On the double critical Maxwell equations

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Abstract

In this paper, we focus on (no)existence and asymptotic behavior of solutions for the double critical Maxwell equation involving with the Hardy, Hardy-Sobolev, Sobolev critical exponents. The existence and noexistence of solutions completely depend on the power exponents and coefficients of equation. On one hand, based on the concentration-compactness ideas, applying the Nehari manifold and the mountain pass theorem, we prove the existence of the ground state solutions for the critical Maxwell equation for three different scenarios. On the other hand, for the case $\lambda < 0$ and $0 \le s_2 < s_1 < 2$, which is a type open problem raised by Li and Lin. Draw support from a changed version of Caffarelli-Kohn-Nirenberg inequality, we find that there exists a constant λ^* which is a negative number having explicit expression, such that the problem has no nontrivial solution as the coefficient $\lambda < \lambda^*$. Moreover, there exists a constant $\lambda^* < \lambda < 0$ such that, as $\lambda^{**} < \lambda < 0$, the equation has a nontrivial solution using truncation methods. Furthermore, we establish the asymptotic behavior of solutions of equation as coefficient converges to zero for the all cases above.

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1 Introduction

In the present paper we focus on the double critical Maxwell equation

$$\nabla \times (\nabla \times \mathbf{u}) = \frac{|\mathbf{u}|^{4-2s_1}\mathbf{u}}{|x|^{s_1}} + \lambda \frac{|\mathbf{u}|^{4-2s_2}\mathbf{u}}{|x|^{s_2}} \quad \text{in } \mathbb{R}^3,$$
(1.1)

where $\nabla \times (\nabla \times \cdot)$ is the curl-curl operator, $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector function, $0 \le s_i \le 2(i = 1, 2)$ are constants, the number $2^*(s) := 6 - 2s$ is named as the Hardy (resp. Hardy-Sobolev, Sobolev) critical exponent as s = 2 (resp. 0 < s < 2, s = 0) due to a reason that the only continuous embedding

$$D^{1,2}(\mathbb{R}^3) \hookrightarrow L^{2^*(s)}(\mathbb{R}^3; |x|^{-s}),$$
 (1.2)

which is noncompact. The general version of the Maxwell equation is formulated as

1	$\partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0$	(Faraday's Law),
	$ abla imes \mathcal{H} = \mathcal{J} + \partial_t \mathcal{D}$	(Ampere's Law),
١	$\operatorname{div}\left(\mathcal{D}\right) = \rho$	(Gauss' Electric Law),
	$\operatorname{div}\left(\mathcal{B}\right) = 0$	(Gauss' Magnetic Law)

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with the electric field \mathcal{E} , the electric displacement field \mathcal{D} , the magnetic field \mathcal{H} , the magnetic induction \mathcal{B} , the current intensity \mathcal{J} and the scalar charge density ρ . These fields are related by the constitutive equations determined by the material. Considering the constitutive relation $\mathcal{D} = \epsilon \mathcal{E} + \mathcal{P}_{NL}(x, \mathcal{E}), \mathcal{B} = \mu \mathcal{H} - \mathcal{M}$, where $\epsilon = \epsilon(x) \in \mathbb{R}^{3\times3}$ is the (linear) permittivity tensor of the material, \mathcal{P}_{NL} is the nonlinear part of the polarization, $\mu = \mu(x) \in \mathbb{R}^{3\times3}$ denotes the magnetic permeability tensor and \mathcal{M} the magnetization of the material. Suppose that there are no currents, charges nor magnetization, i.e. $\mathcal{J} = 0, \rho = 0, \mathcal{M} = 0$. Then combining with Faraday's law and Ampere's Law, we can obtain the nonlinear electromagnetic wave equation of the form

$$\nabla \times (\mu^{-1} \nabla \times \mathcal{E}) + \epsilon(x) \partial_t^2 \mathcal{E} + \partial_t^2 P_{NL}(x, \mathcal{E}) = 0.$$
(1.3)

The equation (1.3) is particularly challenging and in the literature there are several simplifications relying on approximation of the nonlinear electromagnetic wave equation. The most prominent one is the scalar or vector nonlinear Schrödinger equation. In order to justify this approximation one assumes that the term $\nabla(\operatorname{div}(\mathcal{E}))$ in $\nabla \times (\nabla \times \mathcal{E}) = \nabla(\operatorname{div}(\mathcal{E})) - \Delta \mathcal{E}$ is negligible, and that one can use the so-called *slowly varying envelope approximation*. However, this approach may produce non-physical solutions. We can establish the time-harmonic Maxwell equation by

$$\mathcal{E}(x,t) = \mathbf{v}(x)e^{\omega t} \text{ for } x \in \mathbb{R}^3 \text{ and } t \in \mathbb{R}$$
 (1.4)

with frequency $\omega > 0$. We consider a special case in (1.3) with

$$\mathcal{P}_{NL}(x,\mathcal{E}) = -\frac{|\mathbf{v}(x)|^{4-2s_1}}{|x|^{s_1}}\mathcal{E} - \gamma \frac{|\mathbf{v}(x)|^{4-2s_1}}{|x|^{s_1}}\mathcal{E}, \ \mu = I, \ \epsilon = 0.$$

Then (1.3) reduces to the curl-curl equation of the type

$$\nabla \times (\nabla \times \mathbf{v}) = \omega^2 \frac{|\mathbf{v}|^{4-2s_1} \mathbf{v}}{|x|^{s_1}} + \gamma \omega^2 \frac{|\mathbf{v}|^{4-2s_2} \mathbf{v}}{|x|^{s_2}} \text{ in } \mathbb{R}^3.$$
(1.5)

The curl operator $\nabla \times \cdot$ is challenging from the mathematical point of view and is important in mathematical physics: such an operator also appears in the Navier-Stokes equations. Such an operator has several essential features. The kernel of $\nabla \times \cdot$ is of infinite dimensional, which makes that the corresponding energy functionals of the equation (1.1) or (1.5)

$$J_{\omega,\gamma}(\mathbf{v}) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \mathbf{v}|^2 dx - \frac{\omega^2}{6 - 2s_1} \int_{\mathbb{R}^3} \frac{|\mathbf{v}|^{6 - 2s_1}}{|x|^{s_1}} dx - \frac{\gamma \omega^2}{6 - 2s_2} \int_{\mathbb{R}^3} \frac{|\mathbf{v}|^{6 - 2s_2}}{|x|^{s_2}} dx$$

is strongly indefinite, i.e. unbounded from above and below, even on the subspaces of finite codimension, and its critical points have infinite Morse index. Besides, the Fréchet differential of the functionals $J_{\omega,\gamma}(\mathbf{v})$ is not sequentially weak-to weak^{*} continuous, which leads to the limit point of a weakly convergent sequence need not to be a critical point of $J_{\omega,\gamma}(\mathbf{v})$.

It is not difficult to see that using the scaling transformation $\mathbf{u} = \omega^{\frac{2}{4-2s_1}} \mathbf{v}$, the equation (1.5) can be reduced to

$$\nabla \times (\nabla \times \mathbf{u}) = \frac{|\mathbf{u}|^{4-2s_1}\mathbf{u}}{|x|^{s_1}} + \gamma \omega^{2\frac{s_2-s_1}{2-s_1}} \frac{|\mathbf{u}|^{4-2s_2}\mathbf{u}}{|x|^{s_2}} \text{ in } \mathbb{R}^3,$$
(1.6)

which is precisely the equation (1.1) with $\lambda = \gamma \omega^{2\frac{s_2-s_1}{2-s_1}}$. We denote that

$$J_{\lambda}(\mathbf{u}) = J_{1,\lambda}(\mathbf{u}).$$

As far as we know, the initial works researching the exact solutions of the Maxwell's equation are [33, 43]. To the best of our knowledge, the first work dealing with the Maxwell's equation using the variational methods is due to Benci and Fortunato [9], they introduced a series of brilliant ideas, such as splitting the function space into a divergence-free subspace and a curl-free subspace. The second attempt to tackle the Maxwell's equation is due to Azzollini, Benci, D'Aprile and Fortunato [3], they used two group actions of $SO := SO(2) \times \{I\}$ to simplify the curl-curl operator $\nabla \times (\nabla \times \cdot)$ to the vector Laplacian operator $-\Delta \cdot$. The methods used in [3] are to find cylindrically symmetric solutions to (1.5). Let \mathcal{F} be the space of the vector fields $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{u} = \frac{u}{|x'|} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } |x'|^2 = x_1^2 + x_2^2, \tag{1.7}$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ is a SO-invariant scalar function. Let $\mathcal{D}_F := D^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \cap F$. From a direct computations we observe that $\mathbf{u} \in \mathcal{D}_F$ solves the equation

$$\nabla \times (\nabla \times \mathbf{u}) = |\mathbf{u}|^4 \mathbf{u} \quad \text{in } \mathbb{R}^3, \tag{1.8}$$

which is exactly the equation (1.1) with $s_1 = 0$ and $\lambda = 0$, if and only if the function $\phi(x) := u(|x'|, x_3)$ solves the equation

$$-\Delta \phi + \frac{\phi}{|x'|^2} = |\phi|^4 \phi$$
 in \mathbb{R}^3

Surprisingly, the existence of solutions of the equation (1.8) have been proved by Esteban and Lions in [18]. We refer to [5-8, 10, 11, 34-38, 41, 52] for the works on the Maxwell's equation and the references therein. Gaczkowski, Mederski and Schino [20] established a ground state solution in $\mathcal{D}_{\mathcal{F}}$ of the equation (1.8). Mederski and Szulkin [39] innovatively established the optimal constants of Sobolev-type inequality for Curl operator and found the ground state solution without any symmetry assumptions.

The equation

$$-\Delta u = \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} + \lambda \frac{|u|^{2^*(s_2)-2}u}{|x|^{s_2}} \text{ in } \mathbb{R}^N, N \ge 3,$$
(1.9)

on scalar fields has been researched widely with the Hardy(resp. Hardy-Sobolev, Sobolev) critical exponents $2^*(s) := \frac{2(N-s)}{N-2}$ of the embedding

$$D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}(\mathbb{R}^N; |x|^s), N \ge 3$$
 (1.10)

as s = 2 (resp. 0 < s < 2, s = 0), we refer to [2, 13, 16, 17, 19, 22, 24–27, 30, 42, 46, 48–50]. The authors [21–23, 28, 31] considered (1.9) in the half space \mathbb{R}^N_+ . In [31], Li and Lin gave an open problem about (1.9) in the half space \mathbb{R}^N_+ , which has not been fully resolved yet and which is the spindle of problems what has been studied in [50] and we partially answered this question.

The authors [4, 15, 32, 40, 44, 45, 47] studied the equation

$$-\Delta u = \frac{|u|^{2^*(s_1)-2}u}{|\bar{x}|^{s_1}} + \lambda \frac{|u|^{2^*(s_2)-2}u}{|\bar{x}|^{s_2}} \text{ in } \mathbb{R}^N, \ N \ge 3$$
(1.11)

with the so-called Hardy-Maz'ya (resp. Hardy-Sobolev-Maz'ya, Sobolev) critical exponent $2^*(s):=\frac{2(N-s)}{N-2}$ of the embedding

$$D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}(\mathbb{R}^N; |\bar{x}|^s),$$

as s = 2(resp. 0 < s < 2, s = 0), where $x = (\bar{x}, \tilde{x}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $1 \le k \le N$.

For the equation (1.1), based on the transformation (1.7), a direct computations we observe that $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$ solves (1.1) if and only if $u \in X_{SO}$ solves

$$-\Delta u + \frac{u}{|x'|^2} = \frac{|u|^{4-2s_1}u}{|x|^{s_1}} + \lambda \frac{|u|^{4-2s_2}u}{|x|^{s_2}} \quad \text{in } \mathbb{R}^3,$$
(1.12)

$$x := (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}, \ x' := (x_1, x_2) \in \mathbb{R}^2, \ x_3 \in \mathbb{R},$$
(1.13)

where X_{SO} is the closed subspace of the function space X defined by

$$X = \left\{ u \in D^{1,2}(\mathbb{R}^3) \ \Big| \ \int_{\mathbb{R}^N} \frac{|u|^2}{|x'|^2} dx < \infty \right\}$$

consisting of the functions invariant under the usual group action of $SO := O(2) \times \{I\} \subset O(3)$. Note that this is equivalent requiring that such functions be invariant under the action of $O(2) \times \{I\}$ because for every $\xi_1, \xi_2 \in \mathbb{S}^1$, there exists $g \in SO(2)$ such that $\xi_2 = g\xi_1$, where $SO(2) \subset O(2)$ stands for the special orthogonal group in \mathbb{R}^2 . The space X is a Hilbert space endowed with the scalar product

$$\langle u,v\rangle\in X\times X\mapsto \int_{\mathbb{R}^3}\nabla u\cdot\nabla v+\frac{uv}{|x'|^2}dx$$

and the corresponding norm

$$\|u\| := \langle u, u \rangle^{1/2}.$$

We remark that, for any $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$, on the sense of (1.7) with $u \in X_{SO}$, there holds that

$$\int_{\mathbb{R}^{3}} |\nabla \times \mathbf{u}|^{2} dx = \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx =: \mathcal{A}(u),$$

$$\int_{\mathbb{R}^{3}} \frac{|\mathbf{u}|^{6-2s_{1}}}{|x|^{s_{1}}} dx = \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{1}}}{|x|^{s_{1}}} dx =: \mathcal{B}(u),$$

$$\int_{\mathbb{R}^{3}} \frac{|\mathbf{u}|^{6-2s_{2}}}{|x|^{s_{2}}} dx = \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{2}}}{|x|^{s_{2}}} dx =: \mathcal{C}(u).$$
(1.14)

We define the corresponding functional of the equation (1.12) as

$$\mathcal{I}_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|x'|^2} dx - \frac{1}{6 - 2s_1} \int_{\mathbb{R}^3} \frac{|u|^{6 - 2s_1}}{|x|^{s_1}} dx - \frac{\lambda}{6 - 2s_2} \int_{\mathbb{R}^3} \frac{|u|^{6 - 2s_2}}{|x|^{s_2}} dx$$

and define the corresponding Nehari manifold

$$\mathcal{N} := \left\{ u \in X_{\mathcal{SO}} \mid \langle \mathcal{I}'_{\lambda}(u), u \rangle = 0 \right\}.$$

According (1.14), we see that $J_{\lambda}(\mathbf{u}) = \mathcal{I}_{\lambda}(u)$ under the transformation (1.7), for more information and related derivation process one sees [20]. *Thus, we will achieve the solvability of the equation* (1.1) in $\mathcal{D}_{\mathcal{F}}$ by researching the equation (1.12) in X_{SO} . The ground state solutions of the equation (1.12) are the extremal functions of

$$m_{\lambda} := \inf_{u \in \mathcal{N}} \mathcal{I}_{\lambda}(u).$$

We define a number as

$$\bar{\lambda} := \inf_{u \in X_{SO}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|x'|^2} dx}{\int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx},$$
(1.15)

Remark 1.1 It follows from the Hardy inequality that $\bar{\lambda} \geq \frac{1}{4}$. And it is open that whether the best constant $\bar{\lambda}$ is achieve.

For the critical equation with one Sobolev critical exponent, the case that $\lambda = 0$ and s = 0, the existence of ground state solutions of (1.1) in $\mathcal{D}_{\mathcal{F}}$ has been obtained in [20]. Applying the quotient methods, we obtain a result as follows for the critical equation with one Hardy critical exponent and one Hardy-Sobolev critical exponent, the case that $0 < s_1 < 2$, $s_2 = 2$,

Theorem 1.2 Assume that $\lambda < \overline{\lambda}$, $s_2 = 2$ and $0 < s_1 < 2$. Then the equation (1.1) has a nontrivial ground state solution in $\mathcal{D}_{\mathcal{F}}$.

For each $\lambda < \overline{\lambda}$ fixed,

$$\|u\|_{\lambda} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|x'|^2} - \lambda \frac{|u|^2}{|x|^2} dx\right)^{1/2}$$

defines an equivalent norm to ||u|| on X_{SO} .

For the double critical equations, the case $\lambda \neq 0$, we obtain the following results in the cases that $\lambda > 0$ and $\lambda < 0$.

Theorem 1.3 Assume that $\lambda > 0$, $0 \le s_1 < s_2 < 2$. Then the equation (1.1) has a nontrivial ground state solution in $\mathcal{D}_{\mathcal{F}}$.

Theorem 1.4 Assume that $\lambda < 0$, $0 < s_1 < s_2 < 2$. Then the equation (1.1) has a nontrivial ground state solution in $\mathcal{D}_{\mathcal{F}}$.

Define a real number

$$\lambda^* := -\frac{2-s_1}{2-s_2} \bar{S}^{6-2s_1} \left(\frac{s_1-s_2}{2-s_2} \bar{S}^{6-2s_1} \right)^{\frac{(2-s_1)(s_1-s_2)}{(2-s_2)^2}},$$

where \bar{S} is a best constant of Caffarelli-Kohn-Nirenberg type inequality, see Lemma 3.2 in Section 3. We have the following nonexistence and existence results of the nontrivial solutions of (1.1) in the case $\lambda < 0$.

Theorem 1.5 Assume that $\lambda < \lambda^*$, $0 \le s_2 < s_1 < 2$. Then the equation (1.1) has only zero solution in $\mathcal{D}_{\mathcal{F}}$.

Theorem 1.6 Assume that $0 \le s_2 < s_1 < 2$. Then there exists a $\lambda^{**} \in (\lambda^*, 0)$ such that the equation (1.1) has a nontrivial solution in $\mathcal{D}_{\mathcal{F}}$ as $\lambda^{**} < \lambda < 0$.

