*

$$\lim_{n \rightarrow \infty} b_{a_n} f(a_n) = x. \quad (3.17)$$

For $n \in \mathbb{N}$, let k_n be the index such that $a_{k_n} \leq n \leq a_{k_n+1}$. Then if $\lim_{n \rightarrow \infty} \frac{b_{a_{k_n}}}{b_{a_{k_n+1}}} = 1$, we have

$$\lim_{n \rightarrow \infty} b_n f(n) = x. \quad (3.18)$$

Proof. Since b is increasing, f is decreasing and $a_{k_n} \leq n \leq a_{k_n+1}$, we have

$$\frac{b_{a_{k_n+1}}}{b_{a_{k_n}}} b_{a_{k_n}} f(a_{k_n}) \geq b_n f(n) \geq \frac{b_{a_{k_n}}}{b_{a_{k_n+1}}} b_{a_{k_n+1}} f(a_{k_n+1}). \quad (3.19)$$

The result follows by taking the limit $n \rightarrow \infty$. \square

Proof of Lemma 3.4. We prove the statement for \tilde{X}_n^+ , the proof for \tilde{X}_n being identical. We first prove almost sure convergence. We show an upper and lower bound. Let S be the simple random walk such that $\eta = \text{LE}(S[0, \infty))$. Then

$$\begin{aligned} 1 - X_n^+ &\leq (\log n)^{1/3} \mathbb{P}_0^W(W[1, n] \cap \eta[0, n] \neq \emptyset \mid \eta) + (\log n)^{1/3} \mathbb{P}_0^W(W[1, n] \cap \eta[n+1, \infty) \neq \emptyset \mid \eta) \\ &\leq 1 - \tilde{X}_n^+ + (\log n)^{1/3} \mathbb{P}_0^W(W[1, n] \cap S[n+1, \infty) \neq \emptyset \mid S). \end{aligned} \quad (3.20)$$

Let $\varepsilon > 0$. By [18, Theorem 4.3.6],

$$\mathbb{E}_0^S[\mathbb{P}_0^W(W[1, n] \cap S[n+1, \infty) \neq \emptyset \mid S)] \lesssim (\log n)^{-1}. \quad (3.21)$$

Hence, by Markov's inequality,

$$\mathbb{P}_0^S((\log n)^{1/3} \mathbb{P}_0^W(W[1, n] \cap S[n+1, \infty) \neq \emptyset \mid S) > \varepsilon) \lesssim \varepsilon (\log n)^{-2/3}. \quad (3.22)$$

Let $a_n = e^{n^2}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}_0^S((\log a_n)^{1/3} \mathbb{P}_0^W(W[1, a_n] \cap S[a_n+1, \infty) \neq \emptyset \mid S) > \varepsilon) \lesssim \varepsilon \sum_{n=1}^{\infty} (\log a_n)^{-2/3} \leq \varepsilon \sum_{n=1}^{\infty} n^{-4/3} < \infty. \quad (3.23)$$

So by the Borel-Cantelli lemma, $\lim_{n \rightarrow \infty} (\log a_n)^{1/3} \mathbb{P}_0^W(W[1, a_n] \cap S[a_n+1, \infty) \neq \emptyset \mid S) = 0$ almost surely. So almost surely,

$$\limsup_{n \rightarrow \infty} \tilde{X}_{a_n}^+ \leq \limsup_{n \rightarrow \infty} X_{a_n}^+ \leq X_\infty^+. \quad (3.24)$$

⁴It should be possible to prove L^p convergence for all $p > 0$, but we do not need it

Conversely,

$$\begin{aligned} 1 - \tilde{X}_n^+ &\leq (\log n)^{1/3} \mathbb{P}_0^W(W[1, n] \cap \eta[0, n] \neq \emptyset \mid \eta) + (\log n)^{1/3} \mathbb{P}_0^W(W[n+1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta) \\ &\leq 1 - X_n^+ + (\log n)^{1/3} \mathbb{P}_0^W(W[n+1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta). \end{aligned} \quad (3.25)$$

Recall from Lemma 3.2 that $\mathbb{E}[\ell_n] \sim n(\log n)^{1/3}$. So by Markov's inequality,

$$\mathbb{P}(\eta[0, n] \not\subset S[0, n(\log n)^2]) \lesssim (\log n)^{-5/3}. \quad (3.26)$$

