

Quantum Field Measurements in the Fewster-Verch Framework

Jan Mandrysch and Miguel Navascués

November 22, 2024

Abstract

The Fewster-Verch (FV) framework was introduced as a prescription to define local operations within a quantum field theory (QFT) that are free from Sorkin-like causal paradoxes. In this framework the measurement device is modeled via a probe QFT that, after interacting with the target QFT, is subject to an arbitrary local measurement. While the FV framework is rich enough to carry out quantum state tomography, it has two drawbacks. First, it is unclear if the FV framework allows conducting arbitrary local measurements. Second, if the probe field is interpreted as physical and the FV framework as fundamental, then one must demand the probe measurement to be itself implementable within the framework. That would involve a new probe, which should also be subject to an FV measurement, and so on. It is unknown if there exist non-trivial FV measurements for which such an “FV-Heisenberg cut” can be moved arbitrarily far away. In this work, we advance the first problem by proving that measurements of locally smeared fields fit within the FV framework. We solve the second problem by showing that any such field measurement admits a movable FV-Heisenberg cut.

1 Introduction

Quantum Field Theory (QFT) has proven very successful to model particle scattering experiments at high energies. However, as noted by Sorkin [1], the use of seemingly innocuous quantum operations within the QFT formalism would allow three or more separate parties to violate Einstein’s causality. The accepted resolution of Sorkin’s paradox is to acknowledge that the set of quantum operations which can be conducted within a finite region of space-time is certainly smaller than previously envisaged. This raises the question of how to model local operations within QFT.

This problem has been greatly advanced in recent years. In [2], Jubb studies which quantum channels (or update rules) do not lead to Sorkin-type violations of causality in QFT, a property that he calls strong causality. Among other things, he concludes that weak “Gaussian” measurements of locally smeared fields are strongly causal. Oeckl, who calls this property causal transparency, reaches the same conclusion in [3].

Independently, Fewster and Verch proposed in [4, 5] a general framework to model measurements in QFT. In a nutshell, the FV framework consists in making the ‘target’ field theory interact in a compact region with another ‘pointer’ or ‘probe’ field theory. Measurement outcomes are obtained by measuring the pointer in an arbitrary bounded region outside the causal past of the interaction region. The pointer is then discarded, i.e., one is not allowed to measure it a second time. Remarkably, all quantum operations achievable within the FV framework are strongly causal [6]. Moreover, they suffice to carry out full tomography when the target QFT is a scalar field [7].

The FV framework has some drawbacks, though:

1. Given the specification of a quantum measurement, or even the description of the corresponding quantum instrument, it is not straightforward to determine whether the FV framework allows implementing one or the other—even approximately. Doing so involves specifying a probe QFT, its initial state, the form of the interaction and the measurement on the probe, or proving that no such elements exist.
2. If we regard the probe QFT as a physical entity and not as part of a mere mathematical construction to obtain a set of strongly causal operations, then it is unclear how come that one is allowed to measure the probe arbitrarily; a standing assumption of the framework. Note that, if the probe does not magically disappear at the end of the measurement, then allowing arbitrary sequential measurements thereof would run again into Sorkin’s paradox. Consistency therefore demands that the probe measurement itself be modeled within the FV framework. That would involve introducing another probe, whose measurement should be realizable within the FV framework, and so on. In other words, any physical measurement should be expressible as an arbitrarily long chain of consecutive interactions with independent probes, in such a way that we predict the same measurement statistics irrespective of where in the chain we invoke Born’s rule.

In non-relativistic quantum mechanics, the specific probe or pointer within a measurement chain where one applies the collapse postulate is known as “Heisenberg cut”. That the position of the cut can be chosen at will was already observed by von Neumann [8]. What we have argued above is that, if the FV framework is fundamental and its probe field is meant to be physical, then feasible QFT measurements should have a similarly movable “FV-Heisenberg cut”. Which brings us to the question: are there non-trivial FV-realizable measurements with a movable FV-Heisenberg cut?

In this paper, we address the first problem by proving that Gaussian-modulated measurements of a locally smeared quantum field can be modeled within the FV framework. As it turns out, the probe measurement that induces a Gaussian-modulated field measurement can itself be chosen a Gaussian-modulated field measurement. This solves the second problem: Gaussian field measurements are consistent with the FV framework, even if we accept the physicality of the probe field. As

a byproduct, we establish that also projective field measurements can be modeled within the FV framework.

The structure of this paper is as follows. In Section 2 we introduce how to model general as well as Gaussian-modulated measurements in QFT, distinguishing between positive-operator-valued measures (POVMs) and quantum instruments. We also specify the linear QFT setting to which our results apply. In Section 3 we summarize the FV framework and define FV-realizable POVMs and instruments. Finally, in Section 4 we prove our results by presenting a concrete scheme to induce Gaussian-modulated measurements, projective measurements and more—even iteratively.

2 General quantum operations in QFT

We will be dealing with the standard framework of AQFT: namely, starting from a fixed globally hyperbolic spacetime manifold \mathcal{M} we associate to any region $R \subset \mathcal{M}$ a unital $*$ -algebra $\mathcal{A}(R)$. These local algebras, which define our QFT, satisfy the usual axioms: isotony, microcausality, time-slice and Haag’s property; confer [5, Sec. 4] and references therein for details. States of this QFT are linear, positive functionals $\omega : \mathcal{A} \rightarrow \mathbb{C}$ on the global algebra $\mathcal{A} := \mathcal{A}(\mathcal{M})$ with $\omega(1) = 1$. An element $a \in \mathcal{A}$ is said to be localizable in region $R \subset \mathcal{M}$ if $a \in \mathcal{A}(R)$. To express limits, we assume \mathcal{A} and all $\mathcal{A}(R)$ are equipped and completed with respect to a suitable topology. Specifically, we work with von-Neumann algebras; for some fixed Hilbert space \mathcal{H} these are $*$ -subalgebras of $\mathcal{B}(\mathcal{H})$ closed under the strong-operator topology. Limits, infinite sums and integrals pertaining to \mathcal{A} will be taken with respect to this topology. Note that in this setting any density matrix, i.e., $\rho \in \mathcal{B}(\mathcal{H})$ such that $\rho \geq 0$ and $\text{tr } \rho = 1$, gives rise to a state ω_ρ on \mathcal{A} via $\omega_\rho(a) := \text{tr}(\rho a)$.

We wish to model local quantum operations and instruments within this QFT. A quantum instrument is defined by a family of completely positive maps $\Omega := \{\Omega_b\}_{b \in \mathcal{B}}$, where $\Omega_b : \mathcal{A} \rightarrow \mathcal{A}$ is such that

$$\bar{\Omega}(a) := \sum_b \Omega_b(a) \quad \text{satisfies} \quad \bar{\Omega}(1) = 1; \quad (1)$$

assuming for the moment that the index set \mathcal{B} is finite to avoid technicalities. The map $\bar{\Omega} : \mathcal{A} \rightarrow \mathcal{A}$ is called the *measurement channel*. Intuitively, if our QFT is initially in state ω , the instrument $\{\Omega_b\}_b$ describes a measurement that returns an outcome b with probability

$$p(b|\omega) = \omega(\Omega_b(1)), \quad (2)$$

in which case the new state of the QFT is updated to

$$\bar{\omega}_b := \frac{1}{\omega(\Omega_b(1))} \omega \circ \Omega_b. \quad (3)$$

If we ignore or discard the measurement outcome b , then the new QFT state will be

$$\bar{\omega} := \sum_b p(b|\omega) \bar{\omega}_b = \omega \circ \bar{\Omega}. \quad (4)$$

If we are just interested in computing the probabilities of the different outcomes b , it suffices to work with the algebra elements

$$M_b := \Omega_b(1). \quad (5)$$

Then we have that $p(b|\omega) = \omega(M_b)$. Note that the elements $M := \{M_b\}_b$ satisfy

$$\sum_b M_b = 1 \quad \text{and} \quad \forall b : M_b \geq 0. \quad (6)$$

Any such family M is called a positive operator-valued measure (POVM).

The formalism above can also be used to model measurements with infinitely many outcomes. In that case, we distinguish whether \mathcal{B} is discrete or continuous. In the discrete case, we assume that $\Omega_b(a)$ is summable in b for all $a \in \mathcal{A}$ which implies the existence of the measurement channel $\bar{\Omega}(a) := \sum_{b \in \mathcal{B}} \Omega_b(a)$ as a limit of partial sums. In the continuous case, we suppose the set of outcomes \mathcal{B} to be a Borel set within a standard Borel space, e.g. an interval on the real line, and that $\Omega_b(a)$ is integrable in b for all $a \in \mathcal{A}$. This implies the existence of the measurement channel, $\bar{\Omega}(a) := \int_{\mathcal{B}} \Omega_b(a) db$ and we replace \sum_b by $\int_{\mathcal{B}} \bullet db$ whenever it appears; in particular within (1), (4) and (6).

We next illustrate these notions with an example in the setting of linear scalar QFT. In this setting the *smeared field* $\Phi(f)$ corresponds to the quantization of

$$\phi(f) = \int_{\mathcal{M}} \phi(x) f(x) dx =: \langle \phi, f \rangle, \quad (7)$$

a classical point field ϕ satisfying a classical linear differential equation $T\phi(x) = 0$ and smeared over spacetime through a test function f using dx , the volume form on \mathcal{M} . We assume that T is symmetric with respect to $\langle \cdot, \cdot \rangle$ and normally hyperbolic, which implies it has well-defined retarded/advanced Green operators E^\pm and an associated commutator function $E = E^- - E^+$ [9]. This holds e.g. for the Klein-Gordon operator $T = \square_{\mathcal{M}} - m^2$, $m > 0$. We denote the space of real-valued smooth compactly supported functions on \mathcal{M} by $\mathcal{D}(\mathcal{M})$ and by $\mathcal{D}(R)$ if restricting to functions having support in $R \subset \mathcal{M}$. With ϕ having spatially compact support, it is natural to identify $Tf \sim 0$ for all $f \in \mathcal{D}(\mathcal{M})$ since $\phi(Tf) = \langle T\phi, f \rangle = 0$ by partial integration. We denote the resulting ‘‘classical’’ algebra by $\mathcal{C}_T := \mathcal{D}(\mathcal{M})/T\mathcal{D}(\mathcal{M})$ and say that $f \in \mathcal{D}(\mathcal{M})$ is *localizable in* $R \subset \mathcal{M}$, or equivalently $[f] \in \mathcal{C}_T(R)$, if there exists $g \in \mathcal{D}(R)$ such that $f \sim g$.

