Leading Higher Derivative Corrections to Multipole Moments of Kerr-Newman Black Hole

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ABSTRACT

We study the (leading) 4-derivative corrections, including both parity even and odd terms, to electrically-charged Kerr-Newman black holes. The linear perturbative equations are then solved order by order in terms of two dimensionless rotating and charge parameters. The solution allows us to extract the multipole moments of mass and current from the metric as well as the electric and magnetic multipole moments from the Maxwell field. We find that all the multipole moments are invariant under the field redefinition, indicating they are well-defined physical observables in this effective theory approach to quantum gravity. We also find that parity-odd corrections can turn on the multipole moments that vanish in Einstein theory, which may have significant observational implications.

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1 Introduction

A century after its discovery, General Relativity has proven remarkably successful in explaining phenomena across a vast range, from the solar system to the entire universe. On the other hand, it is also widely believed that Einstein gravity should only appear as the leading term in an effective theory of quantum gravity below certain ultra-violate cutoff scale Λ_c. In the simplest scenario, the effective action is given by an infinite derivative expansion controlled by powers of Λ_c

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left(\mathcal{L}_{2\partial} + \Lambda_c^{-2}\mathcal{L}_{4\partial} + \Lambda_c^{-4}\mathcal{L}_{6\partial} + \cdots\right). \tag{1}$$

The higher derivative corrections to infrared physical quantities encode the fine structure of quantum gravity effects, and their observation could shed lights on hidden dynamics at the cutoff scale. Although the correction terms are usually rather small and may not be visible in the near future, one can still ask the interesting question: which physical quantities computed using the effective action (1) are genuine physical observables? The answer to this question is crucial for their potential measurements by future detectors.

In an effective field theory (EFT), the equivalence theorem states that meaningful physical observables should be invariant under field redefinitions, which are invertible and retain the same physical spectrum [1]. In the perturbative higher derivative extensions of Einstein gravity, field redefinitions satisfying the equivalence theorem take the form

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda_1 R_{\mu\nu} + \lambda_2 R g_{\mu\nu} + \cdots \tag{2}$$

where "$\cdots$" refers to matter dependent terms, as well as even higher order terms. It is obvious that coefficients of certain terms in (1) are shifted by (2). Based on the Reall-Santos method [2], we have proven that the Euclidean action of asymptotically flat or AdS black holes are invariant under field redefinitions [3, 4]. Consequently, all the thermodynamic variables derived from the Euclidean action also enjoy this property. In particular, for asymptotically flat black holes, the corrections to thermodynamic variables depend only on the coupling coefficients of the higher order operators that are inert under (2).

Similar to black hole thermodynamics, black hole multipole moments will also receive higher-derivative corrections, except for the mass monopole and the current dipole moment, corresponding to total mass and angular momentum that are typically chosen to be fixed under the perturbation. Black hole multipole moments play a crucial role in describing the external fields and gravitational wave radiation, potentially offering a new window into the footprints of quantum gravity [5–8] beyond standard General Relativity. Upcoming observations of gravitational waves from extreme mass-ratio inspirals (EMRIs) [9–14] may reveal whether black hole mass and spin multipole moments match predictions from classical general relativity or suggest modifications induced by new physics beyond the standard model. It is thus urgent to understand in an effective theory of gravity, whether black hole multipole moments are meaningful physical observables, i.e. invariant under field redefinitions. Based on traditional approach, to obtain black hole multipole moments, one needs to first solve for higher derivative corrections to rotating black holes which is technically quite difficult. Until now, there is no proof that in a generic theory of gravity with higher derivative corrections, the black hole multipole moments are invariant under field redefinitions, except for very few specific examples. For instance, the recent work [15] has computed the multipole moments of Kerr black holes in pure gravity with cubic curvature corrections and showed that they are invariant under field redefinitions of the metric utilizing the Ricci flatness of Kerr solution. For non Ricci flat solutions, such as Kerr-Newman black holes, the situation is unknown. In fact, it was speculated in [15] that for non-Ricci-flat solutions, higher derivative corrections to multipole moments might be affected by field redefinitions.

In this work, we make the first attempt to study leading higher derivative corrections to Kerr-

Newman black hole, with focus on its multipole moments. In four dimensions, the leading higher-derivative extensions of Einstein-Maxwell theory consist of parity even and odd 4-derivative terms built from Riemann tensor and the $U(1)$ field strength. Different from the static black hole solutions, there is no mature method of deriving the complete form of even the leading higher-derivative corrections to rotating black holes. We thus adopt the approximate method proposed in [15–17], by expanding the perturbed metric and $U(1)$ gauge potential in power series of the dimensionless parameters including $\chi_a := a/\mu$ and $\chi_Q := Q/\mu$, where μ, a, Q parametrize the mass, spin and electric charge of the uncorrected black hole solution respectively.

We therefore need to be concerned with two perturbative expansions of the the Kerr-Newman black hole. One is the leading-order perturbation of the 4-derivative couplings. After deriving these linear perturbed field equations, in principle, we should be able to solve them order by order in χ's up to arbitrarily high order. However, in practice, this second procedure is rather time consuming and we have to terminate at certain order.

At the first order in 4-derivative couplings, corrections from the parity even and odd terms to the black hole solution decouple from each other. Thus we can analyze these two cases separately. When parity preserving 4-derivative terms are switched on, we are able to obtain the perturbed solution up to $\mathcal{O}(\chi^7)$, while concerning only parity odd 4-derivative interactions, we can reach $\mathcal{O}(\chi^8)$. From the approximate solution, we could read off corrections to both the gravitational and electromagnetic multipole moments at first few levels. Similar to the pure gravity case, parity preserving 4-derivative interactions only modify multipole moments $\{M_{2n}, \mathcal{S}_{2n+1}, \mathcal{Q}_{2n}, \mathcal{P}_{2n+1}\}$ that are already nonzero at the leading order. In the parity odd case, the 4-derivative interactions contribute to multipole moments $\{M_{2n+1}, \mathcal{S}_{2n}, \mathcal{Q}_{2n+1}, \mathcal{P}_{2n}\}$ that vanish at the leading order. Thus their very presence breaks the equatorial symmetry of the solution and has significant observational implications. Most importantly, we show that these results can be expressed in a way that is manifestly invariant under field redefinitions.

This paper is organized as follows. In section 2, we study the electrically-charged Kerr-Newman black hole and obtain the mass and current multipole moments from the metric and the electric and magnetic multipole moments from the Maxwell field. In section 3, we consider the (leading) 4-derivative corrections, which can be categorized as parity even and parity odd terms. The leading order correction is solved order by order in terms of appropriate two dimensionless parameters (χ_a, χ_Q) of the Kerr-Newman black hole. This allows us to read off the 4-derivative corrections to the multipole moments. We then show that the results are independent of the field redefinition, indicating that they are indeed good physical quantities in our effective theory approach to quantum gravity. We conclude the paper in section 4. In appendix A, we show how the corrections of the solution modify the thermodynamic variables, and the results are consistent with the Reall-Santos method. In appendices B and C, we give the complicated expressions that would be a digression if given in the main text. In appendix D, we discuss briefly the Geroch-Hansen method of multipole moments for general theories of gravity.

2 Multipole moments of Kerr-Newman black hole in 2-derivative theory

Currently, there are three different procedures to calculate black hole gravitational multipole moments, including the Geroch-Hansen formalism [18, 19], the Thorne's formalism [20] and the covariant phase space approach proposed in [21]. Equivalence of the results obtained from three different methods has been discussed in [15, 21]. Here we will adopt Thorne's formalism which directly extracts gravitational multipole moments by recasting the solution in the asymptotically Cartesian and mass-centered (ACMC) coordinate system. In this section, we briefly review how to obtain the gravitaional multipole moments of Kerr-Newman black holes by working in the ACMC coordinate system.

The electrically-charged Kerr-Newman black hole is an exact solution of the Einstein-Maxwell theory, with

$$\mathcal{L}_{2\partial} = R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{3}$$

The solution takes the form

$$\begin{aligned} ds^2 &= -\frac{\Delta_r}{\Sigma}\left(dt - a(1-x^2)d\varphi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{1-x^2}dx^2 + \frac{1-x^2}{\Sigma}\left(adt - (r^2+a^2)d\varphi\right)^2 \\ A_{(1)} &= -\frac{2Qr}{\Sigma}\left(dt - a(1-x^2)d\varphi\right) \ , \\ \Delta_r &= r^2 - 2\mu r + Q^2 + a^2, \qquad \Sigma = r^2 + a^2 x^2 \ . \end{aligned} \tag{4}$$

To compute all the gravitational multipole moments, we must perform a coordinate transformation from Boyer-Lindquist coordinates (r, x) to ACMC-∞ coordinates (r_S, x_S) defined by [15, 22]

$$r_S\sqrt{1-x_S^2} = \sqrt{r^2+a^2}\sqrt{1-x^2}, \qquad r_S x_S = rx \ , \tag{5}$$

in terms of which, the metric takes the form in the far zone [20]

$$\begin{aligned} g_{tt} &= -1 + \frac{2M}{r} + \sum_{\ell\geq 1}^{\infty} \frac{2}{r^{\ell+1}}\left(M_\ell P_\ell + \sum_{\ell'<\ell} c_{\ell\ell'}^{(tt)} P_{\ell'}\right), \\ g_{t\varphi} &= -2r(1-x^2)\left[\sum_{\ell\geq 1}^{\infty} \frac{1}{r^{\ell+1}}\left(\frac{\mathcal{S}_\ell}{\ell} P'_\ell + \sum_{\ell'<\ell} c_{\ell\ell'}^{(t\varphi)} P'_{\ell'}\right)\right], \end{aligned}$$

$$g_{rr} = 1 + \sum_{\ell\geq 0}^{\infty} \frac{1}{r^{\ell+1}} \sum_{\ell'\leq\ell} c^{(rr)}_{\ell\ell'} P_{\ell'}, \quad g_{xx} = \frac{r^2}{1-x^2}\left[1 + \sum_{\ell\geq 0}^{\infty} \frac{1}{r^{\ell+1}} \sum_{\ell'\leq\ell} c^{(xx)}_{\ell\ell'} P_{\ell'}\right],$$
$$g_{\varphi\varphi} = r^2(1-x^2)\left[1 + \sum_{\ell\geq 0}^{\infty} \frac{1}{r^{\ell+1}} \sum_{\ell'\leq\ell} c^{(\varphi\varphi)}_{\ell\ell'} P_{\ell'}\right], \quad g_{rx} = r\left[\sum_{\ell\geq 0}^{\infty} \frac{1}{r^{\ell+1}} \sum_{\ell'\leq\ell} c^{(rx)}_{\ell\ell'} P_{\ell'}\right], \tag{6}$$

where P_ℓ and P'_ℓ denote Legendre polynomial of x and its x-derivative, respectively. Moreover we have removed "S" from the subscript to simplify the notation. The coefficients M_ℓ and $\mathcal{S}_\ell$ are the mass and current multipole moments respectively. On the other hand, the coefficients $c^{(ij)}_{\ell\ell'}$ are gauge dependent and nonphysical [20]. For the Kerr-Newman black hole, the nonvanishing multiplet moments are

$$M_{2n} = \mu(-a^2)^n, \qquad \mathcal{S}_{2n+1} = \mu a(-a^2)^n, \tag{7}$$

where M_0 and $\mathcal{S}_1$ correspond to the total mass and angular momentum. It has been noted in [22, 23] that the multipole moments of the Kerr-Newman black hole share the same form as those of the Kerr black hole in pure Einstein gravity. In other words, they are unaffected by the electric charge.