Furthermore, it follows from [2, Theorem 1.1] that

$$\mathbb{E}_0^S[\mathbb{P}_0^W(W[n+1, \infty) \cap S[0, n(\log n)^2] \neq \emptyset \mid S)] \lesssim \frac{\log \log n}{\log n}. \quad (3.27)$$

Combining this, we have

$$\mathbb{E}_0^S[\mathbb{P}_0^W(W[n+1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta)] \lesssim \frac{\log \log n}{\log n}. \quad (3.28)$$

So by Markov's inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}_0^S((\log a_n)^{1/3} \mathbb{P}_0^W(W[a_n+1, \infty) \cap \eta[0, a_n] \neq \emptyset \mid \eta) > \varepsilon) \lesssim \sum_{n=1}^{\infty} \frac{\log \log a_n}{(\log a_n)^{2/3}} < \infty. \quad (3.29)$$

Another application of Borel-Cantelli gives that

$$\liminf_{n \rightarrow \infty} \tilde{X}_{a_n}^+ \geq \liminf_{n \rightarrow \infty} X_{a_n}^+ = X_\infty^+ \quad (3.30)$$

almost surely, and so,

$$\lim_{n \rightarrow \infty} \tilde{X}_{a_n}^+ = X_\infty^+. \quad (3.31)$$

Convergence of the full sequence \tilde{X}_n^+ now follows by applying Lemma 3.5 with $f(n) = \mathbb{P}_0^W[1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta$ and $b_n = (\log n)^{1/3}$.

We now show convergence in L^p for $1 \leq p < 3$. From the above, we obtain

$$\begin{aligned} \|X_\infty^+ - \tilde{X}_n^+\|_p &\leq \|X_\infty^+ - X_n^+\|_p + \|X_n^+ - \tilde{X}_n^+\|_p \\ &\leq \|X_\infty^+ - X_n^+\|_p + (\log n)^{1/3} (\mathbb{E}_0^\eta [\mathbb{P}_0^W(W[1, n] \cap \eta[n+1, \infty) \neq \emptyset \mid \eta]^p])^{1/p} \\ &\quad + (\log n)^{1/3} (\mathbb{E}_0^\eta [\mathbb{P}_0^W(W[n+1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta]^p])^{1/p}. \\ &\leq \|X_\infty^+ - X_n^+\|_p + (\log n)^{1/3} (\mathbb{E}_0^S [\mathbb{P}_0^W(W[1, n] \cap S[n+1, \infty) \neq \emptyset \mid \eta])^{1/p} \\ &\quad + (\log n)^{1/3} (\mathbb{E}_0^S [\mathbb{P}_0^W(W[n+1, \infty) \cap \eta[0, n] \neq \emptyset \mid \eta])^{1/p}. \\ &\lesssim \|X_\infty^+ - X_n^+\|_p + \frac{(\log n)^{1/3}}{(\log n)^{1/p}} + \frac{(\log \log n)^{1/p} (\log n)^{1/3}}{(\log n)^{1/p}} \rightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (3.32)$$

which settles the proof. \square

Recall that we want to prove a strong law of large numbers for $\text{Cap}(\hat{\eta}[0, n])$, rather than $\eta[0, n]$. In the following lemma, we show that the corresponding escape probabilities also converge for the two-sided LERW.

Lemma 3.6. *There exist random variables \hat{X}_∞ and \hat{X}_∞^+ depending on $\hat{\eta}$ such that*

$$\begin{aligned}\hat{X}_\infty^+ &= \lim_{n \rightarrow \infty} \hat{X}_n^+ := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[1, n] = \emptyset \mid \hat{\eta}), \\ \hat{X}_\infty &= \lim_{n \rightarrow \infty} \hat{X}_n := \lim_{n \rightarrow \infty} (\log n)^{1/3} \mathbb{P}_0^W(W[1, \infty) \cap \hat{\eta}[0, n] = \emptyset \mid \hat{\eta})\end{aligned}\tag{3.33}$$

almost surely and in L^p for every $p < 3$.