For now, we work in a Hilbert space representation and make these assumptions: The *smeared field* Φ is a real-linear map from \mathcal{C}_T to selfadjoint operators on a Hilbert space \mathcal{H} such that

$$\text{[Continuity]} \quad e^{it\Phi(f)} \text{ is strongly operator continuous in } t, \quad (8)$$

$$\text{[Weyl relation]} \quad e^{i\Phi(f)} e^{i\Phi(g)} = e^{-\frac{i}{2}\langle f, Eg \rangle} e^{i\Phi(f+g)}, \quad (9)$$

for all $f, g \in \mathcal{D}(\mathcal{M})$, where we agree to $\Phi(f) := \Phi([f])$ by abuse of notation. Note here that the Weyl relation is a strong and covariant form of the canonical commutation relations. Finally, we obtain a net of von-Neumann algebras $\{\mathcal{A}(R)\}_R \subset \mathcal{B}(\mathcal{H})$

labelled by open causally convex regions $R \subset \mathcal{M}$ by defining $\mathcal{A}(R)$ to be the algebra generated by the *Weyl operators* $\{e^{i\Phi(f)} : f \in \mathcal{D}(R)\}$, equipped and completed with respect to the strong operator topology of $\mathcal{B}(\mathcal{H})$. These local algebras are known to satisfy the usual axioms as listed above.

Moreover, by means of the spectral theorem, for any bounded Borel function $\mu : \mathbb{R} \rightarrow \mathbb{C}$ expressions of the form $\mu(\Phi(f))$ define elements of $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ in terms of (Borel) functional calculus. From this perspective, most statements made in the remainder of this section are direct consequences of the properties of the functions μ , and we defer proofs to Appendix A. Relations to the algebraic setting of QFT are given in Appendix B.

Now, we are ready to discuss instruments and POVMs in this QFT. For $f \in \mathcal{D}(\mathcal{M})$ and $\epsilon > 0$, consider the following continuous-outcome POVM on \mathcal{A} ,

$$M(f, \epsilon) := \{M_b(f, \epsilon)\}_{b \in \mathbb{R}}, \quad M_b(f, \epsilon) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\Phi(f)-b)^2}{2\epsilon^2}}. \quad (10)$$

We will call this POVM a *Gaussian-modulated measurement of the field* $\Phi(f)$, or Gaussian measurement, for short. It can be interpreted as a weak measurement of the field $\Phi(f)$, and indeed one can verify that, in the limit $\epsilon \rightarrow 0$, it corresponds to a projective measurement of the operator $\Phi(f)$ meaning that for any Borel set $B \subset \mathbb{R}$ and with $\Pi_B(A)$ denoting the spectral projection of a selfadjoint operator A onto B ,

$$\lim_{\epsilon \rightarrow 0} \int_B M_b(f, \epsilon) db = \Pi_B(\Phi(f)). \quad (11)$$

Note that there exist many inequivalent instruments realizing the Gaussian POVM (10). One of them is:

$$\Omega_b^\epsilon(\bullet) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\Phi(f)-b)^2}{4\epsilon^2}} \bullet e^{-\frac{(\Phi(f)-b)^2}{4\epsilon^2}}. \quad (12)$$

This instrument appears in the literature of quantum optics and quantum foundations [10, 11]. Its causality properties have been recently studied in QFT, see [2] and [3].

Now, for any $\delta > 0$, define the *dephasing map*

$$D^\delta(\bullet) := \frac{1}{\sqrt{2\pi\delta}} \int e^{-\frac{\nu^2}{2\delta^2}} e^{-i\nu\Phi(f)} \bullet e^{i\nu\Phi(f)} d\nu. \quad (13)$$

This map is completely positive and unit preserving. Intuitively, it introduces decoherence between the generalized eigenstates of $\Phi(f)$.

It is easy to verify that the instrument

$$\Omega^{\epsilon, \delta} := \{\Omega_b^{\epsilon, \delta}\}_{b \in \mathbb{R}}, \quad \Omega_b^{\epsilon, \delta} := \Omega_b^\epsilon \circ D^\delta = D^\delta \circ \Omega_b^\epsilon \quad (14)$$

also induces the POVM (10) for any $\delta > 0$. This example will have a prominent appearance in our measurement scheme and illustrates that different instruments might induce the same POVM.

The instrument $\Omega^{\epsilon, \delta}$ is fully characterized by its action on Weyl operators. In this regard, by a straightforward computation it follows that

$$\Omega_b^{\epsilon, \delta}(e^{i\Phi(h)}) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{\langle f, Eh \rangle^2}{8\epsilon(\epsilon, \delta)^2}} e^{i\Phi(h)} e^{-\frac{(\Phi(f) - b - \frac{\langle f, Eh \rangle}{2})^2}{2\epsilon^2}}, \quad \underline{\epsilon}(\epsilon, \delta)^2 := \frac{1}{4\delta^2 + \frac{1}{\epsilon^2}} \quad (15)$$

for arbitrary $h \in \mathcal{D}(\mathcal{M})$. Integrating over b , we arrive at an expression for the measurement channel of instrument (14), namely:

$$\bar{\Omega}^{\epsilon, \delta}(e^{i\Phi(h)}) = e^{-\frac{\langle f, Eh \rangle^2}{8\epsilon(\epsilon, \delta)^2}} e^{i\Phi(h)}. \quad (16)$$

Hence, the instruments $\Omega^{\underline{\epsilon}(\epsilon, \delta), 0}$ and $\Omega^{\epsilon, \delta}$, despite giving rise to different POVMs, share the same measurement channel.

We conclude this section with some words on tomography. Any family of POVMs $\{M^l\}_l$ with elements spanning the Hermitian part of \mathcal{A} allows one to completely fix the underlying state ω . That is, for any other state ω' on \mathcal{A} , the condition

$$\forall b, l: \quad \omega(M_b^l) = \omega'(M_b^l) \quad (17)$$

implies that $\omega = \omega'$. In that case, we say that the family $\{M^l\}_l$ is *tomographically complete*.

Being able to implement a tomographically complete family of POVMs does not necessarily mean being able to implement, even approximately, any possible POVM. For instance, suppose that, for fixed $\epsilon > 0$, we can measure all POVMs of the form $\{M(f, \epsilon) : f \in \mathcal{D}(\mathcal{M}), \langle Ef, \alpha Ef \rangle = 1\}$, for some positive $\alpha \in L^1(\mathcal{M})$. Now, take $f \in \mathcal{D}(\mathcal{M})$, with $\langle Ef, \alpha Ef \rangle = 1$. Suppose for an ensemble of independent preparations of ω that we measure $M(f, \epsilon)$ and, given the result b , compute $e^{i\lambda b}$, $\lambda \in \mathbb{R}$. Then, the resulting random variable $\beta(f, \lambda)$ has the expectation value

$$\langle \beta(f, \lambda) \rangle = \frac{1}{\sqrt{2\pi\epsilon}} \int \omega \left(e^{-\frac{(\Phi(f) - b)^2}{2\epsilon^2}} \right) e^{i\lambda b} db = e^{-\frac{\epsilon^2 \lambda^2}{2}} \omega(e^{i\lambda \Phi(f)}). \quad (18)$$

Since the linear span of all Weyl operators yields a dense subset of \mathcal{A} , it follows that the given family of POVMs is tomographically complete. Now, consider the projectors

$$\Pi_0 = \Pi_{[-1, 1]}(\Phi(f)), \quad \Pi_1 = 1 - \Pi_0, \quad (19)$$

which form the POVM $\Pi := \{\Pi_0, \Pi_1\}$. Then we have that

$$\frac{1}{\pi} \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} \text{sinc}(\lambda) e^{\frac{\epsilon^2 \lambda^2}{2}} \langle \beta(f, \lambda) \rangle d\lambda = \omega(\Pi_0). \quad (20)$$

That is, given the statistics obtained by measuring independent preparations of ω with $M(f, \epsilon)$, one can estimate with which probability one would have observed outcome 0 (or 1) had we measured Π .

This is, however, not the same as measuring the POVM Π . Indeed, by measuring Π *once*, one can distinguish with certainty if the system was prepared in state $\omega = \omega_\rho$ with $\rho = |\psi_0\rangle\langle\psi_0|$ or $\rho = |\psi_1\rangle\langle\psi_1|$ for any two vectors $|\psi_0\rangle, |\psi_1\rangle$ with $\Pi_j |\psi_j\rangle = |\psi_j\rangle$. In contrast, by measuring $M(f, \epsilon)$, one cannot tell with certainty which of the two states was prepared. Depending on the value of ϵ one would need to measure many independent preparations of ω with $M(f, \epsilon)$ to reach a statistically significant conclusion.

3 The FV framework

It is tempting to think that any instrument of the form $\Omega_b(\bullet) := \sum_i a_{i,b} \bullet a_{i,b}^*$, with $\{a_{i,b}\}_{i,b} \subset \mathcal{A}(R)$ and $\sum_{i,b} a_{i,b} a_{i,b}^* = 1$ can be physically implemented by solely acting on region R or, at most, its causal hull. However, as shown by Sorkin [1], doing so leads to contradictions with Einstein’s causality in scenarios with three or more separate experimenters. For instance $\Omega_j(\bullet) := \Pi_j \bullet \Pi_j$ with POVM Π_j as defined in (19) would be acausal. We thus need to impose further restrictions to arrive at a definition of local instruments that does not clash with causality.

Recently, Fewster and Verch proposed a set quantum instruments that are consistent with Einstein’s causality and moreover sufficiently general to include measurements [4, 5]. Their main idea was to model the measurement of a ‘target’ QFT observable by making it interact with a ‘pointer’ or ‘probe’ QFT in a region K . The probe field is measured using a POVM with elements localized in some processing region L strictly separated from the past of K . After discarding (‘tracing out’) the probe field, the resulting quantum instrument induced in the target theory is shown to be localizable in any causally complete region strictly containing K . Moreover, the composition of such instruments is compatible with Einstein’s causality [6].

