

CONVERGENCE/DIVERGENCE PHENOMENA IN THE VANISHING DISCOUNT LIMIT OF HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We study the asymptotic behavior of solutions of an equation of the form

$$G(x, D_x u, \lambda u(x)) = c_0 \quad \text{in } M \quad (*)$$

on a closed Riemannian manifold M , where $G \in C(T^*M \times \mathbb{R})$ is convex and superlinear in the gradient variable, is globally Lipschitz but not monotone in the last argument, and c_0 is the critical constant associated with the Hamiltonian $H := G(\cdot, \cdot, 0)$. By assuming that $\partial_u G(\cdot, \cdot, 0)$ satisfies a positivity condition of integral type on the Mather set of H , we prove that any equi-bounded family of solutions of $(*)$ uniformly converges to a distinguished critical solution u_0 as $\lambda \rightarrow 0^+$. We furthermore show that any other possible family of solutions uniformly diverges to $+\infty$ or $-\infty$. We then look into the linear case $G(x, p, u) := a(x)u + H(x, p)$ and prove that the family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ of maximal solutions to $(*)$ is well defined and equi-bounded for $\lambda_0 > 0$ small enough. When a changes sign and enjoys a stronger localized positivity assumption, we show that equation $(*)$ does admit other solutions too, and that they all uniformly diverge to $-\infty$ as $\lambda \rightarrow 0^+$. This is the first time that converging and diverging families of solutions are shown to coexist in such a generality.

INTRODUCTION

In this paper we are concerned with the asymptotic behavior of solutions of an equation of the form

$$G(x, D_x u, \lambda u(x)) = c_0 \quad \text{in } M \quad (E_\lambda)$$

posed on a closed Riemannian manifold M , where $G \in C(T^*M \times \mathbb{R})$ is convex and superlinear in the gradient variable, is globally Lipschitz in the last argument, and c_0 is the critical constant associated with the Hamiltonian $H := G(\cdot, \cdot, 0)$. We refer the reader to Section 1.3 for the definition of c_0 and of the other related objects coming from weak KAM Theory that will be mentioned in this introduction. The monotonicity condition on G in the last argument, that is standard for these kind of equations, is dropped here in favor of the following much weaker integral condition

$$(L5) \quad \int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) < 0 \quad \text{for all } \tilde{\mu} \in \widetilde{\mathfrak{M}},$$

where L_G is the convex conjugate function of G , $\widetilde{\mathfrak{M}}$ denotes the set of Mather measures for $L_G(\cdot, \cdot, 0)$, and G (and hence L_G) satisfies a C^1 -type regularity condition near $u = 0$, see conditions (G4) and (L4) in Section 2. Under these assumptions, we prove that any equi-bounded family of solutions of (E_λ) uniformly converges to a distinguished critical

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solution u_0 as $\lambda \rightarrow 0^+$. We furthermore show that any family consisting of other possible solutions uniformly diverges to $+\infty$ or $-\infty$.

We underline that the conditions presented above on G are not sufficient to guarantee the existence and uniqueness of viscosity solutions to (E_λ) . This is due to the fact that, without global strict monotonicity of G with respect to u , there is no comparison principle. This also makes unenforceable Perron's method, which is the technique customarily employed to prove existence of solutions.

The general issue of existence and uniqueness of such solutions is subsequently addressed in the paper in the linear case $G(x, p, u) := a(x)u + H(x, p)$ under the minimal hypotheses on $a \in C(M)$ and $H \in C(T^*M)$ that guarantee that the conditions on G and L_G mentioned above are in force. We prove that the family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ of maximal solutions to (E_λ) is well defined and equi-bounded for $\lambda_0 > 0$ small enough. When a changes sign and $a \geq 0$ in a neighborhood of the projected Aubry set \mathcal{A} , we show that equation (E_λ) does admit other solutions too that uniformly diverge to $-\infty$ as $\lambda \rightarrow 0^+$. If we additionally assume $a > 0$ on \mathcal{A} , we furthermore show that any family made up of solutions to (E_λ) that differ from the maximal ones uniformly diverge to $-\infty$. Incidentally, this completely solves the vanishing discount problem for this model case under the sole assumption that $a > 0$ on \mathcal{A} in view of the results established in [41], where a was additionally assumed nonnegative on M .

Condition (L5) was introduced in [7] and therein employed to solve the vanishing discount problem for an equation of the form (E_λ) under the same set of assumptions considered herein, plus the additional requirement that G is globally non-decreasing in u . Condition (L5) can be read as a strict monotonicity condition on G with respect to u , and this is transparent in the linear case $G(x, p, u) := a(x)u + H(x, p)$. What we find striking about the output of our study is the fact that (L5) is a very weak requirement: it implies that a has to be strictly positive only on some portions of the projected Mather set \mathcal{M} , where the latter is the minimal closed set that contains the projection of the supports of all Mather measures. This set \mathcal{M} can be very small, such as a finite set of points, see Remark 1.10. Furthermore, it has been conjectured by Mañé [33] that for generic Hamiltonians H both \mathcal{A} and \mathcal{M} coincide with the support of a closed curve. Many results have been obtained in this direction, see for example [8] and the references therein, showing that condition (L5) generically leaves a lot of space for a to take negative values.

The main results proved in this paper keep holding when the superlinearity condition on G is relaxed in favor of a simple coercivity. We have decided not to pursue this generalization here since that would add additional technicalities with the drawback of hiding the ideas at the base of our work, see Remark 1.12 for further details.

History of the problem. The so-called *ergodic approximation* is a technique introduced in [32] to study the existence of solutions of the Hamilton-Jacobi equation¹

$$H(x, D_x u) = c \quad \text{in } M \tag{1}$$

on the flat d -dimensional torus $M := \mathbb{T}^d \simeq \mathbb{R}^d / \mathbb{Z}^d$, where the Hamiltonian H is a continuous function on T^*M , coercive in the gradient variable, uniformly with respect to $x \in M$,

¹All solutions in the paper are meant in the viscosity sense. The definition will be provided later.

and c is a real number. Let $\lambda > 0$ and u_λ be the unique solution of

$$\lambda u(x) + H(x, D_x u) = 0 \quad \text{in } M.$$

According to [32], the functions $-\lambda u_\lambda$ uniformly converge on M , as $\lambda \rightarrow 0^+$, to a constant c_0 . Furthermore, the solutions $(u_\lambda)_{\lambda>0}$ are equi-Lipschitz, yielding, by the Arzelá-Ascoli Theorem, that the functions $u_\lambda - \min_{x \in M} u_\lambda$ uniformly converge, along subsequences as $\lambda \rightarrow 0^+$, to a solution of (1) with c equaling c_0 . The constant c_0 is called *critical value* of H and is characterized by the property of being the unique constant $c \in \mathbb{R}$ such that (1) admits solutions. At that time, it was not clear if different converging sequences yield the same limit. Some constraints on the possible limit solutions were subsequently found in [22, 29], but the breakthrough came with the work [12], where the authors proved that the unique solution u_λ of

$$\lambda u(x) + H(x, D_x u) = c_0 \quad \text{in } M \tag{HJ_\lambda}$$

converges to a distinguished solution of

$$H(x, D_x u) = c_0 \quad \text{in } M, \tag{HJ_0}$$

as $\lambda \rightarrow 0^+$ under the sole additional assumption that H is convex in the gradient variable. The proof relies on techniques and tools issued from weak KAM Theory, in particular on the concept of Mather measure, and it works whenever M is a closed Riemannian manifold. This kind of problem is also known as the *vanishing discount problem*. When the convexity condition on H is dropped, the functions u_λ may not converge, as it was pointed out in [44] through a counterexample posed on the 1-dimensional torus.

As a nonlinear generalization (see [6, 5, 23] and [39]), one can study the uniform convergence of the unique solution of

$$H_\lambda(x, D_x u, u(x)) = c_0 \quad \text{in } M,$$

as $\lambda \rightarrow 0^+$, where $H_\lambda(x, p, u)$ is strictly increasing in u , and uniformly converges to $H(x, p)$ on compact sets as $\lambda \rightarrow 0^+$. This kind of problem is called the *vanishing contact structure problem*. The vanishing discount problem falls in this framework as a particular case by choosing $H_\lambda(x, p, u) := \lambda u + H(x, p)$.

The asymptotic convergence result has been subsequently established in many different situations. For the second order case, one can refer to [26, 27, 34, 43]. For the discrete case, one can refer to [11, 38, 42] and also [2] in the context of twist maps. For the similar problem in the mean field game theory, one can refer to [4]. For the weakly coupled Hamilton-Jacobi systems, one can refer to [17, 14, 24, 25]. For the non-compact setting, one can refer to [18, 28].

A natural and challenging question is to weaken the hypothesis on the monotonicity of the Hamiltonian. A first degenerate case was studied in [41], where the author considered the convergence of the solution of

$$\lambda a(x)u(x) + H(x, D_x u) = c_0 \quad \text{in } M$$

as $\lambda \rightarrow 0^+$, where $a(x) \geq 0$ on M , and $a(x) > 0$ on the projected Aubry set of H . Inspired by the works [5, 41], the authors studied in [7] the vanishing discount problem for contact Hamilton-Jacobi equations of the form (E_λ) , where the positivity hypothesis on a assumed

in [41] is weakened and generalized by introducing the non-degeneracy integral condition (L5). This work highlights once more that the concept of Mather measure plays a central role in the convergence result.

It is worth pointing out that all works mentioned above required a global non-decreasing hypothesis of the Hamiltonians in u . If the discounted equation is not increasing in the unknown function u , solutions may even not exist, and, if they exist, they may be not unique. In [15, 40], the authors discussed the uniform convergence of the minimal solution of (HJ_λ) as $\lambda \rightarrow 0^-$. For the non-monotone vanishing discount problem, the second author provided the first example in [35] of nonconvergence. In this example, there exist a convergent family of solutions and a divergent family of solutions at the same time. This phenomenon is new comparing with all the previous works in this direction. In the present paper, we show that the example in [35] is in fact a very general phenomenon when the Hamiltonian is continuous, convex and superlinear in the fibres.

Let us conclude by mentioning that the type of problems we study are also closely linked to optimization problems in economics. The discount factor then models the effect of time through interest rates or inflation. Negative interest rates or deflation have been studied by economists (see for instance [31]). Our results then give possible asymptotics in the presence of coexisting inflation and deflation.

Presentation of our results. We present here our main results. Section 2 contains our analysis on the asymptotic behavior of possible solutions of a general contact Hamilton-Jacobi equation of the form (E_λ) when the discount factor λ goes to 0. The Hamiltonian $G(x, p, u)$ is assumed convex and superlinear in p , and globally Lipschitz in u , see conditions (G1-3) in Section 2. It is also assumed to satisfy a C^1 -type regularity condition in u near $u = 0$, see condition (G4) in Section 2. The latter is for instance satisfied when the map $u \mapsto G(x, p, u)$ is C^1 in a neighborhood of $u = 0$ in the following sense:

(G4') there exists $\varepsilon > 0$ such that $\frac{\partial G}{\partial u}(x, p, u)$ exists for all $(x, p, u) \in T^*M \times (-\varepsilon, \varepsilon)$ and is continuous in $T^*M \times (-\varepsilon, \varepsilon)$.

Let us consider

$$G(x, D_x u, \lambda u(x)) = c_0 \quad \text{in } M, \quad (\text{E}_\lambda)$$

and the limit equation

$$G(x, D_x u, 0) = c_0 \quad \text{in } M, \quad (\text{E}_0)$$

where c_0 is the critical value associated with $H := G(\cdot, \cdot, 0)$. We will furthermore assume the non-degeneracy integral condition (L5), where $\widetilde{\mathfrak{M}}$ denotes the set of Mather measures for $L := L_G(\cdot, \cdot, 0)$.

The main results contained in Section 2 can be summarized as follows, see Theorems 2.2 and 2.5.

THEOREM 1. *Under the previous assumptions, there exist a viscosity solution u_0 of (E_0) and functions $\varphi : (0, 1) \rightarrow (-\infty, +\infty]$, $\psi : (0, 1) \rightarrow [-\infty, +\infty)$ and $\theta : (0, 1) \rightarrow \mathbb{R}$ with*

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = -\infty, \quad \lim_{\lambda \rightarrow 0} \varphi(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow 0} \theta(\lambda) = 0$$

such that, if v_λ is a solution of (E_λ) for some $\lambda > 0$, then either one of the following alternatives occurs:

- (i) $v_\lambda \leq \psi(\lambda)$;
- (ii) $v_\lambda \geq \varphi(\lambda)$;
- (iii) $\|v_\lambda - u_0\|_\infty \leq \theta(\lambda)$.

We stress that all these three behaviors can happen at the same time for a properly chosen Hamiltonian G . Indeed, consider $G(x, p, u) := \sin(u) + \|p\|_x^2$. It is easily seen that G verifies all of the above conditions. The limit Hamiltonian is $H(x, p) := G(x, p, 0) = \|p\|_x^2$, its critical constant is $c_0 = 0$ and the constant functions are the only solutions to $H(x, D_x u) = 0$ in M . Moreover, for all $\lambda > 0$, the three constant functions $u_\lambda = 0$, and $u_\lambda^\pm = \pm \frac{\pi}{\lambda}$ are solutions to the discounted equation.

Theorem 1 shows in great generality that the only possible asymptotic behavior of families of solutions $(v_\lambda)_{\lambda \in (0, \lambda_0)}$ is, up to subsequences, either to uniformly diverge to $\pm\infty$, or to uniformly converge to a specific solution u_0 of (E_0) . We also provide two characterizations of u_0 , see Theorems 2.2 and 2.3.

The general problem of existence and uniqueness of such solutions is addressed in Section 3. Here we consider the linear case $G(x, p, u) := a(x)u + H(x, p)$ under the minimal hypotheses on $a \in C(M)$ and $H \in C(T^*M)$ that guarantee that the conditions on G and L_G presented above are in force. The discounted equation is then

$$\lambda a(x)u(x) + H(x, D_x u) = c_0 \quad \text{in } M, \quad (\text{HJ}_\lambda)$$

with limit equation

$$H(x, D_x u) = c_0 \quad \text{in } M. \quad (\text{HJ}_0)$$

We prove the following existence and convergence result:

THEOREM 2. *Under the previous hypotheses, there is $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, the equation (HJ_λ) admits a maximal viscosity solution $u_\lambda \in C(M)$. Moreover, the family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ is equi-bounded, hence it uniformly converges to u_0 as $\lambda \rightarrow 0^+$.*

The previous theorem excludes the possibility of families of solutions that uniformly diverge to $+\infty$, but leaves open the possibility of families uniformly diverging to $-\infty$. By strengthening the non-degeneracy integral condition (L5) with a pointwise positivity condition on a on the projected Aubry set \mathcal{A} , we are able to improve the statement of Theorem 1 as follows.

THEOREM 3. *Let us additionally assume that $a \geq 0$ in a neighborhood of \mathcal{A} and that there exist $x_0 \in M$ such that $a(x_0) < 0$. Then there exists a family of solutions $(v_\lambda)_{\lambda \in (0, \hat{\lambda})}$ to (HJ_λ) for some $\hat{\lambda} \in (0, 1)$ uniformly diverging to $-\infty$ as $\lambda \rightarrow 0^+$.*

When the previous hypothesis is reinforced, we obtain a stronger conclusion:

THEOREM 4. *Let us additionally assume that $a > 0$ on \mathcal{A} . Then any family $(v_\lambda)_{\lambda \in (0, \lambda')}$ of solutions to (HJ_λ) satisfying $v_\lambda \neq u_\lambda$ for all $\lambda \in (0, \lambda')$, with $\lambda' \in (0, 1)$, uniformly diverges to $-\infty$.*

Namely, in this last case, $(u_\lambda)_\lambda$ is the only converging family of solutions.

Let us conclude this presentation by stressing that, combining this analysis with the results of [41], we fully understand the asymptotic behavior of solutions to (HJ_λ) when $a > 0$ on \mathcal{A} . The output is the following:

- (a) if $a \geq 0$ on the whole M , (this situation was considered in [41]) then, for all $\lambda > 0$, there is a unique solution u_λ to (HJ_λ) , and the family $(u_\lambda)_\lambda$ converges as $\lambda \rightarrow 0^+$.
- (b) if there exist a point $x_0 \in M$ such that $a(x_0) < 0$, (this situation is discussed in the present paper), then we can find a $\lambda_0 > 0$ small enough such that equation (HJ_λ) admits at least two solutions for every $\lambda \in (0, \lambda_0)$. The family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ of maximal solutions uniformly converges to u_0 as $\lambda \rightarrow 0^+$. Any family of other solutions $(v_\lambda)_\lambda$ uniformly diverges to $-\infty$ as $\lambda \rightarrow 0^+$.

1. PRELIMINARIES

1.1. Notation.

- Throughout this paper, we assume that M is a closed, connected and smooth Riemannian manifold.
- We fix g an auxiliary Riemannian metric on M . Let $d(x, y)$ be the distance between x and y in M induced by g . By compactness of M our results are independent on the choice of g .
- Let $\text{diam}(M)$ be the diameter of M .
- We denote by TM and T^*M the tangent and cotangent bundle over M respectively. We denote by (x, p) and (x, v) points of T^*M and TM respectively.
- Let $\pi : TM, T^*M \rightarrow M$ denote both canonical projections, the context will make it clear which one is considered.
- We denote by $\|\cdot\|_x$ the norm on both T_xM and T_x^*M induced by g .
- If N is a smooth manifold, we will denote by $C(N)$ the Polish space of continuous functions from N to \mathbb{R} endowed with the metric of local uniform convergence on N . We will denote by $C_c(N)$ (resp. $C^1(N)$) the set of compactly supported (resp., C^1) functions from N to \mathbb{R} .
- We denote by $\mathcal{P}(TM)$ the space of Borel probability measures on TM endowed with the weak-* topology coming from the dual $(C_c(TM))'$.
- We denote $C_\ell(TM)$ the set of continuous functions $g : TM \rightarrow \mathbb{R}$ with at most linear growth meaning that

$$\sup_{(x,v) \in TM} \frac{|g(x,v)|}{1 + \|v\|_x} < +\infty.$$

This last quantity defines a norm $\|g\|_\ell$ on the vector space $C_\ell(TM)$.

- \mathbb{N} denotes the set of positive integers.

1.2. Viscosity solutions.

We start by recalling the notion of viscosity solution.

