Active Subsampling for Measurement-Constrained M-Estimation of Individualized Thresholds with High-Dimensional Data

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Abstract

The measurement-constrained problems become frequently encountered in many modern applications such as electronic health record studies. In such problems, despite the availability of large datasets, collecting labeled data can be highly costly or time-consuming, and therefore we may be only affordable to observe the labels on a very small portion of the large dataset. This poses a critical question that which data points are most beneficial to label given a budget constraint. In this paper, we focus on the estimation of the optimal individualized threshold in a measurement-constrained M-estimation framework. In particular, our goal is to estimate a high-dimensional parameter $\boldsymbol{\theta}$ in a linear threshold $\boldsymbol{\theta}^T \boldsymbol{Z}$ for a continuous variable X such that the discrepancy between whether X exceeds the threshold $\theta^T Z$ and a binary outcome Y is minimized. In the measurement-constrained setting, we propose a novel K-step active subsampling algorithm to estimate θ , which iteratively samples the most informative observations in the dataset and solves a regularized M-estimator. The theoretical properties of our estimator demonstrate a phase transition phenomenon with respect to $\beta \geq 1$, the smoothness of the conditional density of X given Y and Z. In particular, for $\beta > (1 + \sqrt{3})/2$, we show that the two-step algorithm (with K = 2) yields an estimator with the parametric convergence rate $O_p((s \log d/N)^{1/2})$ in l_2 norm, where d and s are the dimension and sparsity of θ respectively and N is the label budget. The rate of our estimator is strictly faster than the minimax optimal rate $O_p((s \log d/N)^{\beta/(2\beta+1)})$ with N i.i.d. samples drawn from the population, which illustrates the theoretical advantages of the proposed method. However, for the other two scenarios $1 < \beta \leq (1 + \sqrt{3})/2$ and $\beta = 1$, the estimator from the two-step algorithm is sub-optimal. The former requires to run K > 2 steps to attain the same parametric rate, whereas in the latter case only a near parametric rate can be obtained even if K is allowed to scale with N. Furthermore, we formulate a minimax framework for the measurement-constrained M-estimation problem and

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define the *N*-budget minimax risk. We prove that our estimator is minimax rate optimal up to a logarithmic factor. We also provide the practical guidelines for the implementation of our algorithm. Finally, we demonstrate the superior performance of our method in simulation studies and apply the method to analyze a large diabetes dataset from 130 US hospitals.

Keyword: Non-regular models, High-dimensional estimation, Sampling, Measurement constraints, Minimax optimality, Kernel smoothing

1 Introduction

In many applications, the scientific questions can be formulated as identifying and interpreting the optimal individualized threshold value for a continuous variable such that the discrepancy with a binary outcome is minimized. One prominent example is the so-called minimum clinically important difference (MCID), which has attracted increasing interests in medical research over the past decade. The MCID is defined as the smallest difference in post-treatment changes, such that a patient is considered experiencing a clinically meaningful improvement if her/his change exceeds this value. Since introduced by Jaeschke et al. (1989), the MCID has been widely used by clinicians and health policy makers to evaluate the clinical effectiveness of the treatment, because it is tailored to reflect the patient's satisfaction or the improvement of her/his health condition. More recently, to account for the population heterogeneity, it was suggested by Hedayat et al. (2015); Zhou et al. (2020) to incorporate individual patient's clinical profile to construct the individualized MCID (iMCID).

Formally, let X denote a continuous variable representing the measurement of post-treatment change, Y denote a binary outcome in $\{-1, +1\}$, where Y = +1 if the patient's health condition is improved after receiving the treatment and Y = -1 otherwise, and Z denote a vector of d-dimensional covariates such as the patient's demographic information. Zhou et al. (2020) defined the iMCID as a function c(Z) which minimizes

$$\mathbb{P}\left(X < c(\mathbf{Z}) \mid Y = 1\right) + \mathbb{P}\left(X > c(\mathbf{Z}) \mid Y = -1\right).$$

$$(1.1)$$

In other words, $c(\mathbf{Z})$ is the optimal individualized threshold for X which minimizes the disagreement between the estimated patient's health condition and the binary outcome Y. Many other applications can be also formulated as the problem similar to (1.1). For example, in disease diagnosis, the researchers may aim to find the optimal threshold for a continuous biomarker X by maximizing the Youden's index, which is defined as the sum of the sensitivity and specificity of the diagnostic test (Xu et al., 2014). It can be shown that maximizing the Youden's index is exactly equivalent to minimizing (1.1). Other examples of (1.1) include policy learning problems in causal inference (Zhao et al., 2012), binary response models in econometrics (Manski, 1975), and classification in machine learning. The extensions of (1.1) to increasing dimensions and distributed settings are recently studied by Mukherjee et al. (2021); Feng et al. (2022); Chen et al. (2024). From a practical standpoint, a linear structure on the threshold $c(\mathbf{Z})$ is favored for its transparency and ease of interpretation, particularly when dealing with high-dimensional covariates \mathbf{Z} . In this paper, we assume that $c(\mathbf{Z}) = \boldsymbol{\theta}^T \mathbf{Z}$ for some high-dimensional parameter $\boldsymbol{\theta}$. Under these assumptions, we can reformulate (1.1) as the following M-estimation problem

$$\boldsymbol{\theta}^* = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} R(\boldsymbol{\theta}), \text{ where } R(\boldsymbol{\theta}) = \mathbb{E}\left[\gamma(Y)L_{01}\{Y(X - \boldsymbol{\theta}^T \boldsymbol{Z})\}\right],$$
(1.2)

 $L_{01}(u) = \frac{1}{2}\{1 - \operatorname{sign}(u)\}\$ is the 0-1 loss, with $\operatorname{sign}(u) = 1$ if $u \ge 0$ and -1 otherwise, and $\gamma(\cdot)$ is a user-specified weight function. While this model-free formulation offers greater generality and potential for accommodating model misspecifications, minimization of the empirical version of $R(\theta)$ is computationally NP-hard due to the 0-1 loss. Additionally, it is well known that the M-estimator of (1.2) is non-regular, resulting in nonstandard limiting distributions and rates of convergence (Kim and Pollard, 1990). Recently, Feng et al. (2022, 2024) proposed a regularized M-estimation framework with a smoothed surrogate loss to estimate and make inference on the high-dimensional parameter θ^* . In particular, they demonstrated that the finite sample error bound for estimating θ^* in l_2 norm is given by $(s \log d/N)^{\beta/(2\beta+1)}$, where d and s are the dimension and sparsity of θ^* respectively, N is the sample size, and β is the smoothness of the conditional density of X given the response Y and the covariates Z. With the slower-than-classic root-n rate, they also established that the resulting estimator is minimax rate optimal up to a logarithmic factor.

Up to this point, all aforementioned methods rely on the assumptions that the observations are i.i.d. with the estimation process having no influence on the data collection process. However, in practice, the estimation and data collection processes can be intertwined, especially under the measurement-constrained setting, where (X, \mathbb{Z}) are available for all samples but we are only affordable to observe Y on a very small portion of the samples (Wang et al., 2017; Zhang et al., 2021). The measurement-constrained problems become frequently encountered in many modern applications, when acquiring labeled data is highly costly or time-consuming. For instance, in EHR (electronic health records) studies, while the data may contain a tremendous amount of patient's medical and diagnostic information which can be potentially used as (X, \mathbb{Z}) in our problem (1.2), the gold-standard outcome Y is often not immediately available and may require manual chart reviews from medical experts. However, due to very expensive cost, chart reviews are typically conducted only for a small subset of selected patients. Depending on how the patients are sampled in the EHR database, the resulting M-estimator of (1.2) can be very inefficient.

To address this challenge, we propose a novel K-step active subsampling algorithm for estimating the high-dimensional parameter $\theta^* \in \mathbb{R}^d$ in (1.2) under the following measurement-constrained setting. Formally, assume that we are accessible to a very large dataset (e.g., the EHR database) $D = \{X_i, \mathbb{Z}_i\}_{i=1}^n$ with n i.i.d. samples, where the outcome Y_i is unavailable. Let N denote the label budget, that is the expected total number of samples we are allowed to select. Once a specific data point is sampled, we can observe the outcome Y_i (e.g., via chart reviews in the EHR studies). Our goal is to devise a computationally and statistically efficient interactive data sampling and estimation procedure for θ^* , subject to the budget constraint N, in the scenario $N \ll n$ and $N \ll d$. Our proposed algorithm starts from uniformly sampling a set of independent data from D, and solving a regularized M-estimator with a smoothed surrogate loss to construct an initial estimator for θ^* . We then iteratively use the estimator from the previous iteration to guide the selection of a new set of independent samples and solve the corresponding regularized M-estimator. Repeating this process K times yields our final estimator $\hat{\theta}_K$. The underlying principle behind our algorithm is that, as the algorithm iterates, the M-estimators lead to a sequence of intervals with decreasing lengths that contain the true threshold $\theta^{*T}Z$ with high probability, and in return sampling data in the corresponding neighborhoods around the threshold can further improve the estimation accuracy of our M-estimators. One key property of our algorithm is that the probability of the *i*th data point being sampled only depends on (X_i, Z_i) and the previously sampled data, making it applicable to the measurement-constrained setting. From the computational side, by leveraging the smoothness of the surrogate loss, we can design gradient-based algorithms to solve the regularized M-estimator at each iteration. As a result, our proposed K-step active subsampling algorithm is computationally efficient.

To investigate the theoretical results of our estimators, we assume that the conditional density of X given Y and Z satisfies the Hölder smoothness condition with parameter $\beta \geq 1$. Under this assumption, the theoretical properties of our estimator $\hat{\theta}_K$ demonstrate an interesting phase transition phenomenon with respect to β . In particular, for $\beta > (1 + \sqrt{3})/2$, with a proper choice of tuning parameters, we show that the two-step algorithm, i.e., our algorithm with only K = 2 iterations, yields an optimal estimator with the convergence rate $O_p((s \log d/N)^{1/2})$ in l_2 norm, where d and s are the dimension and sparsity of θ^* respectively and N is the label budget. Compared to the minimax optimal rate $O_p((s \log d/N)^{\beta/(2\beta+1)})$ derived in Feng et al. (2022), where the N samples are i.i.d. drawn from the population, our estimator has a faster rate of convergence, given the same number of samples. In other words, the two-step algorithm requires less data to attain the same order of convergence rate, rendering it attractive in scenarios where obtaining labels is highly costly. However, for $1 < \beta \leq (1 + \sqrt{3})/2$, the estimator from the two-step algorithm is sub-optimal. To achieve the same optimal rate $O_p((s \log d/N)^{1/2})$, we need to run at least $K = \left\lceil \log_{\frac{\beta}{2\beta+1}} (1 - \frac{\beta+1}{2\beta^2}) \right\rceil + 1$ iterations in our algorithm, where K is strictly greater than 2 but is fixed and finite. For the last case $\beta = 1$, we can achieve a near optimal rate $O_p((s \log d/N)^{1/2})$ multiplied with some extra logarithmic factors with $K = \lceil \log_3(\log N) \rceil$ iterations, where K diverges slowly as N tends to infinity. The distinct behaviors of our algorithm in above three regimes are driven by the closeness of the sequence of estimators relative to a fast convergence region. Finally, we rigorously formulate a minimax framework to study the optimality of our estimators. Unlike the traditional minimax framework, the distribution of a generic estimator $\hat{\theta}$ depends on the joint distribution P of (X, Z, Y) as well as the sampling distribution Q, where P is determined by the nature but we have the freedom to choose Q. After introducing two proper classes $\mathcal{P}(\beta, s)$ for P and $\mathcal{Q}_N(\mathcal{P}(\beta, s))$ for Q, we define the N-budget minimax risk in l_2 norm as

$$\inf_{Q \in \mathcal{Q}_N(\mathcal{P}(\beta,s))} \inf_{\widehat{\boldsymbol{\theta}}} \sup_{P \in \mathcal{P}(\beta,s)} \mathbb{E}_{P,Q} \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*(P)\|_2,$$

where the supremum is only for the distribution P in $\mathcal{P}(\beta, s)$, the inner infimum is over all possible estimators $\hat{\theta}$ based on the observed data and the outer infimum is over all possible sampling distributions Q in $\mathcal{Q}_N(\mathcal{P}(\beta, s))$, which contains necessary constraints on Q such as the conditional independence assumptions for the sampling mechanism and the budget constraint. We prove in Theorem 5 that the N-budget minimax risk for estimating θ in l_2 norm is lower bounded by $(s \log(d/s)/N)^{1/2}$, which implies that our proposed estimators are indeed rate optimal up to logarithmic factors in the measurement-constrained setting.

1.1 Related Work

Subsampling is an effective method for handling computational constraints when dealing with massive datasets. There's a large literature on subsampling algorithms for regression models, such as linear regression (Drineas et al., 2011; Ma et al., 2014; Wang et al., 2019; Raskutti and Mahoney, 2016) and generalized linear models (Wang et al., 2018). Given the budget constraint, the goal is to construct an estimator based on the sampled data to approximate the least squares or the maximum likelihood estimator from the entire dataset, and find the optimal subsampling weight by minimizing the asymptotic variance. The similar idea has been extended to deal with the measurement-constrained problems (Wang et al., 2017; Zhang et al., 2021). More recently, Zrnic and Candès (2024) proposed to use a machine learning model to identify which data points are most beneficial to label, and then find the optimal sampling weight by minimizing the variance of the estimator or classification uncertainty. For binary data, the case-control subsampling is considered by Fithian and Hastie (2014) among many others.

In recent years, there has been a substantial research focusing on adaptive experimental design, often with the goal of efficiently estimating average treatment effects. For instance, Hahn et al. (2011) developed a two-stage experiment for estimating average treatment effects, with data from the first stage guiding treatment assignment in the second stage. Hadad et al. (2021) considered how to construct confidence intervals for the average treatment effect with adaptively collected data. A recent overview of adaptive design is given by Perera et al. (2020).

We can see that all aforementioned works share the similarity that the data are collected adaptively with the goal of improved asymptotic efficiency for statistical inference. However, our work focuses on the threshold estimation problem which is known as a non-regular problem with nonstandard rate of convergence. Our goal is to design a subsampling procedure to improve the convergence rate of the estimators. Thus, our methodology and theoretical results are completely different from the aforementioned works.

Another closely related area is active learning, see Balcan et al. (2007); Koltchinskii (2010); Balcan and Long (2013); Castro and Nowak (2008); Wang and Singh (2016), among many others.

While our problem setup is similar to the pool-based active learning algorithms, our work is distinct from the active learning literature in both method and theory. Specifically, our algorithm iteratively solves regularized M-estimators with a smoothed surrogate loss via gradient-based methods. which is computationally efficient. However, most of the margin-based active learning algorithms such as Balcan et al. (2007); Wang and Singh (2016) require to minimize the empirical 0-1 loss, which is computationally intractable especially in high-dimensional setting. In addition, when the smoothness parameter β is greater than $(1+\sqrt{3})/2 \approx 1.37$, our proposed algorithm employs a streamlined two-step process (i.e., K = 2 iterations), which significantly enhances implementation simplicity and efficiency. In contrast, the pipeline of an active learning algorithm often involves iterative model updates (usually $\log N$ iterations) until a stopping criterion is met. In theory, the Tsybakov noise condition plays a pivotal role in deriving theoretical guarantees in the active learning literature. Technically, when establishing these theoretical guarantees, the stopping criteria and the number of total iterations in the active learning algorithm are determined to ensure compliance with the assumed Tsybakov noise condition. In contrast, our analysis relies on the smoothness of the conditional density of X given Z and Y rather than such noise conditions, and our estimators show completely different behaviors. Furthermore, the active learning literature focuses on bounding the excess risk of the classifiers, whereas we are interesting in estimating and interpreting the optimal individualized threshold with theoretical guarantees on the estimation error of θ^* . These differences in approach leads to fundamentally distinct theoretical results and proof strategy compared to the active learning literature.

1.2 Organization of the paper

The rest of this paper is organized as follows. Section 2 introduces our proposed active subsampling algorithm and the corresponding estimator. In Section 3 we analyze the theoretical properties of the algorithm, and derive upper bounds for the estimation error, followed by establishing a matching lower bound. Section 4 discusses the practical implementation considerations and presents a datadriven active subsampling algorithm. Simulation studies and a real data application are presented in Sections 5 and 6, respectively.

1.3 Notations

We write 1{} for the indicator function. For any set S, we write |S| for its cardinality. For $\boldsymbol{v} = (v_1, \ldots, v_d)^T \in \mathbb{R}^d$, we use \boldsymbol{v}_S to denote the subvector of \boldsymbol{v} with entries indexed by the set S. For $q = [1, \infty)$, $\|\boldsymbol{v}\|_q = (\sum_{i=1}^d |v_i|^q)^{1/q}$ and $\|\boldsymbol{v}\|_0 = \sum_{i=1}^d 1\{v_i \neq 0\}$. For any $a, b \in \mathbb{R}$, we write $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. For any positive sequences $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$, we write $a_n \lesssim b_n$ or $a_n = O(b_n)$ if there exists a constant c such that $a_n \leq cb_n$ for any n, and $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. Let $\lfloor a \rfloor$ be the greatest integer strictly less than a, and $\lceil a \rceil$ be the smallest integer strictly greater than a. Let $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ be the smallest and largest eigenvalues of M. A random variable X is called sub-Gaussian if there exists a positive constant K such that $\mathbb{P}(|X| \ge t) \le 2\exp(-t^2/K^2)$ for all $t \ge 0$. The sub-Gaussian norm of X is defined as $||X||_{\psi_2} = \inf \{c > 0 : \mathbb{E}[\exp(X^2/c^2)] \le 2\}$. A vector $\mathbf{X} \in \mathbb{R}^d$ is a sub-Gaussian vector if the one-dimensional marginals $\mathbf{v}^T \mathbf{X}$ are sub-Gaussian for all $\mathbf{v} \in \mathbb{R}^d$, and its sub-Gaussian norm is defined as $||\mathbf{X}||_{\psi_2} = \sup_{\|\mathbf{v}\|_2 = 1} \|\mathbf{v}^T \mathbf{X}\|_{\psi_2}$.

2 Proposed Method

2.1 Background and Heuristics for Subsampling

In this section, we briefly review the regularized M-estimation approach proposed by Feng et al. (2022) for estimating θ^* in (1.2) and use this to explain the heuristics for the subsampling method. For now, assume that we observe n i.i.d copies of (X, \mathbf{Z}, Y) . Recall that the risk function $R(\theta)$ is defined in (1.2). While $R(\theta)$ is typically a smooth function of the parameter θ , the empirical version $R_n(\theta) = \frac{1}{n} \sum_{i=1}^n \gamma(Y_i) L_{01}\{Y_i(X_i - \theta^T \mathbf{Z}_i)\}$ is non-smooth, which makes the minimization of $R_n(\theta)$ intractable, especially when the dimension of θ is large, and also leads to nonstandard theoretical properties, such as the cubic root rate of convergence (Kim and Pollard, 1990).

To address these challenges, Feng et al. (2022) proposed to approximate the 0-1 loss by the following smoothed surrogate loss

$$L_{\delta}(u) = \int_{u/\delta}^{\infty} K(t)dt, \qquad (2.1)$$

where K(t) is a proper kernel function defined in Assumption 3.4, and $\delta > 0$ is a bandwidth parameter. As the bandwidth $\delta \to 0$, we have $L_{\delta}(u) \to L_{01}(u)$ for any $u \neq 0$. Thus, it is intuitive to estimate θ^* by the minimizer of the regularized smoothed empirical risk, $\hat{\theta}_{iid} = \operatorname{argmin}\{R^n_{\delta,iid}(\theta) + \lambda \|\theta\|_1\}$, where

$$R_{\delta,iid}^{n}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \gamma(Y_i) L_{\delta}(Y_i(X_i - \boldsymbol{\theta}^T \boldsymbol{Z}_i)),$$

and λ is a tuning parameter. While $R_{\delta,iid}^n(\theta)$ is still non-convex, Feng et al. (2022) showed that the entire solution path for the lasso type estimator $\hat{\theta}_{iid}$ can be computed efficiently via the pathfollowing algorithm. In addition, with a proper choice of δ and λ , the convergence rate of $\hat{\theta}_{iid}$ is faster than the classic cubic root rate.

We note that by the M-estimation theory, since $R^n_{\delta,iid}(\boldsymbol{\theta})$ is differentiable in $\boldsymbol{\theta}$, the gradient of $R^n_{\delta,iid}(\boldsymbol{\theta})$ at $\boldsymbol{\theta}^*$,

$$\nabla R^n_{\delta,iid}(\boldsymbol{\theta}^*) = \frac{1}{n} \sum_{i=1}^n \gamma(Y_i) \frac{Z_i Y_i}{\delta} K\Big(\frac{Y_i(X_i - \boldsymbol{\theta}^{*T} \boldsymbol{Z}_i)}{\delta}\Big),$$

together with some other conditions, determine the convergence rate of $\hat{\theta}_{iid}$. A basic but crucial observation that inspires our subsampling method is that, by the property of the kernel function $K(\cdot)$ in $\nabla R^n_{\delta,iid}(\theta^*)$, the closer $X_i - \theta^{*T} Z_i$ to 0 the higher weight the *i*th data point receives. In

other words, the data points whose $X_i - \theta^{*T} Z_i$ value is close to 0 are the most informative ones for estimating θ^* , and therefore they are most beneficial to label in our subsampling algorithm.

Algorithm 1 $\theta \leftarrow K$ -step Active Subsampling
input: $D = \{X_i, Z_i\}_{i=1}^n$, label budget N, the number of iterations K
parameter: $\{\lambda_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \{\delta_k\}_{k=1}^K$ and $\{N_k\}_{k=1}^K$ with $\sum_{k=1}^K N_k = N$
Randomly split D into K batches: D_1, \dots, D_K , each with batch size n/K .
Draw data (X_i, \mathbf{Z}_i) from D_1 with probability $c_{n,1} = N_1 K/n$. Acquire the label Y_i for each
sampled data and form the dataset $D_1^* = \{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, where $(X_i, \mathbf{Z}_i) \in D_1$.
$\widehat{\boldsymbol{\theta}}_1 \leftarrow \operatorname{argmin}_{\boldsymbol{\theta}} \{ R^{D_1}_{\delta_1}(\boldsymbol{\theta}) + \lambda_1 \ \boldsymbol{\theta} \ _1 \}.$
for $k = 2$ to K do
Compute the active set: $S_k \leftarrow \left\{ (X, \mathbf{Z}) : -b_{k-1} \leq \frac{X - \widehat{\theta}_{k-1}^T \mathbf{Z}}{\sqrt{1 + \ \widehat{\theta}_{k-1}\ _2^2}} \leq b_{k-1} \right\}.$
Given $(X_i, \mathbf{Z}_i) \in S_k$, draw the data point (X_i, \mathbf{Z}_i) from D_k with probability $c_{n,k} =$
$N_k K/(n\mathbb{P}((X, \mathbf{Z}) \in S_k))$. Acquire the label Y_i for each sampled data and form $D_k^* =$
$\{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, where $(X_i, \mathbf{Z}_i) \in D_k$.
$\widehat{\boldsymbol{\theta}}_k \leftarrow \operatorname{argmin}_{\boldsymbol{\theta}} \{ R^{D_k}_{\delta_k}(\boldsymbol{\theta}) + \lambda_k \ \boldsymbol{\theta} \ _1 \}.$
end for
$\mathbf{return} \;\; \widehat{oldsymbol{ heta}}_K$

2.2 Active Subsampling Algorithm

Now, let us consider the measurement-constrained setting. Recall that we are accessible to a very large dataset $D = \{X_i, \mathbf{Z}_i\}_{i=1}^n$ with n i.i.d. samples, where the outcome Y_i is unavailable. We seek to sample $N \ll n$ data points (on average) from the dataset D and collect their outcomes to construct an estimator of θ^* .

We introduce a binary random variable R_i to represent whether the data point (X_i, \mathbf{Z}_i) is sampled or not, where $R_i = 1$ if (X_i, \mathbf{Z}_i) is sampled and $R_i = 0$ otherwise. Now, we introduce our active subsampling approach as outlined in Algorithm 1. The algorithm runs for a total of K iterations, where K is to be specified later on. To ensure that the data distribution during each iteration remains consistent with the original data D, we randomly divide D into K batches D_1, \dots, D_K with equal size n/K. In the first iteration, since there is no prior information on θ^* , we uniformly sample data from D_1 with probability $0 < c_{n,1} < 1$. That is, for each $(X_i, \mathbf{Z}_i) \in D_1$, R_i is generated independently with probability

$$\mathbb{P}(R_i = 1) = c_{n,1} = \frac{N_1 K}{n},$$
(2.2)

where N_1 is the expected number of data points sampled in the first iteration. Given the sampled dataset $D_1^* = \{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, we then minimize the regularized smoothed empirical risk function

to obtain

$$\widehat{\boldsymbol{\theta}}_1 := \operatorname*{argmin}_{\boldsymbol{\theta}} \{ R^{D_1}_{\delta_1}(\boldsymbol{\theta}) + \lambda_1 \| \boldsymbol{\theta} \|_1 \},$$
(2.3)

where $R_{\delta_1}^{D_1}(\boldsymbol{\theta})$ is a special case of (2.4) with a bandwidth parameter δ_1 and $\lambda_1 > 0$ is a regularization parameter. In general, for the sampled data from the dataset D_k with $|D_k| = n/K$, we define

$$R_{\delta_k}^{D_k}(\boldsymbol{\theta}) = \frac{K}{n} \sum_{(X_i, \boldsymbol{Z}_i) \in D_k} \gamma(Y_i) L_{\delta_k}(Y_i(X_i - \boldsymbol{\theta}^T \boldsymbol{Z}_i)) R_i, \qquad (2.4)$$

for all $1 \le k \le K$. For iteration $2 \le k \le K$, we define an active set as

$$S_k := \left\{ (X, \mathbf{Z}) \in \mathbb{R} \times \mathbb{R}^d : -b_{k-1} \le \frac{X - \widehat{\boldsymbol{\theta}}_{k-1}^T \mathbf{Z}}{\sqrt{1 + \|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2}} \le b_{k-1} \right\},\tag{2.5}$$

where $\hat{\theta}_{k-1}$ is the estimator derived from the (k-1)th iteration, and $b_{k-1} > 0$ is the tuning parameter controlling the size of S_k . We propose to uniformly sample the data points from D_k which belong to the active set S_k . Specifically, given $(X_i, \mathbb{Z}_i) \in D_k$ and $\hat{\theta}_{k-1}$, we generate R_i from a Bernoulli distribution with

$$\mathbb{P}(R_i = 1 \mid X_i, \boldsymbol{Z}_i, \widehat{\boldsymbol{\theta}}_{k-1}) = c_{n,k} \cdot \mathbb{1}\{(X_i, \boldsymbol{Z}_i) \in S_k\},$$
(2.6)

where $c_{n,k} = N_k K / (n \mathbb{P}((X, \mathbb{Z}) \in S_k))$ under the label budget constraint N_k for this iteration. By the definition of S_k , the sampling mechanism implies that only the data point whose $X_i - \hat{\theta}_{k-1} \mathbb{Z}_i$ is sufficiently close to 0 is potentially sampled. This matches with our heuristics in Section 2.1.

Given how R_i is generated, we can verify that the independence assumptions $(X_i, \mathbf{Z}_i, Y_i) \perp \bar{H}_{i-1}$ and $R_i \perp Y_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1}$ hold, where \bar{H}_{i-1} denotes all observed data right before we decide whether (X_i, \mathbf{Z}_i) is sampled or not. These two independence assumptions play an important role in the minimax lower bound; see Section 3.3 for more detailed discussions.

Given the dataset D_k and the selection indicators, we derive the estimator

$$\widehat{\boldsymbol{\theta}}_{k} := \operatorname*{argmin}_{\boldsymbol{\theta}} \{ R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}) + \lambda_{k} \| \boldsymbol{\theta} \|_{1} \},$$
(2.7)

where $R_{\delta_k}^{D_k}(\boldsymbol{\theta})$ is defined in (2.4). Repeating this procedure K times, we obtain our final estimator $\hat{\boldsymbol{\theta}}_K$.

In our algorithm, the sampling probability $c_{n,k}$, which depends on $\mathbb{P}((X, \mathbb{Z}) \in S_k)$, is assumed to be known. In practice, provided the active set S_k is given, we can indeed estimate $\mathbb{P}((X, \mathbb{Z}) \in S_k)$ easily as we have a large amount of unlabeled data. Specifically, in Algorithm 1, we can instead randomly divide the dataset D into K + 1 batches $D_0, D_1, ..., D_K$, where we use D_0 to compute an empirical estimator \hat{p}_k of $\mathbb{P}((X, \mathbb{Z}) \in S_k)$, and then construct a plug-in estimator $\hat{c}_{n,k} := N_k K/(n \hat{p}_k)$ of $c_{n,k}$. We draw samples from D_k according to (2.6) with $c_{n,k}$ replaced by $\hat{c}_{n,k}$ and compute the estimator in (2.7). The sample splitting procedure guarantees the desired independence between $\widehat{c}_{n,k}$ and the data in D_k and moreover the estimation error from \widehat{p}_k is shown to be negligible as $\mathbb{P}((X, \mathbb{Z}) \in S_k)$ is estimated based on O(n/K) amount of data with $n \gg N$. The discussion on the other computational aspect of our algorithm is deferred to Section 4.

3 Theoretical Results

We first list the technical assumptions in Section 3.1. The convergence rate of our estimator is studied in Section 3.2, followed by the minimax lower bound in Section 3.3. Without loss of generality, we set the weight function in (1.2) as $\gamma(y) = 1/\mathbb{P}(Y = y)$.

3.1 Assumptions

Assumption 3.1. θ^* is s-sparse with $\|\theta^*\|_0 \leq s$ and $\|\theta^*\|_2 \leq C$ for some constant C.

Assumption 3.1 gives the conditions on θ^* . Besides the sparsity of θ^* , we also assume $\|\theta^*\|_2$ is bounded, which intuitively matches the magnitude of X and its threhold $\theta^{*T}Z$. Technically, this condition is used to verify the restricted strong convexity (RSC) condition (Feng et al., 2022). In particular, we provide a counterexample in Section A.8 that the RSC condition (such as Assumption 3.5 in below) fails when $\|\theta^*\|_2$ diverges to infinity.

Assumption 3.2. (i) There exists a constant 0 < c < 1/2 such that $c \leq \mathbb{P}(Y=1) \leq 1-c$. (ii) Assume that $|Z_{ij}| \leq M_n$ for any $1 \leq i \leq n$ and $1 \leq j \leq d$, where M_n is allowed to increase with n such that

$$M_n \le C \sqrt{\frac{n \min_{1 \le k \le K} \delta_k c_{n,k}}{K \log d}}$$
(3.1)

for some constant C, where K is the number of iterations, δ_k is the bandwidth parameter of the kernel function utilized in the kth iteration and $c_{n,k}$ is defined in (2.6). In addition, it holds that

$$\sup_{\|\boldsymbol{v}\|_{0} \leq s'} \frac{\boldsymbol{v}^{T} \mathbb{E} \left(\boldsymbol{Z} \boldsymbol{Z}^{T} \mid \boldsymbol{Y} = \boldsymbol{y} \right) \boldsymbol{v}}{\|\boldsymbol{v}\|_{2}^{2}} \leq M_{1} < \infty,$$
(3.2)

for some constant $M_1 > 0$, where s' = Cs for some sufficiently large constant C. (iii) \mathbf{Z} given Y = y is a sub-Gaussian vector with a bounded sub-Gaussian norm.

Assumption 3.2 is concerned with the boundedness of \mathbb{Z} and Y. Part (i) ensures that the weight function $\gamma(y) = 1/\mathbb{P}(Y = y)$ is bounded away from infinity. For part (ii), if each component of \mathbb{Z} is sub-Gaussian with bounded sub-Gaussian norm, $\max_{1 \le i \le n, 1 \le j \le d} |Z_{ij}| \le M_n$ holds with high probability with $M_n \asymp (\log(d \lor n))^{1/2}$. Given the choices of δ_k and $c_{n,k}$ presented in Theorem 2, (3.1) reduces to $M_n \le C\sqrt{\frac{N}{K \log d}}$, which is a mild condition provided N is large enough. Furthermore, (3.2) controls the maximal sparse eigenvalues of $\mathbb{E}(\mathbb{Z}\mathbb{Z}^T \mid Y = y)$, we refer to Bühlmann and Van De Geer (2011) for the detailed discussion. Finally, part (iii) is used to provide a sharp bound for the plug-in error of $\hat{\theta}_{k-1}$ in the active set S_k . This condition can be removed with the price of obtaining a sub-optimal rate for our estimator; see Section A.4 for the detailed result.

The following definition and assumption are concerned with the smoothness of the conditional density of X given Y and Z.

Definition 3.1. Let $l = \lfloor \beta \rfloor$ be the greatest integer strictly less than β . We say $P \in \mathcal{P}(\beta, L)$ if the conditional density $f(x \mid y, z)$ of $X \mid Y, Z$ is l times differentiable w.r.t x at $x = \theta^{*T} z$ for any y, z, and satisfies

$$\left| f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} + \Delta \mid \boldsymbol{y}, \boldsymbol{z} \right) - f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{y}, \boldsymbol{z} \right) \right| \le L \left| \Delta \right|^{\beta - l},$$
(3.3)

for any $\Delta \in \mathbb{R}, y \in \{-1, 1\}, z \in \mathbb{R}^d$ and some constant L > 0.

Assumption 3.3. We assume $P \in \mathcal{P}(\beta, L)$, where $\beta \geq 1$ and L > 0 are constants. In addition,

$$\sup_{x \in \mathcal{X}, y \in \{-1,1\}, \mathbf{z} \in \mathcal{Z}} f(x \mid y, \mathbf{z}) < p_{\max} < \infty,$$
(3.4)

and there exists a set $\mathcal{G} \in \mathbb{R}^d$ such that $\mathbb{P}(\mathbf{Z} \in \mathcal{G}) \geq C$ for some constant $0 < C \leq 1$ and

$$\inf_{x \in B(\boldsymbol{\theta}^{*T}\boldsymbol{z},\epsilon_n), \boldsymbol{z} \in \mathcal{G}} f(x \mid \boldsymbol{z}) \ge p_{\min} > 0,$$
(3.5)

where \mathcal{X} and \mathcal{Z} are the support sets of X and \mathbf{Z} , $B(\boldsymbol{\theta}^{*T}\boldsymbol{z}, \epsilon_n) := \{x \in \mathcal{X} : |x - \boldsymbol{\theta}^{*T}\boldsymbol{z}| \leq \epsilon_n\},\ \epsilon_n = C \max_{2 \leq k \leq K} b_{k-1}$ for some constant C large enough, and $p_{\max}, p_{\min} > 0$ are some constants.