More formally, call \mathcal{S} the target QFT and \mathcal{P} the probe QFT, which we assume to be in state σ . The coupling between system and probe is specified in terms of an automorphism Θ acting on the uncoupled theory $\mathcal{S} \otimes \mathcal{P}$ implementing the interaction—referred to as the *scattering morphism*. A key assumption is that the *coupling region* $K \subset \mathcal{M}$ associated with Θ is compact, both in space and time, which allows the identification of the coupled theory, say \mathcal{C} , with the uncoupled one, $\mathcal{S} \otimes \mathcal{P}$, in the so-called ‘in’ and ‘out’ regions $K^\pm := \mathcal{M} \setminus J^\mp(K)$, where $J^\pm(R)$ denotes the causal future/past of a region $R \subset \mathcal{M}$. The *processing region* L is assumed to be precompact such that $\bar{L} \subset K^+$.

If we now implement a POVM $\{M'_b\}_b \subset \mathcal{P}(L)$ on the probe and then discard it, this induces an instrument Ω and a POVM M on the target theory \mathcal{S} . Defining $\eta_\sigma : \mathcal{S} \otimes \mathcal{P} \rightarrow \mathcal{S}$ by linear and continuous extension of $A \otimes B \mapsto \sigma(B)A$ to ‘trace out’ the probe degrees of freedom, the instrument Ω is given by linear maps $\Omega_b : \mathcal{S} \rightarrow \mathcal{S}$,

$$\Omega_b(a) := \eta_\sigma(\Theta(a \otimes M'_b)), \quad a \in \mathcal{S}, \quad (21)$$

where it is easy to verify that these are completely positive and $\sum_b \Omega_b(1) = 1$. Moreover, the induced POVM $M := \{\Omega_b(1)\}$ can be localized in any open causally convex connected region containing K .

One can regard the probe in the FV framework as a mere mathematical artifact to arrive at a well-behaved set of local QFT operations. Alternatively, one could postulate that such a (perhaps, effective) QFT is at the root of any measurement we currently conduct in the lab. In that second case, the measurement of the probe should likewise be realizable within the FV framework. That would require introducing another probe, which in turn should be measured with another probe, and so on.

Now, let us call \mathbb{M}_1 the set of POVMs that can be realized within the FV framework (i.e., of the form M for some measurement scheme) and define the sets of POVMs $\{\mathbb{M}_k\}_k$ by induction: A POVM is in \mathbb{M}_k if it admits an FV realization

such that the probe is subject to a measurement in \mathbb{M}_{k-1} . Analogously, we define \mathbb{I}_k to be the set of instruments which admit an FV realization such that the probe is subject to a measurement in \mathbb{M}_{k-1} . One wonders if there exist non-trivial POVMs M or instruments I —namely, with informative measurement outcomes—such that $M \in \mathbb{M}_k$ or $I \in \mathbb{I}_k$ for all k . Since some quantum operations seem only reachable as limits of other FV-realizable quantum operations, it will be convenient to work with an asymptotic notion of the sets $\{\mathbb{M}_k\}_k$ and $\{\mathbb{I}_k\}_k$.

Definition 3.1. A POVM M on a QFT \mathcal{A} belongs to $\overline{\mathbb{M}}_k$ if there exists a sequence of instruments POVMs $(M^j)_j \subset \mathbb{M}_k$ such that $\lim_{j \rightarrow \infty} M_b^j = M_b$ for all $b \in \mathcal{B}$. Analogously, an instrument I belongs to $\overline{\mathbb{I}}_k$ if there exists a sequence of instruments $(I^j)_j \subset \mathbb{I}_k$ such that $\lim_{j \rightarrow \infty} I_b^j(a) = I_b(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

A problem with the FV framework is that the set of allowed quantum operations is defined implicitly: namely, any instrument of the form (21) is FV-realizable. However, given an instrument $\Omega = \{\Omega_b\}_b$ on the target theory, the problem of deciding whether Ω admits an FV representation is far from trivial: it amounts to determining if there exist a probe theory \mathcal{P} , a probe state σ , a scattering morphism Θ and a POVM $\{M_b\}_b$ such that (21) holds—this we call a *measurement scheme* for Ω . The problem of determining if a given POVM $\{M_b\}_b$ is realizable within the FV framework is similarly challenging. In [7], Fewster, Jubb and Ruet prove that, for the case of a scalar field, the set of FV-realizable instruments is rich enough to carry out state tomography: namely, it allows one to estimate all parameters defining the QFT state of the target theory. As illustrated at the end of the previous section, this is not the same as proving that any POVM can be realized within the FV framework. This begs the question: Can we characterize a class of relevant quantum instruments that are FV-realizable?

In the next section, we will tackle this last question by proving that, in the free scalar theory, Gaussian-modulated local field measurements are asymptotically FV-realizable. Remarkably, the required probe measurement is also a Gaussian field measurement and thus the set of Gaussian field measurements belongs to $\overline{\mathbb{M}}_k$ for all $k \in \mathbb{N}$. Similarly, we will prove that, for arbitrary $\epsilon > 0$ and some range of values of $\delta > 0$, any instrument of the form $\Omega^{\epsilon, \delta}$ belongs to $\overline{\mathbb{I}}_k$ for all $k \in \mathbb{N}$.

4 Implementation of Gaussian measurements within the FV framework

We consider two QFTs, a target \mathcal{S} and a probe \mathcal{P} , induced by, for simplicity the same, classical equation of motion $T\phi(x) = 0$, respectively, $T\psi(x) = 0$, as described in Section 2. We denote the associated quantized fields by Φ for the target and Ψ for the probe. To implement the POVM (10) on \mathcal{S} corresponding to a Gaussian measurement of the target field mode f , we will use weak linear interactions and carefully chosen squeezed probe states.

Following Fewster and Verch [4, Sec. 4], we define a linear and local interaction of strength $\lambda\beta$ between the probe and target field where λ is a real parameter and β a real-valued smooth function on \mathcal{M} with support in a compact region $K \subset \mathcal{M}$.

More precisely, we consider the interaction term

$$\lambda \int_K dx \beta(x) \phi(x) \psi(x), \quad (22)$$

where dx denotes the volume form on \mathcal{M} . This gives rise to a coupled equation of motion for the combined theory,

$$T_\lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0, \quad T_\lambda := \begin{pmatrix} T & \lambda\beta \\ \lambda\beta & T \end{pmatrix}, \quad (23)$$

which has well-defined Green operators for arbitrary $\lambda \geq 0$.

At the QFT level, this interaction generates a scattering morphism of the form

$$\Theta_\lambda (e^{i(\Phi(u)+\Psi(v))}) = e^{i(\Phi(u_\lambda)+\Psi(v_\lambda))}, \quad u, v \in \mathcal{D}(\mathcal{M}); \quad (24)$$

supposing $u, v \in \mathcal{D}(K^+)$ we have the explicit formula

$$\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} = \theta_\lambda \begin{pmatrix} u \\ v \end{pmatrix}, \quad \theta_\lambda := \left(\mathbb{1} - \lambda \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} E_\lambda^- \right), \quad (25)$$

where E_λ^- is the retarded Green operator associated with T_λ . For all such u, v , we find from perturbation theory that

$$\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} = \begin{pmatrix} u - \lambda\beta E^- v + O(\lambda^2) \\ v - \lambda\beta E^- u + O(\lambda^2) \end{pmatrix}, \quad (26)$$

where E^- is the retarded Green operator associated with T .

We next use ideas from the tomographic asymptotic scheme presented in [7]. There it is shown that:

Fact 4.1. *For all $f \in \mathcal{D}(R)$ localizable in a precompact region $R \subset \mathcal{M}$ and all processing regions $L \subset \overline{R}^+$ with $R \subset D^-(L)$, we find a coupling zone K in R , a probe test function $g \in \mathcal{D}(L)$ and an interaction strength $\beta \in \mathcal{D}(K)$ such that*

$$f = -\beta E^- g. \quad (27)$$

Fixing R, L as well as f, β and g as above and considering arbitrary $h \in \mathcal{D}(L)$, we define the functions $f_\lambda, g_\lambda, h_\lambda, p_\lambda$ through the identities:

$$\begin{pmatrix} \lambda f_\lambda \\ g_\lambda \end{pmatrix} = \theta_\lambda \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad \begin{pmatrix} h_\lambda \\ \lambda p_\lambda \end{pmatrix} = \theta_\lambda \begin{pmatrix} h \\ 0 \end{pmatrix}. \quad (28)$$

By (26) we have that $\lim_{\lambda \rightarrow 0} g_\lambda = g$, $\lim_{\lambda \rightarrow 0} h_\lambda = h$ and $\lim_{\lambda \rightarrow 0} f_\lambda = f$. Thus, by analogy we define

$$p := \lim_{\lambda \rightarrow 0} p_\lambda = -\beta E^- h. \quad (29)$$

We remark that

$$\langle p, Eg \rangle = \langle f, Eh \rangle, \quad (30)$$

since

$$\begin{aligned} \langle p, Eg \rangle &= \langle p, (E^- - E^+)g \rangle = \langle p, E^-g \rangle \\ &= -\langle \beta E^-h, E^-g \rangle = -\langle E^-h, \beta E^-g \rangle = \langle E^-h, f \rangle = \langle Eh, f \rangle = \langle f, Eh \rangle, \end{aligned} \quad (31)$$

where we used in the first line that $\text{supp } p \subset K$ while $\text{supp } E^+g \subset J^+(\text{supp } g) = J^+(L)$ such that $\langle p, E^+g \rangle = 0$ and in the second line that $\langle f, E^+h \rangle = 0$ by an analogous argument.