In ACMC-∞ coordinates (r_S, x_S), we find that the Maxwell field in the far zone takes the form, after dropping the subscript "S",

$$\begin{aligned}
A_t &= -\sum_{\ell\geq 0}^{\infty} \frac{4}{r^{\ell+1}}\Big(\mathcal{Q}_\ell P_\ell + \sum_{\ell'<\ell} c^{(t)}_{\ell\ell'} P_{\ell'}\Big),\\
A_\varphi &= \sum_{\ell\geq 0}^{\infty}\Bigg\{\frac{4x}{r^{2\ell}}\Big(\mathcal{P}_{2\ell}P_{2\ell} + \sum_{\ell'<\ell} c^{(\varphi)}_{2\ell,2\ell'}P_{2\ell'}\Big)\\
&\quad -\frac{1-x^2}{r^{2\ell+1}}\frac{4}{2\ell+1}\Big(\mathcal{P}_{2\ell+1}P'_{2\ell+1} + \sum_{\ell'<\ell} c^{(\varphi)}_{2\ell+1,2\ell'+1}P'_{2\ell'+1}\Big)\Bigg\},
\end{aligned} \tag{8}$$

from which we recognize $\mathcal{Q}_\ell$ as the electric multipole moments are given by

$$\mathcal{Q}_{2n} = \frac{1}{2}Q(-a^2)^n, \qquad \mathcal{Q}_{2n+1} = 0. \tag{9}$$

In particular, the electric charge is given by $Q_e = Q_0 = Q/2$. The coefficients $\mathcal{P}_\ell$ take the values below

$$\mathcal{P}_{2n+1} = -\frac{1}{2}Qa(-a^2)^n, \qquad \mathcal{P}_{2n} = 0. \tag{10}$$

As we shall show below that $\mathcal{P}_\ell$ can be interpreted as the magnetic multipole moments, because via electromagnetic duality, they appear in the electric multipole moments of the dual $U(1)$ gauge field. Moreover, in the dual $U(1)$ gauge field, the coefficients in front of $x\mathcal{P}_{2n}$ are in fact non-zero and given by Q_{2n}. This is why we introduce the first line in the expansion of A_φ to make the ansatz more general, even though for the solution given in (4), it is absent in the expansion of A_φ. Note that in this paper, we consider only the electrically charged Kerr-Newman black hole, the magnetic monopole $\mathcal{P}_0$ or the magnetic charge vanishes. The electric charge, under the rotation, can generate odd magnetic multipole moments, namely $\mathcal{P}_{2n+1}$.

To ensure the gauge invariance of the electric multiple moments obtained above, we define electric potential rigorously using the Killing vector $\xi = \partial_t + \Omega_H \partial_\varphi$ that vanishes on the bifurcation horizon of the black hole.

$$\xi^\mu F_{\mu\nu} = \partial_\nu \Phi_e \quad \Longrightarrow \quad \Phi_e = A_t + \Omega_H A_\varphi + \text{const.} \tag{11}$$

Thus in the far-zone, the large r expansion of the gauge invariant electric potential Φ_e acquires not only terms proportional to Legendre polynomials as in the static case, but also terms proportional to derivatives of Legendre polynomials, due to frame dragging effects.

We now examine the magnetic multipole moments more closely. To define them in a rigorous way, we consider the dual $U(1)$ gauge field $\widetilde{A}_\mu$ whose field strength is the Hodge dual of $F_{\mu\nu}$, i.e., $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$. Therefore we have

$$\widetilde{A}_{(1)} = -2\frac{Qax}{\Sigma}dt + 2\Big(\frac{Qa^2(1-x^2)}{\Sigma} + Q\Big)xd\varphi \ , \tag{12}$$

whose components take the form in the ACMC coordinate system

$$\begin{aligned}
\widetilde{A}_t &= -\sum_{\ell\geq 0}^{\infty} \frac{4}{r^{\ell+1}}\Big(\widetilde{\mathcal{Q}}_\ell P_\ell + \sum_{\ell'<\ell} \widetilde{c}^{(t)}_{\ell\ell'} P_{\ell'}\Big) \ , \\
\widetilde{A}_\varphi &= \sum_{\ell\geq 0}^{\infty} \Bigg\{ \frac{4x}{r^{2\ell}}\Big(\widetilde{\mathcal{P}}_{2\ell} P_{2\ell} + \sum_{\ell'<\ell} \widetilde{c}^{(\varphi)}_{2\ell,2\ell'} P_{2\ell'}\Big) \\
&\quad - \frac{1-x^2}{r^{2\ell+1}} \frac{4}{2\ell+1}\Big(\widetilde{\mathcal{P}}_{2\ell+1} P'_{2\ell+1} + \sum_{\ell'<\ell} \widetilde{c}^{(\varphi)}_{2\ell+1,2\ell'+1} P'_{2\ell'+1}\Big)\Bigg\} \ ,
\end{aligned} \tag{13}$$

with

$$\widetilde{\mathcal{Q}}_{2n+1} = -\mathcal{P}_{2n+1}, \quad \widetilde{\mathcal{Q}}_{2n} = -\mathcal{P}_{2n} = 0, \quad \widetilde{\mathcal{P}}_{2n} = \mathcal{Q}_{2n}, \quad \widetilde{\mathcal{P}}_{2n+1} = \mathcal{Q}_{2n+1} = 0 \ . \tag{14}$$

Thus we see that the coefficients $\mathcal{P}_\ell$ that appear in the expansion of A_μ indeed correspond to the magnetic multipole moments. Similarly, we can also define gauge invariant magnetic potential as

$$\xi^\mu \widetilde{F}_{\mu\nu} = \partial_\nu \Phi_m \quad \Longrightarrow \quad \Phi_m = \xi^\mu \widetilde{A}_\mu + \text{const.} \ , \tag{15}$$

which justifies the gauge invariance of the expansion coefficients in (13).

3 Adding 4-derivative corrections

In this section, we will apply the strategy of [15] to compute generic 4-derivative corrections to the multipole moments of $D = 4$ Kerr-Newman black hole. The 4-derivative extension of

Einstein-Maxwell theory contains parity even and odd terms. Up to first order in 4-derivative couplings, their corrections to the black hole solution and multipole moments are disentangled from each other. Thus we shall discuss the parity even and odd cases separately.

3.1 Parity-even case

The general parity-even 4-derivative interactions involving curvature and Maxwell field strength are given by

$$\begin{aligned}\mathcal{L}_4^{(e)} &= c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + c_3 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + c_4 R F^2 + c_5 R^{\mu\nu} F_{\mu\rho} F_\nu{}^\rho \\ &\quad + c_6 R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + c_7 (F^2)^2 + c_8 F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu \,. \end{aligned} \tag{16}$$

The most general redefinition of the metric that preserves the parity is of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda_1 R_{\mu\nu} + \lambda_2 R g_{\mu\nu} + \lambda_3 F_{\mu\rho} F_\nu{}^\rho + \lambda_4 F^2 g_{\mu\nu} \,, \tag{17}$$

which leads to the variation of 4-derivative coefficients as

$$\begin{aligned} &c_1 \to c_1 + \tfrac{1}{2}\lambda_1 + \lambda_2 \,, \qquad c_2 \to c_2 - \lambda_1 \,, \qquad c_3 \to c_3 \,, \\ &c_4 \to c_4 - \tfrac{1}{8}\lambda_1 + \tfrac{1}{2}\lambda_3 + \lambda_4 \,, \quad c_5 \to c_5 + \tfrac{1}{2}\lambda_1 - \lambda_3 \,, \qquad c_6 \to c_6 \,, \\ &c_7 \to c_7 - \tfrac{1}{8}\lambda_3 \,, \qquad c_8 \to c_8 + \tfrac{1}{2}\lambda_3 \,, \end{aligned} \tag{18}$$

under which the combinations below

$$\begin{aligned} \alpha_0 &= 2c_2 + 8c_3 + 4c_5 + 4c_6 + 32c_7 + 16c_8 \,, \\ \alpha_1 &= c_3, \qquad \alpha_2 = c_6, \qquad \alpha_3 = c_2 + 2c_5 + 4c_8 \,, \end{aligned} \tag{19}$$

are invariant [24, 25]. Thus a physical quantity depending only on these combinations satisfy our criteria of being a meaningful observable.