In this assumption, f(x | y, z) is assumed to belong to a β -smooth Hölder class at $x = \boldsymbol{\theta}^{*T} \boldsymbol{z}$ for $\beta \geq 1$. We exclude the case $0 < \beta < 1$, as we show that under mild conditions the non-smoothness of f(x | y, z) leads to diverging curvature of the risk function $R(\boldsymbol{\theta})$, which contradicts with the restricted smoothness (RSM) condition (Feng et al., 2022); see also Assumption 3.5. We defer the detailed results and counterexamples under $0 < \beta < 1$ to Section A.9.

Assumption 3.3 requires that $f(x \mid y, z)$ is upper bounded by some constant. Since we also need to lower bound the probability of (X, Z) belonging to the active set S_k , we further assume that there exists a region \mathcal{G} of z, such that $f(x \mid z)$ is lower bounded by some constant for any $z \in \mathcal{G}$ and $x \in B(\boldsymbol{\theta}^{*T} \boldsymbol{z}, \epsilon_n)$.

Recall that we introduced the kernel function K(t) in the surrogate loss (2.1). The following assumption is concerned with the kernel function K(t).

Assumption 3.4. Assume that K(t) is a proper kernel of order $l = \lfloor \beta \rfloor$ with bounded support, where β is the smoothness parameter in Assumption 3.3. That is K(t) satisfies K(t) = K(-t), $|K(t)| \leq K_{\max} < \infty$, $\int K(t)dt = 1$, $\int K^2(t)dt < \infty$, $\int t^j K(t)dt = 0$, $\forall j = 1, \ldots, l$, and $\int |K(t)||t|^q dt$ is bounded by a constant for any $q \in [l, l+1]$.

Similar to the nonparametric estimation problems, we adopt a kernel of order l to control the higher order bias of the gradient of the smoothed empirical risk $\mathbb{E}(\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) \mid \widehat{\boldsymbol{\theta}}_{k-1})$ as shown

in Proposition A.2. For clarity we present our theoretical results for the kernel with bounded support. With little modification on the proof, our results can be extended to kernels with mild tail conditions such as Gaussian kernels. The detailed results are deferred to Section A.4.

Assumption 3.5. There exists a sequence of sets $\Omega_0 = \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\|_2 \leq R_0\}$ and $\Omega_{k-1} = \{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \boldsymbol{\theta}_{k-1}\|_2 \leq R_{k-1}\}$ for $2 \leq k \leq K$, such that $\boldsymbol{\theta}^* \in \Omega_{k-1}$ and the following restricted strong convexity (RSC) and restricted smoothness (RSM) conditions hold for $R_{\delta_k}^{D_k}(\boldsymbol{\theta})$ over sparse vectors in Ω_{k-1} , for $1 \leq k \leq K$. That is, uniformly over $1 \leq k \leq K$ and $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Omega_{k-1}$ with $(\|\boldsymbol{\theta}'\|_0 \vee \|\boldsymbol{\theta}\|_0) \lesssim s$, we have

$$R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}'\right) \geq R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}\right) + \nabla R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}\right)^{T}\left(\boldsymbol{\theta}'-\boldsymbol{\theta}\right) + \frac{1}{2}\rho_{n,k}^{-}\left\|\boldsymbol{\theta}'-\boldsymbol{\theta}\right\|_{2}^{2},\tag{3.6}$$

and

$$R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}'\right) \leq R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}\right) + \nabla R_{\delta_{k}}^{D_{k}}\left(\boldsymbol{\theta}\right)^{T}\left(\boldsymbol{\theta}'-\boldsymbol{\theta}\right) + \frac{1}{2}\rho_{n,k}^{+}\left\|\boldsymbol{\theta}'-\boldsymbol{\theta}\right\|_{2}^{2},$$
(3.7)

where $\rho_{n,k}^- = C_1 c_{n,k}, \rho_{n,k}^+ = C_2 c_{n,k}$ for some constants $C_1, C_2 > 0$.

The RSC and RSM conditions are commonly used to analyze the statistical rate and computational guarantee of the path-following algorithm for non-convex optimization problems in highdimensional regression. Similar conditions have been discussed extensively in the literature; see Bühlmann and Van De Geer (2011) for the detailed discussion. In our context, we require that the smoothed empirical risk at the kth iteration is $\rho_{n,k}^-$ -strongly convex in (3.6) and $\rho_{n,k}^+$ -smooth in (3.7) when restricted to sparse vectors in Ω_{k-1} , where $\rho_{n,k}^-$ and $\rho_{n,k}^+$ are both proportional to $c_{n,k}$. Since the *i*th data point at the kth iteration contributes to $R_{\delta_k}^{D_k}(\boldsymbol{\theta})$ only when $R_i = 1$, the convexity and smoothness of $R_{\delta_k}^{D_k}(\boldsymbol{\theta})$ is expected to scale with $c_{n,k}$. Assumption 3.5 can be verified in a case by case manner under specific distributional assumptions. When k = 1, the RSC and RSM conditions over Ω_0 have been established by Feng et al. (2022) for some properly chosen constant R_0 . For $k \geq 2$, by mathematical induction, it suffices to conduct a localized analysis over Ω_{k-1} with R_{k-1} being the upper bound for the statistical rate $\|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_2$. With this choice of R_{k-1} , clearly we have $\boldsymbol{\theta}^* \in \Omega_{k-1}$ with high probability. Given $R_{k-1} = o(1)$ and the fact that the smoothed empirical risk $R_{\delta_k}^{D_k}(\boldsymbol{\theta})$ is sufficiently smooth, we can apply the Taylor expansion at $\boldsymbol{\theta}^*$ to verify (3.6) and (3.7). The detailed verification of this assumption is deferred to Section A.7.

3.2 Convergence Rate of the Proposed Estimator

We first present a master theorem that characterizes the effect of subsampling on the convergence rate of our estimators at each iteration.

Theorem 1. Under Assumptions 3.1-3.5, for any $1 \le k \le K$, choose $\lambda_k \asymp \sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$, and $\delta_k \asymp \left(\frac{Ks \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$. With probability greater than $1 - 2d^{-1}$, we have

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{s\log d}{N_1}\right)^{\beta/(2\beta+1)}, \ \|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{s\log d}{N_1}\right)^{\beta/(2\beta+1)}.$$
(3.8)

For $2 \le k \le K$, if we further assume

$$b_{k-1} \ge C\delta_k$$
, the event $\mathcal{W}_{k-1} = \left\{ b_{k-1} \ge C \| \widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^* \|_2 \sqrt{\log \frac{N_k}{s \log d}} \right\}$ holds (3.9)

for some large constant C > 0 and there exists a large constant ζ such that $(\frac{s \log d}{N_k})^{\zeta} \lesssim \delta_k$ with $\delta_k = o(1)$, then with probability greater than $1 - 2d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\|_{2} \lesssim \left(\frac{\mathbb{P}\left((X, \boldsymbol{Z}) \in S_{k}\right) s \log d}{N_{k}}\right)^{\beta/(2\beta+1)}, \\ \|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\|_{1} \lesssim \sqrt{s} \left(\frac{\mathbb{P}\left((X, \boldsymbol{Z}) \in S_{k}\right) s \log d}{N_{k}}\right)^{\beta/(2\beta+1)},$$
(3.10)

where $N_k = \sum_{(X_i, \mathbf{Z}_i) \in D_k} \mathbb{E}(R_i)$ is the expected sample size at the kth iteration.

In this theorem, the tuning parameters λ_k and δ_k are determined by a bias-variance trade-off. Specifically, Proposition A.2 implies that the bias of the smoothed empirical risk $R^{D_k}_{\delta_k}(\boldsymbol{\theta})$ due to kernel smoothing satisfies

$$\left\| \mathbb{E}(\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) | \widehat{\boldsymbol{\theta}}_{k-1}) - \nabla R(\boldsymbol{\theta}^*) \right\|_{\infty} \lesssim c_{n,k} \delta_k^{\beta}.$$

Moreover, Proposition A.1 characterizes the stochastic error of $\nabla R_{\delta_{h}}^{D_{k}}(\boldsymbol{\theta}^{*})$,

$$\left\|\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \mathbb{E}(\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*)|\widehat{\boldsymbol{\theta}}_{k-1})\right\|_{\infty} \lesssim \sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$$

Our analysis reveals that the bandwidth δ_k is chosen to balance the bias $c_{n,k}\delta_k^\beta$ with the stochastic error $\sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$ multiplied by \sqrt{s} . A simple calculation yields $\delta_k \simeq \left(\frac{Ks \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$. The shrinkage parameter λ_k needs to dominate the stochastic error of $\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*)$ to exploit the sparsity of $\boldsymbol{\theta}$. In practice, the tuning parameters can be determined by a cross-validation approach shown in Section 4.

In the following, we comment on the convergence rate of $\hat{\theta}_k$. For k = 1, the convergence rate of $\hat{\theta}_1$ in (3.8) is nonstandard and slower than the typical parametric rate for regular models (e.g., linear/logistic regression with Lasso). Since the estimator $\hat{\theta}_1$ is computed under uniform subsampling, the rate of $\hat{\theta}_1$ matches with Feng et al. (2022), and the rate is minimax optimal (up to a logarithmic factor) when data are i.i.d drawn from the population. In contrast, for $k \geq 2$, the convergence rate of $\hat{\theta}_k$ in (3.10) includes an additional factor $\mathbb{P}((X, \mathbb{Z}) \in S_k)$. By the definition of the active set S_k in (2.5), $\mathbb{P}((X, \mathbb{Z}) \in S_k)$ depends on the regularity of the joint distribution of (X, \mathbb{Z}) , the accuracy of the estimator $\hat{\theta}_{k-1}$ from the previous iteration and the choice of b_{k-1} . Under Assumption 3.3 and when $\hat{\theta}_{k-1} \to 0$ and N_k no smaller than N_1 , the convergence rate of $\hat{\theta}_k$ obtained via active subsampling is faster than $\hat{\theta}_1$ obtained under uniform subsampling. Moreover, since (3.10) holds for any choice of b_{k-1} provided (3.9) is satisfied, there is a trade-off for determining the optimal choice of b_{k-1} and the corresponding optimal rate of $\hat{\theta}_k$. Our previous argument suggests that a smaller value of b_{k-1} is desirable to attain a faster rate of $\hat{\theta}_k$ in (3.10). However, the condition (3.9) prevents us from choosing the value of b_{k-1} too small, see Remark 1 below for interpretation of condition (3.9). These together yield the optimal choice of b_{k-1} and the corresponding optimal rate of $\hat{\theta}_k$.

Remark 1. The condition (3.9) is inherited from Proposition A.2, which is used to control the approximation error of $\mathbb{E}(\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*)|\hat{\boldsymbol{\theta}}_{k-1})$ to $\nabla R(\boldsymbol{\theta}^*)$ (which is 0 by definition). Since we show that $\mathbb{P}((X, \mathbf{Z}) \in S_k) \approx b_{k-1}$, we can interpret b_{k-1} as the size of the active set S_k . The first condition $b_{k-1} \geq C\delta_k$ in (3.9) requires that the bandwidth δ_k should be chosen in a smaller order than the size of the active set. Otherwise, the surrogate risk $R_{\delta}(\cdot)$ is not shrunk to $R(\cdot)$ sufficiently fast. The event \mathcal{W}_{k-1} in (3.9) is concerned with the stability of the active set S_k with respect to the plug-in estimator $\hat{\boldsymbol{\theta}}_{k-1}$. Recall that the sample (X_i, \mathbf{Z}_i) with $X_i - \boldsymbol{\theta}^{*T} \mathbf{Z}_i \approx 0$ is more informative for estimating $\boldsymbol{\theta}$ and therefore such sample is expected to fall into the active set. However, in practice, we need to plug in the estimator $\hat{\boldsymbol{\theta}}_{k-1}$ to compute the active set needs to be large enough to account for the uncertainty of the estimator $\hat{\boldsymbol{\theta}}_{k-1}$.

To obtain the optimal rate for our final estimator $\hat{\theta}_K$ via Theorem 1, we need to optimize the parameters $\{\lambda_k\}_{k=1}^K, \{b_k\}_{k=1}^{K-1}, \{\delta_k\}_{k=1}^K$ and $\{N_k\}_{k=1}^K$ as well as the number of iterations K in Algorithm 1. The choices of the parameters clearly affect the performance of our estimator. In the following, we show that the property of our estimator $\hat{\theta}_K$ demonstrates a phase transition phenomenon with respect to the smoothness parameter β . In particular, there exist two critical points 1 and $(1 + \sqrt{3})/2$ for β , that partition $\beta \in [1, +\infty)$ (see Assumption 3.3) into three cases: (i) $\beta \in ((1 + \sqrt{3})/2, +\infty)$; (ii) $\beta \in (1, (1 + \sqrt{3})/2]$; and (iii) $\beta = 1$. We start from the theoretical result for $\beta \in ((1 + \sqrt{3})/2, +\infty)$.

Theorem 2 (Optimal rate for $\beta > \frac{1+\sqrt{3}}{2}$). Assume that Assumptions 3.1-3.5 hold, $K \ge 2$ and $\beta > \frac{1+\sqrt{3}}{2}$ are both fixed. We set $N_k = N/K$ for $1 \le k \le K$ and

$$\delta_1 = c_1 \left(\frac{s\log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_1 = c_2 \sqrt{\frac{N\log d}{n^2 \delta_1}},$$
$$\delta_k = c_1 \left(\frac{s\log d}{N}\right)^{1/(2\beta)}, \ \lambda_k = c_2 \sqrt{\frac{N\log d}{n^2 b_{k-1} \delta_k}}, \ b_{k-1} = c_3 \left(\frac{s\log d}{N}\right)^{1/(2\beta)}, 2 \le k \le K,$$

for some constants $c_1, c_2, c_3 > 0$. If

$$N \lesssim (s \log d)^{\frac{1}{2\beta+1}} n^{\frac{2\beta}{2\beta+1}},$$
 (3.11)

and $s \log d = o(N)$, then with probability greater than 1 - 2K/d, we have

$$\|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{s\log d}{N}\right)^{1/2}, \quad \|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{s\log d}{N}\right)^{1/2}, \tag{3.12}$$

uniformly over $2 \le k \le K$, where N is the pre-specified label budget.

The tuning parameters δ_k and λ_k in this theorem are chosen in the same way as in Theorem 1, where we plug in $c_{n,k} = \frac{KN_k}{n\mathbb{P}((X,Z)\in S_k)}$ and invoke the intermediate result $\mathbb{P}((X,Z)\in S_k) \approx b_{k-1}$. In addition, when $\beta > \frac{1+\sqrt{3}}{2}$, we choose the smallest value of b_{k-1} such that the condition $b_{k-1} \ge C\delta_k$ in (3.9) holds, and we show that the event \mathcal{W}_{k-1} in (3.9) also holds with high probability for such choice of b_{k-1} . The condition (3.11) requires that the size of the original dataset n should far exceed the desired label budget N, which is reasonable in many applications (e.g., EHR studies). This condition ensures that there exist enough data points in the active set S_k for us to sample. In this theorem, the budget N is evenly divided across K iterations so that the expected sample size at the final iteration is $N_K = N/K$. Intuitively, it may be desirable to allocate more budget to compute $\hat{\theta}_k$ as k increases. However, theoretically, there is no improvement in terms of the convergence rate; see Section A.6 for a variant of Theorem 2 in this case.

The convergence rate of our estimators $\hat{\theta}_k$ at each iterations is shown in (3.12). For any $k \ge 2$, the rate can be viewed as the parametric rate for sparse models and is faster than the minimax optimal rate under uniform subsampling (Feng et al., 2022), see also (3.8), which justifies the theoretical advantage of estimating θ^* via the proposed active subsampling approach. Moreover, as the rate (3.12) stays the same for any $k \ge 2$, it suffices to only run K = 2 iterations in Algorithm 1.

Theorem 3 (Optimal rate for $1 < \beta \leq \frac{1+\sqrt{3}}{2}$). Assume that Assumptions 3.1-3.5 hold, $K = \lceil \log_{\frac{\beta}{2\beta+1}} (1 - \frac{\beta+1}{2\beta^2}) \rceil + 1$ and $1 < \beta \leq \frac{1+\sqrt{3}}{2}$ are fixed. We set $N_k = N/K$ for $1 \leq k \leq K$,

$$\delta_1 = c_1 \left(\frac{s \log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_1 = c_2 \sqrt{\frac{N \log d}{n^2 \delta_1}},$$

for $2 \le k \le K - 1$,

$$b_{k-1} = c_3 \left(\log(\frac{N}{s \log d}) \right)^{\frac{(2\beta+1)(1-(\frac{\beta}{2\beta+1})^{k-1})}{2(\beta+1)}} \left(\frac{s \log d}{N}\right)^{\frac{\beta}{\beta+1}(1-(\frac{\beta}{2\beta+1})^{k-1})}$$
$$\delta_k = c_1 \left(\frac{b_{k-1}s \log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_k = c_2 \sqrt{\frac{N \log d}{n^2 b_{k-1} \delta_k}},$$

and

$$b_{K-1} = c_3 \left(\frac{s \log d}{N}\right)^{1/(2\beta)}, \delta_K = c_1 \left(\frac{s \log d}{N}\right)^{1/(2\beta)}, \lambda_K = c_2 \sqrt{\frac{N \log d}{n^2 b_{K-1} \delta_K}},$$

for some constants $c_1, c_2, c_3 > 0$. If

$$N \lesssim \left(\log(\frac{N}{s\log d})\right)^{\frac{\beta+1}{2(2\beta+1)}} (s\log d)^{\frac{\beta}{2\beta+1}} n^{\frac{\beta+1}{2\beta+1}},$$
(3.13)

(3.11) and $s \log d = o(N)$ hold, then with probability greater than 1 - 2K/d, we have

$$\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{2} \lesssim \left(\frac{s\log d}{N}\right)^{1/2}, \quad \|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{1} \lesssim \sqrt{s} \left(\frac{s\log d}{N}\right)^{1/2}.$$
(3.14)

Compared with the results in Theorem 2, our estimator shows different behaviors when $1 < \beta \leq (1 + \sqrt{3})/2$. While we still obtain the same parametric rate for $\hat{\theta}_K$ in (3.14), the key difference is that we have to run at least $K = \lceil \log_{\frac{\beta}{2\beta+1}} (1 - \frac{\beta+1}{2\beta^2}) \rceil + 1$ iterations in Algorithm 1 to attain (3.14) as opposed to only K = 2 iterations in Theorem 2. Note that $\log_{\frac{\beta}{2\beta+1}} (1 - \frac{\beta+1}{2\beta^2})$ is well defined and is strictly greater than 1 for $1 < \beta \leq (1 + \sqrt{3})/2$. Indeed, a crucial intermediate result in the proof of Theorem 3 is that, for any $2 \leq k \leq K - 1$,

$$\|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_2 \lesssim \left(\log(\frac{N}{s\log d})\right)^{\frac{\beta}{2(1+\beta)}(1-(\frac{\beta}{2\beta+1})^{k-1})} \left(\frac{s\log d}{N}\right)^{(1-(\frac{\beta}{2\beta+1})^k)\frac{\beta}{1+\beta}},\tag{3.15}$$

with high probability. It can be verified that, for any fixed $1 < \beta \leq (1 + \sqrt{3})/2$ and for any $2 \leq k \leq K - 1$, the rate in (3.15) is slower than the parametric rate in (3.14). Consequently, in this case, running Algorithm 1 with the number of iterations less than $\lceil \log_{\frac{\beta}{2\beta+1}} (1 - \frac{\beta+1}{2\beta^2}) \rceil + 1$ leads to the sub-optimal rate (3.15). This phenomenon occurs because, for $1 < \beta \leq (1 + \sqrt{3})/2$, the rate of $\hat{\theta}_1$ is not fast enough so that we have to choose a larger value of b_1 to ensure the event \mathcal{W}_1 in (3.9) holds. This continues to be the case, as the iterations progress until $\hat{\theta}_k$ falls into a fast convergence region

$$\Theta_{fast,\beta} = \left\{ \boldsymbol{\theta} : \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{Ks\log d}{N}\right)^{1/(2\beta)} \sqrt{\log(\frac{Ks\log d}{N})} \right\},\tag{3.16}$$

which is derived by matching the rate of $\hat{\theta}_{k-1}$ with the order of δ_k up to a logarithmic factor (see the two conditions in (3.9)). Once $\hat{\theta}_k \in \Theta_{fast,\beta}$ with high probability, we only need to apply one more iteration to achieve the parametric rate, which is in principle the same as how we analyze the estimator $\hat{\theta}_2$ given $\hat{\theta}_1$ in Theorem 2.

Finally, the following theorem characterizes the convergence rate of the estimator for $\beta = 1$.

Theorem 4 (Optimal rate for $\beta = 1$). Assume that Assumptions 3.1-3.5 hold, $\beta = 1$ and $K = \lceil \log_3(\log N) \rceil$. We set $N_k = N/K$ for $1 \le k \le K$,

$$\delta_1 = c_1 \left(\frac{Ks\log d}{N}\right)^{1/3}, \ \lambda_1 = c_2 \sqrt{\frac{NK\log d}{n^2 \delta_1}},$$

and for $2 \leq k \leq K$,

$$b_{k-1} = c_3 \left(\log(\frac{N}{Ks \log d}) \right)^{\frac{3-1/3^{k-2}}{4}} \left(\frac{Ks \log d}{N} \right)^{\frac{1-1/3^{k-1}}{2}},$$
(3.17)

$$\delta_k = c_1 \left(\frac{b_{k-1} K s \log d}{N}\right)^{1/3}, \ \lambda_k = c_2 \sqrt{\frac{N K \log d}{n^2 b_{k-1} \delta_k}}, \tag{3.18}$$

for some constants $c_1, c_2, c_3 > 0$. If

$$N \lesssim \left(\log(\frac{N}{Ks\log d})\right)^{1/3} (Ks\log d)^{1/3} n^{2/3},\tag{3.19}$$

and $Ks \log d = o(N)$ hold, then with probability greater than $1 - 2Kd^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{2} \lesssim \left(\log(\frac{N}{Ks\log d})\right)^{\frac{1}{4}} \left(\frac{Ks\log d}{N}\right)^{\frac{1}{2}}, \qquad (3.20)$$

$$\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{1} \lesssim \sqrt{s} \left(\log(\frac{N}{Ks\log d})\right)^{\frac{1}{4}} \left(\frac{Ks\log d}{N}\right)^{\frac{1}{2}}.$$
(3.21)

For $\beta = 1$, the convergence rate of $\widehat{\theta}_K$ in (3.20) and (3.21) is nearly parametric (with some extra logarithmic factors). However, unlike the two previous cases (i) $\beta > (1 + \sqrt{3})/2$ and (ii) $1 < \beta \leq (1 + \sqrt{3})/2$, we have to run at least $K = \lceil \log_3(\log N) \rceil$ iterations in Algorithm 1 to attain the near-parametric rate, where K has to grow with N, although, very slowly. The intuition is that for $\beta = 1$, the rate of $\widehat{\theta}_1$ becomes $O_p((\frac{Ks \log d}{N})^{1/3})$ and is too slow so that the sequence of estimators $\widehat{\theta}_1, \widehat{\theta}_2, ..., \widehat{\theta}_k, ...$ can only approach the fast convergence region $\Theta_{fast,\beta=1}$ defined in (3.16) but never fall into this region. Finally, we end the algorithm at $K = \lceil \log_3(\log N) \rceil$ steps when $\widehat{\theta}_K$ is close enough to the fast convergence region. More precisely, we choose K such that the convergence rate of $\widehat{\theta}_{K+1}$ matches with that of $\widehat{\theta}_K$. In other words, there is no further improvement on the statistical rate of the estimators to run more than K iterations in Algorithm 1.

Remark 2. Mallik et al. (2020) studied a general M-estimation problem with multistage sampling procedures. In particular, they considered the following classification problem $d^* = \operatorname{argmin} \mathbb{E}[L_{01}(Y(X-d))]$, where $L_{01}(\cdot)$ is the 0-1 loss. They assumed that $\eta(x) = \mathbb{P}(Y = 1|X = x)$ is continuously differentiable in a neighborhood of d^* . Their main idea is to use an isotonic regression approach to estimate $\eta(x)$ and then invert this function at 1/2 to estimate d^* . Using this approach, their first-stage estimator \hat{d}_1 with N i.i.d data sampled uniformly from the population has the rate $O_p(N^{-1/3})$. In the second stage, they sampled another N i.i.d data in a zoomed-in neighborhood of \hat{d}_1 and the resulting second-stage estimator \hat{d}_2 has the rate $O_p(N^{-(1+\gamma)/3})$ for any $\gamma < 1/3$, whose limiting distribution was also established. Compared with their results, under the assumption that $\beta = 1$ and d, s are fixed, the proof of Theorem 4 shows that our estimator $\hat{\theta}_1$ has the rate $O_p((\log(\log N)/N)^{1/3})$ and $\hat{\theta}_2$ has the rate $O_p((\log N)^{1/6}(\log(\log N)/N)^{4/9})$, which are comparable to the rates of \hat{d}_1 and \hat{d}_2 , respectively. In their paper, they did not pursue the theoretical results of the estimators beyond 2 stages, whereas our results confirm that the convergence rate can be further accelerated until we reach the stage $K = \lceil \log_3(\log N) \rceil$.

Remark 3. It is seen that the choice of tuning parameters and the number of iterations requires the knowledge of the smoothness parameter β . Feng et al. (2022) developed an adaptive estimation procedure for β by applying the Lepski's method, when the data are i.i.d sampled from the population. We expect that a similar approach can be used for adaptive estimation in our context. While this extension is of theoretical interest, the resulting algorithm may become computationally expensive or even infeasible. To make our approach practical, we recommend choosing the tuning parameters by cross-validations and fix K = 2 in practice, see Section 4 for more details about the practical implementations.

Finally, we briefly summarize our main conclusions as follows:

- (i) $\beta > (1 + \sqrt{3})/2$. The first step estimator $\hat{\theta}_1$ belongs to the fast convergence region $\Theta_{fast,\beta}$ with high probability, so that after another iteration $\hat{\theta}_2$ attains the parametric rate.
- (ii) $1 < \beta \leq (1 + \sqrt{3})/2$. After $K 1 = \lceil \log_{\frac{\beta}{2\beta+1}} (1 \frac{\beta+1}{2\beta^2}) \rceil$ iterations, we have $\widehat{\theta}_{K-1} \in \Theta_{fast,\beta}$ with high probability, and therefore $\widehat{\theta}_K$ attains the parametric rate.
- (iii) $\beta = 1$. For any $k \ge 1$, the estimator $\widehat{\theta}_k$ is not necessarily belonging to the fast convergence region $\Theta_{fast,\beta}$, so that the estimator $\widehat{\theta}_K$ only attains the near-parametric rate.

3.3 Minimax Lower Bound

For clarity, we write $R_P(\boldsymbol{\theta})$ for $R(\boldsymbol{\theta})$ in (1.2) to highlight the expectation is taken with respect to P, the joint distribution of (X, Y, \mathbf{Z}) . Similarly, we use $\boldsymbol{\theta}^*(P)$ to denote the unique minimizer of $R_P(\boldsymbol{\theta})$. Let $\mathcal{P}(\beta, L, p_{\min}, p_{\max})$ denote the class of distributions which belong to the Hölder class $\mathcal{P}(\beta, L)$ defined in Definition 3.1 and $\sup_{x \in \mathcal{X}, y \in \{-1,1\}, \mathbf{z} \in \mathcal{Z}} f(x \mid y, \mathbf{z}) < p_{\max}$ and $\inf_{x \in \mathcal{X}, \mathbf{z} \in \mathcal{Z}} f(x \mid \mathbf{z}) \geq p_{\min}$ hold, where \mathcal{X} and \mathcal{Z} are the support sets of X and \mathbf{Z} . We consider the following class of distributions

$$\mathcal{P}(\beta, s) = \left\{ P \in \mathcal{P}(\beta, L, p_{\min}, p_{\max}) : \|\boldsymbol{\theta}^*(P)\|_0 \le s, \|\boldsymbol{\theta}^*(P)\|_2 \le C, \quad (3.2) \text{ holds and} \\ \rho_- \le \lambda_{\min} \left(\nabla^2 R_P \left(\boldsymbol{\theta}^*(P) \right) \right) \le \lambda_{\max} \left(\nabla^2 R_P \left(\boldsymbol{\theta}^*(P) \right) \right) \le \rho_+ \right\}, \quad (3.22)$$

where we treat $L, p_{\min}, p_{\max}, C, M_1, \rho_-$ and ρ_+ as positive constants.

For $1 \leq i \leq n$, assume that (X_i, \mathbf{Z}_i, Y_i) are i.i.d from the distribution P, and we observe data $O_i = (X_i, \mathbf{Z}_i, Y_i)$ if $R_i = 1$ and $O_i = (X_i, \mathbf{Z}_i)$ if $R_i = 0$. Here, the assumed observed data mechanism is more flexible than that considered in Section 2, as we are allowed to keep the data (X_i, \mathbf{Z}_i) even if the *i*th data point is not sampled. Let $\bar{H}_{i-1} = \{O_1, \dots, O_{i-1}\}$ denote the collection of the first i-1 observed data, which can be also viewed as the historical data before we observe the *i*th sample. For

simplicity, denote $\bar{H}_{i-1} = \emptyset$ for i = 1. In terms of the sampling procedure, we allow R_i to depend on (X_i, \mathbf{Z}_i) as well as the first i-1 observed data \bar{H}_{i-1} . We denote $Q_i := \mathbb{P}(R_i = 1 \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1})$ and $Q = (Q_1, ..., Q_n)$. Based on the observed data $\{O_i\}_{i=1}^n$, our goal is to estimate the unknown parameter $\boldsymbol{\theta}^*(P)$. In this setting, the estimator $\hat{\boldsymbol{\theta}}$ is a measurable function of the observed data $(O_1, ..., O_n)$, whose accuracy can be assessed by the following l_q risk

$$\mathbb{E}_{P,Q} \| \widehat{\boldsymbol{\theta}}(O_1, ..., O_n) - \boldsymbol{\theta}^*(P) \|_q$$

where the expectation is taken under (P, Q) which determines the distribution of $(O_1, ..., O_n)$. For any given sampling distribution Q, the minimax risk for estimating $\theta^*(P)$ is defined as

$$\mathcal{M}_n(\mathcal{P}(\beta,s),Q) := \inf_{\widehat{\theta}} \sup_{P \in \mathcal{P}(\beta,s)} \mathbb{E}_{P,Q} \|\widehat{\theta}(O_1,...,O_n) - \theta^*(P)\|_q,$$

where the supremum is only for the distribution P in $\mathcal{P}(\beta, s)$ and the infimum is over all possible estimators $\hat{\theta}$ based on the observed data $\{O_i\}_{i=1}^n$.

Since $\mathcal{M}_n(\mathcal{P}(\beta, s), Q)$ depends on the sampling distribution Q, to characterize the assumptions on Q, we define the following class. For any given distribution $P \in \mathcal{P}(\beta, s)$ and budget N, let

$$\mathcal{Q}_{N}(P) := \left\{ (Q_{1}, \cdots, Q_{n}) : \forall 1 \leq i \leq n, (X_{i}, \mathbf{Z}_{i}, Y_{i}) \perp_{(P,Q)} \bar{H}_{i-1}, \quad R_{i} \perp_{(P,Q)} Y_{i} \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}, \\ \mathbb{E}_{P}\left(\sum_{i=1}^{n} Q_{i}\right) \leq N, \quad \sup_{\|\boldsymbol{v}\|_{0} \leq s} \frac{\boldsymbol{v}^{T} \mathbb{E}_{P}\left(\sum_{i=1}^{n} Q_{i} \mathbf{Z}_{i} \mathbf{Z}_{i}^{T}\right) \boldsymbol{v}}{\|\boldsymbol{v}\|_{2}^{2}} \leq CN \right\}, \quad (3.23)$$

where C is a positive constant. In (3.23), $(X_i, \mathbf{Z}_i, Y_i) \perp_{(P,Q)} \bar{H}_{i-1}$ and $R_i \perp_{(P,Q)} Y_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1}$ formalize the assumptions on the data generating process for R_i . More precisely, $(X_i, \mathbf{Z}_i, Y_i) \perp_{(P,Q)} \bar{H}_{i-1}$ is valid, since we generate $(R_1, ..., R_{i-1})$ without using the future data (X_i, \mathbf{Z}_i, Y_i) . In addition, the assumption $R_i \perp_{(P,Q)} Y_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1}$ is satisfied by the measurement-constrained sampling, that is Y_i is not used to decide whether the *i*th data point is sampled or not. The assumption $\mathbb{E}_P(\sum_{i=1}^n Q_i) \leq N$ is equivalent to $\mathbb{E}_{P,Q}(\sum_{i=1}^n R_i) \leq N$, corresponding to our budget constraint. Similarly, the last assumption in (3.23) can be rewritten as

$$\sup_{\|\boldsymbol{v}\|_{0} \leq s} \frac{\boldsymbol{v}^{T} \mathbb{E}_{P,Q}\left(\sum_{i=1}^{n} R_{i} \boldsymbol{Z}_{i} \boldsymbol{Z}_{i}^{T}\right) \boldsymbol{v}}{\|\boldsymbol{v}\|_{2}^{2}} \lesssim N,$$
(3.24)

which controls the maximal sparse eigenvalues of $\mathbb{E}_{P,Q}\left(\sum_{i=1}^{n} R_i \mathbf{Z}_i \mathbf{Z}_i^T\right)$. To better understand the condition (3.24), let us consider the univariate case $Z \in \mathbb{R}$, in which (3.24) reduces to $\sum_{i=1}^{n} \mathbb{E}_{P,Q}\left(R_i Z_i^2\right) \lesssim N$. When Z has bounded support, it holds that $\sum_{i=1}^{n} \mathbb{E}_{P,Q}\left(R_i Z_i^2\right) \lesssim \sum_{i=1}^{n} \mathbb{E}_{P,Q}(R_i) \leq N$, where the last step follows from the budget constraint. On the other hand, when the support of Z is unbounded, this assumption rules out the sampling procedures, which overly sample the data with extreme values of Z. In practice, such data points often correspond to high leverage points or outliers, which may indeed deteriorate the estimation accuracy.