Next, we assume that $[f] \in \mathcal{C}_T$ is nonzero, which implies that (30) is nonzero. It also follows that $[g]$ is nonzero: If we had $[g] = 0$ then $f = -\beta E^-g = -\beta(E + E^+)Tg' = 0$ since $ET\mathcal{D}(\mathcal{M}) = 0$ and $\text{supp } \beta E^+Tg' \subset \text{supp } \beta \cap J^+(\text{supp } Tg') = K \cap J^+(L) = \emptyset$. Since $[g]$ is nonzero, using that E is nondegenerate on \mathcal{C}_T , we find a *canonical conjugate* to g , i.e., $\bar{g} \in \mathcal{C}_T$ such that $\langle g, E\bar{g} \rangle = 1$. Similarly, we find a canonical conjugate to g_λ : Since $\langle g_\lambda, E\bar{g} \rangle$ is real and tends to 1 for $\lambda \rightarrow 0$, there is an $\lambda_0 > 0$ such that $\langle g_\lambda, E\bar{g} \rangle > 0$ for all $\lambda \in [0, \lambda_0]$. Thus, defining

$$\bar{g}_\lambda := \frac{1}{\langle g_\lambda, E\bar{g} \rangle} \bar{g} \quad (32)$$

yields $\langle g_\lambda, E\bar{g}_\lambda \rangle = 1$ for all $\lambda \in [0, \lambda_0]$.

Now, let us choose the initial state of the probe. A state σ on \mathcal{P} is referred to as *quasi-free* iff there exists a bilinear symmetric form $\Gamma : \mathcal{C}_T^{\times 2} \rightarrow \mathbb{R}$ such that

$$\sigma(e^{i\Psi(q)}) = e^{-\frac{\Gamma(q,q)}{2}}, \quad q \in \mathcal{C}_T. \quad (33)$$

Conversely, (33) defines a state on \mathcal{P} iff

$$\Gamma(q, q)\Gamma(r, r) \geq \frac{1}{4}E(q, r)^2, \quad q, r \in \mathcal{C}_T. \quad (34)$$

We refer to such Γ as *covariance*. At least for stationary spacetimes \mathcal{M} there are many such covariances, resp. states, including ground and thermal states [12].

We will need probe states (or covariances) with specific properties. To construct them, we employ symplectic transformations of our underlying phase space (\mathcal{C}_T, E) , in particular the *squeezing transformation*, and use the following two facts. For completeness, we provide a proof of the first statement in Appendix C and refer to end of Appendix B for the second statement.

Fact 4.2. *Let (X, σ) be a symplectic space and S an arbitrary finite-dimensional symplectic subspace with symplectic complement S^\perp . Given a symplectic map $F : S \rightarrow S$, there is a unique extension of F to a symplectic map on X which acts like the identity map on S^\perp . For any standard basis (u, J) of S , i.e., $J_{lm} := \sigma(u_l, u_m)$, $l, m = 1, \dots, 2n$ with $u = (r_1, s_1, \dots, r_n, s_n)$ such that $\sigma(r_j, r_k) = \sigma(s_j, s_k) = 0$ and $\sigma(r_j, s_k) = \delta_{jk}$ for all $j, k = 1, \dots, n$ the extension is given by*

$$Fv := F(v - v^\perp) + v^\perp, \quad v^\perp := v - \sum_{l,m=1}^{2n} \sigma(v, J_{lm}u_m)u_l. \quad (35)$$

We refer to it as the *canonical extension* of F .

Fact 4.3. *Let \mathcal{A} be a QFT induced by T as described in Section 2. Then, given a symplectic transformation F of (\mathcal{C}_T, E) ,*

$$\alpha_F(e^{i\Phi(q)}) := e^{i\Phi(Fq)}, \quad q \in \mathcal{C}_T \quad (36)$$

fixes uniquely an automorphism α_F on \mathcal{A} since it preserves the Weyl relation and by linear and continuous extension. For any state ω on \mathcal{A} it hence follows that $\omega_F(\bullet) := \omega(\alpha_F(\bullet))$ also defines a state on \mathcal{A} . Moreover, if ω is quasi-free with covariance Γ , then ω_F is quasi-free with covariance $\Gamma_F(q, r) := \Gamma(Fq, Fr)$ for all $q, r \in \mathcal{C}_T$.

For technical reasons, we need a quasi-free state σ on \mathcal{P} with covariance Γ satisfying

$$\Gamma(\bar{g}, g) = 0, \quad \Gamma(g, g) = \Gamma(\bar{g}, \bar{g}) =: c^2. \quad (37)$$

This is easily achieved: We start with an arbitrary quasi-free state σ_0 on \mathcal{P} with covariance Γ_0 . In view of Facts 4.2 and 4.3 the symplectic transformation fixed by $g \mapsto \bar{g}$, $\bar{g} \mapsto -g$ and canonical extension to \mathcal{C}_T induces another quasi-free state $\tilde{\sigma}_0$ on \mathcal{P} . Its covariance $\tilde{\Gamma}_0$ satisfies in particular that $\tilde{\Gamma}_0(g, \bar{g}) = -\Gamma_0(g, \bar{g})$ and the form

$$\Gamma := \frac{1}{2}(\Gamma_0 + \tilde{\Gamma}_0) \quad (38)$$

clearly defines a covariance satisfying $\Gamma(g, g) = \frac{1}{2}(\Gamma_0(g, g) + \Gamma_0(\bar{g}, \bar{g})) = \Gamma(\bar{g}, \bar{g})$ and $\Gamma(g, \bar{g}) = 0$. Thus there exists a quasi-free state σ with covariance Γ satisfying (37).

Now, for any $\mu, \lambda > 0$, by Fact 4.2 the linear map

$$F_{\lambda s} := -\lambda\mu\langle\bar{g}_\lambda, Es\rangle g_\lambda + \frac{1}{\lambda\mu}\langle g_\lambda, Es\rangle\bar{g}_\lambda + s_\lambda^\perp \quad (39)$$

defines a symplectic map, squeezing the mode pair $(g_\lambda, \bar{g}_\lambda)$. Following Fact 4.3, we define the states

$$\sigma_\lambda := \sigma \circ \alpha_{F_\lambda} \quad (40)$$

on \mathcal{P} , which are also quasi-free.

We have provided an interaction and a probe state. The last element we need to specify in the FV measurement scheme is the POVM to be implemented on the probe. We choose it to be Gaussian and dependent on λ : For some $\epsilon > 0$,

$$M_b^\lambda := M_b(\lambda^{-1}g, \epsilon) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\frac{\Psi(g)}{\lambda} - b)^2}{2\epsilon^2}}. \quad (41)$$

We are ready to formulate our first result.

Lemma 4.4. *Consider the instrument $\{\Omega_b^\lambda\}_{b \in \mathbb{R}}$ defined through*

$$\Omega_b^\lambda(a) := \eta_{\sigma_\lambda} \Theta_\lambda(a \otimes M_b^\lambda), \quad a \in \mathcal{S}, \quad (42)$$

with $\Theta_\lambda, \sigma_\lambda, M_b^\lambda$ respectively defined as in (24), (40) (41) for all $\lambda > 0$ with implicit parameters $c, \epsilon, \mu > 0$. Then we have that

$$\lim_{\lambda \rightarrow 0} \Omega_b^\lambda(a) = \Omega_b^{\epsilon, \delta}(a), \quad a \in \mathcal{S}, \quad (43)$$

in particular, $\lim_{\lambda \rightarrow 0} \Omega_b^\lambda(1) = M_b(f, \epsilon')$, with

$$\epsilon' = \sqrt{\epsilon^2 + c^2 \mu^2}, \quad \delta^2 = \frac{c^2}{\mu^2} - \frac{1}{4(\epsilon^2 + c^2 \mu^2)}. \quad (44)$$

Remark 4.5. Due to (34), an analogue of Heisenberg's uncertainty relation, we have $c^2 \geq \frac{1}{2}$. So, even in the limit $\epsilon \rightarrow 0$, δ might be nonzero. More precisely, in that limit, δ vanishes iff $c^2 = \frac{1}{2}$, i.e., if the state has minimum uncertainty for the mode pair g, \bar{g} .

Proof. To characterize the instrument $\{\Omega_b^\lambda\}_b$, it suffices to study its action on Weyl operators $e^{i\Phi(h)}$ with arbitrary $h \in \mathcal{D}(\mathcal{M})$, i.e., we compute

$$\Omega_b^\lambda(e^{i\Phi(h)}) = \frac{1}{\sqrt{2\pi\epsilon}} \eta_{\sigma_\lambda} \left(\Theta_\lambda \left(e^{i\Phi(h)} e^{-\frac{(\frac{\Psi(g)}{\lambda} - b)^2}{2\epsilon^2}} \right) \right) \quad (45)$$

Using from (68) in Appendix A that

$$\frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\frac{\Psi(g)}{\lambda} - b)^2}{2\epsilon^2}} = \int e^{-\frac{\epsilon^2 z^2}{2}} e^{iz(\frac{\Psi(g)}{\lambda} - b)} \frac{dz}{2\pi} \quad (46)$$

we first compute

$$\eta_{\sigma_\lambda} \left(\Theta_\lambda \left(e^{i\Phi(h)} e^{iz(\frac{\Psi(g)}{\lambda} - b)} \right) \right) \quad (47)$$

Since for any $h \in \mathcal{D}(K^+)$ we find a processing region L (subject to the constraints from above) with $\text{supp } h \subset L$ and since $\mathcal{C}_T(K^+) = \mathcal{C}_T$ (by classical timeslice, confer e.g. [13, Thm. 3.3.1]), we assume $h \in \mathcal{D}(L)$ without loss of generality. This allows us to evaluate the action of Θ_λ by (24) and (25). We obtain:

$$\begin{aligned} & \eta_{\sigma_\lambda} \left(\Theta_\lambda \left(e^{i\Phi(h)} e^{iz(\frac{\Psi(g)}{\lambda} - b)} \right) \right) \\ &= \eta_{\sigma_\lambda} \left(e^{i(\Phi(h_\lambda) + \lambda \Psi(p_\lambda))} e^{iz(\frac{\Psi(g_\lambda)}{\lambda} + \Phi(f_\lambda) - b)} \right) \\ &= e^{i\Phi(h_\lambda)} e^{i(\Phi(f_\lambda) - b - \frac{1}{2}\langle p_\lambda, E g_\lambda \rangle)z} \sigma_\lambda \left(e^{i(\Psi(\frac{g_\lambda}{\lambda} z + \lambda p_\lambda))} \right) \\ &= e^{i\Phi(h_\lambda)} e^{i(\Phi(f_\lambda) - b - \frac{1}{2}\langle p_\lambda, E g_\lambda \rangle)z} \sigma \left(e^{i\Psi(\mu g_\lambda(z - \lambda^2 \langle \bar{g}_\lambda, E p_\lambda \rangle) + \frac{\bar{g}_\lambda}{\mu} \langle g_\lambda, E p_\lambda \rangle + \lambda p_\lambda^\perp)} \right), \end{aligned} \quad (48)$$

