POINTWISE CONVERGENCE OF SOLUTIONS OF THE SCHRÖDINGER EQUATION ALONG GENERAL CURVES WITH RADIAL INITIAL DATA ON DAMEK-RICCI SPACES

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ABSTRACT. In this article, we consider the Schrödinger equation corresponding to the Laplace-Beltrami operator with radial initial data on Damek-Ricci spaces and study the Carleson's problem of pointwise convergence of the solution to its initial data along general curves that satisfy certain Hölder conditions and bilipschitz conditions in the distance from the identity. We obtain a sufficient condition on the regularity of the initial data for the above pointwise convergence to hold, which is sharp upto the endpoint. Certain Euclidean analogues are also obtained.

CONTENTS

1. INTRODUCTION

One of the most celebrated problems in Euclidean Harmonic analysis is the Carleson's problem: determining the optimal regularity of the initial condition f of the Schrödinger equation given by

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta u \ , \ (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(0,\cdot) = f \ , \ \text{ on } \mathbb{R}^n \ , \end{cases}$$

in terms of the index β such that f belongs to the inhomogeneous Sobolev space $H^\beta(\mathbb{R}^n)$, so that the solution of the Schrödinger operator u converges pointwise to f, $\lim\limits_{t\to 0+} u(x,t) = f(x)$, almost everywhere. Such a problem was first studied by Carleson [Ca80] and Dahlberg-Kenig [DK82] in dimension 1, followed by several other experts in the field for arbitrary dimension (see [Co83, Sj87, Ve88, Bo16, DGL17, DZ19] and references therein).

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Recently, the author studied the Carleson's problem for solutions of the Schrödinger equation corresponding to the Laplace-Beltrami operator in the setting of Damek-Ricci spaces S [De24]. They are also known as Harmonic NA groups. These spaces S are non-unimodular, solvable extensions of Heisenberg type groups N, obtained by letting $A = \mathbb{R}^+$ act on N by homogeneous dilations. The rank one Riemannian Symmetric spaces of noncompact type are the prototypical examples of (and in fact accounts for a very small subclass of the more general class of) Damek-Ricci spaces [ADY96]. Let Δ be the Laplace-Beltrami operator on S corresponding to the left-invariant Riemannian metric. Its L^2-spectrum is the half line $(-\infty, -Q^2/4]$, where Q is the homogeneous dimension of N. The Schrödinger equation on S is given by

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta u \;, \; (x,t) \in S \times \mathbb{R} \\ u(0,\cdot) = f \;, \;\; \text{on } S \;. \end{cases} \tag{1.1}$$

Then for a radial function f belonging to the L^2-Schwartz class $\mathscr{S}^2(S)_o$ (see (2.6)),

$$S_t f(x) := \int_0^\infty \varphi_\lambda(x)\, e^{it\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \;, \tag{1.2}$$

is the solution to (1.1), where φ_λ are the spherical functions, $\widehat{f}$ is the Spherical Fourier transform of f and $\mathbf{c}(\cdot)$ denotes the Harish-Chandra's $\mathbf{c}$-function.

To quantify the Carleson's problem on S, for $\beta \geq 0$, we recall the fractional L^2-Sobolev spaces on S [APV15]:

$$H^\beta(S) := \left\{ f \in L^2(S) : \|f\|_{H^\beta(S)} := \left(\int_0^\infty \left(\lambda^2 + \frac{Q^2}{4}\right)^\beta \left|\widehat{f}(\lambda)\right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} < \infty \right\}. \tag{1.3}$$

In [De24], the author proved that if f is radial and belongs to the Sobolev space $H^\beta(S)$ for $\beta \geq 1/4$, then the solution to the Schrödinger equation with initial data f converges to f almost everywhere, that is,

$$\lim_{t \to 0+} S_t f(x) = f(x) \;, \tag{1.4}$$

for almost every $x \in S$, with respect to the left Haar measure. A natural generalization of the above pointwise convergence problem (1.4) is to ask for almost everywhere convergence along a wider approach region instead of vertical lines. One of such problems may be the non-tangential convergence to the initial data. On $\mathbb{R}^n$, for $\beta > n/2$, by Sobolev imbedding and standard arguments, the non-tangential convergence to the initial data follows. However, Sjögren and Sjölin showed in [SS89] that the non-tangential convergence fails for $\beta \leq n/2$. Thus it becomes interesting to study the pointwise convergence of solutions along appropriate curves and its connection with the regularity of the initial data. Such problems have received considerable interest in recent years: by Cho-Lee-Vargas [CLV12], Ding-Niu [DN17] in dimension one and by Cao-Miao [CM23], Minguillón [Mi24] in higher dimensions.

Our primary aim in this article is to generalize the result of pointwise convergence of solutions of the Schrödinger equation along vertical lines [De24, Theorem 1.1, Corollary 1.2] to more general curves for radial initial data f on Damek-Ricci spaces with the same regularity, that is, $f \in H^\beta(S)$ for $\beta \geq 1/4$. So for $x \in S$, we consider curves $\gamma_x : \mathbb{R} \to S$ such that $\gamma_x(0) = x$. Now as we will be exclusively dealing with radial initial data, it is both natural and customary to identify all the points x on a geodesic sphere centered at the identity e of S, solely by their geodesic distance, say s from e, that is, $s = d(e,x) \in [0,\infty)$. Here $d(\cdot,\cdot)$ is the inner metric on S corresponding to the left-invariant Riemannian metric. Then the solution of the Schrödinger equation with initial data $f \in \mathscr{S}^2(S)_o$, along a curve γ_s is given by,

$$S_t f(\gamma_s(t)) = S_t f(d(e,\gamma_s(t))) = \int_0^\infty \varphi_\lambda(d(e,\gamma_s(t)))\, e^{it\left(\lambda^2 + \frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,.$$

Again keeping the notion of radiality in mind, we consider curves that satisfy the following natural analogues of the conditions considered in [CLV12]: there exist constants C_j $(j = 1,2,3)$, independent of $s, s' \in [0,\infty)$, $t, t' \in \mathbb{R}$ and $\alpha \geq 0$, such that

$$(\mathscr{H}_1) \qquad |\, d(e,\gamma_s(t)) - d(e,\gamma_s(t'))\,| \leq C_1\, |t - t'|^\alpha\,,$$
$$(\mathscr{H}_2) \qquad C_2\,|s - s'| \leq |\, d(e,\gamma_s(t)) - d(e,\gamma_{s'}(t))\,| \leq C_3\,|s - s'|\,.$$

Here α is essentially the degree of tangential convergence.

For $\beta \geq 0$, defining the homogeneous Sobolev spaces (corresponding to the shifted Laplace-Beltrami operator $\tilde{\Delta} := \Delta + \frac{Q^2}{4}$),

$$\dot{H}^\beta(S) := \left\{ f \in L^2(S) : \|f\|_{\dot{H}^\beta(S)} := \left(\int_0^\infty \lambda^{2\beta}\, \left|\widehat{f}(\lambda)\right|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda \right)^{1/2} < \infty \right\}\,,$$

we state our first result dealing with maximal estimates on annuli (in the following statement $A(\cdot)$ denotes the density function, see section 2 for more details):

Theorem 1.1. *Let $0 < r_1 < r_2 < \infty$. Assume that $\gamma_\cdot(\cdot)$ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, r_2)$ and $t, t' \in [-T, T]$, for some fixed $0 < T < \left(\frac{C_2 r_1}{2C_1}\right)^{1/\alpha}$. Then we have for all $f \in \mathscr{S}^2(S)_o$,*

$$\left(\int_{r_1}^{r_2} \left(\sup_{t \in [-T,T]} |S_t f(d(e,\gamma_s(t)))| \right)^2 A(s)\, ds \right)^{1/2} \leq C\, \|f\|_{\dot{H}^{1/4}(S)}\,, \tag{1.5}$$

where the positive constant C depends only on r_1, r_2 and the dimension of S.