For the time being, there is no well established approach to finding analytical and complete results for 4-derivative corrections to a rotating black hole. Hence we resort to the approximate method proposed by [17]. Up to first order in c_i, we recast the corrected field equations in the form [26]

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R &= \tfrac{1}{2}(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^2) + \tfrac{1}{2}\Delta T_{\mu\nu}[g^{(0)}_{\lambda\sigma}, A^{(0)}_\sigma] \,, \\ \nabla_\mu F^{\mu\nu} &= \Delta J^\nu[g^{(0)}_{\lambda\sigma}, A^{(0)}_\sigma] \,, \end{aligned} \tag{20}$$

where $g^{(0)}_{\mu\nu}$ and $A^{(0)}_\mu$ denote the uncorrected solution and the effective energy-momentum tensor and the effective electric current are defined by

$$\Delta T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_4^{(e)})}{\delta g^{\mu\nu}} \,, \qquad \Delta J^\nu = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_4^{(e)})}{\delta A_\nu} \,. \tag{21}$$

Next, when solving for the perturbed solution, we expand the uncorrected solution in power series of $\chi_a = a/\mu$, $\chi_Q = Q/\mu$ both smaller than 1. Depending on the available computer power, one can solve for the perturbed solution up to certain order in χ. Since the leading order solution is exact in χ and the approximation is only performed in finding the perturbed solution, it has been shown that [15, 17] this procedure can yield a rather accurate approximation to the full perturbed solution, as long as it is carried out to sufficiently high order in χ.

Since the uncorrected Kerr-Newman black hole is stationary and axisymmetric, the effective energy-momentum tensor and electric current will inherit these symmetries. Consequently, the perturbed black hole solution can be parameterized as [17]

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\Sigma}\left(dt - a(1-x^2)d\varphi\right)^2(1+H_1) + (1+H_2)\left(\frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{1-x^2}dx^2\right) \\
&\quad + \frac{1-x^2}{\Sigma}\left((1+H_3)a dt - (1+H_4)(r^2+a^2)d\varphi\right)^2 \\
A_{(1)} &= -\frac{2Qr}{\Sigma}\left((1+H_5)dt - (1+H_6)a(1-x^2)d\varphi\right) \ ,
\end{aligned}
\tag{22}
$$

where the six functions H_i, $i = 1, \ldots 6$, depend only on coordinates r and x. At first glance, the form of the metric ansatz appears different from the original one used in [17], but one can easily show that the two ansätze are related via

$$
\begin{aligned}
H_1^{\rm CR} &= \frac{2a^2H_3(1-x^2) - H_1\Delta_r}{\Sigma}, \quad H_2^{\rm CR} = \frac{(H_3+H_4)(r^2+a^2) - H_1\Delta_r}{2\mu r} \ , \\
H_3^{\rm CR} &= H_2, \quad H_4^{\rm CR} = -\frac{a^2H_1\Delta_r(1-x^2) - 2H_4(r^2+a^2)^2}{\Sigma(r^2+a^2) + 2a^2\mu r(1-x^2)} \ ,
\end{aligned}
\tag{23}
$$

where the superscript "CR" denotes the H_i's defined in [17]. In the next, we shall expand each H_i in power series of parameters

$$
\chi_a = \frac{a}{\mu}, \qquad \chi_Q = \frac{Q}{\mu} \ , \quad \text{with} \quad \chi_a^2 + \chi_Q^2 \leq 1 \ ,
\tag{24}
$$

where the right inequality is saturated by the extremal unperturbed Kerr-Newman black hole. In an infinite series expansion, there are many ways to recollect terms. For simplicity, we choose homogeneous polynomials of χ_a and χ_Q as the expansion basis. To do so, we introduce a bookkeeping parameter ϵ which will be set to 1 at the end of calculation and temporarily replace χ by $\epsilon\chi$. Then the expansion of H_i can be conveniently written as

$$
H_i(r,x) = \sum_{n=0}^{\infty} H_i^{(n)}(r,x)\epsilon^n \ .
\tag{25}
$$

Following [17], $H_i^{(n)}(r,x)$ can always be expressed as a polynomial in x and in $1/r$

$$
H_i^{(n)}(r,x) = \sum_{p=0}^{n}\sum_{k=0}^{k_{\max}} H_i^{(n,p,k)}\frac{x^p}{r^k} \ ,
\tag{26}
$$

where $H_i^{(n,p,k)}$ are constant coefficients and for each undetermined function, the number of $k_{\max}$ depends on n and p. Substituting (22), (25) and (26) into (20), we solve for $H_i^{(n,p,k)}$ order by order in ϵ up to $\mathcal{O}(\epsilon^7)$ and present the results in an accompanying Mathematica notebook. As a double check of our approximate solution, in Appendix A, we first calculate 4-derivative corrections to all the thermodynamic variables of Kerr-Newman black hole using the Reall-Santos method [2], which requires only the knowledge of the uncorrected solution. We then compute the same quantities by applying the standard approach to the corrected solution. It turns out that results obtained from these two methods agree with each other up to the approximation order we have considered.

The higher-derivative terms also modifies the relation between the ACMC-∞ coordinate denoted by $(r_S,\, x_S)$ and the Boyer-Lindquist coordinates (r, x)

$$r = r^{(0)} + r^{(1)}(r_S, x_S), \quad x = x^{(0)} + x^{(1)}(r_S, x_S)\ , \tag{27}$$

where the leading terms $r^{(0)}$ and $x^{(0)}$ are solved from (5) yielding

$$\begin{aligned} r^{(0)} &= \frac{\sqrt{r_S^2 - a^2 + \sqrt{a^4 + 2a^2 r_S^2\left(2x_S^2 - 1\right) + r_S^4}}}{\sqrt{2}}\ , \\ x^{(0)} &= \frac{\sqrt{2} r_S x_S}{\sqrt{r_S^2 - a^2 + \sqrt{a^4 + 2a^2 r_S^2\left(2x_S^2 - 1\right) + r_S^4}}}\ , \end{aligned} \tag{28}$$

while the corrections caused by 4-derivative interactions takes the form

$$r^{(1)}(r_S, x_S) = \sum_{k=-1}^{\infty}\sum_{p=0}^{k+1} b_{k,p}\frac{x_S^p}{r_S^k}\ , \qquad x^{(1)}(r_S, x_S) = (1 - x_S^2)\sum_{k=0}^{\infty}\sum_{p=0}^{k+1} c_{k,p}\frac{x_S^p}{r_S^k}\ . \tag{29}$$

The coefficients $b_{k,p}$ and $c_{k,p}$ are fixed by requiring the modified metric to remain in the standard form (6), from which we read off corrections to the gravitational multipole moments due to the parity even 4-derivative interactions

$$\begin{aligned} \delta M_0^{(e)} &= \frac{1}{35\mu}\Big[(\alpha_0 - 24\alpha_1 - 4\alpha_3)\chi_Q^2 - \alpha_2(35\chi_a^2 + 4\chi_Q^2)\Big] + \frac{1}{13860\mu}\Big[1386\alpha_2\chi_a^4 \\ &\quad +3(65\alpha_0 - 1252\alpha_1 - 3201\alpha_2 - 337\alpha_3)\chi_a^2\chi_Q^2 - 44(3\alpha_0 + 4\alpha_2 + 24\alpha_1 + 4\alpha_3)\chi_Q^4\Big] \\ &\quad + \frac{1}{720720\mu}\Big[27027\alpha_2\chi_a^6 - 6(2148\alpha_0 - 8652\alpha_1 + 883\alpha_2 - 1299\alpha_3)\chi_a^4\chi_Q^2 \\ &\quad +(5999\alpha_0 - 149128\alpha_1 - 339144\alpha_2 - 46562\alpha_3)\chi_a^2\chi_Q^4 - 1040(9\alpha_0 - 4\alpha_2 - 24\alpha_1 \\ &\quad -4\alpha_3)\chi_Q^6\Big] + \frac{2(c_2 + 16c_7)\chi_Q^2}{45045\mu}\Big[290\chi_a^2\chi_Q^2 + 162\chi_a^4 + 65\chi_Q^4 + 117\chi_a^2 - 143\chi_Q^2 \\ &\quad -1287\Big]\ , \\ \delta M_2^{(e)} &= 2\alpha_2\mu\chi_a^2 + \frac{\mu\chi_a^2}{210}\Big[168\alpha_2\chi_a^2 - (13\alpha_0 - 284\alpha_1 - 325\alpha_2 - 59\alpha_3)\chi_Q^2\Big] \end{aligned}$$

$$
\begin{aligned}
&-\frac{\mu\chi_a^2}{138600}\Big[24255\alpha_2\chi_a^4-30(67\alpha_0+1384\alpha_1+3630\alpha_2+370\alpha_3)\chi_a^2\chi_Q^2+22(213\alpha_0\\
&-4176\alpha_1-5393\alpha_2-1004\alpha_3)\chi_Q^4\Big]-\frac{2(c_2+16c_7)\mu\chi_a^2\chi_Q^2}{3465}(9\chi_a^2-11\chi_Q^2-99)\ ,\\
\delta M_4^{(e)}\ =&\ -4\alpha_2\mu^3\chi_a^4-\frac{\mu^3\chi_a^4}{3675}\Big[\alpha_2(2205\chi_a^2+10811\chi_Q^2)-35(10\alpha_0-212\alpha_1-47\alpha_3)\chi_Q^2\Big]\\
&-\frac{2\mu^3\chi_a^4\chi_Q^2}{35}(c_2+16c_7)\ ,\\
\delta M_6^{(e)}\ =&\ 6\alpha_2\mu^5\chi_a^6\ ,\\
\delta\mathcal{S}_1^{(e)}\ =&\ -\alpha_2\chi_a-\frac{\chi_a}{420}\Big[\alpha_2(378\chi_a^2+139\chi_Q^2)-(19\alpha_0-428\alpha_1-83\alpha_3)\chi_Q^2\Big]\\
&+\frac{\chi_a}{55440}\Big[7623\alpha_2\chi_a^4-6(2\alpha_0+2636\alpha_1+7293\alpha_2+707\alpha_3)\chi_a^2\chi_Q^2-22(3\alpha_0+612\alpha_1\\
&+242\alpha_2+137\alpha_3)\chi_Q^4\Big]+\frac{\chi_a}{5765760}\Big[324324\alpha_2\chi_a^6-6(24477\alpha_0+31517\alpha_2-46908\alpha_1\\
&-4815\alpha_3)\chi_a^4\chi_Q^2-(2201\alpha_0+1594568\alpha_1+3531684\alpha_2+472882\alpha_3)\chi_a^2\chi_Q^4-260(189\alpha_0\\
&+1212\alpha_1+763\alpha_2+367\alpha_3)\chi_Q^6\Big]+\frac{2(c_2+16c_7)\chi_a\chi_Q^2}{45045}\Big[290\chi_a^2\chi_Q^2+162\chi_a^4+65\chi_Q^4\\
&+117\chi_a^2-143\chi_Q^2-1287\Big]\ ,\\
\delta\mathcal{S}_3^{(e)}\ =&\ 3\alpha_2\mu^2\chi_a^3+\frac{\mu^2\chi_a^3}{700}\Big[490\alpha_2\chi_a^2-(55\alpha_0-1180\alpha_1-1179\alpha_2-255\alpha_3)\chi_Q^2\Big]\\
&-\frac{\mu^2\chi_a^3}{1940400}\Big[412335\alpha_2\chi_a^4-42(39369\alpha_2+1330\alpha_0+14500\alpha_1+3865\alpha_3)\chi_a^2\chi_Q^2+22(3201\alpha_0\\
&-71916\alpha_1-86188\alpha_2-17419\alpha_3)\chi_Q^4\Big]+\frac{2(c_2+16c_7)\mu^2\chi_a^3\chi_Q^2}{3465}(11\chi_Q^2-9\chi_a^2+99)\ ,\\
\delta\mathcal{S}_5^{(e)}\ =&\ -5\alpha_2\mu^4\chi_a^5-\frac{\mu^4\chi_a^5}{2940}\Big[\alpha_2(1470\chi_a^2+8957\chi_Q^2)-7(47\alpha_0-988\alpha_1-223\alpha_3)\chi_Q^2\Big]\\
&-\frac{2\mu^4\chi_a^5\chi_Q^2}{35}(c_2+16c_7)\ ,\\
\delta\mathcal{S}_7^{(e)}\ =&\ 7\alpha_2\mu^6\chi_a^7\ .
\end{aligned}
\tag{30}
$$