λ	s_1, s_2	Does the equation (1.1) have nontrivial solutions in $\mathcal{D}_\mathcal{F}$
$\lambda = 0$	$s_1 = 0$	Yes(see [20])
$\lambda < \bar{\lambda}$	$0 < s_1 < 2, s_2 = 2$	Yes(see Theorem 1.2)
$\lambda > 0$	$0 \le s_1 < s_2 < 2$	Yes(see Theorem 1.3)
$\lambda < 0$	$0 < s_1 < s_2 < 2$	Yes(see Theorem 1.4)
$\lambda < \lambda^*$	$0 \le s_2 < s_1 < 2$	No(see Theorem 1.5)
$\lambda^{**} < \lambda < 0$	$0 \le s_2 < s_1 < 2$	Yes(see Theorem 1.6)

It follows from results above, we summarize the solvability of the equation (1.1) as follows.

Remark 1.7 According to the table above, it is not difficulty to find that the solvability of the equation (1.1) in $\mathcal{D}_{\mathcal{F}}$ remains unsolved under the following cases,

- $\lambda \ge \bar{\lambda}, 0 \le s_1 < 2, s_2 = 2$,
- $\lambda < \overline{\lambda}$ and $\lambda \neq 0$, $s_1 = 0, s_2 = 2$,
- $\lambda < 0, 0 = s_1 < s_2 < 2.$

Now, we investigate the asymptotic behavior of the solutions of (1.1) as $\lambda \to 0$.

Theorem 1.8 Assume that $0 < s_1 < 2$ and $s_2 = 2$. Then there exist a sequence $\{\lambda_n < \bar{\lambda}\}$ and $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$, a ground state solution of (1.1) in $\mathcal{D}_{\mathcal{F}}$ with $\lambda = 0$, such that the solution sequence $\{\mathbf{u}_n\} \subset \mathcal{D}_{\mathcal{F}}$ of (1.1) corresponding to the sequence $\{\lambda_n < \bar{\lambda}\}$, satisfies that $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{D}_{\mathcal{F}}$ as $\lambda_n \to 0$.

Theorem 1.9 Assume that $0 < s_1 < s_2 < 2, \lambda > 0$. Then there exist a sequence $\{\lambda_n > 0\}$ and $\mathbf{u} \in \mathcal{D}_F$, a ground state solution of (1.1) in \mathcal{D}_F with $\lambda = 0$, such that the solution sequence $\{\mathbf{u}_n\} \subset \mathcal{D}_F$ of (1.1) corresponding to the sequence $\{\lambda_n > 0\}$, satisfies that $\mathbf{u}_n \to \mathbf{u}$ in \mathcal{D}_F as $\lambda_n \to 0^+$.

Theorem 1.10 Assume that $0 < s_1 < s_2 < 2, \lambda < 0$. Then there exist a sequence $\{\lambda_n < 0\}$ and $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$, a ground state solution of (1.1) in $\mathcal{D}_{\mathcal{F}}$ with $\lambda = 0$, such that the solution sequence $\{\mathbf{u}_n\} \subset \mathcal{D}_{\mathcal{F}}$ of (1.1) corresponding to the sequence $\{\lambda_n < 0\}$, satisfies that $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{D}_{\mathcal{F}}$ as $\lambda_n \to 0^-$.

Theorem 1.11 Assume that $0 \le s_2 < s_1 < 2, \lambda < 0$. Then there exist a sequence $\{\lambda_n < 0\}$ and $\mathbf{u} \in \mathcal{D}_F$, a ground state solution of (1.1) in \mathcal{D}_F with $\lambda = 0$, such that the solution sequence $\{\mathbf{u}_n\} \subset \mathcal{D}_F$ of (1.1) corresponding to the sequence $\{\lambda_n < 0\}$, satisfies that $\mathbf{u}_n \to \mathbf{u}$ in \mathcal{D}_F as $\lambda_n \to 0^-$. **Remark 1.12** As arguments above, let \mathcal{F} be the space of the vector fields $\boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\boldsymbol{u} = \frac{u}{|x'|} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } |x'|^2 = x_1^2 + x_2^2,$$
(1.16)

where $u \in X_{SO}$ and $\mathcal{D}_{\mathcal{F}} := D^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \cap \mathcal{F}$, from a direct computations we observe that $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$ solves (1.1) if and only if $u \in X_{SO}$ solves

$$-\Delta u + \frac{u}{|x'|^2} = \frac{|u|^{4-2s_1}u}{|x|^{s_1}} + \lambda \frac{|u|^{4-2s_2}u}{|x|^{s_2}} \text{ in } \mathbb{R}^3.$$
(1.17)

Thus, the proofs of Theorems 1.2-1.11 depend on the corresponding results about the equation (1.17). For more information one can refer to the important paper [20].

Furthermore problems

In this subsection, we give two related problems, based on the proceed of this paper, we can get the similar results of Theorems 1.2-1.11.

Let $N \ge 3, k \ge 2$, we denote

$$x := (\bar{x}, \tilde{x}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \bar{x} = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k, \tilde{x} = (x_{k+1}, \cdots, x_N) \in \mathbb{R}^{N-k}.$$
 (1.18)

The first problem is the following

(**P**₁) The double critical Maxwell equation in higher dimensions.

Consider the case k = 2 in (1.18). In order to find a suitable counterpart for the curl-curl operator $\nabla \times (\nabla \times \cdot)$ in higher dimensions, we can use the identity

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \Delta \mathbf{u}, \ \mathbf{u} \in C^2(\mathbb{R}^N; \mathbb{R}^N)$$

to research the equation

$$\nabla \times (\nabla \times \mathbf{u}) = \frac{|\mathbf{u}|^{2^*(s_1) - 2} \mathbf{u}}{|x|^{s_1}} + \lambda \frac{|\mathbf{u}|^{2^*(s_2) - 2} \mathbf{u}}{|x|^{s_2}} \text{ in } \mathbb{R}^N,$$
(1.19)

where $2^*(s) := \frac{2(N-s)}{N-2}$ is the critical exponent of the embedding (1.10). We can find a solution of (1.19) in $\mathcal{D}_{\mathcal{F}}$, where $\mathcal{D}_{\mathcal{F}} := D^{1,2}(\mathbb{R}^N, \mathbb{R}^N) \cap \mathcal{F}$, \mathcal{F} is the space of the vector fields $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\mathbf{u} = \frac{u}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, x = (x_1, x_2, \tilde{x}) \in \mathbb{R}^N \text{ and } r^2 = x_1^2 + x_2^2$$
(1.20)

for some $SO := O(2) \times \{I_{N-2}\}$ -invariant scalar function $u : \mathbb{R}^N \to \mathbb{R}$ and $u \in X_{SO}$, where X_{SO} is the subspace of

$$X := \left\{ u \in D^{1,2}(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} \frac{|u|^2}{r^2} dx < \infty \right\}$$

consisting of the functions invariant under the usual action of SO. Then, $\mathbf{u} \in D_F$ solves (1.19) if and only if $u \in X_{SO}$ solves

$$-\Delta u + \frac{u}{r^2} = \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} + \lambda \frac{|u|^{2^*(s_2)-2}u}{|x|^{s_2}} \quad \text{in } \mathbb{R}^N.$$
(1.21)

It is worth mentioning that Schino [41, Corollary 4.1.4] researched (1.19) with $s_1 = 0$ and $\lambda = 0$.

The second problem is the following

(P₂) The double critical semilinear equation in higher dimensions

Consider the double critical equation

$$-\Delta u + \frac{u}{|\bar{x}|^2} = \frac{|u|^{2^*(s_1)-2}u}{|x|^{s_1}} + \lambda \frac{|u|^{2^*(s_2)-2}u}{|x|^{s_2}} \quad \text{in } \mathbb{R}^N.$$
(1.22)

As the case k = 2 in (1.18), the equation (1.22) turns into the equation (1.21). Indeed, based on the proceed of the present paper, we are able to verify the similar results of Theorem 1.2-1.11 for the equation (1.22) in the following function space X, without the restriction that the functions are invariant under the usual action of $SO := O(2) \times \{I_{N-2}\}$,

$$X = \left\{ u \in D^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} \frac{|u|^2}{|\bar{x}|^2} dx < \infty \right\}.$$

Since the quantity $\frac{|\phi|^2}{|\bar{x}|^2}$ need not be integrable for $\phi \in C_0^{\infty}(\mathbb{R}^N)$, $C_0^{\infty}(\mathbb{R}^N) \not\subset X$. Thus we can not find a solution of the equation (1.22) in $D^{1,2}(\mathbb{R}^N)$ in the methods in the present paper.

For the case k > 2 in (1.18). On one hand, we also prove the similar results of Theorems 1.2–1.11 for the equation (1.22) in the function space

$$X = \left\{ u \in D^{1,2}(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} \frac{|u|^2}{|\bar{x}|^2} dx < \infty \right\}.$$

On the other hand, since, see [4],

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|\bar{x}|^2} dx \le \left(\frac{2}{k-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \forall \, u \in D^{1,2}(\mathbb{R}^N),$$

the norm

$$\left(\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|\bar{x}|^2} dx\right)^{\frac{1}{2}}$$

of the function space

$$X = \left\{ u \in D^{1,2}(\mathbb{R}^N) \Big| \int_{\mathbb{R}^N} \frac{|u|^2}{|\bar{x}|^2} dx < \infty \right\}.$$

and the norm $\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}$ of $D^{1,2}(\mathbb{R}^N)$ are equivalent. Therefore, for $k \ge 3$, we can prove the existence of solutions for the equation (1.22) in $D^{1,2}(\mathbb{R}^N)$.

Finally, we remark that, for the case $\lambda = 0$, $N \ge 3$, $k \ge 2$ with $s := s_1$, the equation (1.22) becomes

$$-\Delta u + \frac{u}{|\bar{x}|^2} = \frac{|u|^{2^*(s)-2}u}{|x|^s} \quad \text{in } \mathbb{R}^N.$$
(1.23)

The similar result of Theorem 1.2 for the equation (1.23) is new, to the best of our knowledge, none consider the case that different effective dimensions of weight functions, that is, the effective dimensions of the terms $\frac{1}{|\bar{x}|^2}$ and $\frac{1}{|x|^s}$ may be different.

The structure of the paper

In Section 2, we prove Theorem 1.2 by applying the quotient methods and concentration compactness ideas. In Section 3, the existence of the ground state solutions of (1.1), that is the proofs of Theorems 1.3 and 1.4 is confirmed. In Section 4, we focus on the similar open problem raised by Li and Lin [31], we prove the nonexistence (see subsection 4.1) and existence (see subsection 4.2) of nontrivial solutions contained in Theorems 1.5 and 1.6. In the final section, we establish the asymptotic behavior of solutions of (1.1), which is the proofs of Theorems 1.8, 1.9, 1.10 and 1.11.

Now we give some notations description.

- Set $B_R(0)$ is a ball with center $0 \in \mathbb{R}^N$ and radius R in \mathbb{R}^N , specifically, as N = 1 and $0 \in \mathbb{R}$, $B_R(0) := (-R, R)$.
- According to the markings (1.13), $0 := (0_1, 0_2, 0_3) \in \mathbb{R}^3$ and $0' := (0_1, 0_2) \in \mathbb{R}^2$.
- Set b > a > 0,

-
$$B_{a,b}(0) := B_b(0') \times B_b(0_3) \setminus B_a(0') \times B_a(0_3).$$

- $B_{a,b}(0') := B_b(0') \setminus B_a(0').$
- $B_{a,b}(0_3) := B_b(0_3) \setminus B_a(0_3).$

2 **Proof of Theorem 1.2**

In this section, we focus on the proof of Theorem 1.2, that is the equation (1.1) with $\lambda < \overline{\lambda}$, where $\overline{\lambda}$ is in (1.15), to simplify notation we write s in place of s_1 , we consider the equation

$$\nabla \times (\nabla \times \mathbf{u}) - \lambda \frac{\mathbf{u}}{|x|^2} = \frac{|\mathbf{u}|^{4-2s}\mathbf{u}}{|x|^s} \text{ in } \mathbb{R}^3.$$
(2.1)

The existence of ground state solution of (2.1) in $\mathcal{D}_{\mathcal{F}}$ has been obtained in [20] as $s = 0, \lambda = 0$. Next, we will prove the result for the case $0 < s < 2, \lambda < \overline{\lambda}$.

Proof of Theorem 1.2 We observe that $\mathbf{u} \in \mathcal{D}_{\mathcal{F}}$ solves (2.1) if and only if $u \in X_{SO}$ solves

$$-\Delta \phi + \frac{u}{|x'|^2} - \lambda \frac{u}{|x|^2} = \frac{|u|^{4-2s}u}{|x|^s} \quad \text{in } \mathbb{R}^3.$$

We define

$$S_{\lambda,s}(\mathbb{R}^3) = \inf_{u \in X_{SO} \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|x'|^2} - \lambda \frac{|u|^2}{|x|^2} dx}{\left(\int_{\mathbb{R}^3} \frac{|u|^{6-2s}}{|x|^s}\right)^{\frac{1}{3-s}}}.$$
(2.2)

Let $\{\tilde{u}_n\} \subset X_{SO}$ be a minimizing sequence for $S_{\lambda,s}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \frac{|\tilde{u}_n|^{6-2s}}{|x|^s} dx = 1, \quad \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla \tilde{u}_n|^2 + \frac{|\tilde{u}_n|^2}{|x'|^2} - \lambda \frac{|u_n|^2}{|x|^2} dx = S_{\lambda,s}(\mathbb{R}^3).$$

For any n, there exists $r_n > 0$ such that

$$\int_{B_{r_n}(0')\times B_{r_n}(0_3)} \frac{|\tilde{u}_n|^{6-2s}}{|x|^s} dx = \frac{1}{2}.$$

Define $u_n(x) = r_n^{\frac{1}{2}} \tilde{u}_n(r_n x)$, then $u_n \in X_{\mathcal{SO}}$, and we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} - \lambda \frac{|u_n|^2}{|x|^2} dx = S_{\lambda,s}(\mathbb{R}^3).$$
(2.3)

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} - \lambda \frac{|u_n|^2}{|x|^2} dx = 1, \quad \int_{B_1(0') \times B_1(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = \frac{1}{2}.$$
 (2.4)

We first claim that, up to a subsequence,

$$\lim_{R \to \infty} \lim_{n \to \infty} \int_{B_R(0') \times B_R(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = 1,$$
(2.5)

Indeed, for $n \in N$ and r > 0, we define

$$Q_n(r) := \int_{B_r(0') \times B_r(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx.$$

Since $0 \le Q_n \le 1$ and $r \mapsto Q_n(r)$ is nondecreasing for all $n \in N$, then up to a subsequence, there exists $Q : [0, +\infty) \to \mathbb{R}$ nondecreasing such that $Q_n(r) \to Q(r)$ as $n \to +\infty$ for a.e. r > 0. Set

$$\alpha = \lim_{r \to \infty} Q(r).$$

It follows from (2.3) and (2.4) that $1/2 \le \alpha \le 1$. Up to taking another subsequence, there exist $\{r_n\}_n, \{\bar{r}_n\}_n \subset (0, +\infty)$ satisfying

$$\begin{cases} 2r_n \leq \bar{r}_n \leq 3r_n \text{ for any } n \in \mathbb{N}^+, \\ \lim_{n \to \infty} r_n = \lim_{k \to \infty} \bar{r}_n = +\infty, \\ \lim_{n \to \infty} Q_n(r_n) = \lim_{n \to \infty} Q_n(\bar{r}_n) = \alpha. \end{cases}$$

In particular,

$$\lim_{n \to \infty} \int_{B_{r_n}(0') \times B_{r_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = \alpha \text{ and } \lim_{n \to \infty} \int_{\mathbb{R}^3 \setminus B_{\bar{r}_n}(0') \times B_{\bar{r}_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = 1 - \alpha.$$
(2.6)

We claim that

$$\lim_{n \to \infty} r_n^{-2} \int_{B_{r_n}(0') \times B_{r_n}(0_3)} u_n^2 dx = 0.$$
(2.7)

Indeed, for all $x \in B_{r_n,\bar{r}_n}(0)$, we have $r_n \leq |x'| \leq 2r_n$. Therefore, Hölder's inequality yields

$$\int_{B_{r_n,\bar{r}_n}(0)} u_n^2 dx \le Cr_n^2 \left(\int_{B_{r_n,\bar{r}_n}(0)} \frac{|u_n|^{6-2s}}{|x|^s} dx \right)^{\frac{1}{3-s}}$$

for all $n \in N$, conclusion (2.7) then follows from (2.6). We now let $\varphi_1 \in C_0^{\infty}(\mathbb{R}^2), \varphi_2 \in C_0^{\infty}(\mathbb{R})$ and

$$\varphi_1(x) := \begin{cases} 1 \text{ for } x \in B_1(0'), \\ 0 \text{ for } x \in \mathbb{R}^2 \setminus B_2(0'). \end{cases} \quad \varphi_2(x_3) := \begin{cases} 1 \text{ for } x_3 \in B_1(0_3), \\ 0 \text{ for } x_3 \in \mathbb{R} \setminus B_2(0_3). \end{cases}$$

For $n \in N$, we define $\varphi_n(x) = \varphi_{1n}(x')\varphi_{2n}(x_3)$, where

$$\varphi_{1n}(x') := \varphi_1 \left(\frac{|x'|}{\bar{r}_n - r_n} + \frac{\bar{r}_n - 2r_n}{\bar{r}_n - r_n} \right) \text{ for } x' \in \mathbb{R}^2,$$

$$\varphi_{2n}(x_3) := \varphi_2 \left(\frac{|x_3|}{\bar{r}_n - r_n} + \frac{\bar{r}_n - 2r_n}{\bar{r}_n - r_n} \right) \text{ for } x_3 \in \mathbb{R}.$$