Since the class $\mathcal{Q}_N(P)$ depends on the given distribution $P \in \mathcal{P}(\beta, s)$, we define $\mathcal{Q}_N(\mathcal{P}(\beta, s)) := \bigcap_{P \in \mathcal{P}(\beta, s)} \mathcal{Q}_N(P)$, as the set of sampling distributions that satisfy the conditions in (3.23) for all $P \in \mathcal{P}(\beta, s)$.

Remark 4. We give some important examples of sampling distributions in $\mathcal{Q}_N(\mathcal{P}(\beta, s))$.

• Sampling with bounded probability. Consider an arbitrary sampling distribution Q, which satisfies the independence assumptions, the budget constraint $\mathbb{E}_P(\sum_{i=1}^n Q_i) \leq N$ in (3.23) and more importantly a bounded probability assumption $\max_{1 \leq i \leq n} Q_i \leq \frac{CN}{n}$ for some constant C > 0. We can show that $Q \in \mathcal{Q}_N(P)$ for any $P \in \mathcal{P}(\beta, s)$. To see this, for any $\|v\|_0 \leq s$,

$$\boldsymbol{v}^T \mathbb{E}_P \Big(\sum_{i=1}^n Q_i \boldsymbol{Z}_i \boldsymbol{Z}_i^T \Big) \boldsymbol{v} \le \frac{CN}{n} \sum_{i=1}^n \mathbb{E}_P (\boldsymbol{v}^T \boldsymbol{Z}_i)^2 \le CM_1 N \| \boldsymbol{v} \|_2^2$$

where we use the assumption $Q_i \leq \frac{CN}{n}$ in the first inequality, and the second inequality follows from $P \in \mathcal{P}(\beta, s)$ and the corresponding sparse eigenvalue assumption (3.2). Thus, $Q \in \mathcal{Q}_N(\mathcal{P}(\beta, s))$ holds. A specific example of Q is the uniform sampling, where the data are sampled completely at random with $Q_i = \frac{N}{n}$. We note that, unlike the positivity assumption in the missing data literature, we do not impose any lower bound for Q_i , which means we allow $Q_i = 0$ in our scenario. Intuitively, the bounded probability assumption prevents us from dramatically overly sampling some specific samples compared to the uniform sampling.

• Sampling within a region of X. Consider the following sampling distribution

$$Q_{i} = g_{3i}(\bar{H}_{i-1}) \,\mathbb{1}\{f_{i}(\boldsymbol{Z}_{i},\bar{H}_{i-1}) - g_{1i}(\bar{H}_{i-1}) \le X_{i} \le f_{i}(\boldsymbol{Z}_{i},\bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1})\}, \quad (3.25)$$

where $f_i : (\mathbf{Z}_i, \bar{H}_{i-1}) \to \mathbb{R}, g_{1i}, g_{2i} : \bar{H}_{i-1} \to \mathbb{R}^+$ and $g_{3i} : \bar{H}_{i-1} \to [0, 1]$ are user specified functions such that $\mathbb{E}_P(\sum_{i=1}^n Q_i) \leq N$ holds. Apparently, the independence assumptions in (3.23) are guaranteed by the data generating mechanism in (3.25). Moreover, in Section A.10 we show that (3.24) holds for any $P \in \mathcal{P}(\beta, s)$, which implies $Q \in \mathcal{Q}_N(\mathcal{P}(\beta, s))$. Recall that the proposed sampling mechanism in Section 2 can be written as follows, for any $(X_i, \mathbf{Z}_i) \in D_k$ with k > 1,

$$Q_{i} = c_{n,k} \, \mathbb{1}\{-b_{k-1}\sqrt{1+\|\widehat{\theta}(\bar{H}_{i-1})\|_{2}^{2}} \le X_{i} - \widehat{\theta}(\bar{H}_{i-1})^{T} \mathbf{Z}_{i} \le b_{k-1}\sqrt{1+\|\widehat{\theta}(\bar{H}_{i-1})\|_{2}^{2}}\},$$

where we write $\hat{\theta}(\bar{H}_{i-1})$ for $\hat{\theta}_{k-1}$ to highlight that it is function of the historical data. It is seen that our sampling method in Section 2 belongs to the class (3.25). Indeed, this class of sampling methods allow us to overly sample the data within a (small) region of X, which complement the example of sampling with bounded probability. We also note that (3.25) can be generalized to the sampling distributions in multiple regions of X, such as $Q_i = g_{3i}^{(k)}(\bar{H}_{i-1})$ if $X_i \in S_i^{(k)}$ and $Q_i = 0$ otherwise, for some k > 1, where $S_i^{(k)} =$ $[f_i^{(k)}(\mathbf{Z}_i, \bar{H}_{i-1}) - g_{1i}^{(k)}(\bar{H}_{i-1}), f_i^{(k)}(\mathbf{Z}_i, \bar{H}_{i-1}) + g_{2i}^{(k)}(\bar{H}_{i-1})]$ are non-overlapping intervals. Finally, we define the N-budget minimax risk for estimating $\theta^*(P)$ as

$$\mathcal{M}_n(\mathcal{P}(\beta,s),N) := \inf_{Q \in \mathcal{Q}_N(\mathcal{P}(\beta,s))} \inf_{\widehat{\boldsymbol{\theta}}} \sup_{P \in \mathcal{P}(\beta,s)} \mathbb{E}_{P,Q} \|\widehat{\boldsymbol{\theta}}(O_1,...,O_n) - \boldsymbol{\theta}^*(P)\|_q.$$

Theorem 5. Assume that $s(\frac{\log(d/s)}{N})^{1/2} = o(1)$ and the smoothness parameter $\beta \ge 1$. We have

$$\inf_{Q \in \mathcal{Q}_N(\mathcal{P}(\beta,s))} \inf_{\widehat{\boldsymbol{\theta}}} \sup_{P \in \mathcal{P}(\beta,s)} \mathbb{P}_{P,Q} \left[\left\| \widehat{\boldsymbol{\theta}}(O_1, ..., O_n) - \boldsymbol{\theta}^*(P) \right\|_q \ge cs^{\frac{1}{q} - \frac{1}{2}} \left(\frac{s \log(d/s)}{N} \right)^{1/2} \right] \ge c',$$

for q = 1, 2, where c, c' are positive constants.

Compared with the results in Section 3.2, this theorem shows that our proposed estimator via the active subsampling algorithm is minimax rate optimal up to some logarithmic factors. By Markov inequality, we can also obtain the lower bound for the N-budget minimax risk

$$\mathcal{M}_n(\mathcal{P}(\beta, s), N) \ge c' c s^{\frac{1}{q} - \frac{1}{2}} \left(\frac{s \log(d/s)}{N}\right)^{1/2}$$

As a final remark, the same lower bound in Theorem 5 holds with $\hat{\theta}(O_1, ..., O_n)$ replaced by $\hat{\theta}(\{O_i\}_{i:R_i=1})$, that is we take the infimum over all possible estimators based on the selected (and labeled) data. Recall that our proposed estimator that achieves the lower bound is computed with the labeled data only. This implies that the unlabeled data $\{X_i, \mathbf{Z}_i\}_{i:R_i=0}$ does not bring additional information to improve the convergence rate for estimating $\boldsymbol{\theta}^*(P)$ in our model.

4 Practical Considerations

In this section, we discuss several implementation issues in our Algorithm 1.

First, we discuss the computational challenge for solving the optimization problem (2.7). Despite addressing the discontinuity of the 0-1 loss, the smoothed empirical risk function $R_{\delta_k}^{D_k}(\theta)$ remains non-convex. Consequently, obtaining the global solution to (2.7) presents computational challenges. To overcome this issue, we leverage the path-following algorithm outlined in Feng et al. (2022). For any $1 \leq k \leq K$, this algorithm computes approximate local solutions to (2.7) corresponding to a sequence of decreasing regularization parameters λ until the desired regularization parameter is reached. We set $\hat{\theta}_k = \tilde{\theta}_{k,tgt}$, where $\tilde{\theta}_{k,tgt}$ represents the final approximate local solution with the desired regularization parameter $\lambda_{k,tgt}$. By employing this path-following approach, we can efficiently compute the entire solution path for $\hat{\theta}_k$ while preserving sparsity across the sequence. For $k \geq 2$, to speed up the computation of $\hat{\theta}_k$, we can use the estimator $\hat{\theta}_{k-1}$ obtained from the previous iteration as a warm start in the path-following approach. The details of the path-following algorithm for solving (2.7) are provided in Appendix A.1.

Second, we consider how to choose K. Our theoretical results in Section 3.2 reveal that the choice of K may depend on the smoothness parameter β . In particular, when β is greater than

 $(1 + \sqrt{3})/2 \approx 1.37$, the two-step algorithm with K = 2 yields the estimator with the optimal rate. In practice, we recommend choosing K = 2 for the following reasons. First, in many applications, it is often reasonable to assume that the conditional density of X given Y and Z is sufficiently smooth (e.g., second order differentiable), which means β is indeed larger than 1.37. Even if β is not larger than 1.37, the improvement from $\hat{\theta}_1$ to $\hat{\theta}_2$ is far more significant compared to the improvement from $\hat{\theta}_2$ to $\hat{\theta}_3$. Second, compared to $\hat{\theta}_k$ with k > 2, the estimator $\hat{\theta}_2$ depends on a much smaller set of tuning parameters so that the implementation is more convenient and numerically more stable. This makes our algorithm more practical in real world applications than many existing active learning algorithms.

Finally, we discuss how to choose the tuning parameters λ_1, λ_2 and b_1 in our active subsampling algorithm with K = 2. We find that the performance of our algorithm is quite robust to the choice of bandwidth parameters δ_1 and δ_2 . Thus, we set $\delta_1 = \delta_2 = 1$ in both simulation and real data analysis. Indeed, following Feng et al. (2022), we can easily modify our cross-validation algorithm to choose $\lambda_1, \lambda_2, b_1$ and δ_1, δ_2 simultaneously. However, such an approach tends to be computationally far more expensive and may require a very large N to obtain stable results. We do not pursue this approach here.

Similar to Feng et al. (2022), the cross-validation algorithm for λ with one standard error rule is shown in Algorithm 2. Built on this algorithm, the data-driven two-step active subsampling with cross-validation is shown in Algorithm 3. In this algorithm, \tilde{N}_1 , \tilde{N}_{cv} and \tilde{N}_2 stand for the budget for obtaining $\hat{\theta}_1$ in step 1, the budget for selecting b_1 in step 2, and the budget for obtaining the final estimator $\hat{\theta}_2$ in step 3, respectively. In general, we recommend choosing the budget \tilde{N}_2 in the final step relatively large, as \tilde{N}_2 plays the role of sample size for the final estimator $\hat{\theta}_2$.

5 Simulation Studies

We conduct simulations to evaluate the performance of our proposed method. We consider the following two classes of models, and for both models, it can be shown that the parameter θ coincides with the estimand in (1.2).

• Binary response model: We consider $Y = \operatorname{sign}(\widetilde{Y})$, where

$$\widetilde{Y} = X - \boldsymbol{\theta}^T \boldsymbol{Z} + \boldsymbol{\epsilon},$$

 $X \in \mathbb{R}, \mathbb{Z} \in \mathbb{R}^d$, and ϵ is a random noise such that $\operatorname{Median}(\widetilde{Y} \mid X, \mathbb{Z}) = X - \theta^T \mathbb{Z}$. Logistic regression belongs to the class of binary response models by setting ϵ to follow the logistic distribution independent with (X, \mathbb{Z}) . We conduct simulations for both logistic regression and a more general case where we allow ϵ to depend on (X, \mathbb{Z}) .

• Conditional mean model: We consider $Y \in \{-1, 1\}, \mathbf{Z} \in \mathbb{R}^d$, and

$$X = \mu Y + \boldsymbol{\theta}^T \boldsymbol{Z} + \boldsymbol{\epsilon}$$

Algorithm 2 *M*-fold Cross-Validation for λ with one standard error rule

Input: Data $D = \{X_i, Z_i, Y_i\}$, a grid for λ .

Parameters: δ

Randomly split D into M folds, $D_1, ..., D_M$ with equal size. For each λ in the grid, compute

$$\widehat{CV}_{\lambda}^{k} = \frac{1}{|D_{k}|} \sum_{a \in D_{k}} \gamma(Y_{a}) L_{\delta}(Y_{a}(X_{a} - \boldsymbol{Z}_{a}^{T}\widehat{\boldsymbol{\theta}}_{\lambda}^{(-k)})),$$

where $\hat{\theta}_{\lambda}^{(-k)}$ is our estimator with the tuning parameters δ and λ using the data excluding D_k .

Compute the cross-validation error $\widehat{CV}_{\lambda} = \frac{1}{M} \sum_{k=1}^{M} \widehat{CV}_{\lambda}^{k}$.

Define $\widehat{CV}_{\min} = \min_{\lambda} \widehat{CV}_{\lambda}$ as the minimum cross-validation error over this grid, and $\overline{\lambda} = \operatorname{argmin}_{\lambda} \widehat{CV}_{\lambda}$. Define \widehat{SE}_{\min} as the standard error of $\{\widehat{CV}_{\overline{\lambda}}^k\}_{k=1}^M$ over these M folds.

Find $\widehat{\lambda}_{CV}$ via the following "one standard error rule",

$$\widehat{\lambda}_{CV} = \max \ \lambda \in \operatorname{grid} \ s.t. \ \widehat{CV}_{\lambda} \leq \widehat{CV}_{\min} + \widehat{SE}_{\min}$$

return $\widehat{\lambda}_{CV}$ and $\widehat{CV}_{\widehat{\lambda}_{CV}}$.

where $\epsilon \perp Y, Z$ is a random noise and $\mu > 0$ is a constant.

Under each model, we simulate i.i.d. samples with sample size n = 20000 and dimension d = 200. We refer to the collection of these samples as the entire dataset. We set sparsity s = 10, and generated the nonzero elements of θ^* from Uniform(1,2). We then normalize θ^* such that $\|\theta^*\|_2 = 1$. For the logistic regression, we generate $X \sim N(0,1)$, $\mathbf{Z} \sim N_d(0,1)$. For the binary response model where ϵ can depend on (X, \mathbf{Z}) , we generate $X \sim N(0,1)$, $\mathbf{Z} \sim N_d(0,1)$, and $\epsilon \sim N(0, \sigma^2(1 + 2(X - \theta^T \mathbf{Z})^2))$ with $\sigma = 0.5$. We refer to this case as binary response model in the following discussion. For the conditional mean model, we generate $Y \sim \text{Uniform}\{-1,1\}$, $\mathbf{Z} \sim N_d(0,1)$, $\epsilon \sim N(0, (0.1)^2)$ and we set $\mu = 2$.

The label budget is set to be N = 2000 across all scenarios. For our proposed method, we implement the two-step active subsampling algorithm outlined in Algorithm 1 with $N_1 = N/8$ and $N_2 = 7N/8$. Empirically, we find that assigning a larger proportion of the label budget to the second step leads to a more stable final estimator $\hat{\theta}_2$. The numerical results for our algorithm with $N_1 = N/5$ and $N_2 = 4N/5$ are quite similar and are deferred to the Appendix Section A.11. We compare the performance of our proposed method with that of the method proposed in Feng et al. (2022), where the path-following algorithm is applied on a dataset with size N = 2000 uniformly sampled from the entire dataset. We refer to this method as "passive PF", while our proposed method is denoted as "two-step sampling with PF". In the application of the path-following algorithm, we set the bandwidth parameter $\delta = 1$ and use the standard Gaussian density as the

Algorithm 3 Data-Driven Two-Step Active Subsampling with Cross-Validation

Input: $D = \{X_i, \mathbf{Z}_i\}_{i=1}^n$, a grid Δ for b and $\widetilde{N}_1, \widetilde{N}_{cv}, \widetilde{N}_2$ with $\widetilde{N}_1 + \widetilde{N}_{cv} + \widetilde{N}_2 = N$. **Parameters:** δ_1, δ_2

Randomly split D into 3 batches with $|D_1| = \frac{\tilde{N}_1}{N}n$, $|D_{cv}| = \frac{\tilde{N}_{cv}}{N}n$ and $|D_2| = \frac{\tilde{N}_2}{N}n$. Step 1: Obtain Initial Estimator $\hat{\theta}_1$

Draw data from D_1 with probability $c_{n,1} = \frac{N}{n}$. Acquire the label Y_i for each sampled data and form the dataset $D_1^* = \{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, where $(X_i, \mathbf{Z}_i) \in D_1$.

Apply the 5-fold cross-validation Algorithm 2 to D_1^* . Return the optimal parameter $\lambda_{1,opt}$. Compute the initial estimator: $\hat{\theta}_1 \leftarrow \operatorname{argmin}_{\theta} \{ R_{\delta_1}^{D_1}(\theta) + \lambda_{1,opt} \|\theta\|_1 \}.$

Step 2: Select Optimal b_1

For each candidate $b \in \Delta$:

Define the active set: $S_2 \leftarrow \left\{ (X, \mathbf{Z}) : -b \leq \frac{X - \widehat{\theta}_1^T \mathbf{Z}}{\sqrt{1 + \|\widehat{\theta}_1\|_2^2}} \leq b \right\}.$

Given $(X_i, \mathbf{Z}_i) \in S_2$, draw data (X_i, \mathbf{Z}_i) from D_{cv} with probability $\frac{N}{n|\Delta|\mathbb{P}((X, \mathbf{Z}) \in S_2)}$. Acquire the label Y_i for each sampled data and form $D^*_{cv,b} = \{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, where $(X_i, \mathbf{Z}_i) \in D_{cv}$.

Apply the 5-fold cross-validation Algorithm 2 to $D_{cv,b}^*$. Return the optimal parameter $\lambda_{b,opt}$ and the minimum CV error \widehat{CV}_b .

Compute $\hat{b}_1 = \operatorname{argmin}_{b \in \Delta} \widehat{CV}_b$ that minimizes the cross-validation error.

Step 3: Obtain Final Estimator $\hat{\theta}_2$

Define the active set: $S_2 \leftarrow \left\{ (X, \mathbf{Z}) : -\widehat{b}_1 \leq \frac{X - \widehat{\theta}_1^T \mathbf{Z}}{\sqrt{1 + \|\widehat{\theta}_1\|_2^2}} \leq \widehat{b}_1 \right\}.$

Given $(X_i, \mathbf{Z}_i) \in S_2$, draw data (X_i, \mathbf{Z}_i) from D_2 with probability $c_{n,2} = \frac{N}{n\mathbb{P}((X, \mathbf{Z}) \in S_2)}$. Acquire the label Y_i for each sampled data and form $D_2^* = \{X_i, \mathbf{Z}_i, Y_i\}_{R_i=1}$, where $(X_i, \mathbf{Z}_i) \in D_2$.

Apply the 5-fold cross-validation Algorithm 2 to D_2^* . Return the optimal parameter $\lambda_{2,opt}$. Compute the final estimator: $\hat{\theta}_2 \leftarrow \operatorname{argmin}_{\boldsymbol{\theta}} \{ R_{\delta_2}^{D_2}(\boldsymbol{\theta}) + \lambda_{2,opt} \| \boldsymbol{\theta} \|_1 \}.$

return θ_2

kernel function. The number of regularization stages is fixed at T = 20, and we choose $\nu = 1/4$, $\phi = (\lambda_{tgt}/\lambda_0)^{1/T}$ and $\eta = 1$ in the path-following Algorithm 4. For both methods and across all scenarios, the tuning parameter λ_{tqt} is chosen by the 5-fold cross-validation via Algorithm 2.

In our comparison, we also include two simple methods "passive LR" and "two-step sampling with LR". The former refers to the ℓ_1 -penalized logistic regression applied to the dataset with size N = 2000 uniformly sampled from the entire dataset. For the "two-step sampling with LR", we employ a similar two-step approach as in Algorithm 1. However, instead of solving (2.3) using the path-following algorithm, we estimate θ by the ℓ_1 -penalized logistic regression. Similar to our proposed method, the active set in second step has the same form as (2.5), which depends on the choice of b_1 .

Figures 1, 2 and 3 illustrate the pattern of the statistical error $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 norms as the size of the active set b_1 increases under three models. The smallest b_1 value is chosen such that approximately 10% of the entire dataset fall into the active set, while the largest b_1 value corresponds to the case that the active set covers nearly 100% of the entire dataset. The simulation is repeated 50 times.

Since the passive LR and passive PF methods are independent of b_1 , their estimation errors correspond to two horizontal lines. In Figure 1, the passive LR can be viewed as the benchmark estimator, since the data are generated under the logistic regression. We can see that applying the two-step sampling idea to logistic regression generally does not improve the performance of the passive LR. This may be due to the fact that the logistic regression is a regular model and the parameter estimation may not be improvable. However, our proposed two-step sampling with PF method can significantly improve upon the passive PF method, especially for small or moderate values of b_1 . This is consistent with our theoretical results on the convergence rate of the estimators. Similar patterns hold under the conditional mean model in Figure 2. In particular, our proposed method yields the smallest ℓ_1 and ℓ_2 estimation errors among all the competing methods, for small or moderate values of b_1 . For the binary response model in Figure 3, when b_1 is small, our proposed method has larger errors than the passive PF method. However, as b_1 keeps increasing, our proposed method outperforms the passive PF.

We also consider the comparison of the four methods concerning $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction errors, which exhibit similar patterns as $\|\widehat{\theta} - \theta^*\|_1$ and $\|\widehat{\theta} - \theta^*\|_2$. Further details are provided in the Appendix Section A.11.

As discussed above, b_1 plays a significant role in our proposed method, and the optimal choice of b_1 varies depending on the dataset. In the following, we examine the performance of our data-driven two-step sampling method with cross-validation in Algorithm 3. Table 1 presents a comparison between our proposed method and the passive path-following method for the three models. The simulation is repeated 50 times. From the result in Table 1 we can see that our proposed method outperforms the passive path-following method across all three models. For example, the ℓ_1 estimation error of our proposed method is around 45% smaller than the passive path-following method

Error	Conditional mean model		Logistic model		Binary response model	
	Two-step PF	Passive PF	Two-step PF	Passive PF	Two-step PF	Passive PF
ℓ_1	0.918(0.101)	1.648(0.165)	1.514(0.319)	1.625(0.416)	0.835(0.201)	0.937(0.200)
ℓ_2	0.313(0.036)	0.525(0.052)	0.525(0.093)	0.559(0.129)	0.319(0.068)	0.341(0.066)
ℓ_{∞}	0.150(0.026)	0.196(0.022)	0.270(0.047)	0.275(0.050)	0.192(0.044)	0.187(0.038)

Table 1: Comparison of the two-step path-following method with the passive path-following algorithm. The number in the parentheses are standard deviations.

under the conditional mean model.

6 Real Data Analysis

In this section, we apply our proposed method to a dataset of hospitalized patient diagnosed with diabetes, obtained from the UCI Machine Learning Repository (Clore et al., 2014). This dataset contains 101,766 hospital records collected over a decade (1999-2008) from 130 US hospitals. The data includes various attributes such as patient ID, race, gender, age, admission type, readmission status, length of hospital stay, medical specialty of the admitting physician, number of lab tests performed, Hemoglobin A1c (HbA1c) test results, diagnosis, and more. Among these attributes, the HbA1c test result is a key indicator of glucose control and is widely used to evaluate the quality of diabetes care (Baldwin et al., 2005). In an existing work (Strack et al., 2014), researchers were interested in identifying important risk factors that lead to early readmission. They defined the readmission attribute as a binary outcome: "readmitted" if the patient was readmitted within 30 days of discharge, and "otherwise," which includes both readmission after 30 days and no readmission.

In our analysis, we adopt the same definition of response variable Y as in Strack et al. (2014). Specifically, $Y_i = 1$ if a patient was readmitted within 30 days of discharge and $Y_i = -1$ otherwise. We chose the patient's HbA1c test result as the primary measurement X_i , with Z_i representing additional patient demographic statistics and clinical biomarkers. The primary goal of our analysis is to determine the optimal individualized threshold, $\theta^T Z_i$, such that a patient's early readmission can be predicted based on whether the HbA1c test result exceeds this threshold $(X_i \ge \theta^T Z_i)$ or falls below it $(X_i < \theta^T Z_i)$. This problem can be formulated as an estimation task of the form (1.1) or (1.2), with weights $\gamma(y) = 1/\mathbb{P}(Y = y)$.

The original dataset includes multiple inpatient visits from the same patients, making the observations statistically dependent. To address this, as suggested by Strack et al. (2014), we used only the first encounter per patient as the primary admission and determined whether they were readmitted within 30 days. Next, we removed redundant features and those with a high

N	3000		4000		5000	
	Two-step PF	Passive PF	Two-step PF	Passive PF	Two-step PF	Passive PF
ℓ_1	2.422	3.192	0.968	1.276	0.379	0.651
ℓ_2	1.403	2.478	1.649	2.048	0.269	0.543
ℓ_{∞}	0.869	2.369	0.824	2.048	0.203	0.529

Table 2: Comparison of the active path-following method with the passive path-following algorithm.

percentage of missing values. Following the discussion in Strack et al. (2014), we also added pairwise interactions among features as new variables. After these preprocessing steps, the dataset contains 69,984 observations and d = 60 variables, excluding X. Furthermore, we observed significant imbalance in the dataset, with only 6,293 positive instances (Y = 1) compared to 63,691 negative instances (Y = -1). To address this issue, we randomly selected 6,293 samples from the negative class to match the number of positive cases and then combined these with the positive instances. This resulted in a final dataset containing n = 12,586 observations.

Since the true value θ^* is unknown, we first applied the path-following algorithm to the entire dataset (n = 12, 586) to derive an estimator, which is used as the benchmark or equivalently treated as θ^* when evaluating $\|\hat{\theta} - \theta^*\|$ for different methods. Suppose that there is a budget constraint due to study design or administration cost, which prevents us from using the benchmark estimator. Our goal is to illustrate the efficacy of our active subsampling methods under the budget constraint. For our proposed method, we implement the data-driven two-step active subsampling algorithm outlined in Algorithm 3 with label budgets of N = 3000, 4000, and 5000, respectively. We compare the performance of our method with "passive path-following," where the budgeted data is uniformly sampled from the entire dataset to derive the estimator of θ^* . The result is shown in Table 2. It is seen that, as the label budget increases, the estimation errors for both methods decrease, with the two-step path-following method consistently outperforming the passive path-following method across all three settings. From a scientific point of view, our two-step path-following method identified a few important covariates in our model, including 'time in hospital' and 'change of medications'. These findings are consistent with the previous work (Strack et al., 2014), and seem to be clinically meaningful.

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Figure 1: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 norms under the logistic regression. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the label budget is used in the first step for both two-step sampling methods.



Figure 2: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 norms under the conditional mean model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the label budget is used in the first step for both two-step sampling methods.



Figure 3: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 norms under the binary response model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the label budget is used in the first step for both two-step sampling methods.

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A Appendix

A.1 Path-following Algorithm

We employ the path-following algorithm introduced in Feng et al. (2022) to solve the optimization problem $\hat{\theta}_k := \operatorname{argmin}_{\theta} \{ R_{\delta_k}^{D_k}(\theta) + \lambda_k \|\theta\| \}$ for each iteration k. The idea is to compute approximate local solutions corresponding to a sequence of decreasing regularization parameters λ , until the target regularization parameter is reached. To be specific, we firstly choose a sequence of $\lambda_{k,0} > \lambda_{k,1} > \ldots > \lambda_{k,T} = \lambda_{k,tgt}$, where $\lambda_{k,t} = \phi^t \lambda_{k,0}, t = 0, 1, \ldots$ for some constant $\phi \in (0,1)$ and $\lambda_{k,tgt}$ is the target regularization parameter to be specified later. Let T denote the total number of the path-following stages and we set $T = \frac{\log(\lambda_{k,tgt}/\lambda_{k,0})}{\log \phi}$. At each stage $t = 1, \cdots, T$, the goal is to approximately compute the exact local solution $\hat{\theta}_{k,t}$ corresponding to $\lambda_{k,t}$,

$$\widehat{\boldsymbol{\theta}}_{k,t} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} R^{D_k}_{\delta_k}(\boldsymbol{\theta}) + \lambda_{k,t} \|\boldsymbol{\theta}\|_1.$$
(A.1)

To this end, we apply the proximal-gradient method to iteratively approximate $\widehat{\theta}_{k,t}$ by minimizing a sequence of quadratic approximations of $R^{D_k}_{\delta_k}(\theta)$ over a convex constraint set Ω :

$$\boldsymbol{\theta}_{k,t}^{j+1} = \underset{\boldsymbol{\theta}\in\Omega}{\operatorname{argmin}} \left\{ R_{\delta_{k}}^{D_{k}} \left(\boldsymbol{\theta}_{k,t}^{j}\right) + \left\langle \nabla R_{\delta_{k}}^{D_{k}} \left(\boldsymbol{\theta}_{k,t}^{j}\right), \left(\boldsymbol{\theta}-\boldsymbol{\theta}_{k,t}^{j}\right) \right\rangle + \frac{1}{2\eta} \left\| \boldsymbol{\theta}-\boldsymbol{\theta}_{k,t}^{j} \right\|_{2}^{2} + \lambda_{k,t} \|\boldsymbol{\theta}\|_{1} \right\}$$

$$= \underset{\boldsymbol{\theta}\in\Omega}{\operatorname{argmin}} \left\{ \frac{1}{2\eta} \left\| \boldsymbol{\theta}-\boldsymbol{\theta}_{k,t}^{j} + \eta \nabla R_{\delta_{k}}^{D_{k}} \left(\boldsymbol{\theta}_{k,t}^{j}\right) \right\|_{2}^{2} + \lambda_{k,t} \|\boldsymbol{\theta}\|_{1} \right\}$$

$$:= \mathcal{S}_{\lambda_{k,t}\eta} \left(\boldsymbol{\theta}_{k}^{j}, \Omega\right), \qquad (A.2)$$

where η is the step size to be specified later. The proximal-gradient algorithm is described in Algorithm 5.

In the algorithm, we use the stopping criteria $w_{\lambda}(\boldsymbol{\theta})$ defined as

$$\omega_{\lambda}(\boldsymbol{\theta}) = \min_{\boldsymbol{\xi} \in \partial \|\boldsymbol{\theta}\|_{1}} \max_{\boldsymbol{\theta}' \in \Omega} \left\{ \frac{(\boldsymbol{\theta} - \boldsymbol{\theta}')^{T}}{\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|_{1}} \left(\nabla R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}) + \lambda \boldsymbol{\xi} \right) \right\}.$$

At stage t, the proximal-gradient algorithm returns an approximate solution $\tilde{\theta}_{k,t}$ with precision $\epsilon_{k,t} = \nu \lambda_{k,t}, \nu \in (0,1)$ corresponding to $\lambda_{k,t}$. Then we use $\tilde{\theta}_{k,t}$ as a warm start for stage t + 1 and repeat this process. At the final stage, we would compute the approximate solution $\tilde{\theta}_{k,T} = \tilde{\theta}_{k,tgt}$ corresponding to $\lambda_{k,tgt}$ using a high precision $\epsilon_{k,tgt}$. The detail of the path-following algorithm is described in Algorithm 4.

We note that at iteration k, we can use the estimator $\hat{\theta}_{k-1}$ obtained from iteration k-1 as the initial value for θ in Algorithm 4.

A.2 Preliminary Results

Recall that we consider the case $\gamma(y) = 1/\mathbb{P}(Y = y)$ and for simplicity denote $\|\widehat{\omega}_k\|_2 = \sqrt{1 + \|\widehat{\theta}_k\|_2^2}$ in the rest of the proof.

 $\frac{\text{Algorithm 4 } \boldsymbol{\theta} \leftarrow \text{Path-Following } (\lambda_{k,0}, \lambda_{k,tgt}, \nu, T, \epsilon_{k,tgt}, \Omega)}{\text{input:} \quad \boldsymbol{\lambda}_{k,k} \geq 0, \ \boldsymbol{\mu}_{k,k} \geq 0, \ \boldsymbol{$

 $\begin{array}{l} \text{input: } \lambda_{k,tgt} > 0, \nu > 0, \phi > 0, \epsilon_{k,tgt} > 0, \Omega \\ \text{parameter: } \eta > 0 \\ \text{initialize: } \widetilde{\theta}_{k,0} \leftarrow \mathbf{0}, \lambda_{k,0} \leftarrow \left\| \nabla R_{\delta_k}^{D_k}(\mathbf{0}) \right\|_{\infty}, T \leftarrow \frac{\log(\lambda_{k,tgt}/\lambda_{k,0})}{\log \phi} \\ \text{for } t = 1, \ldots, T - 1 \text{ do} \\ \lambda_{k,t} \leftarrow \phi^t \lambda_{k,0} \quad \epsilon_{k,t} \leftarrow \nu \lambda_{k,t} \quad \widetilde{\theta}_{k,t} \leftarrow \text{Proximal-Gradient } \left(\lambda_t, \epsilon_{k,t}, \widetilde{\theta}_{k,t-1}, \Omega \right) \\ \text{end for} \\ \widetilde{\theta}_{k,T} \leftarrow \text{Proximal-Gradient } \left(\lambda_{k,tgt}, \epsilon_{k,tgt}, \widetilde{\theta}_{k,T-1}, \Omega \right) \\ \text{return } \widetilde{\theta}_{k,tgt} = \widetilde{\theta}_{k,T} \end{array}$

 $\begin{array}{l} \textbf{Algorithm 5 } \boldsymbol{\theta} \leftarrow \text{Proximal-Gradient } \left(\lambda, \epsilon, \boldsymbol{\theta}^{0}, \Omega\right) \\ \hline \textbf{input: } \lambda > 0, \epsilon > 0, \boldsymbol{\theta}^{0} \in \mathbb{R}^{d}, \Omega \\ \textbf{parameter: } \eta > 0 \\ \textbf{initialize: } j \leftarrow 0 \\ \textbf{while } w_{\lambda} \left(\boldsymbol{\theta}^{j}\right) > \epsilon \textbf{ do} \\ j \leftarrow j + 1 \quad \boldsymbol{\theta}^{j+1} \leftarrow \mathcal{S}_{\lambda\eta} \left(\boldsymbol{\theta}^{j}, \Omega\right) \\ \textbf{end while} \\ \textbf{return } \boldsymbol{\theta}^{j+1} \end{array}$

We also allow the kernel with unbounded support which satisfies the following tail condition. Assume that there exits a sequence $C_N > 0$ that may depend on N such that

$$\int_{C_N/2}^{\infty} |K(t)| dt \le C\delta_k^{\beta},\tag{A.3}$$

holds for any $2 \le k \le K$, where C is a constant that does not depend on k and δ_k is the bandwidth parameter in the kth iteration.