Then we take (46) and act on it by the map $\eta_{\sigma_\lambda}(\Theta_\lambda(e^{i\Phi(h)} \bullet))$ which we can pull through the z -integral by linearity and continuity of the map. Inserting (48) we obtain

$$\begin{aligned} \Omega_b^\lambda(e^{i\Phi(h)}) &= e^{i\Phi(h_\lambda)} \\ & \cdot \int e^{-\frac{\epsilon^2 z^2}{2}} e^{i(\Phi(f_\lambda) - b - \frac{1}{2}\langle p_\lambda, E g_\lambda \rangle)z} \sigma \left(e^{i\Psi(\mu g_\lambda(z - \lambda^2 \langle \bar{g}_\lambda, E p_\lambda \rangle) + \frac{\bar{g}_\lambda}{\mu} \langle g_\lambda, E p_\lambda \rangle + \lambda p_\lambda^\perp)} \right) \frac{dz}{2\pi}. \end{aligned} \quad (49)$$

Next, let us take the limit $\lambda \rightarrow 0$ on (49). For this note that the integrand in (49) is majorized by $e^{-\frac{\epsilon^2 z^2}{2}}$ uniformly in λ . Thus, by dominated convergence, we

can take the limit inside the integral and obtain:

$$\begin{aligned}
& e^{i\Phi(h)} \int e^{-\frac{\epsilon^2 z^2}{2}} e^{i(\Phi(f)-b-\frac{1}{2}\langle p, Eg \rangle)z} \sigma \left(e^{i\Psi(\mu z g + \frac{\bar{g}}{\mu} \langle g, Ep \rangle)} \right) \frac{dz}{2\pi} \\
&= e^{i\Phi(h)} \int e^{-\frac{\epsilon^2 z^2}{2}} e^{i(\Phi(f)-b-\frac{1}{2}\langle p, Eg \rangle)z} e^{-\frac{c^2}{2} \left(\mu^2 z^2 + \frac{\langle g, Ep \rangle^2}{\mu^2} \right)} \frac{dz}{2\pi} \\
&= \frac{1}{\sqrt{2\pi(\epsilon^2 + c^2 \mu^2)}} e^{i\Phi(h)} e^{-\frac{(\Phi(f)-b-\frac{1}{2}\langle p, Eg \rangle)^2}{2(\epsilon^2 + c^2 \mu^2)}} e^{-\frac{c^2 \langle g, Ep \rangle^2}{2\mu^2}}, \tag{50}
\end{aligned}$$

with p defined as in (29).

Moreover, since $\text{supp } h \subset L$, we have by (30) that $\langle p, Eg \rangle = \langle f, Eh \rangle$ and insertion into (50) yields

$$\lim_{\lambda \rightarrow 0} \Omega_b^\lambda (e^{i\Phi(h)}) = \frac{1}{\sqrt{2\pi(\epsilon^2 + c^2 \mu^2)}} e^{-\frac{c^2 \langle f, Eh \rangle^2}{2\mu^2}} e^{i\Phi(h)} e^{-\frac{(\Phi(f)-b-\frac{1}{2}\langle f, Eh \rangle)^2}{2(\epsilon^2 + c^2 \mu^2)}}. \tag{51}$$

By comparison with (15) we arrive at the statement of the lemma. \square

Note that the POVM $M(f, \epsilon')$ associated with $\Omega^{\epsilon', \delta}$ is independent of δ . Also, by decreasing the free parameters μ and ϵ we can make ϵ' as small as desired. This observation implies that $M(f, \epsilon)$ belongs to $\overline{\mathbb{M}}_1$ for arbitrary $\epsilon > 0$ which leads to our main result.

Theorem 4.6. *Given a QFT \mathcal{S} induced by T as described in Section 2 which admits a quasi-free state. Let $\bar{\epsilon} > 0$ and $f \in \mathcal{C}_T$ be arbitrary. Then, the POVM $M(f, \bar{\epsilon})$ belongs to $\overline{\mathbb{M}}_k$ for all $k \in \mathbb{N}$. Moreover, the FV framework can model the joint interaction of all the probes involved.*

Proof. The proof is easy, but cumbersome, so we omit some of the details. To begin with, we fix arbitrary $\bar{\epsilon} > 0$ and $k \in \mathbb{N}$ and suppose $j \in \{1, \dots, k\}$ throughout the proof. Also, we fix $f \in \mathcal{C}_T(R)$ for some precompact $R \subset \mathcal{M}$. We consider a target QFT \mathcal{S} whose field we denote by Φ and k probe QFTs $\{\mathcal{P}_j\}$ whose fields we denote by $\{\Psi_j\}$, all induced by the same equation of motion operator T .

1. We select interaction regions $\{K_j\}$ and processing regions $\{L_j\}$ within \mathcal{M} as well as probe modes $\{g^j\}$ and interaction strengths $\{\beta_j\}$ within $\mathcal{D}(\mathcal{M})$ suitable to induce the target mode f classically. In particular, we choose $\{L_j\}$ to be arbitrary precompact regions within \overline{R}^+ such that $L_j \subset \overline{L_{j-1}}^+$ and $L_{j-1} \subset D^-(L_j)$ for all j ; here $L_0 = R$. Then we invoke Fact 4.1 iteratively: Starting with $j = 1$ and $g^0 = f$, we find regions $K_j \subset L_{j-1}$, $g^j \in \mathcal{D}(L_j)$ and $\beta_j \in \mathcal{D}(K_j)$ such that

$$g^{j-1} = -\beta_j E^- g^j. \tag{52}$$

By construction, the regions K_j are compact and causally orderable, i.e., $K_{j+1} \subset K_j^+$ for all j .

2. We construct probe states $\{\sigma^j\}$ satisfying our technical condition (37). Note that for $[f] = 0$ within \mathcal{C}_T , the POVM $M(f, \bar{\epsilon})$ and the respective instrument $\Omega^{\bar{\epsilon}, \delta}$ become trivial, and are thus automatically within \mathbb{M}_k , resp., \mathbb{I}_k for all

$k \in \mathbb{N}$. Thus, we assume $[f] \neq 0$ which implies $[g^j] \neq 0$ for all j . Now, for each j , we find functions $\bar{g}^j \in \mathcal{C}_T(L_j)$ such that g^j, \bar{g}^j are canonically conjugate. We also define their λ -deformations $g_\lambda^j, \bar{g}_\lambda^j$ as in (28) and (32). Finally, given an arbitrary quasi-free state σ on \mathcal{P} , in analogy with the construction below (37), we construct quasi-free probe states σ^j with covariance matrices Γ^j satisfying

$$\Gamma^j(g^j, \bar{g}^j) = 0, \quad \Gamma^j(g^j, g^j) = \Gamma^j(\bar{g}^j, \bar{g}^j) =: c_j^2 \quad (53)$$

3. We construct squeezed probe states $\{\sigma_{\lambda_j}^j\}$. To do this, we choose $\epsilon, \mu_1, \dots, \mu_k > 0$ such that

$$\epsilon^2 + \sum_j c_j^2 \mu_j^2 = \bar{\epsilon}^2. \quad (54)$$

Let $\lambda'_j := \prod_{i=1}^j \lambda_i$. Then for each j and $\lambda_1, \dots, \lambda_j > 0$ we fix a symplectic transformation on (\mathcal{C}_T, E) by $F_{\lambda_1, \dots, \lambda_j}^j(g_{\lambda_j}^j) = \lambda'_j \mu_j g_{\lambda_j}^j$, $F_{\lambda_1, \dots, \lambda_j}^j(\bar{g}_\lambda^j) = \frac{1}{\lambda'_j \mu_j} \bar{g}_{\lambda_j}^j$ and canonical extension (Fact 4.2) and use it to define the states

$$\sigma_{\lambda_1, \dots, \lambda_j}^j := \sigma^j \circ \alpha_{F_{\lambda_1, \dots, \lambda_j}^j} \quad (55)$$

on \mathcal{P}_j (Fact 4.3). These will be the initial states of the probes and we may define $\sigma_{\vec{\lambda}} := \sigma_{\lambda_1}^1 \otimes \dots \otimes \sigma_{\lambda_1, \dots, \lambda_k}^k$ for later convenience.

4. We construct the interaction and represent it by the scattering morphism $\Theta_{\vec{\lambda}}$ on $\mathcal{S} \otimes \mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_k$. For each j , we consider the interaction term

$$\lambda_j \int_{K_j} dx \beta_j(x) \psi_{j-1}(x) \psi_j(x) \quad (56)$$

with $\psi_0 = \phi$. Then we define the corresponding scattering morphism $\Theta_{\lambda_j}^j$ on $\mathcal{P}_j \otimes \mathcal{P}_{j+1}$ with $\mathcal{P}_0 = \mathcal{S}$ in analogy with (24); its extension to a scattering morphism on $\mathcal{S} \otimes \mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_k$ is obtained by tensoring identity maps and we denote it by $\hat{\Theta}_{\lambda_j}^j$. Since the regions $\{K_j\}$ are causally orderable, the combination of all those interactions, gives rise to a scattering morphism $\Theta_{\vec{\lambda}}$ on $\mathcal{S} \otimes \mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_k$ which is causally factorizable, i.e.,

$$\Theta_{\vec{\lambda}} = \hat{\Theta}_{\lambda_1}^1 \circ \dots \circ \hat{\Theta}_{\lambda_k}^k; \quad (57)$$

confer e.g. [14].

5. Finally, we subject the k -th probe \mathcal{P}_k to a measurement given by some POVM M^k . Call $M_b^{\vec{\lambda}}$ the induced POVM on \mathcal{S} , i.e.,

$$M_b^{\vec{\lambda}} = \eta_{\sigma_{\vec{\lambda}}}(\Theta_{\vec{\lambda}}(1_k \otimes M_b)), \quad (58)$$

where 1_k denotes the identity on $\mathcal{S} \otimes \mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_{k-1}$. Then, (57) allows us to write

$$M_b^{\vec{\lambda}} = \iota_1 \circ \dots \circ \iota_k(M_b^k), \quad (59)$$

where $\iota_j : \mathcal{P}_j \rightarrow \mathcal{P}_{j-1}$ is given by

$$\iota_j(a) := \eta_{\sigma_{\lambda_j}^j} \left(\Theta_{\lambda_j}^j (1_{\mathcal{P}_{j-1}} \otimes a) \right), \quad a \in \mathcal{P}_j. \quad (60)$$

Thus, $M_b^{\bar{\lambda}} \in \mathbb{M}_k$.

