AN UNCOUNTABLE SUBRING OF R WITH HAUSDORFF DIMENSION ZERO

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Abstract. We construct an uncountable subring of R with Hausdorff dimension zero (and hence of Lebesgue measure zero).

We denote by \mathbb{N}_0 the set of non-negative integers, by $\mathbb N$ the set of positive integers and by $\mathbb Z$ the set of integers. Let

$$
S := \{0\} \cup \{2^n : n \in \mathbb{N}_0\} = \{0, 1, 2, 2^2, 2^3, \ldots\}.
$$

For every $n \in \mathbb{N}$ by nS , we denote the nth sumset of S, i.e.,

$$
nS := \{s_1 + \dots + s_n : s_1, \dots, s_n \in S\}.
$$

For every $n \in \mathbb{N}$ we define

$$
A_n := \left\{ \sum_{k \in nS} \frac{x_k}{2^k} : (x_k)_{k \in nS} \text{ is a bounded sequence of integers} \right\}
$$

.

Theorem 1. Let $n \in \mathbb{N}$. The set A_n is an uncountable subgroup of \mathbb{R} .

Proof. We first show that A_n is a subgroup of R. Trivially, $0 \in A_n$ and $-x \in A_n$ whenever $x \in A_n$. Thus, it suffices to show that A_n is closed under addition. Let $x, y \in A_n$. Therefore,

$$
x = \sum_{k \in nS} \frac{x_k}{2^k} \quad \text{and} \quad y = \sum_{k \in nS} \frac{y_k}{2^k}
$$

where $(x_k)_{k\in nS}$ and $(y_k)_{k\in nS}$ are bounded sequences of integers. It follows that

$$
x + y = \sum_{k \in nS} \frac{x_k + y_k}{2^k}
$$

is also an element of A_n .

Clearly, the set A_n is uncountable, because by the uniqueness of binary expansions which do not end with an infinite string of 1's, the set of real numbers of the form

$$
\sum_{k \in nS} \frac{z_k}{2^k} \quad \text{with } z_k \in \{0, 1\}
$$

is an uncountable subset of A_n .

We now proceed to show that the set A_n has Hausdorff dimension zero for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ write

$$
A_n := \bigcup_{t=1}^{\infty} A_{n,t}
$$

where for every $t \in \mathbb{N}$ we let

$$
A_{n,t} := \left\{ \sum_{k \in nS} \frac{x_k}{2^k} \in \mathbb{R} : x_k \in \mathbb{Z} \text{ and } |x_k| \leq t \text{ for all } k \in nS \right\}.
$$

We will need the following result to show that the set $A_{n,t}$ has Hausdorff dimension zero for every $n, t \in \mathbb{N}$.

Lemma 1. For $n \in \mathbb{N}$ and $z \geq 1$, define

 $g_n(z) := #\{k \in nS : k \leq z\}.$

Then we have the bound

 $g_n(z) \le (2 + \log_2 z)^n$.

Proof. Clearly, for every $z \geq 1$, we have

$$
g_n(z) \le g_1(z)^n
$$
 and $g_1(z) \le 2 + \log_2 z$.

The result follows. \Box

Theorem 2. The set $A_{n,t}$ has Hausdorff dimension zero for all $n, t \in \mathbb{N}$.

Proof. Write $nS = \{k_m : m \in \mathbb{N}\}\$ where $(k_m)_{m \in \mathbb{N}}\$ is an increasing sequence. Let $x \in nS$ and write

$$
x = \sum_{i=1}^{\infty} \frac{a_i}{2^{k_i}}
$$

with $a_i \in \mathbb{Z}$ and $|a_i| \leq t$. Pick $j \in \mathbb{N}$ and split x into

$$
x = m_x + e_x,
$$

where

$$
m_x := \sum_{i=1}^{j-1} \frac{a_i}{2^{k_i}}
$$
 and $e_x := \sum_{i=j}^{\infty} \frac{a_i}{2^{k_i}}$.

Then using the triangle inequality, we have the bound

$$
|e_x| \le t \sum_{i=j}^{\infty} \frac{1}{2^{k_i}} \le t \sum_{h=k_j}^{\infty} \frac{1}{2^h} = \frac{t}{2^{k_j-1}}.
$$

Let $r_j = t/2^{k_j-1}$. It follows that

$$
A_{n,t} \subseteq \bigcup_{m \in X} I_m,
$$

where

$$
X := \left\{ \sum_{i=1}^{j-1} \frac{a_i}{2^{k_i}} : a_i \in \mathbb{Z}, \ |a_i| \le t \right\}
$$

and

$$
I_m := [m - r_j, m + r_j].
$$

Then $\#(X) \leq (2t+1)^{j-1}$ and for every $m \in X$ we see that $\mu(I_m) = 2r_j$ where $\mu(I_m)$ is the length of the interval I_m . Let $d > 0$ be a real number, let $\delta_j := 4r_j$, and let $\mathcal{H}^d_{\delta_j}(A_{n,t})$ be defined as in [\[2\]](#page-3-0). We see that

$$
\mathcal{H}_{\delta_j}^d(A_{n,t}) \leq \sum_{m \in X} \mu(I_m)^d = \#(X)(2r_j)^d \leq \frac{(2t+1)^{j-1}t^d}{2^{(k_j-2)d}}.
$$