The key idea in the proof of Theorem 1.1 is to keep track of the oscillation afforded by both the spherical functions as well as the multiplier corresponding to the Schrödinger operator. But this becomes a challenging task as in the generality of Damek-Ricci spaces, no explicit expression of φ_λ is known. In fact, it only admits certain series expansions depending on the geodesic distance from the identity of the group. Then to capture the aforementioned oscillation from a variation of the classical Harish-Chandra series expansion of φ_λ, to decompose the linearized maximal function into suitable pieces and then to estimate each of them individually, become somewhat technical. In this regard, we also need the improved estimates obtained by Anker-Pierfelice-Vallarino on the coefficients of the above series expansion of the spherical functions (see (2.8)).

Remark 1.2. The sharpness of the regularity threshold in (1.5) at $\beta = 1/4$, follows from the sharpness in the special case of vertical lines [DR24, proof of Theorem 1.1, statement (i), pp. 21-23].

By standard arguments in the literature, Theorem 1.1 implies the following result on almost everywhere pointwise convergence along general curves,

Corollary 1.3. *Let $\alpha \in [\frac{1}{2}, 1]$. Suppose that for every $s_0 \in [0,\infty)$, there exists a neighborhood V ($\subset [0,\infty) \times \mathbb{R}$) of $(s_0, 0)$ such that $(\mathscr{H}_1)$ holds for $(s,t), (s,t') \in V$ and $(\mathscr{H}_2)$ holds for all $(s,t), (s',t) \in V$. Then for all $f \in \dot{H}^{1/4}(S)_o$ and hence for all $f \in H^\beta(S)_o$, with $\beta \geq 1/4$, we have*

$$\lim_{t \to 0} S_t f(\gamma_s(t)) = f(s) ,$$

for almost every $s \in (0,\infty)$, which also means

$$\lim_{t \to 0} S_t f(\gamma_x(t)) = f(x) ,$$

for almost every $x \in S \setminus \{e\}$, with respect to the left Haar measure.

In this article, we include the case when N is abelian and $\mathfrak{n}$ coincides with its center in the definition of H-type groups, as a degenerate case, so that the Real hyperbolic spaces are also included in the class of Damek-Ricci spaces (see [CDKR98, pp. 209-210]).

We now see some consequences of Theorem 1.1 for the classical Euclidean setting, for $n \geq 2$. For a radial function f belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^n)_o$, the solution of the Schrödinger equation on $\mathbb{R}^n$,

$$\begin{cases} i\frac{\partial u}{\partial t} = \Delta_{\mathbb{R}^n} u \, , \; (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(0,\cdot) = f \, , \;\; \text{on } \mathbb{R}^n \, , \end{cases} \tag{1.6}$$

is given by

$$\tilde{S}_t f(s) := \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, e^{it\lambda^2}\, \mathscr{F} f(\lambda)\, \lambda^{n-1}\, d\lambda \, , \tag{1.7}$$

where $\mathscr{J}_{\frac{n-2}{2}}$ is the modified Bessel function of order $\frac{n-2}{2}$ (see section 2 for more details), $\mathscr{F} f$ is the Euclidean Spherical Fourier transform of f and $s = \|x\|$, is the distance of x from the origin o. In this regard, we also formulate the conditions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$, in terms of the Euclidean norm $\|\cdot\|$: there exist constants C_j $(j = 4,5,6)$, independent of $s, s' \in [0,\infty)$, $t, t' \in \mathbb{R}$ and $\alpha \geq 0$, such that

$$\begin{aligned} (\mathscr{H}_3) \quad & |\, \|\gamma_s(t)\| - \|\gamma_s(t')\| \,| \leq C_4\, |t - t'|^\alpha \, , \\ (\mathscr{H}_4) \quad & C_5\, |s - s'| \leq |\, \|\gamma_s(t)\| - \|\gamma_{s'}(t)\| \,| \leq C_6\, |s - s'| \, . \end{aligned}$$

Now it is interesting to observe that carrying out the arguments in the proof of Theorem 1.1 analogously for $\mathbb{R}^n$, seems to be a rather difficult task. Nevertheless, taking a detour using local geometry of Riemannian manifolds and Fourier Analytic tools on Homogeneous spaces, we obtain the following analogue of Theorem 1.1 for small annuli:

Theorem 1.4. *Let $0<\delta_1<\delta_2$ be sufficiently small. Assume that $\gamma_{\cdot}(\cdot)$ satisfies $(\mathscr{H}_3)$ for $\alpha \in [\frac{1}{2},1]$ and $(\mathscr{H}_4)$ for $s,s' \in [0,\delta_2)$ and $t,t' \in [-T,T]$ for some fixed $0<T<\left(\frac{C_5\delta_1}{2C_4}\right)^{1/\alpha}$. Then we have for all $f \in \mathscr{S}(\mathbb{R}^n)_o$,*

$$\left(\int_{\delta_1}^{\delta_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\,ds\right)^{1/2} \leq C\,\|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,, \tag{1.8}$$

where the positive constant C depends only on δ_1,δ_2 and n.

For the proof of Theorem 1.4, we first consider the cases when the Fourier transform of the initial condition is supported in a neighborhood of the origin and when it is supported away from the origin. In the first case, the linearized maximal function is quite straightforward to control. The latter case however, is quite involved. In the large frequency regime, viewing $\mathbb{R}^n$ as the tangent space at the identity of a suitable Damek-Ricci space, we form a connection with Theorem 1.1. Using the fact that the Riemannian exponential map on a connected, simply-connected, complete, non-positively curved Riemannian manifold is a bijective local radial isometry, we can pushforward the curves in the small annuli to the non-flat manifold, preserving the local geometric conditions. The connection between the initial condition and its Schrödinger propagation for $\mathbb{R}^n$ and S, is then obtained by means of the Bessel series expansion of φ_λ, repeated applications of the Abel transform, followed by utilizing a Schwartz multiplier corresponding to the ratio of the weights of the respective Plancherel measures. Then Theorem 1.4 follows by taking a resolution of identity.

Remark 1.5. The sharpness of the regularity threshold in (1.8) at $\beta = 1/4$, follows from the sharpness in the special case of vertical lines [Sj97, pp. 55-58].

As an application of Theorem 1.4, we look at the family of curves whose norms satisfy,

$$\|\gamma_s(t)\| = s + C_7 t^{1/2}\,, \tag{1.9}$$

for some $C_7 \geq 0$, in some neighborhood of every $(s_0,0)$. Then dilating annuli, applying Theorem 1.4 and making use of the fact that the norms of the curves (1.9) are invariant under a certain parabolic dilation in space and time, we get the following:

Theorem 1.6. *Suppose that for every $s_0 \in [0,\infty)$, there exists a neighborhood $V \subset [0,\infty)\times\mathbb{R}$ of $(s_0,0)$ such that (1.9) holds for all $(s,t)\in V$. Then for all $f \in \dot{H}^{1/4}(\mathbb{R}^n)_o$ and hence for all $f \in H^{\beta}(\mathbb{R}^n)_o$, with $\beta \geq 1/4$, we have*

$$\lim_{t\to 0} S_t f(\gamma_s(t)) = f(s)\,,$$

for almost every $s \in (0,\infty)$, which also means

$$\lim_{t\to 0} S_t f(\gamma_x(t)) = f(x)\,,$$

for almost every $x \in \mathbb{R}^n \setminus \{o\}$.