Thus it is evident that the parity even 4-derivative terms will not contribute nontrivially to M_{2n+1} and $\mathcal{S}_{2n}$. In the current choice of integration constants, we noticed that the total mass and angular momentum seem to receive corrections and some of the multipole moments depend on 4-derivative couplings that are variant under field redefinitions. These terms are all proportional to c_2+16c_7.

In fact, the dependence of gravitational multipole moments on c_2+16c_7 is an arifact due to our bad choice of integration constants. The appropriate choice should be that the mass, angular momentum and the electric charge should receive no 4-derivative correction. In terms of new integration constants μ', χ_a', χ_Q' defined below

$$
\mu\to\mu'=\mu+\delta\mu,\quad \chi_a\to\chi_a'=\chi_a+\delta\chi_a,\quad \chi_Q\to\chi_Q'=\chi_Q+\delta\chi_Q\ ,
\tag{31}
$$

where $\delta\mu$, $\delta\chi_a$, $\delta\chi_Q$ are given in Appendix B, we see that the mass, angular momentum, and electric charge of the Kerr-Newman black hole become independent of 4-derivative couplings.

Meanwhile, all the terms that are variant under field redefinitions disappear from the corrections to multipole moments. Below we present the multipole moments written in terms of the new integration constants, and for the tidiness, we remove the "prime" from notation. Using the approximate solution, we are able to obtain the first few gravitational multipole moments listed below

$$
\begin{aligned}
\delta M_2^{(e)} &= \alpha_2\mu\chi_a^2\chi_Q^2 - \frac{\mu\chi_a^2\chi_Q^2}{300}\Big[(8\alpha_0 - 76\alpha_1 - 203\alpha_2 - 19\alpha_3)\chi_Q^2 + 30\alpha_2\chi_a^2\Big] , \\
\delta M_4^{(e)} &= -\frac{2402\mu^3\chi_a^4\chi_Q^2}{1225}\alpha_2 , \qquad \delta M_6^{(e)} = 0 , \\
\delta \mathcal{S}_3^{(e)} &= \frac{23}{25}\alpha_2\mu^2\chi_a^3\chi_Q^2 - \frac{\mu^2\chi_a^3\chi_Q^2}{4900}\Big[\alpha_2(637\chi_a^2 - 3501\chi_Q^2) + (102\alpha_0 - 1172\alpha_1 - 293\alpha_3)\chi_Q^2\Big] , \\
\delta \mathcal{S}_5^{(e)} &= -\frac{453}{245}\alpha_2\mu^4\chi_a^5\chi_Q^2 , \qquad \delta\mathcal{S}_7^{(e)} = 0 .
\end{aligned} \tag{32}
$$

For electric and magnetic multipole moments, we have

$$
\begin{aligned}
\delta \mathcal{Q}_2^{(e)} &= \frac{3}{50}\alpha_2\mu\chi_a^2\chi_Q + \frac{\mu\chi_a^2\chi_Q}{4200}\Big[\alpha_2(90\chi_a^2 + 1519\chi_Q^2) - 49(\alpha_0 + 4\alpha_1 + \alpha_3)\chi_Q^2\Big] \\
&\quad + \frac{\mu\chi_a^2\chi_Q}{94080}\Big[980\alpha_2\chi_a^4 - 3(197\alpha_0 + 1016\alpha_1 + 3288\alpha_2 + 254\alpha_3)\chi_a^2\chi_Q^2 \\
&\quad - 392(4\alpha_0 - 20\alpha_1 - 64\alpha_2 - 4\alpha_3)\chi_Q^4\Big] , \\
\delta \mathcal{Q}_4^{(e)} &= -\frac{97\alpha_2\mu^3\chi_a^4\chi_Q}{1225} + \frac{\mu^3\chi_a^4\chi_Q}{411600}\Big[(6905\alpha_0 + 29256\alpha_1 - 320788\alpha_2 + 7314\alpha_3)\chi_Q^2 \\
&\quad - 10976\alpha_2\chi_a^2\Big] , \\
\delta \mathcal{Q}_6^{(e)} &= \frac{125\alpha_2\mu^5\chi_a^6\chi_Q}{1386} , \\
\delta \mathcal{P}_1^{(e)} &= -\frac{1}{2}\alpha_2\chi_a\chi_Q + \frac{\chi_a\chi_Q}{120}\Big[\alpha_2(6\chi_a^2 - 13\chi_Q^2) + (\alpha_0 - 20\alpha_1 - 5\alpha_3)\chi_Q^2\Big] + \frac{\chi_a\chi_Q}{3360}\Big[63\alpha_2\chi_a^4 \\
&\quad - 6(4\alpha_0 + 4\alpha_1 + 27\alpha_2 + \alpha_3)\chi_a^2\chi_Q^2 + 14(\alpha_0 - 20\alpha_1 - 10\alpha_2 - 5\alpha_3)\chi_Q^4\Big] , \\
\delta \mathcal{P}_3^{(e)} &= \frac{13}{25}\alpha_2\mu^2\chi_a^3\chi_Q - \frac{\mu^2\chi_a^3\chi_Q}{2450}\Big[112\alpha_2\chi_a^2 + (11\alpha_0 - 528\alpha_1 + 755\alpha_2 - 132\alpha_3)\chi_Q^2\Big] , \\
\delta \mathcal{P}_5^{(e)} &= -\frac{781\alpha_2\mu^4\chi_a^5\chi_Q}{1470} .
\end{aligned} \tag{33}
$$

The results depends only on the invariant combinations $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of coupling constants defined in (19).

Similar to the 2-derivative case, from the field equation of A_μ, we can define its magnetic dual vector field $\widetilde{A}_\mu$

$$
\nabla^\mu \mathcal{D}^{(e)}_{\mu\nu} = 0 \quad \Longrightarrow \quad \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\mathcal{D}^{(e)\rho\lambda} = 2\nabla_{[\mu}\widetilde{A}^{(e)}_{\nu]} . \tag{34}
$$

The explicit form of the induction tensor $D^{(e)}_{\mu\nu}$ can be seen in Appendix C from the parity even part of the total $U(1)$ field equation. We find that

$$
\widetilde{A}^{(e)}_{(1)} = -2\Big(\frac{Qax}{\Sigma} + \widetilde{\mathcal{H}}_1\Big)dt - 2\Big(-\frac{Qa^2x(1-x^2)}{\Sigma} - Qx + \widetilde{\mathcal{H}}_2\Big)d\varphi , \tag{35}
$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are also expanded in terms of power series of χ_a and χ_Q. Readers interested in their explicit forms up to $\mathcal{O}(\chi^7)$ are referred to the accompanying Mathematica file. Matching $\widetilde{A}_{(1)}$ to the desired form in (13), we read off the leading expansion coefficients which confirm the relation

$$\delta\widetilde{\mathcal{P}}^{(e)}_{0,\,2,\,4,\,6} = \delta\mathcal{Q}^{(e)}_{0,\,2,\,4,\,6}, \qquad \delta\widetilde{\mathcal{Q}}^{(e)}_{1,\,3,\,5} = -\delta\mathcal{P}^{(e)}_{1,\,3,\,5} \ . \tag{36}$$

3.2 Parity-odd case

We now turn to study the effects of parity-odd 4-derivative interactions on Kerr-Newman black hole and its multipole moments. In $D = 4$, the independent parity odd 4-derivative terms can be parametrized as

$$\mathcal{L}_4^{(o)} = d_1 R_{\mu\nu} F^{\mu\rho} \widetilde{F}^{\nu}{}_{\rho} + d_2 R_{\mu\nu\rho\sigma} F^{\mu\nu} \widetilde{F}^{\rho\sigma} + d_3 \widetilde{F}_{\mu\nu} F^{\mu\nu} F^2 + d_4 \widetilde{F}_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \ , \tag{37}$$

where $\widetilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ and we have used the useful identity that $g_{\mu\nu} F_{\rho\sigma} \widetilde{F}^{\rho\sigma} = 4 F_{\mu\rho} \widetilde{F}_{\nu}{}^{\rho}$. Because of the same identity, the parity odd field redefinition of the metric has only one structure

$$g_{\mu\nu} \to g_{\mu\nu} + \lambda_5 F_{\mu\rho} \widetilde{F}_{\nu}{}^{\rho} \ , \tag{38}$$

which shifts the coupling constants in (37) according to as

$$d_1 \to d_1 + \lambda_5, \quad d_2 \to d_2, \quad d_3 \to d_3 - \frac{\lambda_5}{8}, \quad d_4 \to d_4 + \frac{\lambda_5}{2} \ . \tag{39}$$