Proof. Almost sure convergence follows from absolute continuity of the law of $\hat{\eta}$ combined with Lemma 3.4. Furthermore, let f be the Radon-Nikodym derivative of $\hat{\eta}[0, \infty)$ with respect to η . Let $p < 3$. Let $p' = \frac{3+p}{2}$ and let q' be the Hölder conjugate of p' . Then by Hölder's inequality,

$$\begin{aligned}\|\hat{X}_\infty^+ - \hat{X}_n^+\|_p^p &= \mathbb{E}^\eta[f |\tilde{X}_\infty^+(\eta) - \tilde{X}_n^+(\eta)|^p] \\ &\leq \mathbb{E}^\eta \left[f^{\frac{q'}{p}} \right]^{\frac{p}{q'}} \mathbb{E}^\eta \left[|\tilde{X}_\infty^+ - \tilde{X}_n^+|^{p'} \right]^{\frac{p}{p'}} \rightarrow 0, \quad n \rightarrow \infty.\end{aligned}\tag{3.34}$$

For the final step, we use that f is L^a bounded for all $a > 0$ and Lemma 3.4 in combination with the fact that $p' < 3$. \square

3.2.1 Lower bound

Let $\varepsilon > 0$. By Lemma 3.6, there exists $n_0 = n_0(\hat{\eta})$ such that for all $n \geq n_0$, $X_\infty \leq \tilde{X}_n + \varepsilon$ and $X_\infty^+ \leq \tilde{X}_n^+ + \varepsilon$. Then for all $n \geq m$,

$$\begin{aligned}& \liminf_{n \rightarrow \infty} \frac{(\log n)^{2/3}}{n} \text{Cap}(\hat{\eta}[0, n]) \\ &= \liminf_{n \rightarrow \infty} \frac{(\log n)^{2/3}}{n} \sum_{k=0}^n \mathbb{P}_{\eta(k)}^W(W[1, \infty) \cap \eta[k, n] = \emptyset \mid \eta) \mathbb{P}_{\eta(k)}^W(W[1, \infty) \cap \eta[k+1, n] = \emptyset \mid \eta) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \hat{X}_n(T^k(\hat{\eta})) \hat{X}_n^+(T^k(\hat{\eta})) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \hat{X}_\infty(T^k(\hat{\eta})) \hat{X}_\infty^+(T^k(\hat{\eta})) \mathbf{1}_{\{m \geq n_0\}}(T^k(\hat{\eta})) - \varepsilon \\ &= \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+ \mathbf{1}_{\{m \geq n_0\}}] - \varepsilon \\ &\rightarrow \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+] - \varepsilon, \quad m \rightarrow \infty\end{aligned}\tag{3.35}$$

The first line follows from Lemma 3.3, the fourth line follows from Birkhoff's ergodic theorem, and the last line follows from the monotone convergence theorem. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{n \rightarrow \infty} \frac{(\log n)^{2/3}}{n} \text{Cap}(\hat{\eta}[0, n]) \geq \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+].\tag{3.36}$$

3.2.2 Upper bound

We show an upper bound for $\frac{1}{n} \text{Cap}(\hat{\eta}[0, \lfloor n(\log n)^{2/3} \rfloor])$. The strategy is similar to the upper bound in high dimensions. Let $m \in \mathbb{N}$. By subadditivity of capacity,

$$\begin{aligned} \frac{(\log n)^{2/3}}{n} \text{Cap}(\hat{\eta}[0, n]) &\leq \limsup_{n \rightarrow \infty} \sum_{k=0}^{\lceil \frac{n(\log n)^{-2/3}}{m(\log m)^{-2/3}} \rceil - 1} \text{Cap}(\hat{\eta}[mk, m(k+1)]) \\ &\leq \frac{(\log m)^{2/3}}{m} \frac{1}{\frac{n(\log n)^{-2/3}}{m(\log m)^{-2/3}}} \sum_{k=0}^{\lceil \frac{n(\log n)^{-2/3}}{m(\log m)^{-2/3}} \rceil - 1} \text{Cap}(((T^m)^k \hat{\eta})[0, m]) \end{aligned} \quad (3.37)$$

By the exact same proof as the proof of Proposition 3.1, it follows that every power of T is also ergodic. So by Birkhoff's ergodic theorem,

$$\limsup_{n \rightarrow \infty} \frac{(\log n)^{2/3}}{n} \text{Cap}(\hat{\eta}[0, n]) \leq \frac{(\log m)^{2/3}}{m} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])] \quad (3.38)$$

almost surely for all m . It remains to show that

$$\limsup_{m \rightarrow \infty} \frac{(\log m)^{2/3}}{m} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])] \leq \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+]. \quad (3.39)$$