This article is organized as follows. In section 2, we recall certain aspects of Euclidean Fourier Analysis, the essential preliminaries about Damek-Ricci spaces and Spherical Fourier Analysis thereon and also see an estimate of an oscillatory integral which is useful for the proof of Theorem 1.1. Theorem 1.1 is proved in section 3. Theorems 1.4 and 1.6 are proved in section 4.

Our Euclidean results (Theorems 1.4 and 1.6) can be interpreted as variations of results in [CM23] and [Mi24], as imposing more symmetry on the intial data allows us to work with much lower regularity.

Throughout, the symbols 'c' and 'C' will denote positive constants whose values may change on each occurrence. The enumerated constants $C_1, C_2, \dots$ will however be fixed throughout. $\mathbb{N}$ will denote the set of positive integers. Two non-negative functions f_1 and f_2 will be said to satisfy,

- $f_1 \lesssim f_2$ if there exists a constant $C \geq 1$, so that
$$f_1 \leq C f_2 \,.$$
- $f_1 \asymp f_2$ if there exist constants $C, C' > 0$, so that
$$C f_1 \leq f_2 \leq C' f_1 \,.$$

2. Preliminaries

In this section, we recall some preliminaries and fix our notations.

2.1. **Fourier Analysis on $\mathbb{R}^n$:** In this subsection, we recall some Euclidean Fourier Analysis, most of which can be found in [Gr09, SW90]. On $\mathbb{R}$, for "nice" functions f, the Fourier transform $\tilde{f}$ is defined as

$$\tilde{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx \,.$$

An important inequality in one-dimensional Fourier Analysis is the Pitt's inequality:

Lemma 2.1. [St56, p. 489] *One has the inequality*

$$\left(\int_{\mathbb{R}} \left| \tilde{f}(\xi) \right|^2 |\xi|^{-2\beta} d\xi \right)^{1/2} \lesssim \left(\int_{\mathbb{R}} |f(x)|^p \, |x|^{\beta_1 p} dx \right)^{1/p} ,$$

where $\beta_1 = \beta + \frac{1}{2} - \frac{1}{p}$ and the following two conditions are satisfied:

$$0 \leq \beta_1 < 1 - \frac{1}{p} \,, \quad \textit{and } 0 \leq \beta < \frac{1}{2} \,.$$

A C^∞ function f on $\mathbb{R}$ is called a Schwartz class function if

$$\left| \left(\frac{d}{dx} \right)^M f(x) \right| \lesssim (1 + |x|)^{-N} \,, \text{ for any } M, N \in \mathbb{N} \cup \{0\} \,.$$

We denote by $\mathscr{S}(\mathbb{R})$ the class of all such functions and $\mathscr{S}(\mathbb{R})_e$ will denote the collection of all even Schwartz class functions on $\mathbb{R}$.

Let $\beta \in \mathbb{C}$ with $Re(\beta) > 0$. The Riesz potential of order β is the operator

$$I_\alpha = (-\Delta_{\mathbb{R}})^{-\beta/2} .$$

One can also write as,

$$I_\beta(f)(x) = C_\beta \int_{\mathbb{R}} f(y)\, |x-y|^{\beta-1}\, dy ,$$

for some $C_\beta > 0$, whenever $f \in \mathscr{S}(\mathbb{R})$. For $\beta > 0$, the Fourier transform of the Riesz potential of $f \in \mathscr{S}(\mathbb{R})$ satisfies the following identity:

$$(I_\beta(f))^{\sim}(\xi) = C_\beta |\xi|^{-\beta} \tilde{f}(\xi) . \tag{2.1}$$

For a radial function f in $\mathbb{R}^n$ (for $n \geq 2$), the Euclidean Spherical Fourier transform is defined as

$$\mathscr{F} f(\lambda) := \int_0^\infty f(s)\, \mathscr{J}_{\frac{n-2}{2}}(\lambda s)\, s^{n-1}\, ds ,$$

where for all $\mu \geq 0$,

$$\mathscr{J}_\mu(z) = 2^\mu\, \pi^{1/2}\, \Gamma\left(\mu + \frac{1}{2}\right) \frac{J_\mu(z)}{z^\mu},$$

and J_μ are the Bessel functions [SW90, p. 154].

The class of radial Schwartz class functions on $\mathbb{R}^n$, denoted by $\mathscr{S}(\mathbb{R}^n)_o$, is defined to be the collection of $f \in C^\infty(\mathbb{R}^n)_o$ such that

$$\left| \left(\frac{d}{ds}\right)^M f(s) \right| \lesssim (1+s)^{-N} , \text{ for any } M, N \in \mathbb{N} \cup \{0\} ,$$

where $s = \|x\|$, is the distance of x from the origin. $\mathscr{F} : \mathscr{S}(\mathbb{R}^n)_o \to \mathscr{S}(\mathbb{R})_e$ defines a topological isomorphism.

2.2. **Damek-Ricci spaces and spherical Fourier Analysis thereon:** In this section, we will explain the notations and state relevant results on Damek-Ricci spaces. Most of these results can be found in [ADY96, APV15, As95].

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with an inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. We say that $\mathfrak{n}$ is an H-type algebra if for every $Z \in \mathfrak{z}$ the map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_z X, Y \rangle = \langle [X,Y], Z \rangle, \quad X, Y \in \mathfrak{v}$$

satisfies the condition $J_Z^2 = -|Z|^2 I_{\mathfrak{v}}$, $I_{\mathfrak{v}}$ being the identity operator on $\mathfrak{v}$. A connected and simply connected Lie group N is called an H-type group if its Lie algebra is H-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a diffeomorphism and hence we can parametrize the elements in $N = \exp \mathfrak{n}$ by (X, Z), for $X \in \mathfrak{v}, Z \in \mathfrak{z}$. It follows from the Baker-Campbell-Hausdorff formula that the group law in N is given by

$$(X,Z)\,(X',Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right), \quad X, X' \in \mathfrak{v};\ Z, Z' \in \mathfrak{z}.$$

The group $A = \mathbb{R}^+$ acts on an H-type group N by nonisotropic dilation: $(X, Z) \mapsto (\sqrt{a}X, aZ)$. Let $S = NA$ be the semidirect product of N and A under the above action. Thus the multiplication in S is given by

$$(X, Z, a)\,(X', Z', a') = \left(X + \sqrt{a}X', Z + aZ' + \frac{\sqrt{a}}{2}[X, X'], aa'\right),$$

for $X, X' \in \mathfrak{v}$; $Z, Z' \in \mathfrak{z}; a, a' \in \mathbb{R}^+$. Then S is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket

$$[(X, Z, l)\,, (X', Z', l')] = \left(\frac{1}{2}lX' - \frac{1}{2}l'X, lZ' - lZ + [X, X'], 0\right).$$

We write $na = (X, Z, a)$ for the element $\exp(X + Z)a, X \in \mathfrak{v}, Z \in \mathfrak{z}, a \in A$. We note that for any $Z \in \mathfrak{z}$ with $|Z| = 1$, $J_Z^2 = -I_{\mathfrak{v}}$; that is, J_Z defines a complex structure on $\mathfrak{v}$ and hence $\mathfrak{v}$ is even dimensional. $m_{\mathfrak{v}}$ and m_z will denote the dimension of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. Let n and Q denote dimension and the homogenous dimension of S respectively:

$$n = m_{\mathfrak{v}} + m_{\mathfrak{z}} + 1 \quad and \quad Q = \frac{m_{\mathfrak{v}}}{2} + m_{\mathfrak{z}}.$$

The group S is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, l), (X', Z', l') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + ll'$$

on $\mathfrak{s}$. For $x \in S$, we denote by $s = d(e, x)$, that is, the geodesic distance of x from the identity e. Then the left Haar measure dx of the group S may be normalized so that

$$dx = A(s)\, ds\, d\sigma(\omega)\,,$$

where A is the density function given by,

$$A(s) = 2^{m_{\mathfrak{v}}+m_{\mathfrak{z}}}\,(\sinh(s/2))^{m_{\mathfrak{v}}+m_{\mathfrak{z}}}\,(\cosh(s/2))^{m_{\mathfrak{z}}}\,, \tag{2.2}$$

and $d\sigma$ is the surface measure of the unit sphere.