One can easily check that the combinations of 4-derivative couplings below are invariant under (39)

$$\beta_0 = 4d_3 + d_4, \quad \beta_1 = d_2, \quad \beta_2 = 8d_3 + d_1 \ . \tag{40}$$

Repeating the same procedure as in the parity even case, we solve the perturbed solution up to $\mathcal{O}(\chi^8)$. The modifications to the field equations due to parity odd 4-derivative terms are given in Appendix C. Switching to the ACMC coordinate system, we obtain the parity odd 4-derivative contributions to the first few multipole moments. Interestingly, the M_{2n}, $\mathcal{S}_{2n+1}$, $\mathcal{Q}_{2n}$ and $\mathcal{P}_{2n+1}$ which are nonvanishing at 2-derivative level, do not receive corrections. Instead, non-trivial corrections appear in M_{2n+1}, $\mathcal{S}_{2n}$, $\mathcal{Q}_{2n+1}$ and $\mathcal{P}_{2n}$. This is understandable because the field equations are parity odd, therefore shifting the multipolar index ℓ by 1. The mass dipole remains zero because of the choice of ACMC coordinates. Specifically, for the gravitational multipole moments we have

$$\begin{aligned}
\delta M_3^{(o)} &= -\frac{23}{25}\beta_1 \mu^2 \chi_a^3 \chi_Q^2 + \frac{\mu^2 \chi_a^3 \chi_Q^2}{700}(91\beta_1 \chi_a^2 + 142\beta_0 \chi_Q^2 - 391\beta_1 \chi_Q^2) \ , \\
\delta M_5^{(o)} &= \frac{453}{245}\beta_1 \mu^4 \chi_a^5 \chi_Q^2 \ ,
\end{aligned}$$

$$
\begin{aligned}
\delta\mathcal{S}_2^{(o)} &= \beta_1\mu\chi_a^2\chi_Q^2 - \frac{\mu\chi_a^2\chi_Q^2}{300}(30\beta_1\chi_a^2 + 70\beta_0\chi_Q^2 - 159\beta_1\chi_Q^2) \\
&\quad - \frac{\mu\chi_a^2\chi_Q^2}{560}\Big[28\beta_0\chi_Q^2(5\chi_Q^2 - \chi_a^2) + 3\beta_1(5\chi_a^2\chi_Q^2 + 7\chi_a^4 - 56\chi_Q^4)\Big] , \\
\delta\mathcal{S}_4^{(o)} &= -\frac{2402\beta_1\mu^3\chi_a^4\chi_Q^2}{1225} + \frac{\mu^3\chi_a^4\chi_Q^2}{7350}(1603\beta_1\chi_a^2 + 3331\beta_0\chi_Q^2 - 7845\beta_1\chi_Q^2) , \\
\delta\mathcal{S}_6^{(o)} &= \frac{70667\beta_1\mu^5\chi_a^6\chi_Q^2}{24255} ,
\end{aligned} \tag{41}
$$

and for electric and magnetic multipole moments, we obtain

$$
\begin{aligned}
\delta\mathcal{Q}_1^{(o)} &= -\frac{1}{2}\beta_1\chi_a\chi_Q + \frac{\chi_a\chi_Q}{120}\Big[3\beta_1(2\chi_a^2 + \chi_Q^2) + 14\beta_0\chi_Q^2\Big] + \frac{\chi_a\chi_Q}{480}\Big[4\beta_0\chi_Q^2(7\chi_Q^2 - 3\chi_a^2) \\
&\quad + 3\beta_1(3\chi_a^4 - 2\chi_a^2\chi_Q^2 + 4\chi_Q^4)\Big] + \frac{\chi_a\chi_Q}{640}\Big[\beta_1(9\chi_a^4\chi_Q^2 - 12\chi_a^2\chi_Q^4 + 6\chi_a^6 + 10\chi_Q^6) \\
&\quad + 2\beta_0(10\chi_Q^6 - 3\chi_a^4\chi_Q^2)\Big] , \\
\delta\mathcal{Q}_3^{(o)} &= \frac{13}{25}\beta_1\mu^2\chi_a^3\chi_Q - \frac{\mu^2\chi_a^3\chi_Q}{350}\Big[16\beta_1\chi_a^2 + (44\beta_0 + 177\beta_1)\chi_Q^2\Big] \\
&\quad - \frac{\mu^2\chi_a^3\chi_Q}{50400}\Big[882\beta_1\chi_a^4 - (1142\beta_0 + 3807\beta_1)\chi_a^2\chi_Q^2 - 36(70\beta_0 - 431\beta_1)\chi_Q^4\Big] , \\
\delta\mathcal{Q}_5^{(o)} &= -\frac{781\beta_1\mu^4\chi_a^5\chi_Q}{1470} + \frac{\mu^4\chi_a^5\chi_Q}{52920}\Big[2310\beta_1\chi_a^2 + (6934\beta_0 + 52011\beta_1)\chi_Q^2\Big] , \\
\delta\mathcal{Q}_7^{(o)} &= \frac{121388\beta_1\mu^6\chi_a^7\chi_Q}{225225} , \\
\delta\mathcal{P}_2^{(o)} &= -\frac{3}{50}\beta_1\mu\chi_a^2\chi_Q - \frac{\mu\chi_a^2\chi_Q}{4200}\Big[90\beta_1\chi_a^2 - 49(2\beta_0 - 39\beta_1)\chi_Q^2\Big] \\
&\quad - \frac{\mu\chi_a^2\chi_Q}{3360}\Big[35\beta_1\chi_a^4 - 3(10\beta_0 + 53\beta_1)\chi_a^2\chi_Q^2 - 28(13\beta_0 - 30\beta_1)\chi_Q^4\Big] , \\
\delta\mathcal{P}_4^{(o)} &= \frac{97\beta_1\mu^3\chi_a^4\chi_Q}{1225} + \frac{2\mu^3\chi_a^4\chi_Q}{3675}\Big[49\beta_1\chi_a^2 - 2(29\beta_0 - 843\beta_1)\chi_Q^2\Big] , \\
\delta\mathcal{P}_6^{(o)} &= -\frac{125\beta_1\mu^5\chi_a^6\chi_Q}{1386} .
\end{aligned} \tag{42}
$$

It is evident that all the expressions above are manifestly invariant under field redefinitions. Note that the magnetic charge $\mathcal{P}_0$, which is zero in our original Kerr-Newman black hole, remains uncorrected, whilst the magnetic higher even multipole moments all receive corrections by the party-odd 4-derivative terms.

Similar to the parity-even case, using the $U(1)$ field equations, we can also define the dual 1-form potential $\widetilde{A}_1^{(o)}$ which takes the form

$$
\widetilde{A}_{(1)}^{(o)} = -2\Big(\frac{Qax}{\Sigma} + \widetilde{\mathcal{H}}_3\Big)dt - 2\Big(-\frac{Qa^2x(1-x^2)}{\Sigma} - Qx + \widetilde{\mathcal{H}}_4\Big)d\varphi . \tag{43}
$$

where $\mathcal{H}_3$ and $\mathcal{H}_4$ are also expanded in terms of power series of χ_a and χ_Q. Readers interested in their explicit forms up to $\mathcal{O}(\chi^8)$ are referred to the accompanying Mathematica file. In terms of the ACMC coordinates, by matching the large r expansion of (43) to (13), we confirm that the electric and magnetic multipole moments satisfy the electromagnetic duality relation

$$\widetilde{\mathcal{Q}}^{(o)}_{1,3,5,7} = -\mathcal{P}^{(o)}_{1,3,5,7}\,, \quad \delta\widetilde{\mathcal{Q}}^{(o)}_{2,4,6} = -\delta\mathcal{P}^{(o)}_{2,4,6}\,. \tag{44}$$

4 Conclusion

Black hole multipole moments are potentially useful observables for probing the nonlinear structures of General Relativity and its modifications due to unknown UV physics. In this work, we made progress in addressing a key question: do higher derivative perturbative corrections to black hole multipole moments are invariant under field redefinitions that are not supposed to affect physical quantities in a low energy effective theory of quantum gravity. So far, traditional methods for calculating multipole moments have not explicitly demonstrated this property for general effective theory of gravity. As far as we are aware, the field redefinition invariance of black hole multipole moments has only been shown for $D = 4$ Einstein gravity extended by cubic curvature terms [15], where the Ricci flatness of the leading order solution had played a role in the proof. Here, we further investigated the leading higher-derivative corrections (16,37) to the non-Ricci flat Kerr-Newman black hole (4). We find that the leading higher derivative corrections to black hole multipole moments indeed depend only on the combinations of coupling constants that are inert under field redefinitions.

To achieve this, we first cast the electrically-charged Kerr-Newman black hole in the ACMC coordinates. We read off the nonvanishing mass and current multipole moments $\{M_{2n}, \mathcal{S}_{2n+1}\}$ from the metric, and the nonvanishing electric and magnetic multipole moments $\{\mathcal{Q}_{2n}, \mathcal{P}_{2n+1}\}$ from the Maxwell field. The absence of the magnetic monopole has a consequence that all the even magnetic multiple moments vanish, and the odd ones are generated by the electric charge under rotation. We then adopted an approximate method of solving for the higher derivative corrected metric and $U(1)$ gauge field order by order in dimensionless rotation parameter $\chi_a = a/\mu$ and charge parameter $\chi_Q = q/\mu$. To simplify the computation, we arrange the infinite series expansion in the basis of homogeneous polynomials of χ's.

In $D = 4$, the leading higher derivative corrections to Einstein-Maxwell theory are classified by their properties under parity transformations. For parity preserving 4-derivative interactions, we are able to obtain the perturbed solution up to $\mathcal{O}(\chi^7)$. Similar to the pure gravity case, 4-derivative interactions only add corrections to multipole moments that are actually nonzero at the leading order, namely, $\{M_{2n+2}, \mathcal{S}_{2n+3}, \mathcal{Q}_{2n+2}, \mathcal{P}_{2n+1}\}$ for $n \geq 0$. Concerning the parity odd 4-derivative interactions, we obtain the perturbed solution up to $\mathcal{O}(\chi^8)$. We find that while these higher-derivative terms do not contribute to black hole thermodynamics, they affect the perturbative solutions, thereby modifying the black hole multipole moments. Notably, they contribute to multipole moments $\{M_{2n+1}, \mathcal{S}_{2n}, \mathcal{Q}_{2n-1}, \mathcal{P}_{2n}\}$ for $n \geq 1$ that all vanish at the leading order. This feature has significant observational implications. If they are observed, it would indicate the presence of parity odd higher-derivative corrections to General Relativity.