Let W^1, W^2 be simple random walks and let $\varepsilon > 0$. Then

$$\begin{aligned} &\frac{(\log m)^{2/3}}{m} \mathbb{E}[\text{Cap}(\hat{\eta}[0, m])] \\ &= \frac{(\log m)^{2/3}}{m} \sum_{k=0}^m \mathbb{P}(W^1[1, \infty) \cap \hat{\eta}[0, k] = \emptyset, W^2[1, \infty) \cap \hat{\eta}[1, k] \mid W_0^1 = W_0^2 = \hat{\eta}(k)) \\ &= \frac{1}{m} \sum_{k=0}^m \frac{(\log m)^{2/3}}{(\log k)^{2/3}} \mathbb{E}[\hat{X}_k \hat{X}_k^+] \\ &\leq \frac{1}{m} \sum_{k=\lfloor n^{1-\varepsilon} \rfloor}^m \frac{(\log m)^{1/3}}{(\log m^{1-\varepsilon})^{1/3}} \mathbb{E}[\hat{X}_k \hat{X}_k^+] + \frac{1}{m} m^{1-\varepsilon} (\log m)^{1/3} \\ &\leq \frac{1}{(1-\varepsilon)^{1/3} m} \sum_{k=0}^m \mathbb{E}[\hat{X}_k \hat{X}_k^+] + o(1) \\ &\rightarrow (1-\varepsilon)^{-1/3} \mathbb{E}[\hat{X}_\infty \hat{X}_\infty^+], \quad m \rightarrow \infty. \end{aligned} \quad (3.40)$$

The convergence in the last line follows from Lemma 3.6. Since $\varepsilon > 0$ was arbitrary, the proof is complete.

4 Capacity of LERW in three dimensions

This section is devoted to the proof of Theorem 1.11. We first prove a technical hitting lemma in Section 4.1. In Section 4.2, we prove Theorem 1.11.

4.1 A hitting estimate

As mentioned in Section 1.4, for the proof we need that a SRW started close to $\eta[0, n]$ will hit it with high probability. There are several known results in this direction (most notably [25, Lemma 3.3] and [3, Proposition 3.7]), but none match the exact statement we need.

Lemma 4.1. *There exist $C, \alpha > 0$ such that for all $\varepsilon > 0$, there exists δ_0 such that for all $\delta \leq \delta_0$ and for all n ,*

$$\mathbb{P} \left(\max_{z \in B(\eta[0, n], \delta n^{1/\beta})} \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) > \varepsilon \right) < C\varepsilon^{1/\alpha}. \quad (4.1)$$

For the proof, we reformulate the hitting estimate we need in terms of what was shown in [25, Lemma 3.3] and [3, Proposition 3.7]. This has as advantage that we can use these results as a ‘black box’, at the cost of making the proof somewhat opaque.

Proof of Lemma 4.1. Let $\alpha > 0$ be some constant, whose exact value we will specify later. Let $s = \delta n^{1/\beta}$ and $r = \varepsilon^{-1/\alpha} s$. Then

$$\begin{aligned} & \mathbb{P}^\eta \left(\max_{z \in B(\eta[0, n], \delta n^{1/\beta})} \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) > \varepsilon \right) \\ & \leq \sum_{z \notin B(0, r)} \mathbb{P}^\eta \left(\{\eta[0, n] \cap B(z, s) \neq \emptyset\} \cap \{\mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) > \varepsilon\} \right) \\ & \quad + \mathbb{P}^\eta \left(\max_{z \in B(0, r)} \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) > \varepsilon \right). \end{aligned} \quad (4.2)$$

We bound both terms separately. Let $\tau_A^\eta := \min\{t \geq 0: \eta(t) \in A\}$. If $z \in B(\eta[0, n], s)$, then $\tau_{B(z, s)}^\eta \leq n$. Furthermore, for $\|z\| \geq r$, we have $(\frac{s}{\|z\|})^\alpha \leq (\frac{s}{r})^\alpha = \varepsilon$. Hence, by [25, Lemma 3.3], there exists α and a constant C_1 such that for all n ,

$$\begin{aligned} & \sum_{z \notin B(0, r)} \mathbb{P}^\eta \left(\eta[0, n] \cap B(z, s) \neq \emptyset, \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) > \varepsilon \right) \\ & \leq \sum_{z \notin B(0, r)} \mathbb{P}^\eta \left(\eta[0, \infty) \cap B(z, s) \neq \emptyset, \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, \tau_{B(z, s)}^\eta] = \emptyset \mid \eta) > (\frac{s}{\|z\|})^\alpha \right) \\ & \leq \sum_{z \notin B(0, r)} C_1 \left(\frac{s}{\|z\|} \right)^4 \\ & \lesssim \frac{s}{r} = \varepsilon^{1/\alpha}. \end{aligned} \quad (4.3)$$