DEFINITION 1.1. Let $G : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, $c \in \mathbb{R}$, and consider the equation

$$G(x, D_x u, u(x)) = c \quad \text{in } M. \tag{1.1}$$

(a) We say that $u \in C(M)$ is a *viscosity subsolution* of (1.1), denoted by

$$G(x, D_x u, u(x)) \leq c \quad \text{in } M,$$

if, for all $\varphi \in C^1(M)$ and $x_0 \in M$ such that $u - \varphi$ has a local maximum at x_0 , we have $G(x_0, D_{x_0} \varphi, u(x_0)) \leq c$. Such a function φ is termed *supertangent to u at x_0* .

(b) We say that $u \in C(M)$ is a *viscosity supersolution* of (1.1), denoted by

$$G(x, D_x u, u(x)) \geq c \quad \text{in } M,$$

if, for all $\varphi \in C^1(M)$ and $x_0 \in M$ such that $u - \varphi$ has a local minimum at x_0 , then $G(x_0, D_{x_0} \varphi, u(x_0)) \geq c$. Such a function φ is termed *subtangent to u at x_0* .

(c) We say that $u \in C(M)$ is a *viscosity solution* of (1.1) if it is both a viscosity sub and supersolution.

In this paper, solutions, subsolutions, supersolutions will be always meant in the viscosity sense and implicitly assumed continuous. We recall that, if u is C^1 on an open set U , then it is a viscosity solution (resp. subsolution, supersolution) in U if and only if it is a pointwise solution (resp. subsolution, supersolution) in U .

The following stability result is well known, see for instance [3].

PROPOSITION 1.2. *Let $(G_n)_n$ and $(u_n)_n$ be two sequences of functions in $C(T^*M \times \mathbb{R})$ and $C(M)$, respectively, such that u_n is a subsolution (resp. supersolution, solution) of (1.1) with $G := G_n$, for each $n \in \mathbb{N}$. If $u_n \rightarrow u$ in $C(M)$ and $G_n \rightarrow G$ in $C(T^*M \times \mathbb{R})$ as $n \rightarrow +\infty$, then u is a subsolution (resp. supersolution, solution) of (1.1).*

1.3. Weak KAM solutions and Aubry-Mather theory. We assume $H : T^*M \rightarrow \mathbb{R}$ is a continuous Hamiltonian satisfying

(H1) (Convexity) $H(x, p)$ is convex in p for all $x \in M$.

(H2) (Superlinearity) $\lim_{\|p\|_x \rightarrow +\infty} H(x, p)/\|p\|_x = +\infty$.

Let $L : TM \rightarrow \mathbb{R}$ be the convex conjugate function of H , i.e.,

$$L(x, v) := \sup_{p \in T_x^*M} (p(v) - H(x, p)), \quad (x, v) \in TM.$$

It is well known that the Lagrangian L is a continuous function on TM and it is convex and superlinear in v . The Fenchel inequality is a direct consequence of this definition:

$$L(x, v) + H(x, p) \geq p(v), \quad \text{for all } (x, v, p) \in M \times T_x M \times T_x^* M. \quad (1.2)$$

Moreover, it can be proven that H is itself the convex conjugate of L , i.e.,

$$H(x, p) = \sup_{v \in T_x M} (p(v) - L(x, v)) \quad \text{for all } (x, p) \in T^* M. \quad (1.3)$$

Let $c_0 \in \mathbb{R}$ denote the critical constant defined as follows:

$$c_0 = \min\{c \in \mathbb{R} : H(x, D_x u) = c \text{ in } M \text{ admits subsolutions}\}. \quad (1.4)$$

We present here some facts that we will need about the *critical equation*, i.e.,

$$H(x, D_x u) = c_0 \quad \text{in } M. \quad (\text{HJ}_0)$$

Solutions, subsolutions and supersolutions of (HJ₀) will be termed *critical* in the sequel. Due to the convex character of H , the following holds, see for instance [3, 19].

PROPOSITION 1.3. *Let $u \in C(M)$. The following properties hold:*

- (i) *if u is the pointwise supremum (respectively, infimum) of a family of subsolutions (resp., supersolutions) to (HJ_0) , then u is a subsolution (resp., supersolution) of (HJ_0) ;*
- (ii) *if u is the pointwise infimum of a family of equi-Lipschitz subsolutions to (HJ_0) , then u is a Lipschitz subsolution of (HJ_0) ;*
- (iii) *if u is a convex combination of a family of equi-Lipschitz subsolutions to (HJ_0) , then u is a Lipschitz subsolution of (HJ_0) .*

More precisely, items (ii) and (iii) above require the convexity of H in the momentum, while item (i) is a general fact.

Since we are assuming H to be superlinear (hence coercive, which is enough), we also have the following characterization of critical subsolutions, see for instance [3, 19].

PROPOSITION 1.4. *The following are equivalent facts:*

- (i) *v is a viscosity subsolution of (HJ_0) ;*
- (ii) *v is Lipschitz continuous and an almost everywhere subsolution of (HJ_0) , i.e.,*

$$H(x, D_x v) \leq c_0 \quad \text{for a.e. } x \in M.$$

Moreover, the set of viscosity subsolutions of (HJ_0) is equi-Lipschitz, with $\kappa_{c_0} := \sup\{\|p\|_x : H(x, p) \leq c_0\}$ as a common Lipschitz constant.

For every $t > 0$, we define the minimal action function $h_t : M \times M \rightarrow \mathbb{R}$ as

$$h_t(x, y) = \inf_{\gamma} \int_{-t}^0 [L(\gamma(s), \dot{\gamma}(s)) + c_0] ds,$$

where $\gamma : [-t, 0] \rightarrow M$ is taken among all absolutely continuous curves² satisfying $\gamma(-t) = x$ and $\gamma(0) = y$. The Peierls barrier is the function $h : M \times M \rightarrow \mathbb{R}$ defined by

$$h(x, y) := \liminf_{t \rightarrow +\infty} h_t(x, y). \tag{1.5}$$

It satisfies the following properties, see for instance [16]:

PROPOSITION 1.5.

- (i) *The Peierls barrier h is finite valued and Lipschitz continuous.*
- (ii) *If v is a critical subsolution, then*

$$v(x) - v(y) \leq h(y, x), \quad v(x) - v(y) \leq h_t(y, x) \quad \text{for every } x, y \in M \text{ and } t > 0.$$
- (iii) *For every fixed $y \in M$, the function $h(y, \cdot)$ is a critical solution.*
- (iv) *For every fixed $y \in M$, the function $-h(\cdot, y)$ is a critical subsolution.*

The projected Aubry set \mathcal{A} is the closed set defined by

$$\mathcal{A} := \{y \in M : h(y, y) = 0\}.$$

The following holds, see [19, 21]:

²In the paper, even if not explicitly stated, all curves considered are at least absolutely continuous.

THEOREM 1.6. *There exists a critical subsolution v which is both strict and of class C^∞ in $M \setminus \mathcal{A}$, meaning that*

$$H(x, D_x v) < c_0 \quad \text{for every } x \in M \setminus \mathcal{A}.$$

In particular, the projected Aubry set \mathcal{A} is nonempty.

The last assertion directly follows from the definition of c_0 , see (1.4).

PROPOSITION 1.7. *If u and v are respectively a sub and supersolution such that $u \leq v$ on \mathcal{A} , then $u \leq v$ on the whole of M . In particular, \mathcal{A} is a uniqueness set for (HJ_0) , meaning that if two solutions coincide on \mathcal{A} , then they are equal.*

We will say that a Borel probability measure $\tilde{\mu}$ on TM is *closed* if it satisfies the following conditions:

- (a) $\int_{TM} \|v\|_x d\tilde{\mu}(x, v) < +\infty$;
- (b) for all function $f \in C^1(M)$, we have $\int_{TM} D_x f(v) d\tilde{\mu}(x, v) = 0$.

We will denote by \mathcal{P}_0 the set of such measures.

We will furthermore denote by \mathcal{P}_ℓ the family of probability measures $\tilde{\mu}$ that satisfy condition (a) above. The inclusions $\mathcal{P}_0 \subset \mathcal{P}_\ell \subset (C_\ell(TM))'$ hold. We will endow \mathcal{P}_ℓ with the weak-* topology coming from the dual $(C_\ell(TM))'$. We refer the reader to [9] for more details on these families of measures.

THEOREM 1.8. *The following holds*

$$\min_{\tilde{\mu} \in \mathcal{P}_0} \int_{TM} L(x, v) d\tilde{\mu} = -c_0.$$

Measures realizing the above minimum are called *Mather measures* for L . We denote by $\widetilde{\mathfrak{M}}$ the set of all Mather measures. This set is compact. The *Mather set* and the *projected Mather set* are defined as follows:

$$\widetilde{\mathcal{M}} := \overline{\bigcup_{\tilde{\mu} \in \widetilde{\mathfrak{M}}} \text{supp}(\tilde{\mu})}, \quad \mathcal{M} := \pi(\widetilde{\mathcal{M}}).$$

These sets are also compact, see [19] for a proof in the regular case.³ Furthermore, the following holds, see [41, Proposition 3.13] for a proof in the nonregular case.

THEOREM 1.9. *The following inclusion holds: $\mathcal{M} \subseteq \mathcal{A}$.*

REMARK 1.10. In the example of a mechanical Hamiltonian, i.e., $H(x, p) = \|p\|_x^2/2 + V(x)$, it is well known that $c_0 = \max_M V$, $\mathcal{A} = \{y \in M : V(y) = \max_M V\}$ and the Mather measures are convex combinations of delta Diracs concentrated at points $(y, 0)$ with $y \in \mathcal{A}$, so that \mathcal{M} is also equal to $\{y \in M : V(y) = \max_M V\}$.

We conclude this paragraph by a technical lemma that will be of crucial use (see [9, Theorem 2-4.1.3.]).

³For the present nonregular case, a proof of this can be found in Appendix A in the ArXiv version of [12].

LEMMA 1.11. *Let $a \in \mathbb{R}$. The set $\{\tilde{\mu} \in \mathcal{P}_\ell : \int_{TM} L(x, v) d\tilde{\mu}(x, v) \leq a\}$ is compact in \mathcal{P}_ℓ .*

Note that, as L is bounded below, the quantity $\int_{TM} L(x, v) d\tilde{\mu}(x, v) \in \mathbb{R} \cup \{+\infty\}$ is well defined for any Borel probability measure. We also remark that in [9, Theorem 2-4.1.3.] the result is proved for a particular subclass of measures, however the proof makes no use of this fact and proves the above result.

1.4. Hamiltonians depending on the unknown function. We recall here known results that can be found in [36, 37] and the references therein. In this section, we consider a continuous Hamiltonian $G : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies the following conditions

- (G1) (Lipschitz in u) $u \mapsto G(x, p, u)$ is K -Lipschitz continuous for some $K > 0$, uniformly in $(x, p) \in T^*M$;
- (G2) (Convexity in p) $p \mapsto G(x, p, u)$ is convex for each $(x, u) \in M \times \mathbb{R}$;
- (G3) (Superlinearity in p) $p \mapsto G(x, p, u)$ is superlinear for each $(x, u) \in M \times \mathbb{R}$.

REMARK 1.12. The results of this paper keep holding even when the superlinearity condition (G3) is weakened in favor of a simple coercivity. For instance, Theorems 2.2 and 2.3 can be easily generalized to this setting. Indeed, since we are dealing there with a family of equi-bounded, and hence equi-Lipschitz, solutions, see Lemma 2.6, we could employ the usual trick of modifying G outside a compact subset of $T^*M \times \mathbb{R}$ to make it superlinear. This cannot be done in other parts of the paper since we are dealing with families of solutions that are neither equi-bounded nor equi-Lipschitz in general. And even when they are, as in Section 3.1, this needs to be proved. In fact, this is the core of the analysis performed in Section 3.1, which takes advantage of the fact that the Lagrangian associated with the Hamiltonian via the Fenchel duality is finite-valued. This is no longer true in the purely coercive case, even though the difficulties arising could be handled by showing that all the minimizing curves that come into play in our analysis are indeed supported on the set where the Lagrangian is finite. Yet, we believe that treating this more general case would bring additional technicalities that would have the effect of hiding the ideas at the base of this work. We prefer to leave the coercive case to a possible future work.

Let $L_G : TM \times \mathbb{R} \rightarrow \mathbb{R}$ be the convex conjugate function of G , i.e.,

$$L_G(x, v, u) := \sup_{p \in T_x^*M} (p(v) - G(x, p, u)).$$

Then it can be proven that L_G verifies similar properties (see [5, Lemma 4.1] with easy adaptations):

- (L1) $u \mapsto L_G(x, v, u)$ is K -Lipschitz continuous uniformly in $(x, v) \in TM$;
- (L2) $v \mapsto L_G(x, v, u)$ is convex for each $(x, u) \in M \times \mathbb{R}$;
- (L3) $v \mapsto L_G(x, v, u)$ is superlinear for each $(x, u) \in M \times \mathbb{R}$;

DEFINITION 1.13. Let $G : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ be a Hamiltonian satisfying (G1-3) and let $L_G : TM \times \mathbb{R} \rightarrow \mathbb{R}$ be the associated Lagrangian. Let $c \in \mathbb{R}$. A function $u \in C(M)$ satisfying the following two properties is called a *backward* (resp. *forward*) *weak KAM solution* of

$$G(x, D_x u, u(x)) = c \quad \text{in } M. \tag{1.6}$$

(1) For each absolutely continuous curve $\gamma : [t', t] \rightarrow M$, we have

$$u(\gamma(t)) - u(\gamma(t')) \leq \int_{t'}^t \left[L_G(\gamma(s), \dot{\gamma}(s), u(\gamma(s))) + c \right] ds.$$

The above condition reads as u is dominated by $L_G + c$ and will be denoted by $u \prec L_G + c$.

(2) For each $x \in M$, there exists an absolutely continuous curve $\gamma_- : (-\infty, 0] \rightarrow M$ (resp. $\gamma_+ : [0, +\infty) \rightarrow M$) with $\gamma_-(0) = x$ (resp. $\gamma_+(0) = x$) such that

$$u(x) - u(\gamma_-(t)) = \int_t^0 \left[L_G(\gamma_-(s), \dot{\gamma}_-(s), u(\gamma_-(s))) + c \right] ds, \quad \forall t < 0$$

$$\left(\text{resp. } u(\gamma_+(t)) - u(x) = \int_0^t \left[L_G(\gamma_+(s), \dot{\gamma}_+(s), u(\gamma_+(s))) + c \right] ds, \quad \forall t > 0 \right).$$

The curves satisfying the above equality are called (u, L_G, c) -calibrated curves.

By [37, Appendix D] and [36, Appendix A] we have

LEMMA 1.14.

(i) If $u \in C(M)$ is a backward weak KAM solution of (1.6), it is a viscosity solution of (1.6).

(ii) The function $w \in C(M)$ is a viscosity subsolution of (1.6) if and only if $w \prec L_G + c$. The latter is also equivalent to w being Lipschitz continuous and verifying $G(x, D_x w, w(x)) \leq c$ for almost every $x \in M$.

We then give an approximation result in this setting. It is an easy consequence of [20, Theorem 8.5].

THEOREM 1.15. Assume $G : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous Hamiltonian verifying (G2). Let $w : M \rightarrow \mathbb{R}$ be a Lipschitz function verifying $G(x, D_x w, w(x)) \leq c$ for almost every $x \in M$. Then, for every $\varepsilon > 0$, there is a $w_\varepsilon \in C^\infty(M)$ such that $\|w - w_\varepsilon\|_\infty < \varepsilon$ and

$$G(x, D_x w_\varepsilon, w(x)) \leq c + \varepsilon, \quad G(x, D_x w_\varepsilon, w_\varepsilon(x)) \leq c + \varepsilon \quad \text{for all } x \in M.$$

Let $a \in C(M)$. Assume there are two points x_1 and x_2 such that $a(x_1) > 0$ and $a(x_2) < 0$. Let $c \in \mathbb{R}$ and consider the equation

$$a(x)u(x) + H(x, D_x u) = c \quad \text{in } M. \quad (1.7)$$

This is a particular case of the previous setting for $G(x, p, u) = a(x)u + H(x, p)$. In this case, the associated Lagrangian is $L_G(x, v, u) = L(x, v) - a(x)u$. The following implicit Lax-Oleinik semigroup $(T_t^-)_{t \geq 0}$ is a well defined semigroup of operators $T_t^- : C(M) \rightarrow C(M)$ that verify, for all $\varphi \in C(M)$,

$$T_t^- \varphi(x) = \inf_{\gamma(t)=x} \left\{ \varphi(\gamma(0)) + \int_0^t \left[L(\gamma(\tau), \dot{\gamma}(\tau)) - a(\gamma(\tau))T_\tau^- \varphi(\gamma(\tau)) + c \right] d\tau \right\}, \quad (1.8)$$

where the infimum is taken among absolutely continuous curves $\gamma : [0, t] \rightarrow M$ with $\gamma(t) = x$. A similar property defines and characterizes the forward semigroup $(T_t^+)_{t > 0}$,

where, for all $\varphi \in C(M)$,

$$T_t^+ \varphi(x) = \sup_{\gamma(0)=x} \left\{ \varphi(\gamma(t)) - \int_0^t \left[L(\gamma(\tau), \dot{\gamma}(\tau)) - a(\gamma(\tau)) T_{t-\tau}^+ \varphi(\gamma(\tau)) + c \right] d\tau \right\}. \quad (1.9)$$

LEMMA 1.16. [36, Appendix A] *If u_0 is a subsolution of (1.7), then*

$$T_t^+ u_0 \leq u_0 \leq T_t^- u_0.$$

If u_0 is a strict subsolution⁴ of (1.7), then

$$T_t^+ u_0 < u_0 < T_t^- u_0.$$

LEMMA 1.17. [36, Proposition 2.9, Proposition 3.5, Lemma 5.4] *If u_0 is a subsolution of (1.7), then the limit*

$$u_- := \lim_{t \rightarrow +\infty} T_t^- u_0 \quad (\text{resp. } v_+ := \lim_{t \rightarrow +\infty} T_t^+ u_0)$$

exists, and is a viscosity solution (resp. forward weak KAM solution) of (1.7). In addition,

$$v_- := \lim_{t \rightarrow +\infty} T_t^- v_+$$

is also a viscosity solution of (1.7), and there is a point $x_0 \in M$ such that $v_-(x_0) = v_+(x_0)$.

If u_0 is a strict subsolution of (1.7), then u_- is the maximal solution of (1.7), and v_- is the minimal solution of (1.7).