As for future directions, we would like to push further to consider the next to next leading order higher derivative corrections and check if black hole multipole moments remain invariant under field redefinitions. The current method only allows us to carry out a case by case verification, after solving for the perturbed solution. For more general theories of gravity, in order to prove field redefinition invariance of black hole multipole moments, one may need to resort to a new formalism such as the one based on covariant phase space approach [21]. Inspired by previous work [27], we propose generalizations of Geroch-Hansen formulae to broader gravity models, in which the generalized twist 1-form is derived using the covariant phase space approach. The results are presented in Appendix D. At this moment, we have not succeeded in applying these results to prove the field redefinition invariance of multipole moments and would like to pursue this problem in future study.

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A Black hole thermodynamics

In this appendix, we shall verify our perturbative solution from a thermodynamic point of view. For asymptotically flat black hole like Kerr-Newman, there are at least two ways of computing the thermodynamic variables with leading higher derivative corrections. In the ordinary approach, one needs to first solve for the corrected solution and subsequently computes all the thermodynamic quantities using Wald procedure [28, 29] or quasilocal formalism [30]. The second approach was proposed by Reall and Santos [2], which enables us to derive the leading higher derivative corrections to black hole thermodynamics using only the uncorrected solution. The second approach has been rigorously tested in various problems related to the thermodynamics of rotating black holes [31–34]. Below, we will show that the black hole thermodynamics obtained from the perturbative solution (22) using ordinary method agrees with that derived using the Reall-Santos method, demonstrating the legitimacy of our solution up to the approximation order. It is important to note that the parity odd 4-derivative interactions (37) do not contribute black hole thermodynamics. Thus the results presented below encode corrections only from parity preserving 4-derivative terms.

A.1 Black Hole thermodynamics from ordinary method

For the perturbative solution (22), the black hole outer horizon is located at $r = r_h$, which satisfies

$$(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2)\Big|_{r=r_h} = 0 \quad \Longrightarrow \quad r_h = \mu + \mu\sqrt{1 - \chi^2(\chi_a^2 + \chi_Q^2)} \ . \tag{45}$$

The angular velocity of the black hole is then given by

$$\Omega_H = -\Big(\frac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_{r=r_h} - \frac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_{r=\infty}\Big) \ . \tag{46}$$

Using the Killing vector $\xi = \partial_t + \Omega_H\partial_\varphi$ null at horizon, we compute the surface gravity κ and temperature T as

$$\kappa^2 = -\frac{g^{\mu\nu}\partial_\mu\xi^2\partial_\nu\xi^2}{4\xi^2}\Big|_{r=r_h} , \quad T = \frac{\kappa}{2\pi} \ . \tag{47}$$

We then compute the entropy using Wald formula [28]

$$S = -\frac{1}{8}\int_{\mathcal{B}} d\Omega \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} \ , \tag{48}$$

where $\mathcal{L}$ is the total Lagrangian and $\mathcal{B}$ is the bifurcation horizon. The electric charge and potential are given by The electric charge is defined by the EOM

$$Q_e = \frac{1}{16\pi}\int_{S^2} \star\mathcal{D}_{(2)}, \quad \Phi_e = \xi^\mu A_\mu\Big|_{r=r_h}^{r=\infty} \ . \tag{49}$$

The energy and the angular momentum are obtained using Brown-York quasilocal stress tensor [30]. After performing the redefinition of the integration constants (31), the full set of thermodynamic quantities are summarized below.

$$\begin{aligned}
M &= \mu + \mathcal{O}(c_i^2, \chi^7) \ , \quad Q_e = \frac{1}{2}\mu\chi_Q + \mathcal{O}(c_i^2, \chi^7) \ , \quad J = \mu^2\chi_a + \mathcal{O}(c_i^2, \chi^7) \ , \\
T &= \frac{\sqrt{1-(\chi_a^2+\chi_Q^2)}}{2\pi\mu\big(2\sqrt{1-(\chi_a^2+\chi_Q^2)} - \chi_Q^2 + 2\big)} + \delta T^{(e)} + \mathcal{O}(c_i^2, \chi^7) \ , \\
S &= \pi\mu^2\Big[\big(\sqrt{1-(\chi_a^2+\chi_Q^2)} + 1\big)^2 + \chi_a^2\Big] + \delta S^{(e)} + \mathcal{O}(c_i^2, \chi^7) \ , \\
\Omega_H &= \frac{\chi_a}{\mu\big(2\sqrt{1-(\chi_a^2+\chi_Q^2)} - \chi_Q^2 + 2\big)} + \delta\Omega_H^{(e)} + \mathcal{O}(c_i^2, \chi^7) \ , \\
\Phi_e &= \frac{2\chi_Q\big(\sqrt{1-(\chi_a^2+\chi_Q^2)} + 1\big)}{2\sqrt{1-(\chi_a^2+\chi_Q^2)} - \chi_Q^2 + 2} + \delta\Phi_e^{(e)} + \mathcal{O}(c_i^2, \chi^7) \ .
\end{aligned} \tag{50}$$

where the correction terms take the form

$$
\begin{aligned}
\delta T^{(e)} &= -\frac{\alpha_2\chi_Q^2}{32\pi\mu^3} + \frac{\chi_Q^2}{640\pi\mu^3}\Big[5\alpha_2(5\chi_a^2 - 4\chi_Q^2) + \alpha_0\chi_Q^2\Big] + \frac{\chi_Q^2}{21504\pi\mu^3}\Big[\chi_Q^2\big(\alpha_0(8\chi_a^2 + 63\chi_Q^2) \\
&\quad +128(4\alpha_1 + \alpha_3)\chi_a^2\big) + 2\alpha_2(776\chi_a^2\chi_Q^2 + 231\chi_a^4 - 273\chi_Q^4)\Big] \;, \\
\delta S^{(e)} &= -\pi\alpha_2\chi_Q^2 + \frac{\pi\chi_Q^2}{40}\Big[10\alpha_2\left(\chi_a^2 - 2\chi_Q^2\right) + \alpha_0\chi_Q^2\Big] + \frac{\pi\chi_Q^2}{3360}\Big[\chi_Q^2\big(\alpha_0(31\chi_a^2 + 105\chi_Q^2) \\
&\quad +160(4\alpha_1 + \alpha_3)\chi_a^2\big) + 2\alpha_2(550\chi_a^2\chi_Q^2 + 294\chi_a^4 - 525\chi_Q^4)\Big] \;, \\
\delta\Omega_H^{(e)} &= -\frac{\alpha_2\chi_a\chi_Q^2}{8\mu^3} - \frac{\chi_a\chi_Q^2}{13440\mu^3}\Big[8\alpha_2(42\chi_a^2 + 295\chi_Q^2) + \big(640\alpha_1 + 160\alpha_3 - 11\alpha_0\big)\chi_Q^2\Big] \\
&\quad -\frac{\chi_a\chi_Q^2}{53760\mu^3}\Big[4\alpha_2(1520\chi_a^2\chi_Q^2 + 231\chi_a^4 + 2555\chi_Q^4) + \chi_Q^2\big(32(4\alpha_1 + \alpha_3)(13\chi_a^2 + 35\chi_Q^2) \\
&\quad +\alpha_0(201\chi_a^2 - 35\chi_Q^2)\big)\Big] \;, \\
\delta\Phi_e^{(e)} &= \frac{\alpha_2\chi_Q}{2\mu^2} - \frac{\chi_Q}{40\mu^2}\Big[10\alpha_2(\chi_a^2 - \chi_Q^2) + \alpha_0\chi_Q^2\Big] - \frac{\chi_Q}{6720\mu^2}\Big[\chi_Q^2\big(\alpha_0(20\chi_a^2 + 231\chi_Q^2) \\
&\quad +320(4\alpha_1 + \alpha_3)\chi_a^2\big) + 2\alpha_2(1100\chi_a^2\chi_Q^2 + 399\chi_a^4 - 420\chi_Q^4)\Big] - \frac{\chi_Q}{26880\mu^2}\Big[\chi_Q^2\big((445\chi_a^2\chi_Q^2 \\
&\quad +210\chi_a^4 + 1008\chi_Q^4)\alpha_0 + 32(4\alpha_1 + \alpha_3)(28\chi_a^2 + 65\chi_Q^2)\chi_a^2\big) + 4(2555\chi_a^4\chi_Q^2 + 2630\chi_a^2\chi_Q^4 \\
&\quad +483\chi_a^6 - 420\chi_Q^6)\alpha_2\Big] \;.
\end{aligned}
\tag{51}
$$

In the expressions above, the leading terms in various thermodynamic quantities are exact in χ. The precision level of the corrections is inherited from that of the perturbative solution which is at $\mathcal{O}(\chi^7)$.