A function $f : S \to \mathbb{C}$ is said to be radial if, for all x in S, $f(x)$ depends only on the geodesic distance of x from the identity e. If f is radial, then

$$\int_S f(x)\; dx = \int_0^\infty f(s)\; A(s)\; ds\,.$$

We now recall the spherical functions on Damek-Ricci spaces. The spherical functions φ_λ on S, for $\lambda \in \mathbb{C}$ are the radial eigenfunctions of the Laplace-Beltrami operator Δ, satisfying the following normalization criterion

$$\begin{cases} \Delta\varphi_\lambda = -\left(\lambda^2 + \frac{Q^2}{4}\right)\varphi_\lambda \\ \varphi_\lambda(e) = 1\,. \end{cases}$$

For all $\lambda \in \mathbb{R}$ and $x \in S$, the spherical functions satisfy

$$\varphi_\lambda(x) = \varphi_\lambda(s) = \varphi_{-\lambda}(s)\,.$$

It also satisfies for all $\lambda \in \mathbb{R}$ and all $s \geq 0$:

$$|\varphi_\lambda(s)| \leq 1\,. \tag{2.3}$$

The spherical functions are crucial as they help us define the Spherical Fourier transform of a "nice" radial function f (on S) in the following way:

$$\widehat{f}(\lambda) := \int_S f(x)\varphi_\lambda(x)dx = \int_0^\infty f(s)\varphi_\lambda(s)A(s)ds\,.$$

The Harish-Chandra $\mathbf{c}$-function is defined as

$$\mathbf{c}(\lambda) = \frac{2^{(Q-2i\lambda)}\Gamma(2i\lambda)}{\Gamma\left(\frac{Q+2i\lambda}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m_{\mathfrak{v}}+4i\lambda+2}{4}\right)}\,,$$

for all $\lambda \in \mathbb{R}$. We will need the following pointwise estimates (see [RS09, Lemma 4.8]):

$$|\mathbf{c}(\lambda)|^{-2} \asymp |\lambda|^2(1+|\lambda|)^{n-3}\,, \tag{2.4}$$

and for $j \in \mathbb{N} \cup \{0\}$, the derivative estimates ([As95, Lemma 4.2]):

$$\left|\frac{d^j}{d\lambda^j}|\mathbf{c}(\lambda)|^{-2}\right| \lesssim_j (1+|\lambda|)^{n-1-j}\,,\ \ \lambda \in \mathbb{R}. \tag{2.5}$$

One has the following inversion formula (when valid) for radial functions:

$$f(x) = C\int_0^\infty \widehat{f}(\lambda)\varphi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}d\lambda\,,$$

where C depends only on $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$. Moreover, the Spherical Fourier transform extends to an isometry from the space of radial L^2 functions on S onto $L^2\left((0,\infty), C|\mathbf{c}(\lambda)|^{-2}d\lambda\right)$.

The class of radial L^2-Schwartz class functions on S, denoted by $\mathscr{S}^2(S)_o$, is defined to be the collection of $f \in C^\infty(S)_o$ such that

$$\left|\left(\frac{d}{ds}\right)^M f(s)\right| \lesssim (1+s)^{-N}e^{-\frac{Q}{2}s}\,,\ \text{for any } M,N \in \mathbb{N}\cup\{0\}\,, \tag{2.6}$$

(see [ADY96, p. 652]). One has the following commutative diagram, where every map is a topological isomorphism:

- $\mathscr{A}_{S,\mathbb{R}}$ is the Abel transform defined from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\wedge$ denotes the Spherical Fourier transform from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\sim$ denotes the 1-dimensional Euclidean Fourier transform from $\mathscr{S}(\mathbb{R})_e$ to itself.

$$\begin{array}{ccc}
\mathscr{S}^2(S)_o & \xrightarrow{\mathscr{A}_{S,\mathbb{R}}} & \mathscr{S}(\mathbb{R})_e \\
& {\scriptstyle\wedge}\searrow & \downarrow{\scriptstyle\sim} \\
& & \mathscr{S}(\mathbb{R})_e
\end{array}$$

For more details regarding the Abel transform, we refer to [ADY96, p. 652-653]. These have been generalized to the setting of Chébli-Trimèche Hypergroups [BX98], whose simplest case is that of radial functions on $\mathbb{R}^n$.

For the purpose of our article, we will require both the Bessel series expansion as well as another series expansion (similar to the Harish-Chandra series expansion) of the spherical functions. We first see an expansion of φ_λ in terms of Bessel functions for points near the identity. But before that, let us define the following normalizing constant in terms of the Gamma functions,

$$c_0 = 2^{m_\mathfrak{z}}\, \pi^{-1/2}\, \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\, .$$

Lemma 2.2. [As95, Theorem 3.1] *There exist $R_0, 2 < R_0 < 2R_1$, such that for any $0 \le s \le R_0$, and any integer $M \ge 0$, and all $\lambda \in \mathbb{R}$, we have*

$$\varphi_\lambda(s) = c_0 \left(\frac{s^{n-1}}{A(s)}\right)^{1/2} \sum_{l=0}^{M} a_l(s) \mathscr{J}_{\frac{n-2}{2}+l}(\lambda s) s^{2l} + E_{M+1}(\lambda, s)\, ,$$

where

$$a_0 \equiv 1\, ,\ |a_l(s)| \le C(4R_1)^{-l}\, ,$$

and the error term has the following behaviour

$$|E_{M+1}(\lambda, s)| \le C_M \begin{cases} s^{2(M+1)} & \text{if } |\lambda s| \le 1 \\ s^{2(M+1)} |\lambda s|^{-\left(\frac{n-1}{2}+M+1\right)} & \text{if } |\lambda s| > 1\, . \end{cases}$$

Moreover, for every $0 \le s < 2$, the series

$$\varphi_\lambda(s) = c_0 \left(\frac{s^{n-1}}{A(s)}\right)^{1/2} \sum_{l=0}^{\infty} a_l(s) \mathscr{J}_{\frac{n-2}{2}+l}(\lambda s) s^{2l}\, ,$$

is absolutely convergent.

For the asymptotic behaviour of the spherical functions when the distance from the identity is large, we look at the following series expansion [APV15, pp. 735-736]:

$$\varphi_\lambda(s) = 2^{-m_\mathfrak{z}/2} A(s)^{-1/2} \left\{ \mathbf{c}(\lambda) \sum_{\mu=0}^{\infty} \Gamma_\mu(\lambda) e^{(i\lambda-\mu)s} + \mathbf{c}(-\lambda) \sum_{\mu=0}^{\infty} \Gamma_\mu(-\lambda) e^{-(i\lambda+\mu)s} \right\}\, . \tag{2.7}$$

The above series converges for $\lambda \in \mathbb{R} \setminus \{0\}$, uniformly on compacts not containing the group identity, where $\Gamma_0 \equiv 1$ and for $\mu \in \mathbb{N}$, one has the recursion formula,

$$(\mu^2 - 2i\mu\lambda)\Gamma_\mu(\lambda) = \sum_{j=0}^{\mu-1} \omega_{\mu-j} \Gamma_j(\lambda)\, .$$

Then one has the following estimate on the coefficients [APV15, Lemma 1], for constants $C > 0, d \ge 0$:

$$|\Gamma_\mu(\lambda)| \le C\mu^d (1 + |\lambda|)^{-1}\, , \tag{2.8}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}, \mu \in \mathbb{N}$.