Now, let $\epsilon'_j := \sqrt{\epsilon^2 + \sum_{i=j+1}^k c_i^2 \mu_i^2}$ with $\epsilon'_0 = \bar{\epsilon}$ and $\epsilon'_k = \epsilon$. Further, let $M_b^j := M_b((\lambda'_j)^{-1} g^j, \epsilon'_j)$, then Lemma 4.4 implies

$$\lim_{\lambda_j \rightarrow 0} \iota_j(M_b^j) = M_b^{j-1}. \quad (61)$$

Thus, for an initial measurement M_b^k of the k -th probe, applying Lemma 4.4 iteratively, we obtain

$$\lim_{\lambda_1 \rightarrow 0} \dots \lim_{\lambda_k \rightarrow 0} M_b^{\bar{\lambda}} = M_b(f, \bar{\epsilon}). \quad (62)$$

Thus, there exists a sequence of elements within \mathbb{M}_k that converges to $M(f, \bar{\epsilon})$. We conclude that $M(f, \bar{\epsilon}) \in \overline{\mathbb{M}}_k$ for all $\bar{\epsilon} > 0$ and $k \in \mathbb{N}$. □

Corollary 4.7. *For arbitrary $\bar{\epsilon} > 0$, the instrument $\{\Omega_b^{\bar{\epsilon}, \delta}\}_b$ belongs to $\overline{\mathbb{I}}_k$ for all $k \in \mathbb{N}$ if $\delta > \frac{\sqrt{c^4 - 1}}{2\bar{\epsilon}}$, where $c := \inf_{(g, \bar{g})} \inf_{\Gamma} (\Gamma(g, g) + \Gamma(\bar{g}, \bar{g})) \geq 1$ optimizes over covariances Γ and canonically conjugate mode pairs $g, \bar{g} \in \mathcal{C}_T$, where $g \in \mathcal{D}(L)$ is such that $f = -\beta E^- g$.*

Proof. Note that the corresponding induced instrument in the construction above is $\Omega^{\bar{\epsilon}, \delta}$, with

$$\delta^2 = \frac{c_1^2}{\mu_1^2} - \frac{1}{4\bar{\epsilon}^2} = \frac{1}{4\bar{\epsilon}^2} \left(\frac{4c_1^4}{\mu^2} - 1 \right), \quad (63)$$

where $\bar{\mu} = \frac{c_1 \mu_1}{\bar{\epsilon}} \in (0, 1)$ can be optimized over independent of $\bar{\epsilon}$ such that δ attains arbitrary values greater than $\frac{c_1^4 - 1}{\bar{\epsilon}^2}$ which is obtained in the limit $\bar{\mu} \rightarrow 1$. □

The FV scheme outlined in the proof of Lemma 4.4 allows one to realize many POVMs other than Gaussian-modulated field measurements. Indeed, consider a Gaussian measurement $\{M_b(f, \epsilon)\}_b$ and let $\{p_{\bar{b}} : \mathbb{R} \rightarrow \mathbb{R}^+\}_{\bar{b} \in \bar{\mathcal{B}}}$ be any set of Borel-integrable functions with normalized parameter-dependence $\int_{\bar{\mathcal{B}}} p_{\bar{b}}(b) d\bar{b} = 1$, for all $b \in \mathbb{R}$. Then the POVM $\{\tilde{M}_{\bar{b}}(f, \epsilon)\}_{\bar{b}}$ that results when we measure $\{M_b(f, \epsilon)\}_b$ and, upon obtaining the result b , we sample \bar{b} from the probability distribution $\{p_{\bar{b}}(b)\}_{\bar{b}}$ will have the form

$$\tilde{M}_{\bar{b}}(f, \epsilon) = (p_{\bar{b}} * g_{\epsilon})(\Phi(f)), \quad g_{\epsilon}(x) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon^2}}, \quad (64)$$

using $*$ to denote convolution of functions. Since $\widehat{p_{\bar{b}} * g_{\epsilon}}(z) = e^{-\frac{\epsilon^2 z^2}{2}} \hat{p}_{\bar{b}}(z)$ is clearly integrable in z for all $\epsilon > 0$, the proof of Lemma 4.4 with initial probe measurement $\tilde{M}_{\bar{b}}(\lambda^{-1} g, \epsilon)$ goes through and we obtain $\tilde{M}(f, \epsilon) \in \overline{\mathbb{M}}_1$. Taking the limit $\epsilon \rightarrow 0$, we have that $\tilde{M}_{\bar{b}}(f, \epsilon)$ tends to $p_{\bar{b}}(\Phi(f))$ which is thus also within $\overline{\mathbb{M}}_1$.

Combined with Theorem 4.6, this observation implies that any POVM of the form $\{p_b(\Phi(f))\}_b$ also belongs to $\overline{\mathbb{M}}_k$, for all $k \in \mathbb{N}$. Clearly, also discrete-outcome variants of these POVMs can be modeled; in this case normalization for $\{p_j\}_j$ changes to $\sum_j p_j(b) = 1$. This includes the option $p_j = \chi_{B^j}$, where $\{B^j \subset \mathbb{R}\}_j$ is a discrete family of pairwise disjoint Borel-measurable sets that satisfy $\bigcup_j B^j = \mathbb{R}$. As a result, also the projective measurement $\{\Pi_j^f\}_j$ with $\Pi_j^f := \Pi_{B^j}(\Phi(f))$ is asymptotically FV-realizable to arbitrary degree.

Note, however, that, if we follow the construction of Lemma 4.4, then the measurement channel of the instrument associated to the POVM (64) is always $\Omega^{\epsilon, \delta}$, independently of how we choose to process the “real” outcome b . So, even in the limit $\epsilon \rightarrow 0$ and transforming to discretely many outcomes j , the measurement channel will not resemble anything like $\sum_j \Pi_j^f \bullet \Pi_j^f$. Such “projective measurement” channels are at the root of Sorkin’s causal paradox [1]. The FV formalism neatly avoids them, despite the fact that it allows measuring the projectors Π_j^f up to arbitrary accuracy. The “projection postulate” is restored only for observers in the causal complement region to the field mode f , where they cannot resolve the causal structure of the apparatus which lead to the measurement of f (compare (15) in the limit $\epsilon \rightarrow 0$ and for h supported in the causal complement).

We conclude this section by noting that, even when \mathcal{A} is just a *-algebra and not completed with respect to any topology (as done in the original formulation of the FV framework [4]), it still allows to describe a large class of non-trivial, FV-realizable measurements with a movable FV-Heisenberg cut. Let \mathcal{A}^f denote the *-algebra spanned by all abstract Weyl operators¹ of the form $\{e^{iz\Phi(f)} : z \in \mathbb{R}\}$ and consider a finite-outcome POVM $\{M_j\}_j \subset \mathcal{A}^f$ such that, for some finite family $\{a_j k : j, k\} \subset \mathcal{A}^f$ and every outcome j ,

$$M_j = \sum_k a_j^k (a_j^k)^*. \quad (65)$$

Given the (unambiguous) expression of M_j as a finite linear combination of imaginary exponentials of $\Phi(f)$, define $\hat{M}_j(s)$ by replacing every instance of $\Phi(f)$ by s . Then, for any $\lambda \in \mathbb{R}$, $\{\hat{M}_j(\Psi(g_\lambda/\lambda))\}_j$ defines a POVM in the probe field. Moreover, the induced POVM element

$$\hat{M}_j^\lambda := \eta_{\sigma_\lambda} \circ \Theta_\lambda \left(\hat{M}_j(\Psi(g_\lambda/\lambda)) \right) \quad (66)$$

is also a positive semidefinite element of \mathcal{A} . One can then verify that

$$\lim_{\lambda \rightarrow 0} \omega(\hat{M}_j^\lambda) = \omega(M_j) \quad (67)$$

for any regular state ω on \mathcal{A} , i.e., a state such that $\omega(e^{it\Phi(f)})$ is continuous in t (see also Appendix B for details). Therefore, the FV framework—also in the purely algebraic setting—includes certain “Weyl type” POVMs linearly generated from Weyl operators of a single field mode.

¹In order to keep the notation in line with the main text we write $e^{iz\Phi(f)}$ in place of the abstract symbols $W(zf)$ appearing in Appendix B. However, we don’t suppose that the algebra elements are operators on a Hilbert space or continuous in z or f .

5 Conclusion

We have shown that Gaussian-modulated measurements of a (local) smeared field operator can be asymptotically implemented within the FV scheme. The degree ϵ of the modulation can be chosen arbitrarily low, which allows one to approximate a projective measurement of the field with arbitrary precision. For non-zero ϵ , there is a simple map to update the QFT state conditioned on the measurement outcome: namely, the composition of the Gaussian instrument proposed by Jubb [2] with a single-mode dephasing channel.

That the FV scheme is rich enough to model field measurements does not follow from the tomographic results of [7]. Rather, the authors of this groundbreaking paper prove that averages of the field operator can be estimated asymptotically. As explained in the text, this is not the same as being able to implement a (Gaussian-modulated) field measurement: in fact, their asymptotic tomographic scheme in the limit $\lambda \rightarrow 0$ induces the identity map as a measurement channel on the target QFT.

Notably, we find that the probe measurement required to induce a Gaussian measurement in a QFT is itself a Gaussian measurement. That is, we can model the measurement carried out on this first probe by making it interact with a second probe, which is subsequently measured. This measurement, in turn, can be also conducted by making the second probe QFT interact with a third, and so on *ad infinitum*.

It would be interesting to know if the property of having a movable FV-Heisenberg cut is applicable to all quantum measurements admitting an FV representation. The answer would be a resounding “yes!” if all POVMs with elements localizable within a local algebra happened to be FV realizable.