For example, the Gaussian kernel satisfies $\int_{C_N/2}^{\infty} |K(t)| dt \leq \frac{C}{C_N} e^{-C_N^2/8}$. Note that in Theorem 6, we choose $\delta_k \asymp \left(\frac{C_N Ks \log d}{N}\right)^{1/(2\beta)}$, for $2 \leq k \leq K$. In this case, (A.3) requires $\frac{e^{-C_N^2/8}}{C_N} \leq C\sqrt{\frac{C_N Ks \log d}{N}}$, which holds with $C_N \asymp \sqrt{\log N}$. For the kernel with bounded support, we can simply set $C_N = O(1)$ in the rest of the proof.

Lemma A.1. Under Assumptions 3.1, 3.2 (i) and (ii), 3.3 and 3.4, we have for any $(X_i, Z_i, Y_i) \in D_k$, $2 \le k \le K$, and for all $j = 1 \cdots, d$,

$$\mathbb{E}\left[\left(\gamma(Y_i)\frac{Y_iZ_{ij}}{\delta_k}K\left(\frac{Y_i\left(X_i-\boldsymbol{\theta}^{*T}\boldsymbol{Z}_i\right)}{\delta_k}\right)R_i\right)^2\mid\widehat{\boldsymbol{\theta}}_{k-1}\right]\leq C\frac{c_{n,k}}{\delta_k},$$

for some constant C > 0.

Proof. For simplicity we omit the subscript *i* in the proof. Note that since $\hat{\theta}_{k-1}$ is independent of (X, \mathbb{Z}, Y) , we have

$$\mathbb{E}\left[\left(\gamma(Y)\frac{YZ_{j}}{\delta_{k}}K\left(\frac{Y\left(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}\right)}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1}\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\left(\gamma(Y)\frac{YZ_{j}}{\delta_{k}}K\left(\frac{Y\left(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}\right)}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y\right]\mid\widehat{\boldsymbol{\theta}}_{k-1}\right]$$
$$=\mathbb{E}\left[\left(\frac{Z_{j}}{\delta_{k}}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=1\right]+\mathbb{E}\left[\left(\frac{Z_{j}}{\delta_{k}}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=-1\right].$$

We bound the first term here and the second term follows similarly. Note that

$$\mathbb{E}\left[\left(\frac{Z_j}{\delta_k}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k}\right)R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\left(\frac{Z_j}{\delta_k}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k}\right)R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1, X, \boldsymbol{Z}\right] \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1\right].$$

Recall that

$$S_k := \left\{ (X, \mathbf{Z}) : -b_{k-1} \leq \frac{X - \widehat{\boldsymbol{\theta}}_{k-1}^T \mathbf{Z}}{\|\widehat{\boldsymbol{\omega}}_{k-1}\|_2} \leq b_{k-1} \right\},\$$

and

$$\mathbb{P}(R=1 \mid X, \boldsymbol{Z}, \widehat{\boldsymbol{\theta}}_{k-1}) = c_{n,k} \cdot \mathbb{1}\{(X, \boldsymbol{Z}) \in S_k\},\$$

we have

$$\mathbb{E}\left[\left(\frac{Z_j}{\delta_k}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k}\right)R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1, X, \boldsymbol{Z}\right]$$
$$=c_{n,k}\left(\frac{Z_j}{\delta_k}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k}\right)\right)^2 \cdot \mathbb{1}\{(X, \boldsymbol{Z}) \in S_k\},$$

where we use the fact that $R \perp Y \mid (X, \mathbf{Z}, \widehat{\theta}_{k-1})$. Hence

$$\begin{split} & \mathbb{E}\left[\left(\frac{Z_{j}}{\delta_{k}}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=1\right]\\ =& c_{n,k}\mathbb{E}\left[\left(\frac{Z_{j}}{\delta_{k}}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}}\right)\right)^{2}\cdot\mathbbm{1}\left\{(X,\boldsymbol{Z})\in S_{k}\right\}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=1\right]\\ =& c_{n,k}\int_{\boldsymbol{z}}\frac{z_{j}^{2}}{\delta_{k}^{2}}\int_{-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+\widehat{\boldsymbol{\theta}}_{k-1}^{T}\boldsymbol{z}}K^{2}\left(\frac{x-\boldsymbol{\theta}^{*T}\boldsymbol{z}}{\delta_{k}}\right)f(x\mid\boldsymbol{z},Y=1)dxf(\boldsymbol{z}\mid Y=1)d\boldsymbol{z}\\ =& \frac{c_{n,k}}{\delta_{k}}\int_{\boldsymbol{z}}\int_{(-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+(\widehat{\boldsymbol{\theta}}_{k-1}-\boldsymbol{\theta}^{*})^{T}\boldsymbol{z})/\delta_{k}}z_{j}^{2}K^{2}(\boldsymbol{u})f(\boldsymbol{u}\delta_{k}+\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},Y=1)duf(\boldsymbol{z}\mid Y=1)d\boldsymbol{z}. \end{split}$$

Since $\sup_{x \in \mathbb{R}, y \in \{-1,1\}, z \in \mathbb{R}^d} f(x \mid y, z) < p_{\max} < \infty$, we have

$$\int_{(-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+(\widehat{\boldsymbol{\theta}}_{k-1}-\boldsymbol{\theta}^{*})^{T}\boldsymbol{z})/\delta_{k}}^{(b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+(\widehat{\boldsymbol{\theta}}_{k-1}-\boldsymbol{\theta}^{*})^{T}\boldsymbol{z})/\delta_{k}}K^{2}(u)f(u\delta_{k}+\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},Y=1)du\leq p_{\max}\int K^{2}(u)du.$$

Note that $\mathbb{E}(Z_j^2|Y=1) \leq M_1 < \infty$, we obtain

$$\mathbb{E}\left[\left(\frac{Z_j}{\delta_k}K\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k}\right)R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1\right] \leq \frac{c_{n,k}}{\delta_k}M_1 p_{\max} \int K^2(u) du,$$

...

hence we finish the proof.

Proposition A.1. Denote $\nabla R_{\delta_k,\widehat{\theta}_{k-1}}(\theta^*) = \mathbb{E}\left(\nabla R_{\delta_k}^{D_k}(\theta^*) \mid \widehat{\theta}_{k-1}\right)$ for $2 \le k \le K$. Under Assumptions 3.1, 3.2 (i) and (ii), 3.3 and 3.4, with probability greater than $1 - 2d^{-1}$, we have

$$\|\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \le C_1 \sqrt{\frac{c_{n,k} K \log d}{n\delta_k}},$$

where C_1 is a constant independent of n, d and k.

Proof. Denote $T^k = \nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)$. By definition

$$\|T^k\|_{\infty} = \left\|\frac{K}{n} \sum_{(x_i, \boldsymbol{z}_i) \in D_k} \left(\gamma(y_i) \frac{y_i \boldsymbol{z}_i}{\delta_k} K(\frac{y_i (x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i)}{\delta_k}) R_i\right) - \mathbb{E}\left[\gamma(Y) \frac{Y \boldsymbol{Z}}{\delta_k} K(\frac{Y(X - \boldsymbol{\theta}^{*T} \boldsymbol{Z})}{\delta_k}) R \mid \widehat{\boldsymbol{\theta}}_{k-1}\right]\right\|_{\infty}$$

Note that for some constant C_1 , we have

$$\begin{aligned} |T_{ij}^{k}| &= \left| \gamma(y_{i}) \frac{y_{i} \boldsymbol{z}_{ij}}{\delta_{k}} K(\frac{y_{i}(x_{i} - \boldsymbol{\theta}^{*T} \boldsymbol{z}_{i})}{\delta_{k}}) R_{i} - \mathbb{E} \left[\gamma(Y) \frac{Y Z_{j}}{\delta_{k}} K\left(\frac{Y\left(X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}\right)}{\delta_{k}}\right) R \mid \widehat{\boldsymbol{\theta}}_{k-1} \right] \right| \\ &\leq C_{1} \frac{M_{n} K_{\max}}{\delta_{k}}, \end{aligned}$$

and by Lemma A.1 we know that for some constant C_2 , we have

$$\mathbb{E}((T_{ij}^k)^2 \mid \widehat{\theta}_{k-1}) \le C_2 \frac{c_{n,k}}{\delta_k}.$$

Then by Bernstein inequality we have

$$\mathbb{P}\left(\|T^k\|_{\infty} > t \mid \widehat{\theta}_{k-1}\right) \leq \sum_{j=1}^{d} \mathbb{P}\left(|T_j^k| > t \mid \widehat{\theta}_{k-1}\right)$$
$$\leq 2d \exp\left(-\frac{\frac{1}{2}t^2 n/K}{C_2 \frac{c_{n,k}}{\delta_k} + \frac{t}{3}C_1 M_n K_{\max}/\delta_k}\right).$$

Since the right side doesn't contain $\hat{\theta}_{k-1}$, we obtain that

$$\mathbb{P}\left(\|\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} > t\right) \le 2d \exp\left(-\frac{\frac{1}{2}t^2 n/K}{C_2 \frac{c_{n,k}}{\delta_k} + \frac{t}{3}C_1 M_n K_{\max}/\delta_k}\right)$$

Then note that $M_n \lesssim \sqrt{\frac{c_{n,k}n\delta_k}{K\log d}}$ and take $t = C_3 \sqrt{\frac{c_{n,k}K\log d}{n\delta_k}}$ for some constant C_3 we finish the proof.

Proposition A.2. Recall that $\boldsymbol{\theta}^* = \operatorname{argmin}_{\boldsymbol{\theta}} R(\boldsymbol{\theta})$, where $R(\boldsymbol{\theta}) = \mathbb{E} \left(\gamma(Y) L_{01}(Y(X - \boldsymbol{\theta}^T \boldsymbol{Z})) \right)$, and $\nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*) = \mathbb{E} \left(\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) \mid \widehat{\boldsymbol{\theta}}_{k-1} \right)$ for $2 \leq k \leq K$. Consider the following two cases:

(i) Assumptions 3.1, 3.2 (i) and (ii), 3.3 and 3.4 hold, and

$$b_{k-1} \ge C_N \delta_k$$
 and $b_{k-1} \ge 2 \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n.$ (A.4)

(ii) Assumptions 3.1, 3.2 (i), (ii) and (iii), 3.3 and 3.4 hold, and

$$b_{k-1} \ge C_N \delta_k$$
, and $b_{k-1} \ge C \|\widehat{\theta}_{k-1} - \theta^*\|_2 \sqrt{\log \frac{N}{Ks \log d}}$, (A.5)

where $C_N > 0$ is defined in (A.3) and there exists a large constant ζ such that $(\frac{Ks \log d}{N})^{\zeta} \leq \delta_k$ with $\delta_k = o(1)$.

If either (i) or (ii) holds, we have for any $\boldsymbol{v} \in \mathbb{R}^d$ with $\|\boldsymbol{v}\|_0 \leq s'$,

$$\left| \boldsymbol{v}^T \left(\nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*) - \nabla R(\boldsymbol{\theta}^*) \right) \right| \leq C_2 c_{n,k} \delta_k^\beta \| \boldsymbol{v} \|_2,$$

where s' is defined in Assumption 3.2 and C_2 is a constant independent of n, d and k.

Proof. By definition we have

$$R(\boldsymbol{\theta}) = \mathbb{E}\left(\mathbb{1}(X < \boldsymbol{\theta}^T \boldsymbol{Z}) \mid Y = 1\right) + \mathbb{E}\left(\mathbb{1}(X > \boldsymbol{\theta}^T \boldsymbol{Z}) \mid Y = -1\right),$$

and

$$c_{n,k} \nabla R(\boldsymbol{\theta}^*) = c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{z} f\left(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1\right) f(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z} -c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{z} f\left(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = -1\right) f(\boldsymbol{z} \mid Y = -1) d\boldsymbol{z}$$

Note that

$$\begin{aligned} \nabla R_{\delta_{k},\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^{*}) = & \mathbb{E}\left(\gamma(Y)\frac{Y\boldsymbol{Z}}{\delta_{k}}K(\frac{Y(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z})}{\delta_{k}})R\mid\widehat{\boldsymbol{\theta}}_{k-1}\right) \\ = & \mathbb{E}\left(\frac{\boldsymbol{Z}}{\delta_{k}}K(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}})R\mid Y=1,\widehat{\boldsymbol{\theta}}_{k-1}\right) - \mathbb{E}\left(\frac{\boldsymbol{Z}}{\delta_{k}}K(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}})R\mid Y=-1,\widehat{\boldsymbol{\theta}}_{k-1}\right),\end{aligned}$$
hence

$$\begin{aligned} \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*) &- \nabla R(\boldsymbol{\theta}^*) \\ = \mathbb{E}\left(\frac{\boldsymbol{Z}}{\delta_k}K(\frac{\boldsymbol{X} - \boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k})R \mid \boldsymbol{Y} = 1, \widehat{\boldsymbol{\theta}}_{k-1}\right) - c_{n,k}\int_{\boldsymbol{z}} \boldsymbol{z}f\left(\boldsymbol{\theta}^{*T}\boldsymbol{z} \mid \boldsymbol{z}, \boldsymbol{Y} = 1\right)f(\boldsymbol{z} \mid \boldsymbol{Y} = 1)d\boldsymbol{z} \\ - \mathbb{E}\left(\frac{\boldsymbol{Z}}{\delta_k}K(\frac{\boldsymbol{X} - \boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_k})R \mid \boldsymbol{Y} = -1, \widehat{\boldsymbol{\theta}}_{k-1}\right) + c_{n,k}\int_{\boldsymbol{z}} \boldsymbol{z}f\left(\boldsymbol{\theta}^{*T}\boldsymbol{z} \mid \boldsymbol{z}, \boldsymbol{Y} = -1\right)f(\boldsymbol{z} \mid \boldsymbol{Y} = -1)d\boldsymbol{z}. \end{aligned}$$

We will bound the first term and the second term follows similarly. Denote

$$S_{u} = \{ u : \frac{-b_{k-1} \|\widehat{\boldsymbol{w}}_{k-1}\|_{2} + (\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*})^{T} \boldsymbol{z}}{\delta_{k}} \le u \le \frac{b_{k-1} \|\widehat{\boldsymbol{w}}_{k-1}\|_{2} + (\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*})^{T} \boldsymbol{z}}{\delta_{k}} \}.$$

We have

$$\boldsymbol{v}^{T} \mathbb{E} \left(\frac{\boldsymbol{Z}}{\delta_{k}} K(\frac{\boldsymbol{X} - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_{k}}) R \mid \boldsymbol{Y} = 1, \boldsymbol{\widehat{\theta}}_{k-1} \right)$$

$$= c_{n,k} \int_{\boldsymbol{z}} \int_{-b_{k-1} \parallel \boldsymbol{\widehat{w}}_{k-1} \parallel_{\boldsymbol{z}} + \boldsymbol{\widehat{\theta}}_{k-1}^{T} \boldsymbol{z}} \frac{\boldsymbol{v}^{T} \boldsymbol{z}}{\delta_{k}} K\left(\frac{\boldsymbol{X} - \boldsymbol{\theta}^{*T} \boldsymbol{z}}{\delta_{k}}\right) f(\boldsymbol{x} \mid \boldsymbol{z}, \boldsymbol{Y} = 1) d\boldsymbol{x} f(\boldsymbol{z} \mid \boldsymbol{Y} = 1) d\boldsymbol{z}$$

$$= c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}} K(\boldsymbol{u}) f(\boldsymbol{u} \delta_{k} + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, \boldsymbol{Y} = 1) d\boldsymbol{u} f(\boldsymbol{z} \mid \boldsymbol{Y} = 1) d\boldsymbol{z}$$

$$= c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int K(\boldsymbol{u}) f(\boldsymbol{u} \delta_{k} + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, \boldsymbol{Y} = 1) d\boldsymbol{u} f(\boldsymbol{z} \mid \boldsymbol{Y} = 1) d\boldsymbol{z}$$

$$- c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}^{c}} K(\boldsymbol{u}) f(\boldsymbol{u} \delta_{k} + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, \boldsymbol{Y} = 1) d\boldsymbol{u} f(\boldsymbol{z} \mid \boldsymbol{Y} = 1) d\boldsymbol{z}.$$

Therefore,

$$\boldsymbol{v}^{T}\left(\mathbb{E}\left(\frac{\boldsymbol{Z}}{\delta_{k}}K(\frac{\boldsymbol{X}-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}})\boldsymbol{R}\mid\boldsymbol{Y}=1,\widehat{\boldsymbol{\theta}}_{k-1}\right)-c_{n,k}\int_{\boldsymbol{z}}\boldsymbol{z}f\left(\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},\boldsymbol{Y}=1\right)f(\boldsymbol{z},\boldsymbol{Y}=1)d\boldsymbol{z}\right)$$
$$=c_{n,k}\int_{\boldsymbol{z}}\boldsymbol{v}^{T}\boldsymbol{z}\underbrace{\int_{\boldsymbol{K}}K(\boldsymbol{u})\left(f(\boldsymbol{u}\delta_{k}+\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},\boldsymbol{Y}=1)-f(\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},\boldsymbol{Y}=1)\right)d\boldsymbol{u}}_{(A)}f(\boldsymbol{z}\mid\boldsymbol{Y}=1)d\boldsymbol{z}$$
$$(A.6)$$

Now we look at the term (A). Since $f(x \mid z, y)$ is l times differentiable, by Taylor expansion we have

$$f\left(u\delta_{k} + \boldsymbol{\theta}^{*T}\boldsymbol{z} \mid \boldsymbol{z}, Y = 1\right) - f\left(\boldsymbol{\theta}^{*T}\boldsymbol{z} \mid \boldsymbol{z}, Y = 1\right)$$
$$= \sum_{i=1}^{l-1} \frac{f^{(i)}\left(\boldsymbol{\theta}^{*T}\boldsymbol{z} \mid \boldsymbol{z}, Y = 1\right)}{i!} (u\delta_{k})^{i} + \frac{(u\delta_{k})^{l}}{l!} f^{(l)}\left(\boldsymbol{\theta}^{*T}\boldsymbol{z} + \tau u\delta_{k} \mid \boldsymbol{z}, Y = 1\right)$$

for some $\tau \in [0,1]$ where $l = \lfloor \beta \rfloor$. By the definition of kernel of order l, we obtain

$$(A) = \int K(u) \frac{(u\delta_k)^l}{l!} \left(f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} + \tau u\delta_k \mid \boldsymbol{z}, Y = 1 \right) - f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1 \right) \right) du,$$

hence for any \boldsymbol{v} with $\|\boldsymbol{v}\|_0 \leq s'$,

$$\begin{aligned} \left| c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int K(u) \left(f(u\delta_{k} + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1) - f(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1) \right) duf(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z} \right| \\ &= \left| c_{n,k} \int K(u) \frac{(u\delta_{k})^{l}}{l!} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \left(f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} + \tau u\delta_{k} \mid \boldsymbol{z}, Y = 1 \right) - f^{(l)} \left(\boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1 \right) \right) f(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z} du \\ &\leq c_{n,k} \int |K(u)| \frac{|u\delta_{k}|^{l}}{l!} L \|\boldsymbol{v}\|_{2} |u\delta_{k}|^{\beta-l} du \\ &\leq c_{n,k} L \|\boldsymbol{v}\|_{2} \int |K(u)| \frac{|u\delta_{k}|^{\beta}}{l!} du. \end{aligned}$$

For the second term on the right hand side of (A.6), we first show the result under (A.4), i.e., $b_{k-1} \ge C_N \delta_k$ and $b_{k-1} \ge 2 \|\widehat{\theta}_{k-1} - \theta^*\|_1 M_n$. Then we have

$$\frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 + (\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{z}}{\delta_k} \geq \frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 - \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n}{\delta_k}$$

and

$$\frac{-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 + (\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{z}}{\delta_k} \leq \frac{-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 + \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n}{\delta_k}.$$

Therefore, by (A.3) we have

$$\begin{split} \int_{S_{u}^{c}} |K(u)| du &= \int_{\frac{b_{k-1} \|\widehat{w}_{k-1}\|_{2} + (\widehat{\theta}_{k-1} - \theta^{*})^{T} \mathbf{z}}{\delta_{k}}}^{\infty} |K(u)| du + \int_{-\infty}^{\frac{-b_{k-1} \|\widehat{w}_{k-1}\|_{2} + \|\widehat{\theta}_{k-1} - \theta^{*}\|_{1} M_{n}}{\delta_{k}}} |K(u)| du \\ &\leq \int_{\frac{b_{k-1} \|\widehat{w}_{k-1}\|_{2} - \|\widehat{\theta}_{k-1} - \theta^{*}\|_{1} M_{n}}{\delta_{k}}}^{\infty} |K(u)| du + \int_{-\infty}^{\frac{-b_{k-1} \|\widehat{w}_{k-1}\|_{2} + \|\widehat{\theta}_{k-1} - \theta^{*}\|_{1} M_{n}}{\delta_{k}}} |K(u)| du \\ &= 2\int_{\frac{b_{k-1} \|\widehat{w}_{k-1}\|_{2} - \|\widehat{\theta}_{k-1} - \theta^{*}\|_{1} M_{n}}{\delta_{k}}} |K(u)| du \\ &\leq 2\int_{C_{N}/2}^{\infty} |K(u)| du = O(\delta_{k}^{\beta}), \end{split}$$

where the last inequality follows that $\|\widehat{\boldsymbol{\omega}}_{k-1}\|_2 = \sqrt{1 + \|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2} \ge 1$, hence

$$\frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 - \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n}{\delta_k} \ge \frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 - b_{k-1}/2}{\delta_k} \ge \frac{b_{k-1}}{2\delta_k} \ge C_N/2.$$

Finally, since $\sup_{x \in \mathbb{R}, y \in \{-1,1\}, z \in \mathbb{R}^d} f(x \mid y, z) < p_{\max}$, and $\sup_{\|\boldsymbol{v}\|_0 \leq s'} \frac{\boldsymbol{v}^T \mathbb{E} (\boldsymbol{Z} \boldsymbol{Z}^T \mid \boldsymbol{Y} = \boldsymbol{y}) \boldsymbol{v}}{\|\boldsymbol{v}\|_2^2} \leq L^2$, we have

$$\begin{aligned} |c_{n,k} \int_{\boldsymbol{z}} \boldsymbol{v}^T \boldsymbol{z} \int_{S_u^c} K(u) f(u\delta_k + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1) du f(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z}| \\ \lesssim c_{n,k} p_{\max} |\mathbb{E}(|\boldsymbol{v}^T \boldsymbol{z}| \mid Y = 1)| \delta_k^\beta \\ \lesssim c_{n,k} p_{\max} \sqrt{\mathbb{E}((\boldsymbol{v}^T \boldsymbol{z})^2 \mid Y = 1)} \delta_k^\beta \\ \lesssim c_{n,k} p_{\max} \|\boldsymbol{v}\|_2 L \delta_k^\beta. \end{aligned}$$

Combining the bound for the two terms in (A.6) we finish the proof.

Now we show the result also holds under (A.5), i.e., $b_{k-1} \ge C_N \delta_k$, and $b_{k-1} \ge C \| \widehat{\theta}_{k-1} - \theta^* \|_2 \sqrt{\log \frac{N}{Ks \log d}}$. For the second term on the right hand side of (A.6), we only need to check that

$$\int_{\boldsymbol{z}} \boldsymbol{v}^T \boldsymbol{z} \int_{S_u^c} K(u) f(u\delta_k + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1) du f(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z} = O(\delta_k^\beta).$$

Since $\sup_{x \in \mathbb{R}, y \in \{-1,1\}, z \in \mathbb{R}^d} f(x \mid y, z) < p_{\max}$, we have

$$\int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}^{c}} K(u) f(u\delta_{k} + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y = 1) duf(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z}$$

$$\leq p_{\max} \int_{\boldsymbol{z}} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}^{c}} K(u) duf(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z}$$

$$\lesssim \underbrace{\int_{|(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*})^{T} \boldsymbol{z}| \leq C ||\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*}||_{2} \sqrt{\log(\frac{N}{Ks \log d})}}_{A} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}^{c}} K(u) duf(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z}$$

$$+ \underbrace{\int_{|(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*})^{T} \boldsymbol{z}| > C ||\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*}||_{2} \sqrt{\log(\frac{N}{Ks \log d})}}_{B} \boldsymbol{v}^{T} \boldsymbol{z} \int_{S_{u}^{c}} K(u) duf(\boldsymbol{z} \mid Y = 1) d\boldsymbol{z}.$$

For the term A, we have

$$\begin{split} \int_{S_{u}^{c}} |K(u)| du &= \int_{\frac{b_{k-1} \|\hat{w}_{k-1}\|_{2} + (\hat{\theta}_{k-1} - \theta^{*})^{T} \mathbf{z}}{\delta_{k}}}^{\infty} |K(u)| du + \int_{-\infty}^{\frac{-b_{k-1} \|\hat{w}_{k-1}\|_{2} + (\hat{\theta}_{k-1} - \theta^{*})^{T} \mathbf{z}}{\delta_{k}}} |K(u)| du \\ &\leq \int_{\frac{b_{k-1} \|\hat{w}_{k-1}\|_{2} - C\|\hat{\theta}_{k-1} - \theta^{*}\|_{2} \sqrt{\log(\frac{N}{Ks \log d})/2}}{\delta_{k}}} |K(u)| du + \int_{-\infty}^{\frac{-b_{k-1} \|\hat{w}_{k-1}\|_{2} + C\|\hat{\theta}_{k-1} - \theta^{*}\|_{2} \sqrt{\log(\frac{N}{Ks \log d})/2}}{\delta_{k}}} |K(u)| du \\ &= 2 \int_{\frac{b_{k-1} \|\hat{w}_{k-1}\|_{2} - C\|\hat{\theta}_{k-1} - \theta^{*}\|_{2} \sqrt{\log(\frac{N}{Ks \log d})/2}}{\delta_{k}}} |K(u)| du \\ &\leq 2 \int_{C_{N}/2}^{\infty} |K(u)| du = O(\delta_{k}^{\beta}), \end{split}$$

where the last inequality follows that $\|\widehat{\boldsymbol{\omega}}_{k-1}\|_2 = \sqrt{1 + \|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2} \ge 1$, (A.3) and

$$\frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 - C\|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_2 \sqrt{\log(\frac{N}{Ks\log d})/2}}{\delta_k} \ge \frac{b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_2 - b_{k-1}/2}{\delta_k} \ge \frac{b_{k-1}}{2\delta_k} \ge C_N/2.$$

For the term B, since $(\hat{\theta}_{k-1} - \theta^*)^T \mathbf{Z} \mid Y = 1$ is sub-Gaussian with sub-Gaussian norm that scales with $\|\hat{\theta}_{k-1} - \theta^*\|_2$, we have

$$\mathbb{P}\left(|(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{Z}| > C \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_2 \sqrt{\log(\frac{N}{Ks\log d})} \mid Y = 1\right) \le 2\left(\frac{Ks\log d}{N}\right)^{C'},$$

hence we can choose C sufficiently large such that $C'/2 \ge \zeta\beta$ and thus $\left(\frac{Ks\log d}{N}\right)^{C'/2} \lesssim \delta_k^{\beta}$. Therefore,

$$B \lesssim \mathbb{E} \left[\boldsymbol{v}^T \boldsymbol{Z} \, \mathbb{1} \{ |(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{Z}| > C \| \widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^* \|_2 \sqrt{\log(\frac{N}{Ks \log d})} \} \mid Y = 1 \right]$$

$$\leq \sqrt{\mathbb{E} ((\boldsymbol{v}^T \boldsymbol{Z})^2 \mid Y = 1)} \sqrt{\mathbb{P} \left(|(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{Z}| > C \| \widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^* \|_2 \sqrt{\log(\frac{N}{Ks \log d})} \mid Y = 1 \right)}$$

$$\lesssim \| \boldsymbol{v} \|_2 \left(\frac{Ks \log d}{N} \right)^{C'/2} \lesssim \| \boldsymbol{v} \|_2 \delta_k^{\beta}.$$

Combining the bound for the two terms in (A.6) we finish the proof.

A.3 Proof of the Main Results

Proof of Theorem 1. Note that our estimator is defined as $\hat{\theta}_k := \tilde{\theta}_{k,tgt}$, where $\tilde{\theta}_{k,tgt}$ represents the approximate local solution from the path-following algorithm. Therefore, by Theorem 9, we have that with probability greater than $1 - 2d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{Ks\log d}{nc_{n,k}}\right)^{\beta/(2\beta+1)},\tag{A.7}$$

and

$$\|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{Ks \log d}{nc_{n,k}}\right)^{\beta/(2\beta+1)}.$$
(A.8)

When k = 1, we have

$$\mathbb{P}(R_i = 1) = c_{n,1}, \ \mathbb{P}(R_i = 0) = 1 - c_{n,1}$$

hence $N_1 = nc_{n,1}/K$. Plugging this back to (A.7) and (A.8) we get the result for $\hat{\theta}_1$. For $2 \le k \le K$, recall that

$$\mathbb{P}(R_i = 1 \mid X_i, \boldsymbol{Z}_i, \widehat{\boldsymbol{\theta}}_{k-1}) = c_{n,k} \cdot \mathbb{1}\{(X_i, \boldsymbol{Z}_i) \in S_k\}.$$

We have

$$\mathbb{E}(R_i) = \mathbb{E}\left[\mathbb{E}[R_i \mid X_i, \mathbf{Z}_i, \widehat{\boldsymbol{\theta}}_{k-1}]\right]$$
$$= \mathbb{E}\left[c_{n,k} \cdot \mathbb{1}\left\{(X_i, \mathbf{Z}_i) \in S_k\right\}\right]$$
$$= c_{n,k} \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right),$$

hence $N_k = n\mathbb{E}(R_i)/K = nc_{n,k}\mathbb{P}((X, \mathbb{Z}) \in S_k)/K$. Plugging this back to (A.7) and (A.8) we finish the proof.

Proof of Theorem 2. First, let's consider the case when k = 1. For each $(X_i, Z_i) \in D_1$, we have

$$\mathbb{P}(R_i = 1) = c_{n,1}, \ \mathbb{P}(R_i = 0) = 1 - c_{n,1},$$

and $N/K = N_1 = nc_{n,1}/K$. According to Theorem 1, by selecting

$$\delta_1 = c_1 \left(\frac{Ks \log d}{nc_{n,1}}\right)^{1/(2\beta+1)} = c_1 \left(\frac{Ks \log d}{N}\right)^{1/(2\beta+1)}$$

and $\lambda_1 = c_2 \sqrt{\frac{c_{n,1} K \log d}{n \delta_1}} = c_2 \sqrt{\frac{N K \log d}{n^2 \delta_1}}$, with probability greater than $1 - 2d^{-1}$, we obtain

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{Ks\log d}{N}\right)^{\beta/(2\beta+1)}, \quad \|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{Ks\log d}{N}\right)^{\beta/(2\beta+1)}.$$
(A.9)

Next, we will establish the bound for $\|\widehat{\theta}_k - \theta^*\|_2$. The result for $\|\widehat{\theta}_k - \theta^*\|_1$ follows similarly. According to Assumption 3.3, we have $\sup_{x \in \mathbb{R}, z \in \mathbb{R}^d} f(x \mid z) < p_{\max} < \infty$. Recall that

$$S_k := \left\{ (X, \mathbf{Z}) : -b_{k-1} \leq \frac{X - \widehat{\boldsymbol{\theta}}_{k-1}^T \mathbf{Z}}{\sqrt{1 + \|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2}} \leq b_{k-1} \right\}.$$

Let us start from k = 2. Since $\mathbf{Z}|Y = y$ is sub-Gaussian with a bounded sub-Gaussian norm and independent of $\hat{\theta}_{k-1}$, we have $(\hat{\theta}_{k-1} - \theta^*)^T \mathbf{Z}|Y = y$ is also sub-Gaussian, with a sub-Gaussian norm that scales with $\|\hat{\theta}_{k-1} - \theta^*\|_2$, hence

$$\mathbb{P}\left(|(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*)^T \boldsymbol{Z}| > c \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)}\right) \le 2\left(\frac{Ks\log d}{N}\right)^{c'},$$

where c' is a sufficiently large constant. Consider the following set

$$\mathcal{E} = \Big\{ (\boldsymbol{\theta}, \boldsymbol{Z}) : \|\boldsymbol{\theta}\|_2 \le 2C, |(\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T \boldsymbol{Z}| \le c \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)} \Big\},\$$

where *C* is the constant defined in Assumption 3.1. Note that $\|\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_{k-1}\|_2 \leq C$ with probability greater than $1 - 2d^{-1}$. Since $\|\hat{\boldsymbol{\theta}}_{k-1}\|_2 \leq \|\boldsymbol{\theta}^* - \hat{\boldsymbol{\theta}}_{k-1}\|_2 + \|\boldsymbol{\theta}^*\|_2$, the event $(\hat{\boldsymbol{\theta}}_{k-1}, \boldsymbol{Z}) \in \mathcal{E}$ holds with probability greater than $1 - 2\left(\frac{Ks\log d}{N}\right)^{c'} - 2d^{-1}$. So we have

$$\mathbb{P}\left((X, \mathbf{Z}) \in S_k \mid \widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z} = \mathbf{z}\right) = \int_{-b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2} + \widehat{\boldsymbol{\theta}}_{k-1}^T \mathbf{z}}^{b_{k-1}} f(x \mid \mathbf{z}) dx$$
$$\leq 2b_{k-1}p_{\max}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2}.$$

As a result, we have

$$\mathbb{P}\left((X, \mathbf{Z}) \in S_{k}\right) = \mathbb{E}\left(\mathbb{P}\left((X, \mathbf{Z}) \in S_{k} \mid \widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \mathbb{1}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \in \mathcal{E}\right\}\right) \\
+ \mathbb{E}\left(\mathbb{P}\left((X, \mathbf{Z}) \in S_{k} \mid \widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \mathbb{1}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \notin \mathcal{E}\right\}\right) \\
\leq 2b_{k-1}p_{\max}\sqrt{1 + 4C^{2}} + \mathbb{P}\left(\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \notin \mathcal{E}\right) \\
\leq 2b_{k-1}p_{\max}\sqrt{1 + 4C^{2}} + 2\left(\frac{Ks\log d}{N}\right)^{c'} + 2d^{-1} \\
\lesssim b_{k-1},$$
(A.10)

where $b_{k-1} = c_3 (\frac{Ks \log d}{N})^{1/(2\beta)}$ and the last step follows from the fact that $d \gg N$ and c' is sufficiently large. In addition, it can be shown that

$$\begin{aligned} -b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \widehat{\boldsymbol{\theta}}_{k-1}^{T}\boldsymbol{z} = \boldsymbol{\theta}^{*T}\boldsymbol{z} - b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \left(\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*}\right)^{T}\boldsymbol{z} \\ \geq \boldsymbol{\theta}^{*T}\boldsymbol{z} - C_{1}b_{k-1}, \end{aligned}$$

if the events $(\widehat{\theta}_{k-1}, \mathbb{Z}) \in \mathcal{E}$ and

$$c \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)} \le b_{k-1}/2,$$

hold. By (A.9) and the choice of b_{k-1} , the two events hold with probability greater than $1 - 2\left(\frac{Ks\log d}{N}\right)^{c'} - 2d^{-1}$. By a similar proof, we can show that

$$b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_2^2}+\widehat{\boldsymbol{\theta}}_{k-1}^T\boldsymbol{z}\leq \boldsymbol{\theta}^{*T}\boldsymbol{z}+C_2b_{k-1}.$$

Therefore, with probability greater than $1 - 2\left(\frac{Ks\log d}{N}\right)^{c'} - 2d^{-1}$, the event

$$\mathcal{A}_{k} = \Big\{ \Big[-b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \widehat{\boldsymbol{\theta}}_{k-1}^{T}\boldsymbol{z}, b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \widehat{\boldsymbol{\theta}}_{k-1}^{T}\boldsymbol{z} \Big] \subset B(\boldsymbol{\theta}^{*T}\boldsymbol{z}, \epsilon) \Big\},$$

holds, where $\epsilon = C_3 b_{k-1}$ for some constant C_3 large enough. Following the similar derivations in (A.10), by Assumption 3.3, we can show that

$$\mathbb{P}\left((X, \mathbf{Z}) \in S_{k}\right) \geq \mathbb{E}\left(\mathbb{P}\left((X, \mathbf{Z}) \in S_{k} \mid \widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \mathbb{1}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \in \mathcal{E}, \mathcal{A}_{k}\right\}\right) \\
= \mathbb{E}\left\{\mathbb{1}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \in \mathcal{E}, \mathcal{A}_{k}\right\} \int_{-b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \widehat{\boldsymbol{\theta}}_{k-1}^{T} \mathbf{z}} f(x \mid \mathbf{z}) dx\right\} \\
\geq \mathbb{E}\left\{\mathbb{1}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \in \mathcal{E}, \mathcal{A}_{k}, \mathbf{Z} \in \mathcal{G}\right\} \int_{-b_{k-1}\sqrt{1+\|\widehat{\boldsymbol{\theta}}_{k-1}\|_{2}^{2}} + \widehat{\boldsymbol{\theta}}_{k-1}^{T} \mathbf{z}} f(x \mid \mathbf{z}) dx\right\} \\
\geq 2b_{k-1}p_{\min}\mathbb{P}\left\{\left(\widehat{\boldsymbol{\theta}}_{k-1}, \mathbf{Z}\right) \in \mathcal{E}, \mathcal{A}_{k}, \mathbf{Z} \in \mathcal{G}\right\} \geq Cb_{k-1}, \quad (A.11)$$

for some constant C. Combining (A.10) with (A.11), we have $\mathbb{P}((X, \mathbb{Z}) \in S_k) \simeq b_{k-1}$.