One can easily check that $\varphi_n u_n, (1 - \varphi_n) u_n \in X_{SO}$. It follows that

$$\int_{\mathbb{R}^3} \frac{|\varphi_n u_n|^{6-2s}}{|x|^s} dx \ge \int_{B_{r_n}(0') \times B_{r_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = \alpha + o(1),$$

$$\int_{\mathbb{R}^3} \frac{|(1-\varphi_n)u_n|^{6-2s}}{|x|^s} dx \ge \int_{\mathbb{R}^3 \setminus B_{\bar{r}_n}(0') \times B_{\bar{r}_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = 1 - \alpha + o(1)$$

as $n \to \infty$. The Hardy-Sobolev inequality and (2.7) implies that, as $n \to \infty$,

$$S_{\lambda,s}(\mathbb{R}^{3}) \left(\int_{\mathbb{R}^{N}} \frac{|\varphi_{n}u_{n}|^{6-2s}}{|x|^{s}} dx \right)^{\frac{1}{3-s}} \\ \leq \int_{\mathbb{R}^{3}} |\nabla(\varphi_{n}u_{n})|^{2} + \frac{|\varphi_{n}u_{n}|^{2}}{|x'|^{2}} - \lambda \frac{|\varphi_{n}u_{n}|^{2}}{|x|^{2}} dx \\ \leq \int_{\mathbb{R}^{3}} \varphi_{n}^{2} \left(|\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} - \lambda \frac{|u_{n}|^{2}}{|x|^{2}} \right) dx + o(1)$$

Similarly,

$$S_{\lambda,s}(\mathbb{R}^3) \left(\int_{\mathbb{R}^3} \frac{|(1-\varphi_n)u_n|^{6-2s}}{|x|^s} dx \right)^{\frac{1}{3-s}} \\ \leq \int_{\mathbb{R}^3} (1-\varphi_n)^2 \left(|\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} - \lambda \frac{|u_n|^2}{|x|^2} \right) dx + o(1)$$

as $n \to \infty$. Therefore, we have that

$$\int_{\mathbb{R}^3} \frac{|\varphi_n u_n|^{6-2s}}{|x|^s} dx \ge \int_{B_{r_n}(0') \times B_{r_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = \alpha + o(1).$$

$$\int_{\mathbb{R}^3} \frac{|(1-\varphi_n)u_n|^{6-2s}}{|x|^s} dx \ge \int_{\mathbb{R}^3 \setminus B_{\bar{r}_n}(0') \times B_{\bar{r}_n}(0_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = 1 - \alpha + o(1).$$

To sum up we know,

$$S_{\lambda,s}(\mathbb{R}^{3}) \left(\alpha^{\frac{1}{3-s}} + (1-\alpha)^{\frac{1}{3-s}} + o(1) \right)$$

$$\leq S_{\lambda,s}(\mathbb{R}^{3}) \left(\left(\int_{\mathbb{R}^{3}} \frac{|\varphi_{n}u_{n}|^{6-2s}}{|x|^{s}} dx \right)^{\frac{1}{3-s}} + \left(\int_{\mathbb{R}^{3}} \frac{|(1-\varphi_{n})u_{n}|^{6-2s}}{|x|^{s}} dx \right)^{\frac{1}{3-s}} \right)$$

$$\leq \int_{\mathbb{R}^{3}} \left(\varphi_{n}^{2} + (1-\varphi_{n})^{2} \right) \left(|\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} - \lambda \frac{|u_{n}|^{2}}{|x|^{2}} \right) dx + o(1)$$

$$= \int_{\mathbb{R}^{3}} (1-2\varphi_{n}(1-\varphi_{n})) \left(|\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} - \lambda \frac{|u_{n}|^{2}}{|x|^{2}} \right) dx + o(1)$$

$$\leq S_{\lambda,s}(\mathbb{R}^{3}) + 2|\lambda| \int_{\mathbb{R}^{3}} \varphi_{n}(1-\varphi_{n}) \frac{|u_{n}|^{2}}{|x|^{2}} dx + o(1)$$

$$\leq S_{\lambda,s}(\mathbb{R}^{3}) + r_{n}^{-2} \int_{\mathbb{R}^{3}} u_{n}^{2} dx + o(1)$$

$$\leq S_{\lambda,s}(\mathbb{R}^{3}) + r_{n}^{-2} \int_{\mathbb{R}^{3}} u_{n}^{2} dx + o(1)$$

as $n \to \infty$. Hence $\alpha^{\frac{1}{3-s}} + (1-\alpha)^{\frac{1}{3-s}} \leq 1$, which implies that $\alpha = 1$ since $\frac{1}{2} \leq \alpha \leq 1$. This proves the claim in (2.5).

We now claim that there exist $u_{\infty} \in X_{SO}$ satisfying $u_n \rightharpoonup u_{\infty}$ in X_{SO} as $n \rightarrow \infty$ and $x_0 \neq 0$ such that

either
$$\lim_{n \to \infty} \frac{|u_n|^{6-2s}}{|x|^s} dx = \frac{|u_\infty|^{6-2s}}{|x|^s} dx$$
 and $\int_{\mathbb{R}^N} \frac{|u_\infty|^{6-2s}}{|x|^s} dx = 1,$ (2.8)

or
$$\lim_{n \to \infty} \frac{|u_n|^{6-2s}}{|x|^s} dx = \delta_{x_0}$$
 and $u_{\infty} = 0.$ (2.9)

Arguing as above, we get that for all $x \in \mathbb{R}^N$, we have that

$$\lim_{r \to 0^+} \lim_{n \to \infty} \int_{B_r(x') \times B_r(x_3)} \frac{|u_n|^{6-2s}}{|x|^s} dx = \alpha_x \in \{0, 1\}.$$

Then follows from the second identity of (2.4) that $\alpha_0 \leq \frac{1}{2}$ and therefore $\alpha_0 = 0$. Moreover, it follows from the first identity of (2.4) that there exists at most one point $x_0 \in \mathbb{R}^3$ such that $\alpha_{x_0} = 1$. In particular $x_0 \neq 0$ since $\alpha_0 = 0$. It then follows from Lions's second concentration compactness lemma that, up to a subsequence, there exist $u_{\infty} \in X_{SO}, x_0 \in \mathbb{R}^3 \setminus \{0\}$ and $\nu \in \{0, 1\}$ such that $u_n \rightharpoonup u_{\infty}$ weakly in X_{SO} and

$$\lim_{n \to \infty} \frac{|u_n|^{6-2s}}{|x|^s} dx = \frac{|u_\infty|^{6-2s}}{|x|^s} dx + \nu \delta_{x_0}$$
 in the sense of measures.

In particular, due to (2.4) and (2.5), we have that

$$1 = \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s}}{|x|^s} dx = \int_{\mathbb{R}^3} \frac{|u_\infty|^{6-2s}}{|x|^s} dx + \nu.$$

Since $\nu \in \{0, 1\}$, the claims in (2.8) and (2.9) follow.

We now assume that $u_{\infty} \neq 0$, and we claim that $\lim_{n\to\infty} u_n = u_{\infty}$ strongly in X_{SO} and that u_{∞} is an extremal for $S_{\lambda,s}(\mathbb{R}^3)$.

Indeed, it follows from (2.8) that $\int_{\mathbb{R}^3} \frac{|u_{\infty}|^{6-2s}}{|x|^s} dx = 1$, hence

$$S_{\lambda,s}(\mathbb{R}^3) \le \int_{\mathbb{R}^3} \left(|\nabla u_{\infty}|^2 + \frac{|u_{\infty}|^2}{|x'|^2} - \lambda \frac{|u_{\infty}|^2}{|x|^2} \right) dx.$$
(2.10)

Moreover, since $u_n \rightharpoonup u_\infty$ weakly as $n \rightarrow \infty$, we have that

$$\int_{\mathbb{R}^{3}} \left(|\nabla u_{\infty}|^{2} + \frac{|u_{\infty}|^{2}}{|x'|^{2}} - \lambda \frac{|u_{\infty}|^{2}}{|x|^{2}} \right) dx \\
\leq \liminf_{n \to \infty} \int_{\mathbb{R}^{3}} \left(|\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} - \lambda \frac{|u_{n}|^{2}}{|x|^{2}} \right) dx \\
= S_{\lambda,s}(\mathbb{R}^{3}).$$
(2.11)

Hence, combining with (2.10) and (2.11), u_{∞} is an extremal for $S_{\lambda,s}(\mathbb{R}^3)$ and boundedness yields the weak convergence of u_n to u_{∞} in X_{SO} , furthermore, the fact $\lim_{n\to\infty} u_n = u_{\infty}$ strongly in X_{SO} holds. This proved the claims.

We now assume $u_{\infty} \equiv 0$. According to the fact $u_n \rightarrow u_{\infty} \equiv 0$ weakly in X_{SO} as $n \rightarrow \infty$, then for any $1 \leq q < 2^*(0), u_n \rightarrow 0$ strongly in $L^q_{loc}(\mathbb{R}^3)$ when $n \rightarrow \infty$. It follows from s > 0that $2^*(s) < 2^*(0)$, for $x_0 \neq 0$, we have that

$$\lim_{n \to \infty} \int_{B_{\delta}(x'_0) \times B_{\delta}(x_{03})} \frac{|u_n|^{6-2s}}{|x|^s} dx = 0$$

for $\delta > 0$ small enough, contradicting (2.9). As a result that $u_{\infty} \neq 0$.

Based on the proof above, it can be concluded that there exists a $u_{\lambda,s} \in X_{SO}$ such that

$$S_{\lambda,s}(\mathbb{R}^3) = \frac{\int_{\mathbb{R}^3} |\nabla u_{\lambda,s}|^2 + \frac{|u_{\lambda,s}|^2}{|x'|^2} - \lambda \frac{|u_{\lambda,s}|^2}{|x|^2} dx}{\left(\int_{\mathbb{R}^3} \frac{|u_{\lambda,s}|^{6-2s}}{|x|^s}\right)^{\frac{1}{3-s}}}.$$

According to (1.14), there exists a $\mathbf{u}_{\lambda,s} \in \mathcal{D}_{\mathcal{F}}$ define by (1.16) replacing u with $u_{\lambda,s}$ above such that

$$\frac{\int_{\mathbb{R}^3} |\nabla \times \mathbf{u}_{\lambda,s}|^2 - \lambda \frac{|\mathbf{u}_{\lambda,s}|^2}{|x|^2} dx}{\left(\int_{\mathbb{R}^3} \frac{|\mathbf{u}_{\lambda,s}|^{6-2s}}{|x|^s}\right)^{\frac{1}{3-s}}} = S_{\lambda,s}(\mathbb{R}^3) = \inf_{\mathbf{u}\in\mathcal{D}_{\mathcal{F}}} \frac{\int_{\mathbb{R}^3} |\nabla \times \mathbf{u}|^2 - \lambda \frac{|\mathbf{u}|^2}{|x|^2} dx}{\left(\int_{\mathbb{R}^3} \frac{|\mathbf{u}|^{6-2s}}{|x|^s}\right)^{\frac{1}{3-s}}}.$$

Making a scaling for $\mathbf{u}_{\lambda,s}$, we get a nontrivial ground state solution of (1.1).

3 The case $\lambda \in \mathbb{R} \setminus \{0\}, 0 \le s_1 < s_2 < 2$

In this section, we main consider the case $0 \le s_1 < s_2 < 2, \lambda \in \mathbb{R} \setminus \{0\}$ and prove the Theorems 1.3 and 1.4 based on the Nehari manifold and the results for the case of $\lambda = 0$ in Theorem 1.2. Let us simply denote $S_{0,s}(\mathbb{R}^3)$ as S_s . We first establish the key lemma which is important to show that the least energy is equal to the level of the mountain pass.

Lemma 3.1 Assume $0 \le s_1 < s_2 < 2, \lambda \in \mathbb{R} \setminus \{0\}$ hold. For each $u \in X_{SO} \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $\mathcal{I}_{\lambda}(t_u u) = \max_{t \ge 0} \mathcal{I}_{\lambda}(tu)$. The function $u \mapsto t_u$ is continuous and the map $u \mapsto t_u u$ is a homeomorphism of the unit sphere in X_{SO} with \mathcal{N} .

Proof. For any $u \in X_{SO} \setminus \{0\}, t > 0$,

$$\frac{d\mathcal{I}_{\lambda}(tu)}{dt} = t\mathcal{A}(u) - t^{5-2s}\mathcal{B}(u) - t^{5-2s}\lambda\mathcal{C}(u).$$

The fact $0 \le s_1 < s_2 < 2$ implies that there exists a unique $t_u > 0$ such that $\frac{d\mathcal{I}_{\lambda}(tu)}{dt}|_{t=t_u} = 0$, that is $t_u u \in \mathcal{N}$. [51, Chapter 4] can be referenced for the remaining proof.

We define

$$\bar{c}_{\lambda} := \inf_{u \in X_{SO} \setminus \{0\}} \max_{t \ge 0} \mathcal{I}_{\lambda}(tu), \tag{3.1}$$

$$\hat{c}_{\lambda} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_{\lambda}(\gamma(t)), \tag{3.2}$$

where

$$\Gamma := \{ \gamma \in C([0,1], X_{\mathcal{SO}}) : \ \gamma(0) = 0, \ \mathcal{I}_{\lambda}(\gamma(1)) < 0 \} .$$
(3.3)

By [51, Chapter 4] and Lemma 3.1 we have that

$$m_{\lambda} = \bar{c}_{\lambda} = \hat{c}_{\lambda}. \tag{3.4}$$

As $\lambda = 0$, the ground state solutions have been in Section 3. We will prove the existence of ground state solutions in two different cases as $\lambda \neq 0$.

Now we introduce a interpolation inequality which is a changed version of Caffarelli-Kohn-Nirenberg inequality after a suitable transform of functions in [14].

Lemma 3.2 Assume $0 \le s_1, s_2 < 2$. There exists a constant \overline{S} such that for any $u \in X_{SO}$,

$$\left(\int_{\mathbb{R}^3} \frac{|u|^{6-2s_1}}{|x|^{s_1}} dx\right)^{\frac{1}{6-2s_1}} \le \bar{S} \left(\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|u|^2}{|x'|^2} dx\right)^{\frac{a}{2}} \left(\int_{\mathbb{R}^3} \frac{|u|^{6-2s_2}}{|x|^{s_2}} dx\right)^{\frac{1-a}{6-2s_2}},$$

where

$$1 > a \ge \begin{cases} \frac{3(s_2 - s_1)}{s_2(3 - s_1)} & \text{if } 2 \ge s_2 > s_1 > 0, \\ \frac{s_1 - s_2}{(2 - s_2)(3 - s_1)} & \text{if } 2 > s_1 > s_2 \ge 0. \end{cases}$$

3.1 The case $\lambda > 0, 0 \le s_1 < s_2 < 2$

In this section, we consider the case of $\lambda > 0$ and we always assume $0 \le s_1 < s_2 < 2$.

Lemma 3.3 There holds

$$0 < \hat{c}_{\lambda} < c_{\lambda}^* := \min\left\{\frac{2-s_1}{6-2s_1}S_{s_1}^{\frac{N-s_1}{2-s_1}}, \frac{2-s_2}{6-2s_2}\lambda^{-\frac{1}{2-s_2}}S_{s_2}^{\frac{3-s_2}{2-s_2}}\right\}$$

and there exists a $(PS)_{\hat{c}_{\lambda}}$ sequence $\{u_n\} \in X_{SO}$ of \mathcal{I}_{λ} .

Proof. It is obvious that $\mathcal{I}_{\lambda}(0) = 0$. On one hand, by the inequality (2.2), for any $u \in X_{SO} \setminus \{0\}$, it holds that

$$\mathcal{I}_{\lambda}(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{6 - 2s_1} S_{s_1}^{s_1 - 3} \|u\|^{6 - 2s_1} - \frac{\lambda}{6 - 2s_2} S_{s_2}^{s_2 - 3} \|u\|^{6 - 2s_2}.$$

It will be seen from this that there exists $\rho > 0$ such that

$$\mathcal{I}_{\lambda}(u) \ge \varpi > 0 \text{ as } ||u|| = \rho,$$

where

$$\varpi = \frac{1}{2}\rho^2 - \frac{1}{6 - 2s_1}S_{s_1}^{s_1 - 3}\rho^{6 - 2s_1} - \frac{\lambda}{6 - 2s_2}S_{s_2}^{s_2 - 3}\rho^{6 - 2s_2}.$$

On the other hand, for any fixed $u \in X_{SO} \setminus \{0\}$,

$$\mathcal{I}_{\lambda}(tu) = \frac{t^2}{2}\mathcal{A}(u) - \frac{t^{6-2s_1}}{6-2s_1}\mathcal{B}(u) - \frac{\lambda t^{6-2s_2}}{6-2s_2}\mathcal{C}(u) \to -\infty \text{ as } t \to \infty.$$

Thus there exists $v \in X_{SO}$ satisfying

$$\|v\| > \rho, \quad \mathcal{I}_{\lambda}(v) < 0.$$

Now applying the mountain pass theorem, we obtain a (PS) sequence $\{u_n\} \subset X_{SO}$ of \mathcal{I}_{λ} at the level \hat{c}_{λ} .

Next we prove that

$$\hat{c}_{\lambda} < c_{\lambda}^*.$$

By (3.4) we only need to prove that

$$\bar{c}_{\lambda} < c_{\lambda}^*$$

Choosing the extremum function u_{0,s_1} in (2.2) with $s = s_1$, then we get that

$$\max_{t \ge 0} \mathcal{I}_{\lambda}(tu_{0,s_{1}}) = \frac{t_{s_{1}}^{2}}{2} \mathcal{A}(u_{0,s_{1}}) - \frac{t_{s_{1}}^{6-2s_{1}}}{6-2s_{1}} \mathcal{B}(u_{0,s_{1}}) - \frac{\lambda t_{s_{1}}^{6-2s_{2}}}{6-2s_{2}} \mathcal{C}(u_{0,s_{1}}) \\
\leq \max_{t \ge 0} \left\{ \frac{t^{2}}{2} \mathcal{A}(u_{0,s_{1}}) - \frac{t^{6-2s_{1}}}{6-2s_{1}} \mathcal{B}(u_{0,s_{1}}) \right\} - \frac{\lambda t_{s_{1}}^{6-2s_{2}}}{6-2s_{2}} \mathcal{C}(u_{0,s_{1}}) \\
= \frac{2-s_{1}}{6-2s_{1}} S_{\alpha_{1}}^{\frac{3-s_{1}}{2-s_{1}}} - \frac{\lambda t_{s_{1}}^{6-2s_{2}}}{6-2s_{2}} \mathcal{C}(u_{0,s_{1}}) \\
< \frac{2-s_{1}}{6-2s_{1}} S_{\alpha_{1}}^{\frac{3-s_{1}}{2-s_{1}}},$$
(3.5)

where we have used a fact that

$$\mathcal{C}(u_{0,s_1}) = \int_{\mathbb{R}^N} \frac{|u_{0,s_1}|^{6-2s_2}}{|x|^{s_2}} dx > 0.$$

Similarly, taking the extremum function u_{0,s_2} in (2.2) with $s = s_2$, then we get that

$$\max_{t \ge 0} \mathcal{I}_{\lambda}(tu_{0,s_2}) < \frac{2 - s_2}{6 - 2s_2} \lambda^{-\frac{1}{2 - s_2}} S_{s_2}^{\frac{3 - s_2}{2 - s_2}}.$$
(3.6)

Combining with (3.5) and (3.6), we get that $\hat{c}_{\lambda} = \bar{c}_{\lambda} < c_{\lambda}^*$.