Acknowledgements

The authors thank Maximilian Heinz Ruep, Rainer Verch and Henning Bostelmann for valuable discussions on the subject. J. Mandrysch was funded by the quantA Core project “Local operations on quantum fields”.

Bibliography

- [1] R. D. SORKIN. Impossible Measurements on Quantum Fields.
In: *Directions in General Relativity: An International Symposium in Honor of the 60th Birthdays of Dieter Brill and Charles Misner*. 1993
- [2] I. JUBB. Causal State Updates in Real Scalar Quantum Field Theory.
Physical Review D, **105**:2, 2022. DOI: 10.1103/PhysRevD.105.025003
- [3] R. OECKL. Spectral Decomposition of Field Operators and Causal Measurement in Quantum Field Theory. 2024. DOI: 10.48550/arXiv.2409.08748
- [4] C. J. FEWSTER & R. VERCH. Quantum Fields and Local Measurements.
Communications in Mathematical Physics, **378**:2, 2020.
DOI: 10.1007/s00220-020-03800-6
- [5] C. J. FEWSTER & R. VERCH. Measurement in Quantum Field Theory. 2023.
DOI: 10.48550/arXiv.2304.13356
- [6] H. BOSTELMANN, C. J. FEWSTER & M. H. RUEP.
Impossible Measurements Require Impossible Apparatus.
Physical Review D, **103**:2, 2021. DOI: 10.1103/PhysRevD.103.025017
- [7] C. J. FEWSTER, I. JUBB & M. H. RUEP. Asymptotic Measurement Schemes for Every Observable of a Quantum Field Theory.
Annales Henri Poincaré, **24**:4, 2023. DOI: 10.1007/s00023-022-01239-0
- [8] J. VON NEUMANN. Mathematical Foundations of Quantum Mechanics.
Princeton University Press, 1955
- [9] C. BÄR, N. GINOUX & F. PFÄFFLE.
Wave Equations on Lorentzian Manifolds and Quantization.
ESI Lectures in Mathematics and Physics.
Zürich, Switzerland: European Mathematical Society, 2007
- [10] G. C. GHIRARDI, A. RIMINI & T. WEBER.
Unified Dynamics for Microscopic and Macroscopic Systems.
Physical Review D, **34**:2, 1986. DOI: 10.1103/PhysRevD.34.470
- [11] G. GIEDKE & J. IGNACIO CIRAC.
Characterization of Gaussian Operations and Distillation of Gaussian States.
Physical Review A, **66**:3, 2002. DOI: 10.1103/PhysRevA.66.032316
- [12] K. SANDERS. Thermal Equilibrium States of a Linear Scalar Quantum Field in Stationary Space–Times. *International Journal of Modern Physics A*, **28**:10, 2013.
DOI: 10.1142/S0217751X1330010X
- [13] R. BRUNETTI, C. DAPPIAGGI, K. FREDENHAGEN & J. YNGVASON, eds.
Advances in Algebraic Quantum Field Theory. Mathematical Physics Studies.
Cham: Springer International Publishing, 2015.
DOI: 10.1007/978-3-319-21353-8
- [14] K. REJZNER. Perturbative Algebraic Quantum Field Theory.
Mathematical Physics Studies. Cham: Springer International Publishing, 2016.
DOI: 10.1007/978-3-319-25901-7
- [15] R. M. WALD.
Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.
Chicago Lectures in Physics. Chicago: University of Chicago Press, 1994
- [16] R. HONEGGER & A. RIECKERS. Photons in Fock Space and Beyond.
New Jersey: World Scientific, 2015

A Proofs on Gaussian instruments

In this section, we prove the most relevant statements made in Section 2. Our results apply to an arbitrary (possibly nonseparable) Hilbert space \mathcal{H} . Concerning notation, the vector norm on \mathcal{H} will be denoted by $\|\cdot\|_{\mathcal{H}}$ and on $\mathcal{B}(\mathcal{H})$ and we employ the seminorms $\|\cdot\|_{\varphi} := \|\cdot\|_{\mathcal{H}}$ as well as the (operator) norm $\|\cdot\|$, generating respectively the strong operator and the norm topology on $\mathcal{B}(\mathcal{H})$. Further, we denote by $\mathcal{A}(f) \subset \mathcal{B}(\mathcal{H})$ the von-Neumann algebra generated by $e^{i\Phi(f)}$ for a fixed $f \in \mathcal{D}(\mathcal{M})$. In agreement with the main text a continuous function $g : \mathbb{R} \rightarrow \mathcal{A}(f)$ is termed (*strongly*) *integrable* iff $t \mapsto \|g(t)\|_{\varphi}$ is an integrable function for all $\varphi \in \mathcal{H}$. This implies that $\int g(t)dt \in \mathcal{A}(f)$ defined via $\int g(t)dt \varphi := \int g(t)\varphi dt$ for all $\varphi \in \mathcal{H}$ exists as a Bochner integral on \mathcal{H} . Since $\|a\|_{\varphi} \leq \|a\|$ for all $\varphi \in \mathcal{H}$ with $\|\varphi\|_{\mathcal{H}} = 1$, strong integrability follows also if $\|g(\cdot)\|$ is integrable.

Lemma A.1 (Gaussian measurement). *For any $\epsilon > 0$ and $f \in \mathcal{D}(\mathcal{M})$, $\{M_b(f, \epsilon)\}_{b \in \mathbb{R}}$ as defined in (10) yields a continuous outcome POVM. Further, we find the (strong-operator) limit $\epsilon \rightarrow 0$ as given in (11).*

Proof. We fix arbitrary $\epsilon > 0$ and $g_{\epsilon}(x) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon^2}}$ for any $x \in \mathbb{R}$. Given that $a \mapsto g_{\epsilon}(a-b)$ is a nonnegative bounded smooth function on \mathbb{R} , by functional calculus $M_b(f, \epsilon) = g_{\epsilon}(\Phi(f)-b)$ defines a positive element of $\mathcal{A}(f)$. Note that since $g_{\epsilon} \in L^1(\mathbb{R})$ its Fourier transform is well-defined and yields $\hat{g}_{\epsilon}(z) = \int g_{\epsilon}(x) e^{-izx} dx = e^{-\frac{\epsilon^2 z^2}{2}}$. Thus, we obtain

$$M_b(f, \epsilon) = \int e^{-\frac{\epsilon^2 z^2}{2}} e^{-iz(\Phi(f)-b)} \frac{dz}{2\pi}; \quad (68)$$

where strong integrability follows since the integrand is (strongly operator) continuous wrt z and majorized (in norm) by $z \mapsto e^{-\frac{\epsilon^2 z^2}{2}} \in L^1(\mathbb{R})$. Based on (68), for any bounded Borel set $B \subset \mathbb{R}$, we find

$$\int_B M_b(f, \epsilon) db = \int e^{-\frac{\epsilon^2 z^2}{2}} \chi_B(z) e^{-iz\Phi(f)} \frac{dz}{2\pi}, \quad (69)$$

where χ_B denotes the characteristic function wrt B and the exchange of the order of integrals is based on integrability of $e^{-\frac{\epsilon^2 z^2}{2}}$ majorizing the integral kernel since $\|\hat{\chi}_B(z) e^{-iz\Phi(f)}\| \leq 1$. The latter allows, by dominated convergence, to take the (strong operator) limits $\epsilon \rightarrow 0$, respectively, $\Lambda \rightarrow \infty$ for $B = [-\Lambda, \Lambda]$, $\Lambda > 0$, inside the integral concluding our proof:

$$\lim_{\epsilon \rightarrow 0} \int_B M_b(f, \epsilon) db = \Pi_B(\Phi(f)), \quad \int M_b(f, \epsilon) db = \lim_{\Lambda \rightarrow \infty} \int_{-\Lambda}^{\Lambda} M_b(f, \epsilon) db = \mathbb{1}. \quad (70)$$

□

Lemma A.2 (Gaussian and dephasing instrument). *For any $\epsilon, \delta > 0$ and $f \in \mathcal{D}(\mathcal{M})$, $\{\Omega_b^{\epsilon}(\bullet)\}_{b \in \mathbb{R}}$ as defined in (12) yields a continuous outcome instrument and $D^{\delta}(\bullet)$ as defined in (13) yields a (single outcome) instrument.*

Proof. By definition $\Omega_b^\epsilon(\bullet) = \sqrt{M_b(f, \epsilon)} \bullet \sqrt{M_b(f, \epsilon)}$ and $D^\delta(\bullet) = \int N_\nu(f, \delta) \bullet N_\nu(f, \delta)^\dagger d\nu$, for $N_\nu(f, \delta) := (2\pi)^{-\frac{1}{4}} \delta^{-\frac{1}{2}} e^{-\frac{\nu^2}{4\delta^2}} e^{-i\nu\Phi(f)}$; decomposing them into their Kraus operators $\{\sqrt{M_b(f, \epsilon)}\}_b$ and $\{N_\nu(f, \delta)\}_\nu$ implying complete positivity.

The ν -integral is well-defined as a strong integral since $N_\nu(f, \delta)$ is (strongly operator) continuous in ν and $\|N_\nu(f, \delta) a N_\nu(f, \delta)^\dagger\| \leq (2\pi)^{-\frac{1}{2}} \delta^{-1} e^{-\frac{\nu^2}{2\delta^2}} \|a\|$ is integrable in ν for any $a \in \mathcal{B}(\mathcal{H})$. Pointwise strong integrability of Ω_b^ϵ follows by strong integrability of $M_b(f, \epsilon)$ as proven in the preceding lemma and simple operator estimates: For any $\varphi \in \mathcal{H}$ with $\|\varphi\|_{\mathcal{H}} = 1$ and any $a \in \mathcal{B}(\mathcal{H})$,

$$\|\sqrt{M_b(f, \epsilon)} a \sqrt{M_b(f, \epsilon)}\|_\varphi \leq \|a\| \|\sqrt{M_b(f, \epsilon)}\|_\varphi,$$

based on $\|\sqrt{M_b(f, \epsilon)}\| \leq 1$. Since $\sqrt{M_b(f, \epsilon)} = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sqrt{\epsilon} M_b(f, \sqrt{2}\epsilon)$, the r.h.s. is integrable.