A.2 Black Hole thermodynamics from Reall-Santos method

We now apply the Reall-Santos method [2] method to compute the same set of thermodynamic quantities. In the spirit of [2], we simply plug the uncorrected solution into the total action with 4-derivative terms and integrate from the outer horizon to infinity. The resulting Euclidean action is a function of the uncorrected temperature T_0, angular velocity $\Omega_{H,0}$, and electric potential $\Phi_{e,0}$ whose expressions can be seen in (50). It is also straightforward to see that the parity odd terms in (37) vanish on the purely electric black hole solution. Then similar to the results obtained using ordinary method, only parity even 4-derivative terms contribute to the thermodynamic quantities. The total Euclidean action takes the form

$$
I_E(T_0, \Omega_{H,0}, \Phi_{e,0}) = T_0^{-1} G(T_0, \Omega_{H,0}, \Phi_{e,0}) \;, \quad G = G_0 + \delta G^{(e)} \;, \tag{52}
$$

where the leading order Gibbs free energy and its correction are given by

$$G_0 = \frac{r_h^2 + a^2}{4r_h} + \frac{Q^2}{4r_h}\left(1 - \frac{2r_h^2}{r_h^2 + a^2}\right),$$

$$\delta G^{(e)} = -\frac{Q^4(\alpha_0 + 8\alpha_1 + 12\alpha_2 + 2\alpha_3)}{128a^4 r_h^3}\Big(\frac{3a^4 + 2a^2 r_h^2 + 3r_h^4}{r_h^2 + a^2} + \frac{3\left(a^4 - r_h^4\right)\tan^{-1}\left(\frac{a}{r_h}\right)}{ar_h}\Big)$$
$$-\frac{Q^4 r_h(3a^4 - 10a^2 r_h^2 + 3r_h^4)}{60(r_h^2 + a^2)^5}(\alpha_0 - 8\alpha_1 - 4\alpha_2 - 2\alpha_3) - \frac{2(Q^2 - 2\mu r_h)\big(Q^2 r_h - \mu(r_h^2 - a^2)\big)}{(r_h^2 + a^2)^3}\alpha_1$$
$$-\frac{2Q^2\big(4Q^2 r_h(r_h^2 - a^2) - \mu(a^4 - 10a^2 r_h^2 + 5r_h^4)\big)}{5(r_h^2 + a^2)^4}\alpha_2 \ . \tag{53}$$

Other thermodynamic quantities are derived from the Gibbs free energy according to

$$S = -\frac{\partial G(T_0, \Omega_{H,0}, \Phi_{e,0})}{\partial T_0}\Big|_{(\Omega_{H,0},\Phi_{e,0})}\ , \quad Q_e = -\frac{\partial G^{(e)}(T_0, \Omega_{H,0}, \Phi_{e,0})}{\partial \Phi_{e,0}}\Big|_{(T_0,\Omega_{H,0})}\ ,$$
$$J = -\frac{\partial G^{(e)}(T_0, \Omega_{H,0}, \Phi_{e,0})}{\partial \Omega_{H,0}}\Big|_{(T_0,\Phi_{e,0})}\ , \quad M = G + T_0 S + \Omega_{H,0} J + \Phi_{e,0} Q_e \ . \tag{54}$$

By this way, we obtain all the thermodynamic quantities in a specific choice of the integration constants such that the forms of temperature, angular velocity and electric potential are not modified by the 4-derivative interactions. Instead, the conserved charges such as the mass, angular momentum and electric charge do receive corrections. To compare with the results obtained using ordinary method, we need to redefine the integration constants in terms of which the conserved charges remain the same form as in the 2-derivative theory. After performing the appropriate redefinition of integration constant, we find that

$$M = M_0 + \mathcal{O}(c_i^2), \quad Q_e = Q_{e,0} + \mathcal{O}(c_i^2), \quad J = J_0 + \mathcal{O}(c_i^2)\ , \tag{55}$$

and

$$T = T_0 + \delta T^{(e)}, \quad S = S_0 + \delta S^{(e)}, \quad \Phi_e = \Phi_{e,0} + \delta\Phi_e^{(e)}, \quad \Omega_H = \Omega_{H,0} + \delta\Omega_H^{(e)}\ , \tag{56}$$

where the corrections term indeed agree with those in (51).

B Redefinitions of integration constants

In (31), we have performed a redefinition of the integration constants μ, χ_a and χ_Q so that in terms of the new constants, the mass, angular momentum and electric charge remain the same form as in the 2-derivative theory. Below we list $\delta\mu$, $\delta\chi_a$ and χ_Q up to certain order in χ's

$$\begin{aligned}
\delta\mu =& \frac{1}{35\mu}\Big[35\alpha_2\chi_a^2+(2c_2+32c_7-\alpha_0+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^2\Big]+\frac{1}{13860\mu}\Big[-1386\alpha_2\chi_a^4\\
&+3\chi_a^2(-24c_2-384c_7-65\alpha_0+1252\alpha_1+3201\alpha_2+337\alpha_3)\chi_Q^2+44(2c_2+32c_7+3\alpha_0\\
&+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^4\Big]+\frac{1}{720720\mu}\Big[-6\chi_a^4(864c_2+13824c_7-2148\alpha_0+8652\alpha_1-883\alpha_2\\
&+1299\alpha_3)\chi_Q^2+\chi_a^2(-9280c_2-148480c_7-5999\alpha_0+149128\alpha_1+339144\alpha_2+46562\alpha_3)\chi_Q^4\\
&-27027\alpha_2\chi_a^6-1040(2c_2+32c_7-9\alpha_0+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^6\Big]+\mathcal{O}(\chi^7)\ ,\\
\delta\chi_a =& \frac{\alpha_2\chi_a}{\mu^2}-\frac{1}{420\mu^2}\Big[\chi_a(24c_2+384c_7-5\alpha_0+148\alpha_1-43\alpha_2+13\alpha_3)\chi_Q^2+462\alpha_2\chi_a^3\Big]\\
&+\frac{1}{55440\mu^2}\Big[3465\alpha_2\chi_a^5+6\chi_a^3(48c_2+768c_7+262\alpha_0-2372\alpha_1-5511\alpha_2-641\alpha_3)\chi_Q^2\\
&+22\chi_a(-16c_2-256c_7-45\alpha_0+228\alpha_1+178\alpha_2+73\alpha_3)\chi_Q^4\Big]+\frac{\chi_a}{5765760\mu^2}\Big[108108\alpha_2\chi_a^6\\
&+6\chi_a^4(6912c_2+110592c_7-9891\alpha_0+91524\alpha_1+17389\alpha_2+15969\alpha_3)\chi_Q^2\\
&+5\chi_a^2(14848c_2+237568c_7+19637\alpha_0-158296\alpha_1-378924\alpha_2-54422\alpha_3)\chi_Q^4\\
&+260(64c_2+1024c_7-387\alpha_0+2748\alpha_1+1019\alpha_2+623\alpha_3)\chi_Q^6\Big]+\mathcal{O}(\chi^8)\ ,\\
\delta\chi_Q =& -\frac{1}{35\mu^2}\Big[35\alpha_2\chi_a^2\chi_Q+(2c_2+32c_7-\alpha_0+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^3\Big]+\frac{1}{13860\mu^2}\Big[1386\alpha_2\chi_a^4\chi_Q\\
&+3\chi_a^2(24c_2+384c_7+65\alpha_0-1252\alpha_1-3201\alpha_2-337\alpha_3)\chi_Q^3-44(2c_2+32c_7+3\alpha_0\\
&+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^5\Big]+\frac{1}{720720\mu^2}\Big[1040(2c_2+32c_7-9\alpha_0+4(6\alpha_1+\alpha_2+\alpha_3))\chi_Q^7\\
&+\chi_a^2(9280c_2+148480c_7+5999\alpha_0-149128\alpha_1-339144\alpha_2-46562\alpha_3)\chi_Q^5+6\chi_a^4\big(864c_2\\
&+13824c_7-2148\alpha_0+8652\alpha_1-883\alpha_2+1299\alpha_3\big)\chi_Q^3+27027\alpha_2\chi_a^6\chi_Q\Big]+\mathcal{O}(\chi^8)\ . \qquad (57)
\end{aligned}$$

C Equations of motion

In this section, we derive the field equations from the extended Einstein-Maxwell theory described by the Lagrangian by $\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \widetilde{F}_{\mu\nu}, F_{\mu\nu})$ in $D=4$. Let us first define

$$\widetilde{\mathcal{M}}^{\mu\nu}=-2\frac{\delta\mathcal{L}}{\delta\widetilde{F}_{\mu\nu}}\ ,\quad \mathcal{M}^{\mu\nu}=-2\frac{\delta\mathcal{L}}{\delta F_{\mu\nu}}\ ,\quad P^{\mu\nu\rho\lambda}=\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\lambda}}\ . \qquad (58)$$

Then equations of motion for the metric and $U(1)$ gauge field take the form

$$\begin{aligned}
0=E_{g,\mu\nu}:=&\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}=P_{(\mu}{}^{\alpha\beta\gamma}R_{\nu)\alpha\beta\gamma}-\frac{1}{2}g_{\mu\nu}\mathcal{L}+2\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho}-\frac{1}{2}\widetilde{\mathcal{M}}_{(\mu}{}^{\alpha}\widetilde{F}_{\nu)\alpha}\\
&-\frac{1}{2}\mathcal{M}_{(\mu}{}^{\alpha}F_{\nu)\alpha}+\frac{1}{4}g_{\mu\nu}\widetilde{\mathcal{M}}^{\alpha\beta}\widetilde{F}_{\alpha\beta}+\frac{1}{2}\epsilon_{\alpha\beta\rho(\mu}F_{\nu)}{}^{\rho}\widetilde{\mathcal{M}}^{\alpha\beta}\ ,\\
0=E_A^\mu:=&\frac{\delta\mathcal{L}}{\delta A_\mu}=\nabla_\nu\mathcal{D}^{\nu\mu}\ ,\quad \mathcal{D}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widetilde{\mathcal{M}}_{\rho\sigma}+\mathcal{M}^{\mu\nu}\ . \qquad (59)
\end{aligned}$$

Specific to the 4-derivative extension of Einstein-Maxwell theory considered here, we divide the field equations into the leading pieces and the corrections

$$E_{g,\mu\nu} = E^{(0)}_{g,\mu\nu} + \Delta E_{g,\mu\nu}, \quad E^\mu_A = E^{(0)\mu}_A + \Delta E^\mu_A \ . \tag{60}$$

The effective energy momentum tensor and electric current defined in (21) are given by

$$\Delta T_{\mu\nu} = -2\Delta E_{g,\mu\nu}, \quad \Delta J^\mu = -\Delta E^\mu_A \ . \tag{61}$$

Analogously, the $P_{\mu\nu\rho\sigma}$ and $\mathcal{M}_{\mu\nu}$ can also be separated into 2-and 4- derivative parts

$$\begin{aligned} P_{\mu\nu\rho\sigma} &= P^{(0)}_{\mu\nu\rho\sigma} + \Delta P_{\mu\nu\rho\sigma} \ , \quad \mathcal{M}_{\mu\nu} = \mathcal{M}^{(0)}_{\mu\nu} + \Delta\mathcal{M}_{\mu\nu} \ , \\ P^{(0)}_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \ , \quad \mathcal{M}^{(0)}_{\mu\nu} = F_{\mu\nu} \ . \end{aligned} \tag{62}$$