2. GENERAL CONVERGENCE/DIVERGENCE RESULTS

In this section, we consider a continuous Hamiltonian $G : T^*M \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies the following conditions:

- (G1) (Lipschitz in u) $u \mapsto G(x, p, u)$ is K -Lipschitz continuous uniformly in $(x, p) \in T^*M$ for some $K > 0$;
- (G2) (Convexity in p) $v \mapsto G(x, p, u)$ is convex for each $(x, u) \in M \times \mathbb{R}$;
- (G3) (Superlinearity in p) $p \mapsto G(x, p, u)$ is superlinear for each $(x, u) \in M \times \mathbb{R}$;
- (G4) (Modulus continuity near $u = 0$) The partial derivative $\frac{\partial G}{\partial u}(x, p, 0)$ exists. For every compact subset $S \subset TM$, we can find a modulus of continuity⁵ η_S such that

$$\left| G(x, p, u) - G(x, p, 0) - \frac{\partial G}{\partial u}(x, p, 0)u \right| \leq |u|\eta_S(|u|), \quad \forall (x, p) \in S.$$

The dependence of G in the u -variable is nonlinear, in general, as in [7], but we do not make any global monotonicity assumption. As established in [5, Lemma 4.1] the associated Lagrangian function $L_G : TM \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$L_G(x, v, u) := \sup_{p \in T_x^*M} (p(v) - G(x, p, u)),$$

has similar properties:

- (L1) (Lipschitz in u) $u \mapsto L_G(x, v, u)$ is K -Lipschitz continuous uniformly in $(x, v) \in TM$;

⁴Meaning that c can be replaced with $c - \varepsilon$ for some $\varepsilon > 0$.

⁵A modulus of continuity is a nondecreasing function $\eta : (0, +\infty) \rightarrow (0, +\infty)$ such that $\eta(s) \rightarrow 0$ as $s \rightarrow 0$.

- (L2)** (Convexity in v) $v \mapsto L_G(x, v, u)$ is convex for each $(x, u) \in M \times \mathbb{R}$;
(L3) (Superlinearity in v) $v \mapsto L_G(x, v, u)$ is superlinear for each $(x, u) \in M \times \mathbb{R}$;
(L4) (Modulus continuity near $u = 0$) The partial derivative $\frac{\partial L_G}{\partial u}(x, v, 0)$ exists. For every compact subset $S \subset TM$, we can find a modulus of continuity η_S such that

$$\left| L_G(x, v, u) - L_G(x, v, 0) - \frac{\partial L_G}{\partial u}(x, v, 0)u \right| \leq |u|\eta_S(|u|), \quad \forall (x, v) \in S.$$

The last condition is easier to state with the Lagrangian function as it involves Mather measures defined on TM :

- (L5)** (Non-degeneracy condition) For all Mather measures $\tilde{\mu}$ of $(x, v) \mapsto L_G(x, v, 0)$, we have

$$\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu} < 0.$$

REMARK 2.1. (1) It is classical in convex analysis that for all $(x, v) \in TM$ there is $(x, p) \in T^*M$ verifying $L_G(x, v, 0) + G(x, p, 0) = p(v)$. It is proved in [5, Lemma 4.1] that if $(x, p) \in T^*M$ and $(x, v) \in TM$ verify the previous formula then $\frac{\partial L_G}{\partial u}(x, v, 0) = -\frac{\partial G}{\partial u}(x, p, 0)$. Therefore, an equivalent formulation of (L5) is

- (G5)** (Non-degeneracy condition) For all Mather measures $\tilde{\mu}$ of $(x, v) \mapsto L_G(x, v, 0)$,

$$\int_{TM} \frac{\partial G}{\partial u}(x, p_{(x,v)}, 0) d\tilde{\mu} > 0.$$

where $p_{(x,v)}$ is chosen so to satisfy $L_G(x, v, 0) + G(x, p_{(x,v)}, 0) = p_{(x,v)}(v)$.

(2) A result of Mañé ([33]) asserts that a generic Hamiltonian H has a unique Mather measure.⁶ Hence our condition of integral type is quite loose for such a generic H . We also refer the reader to Remark 1.10 for an explicit example.

Consider

$$G(x, D_x u, \lambda u(x)) = c_0 \quad \text{in } M, \tag{E_\lambda}$$

and the limit equation

$$G(x, D_x u, 0) = c_0 \quad \text{in } M. \tag{E_0}$$

Here we denote by c_0 the critical value of $H(x, p) := G(x, p, 0)$. We denote by $\widetilde{\mathfrak{M}}$ (resp. $\widetilde{\mathcal{M}}$) the set of all Mather measures (resp. the Mather set) corresponding to H . In the sequel, we will always assume that λ belongs to the interval $(0, 1)$.

The main theorems of this section are the following.

THEOREM 2.2. *Let conditions (L1–5) be in force. Let us assume that there exist an equi-bounded family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ of solutions to (E_λ) , for some $\lambda_0 \in (0, 1)$. Then the functions u_λ uniformly converge in M , as $\lambda \rightarrow 0^+$, to a solution u_0 of the critical equation (E_0) . Moreover, u_0 is the largest subsolution w of (E_0) satisfying*

$$\int_{TM} w(x) \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) \geq 0, \quad \forall \tilde{\mu} \in \widetilde{\mathfrak{M}}. \tag{S}$$

⁶It has been conjectured by Mañé that for generic Hamiltonians H this unique Mather measure is supported on a closed curve.

Under an additional pointwise condition on $L_G(\cdot, \cdot, 0)$ on the Mather set, the limit critical solution u_0 can be characterized as follows.

THEOREM 2.3. *Let conditions (L1–5) be in force, and let us furthermore assume that $\frac{\partial L_G}{\partial u}(x, v, 0) \leq 0$ for all $(x, v) \in \mathcal{M}$. Then the function $u_0 = \lim_{\lambda \rightarrow 0^+} u_\lambda$ obtained in Theorem 2.2 above can be characterized as follows:*

$$u_0(x) = \inf_{\tilde{\mu} \in \tilde{\mathfrak{M}}} \frac{\int_{TM} h(y, x) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)}, \quad x \in M,$$

where $h(y, x)$ is the Peierls barrier defined in (1.5).

REMARK 2.4. Let us stress that the previous theorems hold with same proofs even if $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ is replaced by any family $(u_\lambda)_{\lambda \in \Lambda}$ of solutions to (E_λ) , where Λ is a subset of $(0, 1)$ having 0 as accumulation point. In particular, this holds for $\Lambda := (\lambda_n)_n$ with $\lambda_n \rightarrow 0^+$ as $n \rightarrow +\infty$.

Concerning the asymptotic behavior of other possible families of solutions to (E_λ) , we have the following trichotomy result.

THEOREM 2.5. *Let conditions (L1–5) be in force, and let $u_0 = \lim_{\lambda \rightarrow 0^+} u_\lambda$ be the critical solution obtained in Theorem 2.2. There exist $\varphi : (0, 1) \rightarrow (-\infty, +\infty]$, $\psi : (0, 1) \rightarrow [-\infty, +\infty)$ and $\theta : (0, 1) \rightarrow \mathbb{R}$ with*

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = -\infty, \quad \lim_{\lambda \rightarrow 0} \varphi(\lambda) = +\infty, \quad \lim_{\lambda \rightarrow 0} \theta(\lambda) = 0$$

such that, if v_λ is a solution of (E_λ) for some $\lambda > 0$, then either one of the following alternatives occurs:

- (i) $v_\lambda \leq \psi(\lambda)$;
- (ii) $v_\lambda \geq \varphi(\lambda)$;
- (iii) $\|v_\lambda - u_0\|_\infty \leq \theta(\lambda)$.

We start our analysis by remarking that the solutions $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ are equi-Lipschitz continuous. In the remainder of the section, we will denote by $C > 0$ the following constant

$$C := \sup_{\lambda \in (0, \lambda_0)} \|u_\lambda\|_\infty.$$

LEMMA 2.6. *The bounded family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ is equi-Lipschitz continuous.*

Proof. For each $x, y \in M$, we denote by $d := d(x, y)$ the distance between them. Take a geodesic $\zeta : [0, d] \rightarrow M$ satisfying $\zeta(0) = x$ and $\zeta(d) = y$ with constant speed $\|\dot{\zeta}\|_\zeta = 1$. Denote $L_\lambda(x, v) := L_G(x, v, \lambda u_\lambda)$. Since $u_\lambda \prec L_\lambda + c_0$, we have

$$\begin{aligned} u_\lambda(y) - u_\lambda(x) &\leq \int_0^d \left[L_G(\zeta(s), \dot{\zeta}(s), \lambda u_\lambda(\zeta(s))) + c_0 \right] ds \\ &\leq \left(\max_{x \in M, \|v\|_x \leq 1} |L_G(x, v, 0)| + \lambda_0 KC + c_0 \right) d(x, y). \end{aligned}$$

The assertion follows by exchanging the role of x and y . \square

From the fact that u_λ is Lipschitz for every fixed $\lambda \in (0, \lambda_0)$ we deduce the following fact.

LEMMA 2.7. *Let $\lambda \in (0, \lambda_0)$. Then*

$$\int_{TM} L_G(x, v, \lambda u_\lambda(x)) d\tilde{\mu}(x, v) \geq -c_0 \quad \text{for all } \tilde{\mu} \in \mathcal{P}_0.$$

Proof. Pick $\tilde{\mu} \in \mathcal{P}_0$ and choose $\varepsilon > 0$. By applying Theorem 1.15 to the Hamiltonian $(x, p) \mapsto G(x, p, \lambda u_\lambda(x))$ and by choosing $w := u_\lambda$, we infer that there exists $w_\varepsilon \in C^\infty(M)$ such that $G(x, D_x w_\varepsilon, \lambda u_\lambda(x)) \leq c_0 + \varepsilon$ for all $x \in M$. By definition of L_G we have that

$$G(x, D_x w_\varepsilon, \lambda u_\lambda(x)) + L_G(x, v, \lambda u_\lambda(x)) \geq D_x w_\varepsilon(v) \quad \text{for all } (x, v) \in TM.$$

By integrating this inequality with respect to $\tilde{\mu}$ we get

$$\begin{aligned} 0 &= \int_{TM} D_x w_\varepsilon(v) d\tilde{\mu}(x, v) \\ &\leq \int_{TM} \left(L_G(x, v, \lambda u_\lambda(x)) + G(x, D_x w_\varepsilon, \lambda u_\lambda(x)) \right) d\tilde{\mu}(x, v) \\ &\leq \int_{TM} \left(L_G(x, v, \lambda u_\lambda(x)) + c_0 + \varepsilon \right) d\tilde{\mu}(x, v) = \int_{TM} L_G(x, v, \lambda u_\lambda(x)) d\tilde{\mu}(x, v) + c_0 + \varepsilon. \end{aligned}$$

The result follows letting $\varepsilon \rightarrow 0^+$. \square

By the Arzelá-Ascoli Theorem and Lemma 2.6, any sequence $(u_{\lambda_n})_n$ with $\lambda_n \rightarrow 0^+$ admits a subsequence which uniformly converges to a continuous function u^* . By the stability of viscosity solution, see Proposition 1.2, u^* is a solution of (E_0) . In the following, we are going to show the uniqueness of the possible limit u^* , thus establishing the convergence result. Define \mathcal{S} the set of all subsolutions w of (E_0) satisfying condition (S). We define

$$u_0(x) := \sup_{w \in \mathcal{S}} w(x). \tag{2.1}$$

A priori, \mathcal{S} may be empty, and, even if \mathcal{S} is not empty, u_0 might be $+\infty$. Both these circumstances will be excluded under the hypotheses of Theorem 2.2.

LEMMA 2.8. *Any accumulation point u^* of the family $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ as $\lambda \rightarrow 0^+$ satisfies*

$$\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) u^*(x) d\mu(x, v) \geq 0, \quad \forall \tilde{\mu} \in \widetilde{\mathfrak{M}}.$$

In particular, $\mathcal{S} \neq \emptyset$ and $u^ \leq u_0$.*

Proof. Recall that C is the uniform bound of $(u_\lambda)_{\lambda \in (0, \lambda_0)}$. Let $\tilde{\mu} \in \widetilde{\mathfrak{M}}$. For $\lambda \in (0, \lambda_0)$ we have

$$\begin{aligned} -c_0 &\leq \int_{TM} L_G(x, v, \lambda u_\lambda(x)) d\tilde{\mu} \\ &\leq \int_{TM} \left[L_G(x, v, 0) + \lambda \frac{\partial L_G}{\partial u}(x, v, 0) u_\lambda(x) + \lambda C \eta_{\widetilde{\mathfrak{M}}}(\lambda C) \right] d\tilde{\mu} \\ &= -c_0 + \int_{TM} \left[\lambda \frac{\partial L_G}{\partial u}(x, v, 0) u_\lambda(x) + \lambda C \eta_{\widetilde{\mathfrak{M}}}(\lambda C) \right] d\tilde{\mu}, \end{aligned}$$

which implies

$$\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) u_\lambda(x) d\tilde{\mu} \geq -C\eta_{\widetilde{\mathfrak{M}}}(\lambda C).$$

The conclusion follows by sending $\lambda \rightarrow 0^+$. \square

In what follows, we will use the notation $L_\lambda(x, v) := L_G(x, v, \lambda u_\lambda)$.

LEMMA 2.9. *For $x \in M$ and $\lambda \in (0, \lambda_0)$, let $\gamma_\lambda^x : (-\infty, 0] \rightarrow M$ be a $(u_\lambda, L_\lambda, c_0)$ -calibrated curve with $\gamma_\lambda^x(0) = x$. Then there exists $\hat{\kappa} > 0$, independent of $\lambda \in (0, \lambda_0)$ and of $x \in M$, such that γ_λ^x is $\hat{\kappa}$ -Lipschitz continuous for every $\lambda \in (0, \lambda_0)$ and $x \in M$.*

Proof. By Lemma 2.6, there is $\kappa > 0$ independent of λ such that u_λ is κ -Lipschitz continuous. By superlinearity of L_G , for each $T > 0$, there is $C_T \in \mathbb{R}$ such that

$$L_G(x, v, 0) \geq T\|v\|_x + C_T. \quad (2.2)$$

Thus, we have for $0 \geq t > s$

$$\begin{aligned} \kappa d(\gamma_\lambda^x(t), \gamma_\lambda^x(s)) &\geq u_\lambda(\gamma_\lambda^x(t)) - u_\lambda(\gamma_\lambda^x(s)) = \int_s^t \left[L_G(\gamma_\lambda^x(\tau), \dot{\gamma}_\lambda^x(\tau), \lambda u_\lambda(\gamma_\lambda^x(\tau))) + c_0 \right] d\tau \\ &\geq \int_s^t ((\kappa + 1)\|\dot{\gamma}_\lambda^x(\tau)\|_{\gamma_\lambda^x(\tau)} + C_{\kappa+1}) d\tau + (c_0 - \lambda_0 KC)(t - s) \\ &\geq (\kappa + 1)d(\gamma_\lambda^x(t), \gamma_\lambda^x(s)) + (C_{\kappa+1} + c_0 - \lambda_0 KC)(t - s), \end{aligned}$$

which implies

$$d(\gamma_\lambda^x(t), \gamma_\lambda^x(s)) \leq (\lambda_0 KC - c_0 - C_{\kappa+1})(t - s).$$

The proof is now complete. \square

From now on, we denote by γ_λ^x the calibrated curve considered in Lemma 2.9. By the compactness of $\widetilde{\mathfrak{M}}$, there are two constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with

$$-\varepsilon_2 < \inf_{\tilde{\mu} \in \widetilde{\mathfrak{M}}} \int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu} \leq \sup_{\tilde{\mu} \in \widetilde{\mathfrak{M}}} \int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu} < -\varepsilon_1, \quad (2.3)$$

We derive from this the following asymptotic informations on the calibrated curves γ_λ^x , cf. [7, Corollary 7.4].

LEMMA 2.10. *There exist $\bar{\lambda} \in (0, \lambda_0)$ and $T_0 > 0$ such that, for all $\lambda \in (0, \bar{\lambda})$ and for all $x \in M$, we have*

$$-\varepsilon_2 < \frac{1}{b-a} \int_a^b \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds < -\varepsilon_1, \quad (2.4)$$

for all $a < b \leq 0$ with $b - a \geq T_0$.

Proof. Let us prove the left-hand inequality in (2.4). We argue by contradiction. Assume there is a sequence $\lambda_n \rightarrow 0$ and a sequence $b_n - a_n \rightarrow +\infty$ such that

$$\frac{1}{b_n - a_n} \int_{a_n}^{b_n} \frac{\partial L_G}{\partial u}(\gamma_{\lambda_n}^x(s), \dot{\gamma}_{\lambda_n}^x(s), 0) ds \leq -\varepsilon_2. \quad (2.5)$$

Define for all $n \in \mathbb{N}$ a probability measure $\tilde{\mu}_n$ on TM by

$$\int_{TM} g(x, v) d\tilde{\mu}_n := \frac{1}{b_n - a_n} \int_{a_n}^{b_n} g(\gamma_{\lambda_n}^x(s), \dot{\gamma}_{\lambda_n}^x(s)) ds, \quad \forall g \in C_c(TM).$$

Here $C_c(TM)$ is the set of all continuous functions defined on TM with compact supports. By Lemma 2.9, all the measures $\tilde{\mu}_n$ have support in the compact set

$$\{(x, v) \in TM : \|v\|_x \leq \lambda_0 KC - c_0 - C_{\kappa+1}\}.$$

Up to extracting a subsequence if necessary, we can assume that $\tilde{\mu}_n$ converges to $\tilde{\mu}$ in the weak-* topology on $C_c(TM)$. Note that $\tilde{\mu}$ is also compactly supported. For $f \in C^1(M)$, we have

$$\begin{aligned} \int_{TM} D_x f(v) d\tilde{\mu}_n(x, v) &= \frac{1}{b_n - a_n} \int_{a_n}^{b_n} D_{\gamma_{\lambda_n}^x(s)} f(\dot{\gamma}_{\lambda_n}^x(s)) ds \\ &= \frac{1}{b_n - a_n} \left(f(\gamma_{\lambda_n}^x(b_n)) - f(\gamma_{\lambda_n}^x(a_n)) \right) \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$, which implies that $\tilde{\mu}$ is closed. Since $\gamma_{\lambda_n}^x$ is a calibrated curve, we have

$$\begin{aligned} &\int_{TM} \left[L_G(x, v, \lambda_n u_{\lambda_n}(x)) + c_0 \right] d\tilde{\mu}_n(x, v) \\ &= \frac{1}{b_n - a_n} \int_{a_n}^{b_n} \left[L_G(\gamma_{\lambda_n}^x(s), \dot{\gamma}_{\lambda_n}^x(s), \lambda_n u_{\lambda_n}(\gamma_{\lambda_n}^x(s))) + c_0 \right] ds \\ &= \frac{1}{b_n - a_n} \left(u_{\lambda_n}(\gamma_{\lambda_n}^x(b_n)) - u_{\lambda_n}(\gamma_{\lambda_n}^x(a_n)) \right). \end{aligned}$$

Since u_λ is bounded, letting $n \rightarrow +\infty$, we find that

$$\int_{TM} L_G(x, v, 0) d\tilde{\mu} = -c_0.$$

Therefore, the limit $\tilde{\mu}$ is a Mather measure of H . By (2.5), we get

$$\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v) \leq -\varepsilon_2,$$

which contradicts (2.3). The right-hand side inequality in (2.4) can be proved similarly. \square

In the following, we will denote by $\bar{\lambda} > 0$ and $T_0 > 0$ the constants given by Lemma 2.10.