In the following lemmas we investigate the properties of the $(PS)_{\hat{c}_{\lambda}}$ sequence $\{u_n\}$ of \mathcal{I}_{λ} found in Lemma 3.3.

Lemma 3.4 If $u_n \rightarrow 0$ in X_{SO} , then for any domain $B_{a,b}(0)$, up to a subsequence and still denoted by $\{u_n\}$ such that

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx \to 0, \quad \int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_i}}{|x|^{s_i}} dx \to 0, \quad i = 1, 2.$$
(3.7)

Proof. For any R > r > 0, the compactness of the embedding

$$X_{\mathcal{SO}} \hookrightarrow \hookrightarrow L^{6-2s_2}(B_{r,R}(0); |x|^{-s_2})$$
(3.8)

implies that

$$\int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \to 0.$$

Let $\eta = \eta_1 \eta_2$, where $\eta_1 \in C_{0,r}^{\infty}(\mathbb{R}^2)$ such that $0 \leq \eta_1 \leq 1, \eta_1(0') = 0$ and $\eta_1|_{B_{a,b}(0')} \equiv 1$, $\eta_2 \in C_{0,r}^{\infty}(\mathbb{R})$ such that $0 \leq \eta_2 \leq 1, \eta_2(0_3) = 0$ and $\eta_2|_{B_{a,b}(0_3)} \equiv 1$. Since $\eta^2 u_n \in X_{SO}$ for all $n \in \mathbb{N}$, combining with (3.8), we get that

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'_{\lambda}(u_{n}), \eta^{2}u_{n} \rangle \\
&= \int_{\mathbb{R}^{3}} \nabla u_{n} \cdot \nabla(\eta^{2}u_{n}) + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{1}}\eta^{2}}{|x|^{s_{1}}} dx - \lambda \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{2}}\eta^{2}}{|x|^{s_{2}}} dx \\
&= \int_{\mathbb{R}^{3}} \eta^{2} |\nabla u_{n}|^{2} + 2\eta u_{n} \nabla \eta \nabla u_{n} + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{1}}\eta^{2}}{|x|^{s_{1}}} dx.
\end{aligned}$$
(3.9)

We claim that

$$\int_{\mathbb{R}^3} 2\eta u_n \nabla \eta \nabla u_n dx = o(1). \tag{3.10}$$

Based the Sobolev embedding theorem, we can obtain that

$$\int_{\mathrm{supp}|\eta|} |u_n|^2 dx \to 0 \text{ as } n \to \infty,$$

combining with Hölder inequality, then

$$\left| \int_{\mathbb{R}^3} 2\eta u_n \nabla \eta \cdot \nabla u_n dx \right| \le 2 \left(\int_{\mathbb{R}^3} |\nabla \eta \cdot \nabla u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\eta u_n|^2 dx \right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty.$$

Thus, by (3.9) and (3.10), it is easy to see that

$$\int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx - \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1} \eta^2}{|x|^{s_1}} dx \to 0 \text{ as } n \to \infty,$$

which implies that

$$\int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx = \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1} \eta^2}{|x|^{s_1}} dx + o(1) \text{ as } n \to \infty.$$

Using the Hölder inequality, (2.2) and (3.9), (3.10), we have, as $n \to \infty$,

$$\begin{split} \int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx &\leq \left(\int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \right)^{\frac{2-s_1}{3-s_1}} \left(\int_{\mathbb{R}^3} \frac{|\eta u_n|^{6-2s_1}}{|x|^{s_1}} dx \right)^{\frac{1}{3-s_1}} + o(1) \\ &\leq \left(\int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \right)^{\frac{2-s_1}{3-s_1}} S_{s_1}^{-1} \int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx + o(1), \end{split}$$

it follows that

$$\left[1 - \left(\int_{\mathbb{R}^N} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx\right)^{\frac{2-s_1}{3-s_1}} S_{s_1}^{-1}\right] \int_{\mathbb{R}^N} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx \le o(1) \text{ as } n \to \infty.$$
(3.11)

Since $\{u_n\}$ is a $(PS)_{\hat{c}_{\lambda}}$ sequence, it is easy to see that, as $n \to \infty$,

$$\begin{aligned} \hat{c}_{\lambda} + o(1) &= \mathcal{I}_{\lambda}(u_n) - \frac{1}{2} \langle \mathcal{I}'_{\lambda}(u_n), u_n \rangle \\ &= \frac{2 - s_1}{6 - 2s_1} \int_{\mathbb{R}^3} \frac{|u_n|^{6 - 2s_1}}{|x|^{s_1}} dx + \lambda \frac{2 - s_2}{6 - 2s_2} \int_{\mathbb{R}^3} \frac{|u_n|^{6 - 2s_2}}{|x|^{s_2}} dx, \end{aligned}$$

we can deduce that

$$\int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \le \frac{6-2s_1}{4-s_1} \hat{c}_{\lambda}.$$
(3.12)

Combining with (3.11), (3.12) and $\hat{c}_{\lambda} < c_{\lambda}^*$ in Lemma 3.3, we have

$$\int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx \to 0 \text{ as } n \to \infty.$$

Since $\eta|_{B_{a,b}(0)} \equiv 1$, we get

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx \to 0 \text{ as } n \to \infty.$$

We complete the proof.

For any $\delta > 0$, we define

$$\begin{aligned}
\kappa_{1} &:= \limsup_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} \frac{|u_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx, \\
\kappa_{2} &:= \limsup_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} \frac{|u_{n}|^{6-2s_{2}}}{|x|^{s_{2}}} dx, \\
\kappa &:= \limsup_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} |\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} dx.
\end{aligned}$$
(3.13)

It follows from Lemma 3.4 that these three quantities are well defined and independent of the choice of $\delta > 0$.

Lemma 3.5 If $u_n \rightarrow 0$ in X_{SO} , then there exist $\epsilon_0 := \epsilon_0(s_1, s_2, \hat{c}_\lambda, \lambda) > 0$ and subsequence(still denoted by $\{u_n\}$) such that

either
$$\lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_3)} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx = 0$$
 or $\lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_3)} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \ge \epsilon_0$

for all $\delta > 0$.

Proof. Let $\phi = \phi_1 \phi_2$, where $\phi_1 \in C_{0,r}^{\infty}(\mathbb{R}^2)$ is nonnegative and satisfy $\phi_1|_{B_{\delta}(0')} \equiv 1$ with $\delta > 0$, and $\phi_2 \in C_{0,r}^{\infty}(\mathbb{R})$ is nonnegative and satisfy $\phi_2|_{B_{\delta}(0_3)} \equiv 1$ with $\delta > 0$. It follows from $\phi u_n \in X_{SO}$ that as $n \to \infty$,

$$\int_{\mathbb{R}^3} \nabla u_n \nabla(\phi u_n) + \frac{\phi |u_n|^2}{|x'|^2} dx - \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1} \phi}{|x|^{s_1}} dx - \lambda \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2} \phi}{|x|^{s_2}} dx \to 0.$$
(3.14)

By (3.7) in Lemma 3.4, we obtain that

$$\int_{\mathbb{R}^3} \nabla u_n \nabla(\phi u_n) + \frac{\phi |u_n|^2}{|x'|^2} dx = \int_{\mathbb{R}^3} \phi |\nabla u_n|^2 + u_n \nabla u_n \nabla \phi + \frac{\phi |u_n|^2}{|x'|^2} dx$$
$$= \int_{B_{\delta}(0') \times B_{\delta}(0_3)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx + o(1),$$

$$\int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_i} \phi}{|x|^{s_i}} dx \to \int_{B_{\delta}(0') \times B_{\delta}(0_3)} \frac{|u_n|^{6-2s_i}}{|x|^{s_i}} dx \text{ as } n \to \infty, i = 1, 2.$$

The limit (3.14) implies that

$$\kappa = \kappa_1 + \lambda \kappa_2. \tag{3.15}$$

The definition (2.2) leads to

$$\left(\int_{\mathbb{R}^3} \frac{|\phi u_n|^{6-2s_1}}{|x|^{s_1}} dx\right)^{\frac{1}{3-s_1}} \leqslant S_{s_1}^{-1} \int_{\mathbb{R}^3} |\nabla(\phi u_n)|^2 + \frac{\phi |u_n|^2}{|x'|^2} dx.$$

Thus

$$\left(\int_{B_{\delta}(0')\times B_{\delta}(0_{3})}\frac{|u_{n}|^{6-2s_{1}}}{|x|^{s_{1}}}dx\right)^{\frac{1}{3-s_{1}}} \leqslant S_{s_{1}}^{-1}\int_{B_{\delta}(0')\times B_{\delta}(0_{3})}|\nabla u_{n}|^{2} + \frac{\phi|u_{n}|^{2}}{|x'|^{2}}dx \text{ as } n \to \infty.$$

Furthermore,

$$\kappa_1^{\frac{1}{3-s_1}} \leqslant S_{s_1}^{-1} \kappa. \tag{3.16}$$

The conclusions (3.15) and (3.16) lead to

$$\kappa_1^{\frac{1}{3-s_1}} \leqslant S_{s_1}^{-1} \kappa = S_{s_1}^{-1} \kappa_1 + S_{s_1}^{-1} \lambda \kappa_2.$$

It follows that

$$\kappa_{1}^{\frac{1}{3-s_{1}}} \left(1 - S_{s_{1}}^{-1} \kappa_{1}^{\frac{2-s_{1}}{3-s_{1}}}\right) \leqslant S_{s_{1}}^{-1} \lambda \kappa_{2}.$$
(3.17)

Since $\{u_n\}$ is a bounded $(PS)_{\hat{c}_{\lambda}}$ sequence, it is obvious that

$$\mathcal{I}_{\lambda}(u_n) - \frac{1}{2} \langle \mathcal{I}'_{\lambda}(u_n), u_n \rangle = \frac{2 - s_1}{6 - 2s_1} \int_{\mathbb{R}^3} \frac{|u_n|^{6 - 2s_1}}{|x|^{s_1}} dx + \frac{\lambda(2 - s_2)}{6 - 2s_2} \int_{\mathbb{R}^3} \frac{|u_n|^{6 - 2s_2}}{|x|^{s_2}} dx = \hat{c}_{\lambda} + o(1).$$

It is easy to see that

$$\int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \leqslant \frac{6-2s_1}{2-s_1} \hat{c}_{\lambda} + o(1)$$

and

$$\kappa_1 \leqslant \frac{6 - 2s_1}{2 - s_1} \hat{c}_{\lambda}.\tag{3.18}$$

Combining with (3.17) and (3.18), we deduce that

$$\kappa_1^{\frac{1}{3-s_1}} \left(1 - S_{s_1}^{-1} \left(\frac{6-2s_1}{2-s_1} \hat{c}_{\lambda} \right)^{\frac{2-s_1}{3-s_1}} \right) \leqslant S_{s_1}^{-1} \lambda \kappa_2.$$

The fact $\hat{c}_{\lambda} < c^*_{\lambda}$ implies that

$$S_{s_1}^{-1} \left(\frac{6-2s_1}{2-s_1}\hat{c}_{\lambda}\right)^{\frac{2-s_1}{3-s_1}} < 1.$$

Thus there exists $\delta_1 > 0$ depending on $s_1, S_{s_1}, \hat{c}_{\lambda}$ such that $\kappa_1^{\frac{1}{3-s_1}} \leq \delta_1 \kappa_2$. Similarly, we have $\kappa_2^{\frac{1}{3-s_2}} \leq \delta_2 \kappa_1$ for some $\delta_2 > 0$. It follows that there exists $\epsilon_0 := \epsilon_0(s_1, s_2, \hat{c}_{\lambda}, \lambda) > 0$ such that

either
$$\kappa_1 = \kappa_2 = 0$$
 or $\kappa_1 \ge \epsilon_0, \ \kappa_2 \ge \epsilon_0$.

We complete the proof.

We define

$$\tilde{u}_n(x) := r_n^{\frac{1}{2}} u_n(r_n x) \text{ for } x \in \mathbb{R}^3, \ r_n > 0.$$
(3.19)

Then $\{\tilde{u}_n\} \subset X_{SO}$ is also a $(PS)_{\hat{c}_{\lambda}}$ sequence of \mathcal{I}_{λ} .

Lemma 3.6 There exists $\epsilon_1 \in (0, \frac{\epsilon_0}{2}]$ such that for all $\epsilon \in (0, \epsilon_1)$, there exists a sequence $\{r_n > 0\}$ such that $\{\tilde{u}_n\}$ verifies

$$\int_{B_1(0')\times B_1(0_3)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = \epsilon.$$
(3.20)

Proof. Since $\hat{c}_{\lambda} > 0$, it follows from the inequality in Lemma 3.2 that

$$\mathcal{B}_{\infty} := \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx > 0.$$

Let $\epsilon_1 := \min\{\frac{\epsilon_0}{2}, \mathcal{B}_\infty\}$. For fixed $\epsilon \in (0, \epsilon_1)$, up to a subsequence, still denoted by $\{u_n\}$, for any $n \in \mathbb{N}$, there exists $r_n > 0$ such that

$$\int_{B_{r_n}(0')\times B_{r_n}(0_3)}\frac{|u_n|^{6-2s_1}}{|x|^{s_1}}dx=\epsilon.$$

By dilating transformation, it is easy to check that $\{\tilde{u}_n\}$ satisfies (3.20).

Now we are ready to give the proof of Theorem 1.3 for the case of $\lambda > 0, 0 \le s_1 < s_2 < 2$. **Proof of Theorem 1.3.** By Lemma 3.3, \mathcal{I}_{λ} has a (PS)_{\hat{c}_{λ}} sequence $\{u_n\} \subset X_{SO}$. By Lemma 3.6, the sequence $\{\tilde{u}_n\}$ defined by (3.19) satisfies (3.20) and is also a (PS)_{\hat{c}_{λ}} sequence of \mathcal{I}_{λ} . Thus

$$\begin{aligned} \mathcal{I}_{\lambda}(\tilde{u}_{n}) &- \frac{1}{6 - 2s_{2}} \langle \mathcal{I}_{\lambda}'(\tilde{u}_{n}), \tilde{u}_{n} \rangle \\ \geqslant \frac{2 - s_{2}}{6 - 2s_{2}} \int_{\mathbb{R}^{3}} |\nabla \tilde{u}_{n}|^{2} + \frac{|\tilde{u}_{n}|^{2}}{|x'|^{2}} dx + \frac{s_{2} - s_{1}}{2(3 - s_{1})(3 - s_{2})} \mathcal{B}(\tilde{u}_{n}) \\ \geqslant \frac{2 - s_{2}}{6 - 2s_{2}} \int_{\mathbb{R}^{3}} |\nabla \tilde{u}_{n}|^{2} + \frac{|\tilde{u}_{n}|^{2}}{|x'|^{2}} dx. \end{aligned}$$
(3.21)

It follows that $\{\tilde{u}_n\}$ is bounded in X_{SO} . Thus there is $\tilde{u}_0 \in X_{SO}$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}_0 & \text{ in } X_{\mathcal{SO}};\\ \tilde{u}_n \rightharpoonup \tilde{u}_0 & \text{ in } L^{6-2s_i}(\mathbb{R}^3; |x|^{-s_i}), \ i = 1, 2;\\ \tilde{u}_n(x) \rightarrow \tilde{u}_0(x) & \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

From the above it can be concluded that \tilde{u}_0 is a solution of (1.17), furthermore, copying the calculation process of (3.21), we get that $\mathcal{I}_{\lambda}(\tilde{u}_0) \ge 0$. Let $v_n := \tilde{u}_n - \tilde{u}_0$. Then $\{v_n\}$ is bounded in X_{SO} . Define

$$\mathcal{A}(v_n) \to \mathcal{A}_{\infty}, \ \mathcal{B}(v_n) \to \mathcal{B}_{\infty}, \ \mathcal{C}(v_n) \to \mathcal{C}_{\infty}.$$

Then by Brezis-Lieb Lemma [12], we have

$$\mathcal{I}_{\lambda}(v_n) \to \frac{1}{2}\mathcal{A}_{\infty} - \frac{1}{6-2s_1}\mathcal{B}_{\infty} - \frac{\lambda}{6-2s_2}\mathcal{C}_{\infty} = \hat{c}_{\lambda} - \mathcal{I}_{\lambda}(\tilde{u}_0), \qquad (3.22)$$

$$\langle \mathcal{I}'_{\lambda}(v_n), v_n \rangle \to \mathcal{A}_{\infty} - \mathcal{B}_{\infty} - \lambda \mathcal{C}_{\infty} = 0.$$
 (3.23)

If $\mathcal{A}_{\infty} = 0$, then $\mathcal{I}_{\lambda}(\tilde{u}_{0}) = \hat{c}_{\lambda}$ and \tilde{u}_{0} is a ground state solution of (1.17). Assume $\mathcal{A}_{\infty} > 0$ and $\tilde{u}_{0} = 0$. Then Lemma 3.5 implies that either $\lim_{n \to \infty} \int_{B_{1}(0') \times B_{1}(0_{3})} \frac{|\tilde{u}_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx = 0$ or

 $\lim_{n\to\infty} \int_{B_1(0')\times B_1(0_3)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx \ge \epsilon_0. \text{ By Lemma 3.6, this is a contradiction to (3.20) as } 0 < \epsilon < \frac{\epsilon_0}{2}. \text{ It must be } \tilde{u}_0 \ne 0 \text{ and } \tilde{u}_0 \text{ is a nontrivial solution of (1.1). If } \mathcal{I}_{\lambda}(\tilde{u}_0) = \hat{c}_{\lambda} \text{ then we complete the } 1 \le 1 \le 1 \le 1$

 $\frac{-u}{2}$. It must be $u_0 \neq 0$ and u_0 is a nontrivial solution of (1.1). If $\mathcal{L}_{\lambda}(u_0) = c_{\lambda}$ then we complete the proof by (3.4). Otherwise, combining with the key fact (3.4), we deduce that

$$\mathcal{I}_{\lambda}(\tilde{u}_0) > \hat{c}_{\lambda}. \tag{3.24}$$

Since

$$\mathcal{I}_{\lambda}(v_n) - \frac{1}{6 - 2s_2} \langle \mathcal{I}'_{\lambda}(v_n), v_n \rangle \ge \frac{2 - s_2}{6 - 2s_2} \mathcal{A}(v_n) \ge 0,$$

it follows from (3.22) and (3.23) that

$$\mathcal{I}_{\lambda}(\tilde{u}_0) \leqslant \hat{c}_{\lambda},$$

which is a contradiction with (3.24). Thus \tilde{u}_0 is a ground state solution of (1.17). The result of Theorem 1.3 is proved according to Remark 1.12.