We claim that

(1)
$$
\lim_{j \to \infty} \frac{(2t+1)^{j-1}t^d}{2^{(k_j-2)d}} = 0.
$$

Since $\delta_j \to 0$ as $j \to \infty$, this implies that $\mathcal{H}^d(A_{n,t}) = 0$ where $\mathcal{H}^d(A_{n,t})$ is the d-dimensional Hausdorff measure of $A_{n,t}$. (For the definition of this measure and related terms, see [\[2\]](#page-3-0).) Since this holds for all $d > 0$, we conclude that the Hausdorff dimension of $A_{n,t}$ is zero.

Equation [\(1\)](#page-1-0) above is seen as follows. First, it is convenient to note that [\(1\)](#page-1-0) holds if

$$
\lim_{j \to \infty} \frac{(3t)^j}{2^{k_j d}} = 0.
$$

By taking logarithms, we see that [\(1\)](#page-1-0) holds if

$$
\lim_{j \to \infty} (k_j d - j \log_2(3t)) = \infty.
$$

As $d > 0$, we let c be the constant $(1/d) \log_2(3t)$ and we see that [\(1\)](#page-1-0) holds if

$$
\lim_{j \to \infty} (k_j - c_j) = \infty.
$$

Now using Lemma [1,](#page-0-0) we have

$$
j = g_n(k_j) \le (2 + \log_2 k_j)^n,
$$

which implies

$$
k_j \ge 2^{\sqrt[n]{j}-2}
$$

.

Hence, [\(2\)](#page-1-1) follows, which justifies our claim. \square

Let $n \in \mathbb{N}$. From Theorems [1](#page-0-1) and [2](#page-1-2) it follows that the set A_n is an uncountable subgroup of $\mathbb R$ of Hausdorff dimension zero. As $0 \in S$, we have $nS \subseteq (n+1)S$, and hence $A_n \subseteq A_{n+1}$. Let

$$
A := \bigcup_{n=1}^{\infty} A_n.
$$

We deduce that the set A is an uncountable subgroup of $\mathbb R$ of Hausdorff dimension zero. We aim to establish that A is in fact a subring. To this end, we need the following result.

Lemma 2. Let $n \in \mathbb{N}$ and let $a_n(s)$ be the number of representations of an element $s \in \mathbb{N}$ as a sum $s = t_1 + t_2 + \cdots + t_n$ of n elements $t_1, t_2, \ldots, t_n \in S$. Then there exists $c_n \in \mathbb{R}$ such that $a_n(s) \leq c_n$ for every $s \in \mathbb{N}$.

Proof. Clearly $a_1(s) \leq 1$ for every $s \in \mathbb{N}$ and hence, we may take $c_1 := 1$. Now assume that $n \geq 2$ and that there exists $c_{n-1} \in \mathbb{R}$ such that $a_{n-1}(s) \leq c_{n-1}$ for every $s \in \mathbb{N}$. Let $s \in \mathbb{N}$ and suppose

$$
s = t_1 + \dots + t_n
$$

is a representation of s as a sum of n elements of S. Then for at least one $i \in \{1, \ldots, n\}$, we have $t_i \geq s/n$ and hence $t_i \neq 0$. So there exists $k \in \mathbb{N}_0$ such that $t_i = 2^k$. As $s/n \le t_i \le s$, we observe that $\log_2 s - \log_2 n \le k \le \log_2 s$. So if

$$
B_{n,s}:=\big[\log_2 s-\log_2 n,\, \log_2 s\,\big]\cap\mathbb{N}_0,
$$

then we see that

$$
a_n(s) \le n \sum_{k \in B_{n,s}} a_{n-1}(s-2^k) \le n(1 + \log_2 n)c_{n-1}.
$$

So we are done by taking $c_n := n(1 + \log_2 n)c_{n-1}$.

Theorem 3. The set A is a a subring of \mathbb{R} .