We now bound the second term of (4.2). Let $\hat{\alpha}$ be the constant $\hat{\eta}$ from [3, Proposition 3.7]. Then

there exists C_2 such that for all $\delta \leq \varepsilon^{1/\hat{\alpha}}$,

$$\begin{aligned}
& \mathbb{P}^\eta \left(\max_{z \in B(0,r)} \mathbb{P}_z^W (W[0, \infty) \cap \eta[0, n] = \emptyset) > \varepsilon \right) \\
& \leq \mathbb{P}^\eta \left(\left\{ \max_{z \in B(0,r)} \mathbb{P}_z^W (W[0, \infty) \cap \eta[0, n] = \emptyset) > \delta^{\hat{\alpha}} \right\} \cap \left\{ \eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta}) = \emptyset \right\} \right) \\
& \quad + \mathbb{P}^\eta \left(\eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta}) \neq \emptyset \right) \\
& \leq \mathbb{P}^\eta \left(\max_{z \in B(0, \delta \varepsilon^{-1/\alpha} n^{1/\beta})} \mathbb{P}_z^W (W[0, \tau_{B(z, \delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta})^c}^W] \cap \eta[0, \infty) = \emptyset) > \delta^{\hat{\alpha}} \right) \\
& \quad + \mathbb{P}^\eta \left(\eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta}) \neq \emptyset \right) \\
& \leq C_2 \delta + \mathbb{P}^\eta \left(\eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta}) \neq \emptyset \right).
\end{aligned} \tag{4.4}$$

We bound the final probability. Note that

$$\mathbb{P}^\eta \left(\eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha} n^{1/\beta}) \neq \emptyset \right) = \mathbb{P}^\eta \left(n^{-1/\beta} \eta(n, \infty) \cap B(0, 2\delta^{1/2} \varepsilon^{-1/\alpha}) \neq \emptyset \right) \tag{4.5}$$

By convergence of $n^{-1/\beta} \eta(n, \infty)$ to the scaling limit $\gamma(1, \infty)$, the final probability can be made arbitrarily small by choosing δ sufficiently small. Hence, there exists δ_0 such that for all $\delta \leq \delta_0$, and all n ,

$$\mathbb{P}^\eta \left(\max_{z \in B(0,r)} \mathbb{P}_z^W (W[0, \infty) \cap \eta[0, n] = \emptyset) > \varepsilon \right) \leq C_3 \varepsilon^{1/\alpha}. \tag{4.6}$$

We conclude the proof by combining (4.3) and (4.6). \square

4.2 Proof of Theorem 1.11

We now prove Theorem 1.11. The proof is divided into three main steps, plus a preparation step. In each of the main steps we give a rigorous formulation and proof of the approximate equalities (1.18)–(1.20).

Proof of Theorem 1.11. Let $x > 0$ be a continuity point⁵ of the distribution $\mathbb{P}(\text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq \cdot)$. We first show

$$\mathbb{P} \left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \leq x \right) \leq \mathbb{P}(3 \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq x) + o(1), \quad n \rightarrow \infty. \tag{4.7}$$

The proof of the reverse inequality is almost analogous and we briefly remark on it at the end.

⁵Note that the distribution is almost certainly continuous everywhere, but we do not need to prove this for our purposes.

Step 0. We first show that with high probability, η and its scaling limit γ are well-behaved. Let $\varepsilon > 0$ and let $\delta > 0$. We will later choose $\delta > 0$ to be small depending on ε and then n to be sufficiently large depending on δ, ε . Let W be an independent simple random walk on \mathbb{Z}^d and M be an independent Brownian motion on \mathbb{R}^d . Define the events

$$\begin{aligned}
E_1(n, \delta) &:= \left\{ \sup_{0 \leq t \leq 1} \left| \frac{1}{n^{1/\beta}} \eta(\lfloor nt \rfloor) - \gamma(t) \right| \leq \delta \right\}, \\
E_2(n, \delta) &:= \left\{ \sup_{z \in B(\eta[0, n], \delta n^{1/\beta})} \mathbb{P}_z^W(W[0, \infty) \cap \eta[0, n] = \emptyset \mid \eta) < \varepsilon \right\}, \\
E_3(\delta) &:= \left\{ \sup_{z \in B(\gamma[0, 1], \delta)} \mathbb{P}_z^M(M[0, \infty) \cap \gamma[0, 1] = \emptyset \mid \gamma) < \varepsilon \right\}, \\
E_4(\delta) &:= \left\{ \sup_{0 \leq t \leq 1} \|\gamma(t)\| \geq \delta^{-1} \right\}, \\
E(n, \delta) &:= E_1 \cap E_2 \cap E_3 \cap E_4.
\end{aligned} \tag{4.8}$$