3.2 The case $\lambda < 0, 0 < s_1 < s_2 < 2$

In this subsection, we may consider the case of $\lambda < 0$ and $0 \le s_1 < s_2 < 2$. Applying the mountain pass theorem in [1], we have the following lemma.

Lemma 3.7 Let $\lambda < 0, 0 < s_1 < s_2 < 2$. There exists a sequence $\{u_n\} \subset X_{SO}$ such that

$$\mathcal{I}_{\lambda}(u_n) \to \hat{c}_{\lambda} > 0, \quad \mathcal{I}'_{\lambda}(u_n) \to 0, \quad n \to \infty$$
 (3.25)

with \hat{c}_{λ} is in (3.2).

The properties of the $(PS)_{\hat{c}}$ sequence $\{u_n\}$ of \mathcal{I}_{λ} found in Lemma 3.7 will be investigated in the following lemmas.

Lemma 3.8 If $u_n \rightarrow 0$ in X_{SO} , then for any domain $B_{a,b}(0)$, up to a subsequence and still denoted by $\{u_n\}$ such that

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx dx \to 0, \quad \int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_i}}{|x|^{s_i}} dx \to 0, \quad i = 1, 2.$$
(3.26)

Proof. For any R > r > 0, the compactness of the embedding

$$X_{\mathcal{SO}} \hookrightarrow \hookrightarrow L^{6-2s_i}(B_{r,R}(0); |x|^{-s_i}), \ i = 1, 2$$

$$(3.27)$$

implies that

$$\int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_i}}{|x|^{s_i}} dx \to 0 \quad \text{as } n \to \infty, \ i = 1, 2.$$

Applying the function η defined in Lemma 3.4. Then $\eta^2 u_n \in X_{SO}$ for all $n \in \mathbb{N}$, combining with (3.27) and (3.10), we get that

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'_{\lambda}(u_{n}), \eta^{2} u_{n} \rangle \\
&= \int_{\mathbb{R}^{N}} \nabla u_{n} \cdot \nabla(\eta^{2} u_{n}) + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{N}} \frac{|u_{n}|^{6-2s_{1}} \eta^{2}}{|x|^{s_{1}}} dx - \lambda \int_{\mathbb{R}^{N}} \frac{|u_{n}|^{6-2s_{2}} \eta^{2}}{|x|^{s_{2}}} dx \\
&= \int_{\mathbb{R}^{N}} \eta^{2} |\nabla u_{n}|^{2} + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx
\end{aligned}$$
(3.28)

and

$$\int_{\mathbb{R}^N} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx \to 0.$$

Since $\eta|_{B_{a,b}(0)} \equiv 1$, thus we get

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx \to 0 \text{ as } n \to \infty.$$

We complete the proof.

It follows from Lemma 3.8 that those three quantities in (3.13) are well defined and independent of the choice of $\delta > 0$.

Lemma 3.9 If $u_n \rightarrow 0$ in X_{SO} , then there exists subsequence(still denoted by $\{u_n\}$) such that

either
$$\lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} \frac{|u_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx = 0 \quad \text{or} \quad \lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} \frac{|u_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx \ge S_{s_{1}}^{\frac{3-s_{1}}{2-s_{1}}}$$
for all $\delta > 0$.

Proof. Taking the function ϕ and coping the proof in Lemma 3.5, we obtain that

$$\kappa = \kappa_1 + \lambda \kappa_2. \tag{3.29}$$

$$\kappa_1^{\frac{1}{3-s_1}} \leqslant S_{s_1}^{-1} \kappa. \tag{3.30}$$

The conclusions (3.29) and (3.30) with $\lambda < 0$ lead to

$$\kappa_1^{\frac{1}{3-s_1}} \leqslant S_{s_1}^{-1} \kappa \le S_{s_1}^{-1} \kappa_1$$

It follows that

$$\kappa_1 = 0 \text{ or } \kappa_1 \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}.$$

We complete the proof.

Next we consider the transform (3.19), then $\{\tilde{u}_n\} \subset X_{SO}$ is also a (PS) $_{\hat{c}_{\lambda}}$ sequence of \mathcal{I}_{λ} .

Lemma 3.10 There exists $\epsilon_1 \in (0, \frac{1}{2}S_{s_1}^{\frac{3-s_1}{2-s_1}}]$ such that for all $\epsilon \in (0, \epsilon_1)$, there exists a sequence $\{r_n > 0\}$ such that $\{\tilde{u}_n\}$ verifies

$$\int_{B_1(0')\times B_1(0_3)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = \epsilon.$$
(3.31)

Proof. Since $\hat{c}_{\lambda} > 0$, it follows from the interpolation inequality in Lemma 3.2 that

$$\mathcal{B}_{\infty} := \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx > 0.$$

Let $\epsilon_1 \leq \frac{1}{2}S_{s_1}^{\frac{3-s_1}{2-s_1}}$. For fixed $\epsilon \in (0, \epsilon_1)$, up to a subsequence, still denoted by $\{u_n\}$, for any $n \in \mathbb{N}$, there exists $r_n > 0$ such that

$$\int_{B_{r_n}(0')\times B_{r_n}(0_3)}\frac{|u_n|^{6-2s_1}}{|x|^{s_1}}dx=\epsilon.$$

By dilating transformation, it is easy to check that $\{\tilde{u}_n\}$ satisfies (3.31).

Based on the lemmas above, the results of Theorem 1.4 with $\lambda < 0, 0 < s_1 < s_2 < 2$ can be proved by coping the similar proof of Theorem 1.3 in subsection 3.1.

4 The case $\lambda < 0, 0 \le s_2 < s_1 < 2$

In the present section, we focus on the proofs of Theorem 1.5 and Theorem 1.6.

4.1 Non-existence

In this subsection, we prove that the equation (1.1) has only zero solution in $\mathcal{D}_{\mathcal{F}}$ as $\lambda < \lambda^*$.

Proof of Theorem 1.5. Based on the inequality in Lemma 3.2 with $a = a_0 := \frac{s_1 - s_2}{(2 - s_2)(3 - s_1)}$, we can directly obtain

$$\begin{aligned} \langle \mathcal{I}'_{\lambda}(u), u \rangle &= \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{1}}}{|x|^{s_{1}}} dx - \lambda \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{2}}}{|x|^{s_{2}}} dx \\ &\geq \left(1 - \bar{S}^{6-2s_{1}} a_{0}(3-s_{1})\gamma^{\frac{1}{a_{0}(3-s_{1})}}\right) \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx \\ &- \left(\lambda + \bar{S}^{6-2s_{1}} \frac{(1-a_{0})(3-s_{1})}{3-s_{2}}\gamma^{-\frac{(1-a_{0})(3-s_{1})}{(3-s_{2})}}\right) \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{2}}}{|x|^{s_{2}}} dx \end{aligned}$$
(4.1)

and taking

$$\gamma = \frac{1}{2} \left[\left(\frac{(2-s_2)\bar{S}^{-(6-2s_1)}}{s_1 - s_2} \right)^{\frac{s_1 - s_2}{2-s_2}} + \left(-\frac{2-s_1}{\lambda(2-s_2)}\bar{S}^{6-2s_1} \right)^{\frac{2-s_2}{2-s_1}} \right].$$

Since $\lambda < \lambda^*$, we have

$$\begin{aligned} &1 - \bar{S}^{6-2s_1} a_0(3-s_1) \gamma^{\frac{1}{a_0(3-s_1)}} > 0, \\ &\lambda + \bar{S}^{6-2s_1} \frac{(1-a_0)(3-s_1)}{3-s_2} \gamma^{-\frac{(1-a_0)(3-s_1)}{3-s_2}} < 0. \end{aligned}$$

Thus for any $u \in X_{SO} \setminus \{0\}$, as $\lambda < \lambda^*$, we have $\langle \mathcal{I}'_{\lambda}(u), u \rangle > 0$, which implies the problem (1.17) has only zero solution. According to Remark 1.12, we know that the problem (1.1) has only zero solution. And we complete the proof of Theorem 1.5.

4.2 Existence

In this subsection, we prove that the equation (1.1) has solution as λ small enough. According to Remark 1.12, we need to prove the equation (1.17) has solution as λ small enough. The assumptions $\lambda < 0, 0 \le s_2 < s_1 < 2$ make the undesirable obstacle for establishing the mountain pass

structure of equation (1.17). The cut-off method in [29] is applied to overcome this obstacle. For any fixed S > 0, the cut-off functional $\mathcal{J}_S : X_{S\mathcal{O}} \to \mathbb{R}$ is defined as

$$\mathcal{J}_{S}(u) = \frac{1}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx - \frac{1}{6 - 2s_{1}} \int_{\mathbb{R}^{3}} \frac{|u|^{6 - 2s_{1}}}{|x|^{s_{1}}} dx - \frac{\lambda \Psi_{S}(u)}{6 - 2s_{2}} \int_{\mathbb{R}^{3}} \frac{|u|^{6 - 2s_{2}}}{|x|^{s_{2}}} dx, \quad (4.2)$$

where

$$\Psi_S(u) = \psi\left(\frac{\|u\|^2}{S^2}\right) \ge 0 \tag{4.3}$$

and $\psi \in C_0^{\infty}(\mathbb{R}, [0, 1])$ satisfies $\psi(t) = 1$ for $t \in [0, \frac{1}{2}]$ and supp $\psi \subset [0, 1]$. The derivative of \mathcal{J}_S is given by

$$\begin{aligned} \langle \mathcal{J}'_{S}(u),\varphi\rangle &= \int_{\mathbb{R}^{3}} \nabla u \nabla \varphi + \frac{u\varphi}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u|^{4-2s_{1}} u\varphi}{|x|^{s_{1}}} dx - \lambda \Psi_{S}(u) \int_{\mathbb{R}^{N}} \frac{|u|^{4-2s_{2}} u\varphi}{|x|^{s_{2}}} dx \\ &- \frac{2\lambda \int_{\mathbb{R}^{3}} \nabla u \nabla \varphi + \frac{u\varphi}{|x'|^{2}} dx}{2^{*}(s_{2})S^{2}} \psi' \left(\frac{||u||^{2}}{S^{2}}\right) \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{2}}}{|x|^{s_{2}}} dx. \end{aligned}$$

We first show that the functional \mathcal{J}_S has a mountain pass geometry for each fixed S > 0.

Lemma 4.1 Let $\lambda < 0$. We have

- (i) there exist $\tilde{\rho} > 0$ and $\tilde{\delta} > 0$ such that $\mathcal{J}_S(u) \ge \tilde{\delta}$ for any $u \in X_{SO}$ with $||u|| = \tilde{\rho}$;
- (ii) there exists $\tilde{v} \in X_{SO}$ satisfying $\|\tilde{v}\| > \tilde{\rho}$ and $\mathcal{I}_S(\tilde{v}) < 0$.

Proof. (i) Since $\lambda < 0$, by (4.3), using the inequality (2.2), we derive that

$$\mathcal{J}_S(u) \ge \frac{1}{2} \|u\|^2 - S_{s_1}^{s_1 - 3} \|u\|^{6 - 2s_1}$$

It is not difficult to prove the conclusion (i) holds.

(ii) For any fixed $u \in X_{SO} \setminus \{0\}$ and $t > \frac{S}{\|u\|}$, we conclude that

$$\begin{aligned} \mathcal{J}_{S}(tu) &= \frac{t^{2}}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx - \frac{t^{6-2s_{1}}}{6-2s_{1}} \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{1}}}{|x|^{s_{1}}} dx - \frac{\lambda t^{6-2s_{2}}}{6-2s_{2}} \Psi_{S}(tu) \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{2}}}{|x|^{s_{2}}} dx \\ &= \frac{t^{2}}{2} \int_{\mathbb{R}^{3}} |\nabla u|^{2} + \frac{|u|^{2}}{|x'|^{2}} dx - \frac{t^{6-2s_{1}}}{6-2s_{1}} \int_{\mathbb{R}^{3}} \frac{|u|^{6-2s_{1}}}{|x|^{s_{1}}} dx. \end{aligned}$$

Taking $\tilde{v} = t_0 u$, where $t_0 > \frac{S}{\|u\|}$ is large enough. Since $6 - 2s_1 > 2$, it is easy to see that (ii) holds. The proof is complete.

By Lemma 4.1 and $\mathcal{J}_S(0) = 0$, a mountain pass level for \mathcal{J}_S can be defined as

$$\bar{c}_S = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{J}_S(\gamma(t)) > 0, \tag{4.4}$$

where

$$\Gamma_S = \{ \gamma \in C([0,1], X_{SO}) : \ \gamma(0) = 0, \ \gamma(1) = \tilde{v} \}.$$

Applying the mountain pass theorem, there exists $\{u_n\} \subset X_{SO}$ satisfying

$$\mathcal{J}_S(u_n) \to \bar{c}_S, \quad \mathcal{J}'_S(u_n) \to 0, \text{ as } n \to \infty.$$
 (4.5)

Now we investigate the property of the sequence $\{u_n\}$ satisfying (4.5). We have

Lemma 4.2 Let $\{u_n\} \subset X_{SO}$ satisfy (4.5). Then for S > 0 large enough, there exists $\lambda^{**} = \lambda^{**}(S) < 0$ such that for any $\lambda < \lambda^{**}$,

$$\limsup_{n \to \infty} \|u_n\| < \frac{S}{2}.$$
(4.6)

Proof. We first claim that $\{u_n\}$ is bounded. If $||u_n|| \to \infty$ as $n \to \infty$, then it follows from (4.3) that

$$\Psi_S(u_n) = \psi\left(\frac{\|u_n\|^2}{S^2}\right) = 0, \text{ for all large } n \in \mathbb{N},$$

and therefore for all $n \in \mathbb{N}$ large,

$$\mathcal{J}_S(u_n) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{|u|^2}{|x'|^2} dx - \frac{t^{6-2s_1}}{6-2s_1} \int_{\mathbb{R}^3} \frac{|u|^{6-2s_1}}{|x|^{s_1}} dx$$

By (4.5), we have that as $n \in \mathbb{N}$ large,

$$\bar{c}_S + 1 + ||u_n|| \ge \mathcal{J}_S(u_n) - \frac{1}{6 - 2s_1} \langle \mathcal{J}'_S(u_n), u_n \rangle = \frac{2 - s_1}{6 - 2s_1} ||u_n||^2,$$

which is impossible. We have

$$\frac{2-s_1}{6-2s_1} \|u_n\|^2 + \frac{1}{6-2s_1} \langle \mathcal{J}'_S(u_n), u_n \rangle \\
= \mathcal{J}_S(u_n) + \frac{\lambda(s_2-s_1)}{2(3-s_2)(3-s_1)} \Psi_S(u_n) \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \\
- \frac{\lambda \|u_n\|^2}{2(3-s_1)(3-s_2)S^2} \psi' \left(\frac{\|u_n\|^2}{S^2}\right) \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \\
\leqslant \mathcal{J}_S(u_n) + \frac{\lambda(s_2-s_1)}{2(3-s_2)(3-s_1)} \Psi_S(u_n) \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \\
- \frac{\lambda \|u_n\|^2}{2(3-s_1)(3-s_2)S^2} \left| \psi' \left(\frac{\|u_n\|^2}{S^2}\right) \right| \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx.$$
(4.7)

Suppose, up to a subsequence, that

$$\lim_{n \to \infty} \|u_n\| \ge \frac{S}{2}.$$
(4.8)

Since $\{u_n\}$ is bounded, it follows from (4.5) and (4.8) that for n sufficiently large,

$$\frac{2-s_1}{6-2s_1} \|u_n\|^2 + \frac{1}{6-2s_1} \langle \mathcal{J}'_S(u_n), u_n \rangle \ge CS^2 - \frac{\|\mathcal{J}'_S(u_n)\|\|u_n\|}{6-2s_1} \ge CS^2 - S.$$
(4.9)

Notice that if $||u_n|| > S$ then $\Psi_S(u_n) = 0$ and $\psi'\left(\frac{||u_n||^2}{S^2}\right) = 0$. So we can obtain that as n large enough, using (4.7),

$$CS^2 - S \le \bar{c}_S + 1,$$

which is impossible as S large enough. Now we consider the case $||u_n|| \leq S$. Using $\psi \in C_0^{\infty}(\mathbb{R}, [0, 1])$, we deduce that

$$\Psi_S(u_n) \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \leqslant C ||u_n||^{6-2s_2} \Psi_S(u_n) \leqslant CS^{6-2s_2}, \tag{4.10}$$

$$\frac{\|u_n\|^2}{S^2} \left| \psi'\left(\frac{\|u_n\|^2}{S^2}\right) \right| \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx \leqslant C \frac{\|u_n\|^{8-2s_2}}{S^2} \left| \psi'\left(\frac{\|u_n\|^2}{S^2}\right) \right| \leqslant CS^{6-2s_2}.$$
 (4.11)