Proof. It suffices to show that $1 \in A$ and that A is closed under multiplication. We see that

$$
1 = \frac{1}{2^0} + \sum_{k \in S \setminus \{0\}} \frac{0}{2^k} \in A_1.
$$

To establish that A is closed under multiplication, we proceed as follows. Let $x, y \in A$. There exists $m, n \in \mathbb{N}$ such that $x \in A_m$ and $y \in A_n$. Therefore,

$$
x = \sum_{k \in mS} \frac{x_k}{2^k} \quad \text{and} \quad y = \sum_{\ell \in nS} \frac{y_\ell}{2^{\ell}}
$$

where $(x_k)_{k \in mS}$ and $(y_\ell)_{\ell \in nS}$ are bounded sequences of integers. Now

$$
xy = \left(\sum_{k \in mS} \frac{x_k}{2^k}\right) \left(\sum_{\ell \in nS} \frac{y_\ell}{2^\ell}\right) = \sum_{(k,\ell) \in mS \times nS} \frac{x_k y_\ell}{2^{k+\ell}} = \sum_{r \in (m+n)S} \frac{z_r}{2^r}
$$

where

$$
z_r := \sum_{\substack{(k,\ell) \in mS \times nS \\ k+\ell=r}} x_k y_\ell.
$$

The number of representations of an element $r \in (m+n)S$ as a sum $r = k+\ell$ of elements $k \in mS$ and $\ell \in nS$ is bounded by the number of representations of r which are of the form $r = t_1+t_2+\cdots+t_{m+n}$ where $t_1, t_2, \ldots, t_{m+n} \in S$. Hence, we deduce that the sequence $(z_r)_{r \in (m+n)S}$ is bounded as a consequence of Lemma [2](#page-2-0) and the boundedness of the sequences $(x_t)_{t \in [m]}$ and $(y_t)_{t \in [m]}$. It follows that $x y \in A_{m+n}$. This establishes that A is closed under mult $(x_k)_{k \in mS}$ and $(y_\ell)_{\ell \in nS}$. It follows that $xy \in A_{m+n}$. This establishes that A is closed under multiplication.

We now proceed to show that the only rational numbers in A are the dyadic rationals. In particular, it will follow that the ring A is not a field.

Lemma 3. Let $r, n \in \mathbb{N}$. Write $nS = \{k_m : m \in \mathbb{N}\}\$ where $(k_m)_{m \in \mathbb{N}}$ is an increasing sequence. Then there exists $m \in \mathbb{N}$ such that $k_{m+1} - k_m > r$.

Proof. For every $z \in \mathbb{N}$, let $g_n(z) := *{k \in nS : k \leq z}$. If the conclusion is false, then $g_n(z) \geq z/r$ for every $z \in \mathbb{N}$. This contradicts Lemma [1.](#page-0-0)

Proposition 1. Let $x \in A$ and $l \in \mathbb{N}$. Then there exists a binary expansion of x which contains either a string of zeroes or a string of ones having length l.

Proof. Let $x \in A_{n,t}$. Then there exists a sequence of integers $(x_k)_{k\in nS}$ with $|x_k| \leq t$ for all $k \in nS$ such that

$$
x = \sum_{k \in nS} \frac{x_k}{2^k} = \sum_{i=1}^{\infty} \frac{x_{k_i}}{2^{k_i}}.
$$

Let $m \in \mathbb{N}$. We observe that

$$
2^{k_m}x = q_m + r_m
$$

where

$$
q_m := \sum_{i=0}^m 2^{k_m - k_i} x_{k_i}
$$
 and $r_m := \sum_{i=m+1}^{\infty} \frac{x_{k_i}}{2^{k_i - k_m}}$.

Then $q_m \in \mathbb{Z}$ and we have

$$
|r_m| \leq t \sum_{j=k_{m+1}-k_m}^{\infty} \frac{1}{2^j} = \frac{2t}{2^{k_{m+1}-k_m}}.
$$

By Lemma [3,](#page-2-1) there exists $m \in \mathbb{N}$ such that $2^{k_{m+1}-k_m} > 2^{l+1}t$ and so $|r_m| < 1/2^l$. Hence,

$$
\text{frac}(2^{k_m}x) = \begin{cases} r_m, & \text{if } r_m \in \left[0, \frac{1}{2^l}\right); \\ 1 + r_m, & \text{if } r_m \in \left(-\frac{1}{2^l}, 0\right); \end{cases}
$$

and so

$$
\text{frac}(2^{k_m}x) \in [0, \frac{1}{2^l}) \cup (1 - \frac{1}{2^l}, 1),
$$

where frac(z) := $z - [z]$ is the fractional part of $z \in \mathbb{R}$. It follows that the digits after the dot in a binary expansion of x from the $(k_m + 1)$ th position to the $(k_m + l)$ th position are all equal.

Corollary 1. A number $x \in A$ is rational if and only if there exists $a \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ such that $x = a/2^k$.

Proof. From Proposition [1,](#page-2-2) we see that the ring A only contains those rational numbers which have a finite binary expansion. The result follows. \Box

Remark 1. The above construction of an uncountable subring of $\mathbb R$ of Hausdorff dimension zero goes through if the initial set $S = \{0, 1, 2, 2^2, 2^3, \ldots\}$ is replaced by a set $T \cup \{0\}$ where T is a subset of N which has the following property: There exists $b \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, the cardinality of the set $\{\log t : t \in T\} \cap [n, n + 1)$ is less than b.

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