LEMMA 2.11. *Let $\lambda \in (0, \bar{\lambda})$ and $x \in M$. The following holds:*

(i) *for any $t \in (-\infty, -T_0]$, we have*

$$\varepsilon_2 t \leq \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds \leq \varepsilon_1 t.$$

As a consequence, $e^{\lambda \int_{-\infty}^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} = 0$.

(ii) *For any $T \geq T_0$, we have*

$$\frac{e^{-\lambda \varepsilon_2 T_0} - e^{-\lambda \varepsilon_2 T}}{\lambda \varepsilon_2} \leq \int_{-T}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \leq \frac{1}{\lambda \varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{\lambda K}.$$

In particular,

$$\frac{e^{-\lambda\varepsilon_2 T_0}}{\lambda\varepsilon_2} \leq \int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \leq \frac{1}{\lambda\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{\lambda K}. \quad (2.6)$$

Proof. Item (i) is a direct consequence of Lemma 2.10. It remains to prove Item (ii). By Item (i) and (L1) we have

$$\begin{aligned} \int_{-T}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt &= \int_{-T}^{-T_0} e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt + \int_{-T_0}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \\ &\leq \int_{-T}^{-T_0} e^{\lambda\varepsilon_1 t} dt + \int_{-T_0}^0 e^{-\lambda K t} dt = \frac{e^{-\lambda\varepsilon_1 T_0} - e^{-\lambda\varepsilon_1 T}}{\lambda\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{\lambda K} \leq \frac{1}{\lambda\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{\lambda K}. \end{aligned}$$

For the other side, we have

$$\begin{aligned} \int_{-T}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt &\geq \int_{-T}^{-T_0} e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \\ &\geq \int_{-T}^{-T_0} e^{\lambda\varepsilon_2 t} dt = \frac{e^{-\lambda\varepsilon_2 T_0} - e^{-\lambda\varepsilon_2 T}}{\lambda\varepsilon_2}. \end{aligned}$$

Let $T \rightarrow +\infty$, we then get (2.6). \square

We proceed by associating to each calibrated curve a probability measure on TM . These probability measures will play a key role in the proof of the convergence result stated in Theorem 2.2. They can be regarded as a generalization to the case at issue of the analogous measures first introduced in [11, formula (3.5)]. They already appeared in this exact form in [5, 41, 7].

DEFINITION 2.12. We define probability measures $\tilde{\mu}_\lambda^x$ on TM by

$$\int_{TM} f(y, v) d\tilde{\mu}_\lambda^x(y, v) = \frac{\int_{-\infty}^0 f(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t)) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}, \quad \forall f \in C_c(TM).$$

By (2.6), the measure $\tilde{\mu}_\lambda^x$ is well-defined for $\lambda \in (0, \bar{\lambda})$.

The following holds, cf. [11, Proposition 3.6], [5, Proposition 4.5], [41, Proposition 5.8], [7, Proposition 7.5].

LEMMA 2.13. *The family $(\tilde{\mu}_\lambda^x)_{\lambda \in (0, \bar{\lambda})}$ has support contained in a common compact subset of TM , in particular it is relatively compact in $\mathcal{P}(TM)$. Furthermore, if $\tilde{\mu}_{\lambda_n}^x \xrightarrow{*} \tilde{\mu}$ in $\mathcal{P}(TM)$ for $\lambda_n \rightarrow 0^+$, then $\tilde{\mu}$ is a Mather measure.*

Proof. The first part is a direct consequence of Lemma 2.9. It remains to prove that the limit $\tilde{\mu}$ is a Mather measure. We first prove that $\tilde{\mu}$ is closed. For $f \in C^1(M)$, we have, by

integrating by parts,

$$\begin{aligned} \int_{TM} D_x f(v) d\tilde{\mu}_\lambda^x(y, v) &= \frac{\int_{-\infty}^0 \frac{d}{dt} \left(f(\gamma_\lambda^x(t)) \right) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt} \\ &= \frac{f(x) - \int_{-\infty}^0 f(\gamma_\lambda^x(t)) \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}. \end{aligned}$$

By (L1) and $e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} > 0$, we have

$$\begin{aligned} &\left| \int_{-\infty}^0 f(\gamma_\lambda^x(t)) \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt \right| \\ &= \left| \int_{-\infty}^0 \lambda f(\gamma_\lambda^x(t)) \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \right| \\ &\leq \lambda \|f\|_\infty K \int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \leq K \left(\frac{1}{\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{K} \right) \|f\|_\infty, \end{aligned}$$

where, for the last inequality, we have used (2.6). By (2.6) again, we have

$$\int_{TM} D_x f(v) d\tilde{\mu}_\lambda^x(y, v) \leq \frac{\lambda \varepsilon_2}{e^{-\lambda \varepsilon_2 T_0}} K \left(\frac{1}{\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{K} \right) \|f\|_\infty \rightarrow 0,$$

as $\lambda \rightarrow 0^+$.

We then prove that $\tilde{\mu}$ is minimizing. Since $t \mapsto u_\lambda(\gamma_\lambda^x(t))$ is Lipschitz continuous, and γ_λ^x is a $(u_\lambda, L_\lambda, c_0)$ -calibrated curve, for a.e. $t < 0$ we have

$$\frac{d}{dt} u_\lambda(\gamma_\lambda^x(t)) = L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) + c_0.$$

Then

$$\begin{aligned} &\int_{TM} (L_G(x, v, 0) + c_0) d\tilde{\mu}_\lambda^x(x, v) \\ &= \frac{\int_{-\infty}^0 (L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) + c_0) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt} \\ &= \frac{\int_{-\infty}^0 \left(L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) + c_0 - \Delta_\lambda(t) \right) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt} \\ &= \frac{\int_{-\infty}^0 \left(\frac{d}{dt} \left(u_\lambda(\gamma_\lambda^x(t)) \right) - \Delta_\lambda(t) \right) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}, \end{aligned}$$

where

$$\Delta_\lambda(t) = L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) - L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0).$$

Similarly to the first part of the proof, we have

$$\frac{\int_{-\infty}^0 \frac{d}{dt} u_\lambda(\gamma_\lambda^x(t)) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt} \leq \frac{\lambda \varepsilon_2}{e^{-\lambda \varepsilon_2 T_0}} K \left(\frac{1}{\varepsilon_1} + \frac{e^{\lambda K T_0}}{K} \right) C \rightarrow 0,$$

as $\lambda \rightarrow 0^+$, where C is a uniform bound on the $\|u_\lambda\|_\infty$.

Finally, to bound the error term we use (L1) to find

$$|\Delta_\lambda(t)| = \left| L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) - L(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) \right| \leq \lambda K C.$$

Therefore, as $\lambda \rightarrow 0^+$ along the sequence (λ_n) , we conclude that $\tilde{\mu}$ is a Mather measure. \square

The following lemma will be crucial for the proof of the convergence result, cf. [11, Lemma 3.7], [5, Lemma 4.7], [7, Lemma 7.7].

LEMMA 2.14. *Let w be any subsolution of (E_0) . For every $x \in M$ and $\lambda \in (0, \bar{\lambda})$, we have*

$$u_\lambda(x) \geq w(x) - \frac{\int_{TM} w(y) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}_\lambda^x(y, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}_\lambda^x(y, v)} + R_\lambda(x),$$

where $\lim_{\lambda \rightarrow 0^+} R_\lambda(x) = 0$.

Proof. For $\varepsilon > 0$, using Theorem 1.15 we take $w_\varepsilon \in C^\infty(M)$ such that $\|w_\varepsilon - w\|_\infty \leq \varepsilon$ and

$$G(x, D_x w_\varepsilon, 0) \leq c_0 + \varepsilon, \quad \forall x \in M.$$

Using the Fenchel inequality we have for all $(x, v) \in TM$,

$$\begin{aligned} L_G(x, v, 0) &\geq L_G(x, v, 0) + G(x, D_x w_\varepsilon, 0) - c_0 - \varepsilon \\ &\geq D_x w_\varepsilon(v) - c_0 - \varepsilon. \end{aligned}$$

Since $t \mapsto u_\lambda(\gamma_\lambda^x(t))$ is Lipschitz continuous, and γ_λ^x is a $(u_\lambda, L_\lambda, c_0)$ -calibrated curve, for a.e. $t < 0$ we have

$$\begin{aligned} \frac{d}{dt} u_\lambda(\gamma_\lambda^x(t)) &= L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) + c_0 \tag{2.7} \\ &\geq L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) - L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) + D_{\gamma_\lambda^x(t)} w_\varepsilon(\dot{\gamma}_\lambda^x(t)) - \varepsilon \\ &= \frac{d}{dt} w_\varepsilon(\gamma_\lambda^x(t)) + \lambda \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) u_\lambda(\gamma_\lambda^x(t)) - \varepsilon + \Omega_{\lambda, x}(t), \end{aligned}$$

where

$$\begin{aligned} \Omega_{\lambda, x}(t) &:= L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), \lambda u_\lambda(\gamma_\lambda^x(t))) \\ &\quad - L_G(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) - \lambda \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) u_\lambda(\gamma_\lambda^x(t)). \end{aligned}$$

Let us estimate the error term $\Omega_{\lambda, x}(t)$. Let us set $S := \{(x, v) \in TM : \|v\|_x \leq \hat{\kappa}\}$, where $\hat{\kappa} > 0$ is the Lipschitz constant of the curves $\{\gamma_\lambda^x : \lambda \in (0, \lambda_0)\}$, according to Lemma 2.9. By (L4) we have

$$|\Omega_{\lambda, x}(t)| \leq \lambda C \eta_S(\lambda C) \quad \text{for all } t \leq 0 \text{ and } \lambda \in (0, \lambda_0), \tag{2.8}$$

where C is a uniform bound on the $\|u_\lambda\|_\infty$. By multiplying both sides of (2.7) by $e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds}$ and by rearranging terms, we obtain, for a.e. $t < 0$,

$$\frac{d}{dt} \left(u_\lambda(\gamma_\lambda^x(t)) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) \geq \left(\frac{d}{dt} w_\varepsilon(\gamma_\lambda^x(t)) - \varepsilon + \Omega_{\lambda,x}(t) \right) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds}.$$

Integrating the above inequality over the interval $(-T, 0]$ where $T \geq T_0$ as stated in Lemma 2.11, and using an integration by parts, we have

$$\begin{aligned} u_\lambda(x) - u_\lambda(\gamma_\lambda^x(-T)) e^{\lambda \int_{-T}^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \\ \geq w_\varepsilon(x) - w_\varepsilon(\gamma_\lambda^x(-T)) e^{\lambda \int_{-T}^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \\ - \int_{-T}^0 w_\varepsilon(\gamma_\lambda^x(t)) \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt + \int_{-T}^0 (\Omega_{\lambda,x}(t) - \varepsilon) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ it follows that,

$$\begin{aligned} u_\lambda(x) \geq w(x) - (\|w\|_\infty + C) e^{\lambda \int_{-T}^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \\ - \int_{-T}^0 w(\gamma_\lambda^x(t)) \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt + \int_{-T}^0 \Omega_{\lambda,x}(t) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt. \end{aligned}$$

From Lemma 2.11 we infer that the maps

$$t \mapsto e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \quad \text{and} \quad t \mapsto \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right)$$

are in $L^1((-\infty, 0])$ and converge to 0 as $t \rightarrow -\infty$. By taking also into account (2.7), we can send $T \rightarrow +\infty$ in the above inequality, to get, by the Dominated Convergence Theorem,

$$\begin{aligned} u_\lambda(x) \geq w(x) \\ - \int_{-\infty}^0 w(\gamma_\lambda^x(t)) \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt + \int_{-\infty}^0 \Omega_{\lambda,x}(t) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \\ =: w(x) - I_\lambda + R_\lambda(x). \end{aligned}$$

By Definition 2.12, we have

$$\begin{aligned} I_\lambda &= -\lambda \int_{-\infty}^0 w(\gamma_\lambda^x(t)) \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \\ &= -\lambda \left(\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt \right) \int_{TM} w(y) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}_\lambda^x(y, v). \end{aligned}$$

According to Lemma 2.11-(i), for $\lambda \in (0, \lambda_0)$, we derive that

$$\begin{aligned} \lambda \int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt &= \frac{\lambda \int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 \frac{d}{dt} \left(e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \right) dt} \\ &= - \frac{\int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt}{\int_{-\infty}^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(t), \dot{\gamma}_\lambda^x(t), 0) e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} dt} \\ &= - \frac{1}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}_\lambda^x(y, v)}. \end{aligned}$$

By (2.7) we also have

$$|R_\lambda(x)| \leq \lambda C \eta_S(\lambda C) \int_{-\infty}^0 e^{\lambda \int_t^0 \frac{\partial L_G}{\partial u}(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s), 0) ds} \leq \left(\frac{1}{\varepsilon_1} + \frac{e^{\lambda K T_0} - 1}{K} \right) C \eta_S(\lambda C).$$

The assertion follows by sending $\lambda \rightarrow 0^+$. \square

We are now in position to prove the first two main theorems of this section.

Proof of Theorem 2.2. Let u^* be an accumulation point of the $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ as $\lambda \rightarrow 0^+$. By Lemma 2.8, we know that $u^* \in \mathcal{S}$, so $u^* \leq u_0$ in M . Let us prove that $u^* \geq u_0$. Fix $x \in M$ and pick $w \in \mathcal{S}$. From Lemmas 2.13 and 2.14 we infer that

$$u^*(x) \geq w(x) - \frac{\int_{TM} w(y) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}},$$

for some Mather measure $\tilde{\mu}$. Since w satisfies the constraint (S), we get from this that $u^*(x) \geq w(x)$, hence $u^*(x) \geq \sup_{w \in \mathcal{S}} w(x) =: u_0(x)$. By the arbitrariness of the choice of $x \in M$, we get that u_0 is finite-valued and that $u^* \geq u_0$ in M . We conclude that u_0 is the unique accumulation point of the family of functions $(u_\lambda)_{\lambda \in (0, \lambda_0)}$ as $\lambda \rightarrow 0^+$. The proof is complete. \square

Proof of Theorem 2.3. We denote

$$\hat{u}_0(x) := \inf_{\tilde{\mu} \in \tilde{\mathcal{M}}} \frac{\int_{TM} h(y, x) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)}, \quad x \in M.$$

Note that \hat{u}_0 is finite-valued, as $\hat{u}_0 \geq \min_{M \times M} h > -\infty$. We start by remarking that \hat{u}_0 is a subsolution of (E₀). Indeed, for every fixed $\tilde{\mu} \in \tilde{\mathcal{M}}$, the function $h_{\tilde{\mu}} : M \rightarrow \mathbb{R}$, $x \mapsto \int_{TM} h(y, x) d\tilde{\mu}(y)$ is a convex combination of the family of critical solutions $(h_y)_{y \in M}$, where $h_y(x) = h(y, x)$. By the convexity of H in the momentum and the equi-Lipschitz character of the critical subsolutions, see Propositions 1.3 and 1.4, it follows that each $h_{\tilde{\mu}}$ is a critical subsolution. By Proposition 1.3 again, we infer that a finite valued infimum of critical subsolutions is itself a critical subsolution. Therefore \hat{u}_0 is a critical subsolution.

Let us now prove that $u_0 \leq \hat{u}_0$. By Proposition 1.4 we know that $u_0(x) \leq u_0(y) + h(y, x)$ for all $x, y \in M$. Let us integrate this inequality with respect to a Mather measure

$\tilde{\mu} \in \widetilde{\mathfrak{M}}$. By assumption and by definition of Mather set, we have $\frac{\partial L_G}{\partial u}(x, v, 0) \leq 0$ for all $(x, v) \in \text{supp}(\tilde{\mu})$. We infer

$$\begin{aligned} u_0(x) & \int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v) \\ & \geq \int_{TM} u_0(y) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v) + \int_{TM} h(y, x) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v) \\ & \geq \int_{TM} h(y, x) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v), \end{aligned}$$

where, for the last inequality, we used that u_0 satisfies (S). From this we get

$$u_0(x) \leq \frac{\int_{TM} h(y, x) \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(y, v, 0) d\tilde{\mu}(y, v)} \quad \text{for all } x \in M.$$

By taking the inf with respect to $\tilde{\mu} \in \widetilde{\mathfrak{M}}$ of the right-hand side term in the above inequality, we get $u_0 \leq \hat{u}_0$.

Last, we prove that $u_0 \geq \hat{u}_0$. For every fixed $z \in \mathcal{A}$, set

$$U_z(x) := -h(x, z) + \hat{u}_0(z), \quad x \in M.$$

Then U_z is a subsolution of (E_0) , by Proposition 1.5. Furthermore, for every $x \in M$ and $\tilde{\mu} \in \widetilde{\mathfrak{M}}$, we have

$$\frac{\int_{TM} U_z(x) \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v)} = -\frac{\int_{TM} h(x, z) \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v)}{\int_{TM} \frac{\partial L_G}{\partial u}(x, v, 0) d\tilde{\mu}(x, v)} + \hat{u}_0(z) \leq 0,$$

which implies that $U_z \in \mathcal{S}$. Then $u_0(x) \geq U_z(x)$ for all $x \in M$. In particular, by taking $x = z \in \mathcal{A}$, we get

$$u_0(z) \geq U_z(z) = -h(z, z) + \hat{u}_0(z) = \hat{u}_0(z).$$

We have thus shown that $u_0 \geq \hat{u}_0$ on \mathcal{A} , hence $u_0 \geq \hat{u}_0$ on M according to Proposition 1.7. \square

We proceed by proving the third main theorem of this section, namely the trichotomy result stated in Theorem 2.5. We start by establishing a sort of Harnack-type inequality for subsolutions of (E_λ) .