The relevant preliminaries for the degenerate case of the Real hyperbolic spaces can be found in [An91, ST78, AP14].

2.3. **Estimate of an Oscillatory integral.** We will also require the following oscillatory integral estimate, which can be obtained by proceeding exactly as in the proof of Lemma 2.1 in [DN17]:

Lemma 2.3. *Let $0 < \delta_1 < \delta_2 < \infty$. Let $\kappa(\cdot,\cdot) : [0,\infty) \times \mathbb{R} \to [0,\infty)$ satisfy*

$$|\kappa(s,t) - \kappa(s,t')| \lesssim |t-t'|^{\alpha} ,$$
$$|\kappa(s,t) - \kappa(s',t)| \asymp |s-s'| ,$$

for $\alpha \in [\frac{1}{2}, 1]$, $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$ with some fixed $T > 0$. Assume that $t(\cdot) : (\delta_1, \delta_2) \to [-T, T]$ is a measurable function. If $\mu \in C_c^{\infty}[0,\infty)$, then for any $C > 0$,

$$\left| \int_0^{\infty} e^{i\left\{\lambda(\kappa(s',t(s'))-\kappa(s,t(s)))+(t(s')-t(s))\left(\lambda^2+C\right)\right\}} \lambda^{-\frac{1}{2}} \mu\left(\frac{\lambda}{N}\right) d\lambda \right| \lesssim \frac{1}{|s-s'|^{1/2}} ,$$

for all $s, s' \in (\delta_1, \delta_2)$ and $N \in \mathbb{N}$. The implicit constant in the conclusion depends only on μ.

3. Results on Damek-Ricci spaces

In this section, we aim to prove Theorem 1.1. But first we see the following lemma.

Lemma 3.1. *Under the hypothesis of Theorem 1.1, we have for all $s \in (r_1, r_2)$ and $t \in [-T, T]$,*

$$\frac{C_2}{2} s \;<\; d(e, \gamma_s(t)) \;<\; \frac{3C_3}{2} s \,.$$

Proof. We first note by $(\mathscr{H}_2)$ that

$$C_2 s \;\le\; |d(e,\gamma_s(t)) - d(e,\gamma_0(t))| \;\le\; C_3 s \,. \tag{3.1}$$

Next by $(\mathscr{H}_1)$, we have

$$d(e,\gamma_0(t)) \;=\; |d(e,\gamma_0(t)) - d(e,\gamma_0(0))| \;\le\; C_1 |t|^{\alpha} \;\le\; C_1 T^{\alpha} \,. \tag{3.2}$$

Therefore, combining (3.1) and (3.2), we get that

$$C_2 s - C_1 T^{\alpha} \;\le\; d(e,\gamma_s(t)) \;\le\; C_3 s + C_1 T^{\alpha} \,.$$

Now as $0 < T < \left(\frac{C_2 r_1}{2C_1}\right)^{1/\alpha}$, we see that

$$C_2 s - C_1 T^{\alpha} > C_2 \left(s - \frac{r_1}{2}\right) > \frac{C_2}{2} s \,.$$

Similarly,

$$C_3 s + C_1 T^{\alpha} < C_3 s + C_2 \left(\frac{r_1}{2}\right) \le C_3 \left(s + \frac{r_1}{2}\right) < \frac{3C_3}{2} s \,.$$

This completes the proof. □

Proof of Theorem 1.1. In order to prove the theorem, it suffices to prove the following estimate in terms of the linearized maximal function,

$$\left(\int_{r_1}^{r_2} |T_\gamma f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)}\,, \tag{3.3}$$

where,

$$T_\gamma f(s) := \int_0^\infty \varphi_\lambda(d(e,\gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\,, \tag{3.4}$$

and $t(\cdot):(r_1,r_2)\to[-T,T]$ is a measurable function. Now by Lemma 3.1, we have for all $s\in(r_1,r_2)$,

$$\frac{C_2}{2}r_1 < \frac{C_2}{2}s < d(e,\gamma_s(t(s))) < \frac{3C_3}{2}s < \frac{3C_3}{2}r_2\,.$$

Then by invoking the series expansion (2.7) of φ_λ, for $s\in(r_1,r_2)$ and $\lambda>0$, we get that

$$\begin{aligned}\varphi_\lambda(d(e,\gamma_s(t(s)))) &= 2^{-m_\mathfrak{z}/2}A(d(e,\gamma_s(t(s))))^{-1/2}\left\{\mathbf{c}(\lambda)e^{i\lambda d(e,\gamma_s(t(s)))}+\mathbf{c}(-\lambda)e^{-i\lambda d(e,\gamma_s(t(s)))}\right\}\\ &\quad +\mathscr{E}(\lambda,d(e,\gamma_s(t(s))))\,,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\mathscr{E}(\lambda,d(e,\gamma_s(t(s)))) &= 2^{-m_\mathfrak{z}/2}A(d(e,\gamma_s(t(s))))^{-1/2}\\ &\quad\times\left\{\mathbf{c}(\lambda)\sum_{\mu=1}^\infty\Gamma_\mu(\lambda)e^{(i\lambda-\mu)d(e,\gamma_s(t(s)))}+\mathbf{c}(-\lambda)\sum_{\mu=1}^\infty\Gamma_\mu(-\lambda)e^{-(i\lambda+\mu)d(e,\gamma_s(t(s)))}\right\}\,,\end{aligned}$$

and thus again using Lemma 3.1, the local growth asymptotics of the density function (2.2) and the estimate (2.8) on the coefficients Γ_μ, it follows that

$$|\mathscr{E}(\lambda,d(e,\gamma_s(t(s))))| \lesssim A(s)^{-1/2}\,|\mathbf{c}(\lambda)|\,(1+\lambda)^{-1}\,. \tag{3.6}$$

Hence for $s\in(r_1,r_2)$ and $\lambda>0$, in accordance to (3.5), we decompose T_γ as,

$$\begin{aligned}&T_\gamma f(s)\\ =\;& 2^{-m_\mathfrak{z}/2}A(d(e,\gamma_s(t(s))))^{-1/2}\int_0^\infty \mathbf{c}(\lambda)\, e^{i\left\{\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\\ &+2^{-m_\mathfrak{z}/2}A(d(e,\gamma_s(t(s))))^{-1/2}\int_0^\infty \mathbf{c}(-\lambda)\, e^{i\left\{-\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\\ &+\int_0^\infty \mathscr{E}(\lambda,d(e,\gamma_s(t(s))))\,e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\,\widehat{f}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\\ =\;& T_{\gamma,1}f(s)\,+\,T_{\gamma,2}f(s)\,+\,T_{\gamma,3}f(s)\,.\end{aligned}$$

The arguments for $T_{\gamma,1}$ and $T_{\gamma,2}$ are similar and hence we work out the details only for $T_{\gamma,1}$. We aim to show that

$$\left(\int_{r_1}^{r_2}|T_{\gamma,1}f(s)|^2A(s)\,ds\right)^{1/2} \lesssim \left(\int_0^\infty \lambda^{1/2}\,|\widehat{f}(\lambda)|^2|\mathbf{c}(\lambda)|^{-2}d\lambda\right)^{1/2}. \tag{3.7}$$