By Theorem 1.9, there exists a coupling of η and γ such that for all $\delta > 0$ and n sufficiently large depending on δ , $\mathbb{P}(E_1) > 1 - \varepsilon$. Furthermore, by Lemma 4.1, there exist constants $C, \alpha > 0$ such that for all δ sufficiently small and all $n > 0$, $\mathbb{P}(E_2) > 1 - C\varepsilon^{1/\alpha}$. Moreover, since the Hausdorff dimension of $\gamma[0, 1]$ is strictly greater than 1, $\sup_{z \in B(\gamma[0, 1], \delta)} \mathbb{P}_z^M(M[0, \infty) \cap \gamma[0, 1] = \emptyset \mid \gamma)$ tends to 0 as $\delta \downarrow 0$ almost surely. So for δ sufficiently small, $\mathbb{P}(E_3) > 1 - \varepsilon$. Lastly, since $\gamma[0, 1]$ is bounded almost surely, $\mathbb{P}(E_4) > 1 - \varepsilon$ for δ sufficiently small. Combining the above, we obtain that

$$\mathbb{P} \left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \leq x \right) \leq \mathbb{P} \left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \leq x, E(n, \delta) \right) + o(1), \quad \delta \downarrow 0, n \rightarrow \infty. \tag{4.9}$$

Step 1. We now show that on the event E , we can approximate the capacity of $\eta[0, n]$ by the capacity of a sufficiently small sausage around it. Let $A \subset B \subset \mathbb{Z}^d$ and denote by $h(B)$ the harmonic measure on B . It is not difficult to prove that

$$\text{Cap}_{\mathbb{Z}^3}(A) = \text{Cap}_{\mathbb{Z}^3}(B) \mathbb{P}_{h(B)}^W(W[0, \infty) \cap A \neq \emptyset). \tag{4.10}$$

Similarly, for $A \subset B \subset \mathbb{R}^d$, we have

$$\text{Cap}_{\mathbb{R}^3}(A) = \text{Cap}_{\mathbb{R}^3}(B) \mathbb{P}_{h(B)}^M(M[0, \infty) \cap A \neq \emptyset). \tag{4.11}$$

Note that on the event $E \subset E_2$, we have

$$\mathbb{P}_{h(B(\eta[0, n], 3\delta n^{1/\beta}))}^W(W[0, \infty) \cap \eta[0, n] \neq \emptyset) \geq 1 - \varepsilon. \tag{4.12}$$

Thus,

$$\mathbb{P} \left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \leq x, E(n, \delta) \right) \leq \mathbb{P} \left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], 3\delta n^{1/\beta})) \leq (1 - \varepsilon)^{-1} x, E(n, \delta) \right). \tag{4.13}$$

Step 2. We now show that $\text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], 3\delta n^{1/\beta})) \approx 3 \text{Cap}_{\mathbb{R}^3}(B(\eta[0, n], 3\delta n^{1/\beta}))$. We do this by showing that the probability that $B(\eta[0, n], 3\delta n^{1/\beta})$ is hit by a random walk on \mathbb{Z}^3 is close to the probability that $B(\eta[0, n], 3\delta n^{1/\beta})$ is hit by Brownian motion on \mathbb{R}^3 . For this part, we need several standard estimates on the hitting probabilities and displacement of simple random walk and Brownian motion, which we will state without proof.

By [10, Theorem 2], for a SRW W and Brownian motion M started from the same point there exists a coupling of W and M such that almost surely,

$$\sup_{0 \leq k \leq e^{m^{1/4}}} |W_k - M_{k/3}| \leq \sqrt{m} \quad (4.14)$$

for every m . This is a very strong coupling, which is somewhat overkill for our purposes. Recall that on the event E , $n^{1/\beta}\gamma[0, 1] \subset B(0, 2n^{1/\beta}\delta^{-1})$. Now consider W and M started from the harmonic measures on $B(0, 2n^{1/\beta}\delta^{-1})$ in \mathbb{Z}^3 and \mathbb{R}^3 respectively. Since the harmonic measure on $B(0, 2\delta^{-1})$ in $n^{-1/\beta}\mathbb{Z}^3$ converges weakly to the harmonic measure on $B(0, 2\delta^{-1})$ in \mathbb{R}^3 as $n \rightarrow \infty$, we can couple W and M such that $\|W_0 - M_0\| \leq \frac{1}{4}\delta n^{1/\beta}$ and thus

$$\sup_{0 \leq k \leq n^{100/\beta}} |W_k - M_{k/3}| \leq \frac{2}{4}\delta n^{1/\beta}. \quad (4.15)$$