By the definitions of \bar{c}_S and \tilde{v} , we infer that

$$\bar{c}_{S} \leq \max_{t \in [0,1]} \mathcal{J}_{S}(t\tilde{v})
\leq \max_{t \in [0,1]} \left\{ \frac{t^{2}}{2} \|\tilde{v}\|^{2} - \frac{t^{6-2s_{1}}}{6-2s_{1}} \int_{\mathbb{R}^{3}} \frac{|\tilde{v}|^{6-2s_{1}}}{|x|^{s_{1}}} dx \right\}
+ \max_{t \in [0,1]} \left\{ \frac{-\lambda t^{6-2s_{2}}}{6-2s_{2}} \Psi_{S}(t\tilde{v}) \int_{\mathbb{R}^{3}} \frac{|\tilde{v}|^{6-2s_{2}}}{|x|^{s_{2}}} dx \right\}.$$
(4.12)

As in (4.10), we derive that

$$\max_{t \in [0,1]} \left\{ \frac{-\lambda t^{6-2s_2}}{6-2s_2} \Psi_S(t\tilde{v}) \int_{\mathbb{R}^N} \frac{|\tilde{v}|^{6-2s_2}}{|x|^{s_2}} dx \right\} \leqslant \max_{t \in [0,1]} \left\{ \frac{-\lambda C t^{6-2s_2}}{6-2s_2} \|\tilde{v}\|^{6-2s_2} \Psi_S(t\tilde{v}) \right\} \leqslant -\lambda C S^{6-2s_2}$$

Combining this with (4.12), we deduce that

$$\bar{c}_S \leqslant C - C\lambda S^{6-2s_2}.$$

This together with $\mathcal{J}_S(u_n) \to \bar{c}_S$ as $n \to \infty$ imply that for n large enough,

$$\mathcal{J}_S(u_n) \leqslant C - C\lambda S^{6-2s_2}.$$
(4.13)

Substituting (4.10)–(4.13) in (4.7), we have that for *n* sufficiently large,

$$\frac{2-s_2}{6-2s_2} \|u_n\|^2 + \frac{1}{6-2s_2} \mathcal{J}'_S(u_n)[u_n] \leqslant C - C\lambda S^{6-2s_2}.$$
(4.14)

From (4.9) and (4.14), we obtain that

$$C - C\lambda S^{6-2s_2} \geqslant CS^2 - S,\tag{4.15}$$

where C > 0 is independent of S and λ . The inequality (4.15) would not hold for S > 0 sufficiently large and $0 > \lambda > -S^{2s_2-6}$. The proof is complete.

Lemma 4.3 There exists $\lambda^{**} < 0$ such that as $\lambda \in (\lambda^{**}, 0)$, the functional \mathcal{I}_{λ} has a bounded Palais-Smale sequence $\{u_n\}$ at the level \bar{c}_S .

Proof. By Lemma 4.1, the cut-off functional \mathcal{J}_S has a mountain pass level $\bar{c}_S > 0$ given by (4.4) and there exists $\{u_n\} \subset X_{S\mathcal{O}}$ satisfying (4.5) for each fixed S > 0 and $\lambda < 0$. According to Lemma 4.2, we choose S > 0 large enough and $\lambda^{**} = \lambda^{**}(S) < 0$ large such that for any $\lambda < \lambda^{**}$,

$$\limsup_{n \to \infty} \|u_n\| < \frac{S}{2}.$$

Combining this with (4.2) and the definition of Ψ_S given in (4.3), we derive that for *n* large enough, $\mathcal{J}_S(u_n) = \mathcal{I}(u_n)$ and $\mathcal{J}'_S(u_n) = \mathcal{I}'(u_n)$. Therefore we have $\mathcal{I}(u_n) \to \bar{c}_S > 0$ and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. The proof is complete.

By the same proofs of Lemmas 3.8, 3.9, 3.10. We can obtain the following lemmas, in which we investigate the properties of the $(PS)_{\bar{c}_S}$ sequence $\{u_n\}$ of \mathcal{I}_{λ} found in Lemma 4.3.

Lemma 4.4 If $u_n \rightarrow 0$ in X_{SO} , then for any domain $B_{a,b}(0)$, up to a subsequence and still denoted by $\{u_n\}$ such that

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx \to 0, \quad \int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_i}}{|x|^{s_i}} dx \to 0, \quad i = 1, 2.$$
(4.16)

Proof. For any R > r > 0, the compactness of the embedding

$$X_{\mathcal{SO}} \hookrightarrow \hookrightarrow L^{6-2s_1}(B_{r,R}(0); |x|^{-s_1})$$
(4.17)

implies that

$$\int_{B_{a,b}(0)} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \to 0 \quad \text{as } n \to \infty.$$

Applying the function η defined in Lemma 3.4. Since $\eta^2 u_n \in X_{SO}$ for all $n \in \mathbb{N}$, combining with (4.17) and (3.10), we get that

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'_{\lambda}(u_{n}), \eta^{2}u_{n} \rangle \\
&= \int_{\mathbb{R}^{3}} \nabla u_{n} \cdot \nabla(\eta^{2}u_{n}) + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{1}}\eta^{2}}{|x|^{s_{1}}} dx - \lambda \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{2}}\eta^{2}}{|x|^{s_{2}}} dx \\
&= \int_{\mathbb{R}^{3}} \eta^{2} |\nabla u_{n}|^{2} + \frac{|\eta u_{n}|^{2}}{|x'|^{2}} dx - \lambda \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{2}}\eta^{2}}{|x|^{s_{2}}} dx.
\end{aligned}$$
(4.18)

Since $\lambda < 0$, we get, as $n \to \infty$,

$$\int_{\mathbb{R}^3} |\eta \nabla u_n|^2 + \frac{|\eta u_n|^2}{|x'|^2} dx \to 0.$$

The definition $\eta|_{B_{a,b}(0)} \equiv 1$ implies

$$\int_{B_{a,b}(0)} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx \to 0 \text{ as } n \to \infty.$$

We complete the proof.

Lemma 4.5 If $u_n \rightarrow 0$ in X_{SO} , then there exists a subsequence(still denoted by $\{u_n\}$) such that

either
$$\lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_3)} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx = 0 \quad \text{or} \quad \lim_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_3)} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$$

for all $\delta > 0$.

Now, by the transform (3.19), then we find that $\{\tilde{u}_n\} \subset X_{SO}$ is also a $(PS)_{\bar{c}_S}$ sequence of \mathcal{I}_{λ} .

Lemma 4.6 There exists $\epsilon_1 \in (0, S_{s_1}^{\frac{3-s_1}{2-s_1}}]$ such that for all $\epsilon \in (0, \frac{\epsilon_1}{2})$, there exists a sequence $\{r_n > 0\}$ such that $\{\tilde{u}_n\}$ verifies

$$\int_{B_1(0')\times B_1(0_3)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = \epsilon.$$
(4.19)

Proof. Set

$$\mathcal{A}(u_n) \to \mathcal{A}_{\infty}, \ \mathcal{B}(u_n) \to \mathcal{B}_{\infty}, \ \mathcal{C}(u_n) \to \mathcal{C}_{\infty}$$

We claim that $B_{\infty} > 0$, otherwise, since $\{u_n\}$ is a $(PS)_{\overline{c}_S}$ sequence, we obtain

$$\mathcal{A}_{\infty} - \mathcal{B}_{\infty} - \lambda \mathcal{C}_{\infty} = 0. \tag{4.20}$$

Furthermore,

$$\mathcal{A}_{\infty} = \mathcal{B}_{\infty} + \lambda \mathcal{C}_{\infty} \leq \mathcal{B}_{\infty}.$$

Assume $B_{\infty} = 0$, then $A_{\infty} = 0$ and follows from (4.20), there exists a contradiction with the fact $\bar{c}_S > 0$. The remaining proof is similar with the proof of Lemma 3.10.

Now we are ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. By Lemma 4.3, \mathcal{I}_{λ} has a $(PS)_{\bar{c}_S}$ sequence $\{u_n\} \subset X_{SO}$. By Lemma 4.6, the sequence $\{\tilde{u}_n\}$ define by (3.19) is also a bounded $(PS)_{\bar{c}_S}$ sequence of \mathcal{I}_{λ} in X_{SO} . Thus there is $\tilde{u}_0 \in X_{SO}$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}_0 & \text{in } X_{\mathcal{SO}};\\ \tilde{u}_n \rightharpoonup \tilde{u}_0 & \text{in } L^{6-2s_i}(\mathbb{R}^3; |x|^{-s_i}), \ i = 1, 2;\\ \tilde{u}_n(x) \rightarrow \tilde{u}_0(x) & \text{a.e. on } \mathbb{R}^3. \end{cases}$$

It follows that \tilde{u}_0 is a solution of (1.17). Let $v_n := \tilde{u}_n - \tilde{u}_0$. Then $\{v_n\}$ is bounded in X_{SO} . Define

$$\mathcal{A}(v_n) \to \mathcal{A}_{\infty}, \ \mathcal{B}(v_n) \to \mathcal{B}_{\infty}, \ \mathcal{C}(v_n) \to \mathcal{C}_{\infty}.$$

Then by Brezis-Lieb Lemma [12], we get

$$\mathcal{I}_{\lambda}(v_n) \to \frac{1}{2}\mathcal{A}_{\infty} - \frac{1}{6-2s_1}\mathcal{B}_{\infty} - \frac{\lambda}{6-2s_2}\mathcal{C}_{\infty} = \bar{c}_S - \mathcal{I}_{\lambda}(\tilde{u}_0), \tag{4.21}$$

$$\langle \mathcal{I}'_{\lambda}(v_n), v_n \rangle \to \mathcal{A}_{\infty} - \mathcal{B}_{\infty} - \lambda \mathcal{C}_{\infty} = 0.$$
 (4.22)

If $\mathcal{A}_{\infty} = 0$, then $\mathcal{I}_{\lambda}(\tilde{u}_0) = \hat{c}_S$ and \tilde{u}_0 is a nontrivial solution of (1.17). Assume $\mathcal{A}_{\infty} > 0$ and $\tilde{u}_0 = 0$. Then Lemma 4.5 implies that

either
$$\lim_{n \to \infty} \int_{B_1(0)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = 0 \text{ or } \lim_{n \to \infty} \int_{B_1(0)} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$$

By Lemma 4.6, this is a contradiction to (4.19) as $0 < \epsilon < \frac{1}{2}S_{s_1}^{\frac{1}{2-s_1}}$. It must be $\tilde{u}_0 \neq 0$ and \tilde{u}_0 is a nontrivial solution of (1.17). Combining with Remark 1.12, the equation (1.1) has a nontrivial solution.

5 Asymptotic behavior

In this section we focus on the proofs of Theorem 1.8 in subsection 5.1, Theorem 1.9 in subsection 5.2, Theorem 1.10 in subsection 5.3, Theorem 1.11 in subsection 5.4.

According to Theorems 1.2, 1.3, 1.4, we know that

$$m_{\lambda} = \inf_{u \in \mathcal{S}_{\lambda}} \mathcal{I}_{\lambda}$$

with

$$\mathcal{S}_{\lambda} := \left\{ u \in X_{\mathcal{SO}} | \mathcal{I}_{\lambda}'(u) = 0 \right\}$$

is well defined under the conditions of Theorems 1.2, 1.3, 1.4.

5.1 The case of $0 < s_1 < 2, s_2 = 2$

We now prove Theorem 1.8. We first give a key lemma about the asymptotic behavior of energy level.

Lemma 5.1 Assume the assumptions of Theorem 1.8 hold. Then there exists a subsequence $\{\lambda_n\}$ satisfying $\lim_{n\to\infty} \lambda_n = 0$, such that

$$\lim_{n \to \infty} m_{\lambda_n} = m_0 := \frac{2 - s_1}{6 - 2s_1} S_{s_1}^{\frac{3 - s_1}{2 - s_1}}.$$

Proof. We know that there exists a $u_0 \in X_{SO}$ satisfies $m_0 = \mathcal{I}_0(u_0)$ and u_0 belongs to mountain pass type solution of \mathcal{I}_0 . And $m_0 = \bar{c}_0 = \hat{c}_0$, where $\bar{c}_0 = \hat{c}_0$ is defined in (3.1) and (3.2).

Now we prove this lemma in two different situations: $\lambda_n \to 0^+, \lambda_n \to 0^-$.

We first consider the case $\lambda_n > 0$ and $\lambda_n \to 0^+$ as $n \to \infty$. Combining with (3.4) and (3.5), we get

$$m_0 > m_{\lambda_n}.\tag{5.1}$$

It follows from Theorem 1.2 that, for any $0 < \lambda_n < \overline{\lambda}$, there exists a solution u_n satisfies $m_{\lambda_n} = \mathcal{I}_{\lambda_n}(u_n)$ and u_n is mountain pass type, by (5.1), It is not difficult to verify that u_n is bounded in X_{SO} . Now we prove the fact that $\lim_{n\to\infty} m_{\lambda_n} \ge m_0$. Since, as n large,

$$0 \geq \langle \mathcal{I}_{\lambda_{n}}'(u_{n}), u_{n} \rangle$$

= $\int_{\mathbb{R}^{3}} |\nabla u_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} dx - \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx - \lambda_{n} \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{2}}{|x|^{2}} dx$
$$\geq ||u_{n}||^{2} - S_{s_{1}}^{s_{1}-3} ||u_{n}||^{6-2s_{1}} - \frac{\lambda_{n}}{\overline{\lambda}} ||u_{n}||^{2}$$

$$= \left(1 - \frac{\lambda_{n}}{\overline{\lambda}}\right) ||u_{n}||^{2} - S_{s_{1}}^{s_{1}-3} ||u_{n}||^{6-2s_{1}}.$$

It follows from $\lambda_n < \overline{\lambda}$ that there exists $M := M(s_1)$ such that $||u_n|| \ge M$. Furthermore,

$$o(1) + \mathcal{I}_{\lambda_n}(u_n) = \mathcal{I}_{\lambda_n}(u_n) - \frac{1}{6-2s_1} \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle = \frac{2-s_1}{6-2s_1} \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|u_n|^2}{|x|^2} dx \right) \geq \frac{2-s_1}{12-4s_1} M^2$$
(5.2)

as $n \to \infty$. The facts (5.1) and (5.2) imply that $\mathcal{I}_{\lambda_n}(u_n)$ is bounded. Thus there exists a subsequence(still denoted by origin mark) such that

$$m_{\lambda_n} = \mathcal{I}_{\lambda_n}(u_n) \to c > 0, \mathcal{I}'_{\lambda_n}(u_n) = 0.$$

If $c \ge m_0$, then the proof is complete. Otherwise, $c < m_0$, we will construct a contradiction. The boundedness of sequence $\{u_n\}$ implies that there hold,

$$\begin{cases} u_n \rightarrow u_0 \text{ in } X_{\mathcal{SO}}, \\ u_n \rightarrow u_0 \text{ in } L^{6-2s_1}(\mathbb{R}^3; |x|^{-s_1}), \\ u_n \rightarrow u_0 \text{ in } L^2(\mathbb{R}^3; |x|^{-2}), \\ u_n(x) \rightarrow u_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

If follows that u_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(u_0) \ge 0$. Let $v_n = u_n - u_0$, applying the Brezis-Lieb lemma, we can get that

$$\begin{aligned} \mathcal{I}_{\lambda_n}(v_n) &\to c - \mathcal{I}_0(u_0), \\ \|v_n\|^2 - \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^2}{|x|^2} dx \to 0, \\ \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^2}{|x|^2} dx \to 0. \end{aligned}$$

We may therefore assume that

$$||v_n||^2 \to b, \ \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx \to b.$$

The inequality (2.2) implies that $b \ge S_{s_1} b^{\frac{1}{3-s_1}}$, which leads to that either b = 0 or $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. The case b = 0 implies that u_0 is a nontrivial solution and $\mathcal{I}_0(u_0) = c \ge m_0$, which is a contradiction. However, if $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$, we get that

$$c \ge \lim_{n \to \infty} \mathcal{I}_{\lambda_n}(v_n) \ge \frac{2 - s_1}{6 - 2s_1} S_{s_1}^{\frac{3 - s_1}{2 - s_1}} = m_0,$$

which is a contradiction with $c < m_0$. Thus $c \ge m_0$ and the proof of the case $\lambda_n \to 0^+$ is over.

Now we consider the case $\lambda_n < 0$ and $\lambda_n \to 0^-$ as $n \to \infty$. For any $\lambda < 0$, it follows from Theorem 1.2 that there exists a ground state solution u_{λ} of problem (1.17), then we have that

$$m_{\lambda} = \mathcal{I}_{\lambda}(u_{\lambda}) \ge \mathcal{I}_{\lambda}(t_{0}u_{\lambda}) \ge \mathcal{I}_{0}(t_{0}u_{\lambda}) = \frac{2 - s_{1}}{6 - 2s_{1}} \frac{\left(\int_{\mathbb{R}^{3}} |\nabla u_{\lambda}|^{2} + \frac{|u_{\lambda}|^{2}}{|x'|^{2}} dx\right)^{\frac{3 - s_{1}}{2 - s_{1}}}}{\left(\int_{\mathbb{R}^{3}} \frac{|u_{\lambda}|^{6 - 2s_{1}}}{|x|^{s_{1}}} dx\right)^{\frac{1}{2 - s_{1}}}} \ge m_{0}, \quad (5.3)$$

where

$$t_0 = \left(\frac{\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \frac{|u_\lambda|^2}{|x'|^2} dx}{\int_{\mathbb{R}^3} \frac{|u_\lambda|^{6-2s_1}}{|x|^{s_1}} dx}\right)^{\frac{1}{2-s_1}}$$

For $\lambda_1 < \lambda_2 \leq 0$, let u_{λ_1} be a ground state solution, then

$$m_{\lambda_1} = \mathcal{I}_{\lambda_1}(u_{\lambda_1}) \ge \mathcal{I}_{\lambda_1}(t_{\lambda_2}u_{\lambda_1}) \ge \mathcal{I}_{\lambda_2}(t_{\lambda_2}u_{\lambda_1}) \ge m_{\lambda_2}, \tag{5.4}$$

where t_{λ_2} satisfies that $t_{\lambda_2}u_{\lambda_1} \in \mathcal{N}$. Thus for $-1 < \lambda < 0$, we get that $m_{\lambda} < m_{-1}$. Furthermore, combining with (5.3) and (5.4), one has $\lim_{\lambda \to 0^-} m_{\lambda} = m_0$.