That units are preserved follows based on $\int M_b(f, \epsilon) db = 1$ and

$$\int N_\nu(f, \delta) N_\nu(f, \delta)^\dagger d\nu = \frac{1}{\sqrt{2\pi}\delta} \int e^{-\frac{\nu^2}{2\delta^2}} d\nu = 1.$$

□

Lemma A.3 (Commutation relation). *Let $\mu = \hat{l}$ be the Fourier transform of a function $l \in L^1(\mathbb{R})$. Then $\mu(\Phi(f)) \in \mathcal{A}(f)$ exists by functional calculus and it holds that*

$$\mu(\Phi(f)) e^{i\Phi(h)} = e^{i\Phi(h)} \mu(\Phi(f) - \langle f, Eh \rangle). \quad (71)$$

Proof. By Riemann-Lebesgue lemma μ is a bounded continuous function and thus $\mu(\Phi(f)) = \int l(z) e^{-iz\Phi(f)} \frac{dz}{2\pi} \in \mathcal{A}(f)$ exists by Borel functional calculus as a strong integral on $\mathcal{A}(f)$ based on (strong operator) continuity of $e^{-iz\Phi(f)}$ in z and integrability of l . Then, we compute (using the Weyl relation),

$$\begin{aligned} \mu(\Phi(f)) e^{i\Phi(h)} &= \int l(z) e^{-iz\Phi(f)} e^{i\Phi(h)} dz \\ &= e^{i\Phi(h)} \int l(z) e^{iz(\langle f, Eh \rangle - \Phi(f))} dz = e^{i\Phi(h)} \mu(\Phi(f) - \langle f, Eh \rangle \mathbb{1}). \end{aligned} \quad (72)$$

□

Lemma A.4 (Dephased gaussian instrument). *For any $\epsilon, \delta > 0$, $\{\Omega_b^{\epsilon, \delta}\}_{b \in \mathbb{R}}$ as defined in (14) defines a continuous outcome POVM and it holds for arbitrary $h \in \mathcal{C}_T$ that*

$$\Omega_b^{\epsilon, \delta}(e^{i\Phi(h)}) = \frac{1}{\sqrt{2\pi}\epsilon} e^{-\frac{\langle f, Eh \rangle^2}{8\underline{\epsilon}(\epsilon, \delta)^2}} e^{i\Phi(h)} e^{-\frac{(\Phi(f) - b - \frac{\langle f, Eh \rangle}{2})^2}{2\epsilon^2}}, \quad \underline{\epsilon}(\epsilon, \delta)^2 := \frac{1}{4\delta^2 + \frac{1}{\epsilon^2}}. \quad (73)$$

Proof. Since $\Omega_b^{\epsilon, \delta} = D^\delta \circ \Omega_b^\epsilon = \Omega_b^\epsilon \circ D^\delta$ is defined as a concatenation of two instruments (Lemma A.2), it is itself an instrument (there are no questions arising concerning integrability since D^δ is single outcome).

Invoking the identity (71) for μ being a Gaussian function, completing squares and performing Gaussian integrals we find:

$$\begin{aligned}
& \Omega_b^{\epsilon, \delta}(e^{i\Phi(h)}) \\
&= \frac{1}{2\pi\delta\epsilon} \int e^{-\frac{\nu^2}{2\delta^2}} e^{-\frac{(\Phi(f)-b)^2}{4\epsilon^2} - i\nu\Phi(f)} e^{i\Phi(h)} e^{-\frac{(\Phi(f)-b)^2}{4\epsilon^2} + i\nu\Phi(f)} d\nu \\
&= e^{i\Phi(h)} \frac{1}{2\pi\delta\epsilon} \int e^{-\frac{\nu^2}{2\delta^2}} e^{-\frac{(\Phi(f)-b-\langle f, Eh \rangle)^2}{4\epsilon^2} - i\nu(\Phi(f) - \langle f, Eh \rangle)} e^{-\frac{(\Phi(f)-b)^2}{4\epsilon^2} + i\nu\Phi(f)} d\nu \\
&= e^{i\Phi(h)} \left(\frac{1}{\sqrt{2\pi\delta}} \int e^{-\frac{\nu^2}{2\delta^2}} e^{i\nu\langle f, Eh \rangle} d\nu \right) \left(\frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(\Phi(f)-b-\frac{\langle f, Eh \rangle}{2})^2}{2\epsilon^2}} e^{-\frac{\langle f, Eh \rangle^2}{8\epsilon^2}} \right) \\
&= \frac{1}{\sqrt{2\pi\epsilon}} e^{i\Phi(h)} e^{-\frac{\langle f, Eh \rangle^2 \delta^2}{2}} e^{-\frac{(\Phi(f)-b-\frac{\langle f, Eh \rangle}{2})^2}{2\epsilon^2}} e^{-\frac{\langle f, Eh \rangle^2}{8\epsilon^2}} \\
&= \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{\langle f, Eh \rangle^2}{8\epsilon(\epsilon, \delta)^2}} e^{i\Phi(h)} e^{-\frac{(\Phi(f)-b-\frac{\langle f, Eh \rangle}{2})^2}{2\epsilon^2}}. \tag{74}
\end{aligned}$$

□

B Abstract setting for QFT

In this section, to motivate the Hilbert space setting of the main text from the algebraic approach to QFT, we recapitulate a standard construction of QFT for linear fields, but refer to [9] and [13, Chaps. 3, 4] for further details. We also refer to [15, Sec. 4.5] which includes a brief pedagogical account on how to relate measurements in the algebraic approach with the Hilbert space setting.

Definition B.1 (Weyl algebra). Given a normally hyperbolic differential operator T which is symmetric with respect to $\langle \cdot, \cdot \rangle$, let E^\pm denote its associated retarded/advanced Green operators. These are linear continuous maps $\mathcal{D}(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$ such that

$$\text{supp } E^\pm f \subset J^\pm(\text{supp } f), \quad E^\pm T f = T E^\pm f = f, \quad \langle f, E^\pm g \rangle = \langle E^\mp f, g \rangle \tag{75}$$

for all $f, g \in \mathcal{D}(\mathcal{M})$ and such that $E := E^- - E^+$ defines a symplectic form on $\mathcal{C}_T := \mathcal{D}(\mathcal{M})/T\mathcal{D}(\mathcal{M})$. Then let $\tilde{\mathcal{W}}$ denote the unital $*$ -algebra generated by symbols $\{W(u), u \in \mathcal{C}_T\}$ subject to the relations

$$W(u)^* = W(-u), \quad W(u)W(v) = e^{-\frac{i}{2}\langle u, Ev \rangle} W(u+v). \tag{76}$$

We equip $\tilde{\mathcal{W}}$ with its unique C^* -norm and define $\mathcal{W}(R)$ to be the C^* -algebras generated by $\{W(u), u \in \mathcal{C}_T(R)\} \subset \tilde{\mathcal{W}}$ for open causally convex $R \subset \mathcal{M}$ with $\mathcal{W} := \mathcal{W}(\mathcal{M})$.

A state ω on \mathcal{W} is called *regular* iff the orbits $t \mapsto \omega(W(tu))$ are continuous for all $u \in \mathcal{C}_T$. Note that all quasi-free states are regular since $\omega(W(tu)) = e^{-t^2\Gamma(u,u) - itV(u)}$ is continuous in t for all $u \in \mathcal{C}_T$; here V and Γ denote the one- and two-point function of ω . Let $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ denote the GNS representation wrt ω . Then we obtain the

setting of the main text, namely, there is a unique selfadjoint operator $\Phi_\omega(f)$ on \mathcal{H}_ω depending real-linearly on $f \in \mathcal{D}(\mathcal{M})$ such that

$$\pi_\omega(W([f])) = e^{i\Phi_\omega(f)} \quad (77)$$

which satisfies (8) and (9). Define $\mathcal{A}(R) := \pi_\omega(\mathcal{W}(R))' \subset \mathcal{B}(\mathcal{H}_\omega)$, where $'$ denotes the commutant algebra, then by the bicommutant theorem, $\mathcal{A}(R)$ is thus a von-Neumann algebra. The resulting net of von-Neumann algebras $\{\mathcal{A}(R)\}_R$ satisfies the usual axioms as given in the main text. Moreover, note that it is a standard result that *-automorphisms of $\tilde{\mathcal{W}}$ extend to C*-automorphisms on \mathcal{W} and thus any symplectic transformation induces a C*-automorphism on \mathcal{W} (see e.g. [16, Thm. 18.1-11]) implying automorphy of $\mathcal{A}(\mathcal{M})$ and its transformed version (Fact 4.3).

C Proof on symplectic transformations

Lemma C.1. *Fact 4.2 holds.*

Proof. Since (S, σ) is finite-dimensional and symplectic, S is even dimensional and has a standard basis as given in the remark. Further, $J = (J_{kl})_{kl}$ and $K = (F_{kl} := \sigma(u_k, Fu_l))_{kl}$ are $2n \times 2n$ -matrices which are antisymmetric, resp., symmetric and satisfy $J^2 = -1$ and $JKJ^T = K$. Then $p = \sum_{k,l=1}^{2n} J_{kl}\sigma(\cdot, u_l)u_k$ defines an orthogonal projection mapping from X onto S : We compute

$$p \sum_{j=1}^{2n} c_j u_j = \sum_{j,k,l=1}^{2n} c_j J_{kl} J_{jl} u_k = \sum_{k=1}^{2n} c_k u_k, \quad \{c_j\} \subset \mathbb{R} \quad (78)$$

$$p^2 v = \sum_{k,l,k',l'=1}^{2n} \sigma(u_k, J_{k'l'} u_{l'}) \sigma(v, J_{kl} u_l) u_{k'} = pv, \quad v \in X \quad (79)$$

as well as

$$pv = \sum_{k,l=1}^{2n} J_{kl} \sigma(v, u_l) u_k = 0 \quad (80)$$

for all elements $S^\perp := \{v \in X : \sigma(v, S) = 0\}$. Thus, $v^\perp = v - pv \in S^\perp$ and $v - v^\perp = pv \in S$ for any $v \in S$. Thus any linear extension of the map $F : S \rightarrow S$ with $F \upharpoonright_{S^\perp}$ to X has to be of the form $Fv := F(v - v^\perp) + v^\perp$ which yields the canonical extension of F . \square