Now

$$\begin{aligned} & T_{\gamma,1}f(s)\,A(s)^{1/2} \\ = \;& 2^{-m_{\mathfrak{z}}/2}A(d(e,\gamma_s(t(s))))^{-1/2}A(s)^{1/2}\int_0^\infty \mathbf{c}(\lambda)e^{i\left\{\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\widehat{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}d\lambda \\ = \;& 2^{-m_{\mathfrak{z}}/2}A(d(e,\gamma_s(t(s))))^{-1/2}A(s)^{1/2}\int_0^\infty \mathbf{c}(\lambda)e^{i\left\{\lambda d(e,\gamma_s(t(s)))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}g(\lambda)\lambda^{-1/4}|\mathbf{c}(\lambda)|^{-1}d\lambda, \end{aligned}$$

where

$$g(\lambda)=\widehat{f}(\lambda)\,\lambda^{1/4}\,|\mathbf{c}(\lambda)|^{-1}\,.$$

Then setting,

$$P_\gamma g(s):=A(d(e,\gamma_s(t(s))))^{-1/2}A(s)^{1/2}\int_0^\infty \mathbf{c}(\lambda)e^{i\left\{\lambda d(e,\gamma_s(t(s))))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}g(\lambda)\lambda^{-1/4}|\mathbf{c}(\lambda)|^{-1}d\lambda\,,$$

that is,

$$2^{m_{\mathfrak{z}}/2}\,T_{\gamma,1}f(s)\,A(s)^{1/2}=P_\gamma g(s)\,,$$

we note that proving (3.7) is equivalent to proving

$$\left(\int_{r_1}^{r_2}|P_\gamma g(s)|^2\,ds\right)^{1/2}\lesssim\left(\int_0^\infty |g(\lambda)|^2\,d\lambda\right)^{1/2}. \tag{3.8}$$

Taking $\rho\in C_c^\infty[0,\infty)$ real-valued such that $\rho(\lambda)=1$ if $\lambda\le 1$ and $\rho(\lambda)=0$ if $\lambda\ge 2$, for $N>2$ and $s\in(r_1,r_2)$, we set

$$\begin{aligned} P_{\gamma,N}g(s) \;:=\;& A(d(e,\gamma_s(t(s))))^{-1/2}A(s)^{1/2} \\ & \times\int_0^\infty \mathbf{c}(\lambda)e^{i\left\{\lambda d(e,\gamma_s(t(s))))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\rho\left(\frac{\lambda}{N}\right)g(\lambda)\lambda^{-1/4}|\mathbf{c}(\lambda)|^{-1}d\lambda\,. \end{aligned}$$

Now for $G\in C_c^\infty(r_1,r_2)$ and $\lambda>0$, setting

$$\begin{aligned} P^*_{\gamma,N}G(\lambda) \;:=\;& \overline{\mathbf{c}(\lambda)}\,\rho\left(\frac{\lambda}{N}\right)\lambda^{-1/4}\,|\mathbf{c}(\lambda)|^{-1} \\ & \times\int_{r_1}^{r_2} A(d(e,\gamma_s(t(s))))^{-1/2}A(s)^{1/2}e^{-i\left\{\lambda d(e,\gamma_s(t(s))))+t(s)\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}G(s)ds\,, \end{aligned}$$

it is easy to see that

$$\int_{r_1}^{r_2}P_{\gamma,N}F(s)\,\overline{G(s)}\,ds=\int_0^\infty F(\lambda)\,\overline{P^*_{\gamma,N}G(\lambda)}\,d\lambda\,,$$

holds for all $G\in C_c^\infty(r_1,r_2)$ and $F\in L^2(0,\infty)$ having suitable decay at infinity. Thus it suffices to prove that

$$\left(\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda\right)^{1/2}\lesssim\left(\int_{r_1}^{r_2}|h(s)|^2\,ds\right)^{1/2}, \tag{3.9}$$

for all $h \in C_c^\infty(r_1,r_2)$, with the implicit constant independent of N, as then letting $N \to \infty$, we can obtain the estimate (3.8). Now by Fubini's theorem,

$$\begin{aligned}&\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda \\ = &\int_{r_1}^{r_2}\int_{r_1}^{r_2} I_N(s,s')A(d(e,\gamma_s(t(s))))^{-\frac{1}{2}}A(s)^{\frac{1}{2}}A(d(e,\gamma_{s'}(t(s'))))^{-\frac{1}{2}}A(s')^{\frac{1}{2}}h(s)\overline{h(s')}ds\,ds',\end{aligned}$$

where

$$I_N(s,s') = \int_0^\infty e^{i\left\{\lambda(d(e,\gamma_{s'}(t(s')))-d(e,\gamma_s(t(s))))+(t(s')-t(s))\left(\lambda^2+\frac{Q^2}{4}\right)\right\}}\lambda^{-1/2}\rho\left(\frac{\lambda}{N}\right)^2 d\lambda\,.$$

Then by Lemmata 2.3, 3.1 and the local growth asymptotics of the density function, we get that

$$\int_0^\infty \left|P^*_{\gamma,N}h(\lambda)\right|^2 d\lambda \lesssim \int_{r_1}^{r_2}\int_{r_1}^{r_2} \frac{1}{|s-s'|^{1/2}}\,|h(s)|\,|h(s')|\,ds\,ds',$$

with the implicit constant independent of N. Now as $h \in C_c^\infty(r_1,r_2)$, identifying h as an even C_c^∞ function supported in $(-r_2,-r_1)\sqcup(r_1,r_2)$, writing the last integral as a one dimensional Riesz potential and then applying (2.1), we get for some $c>0$,

$$\begin{aligned}\int_{r_1}^{r_2}\int_{r_1}^{r_2} \frac{1}{|s-s'|^{1/2}}\,|h(s)|\,|h(s')|\,ds\,ds' &= \int_0^\infty\int_0^\infty \frac{1}{|s-s'|^{1/2}}\,|h(s)|\,|h(s')|\,ds\,ds' \\ &= c\int_0^\infty I_{1/2}(|h|)(s)\,|h(s)|\,ds \\ &= c\int_{\mathbb{R}} |\xi|^{-\frac{1}{2}}\left|\tilde{h}(\xi)\right|^2 d\xi\,.\end{aligned}$$

Then by Pitt's inequality (as in our case $\beta = \frac{1}{4}, p=2$ and hence $\beta_1 = \beta + \frac{1}{2} - \frac{1}{p} = \frac{1}{4}$, both the conditions of Lemma 2.1 are satisfied),

$$\begin{aligned}\int_{\mathbb{R}} |\xi|^{-\frac{1}{2}}\left|\tilde{h}(\xi)\right|^2 d\xi &\lesssim \int_{\mathbb{R}} |h(x)|^2\,|x|^{\frac{1}{2}}\,dx \\ &= c\int_{r_1}^{r_2} |h(s)|^2\,s^{\frac{1}{2}}\,ds \\ &\leq c\,r_2^{1/2}\,\|h\|^2_{L^2(r_1,r_2)}\,.\end{aligned}$$

Thus we get (3.7). Similarly,

$$\left(\int_{r_1}^{r_2} |T_{\gamma,2}f(s)|^2 A(s)\,ds\right)^{1/2} \lesssim \left(\int_0^\infty \lambda^{1/2}\,|\widehat{f}(\lambda)|^2|\mathbf{c}(\lambda)|^{-2}d\lambda\right)^{1/2}. \tag{3.10}$$