Therefore, we get that there exists a subsequence $\{\lambda_n\}$ satisfying the conclusion of lemma. \Box **Proof of Theorem 1.8.** Let $\{u_n\}$ be a ground state solution of problem (1.17) with $\lambda = \lambda_n$. Then

$$\langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle = 0 \text{ and } \mathcal{I}_{\lambda_n}(u_n) \to m_0,$$
(5.5)

where using Lemma 5.1. Since, for n large,

$$m_{0} + 1 \geq \mathcal{I}_{\lambda_{n}}(u_{n}) - \frac{1}{6-2s_{1}} \langle \mathcal{I}_{\lambda_{n}}'(u_{n}), u_{n} \rangle = \frac{2-s_{1}}{6-2s_{1}} \left(\|u_{n}\|^{2} - \lambda_{n} \int_{\mathbb{R}^{3}} \frac{|u_{n}|^{2}}{|x|^{2}} dx \right) \geq \frac{2-s_{1}}{6-2s_{1}} \left(1 - \frac{\lambda_{n}}{\lambda} \right) \|u_{n}\|^{2}.$$
(5.6)

It follows from $\lambda_n < \overline{\lambda}$ that $\{u_n\}$ is bounded in X_{SO} . Thus there exists a $u_0 \in X_{SO}$ such that

$$\begin{cases} u_n \rightharpoonup u_0 \text{ in } X_{\mathcal{SO}}, \\ u_n \rightharpoonup u_0 \text{ in } L^{6-2s_1}(\mathbb{R}^3; |x|^{-s_1}), \\ u_n \rightharpoonup u_0 \text{ in } L^2(\mathbb{R}^3; |x|^{-2}), \\ u_n(x) \rightarrow u_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

It follows that u_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(u_0) \ge 0$.

If $u_0 \neq 0$, we have $\mathcal{I}_0(u_0) \geq m_0$, set $v_n = u_n - u_0$, applying the Brezis-Lieb lemma, we can get that

$$\mathcal{I}_{\lambda_n}(v_n) \to m_0 - \mathcal{I}_0(u_0), \tag{5.7}$$

$$\|v_n\|^2 - \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^2}{|x|^2} dx \to 0.$$
(5.8)

Similar with (5.6), combining with (5.7) and (5.8), we get that $\mathcal{I}_0(u_0) \leq m_0$. Thus $\mathcal{I}_0(u_0) = m_0$. We may therefore assume that

$$||v_n||^2 \to b, \ \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx \to b$$

Therefore,

$$o(1) = \mathcal{I}_{\lambda_n}(v_n)$$

= $\mathcal{I}_{\lambda_n}(v_n) - \frac{1}{6-2s_1} \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle$
= $\frac{2-s_1}{6-2s_1} \left(||v_n||^2 - \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^2}{|x|^2} dx \right)$
 $\rightarrow \frac{2-s_1}{6-2s_1} b \text{ as } n \rightarrow \infty.$

Thus b = 0, and $u_n \to u_0$, (5.5) implies that $\mathcal{I}_0(u_0) = m_0$, that is u_0 is a ground state solution of equation (1.17) with $\lambda = 0$.

If $u_0 = 0$, since $m_0 > 0$, there exists $\epsilon_1 \in (0, \frac{1}{2}S_{s_1}^{\frac{2-s_1}{3-s_1}}]$ such that for all $\epsilon \in (0, \epsilon_1)$, there exists a sequence $\{r_n > 0\}$ such that $\{\tilde{u}_n := r_n^{\frac{1}{2}}u_n(r_nx)\}$ verifies

$$\int_{B_1(0')\times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = \epsilon$$
(5.9)

and

$$\mathcal{I}_{\lambda_n}(\tilde{u}_n) = \mathcal{I}_{\lambda_n}(u_n) \to m_0, \ \mathcal{I}'_{\lambda_n}(\tilde{u}_n) = 0 \text{ in } (X_{\mathcal{SO}})^*.$$

Moreover, there exists $\tilde{u}_0 \in X_{SO}$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}_0 \text{ in } X_{\mathcal{SO}}, \\ \tilde{u}_n \rightharpoonup \tilde{u}_0 \text{ in } L^{6-2s_1}(\mathbb{R}^3; |x|^{-s_1}), \\ \tilde{u}_n \rightharpoonup \tilde{u}_0 \text{ in } L^2(\mathbb{R}^3; |x|^{-2}), \\ \tilde{u}_n(x) \rightarrow \tilde{u}_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

It follows that \tilde{u}_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(\tilde{u}_0) \ge 0$. Set $\tilde{v}_n = \tilde{u}_n - \tilde{u}_0$, applying the Brezis-Lieb lemma, we can get that

$$\mathcal{I}_{\lambda_n}(\tilde{v}_n) \to m_0 - \mathcal{I}_0(\tilde{u}_0), \|\tilde{v}_n\|^2 - \int_{\mathbb{R}^3} \frac{|\tilde{v}_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|\tilde{v}_n|^2}{|x|^2} dx \to 0.$$
(5.10)

We may therefore assume that

$$\|\tilde{v}_n\|^2 \to b, \ \int_{\mathbb{R}^3} \frac{\|\tilde{v}_n\|^{6-2s_1}}{\|x\|^{s_1}} dx \to b.$$

The inequality (2.2) implies that $b \ge S_{s_1} b^{\frac{1}{3-s_1}}$, which leads to that either b = 0 or $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. The case b = 0 implies that \tilde{u}_0 is a nontrivial solution, which is desired. If $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$ and $\tilde{u}_0 = 0$. Coping the proof of Lemma 3.4 with $s_1 > 0$, we can get that for any domain $B_{a,b}(0)$ and any b > a > 0, up to a subsequence and still denoted by $\{\tilde{u}_n\}$ such that

$$\int_{B_{a,b}(0)} |\nabla \tilde{u}_n|^2 + \frac{|u_n|^2}{|x'|^2} dx \to 0, \quad \int_{B_{a,b}(0)} \frac{|\tilde{u}_n|^{6-2s_i}}{|x|^{s_i}} dx \to 0, \quad i = 1, 2.$$
(5.11)

Set

$$\tilde{\kappa}_{1} := \limsup_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} \frac{|\tilde{u}_{n}|^{6-2s_{1}}}{|x|^{s_{1}}} dx,$$

$$\tilde{\kappa} := \limsup_{n \to \infty} \int_{B_{\delta}(0') \times B_{\delta}(0_{3})} |\nabla \tilde{u}_{n}|^{2} + \frac{|u_{n}|^{2}}{|x'|^{2}} dx.$$
(5.12)

Based on (5.11), similar with Lemma 3.5, for any $\delta > 0$, we get that

$$\tilde{\kappa} = \tilde{\kappa}_1 \tag{5.13}$$

and

$$\tilde{\kappa}_{1}^{\frac{1}{3-s_{1}}} \le S_{s_{1}}^{-1} \tilde{\kappa}.$$
(5.14)

Combining with (5.13) and (5.14), we have

$$\tilde{\kappa}_1^{\frac{1}{3-s_1}} \le S_{s_1}^{-1} \tilde{\kappa}_1.$$
(5.15)

Furthermore, we can obtain that

either
$$\lim_{n \to \infty} \int_{B_1(0') \times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = 0 \text{ or } \lim_{n \to \infty} \int_{B_1(0') \times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx \ge S_{s_1}^{\frac{2-s_1}{3-s_1}}.$$
(5.16)

This is a contradiction with (5.9) as $0 < \epsilon < \frac{1}{2}S_{s_1}^{\frac{2-s_1}{3-s_1}}$. It must be $\tilde{u}_0 \neq 0$. Thus $\mathcal{I}_0(\tilde{u}_0) \ge m_0$. And $\lim_{n\to\infty} \mathcal{I}_{\lambda_n}(\tilde{v}_n) \le 0$, combining with (5.10),

$$\begin{split} o(1) &\geq \mathcal{I}_{\lambda_{n}}(\tilde{v}_{n}) \\ &= \mathcal{I}_{\lambda_{n}}(\tilde{v}_{n}) - \frac{1}{6 - 2s_{1}} \langle \mathcal{I}_{\lambda_{n}}'(\tilde{v}_{n}), \tilde{v}_{n} \rangle \\ &= \frac{2 - s_{1}}{6 - 2s_{1}} \|\tilde{v}_{n}\|^{2} - \frac{\lambda_{n}(2 - s_{1})}{6 - 2s_{1}} \int_{\mathbb{R}^{3}} \frac{|\tilde{v}_{n}|^{2}}{|x|^{2}} dx \\ &\to \frac{2 - s_{1}}{6 - 2s_{1}} b, \end{split}$$

which is a contradiction with $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. Thus $\tilde{u}_n \to \tilde{u}_0$ in X_{SO} . The proof of Theorem 1.8 is over combining with Remark 1.12.

5.2 The case of $\lambda > 0, 0 < s_1 < s_2 < 2$

We now prove Theorem 1.9. We consider the asymptotic behavior of energy level as follows.

Lemma 5.2 Assume $\lambda > 0$ holds. Then there exists a subsequence $\{\lambda_n\}$ satisfying $\lim_{n\to\infty} \lambda_n = 0$, such that

$$\lim_{n \to \infty} m_{\lambda_n} = m_0 := \frac{2 - s_1}{6 - 2s_1} S_{s_1}^{\frac{3 - s_1}{2 - s_1}}$$

Proof. We know that there exists a $u_0 \in X_{SO}$ satisfies $m_0 = \mathcal{I}_0(u_0)$ and u_0 belongs to mountain pass type solution of \mathcal{I}_0 . And $m_0 = \bar{c}_0 = \hat{c}_0$, where \bar{c}_0 , \hat{c}_0 are defined in (3.1) and (3.2). Combining with (3.4) and (3.5), we get

$$m_0 > m_{\lambda_n}.\tag{5.17}$$

It follows from Theorem 1.3 that, for any $\lambda_n > 0$, there exists a solution u_n satisfies $m_{\lambda_n} = \mathcal{I}_{\lambda_n}(u_n)$ and u_n is mountain pass type, by (5.17), It is not difficult to verify that u_n is bounded in X_{SO} . Now we prove the fact that $\lim_{n\to\infty} m_{\lambda_n} \ge m_0$. Since, as n large,

$$1 \geq \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle$$

= $\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{|u_n|^2}{|x'|^2} dx - \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|u_n|^{6-2s_2}}{|x|^{s_2}} dx$
\geq $||u_n||^2 - S_{s_1}^{s_1-3} ||u_n||^{6-2s_1} - S_{s_2}^{s_2-3} ||u_n||^{6-2s_2}.$

It follows from that there exists $M := M(s_1, s_2)$ such that $||u_n|| \ge M$. Furthermore,

$$o(1) + \mathcal{I}_{\lambda_n}(u_n) = \mathcal{I}_{\lambda_n}(u_n) - \frac{1}{6-2s_2} \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle$$

$$= \frac{2-s_2}{6-2s_2} ||u_n||^2 + \frac{s_2-s_1}{2(3-s_1)(3-s_2)} \int_{\mathbb{R}^3} \frac{|u_n|^{2^*(s_1)}}{|x|^{s_1}} dx$$

$$\geq \frac{2-s_2}{12-4s_2} M^2$$
(5.18)

as $n \to \infty$. The facts (5.17) and (5.18) imply that $\mathcal{I}_{\lambda_n}(u_n)$ is bounded. Thus there exists a subsequence(still denoted by origin mark) such that

$$m_{\lambda_n} = \mathcal{I}_{\lambda_n}(u_n) \to c > 0, \mathcal{I}'_{\lambda_n}(u_n) = 0.$$

If $c \ge m_0$, then the proof is complete. Otherwise, $c < m_0$, we will construct a contradiction. The boundedness of sequence $\{u_n\}$ implies that there hold,

$$\begin{cases} u_n \rightharpoonup u_0 \text{ in } X_{\mathcal{SO}}, \\ u_n \rightharpoonup u_0 \text{ in } L^{6-2s_i}(\mathbb{R}^3; |x|^{-s_i}), \ i = 1, 2, \\ u_n(x) \rightarrow u_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

If follows that u_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(u_0) \ge 0$. Let $v_n = u_n - u_0$, applying the Brezis-Lieb lemma, we can get that

$$\begin{aligned} \mathcal{I}_{\lambda_n}(v_n) &\to c - \mathcal{I}_0(u_0), \\ \|v_n\|^2 - \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_2}}{|x|^{s_2}} dx \to 0, \\ \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_2}}{|x|^{s_2}} dx \to 0 \end{aligned}$$

We may therefore assume that

$$||v_n||^2 \to b, \ \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx \to b.$$

The inequality (2.2) implies that $b \ge S_{s_1} b^{\frac{1}{3-s_1}}$, which leads to that either b = 0 or $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. The case b = 0 implies that u_0 is a nontrivial solution and $\mathcal{I}_0(u_0) = c \ge m_0$, which is a contradiction. However, if $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$, we get that

$$c \ge \lim_{n \to \infty} \mathcal{I}_{\lambda_n}(v_n) \ge \frac{2 - s_1}{6 - 2s_1} S_{s_1}^{\frac{3 - s_1}{2 - s_1}} = m_0,$$

which is a contradiction with $c < m_0$. Thus $c \ge m_0$ and the proof is over.

Proof of Theorem 1.9. Let $\{u_n\}$ be a ground state solution of problem (1.17) with $\lambda = \lambda_n > 0$. Then

$$\langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle = 0 \text{ and } \mathcal{I}_{\lambda_n}(u_n) \to m_0,$$
(5.19)

where using Lemma 5.2. Since, for n large,

$$m_0 + 1 \geq \mathcal{I}_{\lambda_n}(u_n) - \frac{1}{6-2s_2} \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle$$

= $\frac{2-s_2}{6-2s_2} ||u_n||^2 + \frac{s_2-s_1}{2(3-s_1)(3-s_2)} \int_{\mathbb{R}^3} \frac{|u_n|^{2^*(s_1)}}{|x|^{s_1}} dx.$ (5.20)

It follows that $\{u_n\}$ is bounded in X_{SO} . Thus there exists a $u_0 \in X_{SO}$ such that

$$\begin{cases} u_n \rightharpoonup u_0 \text{ in } X_{\mathcal{SO}}, \\ u_n \rightharpoonup u_0 \text{ in } L^{6-2s_i}(\mathbb{R}^3; |x|^{-s_i}), \ i = 1, 2, \\ u_n(x) \rightarrow u_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

It follows that u_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(u_0) \ge 0$.

If $u_0 \neq 0$, we have $\mathcal{I}_0(u_0) \geq m_0$, set $v_n = u_n - u_0$, applying the Brezis-Lieb lemma, we can get that

$$\mathcal{I}_{\lambda_n}(v_n) \to m_0 - \mathcal{I}_0(u_0), \tag{5.21}$$

$$\|v_n\|^2 - \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_2}}{|x|^{s_2}} dx \to 0.$$
(5.22)

Similar (5.20), combining with (5.21) and (5.22), we get that $\mathcal{I}_0(u_0) \leq m_0$. Thus $\mathcal{I}_0(u_0) = m_0$. We may therefore assume that

$$||v_n||^2 \to b, \ \int_{\mathbb{R}^3} \frac{|v_n|^{6-2s_1}}{|x|^{s_1}} dx \to b.$$

Therefore,

$$\begin{aligned}
o(1) &= \mathcal{I}_{\lambda_n}(v_n) \\
&= \mathcal{I}_{\lambda_n}(v_n) - \frac{1}{6 - 2s_1} \langle \mathcal{I}'_{\lambda_n}(u_n), u_n \rangle \\
&= \frac{2 - s_1}{6 - 2s_1} \|v_n\|^2 + \frac{\lambda_n(s_1 - s_2)}{2(3 - s_1)(3 - s_2)} \int_{\mathbb{R}^3} \frac{|v_n|^{2^*(s_2)}}{|x|^{s_2}} dx \\
&\to \frac{2 - s_1}{6 - 2s_1} b \quad \text{as } n \to \infty.
\end{aligned}$$

Thus b = 0 and $u_n \to u_0$, (5.19) implies that $\mathcal{I}(u_0) = m_0$, that is u_0 is a ground state solution of equation (1.17) with $\lambda = 0$.