To apply Theorem 1, we need to verify that b_{k-1} satisfies

$$b_{k-1} \ge C\delta_k$$
 and $b_{k-1} \ge C \|\widehat{\theta}_{k-1} - \theta^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)}$. (A.12)

Given $b_{k-1} = c_3 (\frac{Ks \log d}{N})^{1/(2\beta)}$ and from Theorem 1

$$\delta_k \asymp \left(\frac{Ks\log d}{nc_{n,k}}\right)^{1/(2\beta+1)} \asymp \left(\frac{s\log d \ \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right)}{N_k}\right)^{1/(2\beta+1)} \asymp \left(\frac{b_{k-1}Ks\log d}{N}\right)^{1/(2\beta+1)},$$
(A.13)

where it follows from $N_k = n\mathbb{E}(R_i)/K = nc_{n,k}\mathbb{P}\left((X, \mathbf{Z}) \in S_k\right)/K$, we can verify that $b_{k-1} \ge C\delta_k$ holds. In addition, by (A.9), $Ks \log d = o(N)$ and for any fixed $\beta > \frac{1+\sqrt{3}}{2}$ (which implies $\frac{\beta}{2\beta+1} > \frac{1}{2\beta}$), we conclude that $b_{k-1} \ge C \|\widehat{\theta}_{k-1} - \theta^*\|_2 \sqrt{\log(\frac{N}{Ks\log d})}$ holds with probability greater than $1 - 2d^{-1}$. Thus, applying Theorem 1 with δ_k in (A.13) and $\lambda_k = c_2 \sqrt{\frac{NK\log d}{n^2 b_{k-1} \delta_k}}$, we obtain that with probability greater than $1 - 4d^{-1}$

$$\|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\|_{2} \lesssim \left(\frac{\mathbb{P}\left((X, \boldsymbol{Z}) \in S_{k}\right) s \log d}{N_{k}}\right)^{\beta/(2\beta+1)} \lesssim \left(\frac{b_{k-1}Ks \log d}{N}\right)^{\beta/(2\beta+1)} \lesssim \left(\frac{Ks \log d}{N}\right)^{1/2},$$
(A.14)

where we plug in $b_{k-1} = c_3 (\frac{Ks \log d}{N})^{1/(2\beta)}$ in the last step. This completes the proof for k = 2.

By mathematical induction, assuming (A.14) holds for $\widehat{\theta}_k$ with probability greater than $1 - 2kd^{-1}$, we would like to prove (A.14) holds for $\widehat{\theta}_{k+1}$ with probability greater than $1 - 2(k+1)d^{-1}$. Following the similar arguments, we can prove that $\mathbb{P}((X, \mathbb{Z}) \in S_{k+1}) \simeq b_k$. Note that $\delta_{k+1} \simeq (\frac{b_k Ks \log d}{N})^{1/(2\beta+1)}$ and $b_k = c_3(\frac{Ks \log d}{N})^{1/(2\beta)}$. As a result, $b_k \ge C\delta_{k+1}$, and by (A.14) it holds that $b_k \ge C \|\widehat{\theta}_k - \theta^*\|_2 \sqrt{\log(\frac{N}{Ks \log d})}$ with probability greater than $1 - 2kd^{-1}$. Finally, as shown in (A.14), we obtain that with probability greater than $1 - 2(k+1)d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_{k+1} - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{b_k K s \log d}{N}\right)^{\beta/(2\beta+1)} \lesssim \left(\frac{K s \log d}{N}\right)^{1/2}.$$

This completes the proof for k + 1. It is easily seen that, by the union bound argument, the event $\bigcap_{2 \le k \le K} \{ \| \widehat{\theta}_k - \theta^* \|_2 \lesssim (\frac{Ks \log d}{N})^{1/2} \}$ holds with probability greater than $1 - 2Kd^{-1}$.

Finally let's consider the assumption (3.11). By definition, we have $N_k = \sum_{(X_i, \mathbf{Z}_i) \in D_k} \mathbb{E}(R_i) = n\mathbb{E}(R_i)/K = nc_{n,k}\mathbb{P}((X, \mathbf{Z}) \in S_k)/K$. To ensure that $0 < c_{n,k} \leq 1$, we require

$$N_k K \leq n \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right), \ 2 \leq k \leq K.$$

Note that $N_k K = N$, with (A.11), it suffices to ensure that $N \leq C b_{k-1} n$ for some constant C and for all $2 \leq k \leq K$. Some calculation yields (3.11).

Proof of Theorem 3. By Theorem 1, choosing $\delta_1 = c_1 \left(\frac{Ks \log d}{N}\right)^{1/(2\beta+1)}$ and $\lambda_1 = c_2 \sqrt{\frac{NK \log d}{n^2 \delta_1}}$, yields that with probability greater than $1 - 2d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{Ks\log d}{N}\right)^{\beta/(2\beta+1)}, \ \|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{Ks\log d}{N}\right)^{\beta/(2\beta+1)}.$$
(A.15)

Similar as in Theorem 2, we find $\mathbb{P}((X, \mathbb{Z}) \in S_2) \simeq b_1$, leading to

$$\delta_2 \asymp \left(\frac{Ks\log d}{nc_{n,2}}\right)^{1/(2\beta+1)} \asymp \left(\frac{s\log d \mathbb{P}\left((X, \mathbf{Z}) \in S_2\right)}{N_2}\right)^{1/(2\beta+1)} \asymp \left(\frac{b_1 Ks\log d}{N}\right)^{1/(2\beta+1)}.$$
 (A.16)

To invoke Theorem 1 for $\hat{\theta}_2$, we need to verify

$$b_1 \ge C\delta_2$$
 and $b_1 \ge C \|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)}$

For simplicity, we denote $\Delta = \frac{Ks \log d}{N}$ and $\alpha = \frac{\beta}{2\beta+1}$. By (A.16) and (A.15), it suffices to verify

$$b_1 \ge C' \Delta^{1/(2\beta)}$$
 and $b_1 \ge C \Delta^{\alpha} \sqrt{\log\left(\frac{1}{\Delta}\right)}$ (A.17)

for some constant C, C'. For any fixed $1 < \beta \leq \frac{1+\sqrt{3}}{2}$, we have $\alpha \leq \frac{1}{2\beta}$. To satisfy (A.17), we choose $b_1 = C\Delta^{\alpha}\sqrt{\log(\frac{1}{\Delta})}$ for some constant C. With $\lambda_2 = c_2\sqrt{\frac{NK\log d}{n^2b_1\delta_2}}$, Theorem 1 implies, with probability greater than $1 - 4d^{-1}$,

$$\begin{aligned} |\widehat{\boldsymbol{\theta}}_{2} - \boldsymbol{\theta}^{*}||_{2} &\lesssim \left(\frac{\mathbb{P}\left((X, \boldsymbol{Z}) \in S_{2}\right) K s \log d}{N}\right)^{\beta/(2\beta+1)} \\ &\lesssim \left(\frac{b_{1} K s \log d}{N}\right)^{\beta/(2\beta+1)} \lesssim \left(\log(\frac{1}{\Delta})\right)^{\frac{\alpha}{2}} \Delta^{\alpha^{2}+\alpha}. \end{aligned}$$
(A.18)

In the following, we will show that for any $2 \le k \le \left\lceil \log_{\frac{\beta}{2\beta+1}} \left(1 - \frac{\beta+1}{2\beta^2}\right) \right\rceil$

$$\|\widehat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}^*\|_2 \lesssim \left(\log(\frac{1}{\Delta})\right)^{\frac{\alpha - \alpha^k}{2(1-\alpha)}} \Delta^{(1-\alpha^k)\frac{\alpha}{1-\alpha}} := r_k, \tag{A.19}$$

holds with probability greater than $1 - 2kd^{-1}$. Note that $\log_{\frac{\beta}{2\beta+1}}(1 - \frac{\beta+1}{2\beta^2})$ is well defined for $1 < \beta \leq \frac{1+\sqrt{3}}{2}$. Clearly, (A.19) holds for k = 2. Assuming (A.19) holds for k - 1, it suffices to show (A.19) holds for k. Following the same argument above for k = 2, b_{k-1} needs to satisfy

$$b_{k-1} \ge C' \Delta^{1/(2\beta)}$$
 and $b_{k-1} \ge Cr_{k-1} \sqrt{\log\left(\frac{1}{\Delta}\right)}$.

where r_{k-1} is given by (A.19). We note that for any $k \leq \lceil \log_{\frac{\beta}{2\beta+1}} \left(1 - \frac{\beta+1}{2\beta^2}\right) \rceil$ and for any fixed $1 < \beta \leq \frac{1+\sqrt{3}}{2},$ $(1 - \alpha^{k-1})\frac{\alpha}{1-\alpha} = \left(1 - \left(\frac{\beta}{2\beta+1}\right)^{k-1}\right)\frac{\beta}{\beta+1} < \frac{1}{2\beta},$ which implies

$$\Delta^{1/(2\beta)} = O\left(r_{k-1}\sqrt{\log(\frac{1}{\Delta})}\right),\tag{A.20}$$

and therefore we can choose

$$\begin{aligned} b_{k-1} &\asymp r_{k-1} \sqrt{\log(\frac{1}{\Delta})} \\ &\asymp \left(\log(\frac{1}{\Delta})\right)^{\frac{\alpha - \alpha^{k-1}}{2(1-\alpha)} + \frac{1}{2}} \Delta^{(1-\alpha^{k-1})\frac{\alpha}{1-\alpha}} \\ &\asymp \left(\log(\frac{N}{Ks \log d})\right)^{\frac{(2\beta+1)(1-(\frac{\beta}{2\beta+1})^{k-1})}{2(\beta+1)}} \left(\frac{Ks \log d}{N}\right)^{\frac{\beta}{\beta+1}(1-(\frac{\beta}{2\beta+1})^{k-1})} \end{aligned}$$

Similar to (A.18), we have with probability greater than $1 - 2kd^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\|_{2} \lesssim (b_{k-1}\Delta)^{\alpha} \lesssim r_{k-1}^{\alpha} \Big\{ \log(\frac{1}{\Delta}) \Big\}^{\alpha/2} \Delta^{\alpha} = r_{k},$$

which completes the proof of (A.19).

For $k = K = \lceil \log_{\frac{\beta}{2\beta+1}} \left(1 - \frac{\beta+1}{2\beta^2}\right) \rceil + 1$, to satisfy

$$b_{K-1} \ge C' \Delta^{1/(2\beta)}$$
 and $b_{K-1} \ge Cr_{K-1} \sqrt{\log\left(\frac{1}{\Delta}\right)}$.

we set $b_{K-1} = c\Delta^{1/(2\beta)}$. To see this, note that $\Delta = o(1)$, the bound (A.19) holds for r_{K-1} , and thus for any fixed $1 < \beta \leq \frac{1+\sqrt{3}}{2}$, some calculation shows that

$$(1 - \alpha^{K-1})\frac{\alpha}{1 - \alpha} = \left(1 - \left(\frac{\beta}{2\beta + 1}\right)^{K-1}\right)\frac{\beta}{\beta + 1} > \frac{1}{2\beta},\tag{A.21}$$

which implies

$$r_{K-1}\sqrt{\log(\frac{1}{\Delta})} = O(\Delta^{1/(2\beta)})$$

Applying Theorem 1, we select

$$\delta_K \asymp \left(\frac{Kb_{K-1}s\log d}{N}\right)^{1/(2\beta+1)} \asymp \left(\frac{Ks\log d}{N}\right)^{1/(2\beta)},$$

and $\lambda_K = c_2 \sqrt{\frac{NK \log d}{n^2 b_{K-1} \delta_K}}$, to ensure, with probability greater than $1 - 2K d^{-1}$

$$\begin{aligned} \|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{2} &\lesssim \left(\frac{\mathbb{P}\left((X, \boldsymbol{Z}) \in S_{K}\right) s \log d}{N_{K}}\right)^{\beta/(2\beta+1)} \lesssim \sqrt{s} \left(\frac{K b_{K-1} s \log d}{N}\right)^{\beta/(2\beta+1)} \\ &\lesssim \left(\frac{K s \log d}{N}\right)^{1/2}. \end{aligned}$$

The result for $\|\widehat{\theta}_k - \theta^*\|_1$ follows similarly.

Finally, to ensure that $0 < c_{n,k} \leq 1$ for $2 \leq k \leq K$, we require

$$N_k K \leq n \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right), \ 2 \leq k \leq K.$$

Note that $N_k K = N$. It suffices to ensure that $N \leq C b_{k-1} n$ for some constant C, which is implied by (3.13) for $2 \leq k \leq K - 1$ and (3.11) for k = K.

Proof of Theorem 4. Following the proof of Theorem 2 we have by choosing $\delta_1 = c_1 \left(\frac{Ks \log d}{N}\right)^{1/3}$ and $\lambda_1 = c_2 \sqrt{\frac{c_{n,1}K \log d}{n\delta_1}} = c_2 \sqrt{\frac{NK \log d}{n^2\delta_1}}$, with probability greater than $1 - 2d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{Ks\log d}{N}\right)^{1/3}, \quad \|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{Ks\log d}{N}\right)^{1/3}.$$
(A.22)

Similar as in Theorem 2, we find $\mathbb{P}((X, \mathbb{Z}) \in S_2) \approx b_1$. Applying Theorem 1 with

$$\delta_2 \asymp \left(\frac{Ks\log d}{nc_{n,2}}\right)^{1/3} \asymp \left(\frac{s\log d\mathbb{P}\left((X, \mathbf{Z}) \in S_2\right)}{N_2}\right)^{1/3} \asymp \left(\frac{b_1 Ks\log d}{N}\right)^{1/3}, \tag{A.23}$$

and $\lambda_2 = c_2 \sqrt{\frac{NK \log d}{n^2 b_1 \delta_2}}$, we have with probability greater than $1 - 4d^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{K\mathbb{P}\left((X, \boldsymbol{Z}) \in S_2\right) s \log d}{N}\right)^{1/3} \lesssim \left(\frac{Kb_1 s \log d}{N}\right)^{1/3}.$$
 (A.24)

To satisfy

$$b_1 \ge C\delta_2$$
 and $b_1 \ge C \|\widehat{\theta}_1 - \theta^*\|_2 \sqrt{\log\left(\frac{N}{Ks\log d}\right)}$

by (A.23) it suffices to verify

$$b_1 \ge C' \left(\frac{Ks\log d}{N}\right)^{1/2}$$
 and $b_1 \ge C \left(\frac{Ks\log d}{N}\right)^{1/3} \sqrt{\log\left(\frac{N}{Ks\log d}\right)}$ (A.25)

for some constants C, C'. Since $Ks \log d = o(N)$, clearly we have

$$\left(\frac{Ks\log d}{N}\right)^{1/2} = O\left(\left(\frac{Ks\log d}{N}\right)^{1/3}\sqrt{\log\left(\frac{N}{Ks\log d}\right)}\right).$$

Therefore, to satisfy (A.25), we choose $b_1 = C_1 \left(\frac{Ks \log d}{N}\right)^{1/3} \sqrt{\log\left(\frac{N}{Ks \log d}\right)}$ for some constant C_1 . Then by (A.24) and (A.22) we have

$$\|\widehat{\boldsymbol{ heta}}_2 - \boldsymbol{ heta}^*\|_2 \lesssim \left(\log(rac{1}{\Delta})
ight)^{1/6} \Delta^{4/9},$$

where $\Delta = \frac{Ks \log d}{N}$. Using a similar mathematical induction argument, we can show that with probability greater than $1 - 2kd^{-1}$,

$$\begin{aligned} \|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\|_{2} &\lesssim \left(\Delta\sqrt{\log(\frac{1}{\Delta})}\right)^{\sum_{i=1}^{k-1} 1/3^{i}} \|\widehat{\boldsymbol{\theta}}_{1} - \boldsymbol{\theta}^{*}\|_{2}^{1/3^{k-1}} \\ &\lesssim \left(\Delta\sqrt{\log(\frac{1}{\Delta})}\right)^{\frac{1}{2}(1-1/3^{k-1})} \Delta^{1/3^{k}} \\ &\lesssim \left(\log(\frac{1}{\Delta})\right)^{\frac{1}{4}(1-1/3^{k-1})} \Delta^{\frac{1}{2}(1-1/3^{k})}, \end{aligned}$$
(A.26)

for any $2 \le k \le K$. Given $K = \lceil \log_3(\log N) \rceil$, we have $\left(\frac{N}{Ks \log d}\right)^{\frac{1}{2\cdot 3^K}} \le C$ for some constant C > 0, hence (A.26) with k = K can reduce to

$$\|\widehat{\boldsymbol{\theta}}_K - \boldsymbol{\theta}^*\|_2 \lesssim \left(\log(\frac{N}{Ks\log d})\right)^{\frac{1}{4}} \left(\frac{Ks\log d}{N}\right)^{\frac{1}{2}},$$

and the result for $\|\widehat{\theta}_k - \theta^*\|_1$ follows similarly. Since $\frac{1}{2}(1 - 1/3^{k-1}) < \frac{1}{2}$, we can verify that $\Delta^{1/2} \lesssim \left(\log(\frac{1}{\Delta})\right)^{\frac{1}{2} + \frac{1}{4}(1 - 1/3^{k-2})} \Delta^{\frac{1}{2}(1 - 1/3^{k-1})}$ for all $2 \le k \le K$. Therefore, we choose

$$b_{k-1} = C_1 \left(\log(\frac{N}{Ks \log d}) \right)^{\frac{3-1/3^{k-2}}{4}} \left(\frac{Ks \log d}{N} \right)^{\frac{1-1/3^{k-1}}{2}},$$

and $\delta_k = C_2 \left(\frac{b_{k-1}Ks \log d}{N}\right)^{1/3}$ where $C_1, C_2 > 0$ are some constants. Finally, to ensure that $0 < c_{n,k} \leq 1$, we require

$$N_k K \le n \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right), \ 2 \le k \le K - 1.$$

Note that $N_k K = N$. It suffices to ensure that $N \leq C b_{k-1} n$ for some constant C, which is provided in (3.19).

Proof of Theorem 5. The proof consists of the following two steps.

- For any sampling method $Q \in \mathcal{Q}_N(\mathcal{P}(\beta, s))$, construct a set of hypotheses $\mathcal{H} = \{P_j(X, Y, \mathbb{Z})\} \subset \mathcal{P}(\beta, s)$.
- Apply Theorem 2.7 in Tsybakov (2008) by checking the following two conditions:
 - 1. KL $(f^j || f^0) \leq \gamma \log |\mathcal{H}|$ for some $\gamma \in (0, 1/8)$, where KL $(f^j || f^0)$ is the K-L divergence between probability measures f^j and f^0 , and f^j is the probability measure of the random variables $\{O_i\}_{i=1}^n$ under hypothesis j.
 - 2. For all $j \neq k$ and $q = 1, 2, \|\boldsymbol{\theta}_j \boldsymbol{\theta}_k\|_q \ge 2t$, where $t \asymp s^{\frac{1}{q} \frac{1}{2}} \left(\frac{s \log(d/s)}{N}\right)^{1/2}$.

Given the set

$$\mathcal{M} = \left\{ x \in \{0, 1\}^d : \|x\|_0 = s \right\},\$$

there exists a subset \mathcal{H}' of \mathcal{M} such that $\rho_H(x, x') > s/16$ for $x, x' \in \mathcal{H}', x \neq x'$ and $\log |\mathcal{H}'| \geq c's \log \left(\frac{d}{s}\right)$, where ρ_H denotes the Hamming distance and c' is some absolute constant. We let $\omega_0 = \mathbf{0} \in \mathbb{R}^d$ and use ω_j to denote the elements in \mathcal{H}' for $j = 1, \ldots, |\mathcal{H}'|$.

Now we start to construct $P_j(X, Y, \mathbf{Z})$. We choose weight functions such that $\gamma(1) = \gamma(-1)$. For all $j = 0, \ldots, |\mathcal{H}'|$, we assume X and each of Z_1, \cdots, Z_d follows a Uniform distribution on [-1, 1] independently. For each $j = 0, \ldots, |\mathcal{H}'|$, let

$$f_j(y=1 \mid x, \boldsymbol{z}) = \frac{1}{2} + \frac{1}{2\sigma} \left(x - c \left(\frac{s \log(d/s)}{N} \right)^{1/2} \frac{\boldsymbol{\omega}_j^T \boldsymbol{z}}{\sqrt{s}} \right),$$
(A.27)

$$f_j(y = -1 \mid x, \boldsymbol{z}) = \frac{1}{2} - \frac{1}{2\sigma} \left(x - c \left(\frac{s \log(d/s)}{N} \right)^{1/2} \frac{\boldsymbol{\omega}_j^T \boldsymbol{z}}{\sqrt{s}} \right),$$
(A.28)

where c is some sufficiently small constant and σ is some sufficiently large constant. Under the assumption that $s(\frac{\log(d/s)}{N})^{1/2} = o(1)$, we can guarantee that $f_j(y = 1 \mid x, z)$ and $f_j(y = -1 \mid x, z)$ are within [0, 1] for any $(x, z) \in [-1, 1]^{d+1}$, and thus are well defined. By this construction we can conclude that $P_0, \ldots, P_{\mathcal{H}'}$ are well defined probability measures. In the following we present two lemmas which characterize two key properties of P_j .

Lemma A.2. Under the conditions of Theorem 5 and the construction of $P_j = P_j(X, Y, Z)$ above, we have $P_j \in \mathcal{P}(\beta, L, p_{\min}, p_{\max}), \forall j = 0, ..., |\mathcal{H}'|$ and (3.2) holds.

Proof. By the construction of $P(X, Y, \mathbf{Z})$, denoting $\tilde{c} = \frac{c}{\sqrt{s}} \left(\frac{s \log(d/s)}{N}\right)^{1/2}$, we have

$$f_j(y=1,\boldsymbol{z}) = \int_{-1}^1 f_j(x,y=1,\boldsymbol{z})dx$$

$$= \int_{-1}^1 f_j(y=1 \mid x,\boldsymbol{z})f(x)f(\boldsymbol{z})dx$$

$$= \frac{1}{2^{d+1}}(1 - \frac{\widetilde{c}}{\sigma}\boldsymbol{\omega}_j^T\boldsymbol{z}),$$
 (A.29)

and $f_j(y = -1, z) = \frac{1}{2^{d+1}} (1 + \frac{\tilde{c}}{\sigma} \boldsymbol{\omega}_j^T \boldsymbol{z})$. Hence

$$f_j(x \mid y, \boldsymbol{z}) = \frac{f_j(y \mid x, \boldsymbol{z})(1/2)^{d+1}}{f(y, \boldsymbol{z})} = \frac{1 + \frac{y}{\sigma}(x - \tilde{c}\boldsymbol{\omega}_j^T \boldsymbol{z})}{2(1 - y\frac{\tilde{c}}{\sigma}\boldsymbol{\omega}_j^T \boldsymbol{z})} = \frac{1}{2} + \frac{yx}{2\sigma(1 - y\frac{\tilde{c}}{\sigma}\boldsymbol{\omega}_j^T \boldsymbol{z})}$$
(A.30)

is $l = \lfloor \beta \rfloor$ times differentiable w.r.t. x for any y, z. Now we check the condition in Definition 3.1, i.e., $f_j(x \mid y, z), j = 0, \ldots, |\mathcal{H}'|$ satisfies that

$$\left| f_{j}^{(l)} \left(x_{1} \mid y, z \right) - f_{j}^{(l)} \left(x_{2} \mid y, z \right) \right| \leq L \left| x_{1} - x_{2} \right|^{\beta - l}$$
(A.31)

for any $y \in \{-1, 1\}, z \in \mathbb{R}^d, x_1, x_2$, and L > 0 is some constant. When $\beta = 1, l = 0$, by (A.30) we have

$$\left|f_{j}^{(0)}\left(x_{1}\mid y, \boldsymbol{z}\right) - f_{j}^{(0)}\left(x_{2}\mid y, \boldsymbol{z}\right)\right| = \frac{|x_{1} - x_{2}|}{2\sigma(1 - y\frac{\widetilde{c}}{\sigma}\boldsymbol{\omega}_{j}^{T}\boldsymbol{z})} < \frac{|x_{1} - x_{2}|}{\sigma},$$

given σ sufficiently large. Then note that $|x_1 - x_2| \leq 2$, therefore choosing $L = \frac{1}{\sigma}$ we ensure that (A.31) is satisfied. For $\beta > 1$, $l \geq 1$, we have $\left| f_j^{(l)}(x_1 \mid y, z) - f_j^{(l)}(x_2 \mid y, z) \right| = 0$, hence (A.31) holds trivially. This means $P_j \in \mathcal{P}(\beta, L)$. Clearly, $f_j(x|z) = f_j(x) = 1/2 = p_{\min}$, and $f_j(x \mid y, z) \leq 1/2 + 1/\sigma = p_{\max}$. Thus, $P_j \in \mathcal{P}(\beta, L, p_{\min}, p_{\max})$ holds.

Now we check the condition (3.2), i.e.,

$$\sup_{\|\boldsymbol{v}\|_{0} \leq s'} \frac{\boldsymbol{v}^{T} \mathbb{E} \left(\boldsymbol{Z} \boldsymbol{Z}^{T} \mid \boldsymbol{Y} = \boldsymbol{y} \right) \boldsymbol{v}}{\|\boldsymbol{v}\|_{2}^{2}} \leq M_{1}.$$
(A.32)

By (A.29) we have

$$P_j(Y=1) = \int_{\boldsymbol{Z}} \frac{1}{2^{d+1}} (1 - \frac{\widetilde{c}}{\sigma} \boldsymbol{\omega}_j^T \boldsymbol{z}) d\boldsymbol{z} = \frac{1}{2},$$

hence $f_j(z \mid Y = y) = \frac{1}{2^d} (1 - y \frac{\tilde{c}}{\sigma} \boldsymbol{\omega}_j^T \boldsymbol{z})$. We have

$$\boldsymbol{v}^{T} \mathbb{E} \left(\boldsymbol{Z} \boldsymbol{Z}^{T} \mid \boldsymbol{Y} = \boldsymbol{y} \right) \boldsymbol{v} = \int (\boldsymbol{v}^{T} \boldsymbol{z})^{2} \frac{1}{2^{d}} (1 - \boldsymbol{y} \frac{\widetilde{c}}{\sigma} \boldsymbol{\omega}_{j}^{T} \boldsymbol{z}) d\boldsymbol{z}$$
$$= \boldsymbol{v}^{T} \mathbb{E} (\boldsymbol{Z} \boldsymbol{Z}^{T}) \boldsymbol{v} - \frac{\boldsymbol{y} \widetilde{c}}{\sigma} \int \frac{1}{2^{d}} (\boldsymbol{v}^{T} \boldsymbol{z})^{2} \boldsymbol{\omega}_{j}^{T} \boldsymbol{z} d\boldsymbol{z}$$

Note that $\mathbb{E}(\mathbf{Z}\mathbf{Z}^T) = 1/3\mathbb{I}_d$ and $\|\boldsymbol{\omega}_j\|_0 = s$, hence

$$\left|\int \frac{1}{2^d} (\boldsymbol{v}^T \boldsymbol{z})^2 \boldsymbol{\omega}_j^T \boldsymbol{z} d\boldsymbol{z}\right| \leq s \mathbb{E}((\boldsymbol{v}^T \boldsymbol{Z})^2) \leq \frac{s}{3} \|\boldsymbol{v}\|_2^2,$$

and

$$|\boldsymbol{v}^T \mathbb{E}\left(\boldsymbol{Z} \boldsymbol{Z}^T \mid Y = y\right) \boldsymbol{v}| \leq rac{1}{3} \|\boldsymbol{v}\|_2^2 + rac{c}{3\sigma} s \sqrt{rac{\log(d/s)}{N}} \|\boldsymbol{v}\|_2^2.$$

Since $s\sqrt{\frac{\log(d/s)}{N}} = o(1)$, (A.32) holds.

Lemma A.3. Under the conditions of Theorem 5 and the construction of $P_j = P_j(X, Y, Z)$ above, the unique minimizer $\boldsymbol{\theta}_j \in \mathbb{R}^d$ of the risk $R_{P_j}(\boldsymbol{\theta})$ is

$$\boldsymbol{\theta}_{j} = \begin{cases} 0 & \text{if } j = 0, \\ \frac{c}{\sqrt{s}} \left(\frac{s \log(d/s)}{N}\right)^{1/2} \boldsymbol{\omega}_{j} & \text{otherwise,} \end{cases}$$

where c is defined in (A.27). In addition, $\|\boldsymbol{\theta}_j\|_2 \leq C$ for some constant C > 0, and $\rho_- \leq \lambda_{\min} \left(\nabla^2 R_j \left(\boldsymbol{\theta}_j \right) \right) \leq \lambda_{\max} \left(\nabla^2 R_j \left(\boldsymbol{\theta}_j \right) \right) \leq \rho_+$ for some constants $\rho_+ \geq \rho_- > 0$.

Proof. Recall that $R_i(\boldsymbol{\theta}) = \mathbb{E}_i \left[\gamma(Y) \left(1 - \operatorname{sign} \left(Y \left(X - \boldsymbol{\theta}^T \boldsymbol{Z} \right) \right) \right) \right]$, and $\gamma(1) = \gamma(-1) = C$ for some constant C > 0, the difference between $R_j(\boldsymbol{\theta})$ and $R_j(\boldsymbol{\theta}_j)$ can be written as

$$R_{j}(\boldsymbol{\theta}) - R_{j}(\boldsymbol{\theta}_{j}) = \mathbb{E}_{j} \left[\gamma(Y) Y \left(\operatorname{sign} \left(X - \boldsymbol{\theta}_{j}^{T} \boldsymbol{Z} \right) - \operatorname{sign} \left(X - \boldsymbol{\theta}^{T} \boldsymbol{Z} \right) \right) \\ = 2 \int_{\mathcal{G}} \operatorname{sign} \left(x - \boldsymbol{\theta}_{j}^{T} \boldsymbol{z} \right) \mathbb{E}_{j} [\gamma(Y) Y \mid x, \boldsymbol{z}] dP_{j;X,\boldsymbol{Z}} \\ = 2C \int_{\mathcal{G}} \operatorname{sign} \left(x - \boldsymbol{\theta}_{j}^{T} \boldsymbol{z} \right) \mathbb{E}_{j} [Y \mid x, \boldsymbol{z}] dP_{j;X,\boldsymbol{Z}},$$

where

$$\mathcal{G} = \left\{ (x, \boldsymbol{z}) \mid \operatorname{sign} \left(x - \boldsymbol{\theta}^T \boldsymbol{z} \right) \neq \operatorname{sign} \left(x - \boldsymbol{\theta}_j^T \boldsymbol{z} \right) \right\},$$

 $P_{j;X,Z}$ is the joint distribution of (X, Z) under P_j , and

$$\mathbb{E}_{j}[Y \mid x, z] = f_{j}(Y = 1 \mid x, z) - f_{j}(Y = -1 \mid x, z)$$

$$= \frac{1}{\sigma} \left(x - c \left(\frac{s \log(d/s)}{N} \right)^{1/2} \frac{\boldsymbol{\omega}_{j}^{T} z}{\sqrt{s}} \right)$$

$$= \frac{1}{\sigma} \left(x - \boldsymbol{\theta}_{j}^{T} z \right), \qquad (A.33)$$

$$\operatorname{sign}\left(\mathbb{E}_{j}[Y \mid x, z]\right) = \operatorname{sign}\left(x - \boldsymbol{\theta}_{j}^{T} z\right)$$

Therefore,

$$R_{j}(\boldsymbol{\theta}) - R_{j}(\boldsymbol{\theta}_{j}) = 2C \int_{\mathcal{G}} |\mathbb{E}_{j}[Y \mid x, \boldsymbol{z}]| \, dP_{j;X,\boldsymbol{Z}} \ge 0, \tag{A.34}$$

hence $\boldsymbol{\theta}_j$ is a minimizer of $R_j(\boldsymbol{\theta})$. In addition, $\|\boldsymbol{\theta}_j\|_2 = c\sqrt{\frac{\log(d/s)}{N}} \|\boldsymbol{\omega}_j\|_2 = c\sqrt{\frac{s\log(d/s)}{N}} = O(1)$.