Furthermore, it is not difficult to show that $\mathbb{P}_{h(B(0, 2\delta^{-1}n^{1/\beta}))}(W[[n^{100/\beta}], \infty) \cap B(0, 2\delta^{-1}n^{1/\beta})) \rightarrow 0$ and $\mathbb{P}_{h(B(0, 2\delta^{-1}n^{1/\beta}))}(M[[n^{100/\beta}/3], \infty) \cap B(0, 2\delta^{-1}n^{1/\beta})) \rightarrow 0$ as $n \rightarrow \infty$. Also, in between integer times, with high probability, the Brownian motion does not stray too far:

$$\mathbb{P}\left(\exists 0 \leq k \leq n^{100/\beta} : \sup_{k \leq t \leq k+1} \|M_t - M_k\| \geq \frac{1}{4}\delta n^{1/\beta}\right) \rightarrow 0, \quad n \rightarrow \infty. \quad (4.16)$$

Combining the above, we obtain that on the event E , for n sufficiently large depending on δ ,

$$\mathbb{P}_{h(B(0, 2\delta^{-1}n^{1/\beta}))}^W(W[0, \infty) \in B(\eta[0, n], 3\delta n^{1/\beta})) \geq \mathbb{P}_{h(B(0, 2\delta^{-1}n^{1/\beta}))}^M(M[0, \infty) \in B(\eta[0, n], 2\delta n^{1/\beta})). \quad (4.17)$$

Thus, by (4.10) and (4.11), we have

$$\text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], 3\delta n^{1/\beta})) \geq 3 \text{Cap}_{\mathbb{R}^3}(B(\eta[0, n], 3\delta n^{1/\beta})). \quad (4.18)$$

The extra factor 3 comes from the fact that $G_{\mathbb{Z}^d}(0, y) \sim 3G_{\mathbb{R}^d}(0, y)$ as $\|y\| \rightarrow \infty$, see [23, Theorem 3.3] and [19, Theorem 4.3.1]. In conclusion, for n sufficiently large depending on δ ,

$$\begin{aligned} & \mathbb{P}\left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(B(\eta[0, n], 3\delta n^{1/\beta})) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)\right) \\ & \leq \mathbb{P}\left(\frac{3}{n^{1/\beta}} \text{Cap}_{\mathbb{R}^3}(B(\eta[0, n], 2\delta n^{1/\beta})) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)\right) \\ & = \mathbb{P}\left(3 \text{Cap}_{\mathbb{R}^3}(B(n^{-1/\beta}\eta[0, n], 2\delta)) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)\right). \end{aligned} \quad (4.19)$$

Step 3. On the event E , we have $B(n^{-1/\beta}\eta[0, n], 2\delta) \supset B(\gamma[0, 1], \delta) \supset \gamma[0, 1]$, so

$$\begin{aligned} & \mathbb{P}\left(3 \text{Cap}_{\mathbb{R}^3}(B(n^{-1/\beta}\eta[0, n], 2\delta)) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)\right) \\ & \leq \mathbb{P}\left(3 \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)\right). \end{aligned} \quad (4.20)$$

Furthermore, since we assumed x to be a continuity point of the distribution $\mathbb{P}(3 \text{Cap}(\gamma[0, 1]) \leq \cdot)$, we have

$$\mathbb{P}(3 \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq (1 - \varepsilon)^{-1}x, E(n, \delta)) \leq \mathbb{P}(3 \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq x) + o(1), \quad \varepsilon \downarrow 0, \quad (4.21)$$

which completes the proof of (4.7).