LEMMA 2.15. *There exists a constant $A_+ > 0$ such that, if $\lambda \in (0, 1)$ and $w_\lambda : M \rightarrow \mathbb{R}$ is a subsolution to (E_λ) , then*

$$\min_{x \in M} w_\lambda(x) \leq \max_{x \in M} w_\lambda(x) \leq \min_{x \in M} w_\lambda(x) + A_+ \left(1 + \lambda \min_{x \in M} |w_\lambda(x)| \right). \quad (2.9)$$

Proof. The left hand side inequality is obvious. Let us prove the right hand side inequality. Let $x_\lambda \in M$ such that $M_\lambda = w_\lambda(x_\lambda) = \max w_\lambda$. Let $y_\lambda \in M$ such that $m_\lambda = w_\lambda(y_\lambda) = \min w_\lambda$. Denote $d_\lambda := d(y_\lambda, x_\lambda)$. Let $\zeta : [0, d_\lambda] \rightarrow M$ be a geodesic satisfying $\zeta(0) = y_\lambda$

and $\zeta(d_\lambda) = x_\lambda$ with constant speed 1. By $w_\lambda \prec L_\lambda + c_0$ and (L1), we get

$$\begin{aligned} w_\lambda(\zeta(s)) &\leq w_\lambda(y_\lambda) + \int_0^s \left[L_G(\zeta(\tau), \dot{\zeta}(\tau), \lambda w_\lambda(\zeta(\tau))) + c_0 \right] d\tau \\ &\leq m_\lambda + \int_0^s \left[L_G(\zeta(\tau), \dot{\zeta}(\tau), \lambda m_\lambda) + c_0 + \lambda K(w_\lambda(\zeta(\tau)) - m_\lambda) \right] d\tau \\ &\leq m_\lambda + \int_0^s \left[L_G(\zeta(\tau), \dot{\zeta}(\tau), 0) + c_0 + \lambda K|m_\lambda| \right] d\tau + \lambda K \int_0^s \left[w_\lambda(\zeta(\tau)) - m_\lambda \right] d\tau \\ &\leq m_\lambda + (C_{L_G} + \lambda K|m_\lambda|)d_\lambda + \lambda K \int_0^s \left[w_\lambda(\zeta(\tau)) - m_\lambda \right] d\tau \end{aligned}$$

where $C_{L_G} := \max_{x \in M, \|v\|_x \leq 1} |L_G(x, v, 0) + c_0|$. By the Gronwall inequality we infer

$$w_\lambda(\zeta(s)) - m_\lambda \leq (C_{L_G} + \lambda K|m_\lambda|)d_\lambda e^{\lambda K s} \leq (C_{L_G} + \lambda K|m_\lambda|)d_\lambda e^{\lambda K d_\lambda}, \quad \forall s \in (0, d_\lambda].$$

Taking $s = d_\lambda$, and recalling that $\lambda \in (0, 1)$, we have

$$M_\lambda = w_\lambda(x_\lambda) \leq m_\lambda + (C_{L_G} + \lambda K|m_\lambda|)d_\lambda e^{\lambda K d_\lambda} \leq m_\lambda + (C_{L_G} + \lambda K|m_\lambda|)\bar{D}e^{K\bar{D}},$$

where $\bar{D} := \text{diam}(M)$. The result follows taking $A_+ = \max(C_{L_G}, K)\bar{D}e^{K\bar{D}}$. \square

As a consequence, we derive the following key proposition. It will be also used in Section 3 to show the existence of diverging families of solutions.

PROPOSITION 2.16. *Let Λ be a subset of $(0, 1)$ having 0 as accumulation point. Let $(w_\lambda)_{\lambda \in \Lambda}$ be a family of subsolutions of (E_λ) .*

- (i) *If, for each $\lambda \in \Lambda$, there is a point $x_\lambda \in M$ such that $w_\lambda(x_\lambda) \rightarrow -\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$, then w_λ uniformly converges to $-\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$.*
- (ii) *If, for each $\lambda \in \Lambda$, there is a point $x_\lambda \in M$ such that $w_\lambda(x_\lambda) \rightarrow +\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$, then w_λ uniformly converges to $+\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$.*

Proof. As in the previous proof, we denote $M_\lambda = \max w_\lambda$ and $m_\lambda = \min w_\lambda$.

Let us prove assertion (i). The hypothesis $w_\lambda(x_\lambda) \rightarrow -\infty$ is equivalent to $m_\lambda \rightarrow -\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$. Then restricting to $\lambda \in \Lambda \cap (0, 1)$, (2.9) implies that $m_\lambda \sim M_\lambda$ hence $M_\lambda \rightarrow -\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$.

Let us prove item (ii). The hypothesis $w_\lambda(x_\lambda) \rightarrow +\infty$ is equivalent to $M_\lambda \rightarrow +\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$. Then restricting to $\lambda \in \Lambda \cap (0, 1)$, (2.9) implies that $m_\lambda \sim M_\lambda$ hence $m_\lambda \rightarrow +\infty$ as $\lambda \rightarrow 0^+$, $\lambda \in \Lambda$. \square

We are now in position to prove Theorem 2.5. We recall that, in what follows, u_0 still denotes the critical solution provided by Theorem 2.2

Proof of Theorem 2.5. Let us denote by S_λ the set of solutions to (E_λ) . Define

$$S_\lambda^- := \{v_\lambda \text{ is a solution of } (E_\lambda) \text{ with } v_\lambda < u_0 - 1\}$$

and

$$\psi(\lambda) := \sup\{v_\lambda(x) : v_\lambda \in S_\lambda^-, x \in M\}.$$

If $S_\lambda^- = \emptyset$, we set $\psi(\lambda) = -\infty$.

We claim that $\lim_{\lambda \rightarrow 0} \psi(\lambda) = -\infty$. Otherwise, there is $A > 0$, sequences $\lambda_n \rightarrow 0^+$, $v_{\lambda_n} \in S_{\lambda_n}$ and $y_n \in M$ such that

$$v_{\lambda_n}(y_n) > -A, \quad \forall n \in \mathbb{N}. \quad (2.10)$$

We claim that v_{λ_n} is uniformly bounded from below. If not, there is $z_n \in M$ such that

$$v_{\lambda_n}(z_n) \rightarrow -\infty \quad \text{as } n \rightarrow +\infty.$$

By Proposition 2.16, v_{λ_n} uniformly converges to $-\infty$, which contradicts (2.10). Then, according to Theorem 2.2 and Remark 2.4, the sequence v_{λ_n} uniformly converges to the only possible limit u_0 , which contradicts the fact that $v_{\lambda_n} \in S_{\lambda_n}$ for all $n \in \mathbb{N}$.

In a similar manner, define

$$S_\lambda^+ := \{v_\lambda \text{ is a solution of } (E_\lambda) \text{ with } v_\lambda > u_0 + 1\}$$

and

$$\varphi(\lambda) := \inf\{v_\lambda(x) : v_\lambda \in S_\lambda^+, x \in M\}.$$

If $S_\lambda^+ = \emptyset$, we set $\varphi(\lambda) = +\infty$. The same proof yields that $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = +\infty$.

Define

$$\theta(\lambda) := \sup\{|v_\lambda(x) - u_0(x)|, v_\lambda \in S_\lambda \setminus (S_\lambda^- \cup S_\lambda^+), x \in M\}.$$

We claim that $\lim_{\lambda \rightarrow 0} \theta(\lambda) = 0$. Otherwise, there is $\varepsilon > 0$, a sequence of discount factors $\lambda_n \rightarrow 0^+$ and of solutions $v_{\lambda_n} \in S_{\lambda_n} \setminus (S_{\lambda_n}^- \cup S_{\lambda_n}^+)$ and $y_n \in M$ such that

$$|v_{\lambda_n}(y_n) - u_0(y_n)| > \varepsilon, \quad \forall n. \quad (2.11)$$

Since $v_{\lambda_n} \in S_{\lambda_n} \setminus (S_{\lambda_n}^- \cup S_{\lambda_n}^+)$, there is also a point x_n such that

$$|v_{\lambda_n}(x_n) - u_0(x_n)| \leq 1.$$

Similarly to the first step, we infer from Proposition 2.16 that v_{λ_n} is uniformly bounded. Then, according to Theorem 2.2 and Remark 2.4, the sequence v_{λ_n} uniformly converges to the only possible limit u_0 . But again this yields a contradiction as $\|v_{\lambda_n} - u_0\|_\infty > \varepsilon$ for all $n \in \mathbb{N}$. This concludes the proof. \square

3. THE LINEAR CASE

In this section we continue our analysis on the vanishing discount problem in the case when G depends linearly on u . Hence we will consider a Hamilton-Jacobi equation with discount factor $\lambda > 0$ of the form

$$\lambda a(x)u(x) + H(x, D_x u) = c_0 \quad \text{in } M, \quad (\text{HJ}_\lambda)$$

where we assume that $H : T^*M \rightarrow \mathbb{R}$ is a continuous function satisfying (H1)-(H2) (convexity and superlinearity) and the coefficient a is a continuous function on M satisfying the following condition:

$$\int_{TM} a(x) d\tilde{\mu} > 0 \quad \text{for all } \tilde{\mu} \in \widetilde{\mathfrak{M}}, \quad (a1)$$

where $\widetilde{\mathfrak{M}}$ denotes the compact and convex subset of $\mathcal{P}(TM)$ made up by Mather measures associated with H . Without any loss of generality, we shall restrict to the case $\lambda \in (0, 1)$.

Equations of the form (HJ_λ) can be regarded as a model example for the more general Hamilton-Jacobi equations of contact type considered in Section 2, cf. equation (E_λ) . The full Hamiltonian is then given by $G(x, p, u) = a(x)u + H(x, p)$. The Hamiltonian G then fulfills all the hypotheses of the previous section. In particular, it is Lipschitz with respect to u with Lipschitz constant $\|a\|_\infty$.

We will be concerned with the issue of existence of solutions to equation (HJ_λ) , at least for small values of $\lambda \in (0, 1)$, and their asymptotic behavior as $\lambda \rightarrow 0^+$. In Section 3.1 we show the existence of the maximal solution to (HJ_λ) for values of $\lambda \in (0, 1)$ small enough. These solutions are shown to be equi-bounded and equi-Lipschitz, therefore, in view of the results established in Section 2, they converge to a solution of the limit critical equation

$$H(x, Du) = c_0 \quad \text{in } M. \quad (\text{HJ}_0)$$

In Section 3.2, under further natural conditions on the coefficient a , we investigate the existence of possible other families of solutions of (HJ_λ) and we describe their asymptotic behavior as $\lambda \rightarrow 0^+$.

3.1. A converging family of solutions. We recall that $L : TM \rightarrow \mathbb{R}$ is the Lagrangian associated with H via the Legendre transform, namely

$$L(x, v) := \sup_{p \in T_x^* M} (p(v) - H(x, p)) \quad \text{for all } (x, v) \in TM.$$

The constant $c_0 \in \mathbb{R}$ is the critical value of H . We define the following value function, for $x \in M$,

$$V_\lambda(x) := \inf_{\gamma} \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds,$$

where the infimum is taken among absolutely continuous curves $\gamma : (-\infty, 0] \rightarrow M$ with $\gamma(0) = x$. This definition is inspired by the formula given in [41, Theorem 4.8] to represent the unique solution of (HJ_λ) in the case $a \geq 0$ in M and $a > 0$ on \mathcal{A} .

The following holds.

THEOREM 3.1. *There exists $\lambda_0 \in (0, 1)$ such that for every $\lambda \in (0, \lambda_0)$ the following holds:*

- (i) *the value function $V_\lambda : M \rightarrow \mathbb{R}$ is finite-valued;*
- (ii) *the functions $\{V_\lambda : \lambda \in (0, \lambda_0)\}$ are equi-bounded and equi-Lipschitz;*
- (iii) *the value function V_λ is the maximal subsolution of (HJ_λ) . In particular, it is the maximal solution of (HJ_λ) ;*
- (iv) *for every $x \in M$, there exists a curve $\gamma_\lambda^x : (-\infty, 0] \rightarrow M$ such that*

$$V_\lambda(x) = \int_{-\infty}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds.$$

Furthermore, the curve γ_λ^x is $\hat{\kappa}$ -Lipschitz, for some $\hat{\kappa} > 0$ independent of $\lambda \in (0, \lambda_0)$ and $x \in M$.

REMARK 3.2. For every $\lambda > 0$, define

$$c_\lambda := \inf_{u \in C^\infty(M)} \sup_{x \in M} \{H(x, Du) + \lambda a(x)u\}.$$

According to [30, 36], if (HJ_λ) has solutions, then $c_0 \geq c_\lambda$. Theorem 3.1 implies that $c_0 \geq c_\lambda$ for every $\lambda > 0$ small enough if (a1) holds. We also point out that the value provided by the above inf-sup with $\lambda = 0$ agrees with the critical constant c_0 , as it is well known, see for instance [21, 10].

According to Theorem 2.2, the functions V_λ uniformly converge to a critical solution u_0 as $\lambda \rightarrow 0^+$, where u_0 is the maximal subsolution w of (HJ_0) satisfying

$$\int_{TM} w(x)a(x)d\tilde{\mu}(x, v) \leq 0, \quad \forall \tilde{\mu} \in \widetilde{\mathfrak{M}}.$$

We can furthermore strengthen the conclusion of Theorem 2.5 on the asymptotic behavior, as $\lambda \rightarrow 0^+$, of all possible families of solutions to equation (HJ_λ) . Theorem 3.1 in fact rules out the possibility that there exist families of solutions that diverge to $+\infty$. The precise statement is the following.

THEOREM 3.3. *There exist $\psi : (0, 1) \rightarrow [-\infty, +\infty)$ and $\theta : (0, 1) \rightarrow \mathbb{R}$ with*

$$\lim_{\lambda \rightarrow 0} \psi(\lambda) = -\infty, \quad \lim_{\lambda \rightarrow 0} \theta(\lambda) = 0$$

such that, if v_λ is a solution of (HJ_λ) , then either $v_\lambda \leq \psi(\lambda)$ or $\|v_\lambda - u_0\|_\infty \leq \theta(\lambda)$ for all $\lambda \in (0, 1)$.

The remainder of this subsection is devoted to prove the statement of Theorem 3.1. This will be obtained via a series of intermediate results.

A key step in order to establish that the value function is a (Lispchitz) viscosity subsolution to (HJ_0) is to prove that it satisfies the Dynamic Programming Principle. Please note that the next proposition is valid even when V_λ is not finite-valued.

PROPOSITION 3.4. *(Dynamic Programming Principle). Let $\lambda \in (0, 1)$. For each absolutely continuous curve $\gamma : [b_1, b_2] \rightarrow M$, we have*

$$e^{-\lambda \int_{b_2}^0 a(\gamma(\tau))d\tau} V_\lambda(\gamma(b_2)) \leq e^{-\lambda \int_{b_1}^0 a(\gamma(\tau))d\tau} V_\lambda(\gamma(b_1)) + \int_{b_1}^{b_2} e^{-\lambda \int_s^0 a(\gamma(\tau))d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds.$$

Proof. We start by claiming that we can reduce to the case $b_2 = 0$, without any loss of generality. Indeed, by multiplying the above inequality for $e^{\lambda \int_{b_2}^0 a(\gamma(\tau))d\tau}$, we get

$$V_\lambda(\gamma(b_2)) \leq e^{-\lambda \int_{b_1}^{b_2} a(\gamma(\tau))d\tau} V_\lambda(\gamma(b_1)) + \int_{b_1}^{b_2} e^{-\lambda \int_s^{b_2} a(\gamma(\tau))d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds.$$

The claim follows by making the change of variables $\tau' := \tau - b_2$, $s' := s - b_2$ and by replacing γ with $\gamma_{-b_2}(\cdot) := \gamma(\cdot + b_2)$.

Let us then prove the assertion with $b_2 = 0$. For each $\xi : (-\infty, 0] \rightarrow M$ with $\xi(0) = \gamma(b_1)$, we define

$$\bar{\gamma}(t) := \begin{cases} \gamma(t) & \text{if } t \in [b_1, 0] \\ \xi(t - b_1) & \text{if } t \in (-\infty, b_1) \end{cases}$$

Then, by definition of the value function, we have

$$\begin{aligned} V_\lambda(\gamma(0)) &\leq \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\bar{\gamma}(\tau)) d\tau} \left(L(\bar{\gamma}(s), \dot{\bar{\gamma}}(s)) + c_0 \right) ds \\ &= \int_{b_1}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \\ &\quad + e^{-\lambda \int_{b_1}^0 a(\gamma(\tau)) d\tau} \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\xi(\tau)) d\tau} \left(L(\xi(s), \dot{\xi}(s)) + c_0 \right) ds. \end{aligned}$$

Taking the infimum among all ξ , we get the assertion. \square

We now proceed to show V_λ is a bounded function on M , at least for $\lambda \in (0, 1)$ small enough. We start by proving the following upper bound.

LEMMA 3.5. *There is a constant $\widehat{C}_v^+ > 0$ such that $V_\lambda(x) \leq \widehat{C}_v^+/\lambda$ for all $x \in M$ and $\lambda \in (0, 1)$.*

Proof. By condition (a1), there is a point $x_0 \in M$ such that $a(x_0) > 0$. We denote $d := d(x_0, x)$, $\overline{D} := \text{diam}(M)$. We take a geodesic $\zeta : [-d, 0] \rightarrow M$ with $\zeta(-d) = x_0$ and $\zeta(0) = x$. Define

$$\bar{\gamma}(t) = \begin{cases} \zeta(t), & -d \leq t \leq 0, \\ x_0, & t < -d \end{cases}$$

and set $C_L := \max_{x \in M, \|v\|_x \leq 1} |L(x, v) + c_0|$. Then

$$\begin{aligned} V_\lambda(x) &\leq \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\bar{\gamma}(\tau)) d\tau} \left(L(\bar{\gamma}(s), \dot{\bar{\gamma}}(s)) + c_0 \right) ds \\ &= \int_{-d}^0 e^{-\lambda \int_s^0 a(\zeta(\tau)) d\tau} \left(L(\zeta(s), \dot{\zeta}(s)) + c_0 \right) ds \\ &\quad + e^{-\lambda \int_{-d}^0 a(\zeta(\tau)) d\tau} \limsup_{t \rightarrow +\infty} \int_{-t}^{-d} e^{-\lambda \int_s^{-d} a(x_0) d\tau} \left(L(x_0, 0) + c_0 \right) ds \\ &\leq C_L \left(\int_{-d}^0 e^{-\lambda \|a\|_\infty s} ds + e^{\lambda \|a\|_\infty d} \int_{-\infty}^{-d} e^{\lambda a(x_0)(d+s)} ds \right) \\ &= C_L \left(\frac{e^{\lambda \|a\|_\infty d} - 1}{\lambda \|a\|_\infty} + \frac{e^{\lambda \|a\|_\infty d}}{\lambda a(x_0)} \right) \leq \frac{2C_L e^{\lambda \|a\|_\infty \overline{D}}}{\lambda a(x_0)}. \end{aligned} \quad \square$$

Now we know that the infimum in $V_\lambda(x)$ is taken among curves in

$$\Gamma_\lambda := \left\{ \gamma : (-\infty, 0] \rightarrow M : \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \leq \frac{\widehat{C}_v^+}{\lambda} \right\}.$$

For $\gamma : (-\infty, 0] \rightarrow M$ and $t > 0$, we define

$$\alpha(\lambda, \gamma, t) := \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} ds$$

and

$$\beta(\lambda, \gamma, t) := e^{-\lambda \int_{-t}^0 a(\gamma(s)) ds}.$$

Since $\frac{d}{dt}\alpha(\lambda, \gamma, t) = \beta(\lambda, \gamma, t) > 0$, $t \mapsto \alpha(\lambda, \gamma, t)$ is increasing. Then $\lim_{t \rightarrow +\infty} \alpha(\lambda, \gamma, t)$ exists for each λ and γ , and may equal $+\infty$. We will also need the following auxiliary remark.