Now we check the uniqueness. By (A.33) we have

$$R_{j}(\boldsymbol{\theta}) - R_{j}(\boldsymbol{\theta}_{j}) = \frac{2C}{\sigma} \int_{\mathcal{G}} \left| \boldsymbol{x} - \boldsymbol{\theta}_{j}^{T} \boldsymbol{z} \right| dP_{j;\boldsymbol{X},\boldsymbol{Z}}.$$
 (A.35)

For any $\theta \neq \theta_j$, consider the set

$$\mathcal{G}_{\boldsymbol{z}} = \left\{ \boldsymbol{z} : (\boldsymbol{\theta} - \boldsymbol{\theta}_j)^T \, \boldsymbol{z} \neq 0, \left| \boldsymbol{\theta}^T \boldsymbol{z} \right| \le 1, \left| \boldsymbol{\theta}_j^T \boldsymbol{z} \right| \le 1 \right\}.$$

Note that there exists an open neighborhood in \mathcal{G}_z , hence \mathcal{G}_z has nonzero measure. Then define $\bar{\mathcal{G}} = \left\{ (x, \boldsymbol{z}) : \boldsymbol{\theta}^T \boldsymbol{z} < x < \boldsymbol{\theta}_j^T \boldsymbol{z} \text{ or } \boldsymbol{\theta}_j^T \boldsymbol{z} < x < \boldsymbol{\theta}^T \boldsymbol{z}, \boldsymbol{z} \in \mathcal{G}_{\boldsymbol{z}} \right\}, \text{ we have } \bar{\mathcal{G}} \subset \mathcal{G}_{\boldsymbol{z}} \text{ and } \bar{\mathcal{G}} \text{ has nonzero measure as well. Therefore, (A.35) implies that } R_j(\boldsymbol{\theta}) - R_j(\boldsymbol{\theta}_j) > 0, \text{ which completes the proof of }$ the uniqueness. Note that

$$\begin{aligned} \nabla R_j(\boldsymbol{\theta}) &= \sum_{y=\pm 1} \gamma(y) \int_{\boldsymbol{Z}} \boldsymbol{z} y f\left(\boldsymbol{\theta}^T \boldsymbol{z} \mid \boldsymbol{z}, y\right) f_j(\boldsymbol{z}, y) d\boldsymbol{z} \\ &= \sum_{y=\pm 1} \gamma(y) \int_{\boldsymbol{Z}} \boldsymbol{z} y f_j\left(y \mid \boldsymbol{\theta}^T \boldsymbol{z}, \boldsymbol{z}\right) f_{X, \boldsymbol{Z}}(\boldsymbol{\theta}^T \boldsymbol{z}, \boldsymbol{z}) d\boldsymbol{z} \\ &= \frac{1}{2} \sum_{y=\pm 1} \gamma(y) \int_{\boldsymbol{Z}} \boldsymbol{z} y\left(\frac{1}{2} + \frac{y}{2\sigma} \left(\boldsymbol{\theta}^T \boldsymbol{z} - \boldsymbol{\theta}_j^T \boldsymbol{z}\right)\right) f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z}, \end{aligned}$$

hence

$$\nabla^2 R_j(\boldsymbol{\theta}) = \frac{1}{4\sigma} \sum_{y=\pm 1} \gamma(y) \int_{\boldsymbol{Z}} \boldsymbol{z} \boldsymbol{z}^T f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} = \frac{C}{2\sigma} \mathbb{E}(\boldsymbol{Z} \boldsymbol{Z}^T) = \frac{C}{6\sigma} \mathbb{I}_d,$$

and $\lambda_{\min}\left(\nabla^2 R_j\left(\boldsymbol{\theta}_j\right)\right) = \lambda_{\max}\left(\nabla^2 R_j\left(\boldsymbol{\theta}_j\right)\right) = \frac{C}{6\sigma} > 0$. This completes the proof.

For the second step of the proof, we check both the two conditions in the following. Recall that f^j is the probability measure of the random variables $\{(X_i, \mathbf{Z}_i, Y_i, R_i)^{\mathbb{1}\{R_i=1\}}, (X_i, \mathbf{Z}_i, R_i)^{\mathbb{1}\{R_i=0\}}\}_{i=1}^n$ under hypothesis j, and $\bar{H}_{i-1} = \{(X_j, \mathbf{Z}_j, Y_j, R_j)^{\mathbb{1}\{R_j=1\}}, (X_j, \mathbf{Z}_j, R_j)^{\mathbb{1}\{R_j=0\}}\}_{j=1}^{i-1}$. We have

$$\begin{split} f^{j} &= \prod_{i=1}^{n} \left\{ f(R_{i} = 1 \mid X_{i}, \mathbf{Z}_{i}, Y_{i}, \bar{H}_{i-1}) f_{j}(X_{i}, \mathbf{Z}_{i}, Y_{i} \mid \bar{H}_{i-1}) \right\}^{\mathbb{1}\{R_{i} = 1\}} \\ &\cdot \left\{ f(R_{i} = 0 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}) f_{j}(X_{i}, \mathbf{Z}_{i} \mid \bar{H}_{i-1}) \right\}^{\mathbb{1}\{R_{i} = 0\}} \\ &= \prod_{i=1}^{n} \left\{ f(R_{i} = 1 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}) f_{j}(X_{i}, \mathbf{Z}_{i}, Y_{i}) \right\}^{\mathbb{1}\{R_{i} = 1\}} \cdot \left\{ f(R_{i} = 0 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}) f_{j}(X_{i}, \mathbf{Z}_{i}) \right\}^{\mathbb{1}\{R_{i} = 0\}} \\ &= \prod_{i=1}^{n} \left\{ f(R_{i} = 1 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}) \right\}^{\mathbb{1}\{R_{i} = 1\}} \left\{ f(R_{i} = 0 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}) \right\}^{\mathbb{1}\{R_{i} = 0\}} \\ &\cdot f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})^{\mathbb{1}\{R_{i} = 1\}} f_{j}(X_{i}, \mathbf{Z}_{i}), \end{split}$$

where in the second last equation we used the fact that $R_i \perp Y_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1}$ and $(X_i, \mathbf{Z}_i, Y_i) \perp \bar{H}_{i-1}$ as specified in the class of sampling method in (3.23). Since the sampling method $f(R_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1})$ and the joint distribution of (X, \mathbf{Z}) keep invariant under different hypotheses, we have

$$\begin{aligned} \operatorname{KL}\left(f^{j} \| f^{0}\right) \\ = \mathbb{E}\left[\log \frac{\prod_{i=1}^{n} f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})^{1\{R_{i}=1\}} f_{j}(X_{i}, \mathbf{Z}_{i})}{\prod_{i=1}^{n} f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})^{1\{R_{i}=1\}} f_{0}(X_{i}, \mathbf{Z}_{i})}\right] \\ = \mathbb{E}\left[\log \frac{\prod_{i=1}^{n} f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})^{1\{R_{i}=1\}}}{\prod_{i=1}^{n} f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})^{1\{R_{i}=1\}}}\right] \\ = \mathbb{E}\left[\log \prod_{i=1}^{n} \left(\frac{f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}{f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}\right)^{1\{R_{i}=1\}}\right] \\ = \sum_{i=1}^{n} \mathbb{E}\left[1\{R_{i}=1\} \cdot \log \frac{f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}{f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}\right] \\ = \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[1\{R_{i}=1\} \cdot \log \frac{f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}{f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})} \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}\right]\right] \\ = \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[\log \frac{f_{j}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})}{f_{0}(Y_{i} \mid X_{i}, \mathbf{Z}_{i})} \mid X_{i}, \mathbf{Z}_{i}\right] \cdot \mathbb{P}\left(R_{i}=1 \mid X_{i}, \mathbf{Z}_{i}, \bar{H}_{i-1}\right)\right], \tag{A.36}$$

where in the last equation we used the fact that $R_i \perp Y_i \mid X_i, \mathbf{Z}_i, \bar{H}_{i-1}$ and $(X_i, \mathbf{Z}_i, Y_i) \perp \bar{H}_{i-1}$, and the expectation is with respect to f^j . Note that $\mathbb{E}\left[\log \frac{f_j(Y_i|X_i, \mathbf{Z}_i)}{f_0(Y_i|X_i, \mathbf{Z}_i)} \mid X_i, \mathbf{Z}_i\right]$ is the K-L divergence between two Bernoulli distributions $f_j(Y_i \mid X_i, \mathbf{Z}_i)$ and $f_0(Y_i \mid X_i, \mathbf{Z}_i)$. Since $\delta_j := c\left(\frac{s\log(d/s)}{N}\right)^{1/2} \frac{\omega_j^T \mathbf{z}}{\sqrt{s}} = o(1)$ uniformly over \mathbf{z} , we have

$$\mathbb{E}\left[\log\frac{f_j(Y_i \mid X_i, \mathbf{Z}_i)}{f_0(Y_i \mid X_i, \mathbf{Z}_i)} \mid X_i = x, \mathbf{Z}_i = \mathbf{z}\right]$$

= $f_j(y = 1 \mid x, \mathbf{z})\log\frac{f_j(y = 1 \mid x, \mathbf{z})}{f_0(y = 1 \mid x, \mathbf{z})} + f_j(y = -1 \mid x, \mathbf{z})\log\frac{f_j(y = -1 \mid x, \mathbf{z})}{f_0(y = -1 \mid x, \mathbf{z})}$
= $\left(\frac{1}{2} + \frac{1}{2\sigma}(x - \delta_j)\right)\log\left(1 - \frac{\delta_j}{\sigma + x}\right) + \left(\frac{1}{2} - \frac{1}{2\sigma}(x - \delta_j)\right)\log\left(1 + \frac{\delta_j}{\sigma - x}\right)$
= $\frac{\delta_j^2}{2(\sigma + x)(\sigma - x)} + o(\delta_j^2) \le \delta_j^2.$

Plugging this back to (A.36), we have

$$\operatorname{KL}\left(f^{j} \| f^{0}\right) \leq \sum_{i=1}^{n} \mathbb{E}\left[\delta_{j}^{2} \mathbb{P}\left(R_{i}=1 \mid X_{i}, \boldsymbol{Z}_{i}, \bar{H}_{i-1}\right)\right]$$
$$= c^{2} \frac{s \log(d/s)}{N} \frac{\boldsymbol{\omega}_{j}^{T}}{s} \sum_{i=1}^{n} \mathbb{E}\left[Q_{i} \boldsymbol{Z}_{i} \boldsymbol{Z}_{i}^{T}\right] \boldsymbol{\omega}_{j}.$$
(A.37)

Since $Q \in \mathcal{Q}_N(\mathcal{P}(\beta, s))$, we have

$$\boldsymbol{\omega}_{j}^{T}\sum_{i=1}^{n} \mathbb{E}\left[\boldsymbol{Z}_{i}\boldsymbol{Z}_{i}^{T} \cdot \mathbb{P}\left(\boldsymbol{R}_{i}=1 \mid \boldsymbol{X}_{i}, \boldsymbol{Z}_{i}, \bar{\boldsymbol{H}}_{i-1}\right)\right] \boldsymbol{\omega}_{j} \leq C \|\boldsymbol{\omega}_{j}\|_{2}^{2} N.$$

Giving this back to (A.37), and choosing c in (A.27) sufficiently small, we can ensure C_2 small enough such that

$$\operatorname{KL}\left(f^{j} \| f^{0}\right) \leq C_{2} \frac{s \log(d/s)}{N} \frac{\|\boldsymbol{\omega}_{j}\|_{2}^{2} N}{s} = C_{2} s \log(d/s) \leq \gamma c' s \log(d/s) \leq \gamma \log|\mathcal{H}'|$$

for some $\gamma \in (0, 1/8)$, hence condition 1 is satisfied.

For condition 2, by Lemma A.3, we have when $j \neq 0$,

$$\|\boldsymbol{\theta}_{0} - \boldsymbol{\theta}_{j}\|_{2} = c \left(\frac{s \log(d/s)}{N}\right)^{1/2} \|\boldsymbol{\omega}_{j}\|_{2} / \sqrt{s} = c \left(\frac{s \log(d/s)}{N}\right)^{1/2},$$
$$\|\boldsymbol{\theta}_{0} - \boldsymbol{\theta}_{j}\|_{1} = c \left(\frac{s \log(d/s)}{N}\right)^{1/2} \|\boldsymbol{\omega}_{j}\|_{1} / \sqrt{s} = c \sqrt{s} \left(\frac{s \log(d/s)}{N}\right)^{1/2}.$$

For all $j, k \neq 0$, we have

$$\left\|\boldsymbol{\theta}_{j}-\boldsymbol{\theta}_{k}\right\|_{2}=c\left(\frac{s\log(d/s)}{N}\right)^{1/2}\left\|\boldsymbol{\omega}_{j}-\boldsymbol{\omega}_{k}\right\|_{2}/\sqrt{s}\geq\frac{c}{4}\left(\frac{s\log(d/s)}{N}\right)^{1/2},$$

$$\left\|\boldsymbol{\theta}_{j}-\boldsymbol{\theta}_{k}\right\|_{1}=c\left(\frac{s\log(d/s)}{N}\right)^{1/2}\left\|\boldsymbol{\omega}_{j}-\boldsymbol{\omega}_{k}\right\|_{1}/\sqrt{s}\geq\frac{c\sqrt{s}}{16}\left(\frac{s\log(d/s)}{N}\right)^{1/2}$$

Therefore condition 2 holds. Then we apply Theorem 2.7 in Tsybakov (2008) and finish the proof. $\hfill \Box$

A.4 Additional theoretical results

In this section, we assume that there exits a sequence $C_N > 0$ that may depend on N such that

$$\int_{C_N/2}^{\infty} |K(t)| dt \le C\delta_k^{\beta},\tag{A.38}$$

holds for any $2 \le k \le K$, where C is a constant that does not depend on k and δ_k is the bandwidth parameter in the kth iteration. In addition, we also remove the sub-Gaussian vector assumption for Z in Assumption 3.2 in this section.

The proof of the following theorems is similar to the proof of the main results in the previous section, and is omitted to avoid repetition. We note that without the sub-Gaussian vector assumption, it becomes complicated to pinpoint the critical points for β at which the transition of the property of the algorithm occurs. Indeed, the conditions (A.39), (A.42) and (A.46) in the following three theorems correspond to the cases (i) $\beta \in ((1 + \sqrt{3})/2, +\infty)$; (ii) $\beta \in (1, (1 + \sqrt{3})/2]$; and (iii) $\beta = 1$ considered in the main paper.

Theorem 6. Assume that Assumptions 3.1, 3.2 (i) and (ii), 3.3-3.5 hold, and $K \ge 2$. We set $N_k = N/K$ for $1 \le k \le K$, and

$$\delta_{1} = c_{1} \left(\frac{Ks \log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_{1} = c_{2} \sqrt{\frac{NK \log d}{n^{2} \delta_{1}}},$$
$$= c_{1} \left(\frac{C_{N}Ks \log d}{N}\right)^{1/(2\beta)}, \ \lambda_{k} = c_{2} \sqrt{\frac{NK \log d}{n^{2} b_{k-1} \delta_{k}}}, \ b_{k-1} = c_{3} \left(\frac{C_{N}^{2\beta+1}Ks \log d}{N}\right)^{1/(2\beta)}, 2 \le k \le K,$$

for some constants $c_1, c_2, c_3 > 0$ and $c_3 \ge c_1$. If

$$\sqrt{s}M_n\left(\left(\frac{Ks\log d}{N}\right)^{\beta/(2\beta+1)} \vee \left(\frac{C_NKs\log d}{N}\right)^{1/2}\right) = O\left(\left(\frac{C_N^{2\beta+1}Ks\log d}{N}\right)^{1/(2\beta)}\right), \quad (A.39)$$

and

 δ_k

$$N \le C n^{2\beta/(2\beta+1)} (Ks \log d)^{1/(2\beta+1)} C_N$$
(A.40)

hold for some constant C, then with probability greater than 1 - 2K/d, we have

$$\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{2} \lesssim \left(\frac{C_{N}Ks\log d}{N}\right)^{1/2}, \ \|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{1} \lesssim \sqrt{s} \left(\frac{C_{N}Ks\log d}{N}\right)^{1/2}, \tag{A.41}$$

where N is the given label budget.

Theorem 7. Assume that Assumptions 3.1, 3.2 (i) and (ii), 3.3-3.5 hold, and suppose there exists an integer K^* such that

$$\frac{M_n^{\frac{2\beta+1}{\beta+1}-\frac{\beta^{K^*-1}}{(\beta+1)(2\beta+1)K^*-2}}s^{\frac{2\beta+1}{2(\beta+1)}-\frac{\beta^{K^*-1}}{2(\beta+1)(2\beta+1)K^*-2}}}{C_N^{(2\beta+1)/(2\beta)}} = O\left(\left(\frac{K\log d}{N}\right)^{\frac{1}{2\beta}-\frac{\beta}{\beta+1}+\frac{\beta^{K^*}}{(\beta+1)(2\beta+1)K^*-1}}\right).$$
 (A.42)

We set $K = K^*, N_k = N/K$ for $1 \le k \le K$. We set

$$\delta_1 = c_1 \left(\frac{sK \log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_1 = c_2 \sqrt{\frac{NK \log d}{n^2 \delta_1}},$$

for $2 \le k \le K - 1$,

$$\delta_k = c_1 \left(\frac{Ks\log db_{k-1}}{N}\right)^{1/(2\beta+1)}, \ \lambda_k = c_2 \sqrt{\frac{NK\log d}{n^2 b_{k-1} \delta_k}},$$
$$b_{k-1} = c_3 M_n^{\frac{2\beta+1}{\beta+1} - \frac{\beta^{k-1}}{(\beta+1)(2\beta+1)^{k-2}}} s^{\frac{2\beta+1}{2(\beta+1)} - \frac{\beta^{k-1}}{2(\beta+1)(2\beta+1)^{k-2}}} \left(\frac{Ks\log d}{N}\right)^{\frac{\beta}{\beta+1} - \frac{\beta^k}{(\beta+1)(2\beta+1)^{k-1}}}$$

for some constants $c_1, c_2, c_3 > 0$, and

$$\delta_K = c_1' \left(\frac{C_N K s \log d}{N}\right)^{1/(2\beta)}, \ \lambda_K = c_2' \sqrt{\frac{N K \log d}{n^2 b_{K-1} \delta_K}}, \ b_{K-1} = c_3' \left(\frac{C_N^{2\beta+1} K s \log d}{N}\right)^{1/(2\beta)},$$

for some constants $c_1',c_2',c_3'>0$ and $c_3'\geq c_1'.$ If

$$\left(\frac{K\log d}{N}\right)^{\frac{1}{2\beta}-\frac{\beta}{\beta+1}+\frac{\beta^k}{(\beta+1)(2\beta+1)^{k-1}}} = O\left(\frac{M_n^{\frac{2\beta+1}{\beta+1}-\frac{\beta^{k-1}}{(\beta+1)(2\beta+1)^{k-2}}}s^{\frac{2\beta+1}{2(\beta+1)}-\frac{\beta^{k-1}}{2(\beta+1)(2\beta+1)^{k-2}}}}{C_N^{(2\beta+1)/(2\beta)}}\right)$$
(A.43)

for $2 \leq k \leq K - 1$, and

$$N \le C_1 M_n^{\frac{\beta+1}{2\beta+1}} s^{\frac{\beta+1}{2(2\beta+1)}} (Ks \log d)^{\frac{\beta}{2\beta+1}},$$
(A.44)

,

$$N \le C_2 n^{2\beta/(2\beta+1)} (Ks \log d)^{1/(2\beta+1)} C_N \tag{A.45}$$

hold for some constants C_1, C_2 , then with probability greater than $1 - 2Kd^{-1}$,

$$\|\widehat{\boldsymbol{\theta}}_K - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{C_N K s \log d}{N}\right)^{1/2}, \ \|\widehat{\boldsymbol{\theta}}_K - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{C_N K s \log d}{N}\right)^{1/2}.$$

Theorem 8. Assume that Assumptions 3.1, 3.2 (i) and (ii), 3.3-3.5 hold. We set $N_k = N/K$ for $1 \le k \le K$, and

$$\delta_1 = c_1 \left(\frac{Ks\log d}{N}\right)^{1/(2\beta+1)}, \ \lambda_1 = c_2 \sqrt{\frac{NK\log d}{n^2 \delta_1}},$$
$$2 \le k \le K: \ \delta_k = c_1 \left(\frac{Ks\log db_{k-1}}{N}\right)^{1/(2\beta+1)}, \ \lambda_k = c_2 \sqrt{\frac{NK\log d}{n^2 b_{k-1} \delta_k}},$$

$$b_{k-1} = c_3 M_n^{\frac{2\beta+1}{\beta+1} - \frac{\beta^{k-1}}{(\beta+1)(2\beta+1)^{k-2}}} s^{\frac{2\beta+1}{2(\beta+1)} - \frac{\beta^{k-1}}{2(\beta+1)(2\beta+1)^{k-2}}} \left(\frac{Ks\log d}{N}\right)^{\frac{\beta}{\beta+1} - \frac{\beta^k}{(\beta+1)(2\beta+1)^{k-1}}}$$

for some constants $c_1, c_2, c_3 > 0$. If

$$\left(\frac{K\log d}{N}\right)^{\frac{1}{2\beta}-\frac{\beta}{\beta+1}+\frac{\beta^k}{(\beta+1)(2\beta+1)^{k-1}}} = O\left(\frac{M_n^{\frac{2\beta+1}{\beta+1}-\frac{\beta^{k-1}}{(\beta+1)(2\beta+1)^{k-2}}}s^{\frac{2\beta+1}{2(\beta+1)}-\frac{\beta^{k-1}}{2(\beta+1)(2\beta+1)^{k-2}}}}{C_N^{(2\beta+1)/(2\beta)}}\right)$$
(A.46)

for all $2 \leq k \leq K$, and

$$N \le CM_n^{\frac{\beta+1}{2\beta+1}} s^{\frac{\beta+1}{2(2\beta+1)}} (Ks \log d)^{\frac{\beta}{2\beta+1}}, \tag{A.47}$$

,

hold for some constant C, then with probability greater than $1 - 2Kd^{-1}$,

$$\begin{aligned} \|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{2} &\lesssim M_{n}^{\frac{\beta}{\beta+1} - \frac{\beta^{K}}{(\beta+1)(2\beta+1)^{K-1}}} s^{\frac{\beta}{2(\beta+1)} - \frac{\beta^{K}}{2(\beta+1)(2\beta+1)^{K-1}}} \left(\frac{Ks \log d}{N}\right)^{\frac{\beta}{\beta+1} - \frac{\beta^{K+1}}{(\beta+1)(2\beta+1)^{K}}}, \\ \|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\|_{1} &\lesssim M_{n}^{\frac{\beta}{\beta+1} - \frac{\beta^{K}}{(\beta+1)(2\beta+1)^{K-1}}} s^{\frac{2\beta+1}{2(\beta+1)} - \frac{\beta^{K}}{2(\beta+1)(2\beta+1)^{K-1}}} \left(\frac{Ks \log d}{N}\right)^{\frac{\beta}{\beta+1} - \frac{\beta^{K+1}}{(\beta+1)(2\beta+1)^{K}}}. \end{aligned}$$

A.5 Supplementary Results for Path-following Algorithm

The analysis of the path-following algorithm follows the same line as Feng et al. (2022) and the references therein. We only provide a sketch of the proof and refer the details to the original paper. In fact, the key difference between our proof and Feng et al. (2022) is established in Proposition A.2 and Proposition A.1.

Lemma A.4. Assume the conditions of Proposition A.2 and Assumption 3.5 hold. For $\lambda \geq \lambda_{k,tgt}$, if $\boldsymbol{\theta} \in \Omega$, $\|\boldsymbol{\theta}_{S^{*c}}\|_0 \leq \tilde{s}, \omega_{\lambda}(\boldsymbol{\theta}) \leq \frac{1}{2}\lambda$, and $\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\hat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \leq \lambda/8$, we have

$$\begin{split} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 &\leq \frac{\bar{C}_1}{\rho_-} \left(c_{n,k} \delta_k^\beta \vee \sqrt{s} \lambda \right), \\ \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 &\leq \frac{\bar{C}_2}{\rho_-} \left(\frac{c_{n,k}^2 \delta_k^{2\beta}}{\lambda} \vee \sqrt{s} c_{n,k} \delta_k^\beta \vee s \lambda \right), \\ f_\lambda(\boldsymbol{\theta}) - f_\lambda\left(\boldsymbol{\theta}^*\right) &\leq \frac{\bar{C}_2}{2\rho_-} \left(c_{n,k}^2 \delta_k^{2\beta} \vee \sqrt{s} c_{n,k} \delta_k^\beta \lambda \vee s \lambda^2 \right), \end{split}$$

where $f_{\lambda}(\boldsymbol{\theta})$ denotes the objective function $R_{\delta_k}^{D_k}(\boldsymbol{\theta}) + \lambda \|\boldsymbol{\theta}\|_1$ and $\bar{C}_1, \bar{C}_2 > 0$ are constants that depend on C_2 in Proposition A.2.

Proof. Combining (3.6) in Assumption 3.5 with the definition of $w_{\lambda}(\boldsymbol{\theta})$, we can derive

$$\frac{3}{2}\lambda \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^*}\|_1 - (\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T \nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) \ge \frac{1}{2}\lambda \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_1 + \rho_- \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2^2.$$
(A.48)

By Proposition A.2, Proposition A.1, and notice the sparsity of θ, θ^* , we have

$$(\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T \nabla R_{\delta_k}^{D_k} (\boldsymbol{\theta}^*) \Big| = | (\boldsymbol{\theta} - \boldsymbol{\theta}^*)^T (\underbrace{\nabla R_{\delta_k}^{D_k} (\boldsymbol{\theta}^*) - \nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}} (\boldsymbol{\theta}^*)}_{E_1} + \underbrace{\nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}} (\boldsymbol{\theta}^*) - \nabla R(\boldsymbol{\theta}^*)}_{E_2}) |)$$

$$\leq ||\boldsymbol{\theta} - \boldsymbol{\theta}^*||_1 ||E_1||_{\infty} + C_2 c_{n,k} \delta_k^{\beta} ||\boldsymbol{\theta} - \boldsymbol{\theta}^*||_2,$$

$$(A.49)$$

where C_2 is the constant defined in Proposition A.2. Combining (A.49) with (A.48) we have

$$\rho_{-} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}^{2} \leq C_{2} c_{n,k} \delta_{k}^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2} + \left(\frac{3}{2}\lambda + \|E_{1}\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^{*})_{S^{*}}\|_{1} - \left(\frac{1}{2}\lambda - \|E_{1}\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^{*})_{S^{*c}}\|_{1}.$$
(A.50)

Now we discuss two cases. If $\rho_{-} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}^{2} \leq 3C_{2}c_{n,k}\delta_{k}^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}$, then

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \le \frac{3}{\rho_-} C_2 c_{n,k} \delta_k^{\beta}.$$
(A.51)

If $\rho_{-} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}^{2} > 3C_{2}c_{n,k}\delta_{k}^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}$, then we have

$$\frac{2\rho_{-}}{3} \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{2}^{2} \leq \left(\frac{3}{2}\lambda + \|E_{1}\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^{*})_{S^{*}}\|_{1} - \left(\frac{1}{2}\lambda - \|E_{1}\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^{*})_{S^{*c}}\|_{1}.$$
(A.52)

Note that the condition of λ ensures that $\frac{1}{2}\lambda - ||E_1||_{\infty} \ge 0$, hence we have

$$\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\right\|_2 \le \frac{3\sqrt{s}\left(\frac{3}{2}\lambda + \|E_1\|_{\infty}\right)}{2\rho_-} \le \frac{3}{\rho_-}\sqrt{s}\lambda.$$
(A.53)

Combining the above two cases, we conclude that

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \le \frac{3}{\rho_-} \left(C_2 c_{n,k} \delta_k^\beta \vee \sqrt{s} \lambda \right).$$
(A.54)

For $\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1$, define $\gamma = \frac{\frac{3}{2}\lambda + \|E_1\|_{\infty}}{\frac{1}{2}\lambda - \|E_1\|_{\infty}} \leq \frac{13}{3}$, and we consider two cases below. If $\|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_1 > 2\gamma \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^*}\|_1$, by (A.54) we obtain

$$\begin{aligned} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 &\leq \sqrt{s}(1+2\gamma) \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \\ &\leq \frac{29}{3}\sqrt{s} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \\ &\leq \frac{29}{\rho_-} \left(C_2 \sqrt{s} c_{n,k} \delta_k^\beta \vee s \lambda \right). \end{aligned}$$
(A.55)

If $\|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_1 > 2\gamma \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^*}\|_1$, we have

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 \le \left(1 + \frac{1}{2\gamma}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_1.$$
(A.56)

By (A.50) we have

$$\begin{pmatrix} \frac{1}{2}\lambda - \|E_1\|_{\infty} \end{pmatrix} \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_{1} \leq C_2 c_{n,k} \delta_k^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_{2} - \rho_{-} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_{2}^{2} + \left(\frac{3}{2}\lambda + \|E_1\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^*}\|_{1} \\ \leq C_2 c_{n,k} \delta_k^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_{2} - \rho_{-} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_{2}^{2} + \left(\frac{3}{2}\lambda + \|E_1\|_{\infty}\right) \frac{1}{2\gamma} \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^*}\|_{1}$$

By the definition of γ we have

$$\left(\frac{1}{2}\lambda - \|E_1\|_{\infty}\right) \|(\boldsymbol{\theta} - \boldsymbol{\theta}^*)_{S^{*c}}\|_1 \le 2\left(C_2 c_{n,k} \delta_k^\beta \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 - \rho_- \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2^2\right).$$

Combine this with (A.56) and note that $||E_1||_{\infty} \leq \lambda/8$ we have

$$\begin{aligned} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 &\leq \frac{\left(2 + \frac{1}{\gamma}\right)}{\frac{1}{2}\lambda - \|E_1\|_{\infty}} C_2 c_{n,k} \delta_k^{\beta} \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \\ &\leq \frac{56C_2 c_{n,k} \delta_k^{\beta}}{3\rho_-} \left(\frac{C_2 c_{n,k} \delta_k^{\beta}}{\lambda} \vee \sqrt{s}\right). \end{aligned}$$
(A.57)

Conclude the two cases we have

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 \le \frac{1}{\rho_-} \left(29\sqrt{s}C_2 c_{n,k} \delta_k^\beta \vee 29s\lambda \vee \frac{56C_2^2 c_{n,k} \delta_k^{2\beta}}{3\lambda} \right).$$
(A.58)

Finally, note that $\boldsymbol{\xi} \in \partial \|\boldsymbol{\theta}\|_1$ implies $\boldsymbol{\xi}^T(\boldsymbol{\theta} - \boldsymbol{\theta}^*) \geq \|\boldsymbol{\theta}\|_1 - \|\boldsymbol{\theta}^*\|_1$, and Assumption 3.5 gives $\nabla R^{D_k}_{\delta_k}(\boldsymbol{\theta})^T(\boldsymbol{\theta} - \boldsymbol{\theta}^*) \geq R^{D_k}_{\delta_k}(\boldsymbol{\theta}) - R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*)$, therefore

$$\begin{split} f_{\lambda}(\boldsymbol{\theta}) - f_{\lambda}\left(\boldsymbol{\theta}^{*}\right) &= R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}) - R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}) + \lambda \|\boldsymbol{\theta}\|_{1} - \lambda \|\boldsymbol{\theta}^{*}\|_{1} \\ &\leq (\boldsymbol{\theta} - \boldsymbol{\theta}^{*})^{T} \left(\nabla R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}) + \lambda \boldsymbol{\xi} \right) \\ &\leq \frac{1}{2} \lambda \|\boldsymbol{\theta} - \boldsymbol{\theta}^{*}\|_{1} \\ &\leq \frac{1}{2\rho_{-}} \left(29\sqrt{s}C_{2}c_{n,k}\delta_{k}^{\beta}\lambda \vee 29s\lambda^{2} \vee \frac{56C_{2}^{2}c_{n,k}\delta_{k}^{2\beta}}{3} \right). \end{split}$$

Rewriting the constants we finish the proof.

The next two lemmas characterize the properties of the iterates $\theta_{k,t}^1, \cdots$ at stage t.

Lemma A.5 (Lemma 2 in Feng et al. (2022)). Assume the conditions of Proposition A.2 and Assumption 3.5 hold. For $\lambda \geq \lambda_{k,tgt}$, if $\boldsymbol{\theta} \in \Omega$, $\|\boldsymbol{\theta}_{S^{*c}}\|_0 \leq \tilde{s}$, and $\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \leq \lambda/8$, and $f_{\lambda}(\boldsymbol{\theta}) - f_{\lambda}(\boldsymbol{\theta}^*) \leq \frac{\bar{C}_2}{2\rho_-} \left(c_{n,k}^2 \delta_k^{2\beta} \vee \sqrt{s} c_{n,k} \delta_k^{\beta} \lambda \vee s \lambda^2\right)$, then we have

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_2 \leq \frac{\bar{C}_1'}{\rho_-} \left(c_{n,k} \delta_k^\beta \vee s^{1/4} \sqrt{c_{n,k} \delta_k \lambda} \vee \sqrt{s} \lambda \right),$$

$$\left\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\right\|_1 \leq \frac{2\bar{C}_2}{\rho_-} \left(\frac{c_{n,k}^2 \delta_k^{2\beta}}{\lambda} \vee \sqrt{s} c_{n,k} \delta_k^\beta \vee s \lambda\right),$$

where \bar{C}'_1 depends on \bar{C}_1, \bar{C}_2 , which are constants defined in Lemma A.4.