Finally for $T_{\gamma,3}$, using the estimate on the error term (3.6) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|T_{\gamma,3}f(s)| &\lesssim A(s)^{-1/2} \int_0^\infty (1+\lambda)^{-1} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-1}\, d\lambda \\
&\leq A(s)^{-1/2} \left[\int_0^1 \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-1}\, d\lambda + \int_1^\infty \lambda^{-1} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-1}\, d\lambda\right] \\
&\leq A(s)^{-1/2} \|f\|_{\dot{H}^{1/4}(S)} \left[\left(\int_0^1 \frac{d\lambda}{\lambda^{1/2}}\right)^{1/2} + \left(\int_1^\infty \frac{d\lambda}{\lambda^{5/2}}\right)^{1/2}\right] \\
&\lesssim A(s)^{-1/2} \|f\|_{\dot{H}^{1/4}(S)} .
\end{aligned}$$

Thus

$$\left(\int_{r_1}^{r_2} |T_{\gamma,3}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim (r_2-r_1)^{1/2} \|f\|_{\dot{H}^{1/4}(S)} . \tag{3.11}$$

Then combining (3.7), (3.10) and (3.11), we get the result. □

We now focus on 'small annuli'. Let $0 < \delta_1 < \delta_2 \leq (2R_0)/(3C_3)$, where R_0 is as in Lemma 2.2. As in Theorem 1.1, we also assume that $\gamma_\cdot(\cdot)$ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$, for some fixed $T > 0$, such that $T^\alpha < (C_2\delta_1)/(2C_1)$. Then by Lemma 3.1, we have for all $s \in (\delta_1, \delta_2)$ and $t \in [-T, T]$,

$$d(e, \gamma_s(t)) < \frac{3C_3}{2} s < \frac{3C_3}{2}\delta_2 \leq R_0 .$$

Thus for $f \in \mathscr{S}^2(S)_o$, we can also decompose the linearized maximal function (3.4) in terms of the Bessel series expansion of φ_λ (by taking $M = 0$ in Lemma 2.2),

$$\begin{aligned}
T_\gamma f(s) &= \int_0^\infty \varphi_\lambda(d(e, \gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= c_0\left(\frac{d(e,\gamma_s(t(s)))^{n-1}}{A(d(e,\gamma_s(t(s))))}\right)^{1/2} \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda d(e,\gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&\quad + \int_0^\infty E(\lambda, d(e,\gamma_s(t(s))))\, e^{it(s)\left(\lambda^2+\frac{Q^2}{4}\right)}\, \widehat{f}(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\
&= T_{\gamma,4}f(s) + T_{\gamma,5}f(s) .
\end{aligned}$$

We obtain the following boundedness result for $T_{\gamma,4}$, which will be crucial for the proof of Theorem 1.4 in the next section:

Corollary 3.2. *Let $0 < \delta_1 < \delta_2 \leq (2R_0)/(3C_3)$, where R_0 is as in Lemma 2.2. Assume that $\gamma_\cdot(\cdot)$ satisfies $(\mathscr{H}_1)$ for $\alpha \in [\frac{1}{2}, 1]$ and $(\mathscr{H}_2)$ for $s, s' \in [0, \delta_2)$ and $t, t' \in [-T, T]$, for some fixed $T > 0$, such that $T^\alpha < (C_2\delta_1)/(2C_1)$. Then we have for all $f \in \mathscr{S}^2(S)_o$,*

$$\left(\int_{\delta_1}^{\delta_2} |T_{\gamma,4}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)} ,$$

where the implicit constant depends only on δ_1, δ_2 *and the dimension of* S.

Proof. By Theorem 1.1, it suffices to prove that

$$\left(\int_{\delta_1}^{\delta_2} |T_{\gamma,5}f(s)|^2 A(s)\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(S)} , \tag{3.12}$$

where the implicit constant depends only on δ_1, δ_2 and the dimension of S. For $\lambda \ge 0$, we recall the estimates on the error term in Lemma 2.2 (for $M = 0$),

$$|E(\lambda, d(e, \gamma_s(t(s))))| \lesssim \begin{cases} d(e,\gamma_s(t(s)))^2 & \text{if } \lambda d(e,\gamma_s(t(s))) \le 1 \\ d(e,\gamma_s(t(s)))^2 \left\{\lambda d(e,\gamma_s(t(s)))\right\}^{-\left(\frac{n+1}{2}\right)} & \text{if } \lambda d(e,\gamma_s(t(s))) > 1 . \end{cases}$$

Then using the above pointwise bounds, Lemma 3.1 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |T_{\gamma,5}f(s)| &\lesssim d(e,\gamma_s(t(s)))^2 \int_0^{1/d(e,\gamma_s(t(s)))} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\quad + \frac{1}{(d(e,\gamma_s(t(s))))^{\left(\frac{n-3}{2}\right)}} \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \lambda^{-\left(\frac{n+1}{2}\right)} \left|\widehat{f}(\lambda)\right| |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\lesssim \|f\|_{\dot{H}^{1/4}(S)} \left[s^2 I_1(s)^{1/2} + \frac{1}{s^{\left(\frac{n-3}{2}\right)}} I_2(s)^{1/2}\right] , \end{aligned}$$

where

$$I_1(s) = \int_0^{1/d(e,\gamma_s(t(s)))} \frac{|\mathbf{c}(\lambda)|^{-2}\, d\lambda}{\lambda^{1/2}} , \quad \text{and} \quad I_2(s) = \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \frac{|\mathbf{c}(\lambda)|^{-2}\, d\lambda}{\lambda^{n+\frac{3}{2}}} .$$

Now first using the asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ (2.4) for small frequency and Lemma 3.1, we get that

$$I_1(s) \lesssim \int_0^{1/d(e,\gamma_s(t(s)))} \lambda^{3/2}\, d\lambda \lesssim \frac{1}{s^{5/2}} .$$

Similarly the asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ (2.4) for large frequency and Lemma 3.1 yield

$$I_2(s) \lesssim \int_{1/d(e,\gamma_s(t(s)))}^{\infty} \frac{d\lambda}{\lambda^{5/2}} \lesssim s^{3/2} .$$

Hence, for $s \in (\delta_1, \delta_2)$,

$$|T_{\gamma,5}f(s)| \lesssim \|f\|_{\dot{H}^{1/4}(S)}\, s^{\left(\frac{9-2n}{4}\right)} ,$$

and thus using the local growth asymptotics of the density function, we obtain

$$\int_{\delta_1}^{\delta_2} |T_{\gamma,5}f(s)|^2 A(s)\, ds \lesssim \|f\|^2_{\dot{H}^{1/4}(S)} \int_{\delta_1}^{\delta_2} s^{7/2}\, ds \lesssim_{\delta_1,\delta_2} \|f\|^2_{\dot{H}^{1/4}(S)} .$$

This completes the proof of Corollary 3.2. □

4. Results on $\mathbb{R}^n$

Our aim in this section is to prove Theorems 1.4 and 1.6.

For $f \in \mathscr{S}(\mathbb{R}^n)_o$, with $n \geq 2$, we first consider the linearization of the maximal function appearing in Theorem 1.4:

$$\tilde{T}_\gamma f(s) := \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda \|\gamma_s(t(s))\|)\, e^{it(s)\lambda^2}\, \mathscr{F}f(\lambda)\, \lambda^{n-1}\, d\lambda\,, \tag{4.1}$$

where $t(\cdot) : (\delta_1, \delta_2) \to [-T, T]$ is a measurable function. We first consider the case when $\mathscr{F}f$ is supported near the origin.

Lemma 4.1. *Under the hypothesis of Theorem 1.4, we have for all $f \in \mathscr{S}(\mathbb{R}^n)_o$ with Supp $(\mathscr{F}f) \subset [0, \Lambda]$ for some $\Lambda > 0$,*

$$\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,,$$

where the implicit constant depends only on $\Lambda, \delta_1, \delta_2$ and n.