LEMMA 3.6. *For every $\lambda > 0$ and for all curves γ , we have*

$$-\lambda \|a\|_\infty + \frac{1}{\alpha(\lambda, \gamma, t)} \leq \frac{\beta(\lambda, \gamma, t)}{\alpha(\lambda, \gamma, t)} \leq \lambda \|a\|_\infty + \frac{1}{\alpha(\lambda, \gamma, t)}, \quad \forall t > 0.$$

Proof. A direct calculation shows that

$$\frac{\frac{d}{dt}\beta(\lambda, \gamma, t)}{\frac{d}{dt}\alpha(\lambda, \gamma, t)} = -\lambda a(\gamma(-t)).$$

By the Cauchy mean value theorem, there is $\xi \in (0, t)$ such that

$$\frac{\beta(\lambda, \gamma, t) - \beta(\lambda, \gamma, 0)}{\alpha(\lambda, \gamma, t) - \alpha(\lambda, \gamma, 0)} = \frac{\beta(\lambda, \gamma, t) - 1}{\alpha(\lambda, \gamma, t)} = \frac{\frac{d}{dt}\beta(\lambda, \gamma, \xi)}{\frac{d}{dt}\alpha(\lambda, \gamma, \xi)} = -\lambda a(\gamma(-\xi)),$$

which implies the conclusion. \square

We distill in the next lemma an argument that we will repeatedly use in the sequel.

LEMMA 3.7. *Assume there are sequences $\lambda_n \rightarrow 0^+$, $\gamma_n : (-\infty, 0] \rightarrow M$, $t_n \in (0, +\infty)$ and $\theta_n \rightarrow 0^+$ such that*

$$\alpha(\lambda_n, \gamma_n, t_n) := \int_{-t_n}^0 e^{-\lambda_n \int_s^0 a(\gamma_n(\tau)) d\tau} ds \rightarrow +\infty, \quad \text{as } n \rightarrow +\infty, \quad (3.1)$$

$$\frac{1}{\alpha(\lambda_n, \gamma_n, t_n)} \int_{-t_n}^0 e^{-\lambda_n \int_s^0 a(\gamma_n(\tau)) d\tau} \left(L(\gamma_n(s), \dot{\gamma}_n(s)) + c_0 \right) ds \leq \theta_n, \quad \forall n \in \mathbb{N}. \quad (3.2)$$

Define a probability measure $\tilde{\mu}_n \in \mathcal{P}(TM)$ by

$$\int_{TM} f d\tilde{\mu}_n := \frac{\int_{-t_n}^0 e^{-\lambda_n \int_s^0 a(\gamma_n(\tau)) d\tau} f(\gamma_n(s), \dot{\gamma}_n(s)) ds}{\alpha(\lambda_n, \gamma_n, t_n)}, \quad \forall f \in C_\ell(TM). \quad (\star)$$

Then the set $(\tilde{\mu}_n)_n$ is relatively compact in \mathcal{P}_ℓ (for the weak-* topology coming from $C_\ell(TM)$) and any of its accumulation points is a Mather measure associated with L .

Proof. Note that, due to (3.1), we have $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Since L is uniformly superlinear in the fibers, there exists a constant $C_1 \geq 0$ such that

$$\max_{n \in \mathbb{N}} \theta_n \geq \int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n(x, v) \geq \int_{TM} (\|v\|_x - C_1) d\tilde{\mu}_n \quad \text{for all } n \in \mathbb{N},$$

which readily implies that the sequence $(\tilde{\mu}_n)_n$ is a well defined sequence in \mathcal{P}_ℓ . The asserted precompactness of $(\tilde{\mu}_n)_n$ in $\mathcal{P}(TM)$ follows from Lemma 1.11.

Let $\tilde{\mu}$ be a limit point of a subsequence of $(\tilde{\mu}_n)_n$, that we will not relabel to ease notations. We are going to show that $\tilde{\mu}$ is a Mather measure.

The measure $\tilde{\mu}$ is closed: we pick $\phi \in C^1(M)$. An integration by parts shows that

$$\begin{aligned} & \int_{TM} D_x \phi(v) d\tilde{\mu}_n(x, v) \\ &= \frac{\int_{-t_n}^0 \frac{d\phi}{dt}(\gamma_n(s)) e^{-\lambda_n \int_s^0 a(\gamma_n(\tau)) d\tau} ds}{\alpha(\lambda_n, \gamma_n, t_n)} \\ &= \frac{\phi(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau} \phi(\gamma_n(-t_n))}{\alpha(\lambda_n, \gamma_n, t_n)} - \lambda_n \int_{TM} a(x) \phi(x) d\tilde{\mu}_n(x, v). \end{aligned} \quad (3.3)$$

We infer

$$\begin{aligned} \left| \int_{TM} D_x \phi(v) d\tilde{\mu}_n \right| &\leq \frac{\|\phi\|_\infty}{\alpha(\lambda_n, \gamma_n, t_n)} + \frac{e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau}}{\alpha(\lambda_n, \gamma_n, t_n)} \|\phi\|_\infty + \lambda_n \|a\|_\infty \|\phi\|_\infty \\ &= \left(\frac{1}{\alpha(\lambda_n, \gamma_n, t_n)} + \frac{\beta(\lambda_n, \gamma_n, t_n)}{\alpha(\lambda_n, \gamma_n, t_n)} + \lambda_n \|a\|_\infty \right) \|\phi\|_\infty. \end{aligned}$$

By sending $n \rightarrow +\infty$ and by using Lemma 3.6, hypothesis (3.1) and again Lemma 1.11, we derive

$$\int_{TM} D_x \phi(v) d\tilde{\mu}(x, v) = \lim_{n \rightarrow +\infty} \int_{TM} D_x \phi(v) d\tilde{\mu}_n(x, v) = 0.$$

The measure $\tilde{\mu}$ is minimizing: by (3.2) we get

$$0 = \lim_{n \rightarrow +\infty} \theta_n \geq \liminf_{n \rightarrow +\infty} \int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n(x, v) \geq \int_{TM} (L(x, v) + c_0) d\tilde{\mu}(x, v),$$

where the last inequality follows from the fact that L is continuous and bounded from below and the measures $\tilde{\mu}_n$ are weakly-* converging to $\tilde{\mu}$ in \mathcal{P}_ℓ (hence, narrowly converging), see for instance [1, Section 5.1.1].

In view of Theorem 1.8, we conclude that $\tilde{\mu}$ is a Mather measure. \square

Let $C > 0$ be a fixed constant and define, for any fixed integer $p \geq 0$,

$$\Gamma_\lambda(p) := \left\{ \gamma : (-\infty, 0] \rightarrow M : \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} (L(\gamma(s), \dot{\gamma}(s)) + c_0) ds \leq \frac{C}{\lambda^p} \right\}.$$

The information provided by the next lemma will be crucial for our upcoming analysis.

LEMMA 3.8. *Let $p \geq 0$ be a fixed integer and assume that $\Gamma_\lambda(p) \neq \emptyset$ for every $\lambda \in (0, 1)$ small enough. Then there exists $\lambda(p) \in (0, 1)$ such that*

$$A(p) := \sup \left\{ \lambda^{p+1} \alpha(\lambda, \gamma, t) : \lambda \in (0, \lambda(p)), \gamma \in \Gamma_\lambda(p), t > 0 \right\} < +\infty.$$

REMARK 3.9. Note that $\Gamma_\lambda(p+1) \supseteq \Gamma_\lambda(p)$ for every integer $p \geq 0$ and $\lambda \in (0, 1)$. In particular, when $p \geq 1$ and $C := \widehat{C}_v^+$, we have $\Gamma_\lambda(p) \neq \emptyset$ for every $\lambda \in (0, 1)$ in view of Lemma 3.5.

Proof. Let $p \geq 0$ be a fixed integer. We argue by contradiction. Assume there exist sequences $\lambda_n \rightarrow 0^+$, $\gamma_n \in \Gamma_{\lambda_n}(p)$ and $t_n \rightarrow +\infty$ such that

$$\lambda_n^{p+1} \alpha(\lambda_n, \gamma_n, t_n) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty,$$

with

$$\int_{-t_n}^0 e^{-\lambda_n \int_s^0 a(\gamma_n(\tau)) d\tau} \left(L(\gamma_n(s), \dot{\gamma}_n(s)) + c_0 \right) ds \leq \frac{C}{\lambda_n^p} + 1.$$

In particular, conditions (3.1) and (3.2) in Lemma 3.7 are satisfied with

$$\theta_n := \frac{C/\lambda_n^p + 1}{\alpha(\lambda_n, \gamma_n, t_n)}.$$

Let $\tilde{\mu}_n$ be the probability measure defined in (\star) . According to Lemma 3.7, up to extraction of a subsequence (not relabeled), the measures $\tilde{\mu}_n$ weakly converge to a Mather measure $\tilde{\mu}$.

Now we choose a subsolution $\varphi \leq -1$ of (HJ₀). For every $\varepsilon > 0$, there exists, in view of Theorem 1.15, a function $\varphi_\varepsilon \in C^\infty(M)$ satisfying

$$\|\varphi_\varepsilon - \varphi\|_\infty \leq \varepsilon \quad \text{and} \quad H(x, D_x \varphi_\varepsilon) \leq c_0 + \varepsilon \quad \text{for all } x \in M. \quad (3.4)$$

For $\varepsilon > 0$ small, we have $\varphi_\varepsilon < 0$. We get

$$\begin{aligned} \theta_n &\geq \int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n(x, v) \\ &\geq \int_{TM} (D_x \varphi_\varepsilon(v) - H(x, D_x \varphi_\varepsilon) + c_0) d\tilde{\mu}_n(x, v) \geq \int_{TM} (D_x \varphi_\varepsilon(v) - \varepsilon) d\tilde{\mu}_n(x, v) \\ &= \frac{\varphi_\varepsilon(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau} \varphi_\varepsilon(\gamma_n(-t_n))}{\alpha(\lambda_n, \gamma_n, t_n)} - \lambda_n \int_{TM} a(x) \varphi_\varepsilon(x) d\tilde{\mu}_n(x, v) - \varepsilon \\ &> \frac{\varphi_\varepsilon(\gamma_n(0))}{\alpha(\lambda_n, \gamma_n, t_n)} - \lambda_n \int_{TM} a(x) \varphi_\varepsilon(x) d\tilde{\mu}_n(x, v) - \varepsilon. \end{aligned}$$

The equality appearing above is obtained via an integration by parts (cf. proof of Lemma 3.7 equation (3.3), when we prove that $\tilde{\mu}$ is closed), while for the last inequality we have used that fact that $\varphi_\varepsilon < 0$. Now we send $\varepsilon \rightarrow 0^+$ and divide the above inequality by λ_n . We get

$$\frac{C}{\lambda_n^{p+1} \alpha(\lambda_n, \gamma_n, t_n)} + \frac{1}{\lambda_n \alpha(\lambda_n, \gamma_n, t_n)} \geq \frac{\varphi(\gamma_n(0))}{\lambda_n \alpha(\lambda_n, \gamma_n, t_n)} - \int_{TM} a(x) \varphi(x) d\tilde{\mu}_n.$$

Since $\lambda_n \alpha(\lambda_n, \gamma_n, t_n) \geq \lambda_n^{p+1} \alpha(\lambda_n, \gamma_n, t_n) \rightarrow +\infty$ as $n \rightarrow +\infty$, we get

$$\int_{TM} a(x) \varphi(x) d\tilde{\mu}(x, v) \geq 0.$$

Since for all $m \geq 0$ the function $\varphi - m \leq -1$ is also a negative subsolution of (HJ₀), by replacing φ with $\varphi - m$ in the inequality above we obtain

$$\|\varphi\|_\infty \|a\|_\infty \geq \int_{TM} a(x) \varphi(x) d\tilde{\mu}(x, v) \geq m \int_{TM} a(x) d\tilde{\mu}(x, v), \quad \text{for all } m \geq 0,$$

which implies that $\int_{TM} a(x) d\tilde{\mu} \leq 0$. This contradicts assumption (a1). \square

We now proceed to show that the value function V_λ is bounded from below, for every fixed $\lambda \in (0, \lambda(1))$, where $\lambda(1) > 0$ is the value obtained according to Lemma 3.7 with $p = 1$ and $C = \widehat{C}_v^+$, where \widehat{C}_v^+ is the constant provided by Lemma 3.5.

LEMMA 3.10. *There exists a constant $\widehat{C}_v^- > 0$ independent of λ such that*

$$V_\lambda(x) \geq -\widehat{C}_v^- \lambda^{-2} \quad \text{for all } x \in M \text{ and } \lambda \in (0, \lambda(1)).$$

Proof. Let us fix $\gamma \in \Gamma_\lambda$. For every $t > 0$ we have

$$\int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \geq -C_L^- \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} ds,$$

where $C_L^- := -\min \{ \min_{TM} (L(x, v) + c_0), 0 \} \geq 0$. We are now going to apply Lemma 3.8 with $p = 1$ and by choosing $C := \widehat{C}_v^+$ in the definition of $\Gamma_\lambda(1)$, so that $\Gamma_\lambda(1) = \Gamma_\lambda$: from the above inequality we infer

$$\int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \geq -\frac{C_L^- A(1)}{\lambda^2} \quad \text{for all } t > 0.$$

The assertion readily follows from the definition of V_λ . \square

From the information gathered so far, we know that the value function V_λ is finite-valued on M for every $\lambda \in (0, \lambda(1))$. We now proceed to get bounds for V_λ from above and from below on M independent of λ .

LEMMA 3.11. *There is a constant $\overline{C}_v^+ > 0$ such that*

$$V_\lambda(x) \leq \overline{C}_v^+ \quad \text{for all } x \in \mathcal{A} \text{ and } \lambda \in (0, \lambda(1)).$$

Proof. According to [13, Theorem 4.14 and Proposition 4.4], see also [16, Theorem 3.3], for every $x \in \mathcal{A}$ there exists a curve $\eta : \mathbb{R} \rightarrow \mathcal{A}$ with $\eta(0) = x$ such that, for every subsolution φ of (HJ₀),

$$L(\eta(s), \dot{\eta}(s)) + c_0 = \frac{d}{ds}(\varphi \circ \eta)(s) \quad \text{for a.e. } s \in \mathbb{R}. \quad (3.5)$$

Let us denote by \mathcal{K} the family of curves $\eta : \mathbb{R} \rightarrow \mathcal{A}$ satisfying (3.5). We remark for further use that these curves are equi-Lipschitz, see for instance [13, Lemma 4.9].

Pick a subsolution $\varphi \leq 0$ of (HJ₀) and fix $x \in \mathcal{A}$. From the definition of V_λ we get

$$\begin{aligned} V_\lambda(x) &\leq \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} \left(L(\eta(s), \dot{\eta}(s)) + c_0 \right) ds \\ &= \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} \frac{d}{ds}(\varphi \circ \eta)(s) ds \\ &= \limsup_{t \rightarrow +\infty} \left\{ \varphi(\eta(0)) - e^{-\lambda \int_{-t}^0 a(\eta(\tau)) d\tau} \varphi(\eta(-t)) - \lambda \alpha(\lambda, \eta, t) \int_{TM} a(x) \varphi(x) d\tilde{\mu}_t^\eta \right\}, \end{aligned} \quad (3.6)$$

where $\tilde{\mu}_t^\eta \in \mathcal{P}(TM)$ is defined by

$$\int_{TM} f(x, v) d\tilde{\mu}_t^\eta(x, v) := \frac{\int_{-t}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} f(\eta(s), \dot{\eta}(s)) ds}{\int_{-t}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} ds}, \quad \forall f \in C_c(TM).$$

The second equality in (3.6) is derived via an integration by parts as for (3.3). By assumption (a1) and compactness of the family of Mather measures $\widetilde{\mathfrak{M}}$, there exists $\varepsilon > 0$

such that

$$\int_{TM} a(x) d\tilde{\mu}(x, v) > \varepsilon \quad \text{for all } \tilde{\mu} \in \widetilde{\mathfrak{M}}. \quad (3.7)$$

We show that there is $T_0 > 0$ such that, for all curves $\eta \in \mathcal{K}$, we have

$$\int_{-t}^0 a(\eta(s)) ds > \varepsilon t \quad \text{for all } t \geq T_0.$$

We argue by contradiction. Assume there exist sequences $\eta_n \in \mathcal{K}$ and $t_n \rightarrow 0$ such that

$$\int_{-t_n}^0 a(\eta_n(s)) ds \leq \varepsilon t_n. \quad (3.8)$$

Define $\tilde{\mu}_n \in \mathcal{P}(TM)$ by

$$\int_{TM} f(x, v) d\tilde{\mu}_n := \frac{1}{t_n} \int_{-t_n}^0 f(\eta_n(s), \dot{\eta}_n(s)) ds, \quad \forall f \in C_c(TM).$$

Due to the fact that the curves $(\eta_n)_n$ are equi-Lipschitz, the measures $(\tilde{\mu}_n)_n$ have equi-compact support. In particular, up to extracting a subsequence (not relabeled), they weakly converge to a probability measure $\tilde{\mu} \in \mathcal{P}(TM)$. Furthermore

$$\lim_n \int_{TM} f(x, v) d\tilde{\mu}_n(x, v) = \int_{TM} f(x, v) d\tilde{\mu}(x, v) \quad \forall f \in C(TM).$$