Lemma A.6 (Lemma 3 in Feng et al. (2022)). Under the same conditions of Lemma A.5, if we choose $\lambda_{k,tgt} = C \sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$ for some large enough constant C, and $\delta_k = c \left(\frac{sK \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$ for some constant c > 0, then

$$\left\| \mathcal{S}_{\lambda_{k,t}\eta} \left(\boldsymbol{\theta}, \mathbb{R}^d \right)_{S^{*c}} \right\|_0 \le \widetilde{s}, \tag{A.59}$$

where $\tilde{s} = 8\left(\frac{\bar{C}_2}{\eta\rho_-} + \frac{2\bar{C}_1'^2\rho_+^2}{\rho_-^2} + 2C_2^2\right) \cdot s$, and $\bar{C}_1', \bar{C}_2, C_2$ are constants defined in Lemma A.5, Lemma A.4 and Proposition A.2, respectively.

Lemma A.5 and Lemma A.6 together imply that if the initialization at stage t is sparse and satisfies $\omega_{\lambda_{k,t}}\left(\boldsymbol{\theta}_{k,t}^{0}\right) \leq \frac{1}{2}\lambda_{k,t}$, then the subsequent iterate should also retain sparsity and exhibit favorable properties. Furthermore, under Assumption 3.5, Lemma A.7 ensures that $f_{\lambda}(\boldsymbol{\theta}_{k,t}^{0}), f_{\lambda}(\boldsymbol{\theta}_{k,t}^{1}), \cdots$ are decreasing. Consequently, the conditions delineated in Lemma A.5 and Lemma A.6 persist throughout the entire path $\boldsymbol{\theta}_{k,t}^{1}, \cdots$, guaranteeing both sparsity and convergence towards a local solution. This proposition is formally articulated as follows.

Proposition A.3 (Proposition 3 in Feng et al. (2022)). Under Assumptions 3.1-3.5, suppose $\delta_k = c \left(\frac{sK \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$ for some constant c > 0. If $\left\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\right\|_{\infty} \leq \lambda_{k,tgt}/8 \leq \lambda_{k,t}/8$ and at stage t, the proximal gradient method is initialized with $\boldsymbol{\theta}_{k,t}^0 \in \Omega$ satisfying

$$\left\| \left(\boldsymbol{\theta}_{k,t}^{0}\right)_{S^{*c}} \right\|_{0} \leq \widetilde{s} \text{ and } \omega_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{0}\right) \leq \frac{1}{2} \lambda_{k,t}, \tag{A.60}$$

then for $j = 1, 2, \cdots$, we have

- $\left\| \left(\boldsymbol{\theta}_{k,t}^{j} \right)_{S^{*c}} \right\|_{0} \leq \widetilde{s}$
- The sequence $\left\{ \boldsymbol{\theta}_{k,t}^{j} \right\}_{j=0}^{\infty}$ converges towards a unique local solution $\widehat{\boldsymbol{\theta}}_{k,t}$ satisfying the first-order optimality $\omega_{\lambda_{k,t}} \left(\widehat{\boldsymbol{\theta}}_{k,t} \right) \leq 0$ with $\left\| \left(\boldsymbol{\theta}_{k,t}^{j} \right)_{S^{*c}} \right\|_{0} \leq \widetilde{s}$

•
$$f_{\lambda_{k,t}}\left(\boldsymbol{\theta}_{k,t}^{j}\right) - f_{\lambda_{k,t}}\left(\widehat{\boldsymbol{\theta}}_{k,t}\right) \leq \left(1 - \frac{\eta\rho_{-}}{4}\right)^{j} \left(f_{\lambda_{k,t}}\left(\boldsymbol{\theta}_{k,t}^{0}\right) - f_{\lambda_{k,t}}\left(\widehat{\boldsymbol{\theta}}_{k,t}\right)\right)$$

Lemma A.7. (Nesterov (2013), Theorem 1) Under the same conditions of Lemma A.6, we have

$$f_{\lambda}\left(\mathcal{S}_{\lambda\eta}(\boldsymbol{\theta},\Omega)\right) \leq f_{\lambda}(\boldsymbol{\theta}) - \frac{1}{2\eta} \left\|\mathcal{S}_{\lambda\eta}(\boldsymbol{\theta},\Omega) - \boldsymbol{\theta}\right\|_{2}^{2}$$

Lemma A.8. (Nesterov (2013), Corollary 1) Under the same conditions of Lemma A.6, we have

$$\omega_{\lambda_t} \left(\mathcal{S}_{\lambda\eta}(\boldsymbol{\theta}, \Omega) \right) \leq \left(\frac{1}{\eta} + \rho_+ \right) \left\| \mathcal{S}_{\lambda\eta}(\boldsymbol{\theta}, \Omega) - \boldsymbol{\theta} \right\|_2$$

Lemma A.9 (Lemma 10 in Feng et al. (2022)). Suppose Assumption 3.5 holds. If $\lambda \geq \lambda_{k,tgt}$, $\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \leq \lambda_{k,tgt}/8$, $\omega_{\lambda}(\boldsymbol{\theta}) \leq \frac{1}{2}\lambda$, $\|\boldsymbol{\theta}_{S^*}\|_0 \leq \tilde{s}$ and $\widehat{\boldsymbol{\theta}}_{\lambda} \in \Omega$ is a minimizer of f_{λ} satisfying $\|(\widehat{\boldsymbol{\theta}}_{\lambda})_{S^{*c}}\|_0 \leq \tilde{s}$, then we have

$$f_{\lambda}(\boldsymbol{\theta}) - f_{\lambda}\left(\widehat{\boldsymbol{\theta}}_{\lambda}\right) \leq \frac{\bar{C}_2}{\rho_-} \left(\delta^{2\beta} \vee \sqrt{s}\delta^{\beta}\lambda \vee s\lambda^2\right).$$

Theorem 9. Assume the conditions of Proposition A.2 and Assumption 3.5 hold. By choosing $\nu = 0.25, \phi = 0.9, \eta \leq \frac{1}{\rho_+}, \lambda_{k,tgt} = 8C_1 \sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$, where C_1 is defined in Proposition A.1 and $\delta_k = c \left(\frac{sK \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$ for some constant c > 0, with probability greater than $1 - 2d^{-1}$, the final approximate local solution $\tilde{\theta}_{k,tgt}$ from the path-following algorithm satisfies

$$\begin{split} \left\| \widetilde{\boldsymbol{\theta}}_{k,tgt} - \boldsymbol{\theta}^* \right\|_2 &\lesssim \left(\frac{Ks \log d}{nc_{n,k}} \right)^{\beta/(2\beta+1)}, \\ \left\| \widetilde{\boldsymbol{\theta}}_{k,tgt} - \boldsymbol{\theta}^* \right\|_1 &\lesssim \sqrt{s} \left(\frac{Ks \log d}{nc_{n,k}} \right)^{\beta/(2\beta+1)} \end{split}$$

Proof. By Proposition A.1 we have

$$\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k,\widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \le C_1 \sqrt{\frac{c_{n,k} K \log d}{n\delta_k}}$$

hence with probability greater than $1 - 2d^{-1}$, $\|\nabla R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla R_{\delta_k, \widehat{\boldsymbol{\theta}}_{k-1}}(\boldsymbol{\theta}^*)\|_{\infty} \leq \lambda_{k, tgt}/8$ holds. We prove this theorem by induction. Note that the initialization in Algorithm 4 guarantees that

$$\left\| \left(\boldsymbol{\theta}_{k,0}^{0}\right)_{S^{*c}} \right\|_{0} \leq \widetilde{s} \text{ and } \omega_{\lambda_{k,0}} \left(\boldsymbol{\theta}_{k,0}^{0}\right) \leq \frac{1}{2} \lambda_{k,0},$$

where \tilde{s} is defined in Proposition A.3. Suppose at stage $t = 1, \dots, T - 1$, we have

$$\left\| \left(\boldsymbol{\theta}_{k,t}^{0}\right)_{S^{*c}} \right\|_{0} \leq \widetilde{s} \text{ and } \omega_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{0}\right) \leq \frac{1}{2} \lambda_{k,t}.$$

By Proposition A.3, we know $\left\| \left(\boldsymbol{\theta}_{k,t}^{j} \right)_{S^{*c}} \right\|_{0} \leq \tilde{s}$ for $j = 1, \cdots$, which implies that $\left\| \left(\tilde{\boldsymbol{\theta}}_{k,t} \right)_{S^{*c}} \right\|_{0} \leq \tilde{s}$ if exists. Recall that at stage t, the stopping criteria requires $\omega_{\lambda_{k,t}} \left(\boldsymbol{\theta} \right) \leq \frac{1}{4} \lambda_{k,t}$, therefore it suffices to find j such that $\omega_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{j} \right) \leq \frac{1}{4} \lambda_{k,t}$ to finish stage t. By Lemma A.8, we have

$$\omega_{\lambda_{k,t}}\left(\boldsymbol{\theta}_{k,t}^{j}\right) \leq \left(\frac{1}{\eta} + \rho_{+}\right) \left\|\boldsymbol{\theta}_{k,t}^{j} - \boldsymbol{\theta}_{k,t}^{j-1}\right\|_{2}$$

Recall that $\widehat{\theta}_{k,t}$ is defined as (A.1), from Lemma A.7 we obtain

$$\frac{1}{2\eta} \left\| \boldsymbol{\theta}_{k,t}^{j} - \boldsymbol{\theta}_{k,t}^{j-1} \right\|_{2}^{2} \leq f_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{j-1} \right) - f_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{t}^{j} \right) \\
\leq f_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{j-1} \right) - f_{\lambda_{k,t}} \left(\boldsymbol{\widehat{\theta}}_{k,t} \right) \\
\leq \left(1 - \frac{\eta \rho_{-}}{4} \right)^{j-1} \left(f_{\lambda_{k,t}} \left(\boldsymbol{\theta}_{k,t}^{0} \right) - f_{\lambda_{k,t}} \left(\boldsymbol{\widehat{\theta}}_{k,t} \right) \right) \\
\leq \left(1 - \frac{\eta \rho_{-}}{4} \right)^{j-1} \frac{\bar{C}_{2}}{\rho_{-}} \left(\delta_{k}^{2\beta} \vee \sqrt{s} \delta_{k}^{\beta} \lambda_{k,t} \vee s \lambda_{k,t}^{2} \right),$$
(A.61)

where the second last inequality is from Proposition A.3 and the last inequality follows from Lemma A.9. Now it suffices to guarantee that

$$\left(\frac{1}{\eta} + \rho_{+}\right)\sqrt{2\eta\left(1 - \frac{\eta\rho_{-}}{4}\right)^{j-1}\frac{\bar{C}_{2}}{\rho_{-}}\left(\delta^{2\beta} \vee \sqrt{s}\delta^{\beta}\lambda_{k,t} \vee s\lambda_{k,t}^{2}\right)} \leq \frac{1}{4}\lambda_{k,t}.$$

Recall that we choose $\delta_k = c \left(\frac{sK \log d}{nc_{n,k}}\right)^{1/(2\beta+1)}$, $\lambda_{k,t} > \lambda_{k,tgt}$ and $\lambda_{k,tgt} = 8C_1 \sqrt{\frac{c_{n,k}K \log d}{n\delta_k}}$. With some algebra we can show that it suffices to guarantee that

$$j \ge \log\left(\frac{32\left(\frac{1}{\eta} + \rho_+\right)^2 \eta \bar{C}_2 s}{\rho_-}\right) / \log\left(\frac{4}{4 - \eta \rho_-}\right) + 1,$$

where the RHS is independent of λ .

A.6 A Variation of Theorem 6

Theorem 10. Under Assumptions 3.1-3.5, assume that the number of iterations $K \ge 2$ and $\beta \ge 2$. We set $N_K = N/2$, $N_k = (tN_{k+1})^{1/\beta}$ for $2 \le k \le K-1$, $N_1 = \left(\frac{tN_2}{(s\log d)^{\beta/(2\beta+1)}C_N^{\beta}}\right)^{(2\beta+1)/(2\beta^2)}$, and

$$\delta_1 = c_1 \left(\frac{s \log d}{N_1}\right)^{1/(2\beta+1)}, \ \lambda_1 = c_2 \sqrt{\frac{N_1 K^2 \log d}{n^2 \delta_1}},$$
$$\delta_k = c_1 \left(\frac{C_N s \log d}{N_k}\right)^{1/(2\beta)}, \ \lambda_k = c_2 \sqrt{\frac{N_k K^2 \log d}{n^2 b_{k-1} \delta_k}}, \ b_{k-1} = c_3 \left(\frac{C_N^{2\beta+1} s \log d}{N_k}\right)^{1/(2\beta)}, 2 \le k \le K,$$

for some constants c_1, c_2, c_3 and $c_3 \ge c_1$. If

$$N \ge t^{\frac{1}{\beta-1}} 2^{K} \vee \left(\frac{t^{\frac{\beta-1/\beta^{(K-2)}}{\beta-1}}}{(s\log d)^{\beta/(2\beta+1)} C_{N}^{\beta}} \right)^{\frac{\beta^{K-2}(2\beta+1)}{2\beta^{K}-(2\beta+1)}} 2^{\frac{2(K-1)\beta^{K}-(2\beta+1)}{2\beta^{K}-(2\beta+1)}},$$
(A.62)

where $t = s^{2\beta-1} M_n^{2\beta} (\log d)^{\beta-1} / C_N^{\beta+1}$ and

$$N \le C \left(n/K \right)^{2\beta/(2\beta+1)} (s \log d)^{1/(2\beta+1)} C_N$$
(A.63)

hold for some constant C, then $\sum_{k=1}^{K} N_k \leq N$ and with probability greater than 1 - 2K/d,

$$\left\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\right\|_{2} \lesssim \left(\frac{C_{N}s\log d}{N}\right)^{1/2}, \ \left\|\widehat{\boldsymbol{\theta}}_{K} - \boldsymbol{\theta}^{*}\right\|_{1} \lesssim \sqrt{s}\left(\frac{C_{N}s\log d}{N}\right)^{1/2}$$

Proof. First, let's consider the case when k = 1. For each $(X_i, Z_i) \in D_1$, we have

$$\mathbb{P}(R_i = 1) = c_{n,1}, \ \mathbb{P}(R_i = 0) = 1 - c_{n,1}.$$

and $N_1 = nc_{n,1}/K$. According to Theorem 1, by selecting $\delta_1 = c_1 \left(\frac{s \log d}{N_1}\right)^{1/(2\beta+1)}$ and $\lambda_1 = c_2 \sqrt{\frac{N_1 K^2 \log d}{n^2 \delta_1}}$, with probability greater than $1 - 2d^{-1}$, we obtain

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_2 \lesssim \left(\frac{s\log d}{N_1}\right)^{\beta/(2\beta+1)},\tag{A.64}$$

$$\|\widehat{\boldsymbol{\theta}}_1 - \boldsymbol{\theta}^*\|_1 \lesssim \sqrt{s} \left(\frac{s\log d}{N_1}\right)^{\beta/(2\beta+1)}.$$
(A.65)

Next, we will establish the bound for $\|\widehat{\theta}_K - \theta^*\|_1$. The result for $\|\widehat{\theta}_K - \theta^*\|_2$ follows similarly. Following the same proof as in Theorem 6, we can show that $\mathbb{P}((X, \mathbb{Z}) \in S_k) \simeq b_{k-1}$. Note that $N_k = n\mathbb{E}(R_i)/K = nc_{n,k}\mathbb{P}((X, \mathbb{Z}) \in S_k)/K$. Applying Theorem 1, we select

$$\delta_k \asymp \left(\frac{Ks\log d}{nc_{n,k}}\right)^{1/(2\beta+1)} \asymp \left(\frac{s\log d\mathbb{P}\left((X, \mathbf{Z}) \in S_k\right)}{N_k}\right)^{1/(2\beta+1)} \asymp \left(\frac{b_{k-1}s\log d}{N_k}\right)^{1/(2\beta+1)}, \quad (A.66)$$

and $\lambda_k = c_{k,2} \sqrt{\frac{N_k K^2 \log d}{n^2 b_{k-1} \delta_k}}$, to ensure, with probability greater than $1 - 2d^{-1}$

$$\left\|\widehat{\boldsymbol{\theta}}_{k}-\boldsymbol{\theta}^{*}\right\|_{1} \lesssim \sqrt{s} \left(\frac{\mathbb{P}\left((X,\boldsymbol{Z})\in S_{k}\right)s\log d}{N_{k}}\right)^{\beta/(2\beta+1)} \lesssim \sqrt{s} \left(\frac{b_{k-1}s\log d}{N_{k}}\right)^{\beta/(2\beta+1)}.$$
 (A.67)

It's important to note that for Proposition A.2 to hold which enables us to apply Theorem 1, we need to choose b_{k-1} such that

$$b_{k-1} \ge C_N \delta_k$$
 and $b_{k-1} \ge 2 \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n.$ (A.68)

To meet the first condition and by (A.66), we have $\left(\frac{C_N^{2\beta+1}s\log d}{N_k}\right)^{1/(2\beta)} = O(b_{k-1})$. Therefore, we choose $b_{k-1} = c_3 \left(\frac{C_N^{2\beta+1}s\log d}{N_k}\right)^{1/(2\beta)}$ for some constant c_3 . Substituting this into (A.67), we get

$$\left\|\widehat{\boldsymbol{\theta}}_{k} - \boldsymbol{\theta}^{*}\right\|_{1} \lesssim \sqrt{s} \left(\frac{C_{N} s \log d}{N_{k}}\right)^{1/2},\tag{A.69}$$

which implies $\left\|\widehat{\theta}_{K} - \theta^{*}\right\|_{1} \lesssim \sqrt{s} \left(\frac{C_{N}s \log d}{N}\right)^{1/2}$, and $\left\|\widehat{\theta}_{K} - \theta^{*}\right\|_{2} \lesssim \left(\frac{C_{N}s \log d}{N}\right)^{1/2}$ follows similarly. Now, let's explore the assumption (A.63). By definition, we have $N_{k} = \sum_{(X_{i}, \mathbf{Z}_{i}) \in D_{k}} \mathbb{E}(R_{i}) =$

 $n\mathbb{E}(R_i)/K = nc_{n,k}\mathbb{P}\left((X, \mathbf{Z}) \in S_k\right)/K$, where $0 < c_{n,k} < 1$ is defined as

$$\mathbb{P}(R_i = 1 \mid Y_i, X_i, \mathbf{Z}_i, \widehat{\boldsymbol{\theta}}_{k-1}) = \mathbb{P}(R_i = 1 \mid X_i, \mathbf{Z}_i, \widehat{\boldsymbol{\theta}}_{k-1}) = c_{n,k} \cdot \mathbb{1}\{(X_i, \mathbf{Z}_i) \in S_k\}.$$

To ensure that $0 < c_{n,k} \leq 1$, we require

$$N_k K \leq n \mathbb{P}\left((X, \mathbf{Z}) \in S_k\right), \ 2 \leq k \leq K,$$

and it suffices to ensure that $N_k K \leq C b_{k-1} n$, where $b_{k-1} = c_3 \left(\frac{C_N^{2\beta+1} s \log d}{N_k}\right)^{1/(2\beta)}$. Subsequently, some calculation yields $N_k \leq C (n/K)^{2\beta/(2\beta+1)} (s \log d)^{1/(2\beta+1)} C_N$ for some constant C, which is provided in (A.63).

Next we check that the second condition in (A.68), i.e., $b_{k-1} \ge 2 \|\widehat{\theta}_{k-1} - \theta^*\|_1 M_n$ holds for the chosen b_{k-1} and N_1, \dots, N_K . Recall that we choose $N_K = N/2$, $N_k = (tN_{k+1})^{1/\beta}$ for $2 \le k \le K-1$, $N_1 = \left(\frac{tN_2}{(s\log d)^{\beta/(2\beta+1)}C_N^\beta}\right)^{(2\beta+1)/2\beta^2}$, where $t = s^{2\beta-1}M_n^{2\beta}(\log d)^{\beta-1}/C_N^{\beta+1}$. When k = 2, by (A.65) it suffices to show that

$$N_{2} = O\left(\frac{C_{N}^{2\beta+1}\log d}{s^{\beta-1}M_{n}^{2\beta}}\left(\frac{N_{1}}{s\log d}\right)^{2\beta^{2}/(2\beta+1)}\right),$$
(A.70)

and when $2 \le k \le K$, by (A.69) we need to ensure that

$$N_k = O\left(\frac{C_N^{\beta+1}\log d}{s^{\beta-1}M_n^{2\beta}} \left(\frac{N_{k-1}}{s\log d}\right)^{\beta}\right).$$
(A.71)

It is easy to check that the choice of N_k above satisfies both (A.70) and (A.71).

Lastly, we demonstrate that the selected values N_1, \dots, N_K satisfy the condition $\sum_{k=1}^K N_k \leq N$. Our objective is to prove that for all $0 \leq j \leq K-2$,

$$N_{K-j} \le \frac{N}{2^{j+1}},$$
 (A.72)

and $N_1 \leq \frac{N}{2^{K-1}}$. These inequalities together imply that $\sum_{j=1}^{K} N_j \leq \frac{N}{2^{K-1}} + \sum_{j=1}^{K-1} \frac{N}{2^j} = N$. Given $N_K = N/2, N_k = (tN_{k+1})^{1/\beta}$ for $2 \leq k \leq K-1$, we can derive

$$N_{K-j} = t^{\sum_{i=1}^{j} 1/\beta^{j}} \left(\frac{N}{2}\right)^{1/\beta^{j}} = t^{\frac{1-1/\beta^{j}}{\beta-1}} \left(\frac{N}{2}\right)^{1/\beta^{j}}$$

Thus to show (A.72), it suffices to prove $t^{\frac{1-1/\beta^j}{\beta-1}} \left(\frac{N}{2}\right)^{1/\beta^j} \leq \frac{N}{2^{j+1}}$ for all $0 \leq j \leq K-2$. Some calculations yield

$$N \ge t^{\frac{1}{\beta-1}} 2^{\frac{(j+1)\beta^j - 1}{\beta^j - 1}}.$$

Noting that $\frac{(j+1)\beta^j-1}{\beta^j-1} = j+1+\frac{j}{\beta^j-1} \leq K$, we have $N \geq t^{\frac{1}{\beta-1}}2^K$ in (A.62), ensuring the result. To show that $N_1 \leq \frac{N}{2^{K-1}}$, first note that $N_1 = \left(\frac{tN_2}{(s\log d)^{\beta/(2\beta+1)}C_N^{\beta}}\right)^{(2\beta+1)/2\beta^2}$ and $N_2 = t^{\frac{1-1/\beta^{(K-2)}}{\beta-1}} \left(\frac{N}{2}\right)^{1/\beta^{(K-2)}}$. We need to verify that

$$\left(\frac{t^{\frac{\beta-1/\beta^{(K-2)}}{\beta-1}}\left(\frac{N}{2}\right)^{1/\beta^{(K-2)}}}{(s\log d)^{\beta/(2\beta+1)}C_N^{\beta}}\right)^{\frac{2\beta+1}{2\beta^2}} = \left(\frac{t^{\frac{\beta-1/\beta^{(K-2)}}{\beta-1}}}{(s\log d)^{\beta/(2\beta+1)}C_N^{\beta}}\right)^{\frac{2\beta+1}{2\beta^2}} \left(\frac{N}{2}\right)^{\frac{2\beta+1}{2\beta^K}} \le \frac{N}{2^{K-1}}$$

which implies $N \geq 2^{\frac{2(K-1)\beta^K - (2\beta+1)}{2\beta^K - (2\beta+1)}} \left(\frac{t^{\frac{\beta-1/\beta^{(K-2)}}{\beta-1}}}{(s\log d)^{\beta/(2\beta+1)}C_N^{\beta}}\right)^{\frac{\beta^{K-2}(2\beta+1)}{2\beta^K - (2\beta+1)}}$, and this is provided by the second part of (A.62).

A.7 Justification of Assumption 3.5

In this section, we verify that the restricted strong convexity and restricted smoothness condition hold w.h.p for the class of conditional mean model. For simplicity, we assume $Y \sim \text{Uniform}(\{-1,1\})$ and $\gamma(y) = 1/\mathbb{P}(Y = y) = 2$ is known. The model is defined as

$$X = \boldsymbol{\theta}^{*T} \boldsymbol{Z} + \mu Y + u,$$

where we assume $\mathbf{Z} \in \mathbb{R}^d$ is a zero-mean sub-Gaussian vector with parameter σ^2 , $\mathbf{Z} \perp u$, and $\mu > 0$.

Recall that in Theorem 2, we showed that with high probability, $\|\widehat{\theta}_1 - \theta^*\|_2 \lesssim \left(\frac{s \log d}{N}\right)^{\beta/(2\beta+1)}$ and $\|\widehat{\theta}_k - \theta^*\|_2 \lesssim \left(\frac{s \log d}{N}\right)^{1/2}$ for $2 \le k \le K$. For k = 1 when data are uniformly sampled from D_1 , Feng et al. (2022) have shown that under some regularity conditions, Assumption 3.5 holds for the conditional mean model specified above with high probability. Therefore, it suffices to verify Assumption 3.5 on the set $\Omega = \{\theta : \|\theta - \widehat{\theta}_{k-1}\|_2 \lesssim \left(\frac{s \log d}{N}\right)^{\beta/(2\beta+1)}\}$ at the *k*th iteration for any $2 \le k \le K$. We apply a proper kernel function K satisfying (i) K has bounded support on [-1, 1], (ii) $\|K\|_{\infty}, \|K'\|_{\infty}, \|K''\|_{\infty}$ and $\widetilde{K} = -\int K'(t)tdt > 0$ are bounded above by universal constants. To verify that Assumption 3.5 holds w.h.p, it suffices to show that the following sparse eigenvalue condition holds w.h.p:

$$\rho_{\max} = \sup\left\{ \boldsymbol{v}^T \nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}) \boldsymbol{v} : \|\boldsymbol{v}\|_2 = 1, \|\boldsymbol{v}\|_0 \le Cs, \boldsymbol{\theta} \in \Omega, \|\boldsymbol{\theta}\|_0 \le Cs \right\} < C_1 c_{n,k},$$
(A.73)

$$\rho_{\min} = \inf\left\{ \boldsymbol{v}^T \nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}) \boldsymbol{v} : \|\boldsymbol{v}\|_2 = 1, \|\boldsymbol{v}\|_0 \le Cs, \boldsymbol{\theta} \in \Omega, \|\boldsymbol{\theta}\|_0 \le Cs \right\} > C_2 c_{n,k},$$
(A.74)

for some constants $C_1, C_2 > 0$. We let $\Sigma_{\mathbf{Z}} = \mathbf{Cov}(\mathbf{Z})$ and $g(\cdot)$ be the p.d.f of the error u. Denote $\nabla^2 R_{\delta_k}(\boldsymbol{\theta}) = \mathbb{E}\left[\nabla^2 R_{\delta_k}^{D_k}(\boldsymbol{\theta}) | \widehat{\boldsymbol{\theta}}_{k-1}\right]$ as the population Hessian.

Proposition A.4. Under the setup above, suppose $\tilde{g'} = \min_{u \in \left[\frac{-3\mu}{2}, \frac{-\mu}{2}\right]} g'(u) > 0$. If $M_n^2 \lesssim \sqrt{\frac{n \min_{1 \le k \le K} \delta_k c_{n,k}}{K \log d}}$,

$$s^{\frac{2\beta-1}{\beta-1}}\log d = o(N),$$
 (A.75)

and

$$s\frac{M_n^3\sqrt{s}}{\delta_k^3} \left(\frac{s\log d}{N}\right)^{\beta/(2\beta+1)} = o\left(1\right),\tag{A.76}$$

then with probability greater than $1 - 4d^{-1}$, it holds that

$$\rho_{\min} \ge c_{n,k} \widetilde{g'} \widetilde{K} \lambda_{\min} \left(\boldsymbol{\Sigma}_{\boldsymbol{Z}} \right), \quad \rho_{\max} \le 3 c_{n,k} \widetilde{K} \left\| g' \right\|_{\infty} \lambda_{\max} \left(\boldsymbol{\Sigma}_{\boldsymbol{Z}} \right).$$

Proof. Write

$$\boldsymbol{v}^{T}\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta})\boldsymbol{v} = \boldsymbol{v}^{T}\nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*})\boldsymbol{v} + \boldsymbol{v}^{T}(\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}) - \nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*}))\boldsymbol{v} + \boldsymbol{v}^{T}(\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}) - \nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}))\boldsymbol{v}.$$
(A.77)

By Lemma A.10 we have

$$\boldsymbol{v}^{T}\nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*})\boldsymbol{v} \leq 2c_{n,k}\widetilde{K}\left\|\boldsymbol{g}'\right\|_{\infty}\lambda_{\max}\left(\boldsymbol{\Sigma}_{\boldsymbol{Z}}\right), \ \boldsymbol{v}^{T}\nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*})\boldsymbol{v} \geq 2c_{n,k}\widetilde{\boldsymbol{g}'}\widetilde{K}\lambda_{\min}\left(\boldsymbol{\Sigma}_{\boldsymbol{Z}}\right).$$
(A.78)

By Lemma A.12 we have with probability greater than $1 - d^{-1}$,

$$\|\nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*)\|_{\infty} \le C_2 \sqrt{\frac{c_{n,k} K \log d}{n\delta_k^3}}$$

Therefore, for any \boldsymbol{v} such that $\|\boldsymbol{v}\|_2 = 1, \|\boldsymbol{v}\|_0 \leq Cs$, with probability greater than $1 - d^{-1}$,

$$\boldsymbol{v}^{T}(\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}) - \nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*}))\boldsymbol{v} = \sum_{j=1}^{d}\sum_{k=1}^{d}v_{j}(\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}) - \nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*}))_{jk}v_{k}$$
$$\leq \|\nabla^{2}R_{\delta_{k}}^{D_{k}}(\boldsymbol{\theta}^{*}) - \nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*})\|_{\infty}\|\boldsymbol{v}\|_{1}^{2} \leq C_{1}s\sqrt{\frac{c_{n,k}K\log d}{n\delta_{k}^{3}}} \quad (A.79)$$

for some constant $C_1 > 0$. Recall that $c_{n,k} \asymp \frac{N}{nb_{k-1}}$, and by Theorem 2 we have

$$\delta_k \asymp \left(\frac{s\log d}{N}\right)^{1/(2\beta)}, \ b_{k-1} \asymp \left(\frac{s\log d}{N}\right)^{1/(2\beta)},$$

then by (A.75) we can show that

$$\boldsymbol{v}^{T}(\nabla^{2}R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}^{*}) - \nabla^{2}R_{\delta_{k}}(\boldsymbol{\theta}^{*}))\boldsymbol{v} = O\left(s\sqrt{\frac{c_{n,k}K\log d}{n\delta_{k}^{3}}}\right) = o(c_{n,k}).$$
(A.80)

Recall that with probability greater than $1 - 2d^{-1}$, $\|\widehat{\theta}_1 - \theta^*\|_1 \lesssim \sqrt{s} \left(\frac{s\log d}{N}\right)^{\beta/(2\beta+1)}$. Hence uniformly over $\theta \in \Omega = \{\theta : \|\theta - \widehat{\theta}_{k-1}\|_2 \lesssim \left(\frac{s\log d}{N}\right)^{\beta/(2\beta+1)}\}$ and $\|\theta\|_0 \leq Cs$, we have with probability greater than $1 - 2d^{-1}$,

$$\begin{split} \left| K'(\frac{y_i(x_i - \boldsymbol{\theta}^T \boldsymbol{z}_i)}{\delta_k}) - K'(\frac{y_i(x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i)}{\delta_k}) \right| &= \left| \frac{(\boldsymbol{\theta}^* - \boldsymbol{\theta})^T \boldsymbol{z}_i}{\delta_k} K''(\frac{y_i(x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i + \kappa(\boldsymbol{\theta}^* - \boldsymbol{\theta})^T \boldsymbol{z}_i)}{\delta_k}) \right| \\ &\leq \frac{\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|_1 M_n}{\delta_k} |K''_{\max}| \\ &\leq \frac{\left(\|\boldsymbol{\theta} - \boldsymbol{\theta}_{k-1}\|_1 + \|\boldsymbol{\theta}_{k-1} - \boldsymbol{\theta}^*\|_1\right) M_n}{\delta_k} |K''_{\max}| \\ &\lesssim \frac{M_n}{\delta_k} \sqrt{s} \left(\frac{s \log d}{N}\right)^{\beta/(2\beta+1)} \end{split}$$

where $0 < \kappa < 1$. Then note that $\mathbb{E}R_i = \mathbb{E}(c_{n,k} \mathbb{1}\{(X_i, \mathbf{Z}_i) \in S_k\}) \leq c_{n,k}, |R_i - \mathbb{E}R_i| \leq 1, \mathbb{E}R_i^2 \leq c_{n,k}$, by Bernstein inequality we have

$$\mathbb{P}\left(\left\|\frac{K}{n}\sum_{(X_i, \mathbf{Z}_i)\in D_k} R_i - \mathbb{E}R_i\right\|_{\infty} > t\right) \le \exp\left(-\frac{\frac{1}{2}t^2n/K}{c_{n,k} + \frac{t}{3}}\right).$$

Note that $c_{n,k} = \frac{N}{n\mathbb{P}((X, \mathbb{Z}) \in S_k)} \geq \frac{K}{n}$, hence take $t = C\sqrt{\frac{Kc_{n,k}}{n}}$ for some constant C, and with probability greater than $1 - d^{-1}$,

$$\left\|\frac{K}{n}\sum_{(x_i,\boldsymbol{z}_i)\in D_k}R_i - \mathbb{E}R_i\right\|_{\infty} \lesssim c_{n,k}.$$

Therefore, with probability greater than $1 - 3d^{-1}$,

$$\begin{split} \sup_{\boldsymbol{\theta}\in\Omega} \|\nabla^2 R_{\delta_k}^{D_k}(\boldsymbol{\theta}) - \nabla^2 R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*)\|_{\infty} \\ &= \sup_{\boldsymbol{\theta}\in\Omega} \left\| \frac{K}{n} \sum_{(x_i, \boldsymbol{z}_i)\in D_k} \gamma(y_i) \frac{\boldsymbol{z}_i \boldsymbol{z}_i^T}{\delta_k^2} \left(K'(\frac{y_i(x_i - \boldsymbol{\theta}^T \boldsymbol{z}_i)}{\delta_k}) - K'(\frac{y_i(x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i)}{\delta_k}) \right) R_i \right\|_{\infty} \\ &\lesssim \frac{M_n^2}{\delta_k^2} \cdot \frac{M_n}{\delta_k} \sqrt{s} \left(\frac{s \log d}{N} \right)^{\beta/(2\beta+1)} \left\| \frac{K}{n} \sum_{(x_i, \boldsymbol{z}_i)\in D_k} R_i \right\|_{\infty} \\ &= O\left(\frac{M_n^3 \sqrt{s}}{\delta_k^3} \left(\frac{s \log d}{N} \right)^{\beta/(2\beta+1)} c_{n,k} \right), \end{split}$$

and by (A.76) we have

$$\boldsymbol{v}^{T}(\nabla^{2}R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}) - \nabla^{2}R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}^{*}))\boldsymbol{v} \leq \|\nabla^{2}R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}) - \nabla^{2}R^{D_{k}}_{\delta_{k}}(\boldsymbol{\theta}^{*})\|_{\infty}\|\boldsymbol{v}\|_{1}^{2}$$
$$\leq C_{2}s\frac{M^{3}_{n}\sqrt{s}}{\delta^{3}_{k}}\left(\frac{s\log d}{N}\right)^{\beta/(2\beta+1)}c_{n,k} = o(c_{n,k}), \quad (A.81)$$

where $C_2 > 0$ is some constant. Combining (A.78), (A.80), (A.81) and giving them back to (A.77) we finish the proof.