On the other hand, $\widetilde{M}_{\mu\nu}$ receive contributions only from parity odd 4-derivative terms. Corresponding to our parametrization of the 4-derivative actions (16) and (37), we have

$$\begin{aligned} \Delta P_{\mu\nu\rho\sigma} &= \sum_{i=1}^{8} c_i P^{(e,i)}_{\mu\nu\rho\sigma} + \sum_{i=1}^{4} d_i P^{(o,i)}_{\mu\nu\rho\sigma}, \\ P^{(e,1)}_{\mu\nu\rho\sigma} &= R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \ , \quad P^{(e,2)}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\mu\rho}R_{\nu\sigma}\right) \ , \\ P^{(e,3)}_{\mu\nu\rho\sigma} &= 2R_{\mu\nu\rho\sigma}, \quad P^{(e,4)}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)F^2 \ , \\ P^{(e,5)}_{\mu\nu\rho\sigma} &= g_{\nu\sigma}F^2_{\mu\rho} - g_{\nu\rho}F^2_{\mu\sigma} - g_{\mu\sigma}F^2_{\nu\rho} + g_{\mu\rho}F^2_{\nu\sigma}, \quad P^{(e,6)}_{\mu\nu\rho\sigma} = F_{\mu\nu}F_{\rho\sigma}, \quad P^{(e,7,8)}_{\mu\nu\rho\sigma} = 0 \ , \\ P^{(o,1)}_{\mu\nu\rho\sigma} &= -\frac{1}{4}g_{\nu[\rho}\widetilde{F}^2_{\sigma]\mu} + \frac{1}{4}g_{\mu[\rho}\widetilde{F}^2_{\sigma]\nu} - \frac{1}{4}g_{\sigma[\mu}\widetilde{F}^2_{\nu]\rho} + \frac{1}{4}g_{\rho[\mu}\widetilde{F}^2_{\nu]\sigma}, \\ P^{(o,2)}_{\mu\nu\rho\sigma} &= \frac{1}{2}F_{\rho\sigma}\widetilde{F}_{\mu\nu} + \frac{1}{2}F_{\mu\nu}\widetilde{F}_{\rho\sigma}, \quad P^{(o,3)}_{\mu\nu\rho\sigma} = 0, \quad P^{(o,4)}_{\mu\nu\rho\sigma} = 0 \ . \end{aligned} \tag{63}$$

and

$$\begin{aligned} \Delta\mathcal{M}_{\mu\nu} &= \sum_{i=1}^{8} c_i \mathcal{M}^{(e,i)}_{\mu\nu}, \quad \widetilde{\mathcal{M}}_{\mu\nu} = \sum_{i=1}^{4} d_i \widetilde{\mathcal{M}}^{(o,i)}_{\mu\nu} \ , \\ \mathcal{M}^{(e,1)}_{\mu\nu} &= 0, \quad \mathcal{M}^{(e,2)}_{\mu\nu} = 0, \quad \mathcal{M}^{(e,3)}_{\mu\nu} = 0, \quad \mathcal{M}^{(e,4)}_{\mu\nu} = -4F_{\mu\nu}R, \quad \mathcal{M}^{(e,5)}_{\mu\nu} = -4F_{[\mu}{}^{\rho}R_{\nu]\rho} \ , \\ \mathcal{M}^{(e,6)}_{\mu\nu} &= -4R_{\mu\nu\rho\sigma}F^{\rho\sigma} \ , \quad \mathcal{M}^{(e,7)}_{\mu\nu} = -4F_{\mu\nu}F^2 \ , \quad \mathcal{M}^{(e,8)}_{\mu\nu} = -8F_\mu{}^\rho F_\nu{}^\sigma F_{\rho\sigma} \ , \\ \mathcal{M}^{(o,1)}_{\mu\nu} &= -2\widetilde{F}_{[\mu}{}^{\alpha}R_{\nu]\alpha}, \quad \mathcal{M}^{(o,2)}_{\mu\nu} = -2R_{\mu\nu\rho\sigma}\widetilde{F}^{\rho\sigma}, \quad \mathcal{M}^{(o,3)}_{\mu\nu} = -4\widetilde{F}^2F_{\mu\nu} - 2F^2\widetilde{F}_{\mu\nu}, \\ \mathcal{M}^{(o,4)}_{\mu\nu} &= -2F_{\mu\rho}F_{\nu\sigma}\widetilde{F}^{\rho\sigma} + 2F^{\rho\sigma}F_{\nu\rho}\widetilde{F}_{\mu\sigma} - 2F^{\rho\sigma}F_{\mu\rho}\widetilde{F}_{\nu\sigma}, \\ \widetilde{\mathcal{M}}^{(o,1)}_{\mu\nu} &= -2F_{[\mu}{}^{\alpha}R_{\nu]\alpha}, \quad \widetilde{\mathcal{M}}^{(o,2)}_{\mu\nu} = -2R_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad \widetilde{\mathcal{M}}^{(o,3)}_{\mu\nu} = -2F^2F_{\mu\nu} \ , \\ \widetilde{\mathcal{M}}^{(o,4)}_{\mu\nu} &= -2F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma} \ , \end{aligned} \tag{64}$$

where we introduced the notation

$$\widetilde{F}^2_{\mu\nu} = F_{\mu\rho}\widetilde{F}_\nu{}^\rho, \quad \widetilde{F}^2 = F_{\mu\nu}\widetilde{F}^{\mu\nu}. \tag{65}$$

D Geroch-Hansen method for general theories of gravity

The gravitational multipole moments for asymptotically flat spacetimes in $D = 4$ were defined by Geroch [18] and Hansen [19]. Utilizing the timelike Killing vector $\xi = \partial_t$, they constructed two scalars which encode the information about gravitational multipole moments. The first scalar is simply $\lambda = \xi^2 = g^{tt}$. The second scalar arises from the twist 1-form of ξ defined as

$$\omega_{(1)} = i_\xi * d\xi \ , \tag{66}$$

which has the property that

$$d\omega_{(1)} = -\epsilon_{\mu\nu\rho\sigma}\xi^\rho R^\sigma{}_\lambda \xi^\lambda dx^\mu \ . \tag{67}$$

Thus for Ricci-flat spacetimes, such as Kerr black hole, we can define the second scalar ω from the closure of $\omega_{(1)}$

$$\omega_\mu = \partial_\mu \omega \ . \tag{68}$$

One can now combine λ and ω into two new scalars

$$\Phi_M = \frac{1}{4\lambda}(\lambda^2 + \omega^2 - 1), \qquad \Phi_J = \frac{\omega}{2\lambda} \ , \tag{69}$$

which play the role of generating functions for the mass and current multipole moments respectively. Moreover, for Ricci-flat spacetimes, the equivalence between the Geroch-Hansen formalism and Thorne ACMC formalism was proven by [35].

In [27], the author extended the Geroch-Hansen method to Einstein gravity with matter couplings in $D = 4$. In particular, a closed 1-form was identified for a class of $N = 2$ supergravity models, which generalizes the twist 1-form mentioned above. Below we will use covariant phase space approach [28,29] to extend the result of [27] to more general theories of gravity with higher derivative corrections.

From the above definition, we see that the key point of the Geroch-Hansen method is the construction of a closed 1-form $\omega_{(1)}$ from the Killing vector ξ, which further defines the second scalar ω. We start from a general matter coupled theory of gravity described by a diffeomorphism invariant Lagrangian 4-form $\mathbf{L}(\Phi)$, where Φ is a shorthand notation for all the fields involved. Consider an infinitesimal coordinate transformation generated by a local vector field η

$$\delta_\eta \mathbf{L}[\Phi] = \mathbf{E}_\Phi \delta_\eta \Phi + d\mathbf{\Theta}(\Phi, \delta_\eta \Phi) \ . \tag{70}$$

Using the on-shell condition $\mathbf{E}_\Phi = 0$ and Cartan magic formula $\delta_\eta = di_\eta + i_\eta d$, we can define a conserved Noether current $\mathbf{J}_\eta$ whose closure implies the existence of the Noether charge $\mathbf{Q}_\eta$

$$\begin{aligned} \mathbf{J}_\eta &= \mathbf{\Theta}(\Phi, \delta_\eta \Phi) - i_\eta \mathbf{L}[\Phi], \\ d\mathbf{J}_\eta &= 0 \quad \Longrightarrow \quad \mathbf{J}_\eta = d\mathbf{Q}_\eta \ . \end{aligned} \tag{71}$$

Choosing η to be the Killing vector ξ, we have $\delta_\xi \Phi = 0$ and thus $\mathbf{\Theta}(\Phi, \delta_\xi \Phi) = 0$. Consequently, onshell we have

$$d i_\xi \mathbf{L}[\Phi] = 0 \quad \Longrightarrow \quad i_\xi \mathbf{L}[\Phi] = -d\Omega. \tag{72}$$

The definition of $\mathbf{J}_\xi$ implies that we can form a closed 2-form from $\mathbf{Q}_\xi$ and Ω as

$$d(\mathbf{Q}_\xi - \Omega) = 0 \quad \Longrightarrow \quad \widetilde{\mathbf{Q}}_\xi = \mathbf{Q}_\xi - \Omega + d\mathbf{Y} \ , \tag{73}$$

where $\mathbf{Y}$ is ambiguity that one can play with. The generalized twist 1-form (66) and its potential are then given by

$$\omega_{(1)} = i_\xi \widetilde{\mathbf{Q}}_\xi = d\omega. \tag{74}$$

For Einstein-Maxwell theory and the Einstein-Maxwell-dilaton theory, the expression of $\omega_{(1)}$ is given in [27,36]. The author has pointed out, one can choose the ambiguous term $\mathbf{Y}$ to ensure that the matter contributions to ω will not affect the multipole moments. This choice of $\mathbf{Y}$ is also consistent with the requirement of ACMC coordinate system. So far, since the scalar λ appearing in (69) does not have a direct origin from the Lagrangian, we have not been able to use these results to prove that multipole moments are invariant under field redefinitions.

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