If $u_0 = 0$, since $m_0 > 0$, there exists $\epsilon_1 \in (0, \frac{1}{2}S_{s_1}^{\frac{3-s_1}{2-s_1}}]$ such that for all $\epsilon \in (0, \epsilon_1)$, there exists a sequence $\{r_n > 0\}$ such that $\{\tilde{u}_n := r_n^{\frac{1}{2}}u_n(r_nx)\}$ verifies

$$\int_{B_1(0')\times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = \epsilon$$
(5.23)

and

$$\mathcal{I}_{\lambda_n}(\tilde{u}_n) = \mathcal{I}_{\lambda_n}(u_n) \to m_0, \mathcal{I}'_{\lambda_n}(\tilde{u}_n) = 0 \text{ in } (X_{\mathcal{SO}})^*.$$

Moreover, there exists $\tilde{u}_0 \in X_{\mathcal{SO}}$ such that

$$\begin{cases} \tilde{u}_n \to \tilde{u}_0 \text{ in } X_{\mathcal{SO}}, \\ \tilde{u}_n \to \tilde{u}_0 \text{ in } L^{6-2s_1}(\mathbb{R}^3; |x|^{-s_1}), \\ \tilde{u}_n(x) \to \tilde{u}_0(x) \text{ a.e. on } \mathbb{R}^3. \end{cases}$$

It follows that \tilde{u}_0 is a critical point of \mathcal{I}_0 and $\mathcal{I}_0(\tilde{u}_0) \ge 0$. Set $\tilde{v}_n = \tilde{u}_n - \tilde{u}_0$, applying the Brezis-Lieb lemma, we can get that

$$\mathcal{I}_{\lambda_n}(\tilde{v}_n) \to m_0 - \mathcal{I}_0(\tilde{u}_0), \|\tilde{v}_n\|^2 - \int_{\mathbb{R}^3} \frac{|\tilde{v}_n|^{6-2s_1}}{|x|^{s_1}} dx - \lambda_n \int_{\mathbb{R}^3} \frac{|\tilde{v}_n|^{6-2s_2}}{|x|^{s_2}} dx \to 0.$$
(5.24)

We may therefore assume that

$$\|\tilde{v}_n\|^2 \to b, \ \int_{\mathbb{R}^3} \frac{|\tilde{v}_n|^{6-2s_1}}{|x|^{s_1}} dx \to b.$$

The inequality (2.2) implies that $b \ge S_{s_1}b^{\frac{1}{3-s_1}}$, which leads to that either b = 0 or $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. The case b = 0 implies that \tilde{u}_0 is a nontrivial solution, which is desired. If $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$ and $\tilde{u}_0 = 0$. Similar with (5.12)-(5.16) in the proof of Theorem 1.8, we can obtain that

either
$$\lim_{n \to \infty} \int_{B_1(0') \times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx = 0 \text{ or } \lim_{n \to \infty} \int_{B_1(0') \times B_1(\tilde{0})} \frac{|\tilde{u}_n|^{6-2s_1}}{|x|^{s_1}} dx \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}.$$

This contradicts to (5.23) as $0 < \epsilon < \frac{1}{2}S_{s_1}^{\frac{3-s_1}{2-s_1}}$. It must be $\tilde{u}_0 \neq 0$. Thus $\mathcal{I}_0(\tilde{u}_0) \geq m_0$. And $\lim_{n\to\infty} \mathcal{I}_{\lambda_n}(\tilde{v}_n) \leq 0$, combining with (5.24),

$$\begin{array}{lll}
o(1) & \geq & \mathcal{I}_{\lambda_{n}}(\tilde{v}_{n}) \\
& = & \mathcal{I}_{\lambda_{n}}(\tilde{v}_{n}) - \frac{1}{6 - 2s_{1}} \langle \mathcal{I}_{\lambda_{n}}'(\tilde{v}_{n}), \tilde{v}_{n} \rangle \\
& = & \frac{2 - s_{1}}{6 - 2s_{1}} \|\tilde{v}_{n}\|^{2} + \frac{\lambda_{n}(s_{1} - s_{2})}{2(3 - s_{1})(3 - s_{2})} \int_{\mathbb{R}^{3}} \frac{|\tilde{v}_{n}|^{6 - 2s_{2}}}{|x|^{s_{2}}} dx \\
& \to & \frac{2 - s_{1}}{6 - 2s_{1}} b,
\end{array}$$

which is a contradiction with $b \ge S_{s_1}^{\frac{3-s_1}{2-s_1}}$. Thus $\tilde{u}_n \to \tilde{u}_0$ in X_{SO} . Finally, according to Remark 1.12 we can prove the Theorem 1.9.

5.3 The case of $\lambda < 0, 0 < s_1 < s_2 < 2$

In this section we focus on the proof of Theorem 1.10, that is the case $\lambda < 0, s_1 < s_2$, we first give a asymptotic behavior of energy level.

Lemma 5.3 Assume $\lambda < 0, 0 < s_1 < s_2 < 2$. Then there exists sequence $\{\lambda_n\}$ satisfying $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\lim_{n \to \infty} m_{\lambda_n} = m_0 := \frac{2 - s_1}{6 - 2s_1} S_{s_1}^{\frac{3 - s_1}{2 - s_1}}.$$

Proof. We know that there exists a $u_0 \in X_{SO}$ satisfies $m_0 = \mathcal{I}_0(u_0)$ and u_0 belongs to mountain pass type solution of \mathcal{I}_0 . And $m_0 = \bar{c}_0 = \hat{c}_0$, where \bar{c}_0, \hat{c}_0 are defined in (3.1) and (3.2). For any $\lambda < 0$, it follows from Theorem 1.4 that there exists a mountain pass type ground state solution u_λ of problem (1.17), then we have that

$$m_{\lambda} = \mathcal{I}_{\lambda}(u_{\lambda}) \ge \mathcal{I}_{\lambda}(t_{0}u_{\lambda}) \ge \mathcal{I}_{0}(t_{0}u_{\lambda}) = \frac{2 - s_{1}}{6 - 2s_{1}} \frac{\left(\int_{\mathbb{R}^{3}} |\nabla u_{\lambda}|^{2} + \frac{|u_{\lambda}|^{2}}{|x'|^{2}} dx\right)^{\frac{3 - s_{1}}{2 - s_{1}}}}{\left(\int_{\mathbb{R}^{3}} \frac{|u_{\lambda}|^{6 - 2s_{1}}}{|x|^{s_{1}}} dx\right)^{\frac{1}{2 - s_{1}}}} \ge m_{0},$$
(5.25)

where

$$t_0 = \left(\frac{\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \frac{|u_\lambda|^2}{|x'|^2} dx}{\int_{\mathbb{R}^3} \frac{|u_\lambda|^{6-2s_1}}{|x|^{s_1}} dx}\right)^{\frac{1}{2-s_1}}.$$

For $\lambda_1 < \lambda_2 \leq 0$, let u_{λ_1} be a ground state solution, then

$$m_{\lambda_1} = \mathcal{I}_{\lambda_1}(u_{\lambda_1}) \ge \mathcal{I}_{\lambda_1}(t_{\lambda_2}u_{\lambda_1}) \ge \mathcal{I}_{\lambda_2}(t_{\lambda_2}u_{\lambda_1}) \ge m_{\lambda_2}, \tag{5.26}$$

where t_{λ_2} satisfies that $t_{\lambda_2}u_{\lambda_1} \in \mathcal{N}$. Thus for $-1 < \lambda < 0$, we get that $m_{\lambda} < m_{-1}$. Thus, combining with (5.25) and (5.26), one has $\lim_{\lambda \to 0^-} m_{\lambda} = m_0$.

Proofs of Theorem 1.10. The proof is similar with Theorem 1.9.

5.4 Proof of Theorem 1.11

In the present section, we only prove Theorem 1.11. According to Theorem 1.5, we get that for large S > 0, there exists a $\lambda^* < 0$ such that as $\lambda^* < \lambda < 0$, the equation (1.17) has a nontrivial solution \tilde{u}_{λ} satisfying

$$\mathcal{J}_S(\tilde{u}_\lambda) = \mathcal{I}_\lambda(\tilde{u}_\lambda) = \hat{c}_\lambda,$$

where

$$\hat{c}_{\lambda} = \inf_{\gamma \in \Gamma_S^{\lambda}} \max_{t \in [0,1]} \mathcal{J}_S(\gamma(t)) > 0$$

and

$$\Gamma_{S}^{\lambda} = \{ \gamma \in C([0,1], X_{\mathcal{SO}}) : \gamma(0) = 0, \ \mathcal{J}_{S}(\gamma(1)) < 0 \}$$

Fixed S large, if $\lambda^1 < \lambda^2$ and $\gamma \in \Gamma_S^{\lambda^1}$, it follows from

$$\mathcal{J}_{S}(\gamma(1)) = \frac{1}{2}\mathcal{A}(\gamma(1)) - \frac{1}{6 - 2s_{1}}\mathcal{B}(\gamma(1)) - \frac{\lambda^{1}}{6 - 2s_{2}}\Psi_{S}(\gamma(1))\mathcal{C}(\gamma(1)) < 0$$

that

$$\mathcal{J}_S(\gamma(1)) = \frac{1}{2}\mathcal{A}(\gamma(1)) - \frac{1}{6-2s_1}\mathcal{B}(\gamma(1)) - \frac{\lambda^2}{6-2s_2}\Psi_S(\gamma(1))\mathcal{C}(\gamma(1)) < 0.$$

Thus $\Gamma_S^{\lambda^1} \subset \Gamma_S^{\lambda^2}$ and so $\hat{c}_{\lambda^2} \leq \hat{c}_{\lambda^1}$. Thus there exists a sequence $\{\lambda_n > 0\}$ satisfying $\lim_{n \to +\infty} \lambda_n = 0$ such that $\lim_{n \to +\infty} \hat{c}_{\lambda_n} = \hat{c}_0 > 0$. We use \tilde{u}_{λ_n} to denote the solutions of (1.17) corresponding to the energy \hat{c}_{λ_n} , that is

$$\mathcal{I}_{\lambda_n}(\tilde{u}_{\lambda_n}) = \hat{c}_{\lambda_n}, \ \mathcal{I}'_{\lambda_n}(\tilde{u}_{\lambda_n}) = 0.$$

Coping the proof of Theorem 1.2, we see that $\{u_n = r_n^{\frac{1}{2}} \tilde{u}_{\lambda_n}(r_n x)\}$ also a solution of (1.17) and there exists a $u \in X_{SO} \setminus \{0\}$ such that $u_n \to u$ in X_{SO} . As a consequence, u satisfies (1.17) with $\lambda = 0$ and the energy \hat{c}_0 . Finally, according to Remark 1.12, the Theorem 1.11 is proved.

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Data availibility

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest

The authors have no Conflict of interest to declare that are relevant to the content of this article.

References

- [1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications. J. Funct. Anal., **14**(1973) 349–381.
- [2] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl., 55(1976) 269–296.
- [3] A. Azzollini, V. Benci, T. D'Aprile, D. Fortunato, Existence of static solutions of the semilinear Maxwell equations. *Ric. Mat.*, 55(2006) 283–297.
- [4] M. Badiale, G. Tarantello, A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal.*, 163(2002) 259–293.
- [5] T. Bartsch, T. Dohnal, M. Plum, W. Reichel, Ground states of a nonlinear curl-curl problem in cylindrically symmetric media. *NoDEA Nonlinear Differential Equations Appl.*, 23(2016) 34 pp.
- [6] T. Bartsch, J. Mederski, Ground and bound state solutions of semilinear time-harmonic Maxwell equations in a bounded domain. Arch. Ration. Mech. Anal., 215(2015) 283–306.
- [7] T. Bartsch, J. Mederski, Nonlinear time-harmonic Maxwell equations in domains. J. Fixed Point Theory Appl., 19(2017) 959–986.
- [8] T. Bartsch, J. Mederski, Nonlinear time-harmonic Maxwell equations in an anisotropic bounded medium. J. Funct. Anal., 272(2017) 4304–4333.
- [9] V. Benci, D. Fortunato, Towards a unified field theory for classical electrodynamics. Arch. Ration. Mech. Anal., 173(2004) 379–414.
- [10] V. Benci, D. Fortunato, A unitarian approach to classical electrodynamics: the semilinear Maxwell equations. *Progr. Nonlinear Differential Equations Appl.*, 66, *Birkhäuser, Basel*, 2006.
- [11] B. Bieganowski, Solutions to a nonlinear Maxwell equation with two competing nonlinearities in \mathbb{R}^3 . Bull. Pol. Acad. Sci. Math., **69**(2021) 37–60.
- [12] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(1983) 486–490.
- [13] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(1983) 437–477.
- [14] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights. *Compositio Math.*, 53(1984) 259–275.
- [15] P. C. Carrião, R. Demarque, O. H. Miyagaki, Existence and non-existence of solutions for *p*-Laplacian equations with decaying cylindrical potentials. *J. Differential Equations*, 255(2013) 3412–3433.

- [16] F. Catrina, Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2001) 229–258.
- [17] K. S. Chou, C. W. Chu, On the best constant for a weighted Sobolev-Hardy inequality. J. London Math. Soc., 48(1993) 137–151.
- [18] M. J. Esteban, P.-L. Lions, Stationary solutions of nonlinear Schrödinger equations with an external magnetic field. *Partial differential equations and the calculus of variations, Vol. I,* 401–449, Progr. Nonlinear Differential Equations Appl., 1, *Birkhäuser Boston, Boston, MA*, 1989.
- [19] R. Filippucci, P. Pucci, F. Robert, On a p-Laplace equation with multiple critical nonlinearities. J. Math. Pures Appl., 91(2009) 156–177.
- [20] M. Gaczkowski, J. Mederski, J. Schino, Multiple solutions to cylindrically symmetric curl-curl problems and related Schrödinger equations with singular potentials. *SIAM J. Math. Anal.*, 55(2023) 4425–4444.
- [21] N. Ghoussoub, X. S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities. Ann. Inst. H. Poincaré C Anal. Non Linéaire, 21(2004) 767–793.
- [22] N. Ghoussoub, F. Robert, Sobolev inequalities for the Hardy-Schrödinger operator: extremals and critical dimensions. Bull. Math. Sci., 6(2016) 89–144.
- [23] N. Ghoussoub, F. Robert, The effect of curvature on the best constant in the Hardy-Sobolev inequalities. *Geom. Funct. Anal.*, 16(2006) 1201–1245.
- [24] N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Amer. Math. Soc.*, 352(2000) 5703–5743.
- [25] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 24(1981) 525–598.
- [26] F. Gladiali, M. Grossi, S. L. N. Neves, Nonradial solutions for the Hénon equation in \mathbb{R}^N . Adv. Math., 249(2013) 1–36.
- [27] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin. *J. Inequal. and Appl.*, 1(1997) 275–292.
- [28] C.-H. Hsia, C.-S. Lin, H. Wadade, Revisiting an idea of Brézis and Nirenberg. J. Funct. Anal., 259(2010) 1816– 1849.
- [29] L. Jeanjean, S. Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations. *Adv. Differential Equations*, **11** (2006) 813–840.
- [30] E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. Ann. of Math. 118(1983) 349–374.
- [31] Y. Y. Li, C.-S. Lin, A nonlinear elliptic PDE and two Sobolev-Hardy critical exponents. Arch. Ration. Mech. Anal., 203(2012) 943–968.
- [32] V. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations. Second, revised and augmented edition. Springer, Heidelberg, 2011.
- [33] J. B. McLeod, C. A. Stuart, W. C. Troy, An exact reduction of Maxwell's equations. *Nonlinear diffusion equations and their equilibrium states*, 3(Gregynog, 1989), 391–405, Progr. Nonlinear Differential Equations Appl., 7, *Birkhäuser Boston, Boston, MA*, 1992.
- [34] J. Mederski, Ground states of time-harmonic semilinear Maxwell equations in \mathbb{R}^3 with vanishing permittivity. *Arch. Ration. Mech. Anal.*, **218**(2015) 825–861.
- [35] J. Mederski, Nonlinear time-harmonic Maxwell equations in a bounded domain: lack of compactness. Sci. China Math., 61(2018) 1963–1970.

- [36] J. Mederski, Nonlinear time-harmonic Maxwell equations in \mathbb{R}^3 : recent results and open questions. *Recent advances in nonlinear PDEs theory*, 47–57, *Semin. Interdiscip. Mat. (S.I.M.), Potenza*, 2016.
- [37] J. Mederski, J. Schino, Nonlinear curl-curl problems in \mathbb{R}^3 , *Minimax Theory Appl.*, 7(2022) 339–364.
- [38] J. Mederski, J. Schino, A. Szulkin, Multiple solutions to a nonlinear curl-curl problem in \mathbb{R}^3 . Arch. Ration. Mech. Anal., **236**(2020) 253–288.
- [39] J. Mederski, A. Szulkin, A Sobolev-type inequality for the curl operator and ground states for the curl-curl equation with critical Sobolev exponent. *Arch. Ration. Mech. Anal.*, 241(2021) 1815–1842.
- [40] R. Musina, Ground state solutions of a critical problem involving cylindrical weights. *Nonlinear Anal.*, 68(2008) 3972–3986.
- [41] J. Schino, Ground state, bound state, and normalized solutions to semilinear Maxwell and Schrödinger equations. 2022. arXiv:2207.07461.
- [42] S. Secchi, D. Smets, M. Willem, Remarks on a Hardy-Sobolev inequality. C. R. Math. Acad. Sci. Paris, 336(2003) 811–815.
- [43] C. A. Stuart, Self-trapping of an electromagnetic field and bifurcation from the essential spectrum. Arch. Rational Mech. Anal., 113(1990) 65–96.
- [44] X. Sun, *p*-Laplace equations with multiple critical exponents and singular cylindrical potential. *Acta Math. Sci. Ser. B (Engl. Ed.)*, **33**(2013) 1099–1112.
- [45] X. Sun, Y. Zhang, Elliptic equations with cylindrical potential and multiple critical exponents. *Commun. Pure Appl. Anal.*, **12**(2013) 1943–1957.
- [46] G. Talenti, Best constant in Sobolev inequality. Ann. Mat. Pura Appl., 110(1976) 353–372.
- [47] A. Tertikas, K. Tintarev, On existence of minimizers for the Hardy-Sobolev-Maz'ya inequality. *Ann. Mat. Pura Appl.*, **186**(2007) 645–662.
- [48] C. Wang, J. Su, The ground states of quasilinear Hénon equation with double weighted critical exponents. Proc. Roy. Soc. Edinburgh Sect. A, 153(2023) 1037–1044.
- [49] C. Wang, J. Su, The semilinear elliptic equations with double weighted critical exponents. J. Math. Phys., 63(2022) Paper No. 041505, 21 pp.
- [50] C. Wang, J. Su, On the double weighted critical quasilinear Hénon problems. 2023. Submitted.
- [51] M. Willem, Minimax Theorems. Birkhäuser Boston, Inc. Boston, 1996.
- [52] X. Zeng, Cylindrically symmetric ground state solutions for curl-curl equations with critical exponent. Z. Angew. Math. Phys., 68(2017) Paper No.135, 12 pp.