Reverse inequality. The proof that

$$\mathbb{P}(3 \text{Cap}_{\mathbb{R}^3}(\gamma[0, 1]) \leq x) \leq \mathbb{P}\left(\frac{1}{n^{1/\beta}} \text{Cap}_{\mathbb{Z}^3}(\eta[0, n]) \leq x\right), \quad n \rightarrow \infty \quad (4.22)$$

is entirely analogous. Steps 0, 2 and 3 are the same. Step 1 is also the same, only using that $E \subset E_3$, instead of E_2 . \square

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References

- [1] Sharp one-point estimates and Minkowski content for the scaling limit of three-dimensional loop-erased random walk, author=Hernandez-Torres, S. and Li, Xinyi and Shiraishi, Daisuke, journal=arXiv preprint arXiv:2403.07256, year=2024.
- [2] S. Albeverio and X. Y. Zhou. Intersections of random walks and Wiener sausages in four dimensions. *Acta Appl. Math.*, 45:195–237, 1996.
- [3] O. Angel, D. A. Croydon, S. Hernandez-Torres, and D. Shiraishi. Scaling limits of the three-dimensional uniform spanning tree and associated random walk. *Ann. Probab.*, 49(6):3032 – 3105, 2021.
- [4] E. Archer, A. Nachmias, and M. Shalev. The GHP scaling limit of uniform spanning trees in high dimensions. *Communications in Mathematical Physics*, 405(3):73, 2024.
- [5] E. Archer and M. Shalev. The GHP scaling limit of uniform spanning trees of dense graphs. *Random Structures & Algorithms*, 65(1):149–190, 2024.
- [6] A. Asselah. Private communication, 2023.
- [7] A. Asselah, B. Schapira, and P. Sousi. Capacity of the range of random walk on \mathbb{Z}^d . *Transactions of the American Mathematical Society*, 370(11):7627–7645, 2018.
- [8] S. Asselah, B. Schapira, and P. Sousi. Capacity of the range of the random walk on \mathbb{Z}^4 . *Ann. Probab.*, 47(3):1447–1497, 2019.
- [9] Y. Chang. Two observations on the capacity of the range of simple random walks on \mathbb{Z}^3 and \mathbb{Z}^4 . *Electron. Commun. Probab.*, 25:1–9, 2017.

- [10] U. Einmahl. Extensions of results of Komlós, Major, and Tusnády to the multivariate case. *J. Multivar. Anal.*, 28(1):20–68, 1989.
- [11] R. Gray. *Probability, random processes, and ergodic properties*. Springer Science & Business Media, 2009.
- [12] N. Halberstam and T. Hutchcroft. Logarithmic corrections to the Alexander-Orbach conjecture for the four-dimensional uniform spanning tree. *arXiv:2211.01307*, 2022.
- [13] T. Hutchcroft. Universality of high-dimensional spanning forests and sandpiles. *Probab. Theory Relat. Fields*, 176(1):533–597, 2020.
- [14] T. Hutchcroft and P. Sousi. Logarithmic corrections to scaling in the four-dimensional uniform spanning tree. 401:2115–2191, 2024.
- [15] N. Jain and S. Orey. On the range of random walk. *Israel J. Math.*, 6:373–380, 1968.
- [16] G. Kozma. The scaling limit of loop-erased random walk in three dimensions. *Acta Math.*, 199(1):29–152, 2007.
- [17] G. Lawler. A self-avoiding random walk. *Duke Math. J.*, 47(3):655–693, 1980.
- [18] G. Lawler. *Intersections of random walks*. Birkhäuser Boston, 1991.
- [19] G. Lawler and V. Limic. *Random walk: a modern introduction*, volume 123. Cambridge University Press, 2010.
- [20] G. Lawler, X. Sun, and W. X. Four-dimensional loop-erased random walk. *Ann. Probab.*, 47(6):3866–3910, 2019.
- [21] I. Losev and S. Smirnov. How long are the arms in DBM? *arXiv:2307.14931*, 2023.
- [22] P. Michaeli, A. Nachmias, and M. Shalev. The diameter of uniform spanning trees in high dimensions. *Probab. Theory Relat. Fields*, 179:261–294, 2021.
- [23] P. Mörters and Y. Peres. *Brownian motion*, volume 30. Cambridge University Press, 2010.
- [24] Y. Peres and D. Revelle. Scaling limits of the uniform spanning tree and loop-erased random walk on finite graphs. *arXiv preprint math/0410430*, 2004.
- [25] A. Sapozhnikov and D. Shiraishi. On Brownian motion, simple paths, and loops. *Probability Theory and Related Fields*, 172:615–662, 2018.
- [26] J. Schweinsberg. The loop-erased random walk and the uniform spanning tree on the four-dimensional discrete torus. *Probability Theory and Related Fields*, 144(3):319–370, 2009.
- [27] D. Shiraishi. Growth exponent for loop-erased random walk in three dimensions. *Ann. Probab.*, 46(2):687–774, 2018.