We claim that $\tilde{\mu}$ is closed. Indeed, for every $\phi \in C^1(M)$ we have

$$\begin{aligned} \int_{TM} D_x \phi(v) d\tilde{\mu}(x, v) &= \lim_n \int_{TM} D_x \phi(v) d\tilde{\mu}_n(x, v) \\ &= \lim_n \frac{1}{t_n} \int_{-t_n}^0 \frac{d}{ds} (\phi \circ \eta_n)(s) ds \leq \lim_n \frac{2\|\phi\|_\infty}{t_n} = 0. \end{aligned}$$

We proceed to show that $\tilde{\mu}$ is minimizing, namely, a Mather measure. Pick a subsolution φ to (HJ₀). By exploiting (3.5), we get

$$\begin{aligned} \int_{TM} (L(x, v) + c_0) d\tilde{\mu} &= \lim_n \int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n = \lim_n \frac{1}{t_n} \int_{-t_n}^0 (L(\eta_n(s), \dot{\eta}_n(s)) + c_0) ds \\ &= \lim_n \frac{1}{t_n} \int_{-t_n}^0 \frac{d}{ds} (\varphi \circ \eta_n)(s) ds = \lim_n \frac{\varphi(\eta_n(0)) - \varphi(\eta_n(-t_n))}{t_n} = 0. \end{aligned}$$

By (3.8), we also have $\int_{TM} a(x) d\tilde{\mu}(x, v) \leq \varepsilon$, which leads to a contradiction with (3.7). Then for $t > T_0$ we have

$$e^{-\lambda \int_{-t}^0 a(\eta(\tau)) d\tau} \leq e^{-\lambda \varepsilon t}$$

and

$$\begin{aligned}
0 < \lambda\alpha(\lambda, \eta, t) &= \lambda \int_{-t}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} ds \\
&= \lambda \int_{-T_0}^0 e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} ds + \lambda \int_{-t}^{-T_0} e^{-\lambda \int_s^0 a(\eta(\tau)) d\tau} ds \\
&\leq \lambda \int_{-T_0}^0 e^{-\lambda \|a\|_\infty s} ds + \lambda \int_{-t}^{-T_0} e^{\lambda \varepsilon s} ds \\
&\leq \frac{e^{\lambda \|a\|_\infty T_0} - 1}{\|a\|_\infty} + \frac{e^{-\lambda \varepsilon T_0}}{\varepsilon} \leq \frac{e^{\|a\|_\infty T_0}}{\|a\|_\infty} + \frac{1}{\varepsilon} =: C_\eta.
\end{aligned}$$

By using these inequalities in (3.6) and recalling that $\varphi \leq 0$, we finally get

$$V_\lambda(x) \leq C_\eta \|a\|_\infty \|\varphi\|_\infty,$$

which gives the upper bound of V_λ on \mathcal{A} independent of λ . \square

By exploiting the fact that V_λ satisfies the Dynamic Programming Principle, we show that the partial upper bound obtained in Lemma 3.11 actually entails a uniform upper bound on the whole M .

PROPOSITION 3.12. *There is $C_v^+ > 0$ independent of λ such that*

$$V_\lambda(x) \leq C_v^+ \quad \text{for all } x \in M \text{ and } \lambda \in (0, \lambda(1)).$$

Proof. Fix $x \in M$ and pick a point $y \in \mathcal{A}$. Set $d := d(x, y)$, $\bar{D} := \text{diam}(M)$ and $C_L := \max_{x \in M, \|v\|_x \leq 1} |L(x, v) + c_0|$. Take a geodesic $\zeta : [-d, 0] \rightarrow M$ with $\zeta(-d) = y$, $\zeta(0) = x$ and $\|\dot{\zeta}\|_\zeta = 1$. By Proposition 3.4 we have

$$\begin{aligned}
V_\lambda(x) &\leq e^{-\lambda \int_{-d}^0 a(\zeta(\tau)) d\tau} V_\lambda(y) + \int_{-d}^0 e^{-\lambda \int_s^0 a(\zeta(\tau)) d\tau} \left(L(\zeta(s), \dot{\zeta}(s)) + c_0 \right) ds \\
&\leq e^{\lambda \|a\|_\infty d} \bar{C}_v^+ + C_L d e^{\lambda \|a\|_\infty d} \leq e^{\|a\|_\infty \bar{D}} (\bar{C}_v^+ + C_L \bar{D}),
\end{aligned}$$

which gives the sought uniform upper bound of V_λ . \square

Now that we know that V_λ is uniformly bounded from above, we can prove that V_λ is also uniformly bounded from below.

PROPOSITION 3.13. *There exist $\bar{\lambda} \in (0, \lambda(1))$ and a constant $C_v^- > 0$ independent of λ such that*

$$V_\lambda(x) \geq -C_v^- \quad \text{for all } x \in M \text{ and } \lambda \in (0, \bar{\lambda}).$$

In particular, $|V_\lambda(x)| \leq C_v := \max\{C_v^-, C_v^+\}$ for all $x \in M$ and $\lambda \in (0, \bar{\lambda})$.

Proof. By Proposition 3.12, we know that, for $\lambda < \lambda(1)$, the infimum in V_λ is taken among the set

$$\Gamma := \left\{ \gamma : (-\infty, 0] \rightarrow M : \limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \leq C_v^+ \right\}.$$

By Lemma 3.8, there exists $\lambda(0) \in (0, 1)$ (depending on C_v^+) such that

$$A(0) := \sup \left\{ \lambda \alpha(\lambda, \gamma, t) : \lambda \in (0, \lambda(0)), \gamma \in \Gamma, t > 0 \right\} < +\infty.$$

Let us set $\bar{\lambda} := \min\{\lambda(0), \lambda(1)\}$.⁷ Let $\varphi \leq -1$ be a subsolution of (HJ₀). For each $\varepsilon > 0$, we can take $\varphi_\varepsilon \in C^\infty(M)$ satisfying $\|\varphi_\varepsilon - \varphi\|_\infty \leq \varepsilon$ and $H(x, D_x \varphi_\varepsilon) \leq c_0 + \varepsilon$ for all $x \in M$, given by Theorem 1.15. For $\varepsilon > 0$ small, we have $\varphi_\varepsilon < 0$. For each $\gamma \in \Gamma$, we have

$$\begin{aligned} & \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \\ & \geq \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(D_{\gamma(s)} \varphi_\varepsilon(\dot{\gamma}(s)) - H(\gamma(s), D_{\gamma(s)} \varphi_\varepsilon) + c_0 \right) ds \\ & \geq \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(\frac{d\varphi_\varepsilon}{dt}(\gamma(s)) - \varepsilon \right) ds \\ & = \varphi_\varepsilon(\gamma(0)) - e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \varphi_\varepsilon(\gamma(-t)) - \alpha(\lambda, \gamma, t) \int_{TM} (\lambda a(x) \varphi_\varepsilon(x) + \varepsilon) d\tilde{\mu}_t^\gamma \\ & \geq \varphi_\varepsilon(\gamma(0)) - \alpha(\lambda, \gamma, t) \int_{TM} (\lambda a(x) \varphi_\varepsilon(x) + \varepsilon) d\tilde{\mu}_t^\gamma, \end{aligned}$$

where

$$\int_{TM} f(x, v) d\tilde{\mu}_t^\gamma(x, v) := \frac{\int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} f(\gamma(s), \dot{\gamma}(s)) ds}{\alpha(\lambda, \gamma, t)}, \quad \forall f \in C_c(TM).$$

We recall that $\alpha(\lambda, \gamma, t) := \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} ds$. Now let $\varepsilon \rightarrow 0^+$ to get

$$\int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \geq \varphi(\gamma(0)) - \lambda \alpha(\lambda, \gamma, t) \int_{TM} a(x) \varphi(x) d\tilde{\mu}_t^\gamma.$$

Sending $t \rightarrow +\infty$ we get

$$\limsup_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \geq -(1 + A(0) \|a\|_\infty) \|\varphi\|_\infty.$$

The bound from below readily follows from this by definition of V_λ . The last assertion is a consequence of Proposition 3.12 and of what we have just shown above. \square

We now proceed to show that the value function is Lipschitz continuous. This is indeed a consequence of this more general result.

PROPOSITION 3.14. *Let $w : M \rightarrow \mathbb{R}$ be a bounded function. Assume that w satisfies the Dynamic Programming Principle, i.e., for each absolutely continuous curve $\gamma : [-t, 0] \rightarrow M$, we have*

$$w(\gamma(0)) - e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} w(\gamma(-t)) \leq \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds. \quad (\text{DPP})$$

Then w is Lipschitz continuous, with a Lipschitz constant that only depends on L , $\text{diam}(M)$ and $\|w\|_\infty$.

⁷We recall that $\lambda(1) \in (0, 1)$ is the value obtained according to Lemma 3.7 with $p = 1$ and $C = \widehat{C}_v^+$, where \widehat{C}_v^+ is the constant provided by Lemma 3.5.

Proof. Pick $x, y \in M$. Let $\zeta : [-d, 0] \rightarrow M$ be a geodesic with $\zeta(-d) = y$ and $\zeta(0) = x$, where $d := d(x, y)$. By (DPP) we have

$$w(x) - w(y) \leq -w(y) \left(1 - e^{-\lambda \int_{-d}^0 a(\zeta(\tau)) d\tau} \right) + \int_{-d}^0 e^{-\lambda \int_s^0 a(\zeta(\tau)) d\tau} (L(\zeta(s), \dot{\zeta}(s)) + c_0) ds.$$

There is $\tau_0 \in (-d, 0)$ such that $\int_{-d}^0 a(\zeta(\tau)) d\tau = a(\zeta(\tau_0))d$. If $a(\zeta(\tau_0)) \geq 0$, we have

$$0 \leq 1 - e^{-\lambda a(\zeta(\tau_0))d} \leq e^{\lambda a(\zeta(\tau_0))d} - 1 \leq e^{\lambda \|a\|_\infty d} - 1.$$

If $a(\zeta(\tau_0)) < 0$, we have

$$0 \geq 1 - e^{-\lambda a(\zeta(\tau_0))d} \geq 1 - e^{\lambda \|a\|_\infty d}.$$

Then

$$w(x) - w(y) \leq \lambda \|w\|_\infty \frac{e^{\lambda \|a\|_\infty d} - 1}{\lambda} + C_L \int_{-d}^0 e^{-\lambda \|a\|_\infty s} ds \leq (\|a\|_\infty \|w\|_\infty + C_L) \frac{e^{\lambda \|a\|_\infty d} - 1}{\lambda \|a\|_\infty}.$$

Since $\lambda \in (0, 1)$ and $d \leq \bar{D}$, there is $C_{\bar{D}} > 0$ such that $e^{\lambda \|a\|_\infty d} - 1 \leq C_{\bar{D}} \lambda \bar{d}$. Exchanging the role of x and y , we get the conclusion. \square

As a consequence of the previous proposition and Proposition 3.13, we derive the following information.

COROLLARY 3.15. *There is $\kappa > 0$ independent of λ such that the functions $\{V_\lambda : \lambda \in (0, \bar{\lambda})\}$ are κ -Lipschitz continuous.*

Next, we show that the value function is a viscosity subsolution of the equation (HJ $_\lambda$). Indeed, the following result holds.

PROPOSITION 3.16. *Let $w \in C(M)$. Then w is a subsolution of (HJ $_\lambda$) if and only if (DPP) holds for every absolutely curve $\gamma : [-t, 0] \rightarrow M$ and every $t > 0$.*

Proof. Let us first assume that w is a viscosity subsolution of (HJ $_0$). By Proposition 1.4, we derive that w is Lipschitz continuous. Using Theorem 1.15, we take a sequence $w_n \in C^1(M)$ such that $\|w_n - w\|_\infty \leq 1/n$ and

$$\lambda a(x)w_n(x) + H(x, D_x w_n) \leq c_0 + \frac{1}{n} \quad \text{for all } x \in M.$$

For $\gamma : [-t, 0] \rightarrow M$, we have, performing the now usual integration by parts and the Fenchel inequality (1.2),

$$\begin{aligned}
 w_n(\gamma(0)) - e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} w_n(\gamma(-t)) &= \int_{-t}^0 \frac{d}{ds} \left(e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} w_n(\gamma(s)) \right) ds \\
 &= \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(\lambda a(\gamma(s)) w_n(\gamma(s)) + D_{\gamma(s)} w_n(\dot{\gamma}(s)) \right) ds \\
 &\leq \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(\lambda a(\gamma(s)) w_n(\gamma(s)) + H(\gamma(s), D_{\gamma(s)} w_n) + L(\gamma(s), \dot{\gamma}(s)) \right) ds \\
 &\leq \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(c_0 + \frac{1}{n} + L(\gamma(s), \dot{\gamma}(s)) \right) ds \\
 &\leq \frac{1}{n} \int_{-t}^0 e^{-\lambda \|a\|_\infty s} ds + \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \\
 &= \frac{e^{\lambda \|a\|_\infty t} - 1}{n \|a\|_\infty \lambda} + \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds.
 \end{aligned}$$

The assertion follows by sending $n \rightarrow +\infty$.

Conversely, let us assume that w satisfies (DPP). According to Proposition 3.14, w is Lipschitz continuous. We only need to check if w is a subsolution where w is differentiable, thanks to Proposition 1.4. Let w be differentiable at x . We take a C^1 curve $\gamma : [-t, 0] \rightarrow M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then

$$\frac{w(\gamma(0)) - e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} w(\gamma(-t))}{t} \leq \frac{1}{t} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds.$$

Letting $t \rightarrow 0^+$, we get

$$\lambda a(x)w(x) + D_x w(v) - L(x, v) \leq c_0.$$

Taking the supremum with respect to v , we get, thanks to (1.3),

$$\lambda a(x)w(x) + H(x, D_x w) \leq c_0,$$

which implies that w is a subsolution. \square

We proceed to show that the value function is the maximal viscosity subsolution of (HJ $_\lambda$), and hence a solution by maximality. We need an auxiliary lemma first.

LEMMA 3.17. *Let $\lambda > 0$ and let $\gamma : (-\infty, 0] \rightarrow M$ be an absolutely continuous curve. Let us assume that*

$$\sup_{t>0} \alpha(\lambda, \gamma, t) < +\infty. \tag{3.9}$$

Then $e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \rightarrow 0$ as $t \rightarrow +\infty$.

Proof. We argue by contradiction. Assume there exist an increasing sequence $t_n \rightarrow +\infty$ and a $\delta \in (0, 1)$ small enough such that

$$e^{-\lambda \int_{-t_n}^0 a(\gamma(\tau)) d\tau} \geq \delta \quad \text{for all } n \in \mathbb{N}.$$

Then, for all $t \in (t_n, t_n + \frac{\ln(\delta^{-1})}{\lambda \|a\|_\infty})$, we have

$$-\lambda \int_{-t}^{-t_n} a(\gamma(\tau)) d\tau \geq -\lambda \|a\|_\infty |t - t_n| \geq \ln \delta,$$

hence

$$e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} = e^{-\lambda \int_{-t}^{-t_n} a(\gamma(\tau)) d\tau} e^{-\lambda \int_{-t_n}^0 a(\gamma(\tau)) d\tau} \geq \delta^2, \quad \text{for all } n \in \mathbb{N}.$$

Let us pick $r > 0$ with $r \leq \frac{\ln(\delta^{-1})}{\lambda \|a\|_\infty}$. Up to extracting a subsequence, we can assume that $r \leq |t_{n+1} - t_n|$ for all $n \in \mathbb{N}$. Let us set $t_0 := 0$. We have

$$\int_{-t_n}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} ds \geq \sum_{i=0}^{n-1} \int_{-t_i-r}^{-t_i} e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} ds \geq r \delta^2 n \rightarrow +\infty$$

as $n \rightarrow +\infty$, which contradicts (3.9). \square

Let us prove the result announced above. We recall that the $\bar{\lambda}$ appearing in the next two statements is the real number in $(0, 1)$ provided by Proposition 3.13.

PROPOSITION 3.18. *For every fixed $\lambda \in (0, \bar{\lambda})$, the value function V_λ is the maximal subsolution of (HJ_λ) . In particular, it is a viscosity solution of (HJ_λ) .*

Proof. Let w be a subsolution of (HJ_λ) . Let us fix $x \in M$. By definition, for all $\varepsilon > 0$, there is $\gamma \in \Gamma$ such that

$$\begin{aligned} V_\lambda(x) + \varepsilon &> \limsup_{T \rightarrow +\infty} \int_{-T}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \\ &= \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \\ &\quad + e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \limsup_{T \rightarrow +\infty} \int_{-T}^0 e^{-\lambda \int_s^0 a(\xi(\tau)) d\tau} \left(L(\xi(s), \dot{\xi}(s)) + c_0 \right) ds, \end{aligned}$$

where $\xi(s) := \gamma(s - t)$ for $s \leq 0$. By Proposition 3.16, we have

$$\int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma(\tau)) d\tau} \left(L(\gamma(s), \dot{\gamma}(s)) + c_0 \right) ds \geq w(x) - e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} w(\gamma(-t)).$$

By the definition of V_λ , we have

$$e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \limsup_{T \rightarrow +\infty} \int_{-T}^0 e^{-\lambda \int_s^0 a(\xi(\tau)) d\tau} \left(L(\xi(s), \dot{\xi}(s)) + c_0 \right) ds \geq e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} V_\lambda(\gamma(-t)).$$

Combining all the above inequalities, we conclude

$$V_\lambda(x) + \varepsilon > w(x) + e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \left(V_\lambda(\gamma(-t)) - w(\gamma(-t)) \right).$$

We know that $\sup_{t>0} \alpha(\lambda, \gamma, t) < +\infty$, cf. proof of Proposition 3.13. By Lemma 3.17, we derive that $e^{-\lambda \int_{-t}^0 a(\gamma(\tau)) d\tau} \rightarrow 0$ as $t \rightarrow +\infty$. By finally sending $\varepsilon \rightarrow 0^+$ we conclude that $V_\lambda(x) \geq w(x)$. This, together with Propositions 3.4 and 3.16, proves the asserted maximality of V_λ .