By definition

$$\nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}) = -\frac{K}{n} \sum_{(x_i, \boldsymbol{z}_i) \in D_k} \gamma(y_i) \frac{y_i^2 \boldsymbol{z}_i \boldsymbol{z}_i^T}{\delta_k^2} K'(\frac{y_i(x_i - \boldsymbol{\theta}^T \boldsymbol{z}_i)}{\delta_k}) R_i$$

Denoting $\frac{-b_{k-1}\|\widehat{w}_{k-1}\|_2 + (\widehat{\theta}_{k-1} - \theta)^T z}{\delta_k} := S_u^-$ and $\frac{b_{k-1}\|\widehat{w}_{k-1}\|_2 + (\widehat{\theta}_{k-1} - \theta)^T z}{\delta_k} := S_u^+$, then we have

$$\begin{aligned} \boldsymbol{v}^{T} \nabla^{2} R_{\delta_{k}}(\boldsymbol{\theta}) \boldsymbol{v} \\ &= -\frac{c_{n,k}}{\delta_{k}^{2}} \int_{\boldsymbol{z}} \int_{-b_{k-1} \|\widehat{\boldsymbol{w}}_{k-1}\|_{2} + \widehat{\boldsymbol{\theta}}_{k-1}^{T} \boldsymbol{z}} \left(\boldsymbol{v}^{T} \boldsymbol{z} \right)^{2} \\ &\cdot \left(K' \left(\frac{x - \boldsymbol{\theta}^{T} \boldsymbol{z}}{\delta_{k}} \right) g \left(x - \boldsymbol{\theta}^{*T} \boldsymbol{z} - \boldsymbol{\mu} \right) + K' \left(\frac{-(x - \boldsymbol{\theta}^{T} \boldsymbol{z})}{\delta_{k}} \right) g \left(x - \boldsymbol{\theta}^{*T} \boldsymbol{z} + \boldsymbol{\mu} \right) \right) f(\boldsymbol{z}) dx d\boldsymbol{z} \\ &= -\frac{c_{n,k}}{\delta_{k}} \int_{\boldsymbol{z}} (\boldsymbol{v}^{T} \boldsymbol{z})^{2} \int_{S_{u}^{-}}^{S_{u}^{+}} \left(K'(t) g \left(\delta_{k} t + \boldsymbol{\theta}^{T} \boldsymbol{z} - \boldsymbol{\theta}^{*T} \boldsymbol{z} - \boldsymbol{\mu} \right) + K'(-t) g \left(\delta_{k} t + \boldsymbol{\theta}^{T} \boldsymbol{z} - \boldsymbol{\theta}^{*T} \boldsymbol{z} + \boldsymbol{\mu} \right) \right) f(\boldsymbol{z}) dt d\boldsymbol{z} \end{aligned}$$

Note that

$$g\left(\delta_{k}t + \boldsymbol{\theta}^{T}\boldsymbol{z} - \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right) = g\left(\boldsymbol{\theta}^{T}\boldsymbol{z} - \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right) + \delta_{k}tg'\left(\kappa\delta_{k}t + \boldsymbol{\theta}^{T}\boldsymbol{z} - \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right),$$

and

$$g\left(\delta_{k}t + \boldsymbol{\theta}^{T}\boldsymbol{z} - \boldsymbol{\theta}^{*T}\boldsymbol{z} + \mu\right) = g\left(-\delta_{k}t - \boldsymbol{\theta}^{T}\boldsymbol{z} + \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right)$$
$$= g\left(-\boldsymbol{\theta}^{T}\boldsymbol{z} + \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right) - \delta_{k}tg'\left(-\kappa'\delta_{k}t - \boldsymbol{\theta}^{T}\boldsymbol{z} + \boldsymbol{\theta}^{*T}\boldsymbol{z} - \mu\right),$$

where $0 < \kappa, \kappa' < 1$. We choose K(t) such that K'(t) is an odd function, then we have

$$\boldsymbol{v}^T \nabla^2 R_{\delta_k}(\boldsymbol{\theta}) \boldsymbol{v} \tag{A.82}$$

$$= -\frac{c_{n,k}}{\delta_k} \int_{\boldsymbol{z}} (\boldsymbol{v}^T \boldsymbol{z})^2 \int_{S_u^-}^{S_u^-} K'(t) dt \left(g \left(\boldsymbol{\theta}^T \boldsymbol{z} - \boldsymbol{\theta}^{*T} \boldsymbol{z} - \boldsymbol{\mu} \right) - g \left(-\boldsymbol{\theta}^T \boldsymbol{z} + \boldsymbol{\theta}^{*T} \boldsymbol{z} - \boldsymbol{\mu} \right) \right) f(\boldsymbol{z}) d\boldsymbol{z}$$
(A.83)

$$-c_{n,k} \int_{\boldsymbol{z}} (\boldsymbol{v}^T \boldsymbol{z})^2 \int_{S_u^-}^{S_u^+} K'(t) t \left(g' \left(\kappa \delta_k t + \boldsymbol{\theta}^T \boldsymbol{z} - \boldsymbol{\theta}^{*T} \boldsymbol{z} - \mu \right) + g' \left(-\kappa' \delta_k t - \boldsymbol{\theta}^T \boldsymbol{z} + \boldsymbol{\theta}^{*T} \boldsymbol{z} - \mu \right) \right) dt f(\boldsymbol{z}) d\boldsymbol{z}.$$
(A.84)

The following lemma shows that $\nabla^2 R_{\delta_k}(\boldsymbol{\theta^*})$ satisfies the sparse eigenvalue condition.

Lemma A.10. Suppose $\widetilde{K} = -\int K'(t)tdt > 0$, $\widetilde{g'} = \min_{u \in \left[\frac{-3\mu}{2}, \frac{-\mu}{2}\right]} g'(u) > 0$, then for all unit vector $\boldsymbol{v} \in \mathbb{R}^d$ we have

$$\boldsymbol{v}^T \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*) \boldsymbol{v} \leq 2c_{n,k} \widetilde{K} \left\| g' \right\|_{\infty} \lambda_{\max}\left(\boldsymbol{\Sigma}_{\boldsymbol{Z}}\right), \ \boldsymbol{v}^T \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*) \boldsymbol{v} \geq 2c_{n,k} \widetilde{g'} \widetilde{K} \lambda_{\min}\left(\boldsymbol{\Sigma}_{\boldsymbol{Z}}\right).$$

Proof. Denote $\frac{-b_{k-1}\|\widehat{w}_{k-1}\|_2 + (\widehat{\theta}_{k-1} - \theta^*)^T z}{\delta_k} := S_u^{*-}$ and $\frac{b_{k-1}\|\widehat{w}_{k-1}\|_2 + (\widehat{\theta}_{k-1} - \theta^*)^T z}{\delta_k} := S_u^{*+}$. Recall that in Proposition A.2 we showed that

$$S_u^{*+} \ge \frac{b_{k-1} \|\widehat{\boldsymbol{w}}_{k-1}\|_2 - \|\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^*\|_1 M_n}{\delta_k} \ge C_N/2,$$

and similarly,

$$S_u^{*-} \le -C_N/2.$$

By (A.82) we have

$$\boldsymbol{v}^{T} \nabla^{2} R_{\delta_{k}}(\boldsymbol{\theta}^{*}) \boldsymbol{v}$$

= $-c_{n,k} \int_{\boldsymbol{z}} (\boldsymbol{v}^{T} \boldsymbol{z})^{2} \int_{S_{u}^{*^{-}}}^{S_{u}^{*^{+}}} K'(t) t \left(g'(\kappa \delta_{k} t - \mu) + g'(-\kappa' \delta_{k} t - \mu)\right) dt f(\boldsymbol{z}) d\boldsymbol{z},$

where $0 < \kappa, \kappa' < 1$. Note that $\widetilde{K} = -\int K'(t)tdt > 0$, g' and \widetilde{K} are bounded, we obtain that $\boldsymbol{v}^T \nabla^2 R_{\delta}(\boldsymbol{\theta}^*) \boldsymbol{v} \leq 2c_{n,k} \widetilde{K} \|g'\|_{\infty} \lambda_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{Z}})$. By the choice of the kernel function we can ensure that

$$\kappa \delta_k t - \mu, -\kappa' \delta_k t - \mu \in [\frac{-3\mu}{2}, \frac{-\mu}{2}],$$

therefore, $\boldsymbol{v}^T \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*) \boldsymbol{v} \geq 2c_{n,k} \widetilde{g'} \widetilde{K} \lambda_{\min}(\boldsymbol{\Sigma}_{\boldsymbol{Z}})$.

Lemma A.11. For any $(X, \mathbb{Z}, Y) \in D_k$, $2 \le k \le K$, for all $j, k = 1 \cdots, d$,

$$\mathbb{E}\left[\left(\gamma(Y)\frac{Z_j Z_k}{\delta_k^2} K'(\frac{Y(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z})}{\delta_k})R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}\right] \le C\frac{c_{n,k}}{\delta_k^3},$$

for some constant C > 0.

Proof. We have

$$\mathbb{E}\left[\left(\gamma(Y)\frac{Z_{j}Z_{k}}{\delta_{k}^{2}}K'\left(\frac{Y(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z})}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1}\right]$$
$$=\mathbb{E}\left[\mathbb{E}\left[\left(\gamma(Y)\frac{Z_{j}Z_{k}}{\delta_{k}^{2}}K'\left(\frac{Y(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z})}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y\right]\mid\widehat{\boldsymbol{\theta}}_{k-1}\right]$$
$$=\mathbb{E}\left[\left(\frac{Z_{j}Z_{k}}{\delta_{k}^{2}}K'\left(\frac{X-\boldsymbol{\theta}^{*T}\boldsymbol{Z}}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=1\right]+\mathbb{E}\left[\left(\frac{Z_{j}Z_{k}}{\delta_{k}^{2}}K'\left(\frac{-(X-\boldsymbol{\theta}^{*T}\boldsymbol{Z})}{\delta_{k}}\right)R\right)^{2}\mid\widehat{\boldsymbol{\theta}}_{k-1},Y=-1\right]$$

We bound the first term here and the second term follows similarly. Note that

$$\mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right) R\right)^2 | \, \widehat{\boldsymbol{\theta}}_{k-1}, Y = 1\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right) R\right)^2 | \, \widehat{\boldsymbol{\theta}}_{k-1}, Y = 1, X, \boldsymbol{Z}\right] | \, \widehat{\boldsymbol{\theta}}_{k-1}, Y = 1\right].$$

Recall that

$$S_k := \left\{ (X, \mathbf{Z}) : -b_{k-1} \le \frac{X - \widehat{\boldsymbol{\theta}}_{k-1}^T \mathbf{Z}}{\|\widehat{\boldsymbol{\omega}}_{k-1}\|_2} \le b_{k-1} \right\},\$$

and

$$\mathbb{P}(R=1 \mid X, \boldsymbol{Z}, \widehat{\boldsymbol{\theta}}_{k-1}) = c_{n,k} \cdot \mathbb{1}\{(X, \boldsymbol{Z}) \in S_k\},\$$

we have

$$\mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right) R\right)^2 | \, \widehat{\boldsymbol{\theta}}_{k-1}, Y = 1, X, \boldsymbol{Z}\right]$$
$$= c_{n,k} \left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right)\right)^2 \cdot \mathbb{1}\{(X, \boldsymbol{Z}) \in S_k\},$$

where we use the fact that $R \perp Y \mid (X, \mathbf{Z}, \widehat{\theta}_{k-1})$. Hence

$$\begin{split} & \mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X-\boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right) R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1\right] \\ = & c_{n,k} \mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X-\boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right)\right)^2 \cdot \mathbb{1}\{(X, \boldsymbol{Z}) \in S_k\} \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y=1\right] \\ = & c_{n,k} \int_{\boldsymbol{z}} \frac{z_j^2 z_k^2}{\delta_k^4} \int_{-b_{k-1} \parallel \widehat{\boldsymbol{w}}_{k-1} \parallel_2 + \widehat{\boldsymbol{\theta}}_{k-1}^T \boldsymbol{z}}{\delta_{k-1} - b_{k-1} \parallel \widehat{\boldsymbol{w}}_{k-1} \parallel_2 + \widehat{\boldsymbol{\theta}}_{k-1}^T \boldsymbol{z}} K'^2\left(\frac{x-\boldsymbol{\theta}^{*T} \boldsymbol{z}}{\delta_k}\right) f(x \mid \boldsymbol{z}, Y=1) dx f(\boldsymbol{z} \mid Y=1) d\boldsymbol{z} \\ = & \frac{c_{n,k}}{\delta_k^3} \int_{\boldsymbol{z}} \int_{(-b_{k-1} \parallel \widehat{\boldsymbol{w}}_{k-1} \parallel_2 + (\widehat{\boldsymbol{\theta}}_{k-1} - \boldsymbol{\theta}^{*})^T \boldsymbol{z}) / \delta_k} z_j^2 z_k^2 K'^2(\boldsymbol{u}) f(\boldsymbol{u}\delta_k + \boldsymbol{\theta}^{*T} \boldsymbol{z} \mid \boldsymbol{z}, Y=1) du f(\boldsymbol{z} \mid Y=1) d\boldsymbol{z}. \end{split}$$

Since $\sup_{x \in \mathbb{R}, y \in \{-1,1\}, \mathbf{z} \in \mathbb{R}^d} f(x \mid y, \mathbf{z}) < p_{\max} < \infty$, we have

$$\int_{(-b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+(\widehat{\boldsymbol{\theta}}_{k-1}-\boldsymbol{\theta}^{*})^{T}\boldsymbol{z})/\delta_{k}}^{(b_{k-1}\|\widehat{\boldsymbol{w}}_{k-1}\|_{2}+(\widehat{\boldsymbol{\theta}}_{k-1}-\boldsymbol{\theta}^{*})^{T}\boldsymbol{z})/\delta_{k}}K'^{2}(u)f(u\delta_{k}+\boldsymbol{\theta}^{*T}\boldsymbol{z}\mid\boldsymbol{z},Y=1)du \leq p_{\max}\int K'^{2}(u)du.$$

Note that $\mathbb{E}(Z_j^2 Z_k^2) \leq M_2 < \infty$, we obtain

$$\mathbb{E}\left[\left(\frac{Z_j Z_k}{\delta_k^2} K'\left(\frac{X - \boldsymbol{\theta}^{*T} \boldsymbol{Z}}{\delta_k}\right) R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}, Y = 1\right] \le \frac{c_{n,k}}{\delta_k^3} M_2 p_{\max} \int K'^2(u) du,$$

hence we finish the proof.

Lemma A.12. Suppose $M_n^2 \lesssim \sqrt{\frac{n \min_{1 \le k \le K} \delta_k c_{n,k}}{K \log d}}$, then with probability greater than $1 - d^{-1}$, we have

$$\|\nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*)\|_{\infty} \le C_2 \sqrt{\frac{c_{n,k} K \log d}{n\delta_k^3}},$$

where C_2 is a constant independent of n, d.

Proof. Denote $T = \|\nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*)\|_{\infty}$. By definition,

$$\|T\|_{\infty} = \left\| \frac{K}{n} \sum_{(x_i, \boldsymbol{z}_i) \in D_k} \gamma(y_i) \frac{\boldsymbol{z}_i \boldsymbol{z}_i^T}{\delta_k^2} K'(\frac{y_i(x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i)}{\delta_k}) R_i - \mathbb{E} \left[\gamma(Y) \frac{\boldsymbol{Z} \boldsymbol{Z}^T}{\delta_k^2} K'(\frac{Y(X - \boldsymbol{\theta}^{*T} \boldsymbol{Z})}{\delta_k}) R \mid \widehat{\boldsymbol{\theta}}_{k-1} \right] \right\|_{\infty}$$

We have

$$\begin{split} T_{ijk} &= \left| \gamma(y_i) \frac{z_{ij} z_{ik}}{\delta_k^2} K'(\frac{y_i(x_i - \boldsymbol{\theta}^{*T} \boldsymbol{z}_i)}{\delta_k}) R_i - \mathbb{E} \left[\gamma(Y) \frac{Z_j Z_k}{\delta_k^2} K'(\frac{Y(X - \boldsymbol{\theta}^{*T} \boldsymbol{Z})}{\delta_k}) R \mid \widehat{\boldsymbol{\theta}}_{k-1} \right] \right| \\ &\leq \frac{4K'_{\max} M_n^2}{\delta_k^2}, \end{split}$$

and by Lemma A.11 we have

$$E[T_{ijk}^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}] \leq \mathbb{E}\left[\left(\gamma(Y) \frac{Z_j Z_k}{\delta_k^2} K'(\frac{Y(X - \boldsymbol{\theta}^{*T} \boldsymbol{Z})}{\delta_k})R\right)^2 \mid \widehat{\boldsymbol{\theta}}_{k-1}\right] \leq C \frac{c_{n,k}}{\delta_k^3}$$

for some constant C > 0. Then by Bernstein inequality we have

$$\mathbb{P}\left(\|T\|_{\infty} > t \mid \widehat{\boldsymbol{\theta}}_{k-1}\right) \leq \sum_{j=1}^{d} \sum_{k=1}^{d} \mathbb{P}\left(|T_{jk}| > t \mid \widehat{\boldsymbol{\theta}}_{k-1}\right)$$
$$\leq 2d^{2} \exp\left(-\frac{\frac{1}{2}t^{2}n/K}{C\frac{c_{n,k}}{\delta_{k}^{3}} + \frac{4t}{3}M_{n}^{2}K_{\max}'/\delta_{k}^{2}}\right).$$

Since the right side doesn't contain $\widehat{\theta}_{k-1}$, we obtain that

$$\mathbb{P}\left(\|\nabla^2 R^{D_k}_{\delta_k}(\boldsymbol{\theta}^*) - \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*)\|_{\infty} > t\right) \le 2d^2 \exp\left(-\frac{\frac{1}{2}t^2 n/K}{C\frac{c_{n,k}}{\delta_k^3} + \frac{4t}{3}M_n^2 K_{\max}'/\delta_k^2}\right)$$

Then note that $M_n^2 \lesssim \sqrt{\frac{n \min_{1 \le k \le K} \delta_k c_{n,k}}{K \log d}}$ and take $t = C_2 \sqrt{\frac{c_{n,k} K \log d}{n \delta_k^3}}$ for some constant C_2 we finish the proof.

A.8 Binary Response Model with Diverging $\|\theta^*\|_2$

For the binary response model, we can construct a counterexample that shows that the eigenvalues of the Hessian matrix of $R(\boldsymbol{\theta}^*)$ at the true value $\boldsymbol{\theta}^*$ are of order $1/\|\boldsymbol{\theta}^*\|_2$. When $\|\boldsymbol{\theta}^*\|_2$ is diverging, the eigenvalues of $\nabla^2 R(\boldsymbol{\theta}^*)$ shrink to 0 and thus the RSC condition would fail even if we only consider $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. Thus, the statistical rate presented in Section 3 is generally not applicable to the case with diverging $\|\boldsymbol{\theta}^*\|_2$. In the following, we provide the counterexample.

Recall that the binary response model takes the form $Y = \operatorname{sign}(X - \theta^{*T} \mathbf{Z} + u)$. For simplicity we assume $X \sim N(0, 1)$, $\mathbf{Z} \sim N(0, \mathbf{I})$ independent of X and $\gamma(y) \equiv 1$. In addition, we assume the error u is independent of X (but possibly depends on \mathbf{Z}), and the conditional density given $\mathbf{Z} = \mathbf{z}$, denoted by $f_{u|\mathbf{z}}(\cdot)$, is bounded by a constant for all \mathbf{z} . The median of u given \mathbf{Z} is 0.

By the definition of $R(\boldsymbol{\theta})$, it is easily shown that for any $\|\boldsymbol{v}\|_2 = 1$

$$\boldsymbol{v}^T \nabla^2 R(\boldsymbol{\theta}^*) \boldsymbol{v} = 2 \int (\boldsymbol{z}^T \boldsymbol{v})^2 f_{u|\boldsymbol{z}}(0) \phi(\boldsymbol{\theta}^{*T} \boldsymbol{z}) \phi(z_1) ... \phi(z_n) d\boldsymbol{z}$$

where ϕ is the pdf of N(0,1). Since $f_{u|z}(\cdot)$ is upper bounded by a constant C, we have

$$oldsymbol{v}^T
abla^2 R(oldsymbol{ heta}^*) oldsymbol{v} \leq 2C \int (oldsymbol{z}^T oldsymbol{v})^2 rac{1}{(2\pi)^{1/2}} rac{1}{(2\pi)^{d/2}} \exp(-rac{oldsymbol{z}^T (oldsymbol{I} + oldsymbol{ heta}^* oldsymbol{ heta}^* T) oldsymbol{z}}{2}) doldsymbol{z}$$

$$= 2C rac{oldsymbol{v}^T (oldsymbol{I} + oldsymbol{ heta}^* oldsymbol{ heta}^* T)^{-1} oldsymbol{v}}{(2\pi)^{1/2} |oldsymbol{I} + oldsymbol{ heta}^* oldsymbol{ heta}^* T)^{1/2}},$$

where in the last step we use the integral of a normal distribution with variance $(\mathbf{I} + \boldsymbol{\theta}^* \boldsymbol{\theta}^{*T})^{-1}$. Since the matrix $\mathbf{I} + \boldsymbol{\theta}^* \boldsymbol{\theta}^{*T}$ has d - 1 eigenvalues 1 and 1 eigenvalue $1 + \|\boldsymbol{\theta}^*\|_2^2$, we obtain that

$$v^T \nabla^2 R(\theta^*) v \le rac{2C}{(2\pi)^{1/2} (1 + \|\theta^*\|_2^2)^{1/2}}$$

Thus, when $\|\boldsymbol{\theta}^*\|_2$ is diverging, $\boldsymbol{v}^T \nabla^2 R(\boldsymbol{\theta}^*) \boldsymbol{v}$ is of order $1/\|\boldsymbol{\theta}^*\|_2$. Note that we can make the approximation error $\boldsymbol{v}^T \nabla^2 (R_{\delta_k}(\boldsymbol{\theta}^*) - c_{n,k}R(\boldsymbol{\theta}^*))\boldsymbol{v}$ ignorable with a small bandwidth δ_k and the stochastic error $\boldsymbol{v}^T (\nabla^2 R_{\delta_k}^{D_k}(\boldsymbol{\theta}^*) - \nabla^2 R_{\delta_k}(\boldsymbol{\theta}^*))\boldsymbol{v}$ ignorable via concentration inequality. Thus, we can show that $\boldsymbol{v}^T \nabla^2 R_{\delta_k}^{D_k}(\boldsymbol{\theta})\boldsymbol{v}$ is also of order $c_{n,k}/\|\boldsymbol{\theta}^*\|_2$, violating our RSC condition (even at the true value $\boldsymbol{\theta}^*$) when $\|\boldsymbol{\theta}^*\|_2$ is diverging.

A.9 Remark on $\beta < 1$

In this section, we argue that the density function with $\beta < 1$ in Assumption 3.3 may contradict with the RSM condition in Assumption 3.5. As shown in Feng et al. (2022), to verify the RSM condition under the surrogate loss $R_{\delta}^{n}(\cdot)$, the key step is to verify the population-level RSM condition under the 0-1 loss. For simplicity, we consider the model with no covariate Z, and define $\widetilde{R}(c) = \mathbb{E} \left[\gamma(Y) L_{01} \left(Y \left(X - c \right) \right) \right]$ and $c^* = \operatorname{argmin}_{c} \widetilde{R}(c)$. Under this model, the population-level RSM condition is given by the following: there exits a small neighborhood around c^* such that for any c in this neighborhood it holds that

$$\widetilde{R}(c) - \widetilde{R}(c^*) - (c - c^*)\widetilde{R}'(c^*) \le \rho^+ (c - c^*)^2$$
(A.85)

for some constant $\rho^+ > 0$.

Proposition A.5. Let f(c|y) be the conditional density function of X at c given Y = y. Assume that, for any constant L > 0, there exists some u > 0, such that for any $|\Delta| < u, y \in \{-1, 1\}$,

$$|f(c^* + \Delta | y) - f(c^* | y)| > L|\Delta|.$$
(A.86)

In addition, assume that for any $c \in (c^* - u, c^* + u)$,

$$[f(c|y=1) - f(c^*|y=1)](c-c^*) > 0, \text{ and } [f(c|y=-1) - f(c^*|y=-1)](c-c^*) < 0.$$
(A.87)

Then for any $c \in (c^* - u, c^* + u)$, we have

$$\widetilde{R}(c) - \widetilde{R}(c^*) - (c - c^*)\widetilde{R}'(c^*) > L(c - c^*)^2.$$

Proof. By definition, we have

$$\widetilde{R}(c) = \mathbb{P}(x < c \mid Y = 1) + \mathbb{P}(x > c \mid Y = -1) = \int_{-\infty}^{c} f(x \mid y = 1) dx + \int_{c}^{+\infty} f(x \mid y = -1) dx,$$

and

$$\tilde{R}'(c) = f(c \mid y = 1) - f(c \mid y = -1).$$

For any $c \in (c^* - u, c^* + u)$,

$$\begin{split} \widetilde{R}(c) &- \widetilde{R}(c^*) - (c - c^*) \widetilde{R}'(c^*) \\ &= \int_{c^*}^c f(x \mid y = 1) dx - \int_{c^*}^c f(x \mid y = -1) dx - (c - c^*) \left(f(c^* \mid y = 1) - f(c^* \mid y = -1) \right) \\ &= \int_{c^*}^c [f(x \mid y = 1) - f(c^* \mid y = 1)] dx - \int_{c^*}^c [f(x \mid y = -1) - f(c^* \mid y = -1)] dx. \end{split}$$

Without loss of generality, we assume $c > c^*$. By (A.86) and (A.87), we have

$$\int_{c^*}^{c} [f(x \mid y=1) - f(c^* \mid y=1)] dx > L \int_{c^*}^{c} (x-c^*) dx = \frac{L}{2} (c-c^*)^2,$$

and

$$\int_{c^*}^{c} [f(c^* \mid y = -1) - f(x \mid y = -1)] dx > L \int_{c^*}^{c} (x - c^*) dx = \frac{L}{2} (c - c^*)^2.$$

This implies $\widetilde{R}(c) - \widetilde{R}(c^*) - (c - c^*)\widetilde{R}'(c^*) > L(c - c^*)^2$.

Given this proposition, by taking a sequence $\{L_j : j = 1, 2...\}$ diverging to infinity, we can find a sequence $\{c_j : j = 1, 2...\}$ converging to c^* such that

$$\frac{\widetilde{R}(c_j) - \widetilde{R}(c^*) - (c_j - c^*)\widetilde{R}'(c^*)}{(c_j - c^*)^2} > L_j \to \infty,$$

which contradicts with the population-level RSM condition (A.85). In the following, we give an example of the conditional mean model where (A.86) and (A.87) hold, and f(c|y) is β -smooth at $c = c^*$ with $\beta < 1$. The example can be easily generalized to a large class of β -smooth functions. To avoid imposing contradictory conditions, we have to exclude the case $\beta < 1$ when deriving the convergence rate of our estimators.

Remark 5. For any $0 < \beta < 1$, consider the conditional mean model $X = \mu Y + \epsilon$, where μ is a positive constant and the pdf of ϵ is

$$f_{\epsilon}(x) = \begin{cases} a_1 + a_2(x+\mu)^{\beta}, & -\mu \le x \le 0\\ a_1 - a_2(-x-\mu)^{\beta}, & -a_3 \le x < -\mu, \end{cases}$$
(A.88)

and $f_{\epsilon}(x) = f_{\epsilon}(-x)$ for $0 < x \le a_3$, where a_1, a_2, a_3 are some proper positive constants which makes $f_{\epsilon}(x)$ a valid pdf. Under this model, it is easily seen that

$$f(x|y=1) = f_{\epsilon}(x-\mu)$$
, and $f(x|y=-1) = f_{\epsilon}(x+\mu)$.

Since $f_{\epsilon}(x)$ is symmetric, there is a unique solution of $\widetilde{R}'(c) = f(c \mid y = 1) - f(c \mid y = -1) = 0$, that is $c^* = 0$.

First, we will show that f(c|y) is β -smooth at $c = c^* = 0$. For any $\Delta \in (0, \mu)$,

$$|f(\Delta \mid y=1) - f(0 \mid y=1)| = |f_{\epsilon}(\Delta - \mu) - f_{\epsilon}(-\mu)| = a_2 |\Delta|^{\beta},$$
(A.89)

and the same bound holds for $\Delta \in (-a_3 + \mu, 0)$. Thus, f(c|y = 1) is β -smooth at c = 0 and so is f(c|y = -1).

Second, to verify (A.86), from (A.89) we have that, for any constant L > 0, and any $|\Delta| < u$ with $u = \min(a_3 - \mu, \mu, (\frac{a_2}{L})^{\frac{1}{1-\beta}})$,

$$|f(\Delta \mid y = 1) - f(0 \mid y = 1)| = a_2 |\Delta|^{\beta} > L |\Delta|.$$

Since $f_{\epsilon}(x)$ is symmetric, the same bound holds for X given Y = -1. This verifies (A.86).

Finally, for (A.87), it is easily seen that f(x|y=1) in monotonically increasing for $x \in (-u, u)$ and f(x|y=-1) in monotonically decreasing for $x \in (-u, u)$. So (A.87) holds.

A.10 Analysis of Sampling Within a region of X

Recall that

$$Q_i = g_{3i}(\bar{H}_{i-1}) \, \mathbb{1}\{f_i(\mathbf{Z}_i, \bar{H}_{i-1}) - g_{1i}(\bar{H}_{i-1}) \le X_i \le f_i(\mathbf{Z}_i, \bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1})\}.$$

Since $\mathbb{E}_P(\sum_{i=1}^n Q_i) \leq N$ holds and $\inf_{x \in \mathbb{R}, z \in \mathbb{R}^d} f(x \mid z) \geq p_{\min}$ for $P \in \mathcal{P}(\beta, s)$, we have

$$\sum_{i=1}^{n} \mathbb{E}_{P}(g_{3i}(\bar{H}_{i-1})(g_{1i}(\bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1}))) = O(N)$$

Finally, we obtain that

$$\begin{split} &\sum_{i=1}^{n} \mathbb{E} \left[\boldsymbol{Z}_{i} \boldsymbol{Z}_{i}^{T} \cdot Q_{i} \right] \\ &= \sum_{i=1}^{n} \mathbb{E} \left[\boldsymbol{Z}_{i} \boldsymbol{Z}_{i}^{T} \cdot g_{3i}(\bar{H}_{i-1}) \,\mathbb{1} \{ f_{i}(\boldsymbol{Z}_{i}, \bar{H}_{i-1}) - g_{1i}(\bar{H}_{i-1}) \leq X_{i} \leq f_{i}(\boldsymbol{Z}_{i}, \bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1}) \} \right] \\ &\leq & p_{\max} \sum_{i=1}^{n} \mathbb{E} \left[\boldsymbol{Z}_{i} \boldsymbol{Z}_{i}^{T} \cdot g_{3i}(\bar{H}_{i-1}) (g_{1i}(\bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1})) \right] \\ &= & p_{\max} \mathbb{E} (\boldsymbol{Z} \boldsymbol{Z}^{T}) \sum_{i=1}^{n} \mathbb{E} \left[g_{3i}(\bar{H}_{i-1}) (g_{1i}(\bar{H}_{i-1}) + g_{2i}(\bar{H}_{i-1})) \right] \\ &= & O(N) \cdot \mathbb{E} (\boldsymbol{Z} \boldsymbol{Z}^{T}), \end{split}$$


Figure 4: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 for logistic regression. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the label budget is used in the first step for both two-step sampling methods.

where we used the fact that $(X_i, \mathbf{Z}_i) \perp \overline{H}_{i-1}$. Since the sparse eigenvalue assumption (3.2) holds, the last condition in (3.23) holds as well.

A.11 Additional simulation results

In this section, we consider the numerical results for our algorithm with $N_1 = N/5$ and $N_2 = 4N/5$ and the comparison of the four methods concerning $\|\hat{\theta} - \theta^*\|_{\infty}$ and prediction errors, which exhibit similar patterns as $\|\hat{\theta} - \theta^*\|_1$ and $\|\hat{\theta} - \theta^*\|_2$.



Figure 5: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 for conditional mean model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the label budget is used in the first step for both two-step sampling methods.



Figure 6: $\|\widehat{\theta} - \theta^*\|$ in ℓ_1 and ℓ_2 for binary response model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the label budget is used in the first step for both two-step sampling methods.



Figure 7: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for logistic regression. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the total budget of labeled data is used in the first step for both two-step sampling methods.



Figure 8: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for conditional mean model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the total budget of labeled data is used in the first step for both two-step sampling methods.



Figure 9: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for binary response model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/8 of the total budget of labeled data is used in the first step for both two-step sampling methods.



Figure 10: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for logistic regression. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the total budget of labeled data is used in the first step for both two-step sampling methods.



Figure 11: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for conditional mean model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the total budget of labeled data is used in the first step for both two-step sampling methods.



Figure 12: $\|\widehat{\theta} - \theta^*\|_{\infty}$ and prediction error for binary response model. LR: ℓ_1 penalized logistic regression; PF: path-following algorithm. 1/5 of the total budget of labeled data is used in the first step for both two-step sampling methods.