Now we show that V_λ is a viscosity solution of (HJ_λ) . Since V_λ is the maximal subsolution, we only need to show that it is a supersolution. The argument is standard and

depends on the bump construction. We give it below for the reader's convenience. Assume, by contradiction, that the supersolution test fails at some point $z \in M$. This means that there is a strict subgradient $\phi \in C^1(M)$ of V_λ at z ⁸ such that

$$\lambda a(z)V_\lambda(z) + H(z, D_z\phi) < c_0.$$

Up to adding a constant to ϕ , we can assume that $V_\lambda(z) = \phi(z)$. Let $B_r(z)$ be the open ball centered at z with the radius r . Let us choose $r > 0$ and $\varepsilon > 0$ small enough so that

$$\phi + \varepsilon < V_\lambda \text{ on } \partial B_r(z) \quad \text{and} \quad \lambda a(x)(\phi(x) + \varepsilon) + H(x, D_x\phi) < c_0 \quad \forall x \in B_r(z). \quad (3.10)$$

Set

$$\tilde{V}_\lambda(x) := \begin{cases} \max\{V_\lambda(x), \phi(x) + \varepsilon\} & \text{if } x \in B_r(z). \\ V_\lambda(x) & \text{if } x \in M \setminus B_r(z) \end{cases}$$

Due to (3.10), the function \tilde{V}_λ is a subsolution of (HJ $_\lambda$) in $B_r(z)$ as the maximum of two subsolutions in that open ball, and it agrees with V_λ in an open neighborhood of $M \setminus B_r(z)$. This readily implies that \tilde{V}_λ is a subsolution of (HJ $_\lambda$) on the whole M . Yet, we have $\tilde{V}_\lambda(z) = \phi(z) + \varepsilon > V_\lambda(z)$, contradicting the fact that V_λ is the maximal subsolution of (HJ $_\lambda$). This shows that V_λ is indeed a solution to (HJ $_\lambda$) in M . \square

From now on, we denote by $u_\lambda(x)$ the value function $V_\lambda(x)$, since it is the maximal solution.

PROPOSITION 3.19. *There exists $\lambda_0 \in (0, \bar{\lambda})$ such that, for every fixed $x \in M$ and $\lambda \in (0, \lambda_0)$, we can find a curve $\gamma_\lambda^x : (-\infty, 0] \rightarrow M$ with $\gamma_\lambda^x(0) = x$ such that*

$$u_\lambda(x) = \int_{-\infty}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds.$$

Furthermore, the curve γ_λ^x is $\hat{\kappa}$ -Lipschitz continuous, for some constant $\hat{\kappa} > 0$ independent of $\lambda \in (0, \lambda_0)$ and $x \in M$.

Proof. Let us fix $\lambda \in (0, \bar{\lambda})$. According to Proposition 3.18, u_λ is a viscosity, hence a weak KAM, solution of

$$\lambda a(x)u_\lambda(x) + H(x, D_x u) = c_0 \quad \text{in } M.$$

By Lemma 2.9, we know that, for every fixed $x \in M$, there is a Lipschitz curve $\gamma_\lambda^x : (-\infty, 0] \rightarrow M$ with $\gamma_\lambda^x(0) = x$ such that

$$u_\lambda(\gamma_\lambda^x(b_2)) - u_\lambda(\gamma_\lambda^x(b_1)) = \int_{b_1}^{b_2} \left[L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 - \lambda a(\gamma_\lambda^x(s))u_\lambda(\gamma_\lambda^x(s)) \right] ds \quad (3.11)$$

for all $b_1 < b_2 \leq 0$. The above equality with $b_1 = -t$ and $b_2 = 0$ can be restated as

$$u(t) = u(0) + \int_0^t h(s)u(s)ds + \int_0^t \ell(s)ds \quad \text{for all } t > 0, \quad (3.12)$$

where

$$u(s) := -u_\lambda(\gamma_\lambda^x(-s)), \quad h(s) := \lambda a(\gamma_\lambda^x(-s)), \quad \ell(s) := L(\gamma_\lambda^x(-s), \dot{\gamma}_\lambda^x(-s)) + c_0.$$

⁸Meaning that $V_\lambda - \phi$ has a strict local minimum at z .

Note that the function u and h are in $L^\infty([0, t])$. Since ℓ is uniformly bounded from below, this implies, in view of (3.12), that ℓ is in $L^1([0, t])$. The operator \mathcal{T} defined by

$$\mathcal{T}f(t) = u(0) + \int_0^t h(s)f(s)ds + \int_0^t \ell(s)ds$$

is a contraction when $\lambda\|a\|_\infty t < 1$, in particular there is a unique fixed point of \mathcal{T} . This gives the existence of the local unique solution of (3.12), which is defined, for example, on $[0, t_0]$ with $t_0 = \frac{1}{2\lambda\|a\|_\infty}$. Since the time of local existence is independent of the initial data, the maximal solution is defined for all $t > 0$, and is given by

$$u(t) = u(0)e^{\int_0^t h(s)ds} + \int_0^t \ell(s)e^{\int_s^t h(\tau)d\tau} ds,$$

as an explicit computation shows. This gives directly

$$u_\lambda(x) = e^{-\lambda \int_{-t}^0 a(\gamma_\lambda^x(\tau))d\tau} u_\lambda(\gamma_\lambda^x(-T)) + \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau))d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds \quad (3.13)$$

for all $t > 0$. Due to the fact that the functions $(u_\lambda)_{\lambda \in (0, \bar{\lambda})}$ are equi-bounded and equi-Lipschitz, this implies that the curve γ_λ^x is $\hat{\kappa}$ -Lipschitz, with a Lipschitz constant $\hat{\kappa}$ that is independent of $\lambda \in (0, \bar{\lambda})$ and $x \in M$, see Lemma 2.9.

We want to show that there exists $\lambda_0 \in (0, \bar{\lambda}]$ such that $e^{-\lambda \int_{-t}^0 a(\gamma_\lambda^x(\tau))d\tau} \rightarrow 0$ as $t \rightarrow +\infty$ whenever $\lambda \in (0, \lambda_0)$. According to Lemma 3.17, it suffices to show that there exists a $\lambda_0 \in (0, \bar{\lambda}]$ such that

$$\sup \left\{ \lambda \alpha(\lambda, \gamma, t) : \lambda \in (0, \lambda_0), \gamma \in \mathcal{C}_\lambda^x, x \in M, t > 0 \right\} < +\infty, \quad (3.14)$$

where we have denoted by \mathcal{C}_λ^x the family of absolutely continuous curves $\gamma : (-\infty, 0] \rightarrow M$ with $\gamma(0) = x$ that satisfy (3.11). Notice in fact that (3.14) implies in particular that, for every fixed $\lambda \in (0, \lambda_0)$, condition (3.9) in Lemma 3.17 is met. We argue by contradiction. Let us assume the claim false. Then there exist sequences $\lambda_n \rightarrow 0^+$, $t_n \in (0, +\infty)$, $x_n \in M$ and $\gamma_n \in \mathcal{C}_{\lambda_n}^{x_n}$ such that $\lambda_n \alpha(\lambda_n, \gamma_n, t_n) \rightarrow +\infty$ as $n \rightarrow +\infty$. Notice that the latter implies that $t_n \rightarrow +\infty$ and $\alpha_n := \alpha(\lambda_n, \gamma_n, t_n) \rightarrow +\infty$ as $n \rightarrow +\infty$. Let $\tilde{\mu}_n$ be the probability measure defined in (\star) . Then (3.13) can be restated as

$$\int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n(x, v) = \frac{1}{\alpha_n} \left(u_{\lambda_n}(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau))d\tau} u_{\lambda_n}(\gamma_n(-t_n)) \right) \quad (3.15)$$

for all $n \in \mathbb{N}$. With the aid of Lemma 3.6, it is easy to check that the right-hand side of (3.15) goes to 0 as $n \rightarrow +\infty$. We can therefore apply Lemma 3.7 to infer that, up to extracting a subsequence, $\tilde{\mu}_n$ weakly converges to a Mather measure $\tilde{\mu}$.

Now we choose a subsolution $\varphi \leq -1$ of (HJ_0) . For $\varepsilon > 0$, we take $\varphi_\varepsilon \in C^\infty(M)$ satisfying $\|\varphi_\varepsilon - \varphi\|_\infty \leq \varepsilon$ and $H(x, D_x \varphi_\varepsilon) \leq c_0 + \varepsilon$ for all $x \in M$, given by Theorem 1.15.

For ε small enough, we have $\varphi_\varepsilon < 0$. By (3.15) we get

$$\begin{aligned}
 & \frac{1}{\alpha_n} \left(u_{\lambda_n}(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau} u_{\lambda_n}(\gamma_n(-t_n)) \right) = \int_{TM} (L(x, v) + c_0) d\tilde{\mu}_n \\
 & \geq \int_{TM} (D_x \varphi_\varepsilon(v) - H(x, D_x \varphi_\varepsilon) + c_0) d\tilde{\mu}_n \geq \int_{TM} (D_x \varphi_\varepsilon(v) - \varepsilon) d\tilde{\mu}_n \\
 & = \frac{\varphi_\varepsilon(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau} \varphi_\varepsilon(\gamma_n(-t_n))}{\alpha(\lambda_n, \gamma_n, t_n)} - \lambda_n \int_{TM} a(x) \varphi_\varepsilon(x) d\tilde{\mu}_n - \varepsilon \\
 & \geq \frac{\varphi_\varepsilon(\gamma_n(0))}{\alpha_n} - \lambda_n \int_{TM} a(x) \varphi_\varepsilon(x) d\tilde{\mu}_n - \varepsilon,
 \end{aligned}$$

where for the first inequality we have used Fenchel's inequality (1.2) and for the subsequent equality an integration by parts. Let $\varepsilon \rightarrow 0^+$ to get

$$\frac{1}{\alpha_n} \left(u_{\lambda_n}(\gamma_n(0)) - e^{-\lambda_n \int_{-t_n}^0 a(\gamma_n(\tau)) d\tau} u_{\lambda_n}(\gamma_n(-t_n)) \right) \geq \frac{\varphi(\gamma_n(0))}{\alpha_n} - \lambda_n \int_{TM} a(x) \varphi(x) d\tilde{\mu}_n.$$

Now we divide by λ_n , we use the fact that $\lambda_n \alpha_n \rightarrow +\infty$ and Lemma 3.6 to get

$$\int_{TM} a(x) \varphi(x) d\tilde{\mu}(x, v) \geq -\|a\|_\infty \|u_\lambda\|_\infty.$$

Since for, all $m \geq 0$, $\varphi - m \leq 0$ is also a subsolution of (HJ₀), we get, by applying the previous inequality to $\varphi - m$, that

$$\|\varphi\|_\infty \geq \int_{TM} a(x) \varphi(x) d\tilde{\mu}(x, v) \geq m \int_{TM} a(x) d\tilde{\mu}(x, v) - \|a\|_\infty \|u_\lambda\|_\infty, \quad \forall m \geq 0,$$

which implies that $\int_{TM} a(x) d\tilde{\mu} \leq 0$. This leads to a contradiction with (a1) being $\tilde{\mu}$ a Mather measure, as we recalled above.

Let us fix $\lambda \in (0, \lambda_0)$ and $x \in M$. By sending $t \rightarrow +\infty$ in (3.13) and by exploiting the information just gathered, we derive that

$$u_\lambda(x) = \lim_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds.$$

We conclude that

$$\begin{aligned}
 u_\lambda(x) &= \lim_{t \rightarrow +\infty} \int_{-t}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds & (3.16) \\
 &= \int_{-\infty}^0 e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau} \left(L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s)) + c_0 \right) ds.
 \end{aligned}$$

Indeed (3.14) means that the positive function $s \mapsto e^{-\lambda \int_s^0 a(\gamma_\lambda^x(\tau)) d\tau}$ is in $L^1((-\infty, 0])$. Furthermore, the fact that the curve γ_λ^x is Lipschitz implies that the function $s \mapsto L(\gamma_\lambda^x(s), \dot{\gamma}_\lambda^x(s))$ is in $L^\infty((-\infty, 0])$. Equality (3.16) follows from this by making use of the Dominated Convergence Theorem. \square

3.2. Diverging families of solutions. We have seen in Theorem 3.1 that, under the assumption (a1), the maximal solution u_λ of (HJ_λ) uniformly converges as $\lambda \rightarrow 0^+$ to a critical solution u_0 . In this section, we will show that if we furthermore assume

(a2) there is a point $x_0 \in M$ such that $a(x_0) < 0$;

(a3) $a(x) \geq 0$ in an open neighborhood U of the projected Aubry set \mathcal{A} ,

then there exists a family of solutions to (HJ_λ) that uniformly diverges to $-\infty$. Furthermore, when condition (a3) is reinforced in favor of the following one

(a3') $a(x) > 0$ on \mathcal{A} ,

then all solutions to (HJ_λ) different from the maximal ones uniformly diverge to $-\infty$ as $\lambda \rightarrow 0^+$. We summarize all this in the following statement. We refer the reader to Section 1.3 for the definition of the projected Aubry set. We underline that, in view of Theorem 1.9, condition (a3) is stronger than the integral condition (a1). Throughout this section, we will denote by u_λ the maximal solution of (HJ_λ) .

THEOREM 3.20. *Let us assume (a2).*

(i) *If (a3) holds, there is a family of solutions $(v_\lambda)_{\lambda \in (0, \hat{\lambda})}$ of (HJ_λ) for some $\hat{\lambda} \in (0, 1)$ uniformly diverging to $-\infty$ as $\lambda \rightarrow 0^+$.*

(ii) *If (a3') holds, then any family of solutions $(v_\lambda)_{\lambda \in (0, \lambda')}$ of (HJ_λ) with $\lambda' \in (0, 1)$ satisfying $v_\lambda \neq u_\lambda$ for all $\lambda \in (0, \lambda')$ uniformly diverges to $-\infty$ as $\lambda \rightarrow 0^+$.*

Proof. By Theorem 1.6, there is a subsolution φ of (HJ_0) which is C^∞ and strict in $M \setminus \mathcal{A}$. Therefore, for every open neighborhood U of \mathcal{A} there is a $\delta = \delta(U) > 0$ depending on U such that

$$H(x, D_x \varphi) \leq c_0 - \delta < c_0 \quad \text{for all } x \in M \setminus U.$$

Let us prove (i). Let U be the open neighborhood of \mathcal{A} given by condition (a3). Define

$$\varphi_\lambda := \varphi - \frac{1}{\sqrt{\lambda}}.$$

For $\hat{\lambda} > 0$ small enough, we have $\varphi_\lambda < 0$ for all $\lambda \in (0, \hat{\lambda})$. Up to choosing a smaller $\hat{\lambda} > 0$ if necessary, we have

$$\lambda a(x) \varphi_\lambda + H(x, D_x \varphi_\lambda) \leq \lambda a(x) \varphi_\lambda(x) + c_0 \leq c_0 \quad \text{a.e. in } U,$$

and

$$\lambda a(x) \varphi_\lambda + H(x, D_x \varphi_\lambda) \leq \lambda \|a\|_\infty \|\varphi\|_\infty + \|a\|_\infty \sqrt{\lambda} + c_0 - \delta < c_0 \quad \text{in } M \setminus U,$$

for all $\lambda \in (0, \hat{\lambda})$. Therefore, φ_λ is a subsolution of (HJ_λ) . Now we denote by $T_t^{\lambda, -}$ (resp. $T_t^{\lambda, +}$) the Lax-Oleinik semigroup defined in (1.8) (resp. the forward semigroup defined in (1.9)) associated to $L(x, v) - \lambda a(x)u$. Define

$$v_\lambda^+ := \lim_{t \rightarrow +\infty} T_t^{\lambda, +} \varphi_\lambda, \quad v_\lambda := \lim_{t \rightarrow +\infty} T_t^{\lambda, -} v_\lambda^+,$$

By Lemma 1.16, $v_\lambda^+ \leq \varphi_\lambda$, in particular v_λ^+ uniformly diverges to $-\infty$ as $\lambda \rightarrow 0^+$. By Lemma 1.17, v_λ is a solution of (HJ_λ) , and there is a point $x_\lambda \in M$ such that $v_\lambda(x_\lambda) = v_\lambda^+(x_\lambda)$ for each $\lambda \in (0, \hat{\lambda})$. We derive that $v_\lambda(x_\lambda) \rightarrow -\infty$ as $\lambda \rightarrow 0^+$. By Proposition 2.16, we conclude that v_λ uniformly diverges to $-\infty$ as $\lambda \rightarrow 0^+$.

Let us prove (ii). From assumption (a3') we infer that there is an open neighborhood U of \mathcal{A} and $\theta > 0$ such that $a(x) \geq \theta$ for all $x \in U$. One can easily check that there exist $\lambda' \in (0, 1)$ small enough and $\varepsilon > 0$ such that

$$\lambda a(x)\varphi_\lambda + H(x, D_x\varphi_\lambda) \leq \lambda a(x)\varphi_\lambda(x) + c_0 \leq \lambda \|a\|_\infty \|\varphi\|_\infty - \sqrt{\lambda}\theta + c_0 \leq c_0 - \varepsilon \quad \text{a.e. in } U$$

and

$$\lambda a(x)\varphi_\lambda + H(x, D_x\varphi_\lambda) \leq \lambda \|a\|_\infty \|\varphi\|_\infty + \|a\|_\infty \sqrt{\lambda} + c_0 - \delta \leq c_0 - \varepsilon \quad \text{in } M \setminus U,$$

for all $\lambda \in (0, \lambda')$. Therefore, φ_λ is a strict subsolution of (HJ $_\lambda$).

CLAIM: *Let $\lambda \in (0, \lambda')$. There is no solution $v_\lambda \neq u_\lambda$ of (HJ $_\lambda$) satisfying $\varphi_\lambda \leq v_\lambda \leq u_\lambda$ in M .*

We argue by contradiction. Assume there is such a solution v_λ . We have

$$T_t^{\lambda,-} \varphi_\lambda \leq T_t^{\lambda,-} v_\lambda \leq T_t^{\lambda,-} u_\lambda = u_\lambda \quad \forall t > 0.$$

Since φ_λ is a strict subsolution of (HJ $_\lambda$), by Lemma 1.17,

$$\lim_{t \rightarrow +\infty} T_t^{\lambda,-} \varphi_\lambda = u_\lambda.$$

We get

$$\lim_{t \rightarrow +\infty} T_t^{\lambda,-} v_\lambda = u_\lambda,$$

which contradicts the fact that v_λ is a fixed point of $T_t^{\lambda,-}$.

Therefore, for each solution v_λ of (HJ $_\lambda$), there is a point $x_\lambda \in M$ such that

$$v_\lambda(x_\lambda) \leq \varphi_\lambda(x_\lambda) \rightarrow -\infty \quad \text{as } \lambda \rightarrow 0^+.$$

By Proposition 2.16, v_λ uniformly converges to $-\infty$ as $\lambda \rightarrow 0^+$. □

REMARK 3.21. We note that the proof above also shows that the solutions v_λ diverge to $-\infty$ with speed (at least) of order $-1/\sqrt{\lambda}$.

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