Proof. The Lemma follows from the boundedness of the modified Bessel functions and Cauchy-Schwarz inequality. Indeed, for $s \in (\delta_1, \delta_2)$,

$$\left|\tilde{T}_\gamma f(s)\right| \lesssim \int_0^\Lambda |\mathscr{F}f(\lambda)|\ \lambda^{n-1}\, d\lambda \leq \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} \left(\int_0^\Lambda \lambda^{n-\frac{3}{2}}\, d\lambda\right)^{1/2} \lesssim_\Lambda \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.$$

Hence, the result follows. □

We now turn our attention to the situation when $\mathscr{F}f$ is supported away from the origin. In this case, we need to do some local geometry. For each $n \geq 2$, there exists at least one n-dimensional Damek-Ricci space S (in particular, the degenerate case of Real hyperbolic spaces). We now consider the Riemannian exponential map at e, denoted by $\exp_e$ defined on T_eS, the tangent space of S at the identity e. As Damek-Ricci spaces are connected, simply-connected, complete Riemannian manifolds of non-positive sectional curvature, $\exp_e$ defines a global diffeomorphism from T_eS onto S. Now for a general Riemannian manifold, $\exp_e$ is always a local radial isometry. Hence, there exists $\varepsilon > 0$, such that $\exp_e$ maps the ball $\mathcal{B}(0, \varepsilon) \subset T_eS$ (with respect to the norm $\tilde{d}(\cdot)$ induced by the Riemannian metric) diffeomorphically onto the geodesic ball $\mathscr{B}(e, \varepsilon) \subset S$, with the property that for any $p \in \mathcal{B}(0, \varepsilon)$, we have

$$d(e, exp_e(p)) = \tilde{d}(p)\,. \tag{4.2}$$

Moreover, since any two finite dimensional inner product spaces with the same dimension, are isometrically isomorphic, there exists a linear isometry $\mathcal{I}$ from $\mathbb{R}^n$ equipped with the flat metric onto T_eS equipped with the Riemannian metric. Hence for any $x \in \mathbb{R}^n$, we have

$$\|x\| = \tilde{d}\left(\mathcal{I}(x)\right)\,. \tag{4.3}$$

We now restrict to the Euclidean ball $B(o, \varepsilon)$ and consider the composition,

$$\mathcal{E} := \exp_e \circ \mathcal{I}\,.$$

Thus combining (4.2) and (4.3), it follows that $\mathcal{E}$ maps the ball $B(o,\varepsilon) \subset \mathbb{R}^n$ diffeomorphically onto the geodesic ball $\mathscr{B}(e,\varepsilon) \subset S$, with the property that for any $x \in B(o,\varepsilon)$, we have,

$$d\left(e, \mathcal{E}(x)\right) = \|x\| \,. \tag{4.4}$$

As we are only interested in radial functions, identifying all the points on a geodesic sphere centered at e, contained in the geodesic ball $\mathscr{B}(e,\varepsilon)$, solely by their distance from e and using (4.4), we make the the abuse of notation,

$$\mathcal{E}(x) \sim d\left(e, \mathcal{E}(x)\right) = \|x\|, \quad \text{for } x \in B(o,\varepsilon). \tag{4.5}$$

Now under the hypothesis of Theorem 1.4, a computation similar to Lemma 3.1 using $(\mathscr{H}_3)$ and $(\mathscr{H}_4)$ yields that for all $s \in [0,\delta_2)$ and $t \in [-T,T]$,

$$\|\gamma_s(t)\| < \frac{3C_6}{2}\delta_2 \,.$$

Then furthermore assuming $\delta_2 \leq (2\varepsilon)/(3C_6)$, we get that $\|\gamma_s(t)\| < \varepsilon$, for all $s \in [0,\delta_2)$ and $t \in [-T,T]$. Hence it makes sense to consider the pushforwards (by the map $\mathcal{E}$) of the curves $\gamma_\cdot(\cdot)$ for $s \in [0,\delta_2)$ and $t \in [-T,T]$, which we write by (4.5) as,

$$\left(\mathcal{E}^*\gamma\right)_s(t) := \mathcal{E}(\gamma_s(t)) \,.$$

We now summarize some important properties of these pushforwarded curves:

Lemma 4.2. *Under the hypothesis of Theorem 1.4, with $\delta_2 \leq (2\varepsilon)/(3C_6)$, where ε is as in (4.4), for $s, s' \in [0,\delta_2)$ and $t, t' \in [-T,T]$, we have for $\alpha \in [\frac{1}{2}, 1]$,*

$$(i)\; \left|d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t)\right) - d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t')\right)\right| \leq C_4 |t-t'|^{\alpha} \,,$$
$$(ii)\; C_5|s-s'| \leq \left|d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t)\right) - d\left(e, \left(\mathcal{E}^*\gamma\right)_{s'}(t)\right)\right| \leq C_6|s-s'| \,.$$

Proof. Using (4.4) and $(\mathscr{H}_3)$, property (i) is proved as follows,

$$\begin{aligned} \left|d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t)\right) - d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t')\right)\right| &= \left|d\left(e, \mathcal{E}(\gamma_s(t))\right) - d\left(e, \mathcal{E}(\gamma_s(t'))\right)\right| \\ &= \left|\|\gamma_s(t)\| - \|\gamma_s(t')\|\right| \\ &\leq C_4 |t-t'|^{\alpha} \,. \end{aligned}$$

Property (ii) can be verified similarly. □

We next discuss some ideas on how to obtain a natural connection between spaces of radial Schwartz class functions on $\mathbb{R}^n$ and S with Spherical Fourier transforms supported away from the origin. These ideas have also been explored recently in [DR24] by the author and Ray. For the sake of completion, we briefly mention them here.

By the well-known properties of the Abel transform, we have the following commutative diagram, where

- $\mathscr{A}_{S,\mathbb{R}}$ is the Abel transform defined from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\mathscr{A}_{\mathbb{R}^n,\mathbb{R}}$ is the Abel transform defined from $\mathscr{S}(\mathbb{R}^n)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\wedge$ denotes the Spherical Fourier transform from $\mathscr{S}^2(S)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\mathscr{F}$ denotes the Euclidean Spherical Fourier transform from $\mathscr{S}(\mathbb{R}^n)_o$ to $\mathscr{S}(\mathbb{R})_e$.
- $\sim$ denotes the 1-dimensional Euclidean Fourier transform from $\mathscr{S}(\mathbb{R})_e$ to itself.

$$\begin{array}{ccccc} \mathscr{S}^2(S)_o & \xrightarrow{\mathscr{A}_{S,\mathbb{R}}} & \mathscr{S}(\mathbb{R})_e & \xleftarrow{\mathscr{A}_{\mathbb{R}^n,\mathbb{R}}} & \mathscr{S}(\mathbb{R}^n)_o \\ & \searrow \wedge & \downarrow \sim & \swarrow \mathscr{F} & \\ & & \mathscr{S}(\mathbb{R})_e & & \end{array}$$

Then, since all the maps above are topological isomorphisms, defining $\mathscr{A} := \mathscr{A}_{\mathbb{R}^n,\mathbb{R}}^{-1} \circ \mathscr{A}_{S,\mathbb{R}}$, one can reduce matters to the following simplified commutative diagram:

$$\begin{array}{ccc} \mathscr{S}^2(S)_o & \xrightarrow{\mathscr{A}} & \mathscr{S}(\mathbb{R}^n)_o \\ & \searrow \wedge & \downarrow \mathscr{F} \\ & & \mathscr{S}(\mathbb{R})_e \end{array}$$

Hence, for $g \in \mathscr{S}^2(S)_o$, we have

$$\widehat{g}(\lambda) = \mathscr{F}(\mathscr{A} g)(\lambda) \, . \tag{4.6}$$

Next let us denote by $\mathscr{S}(\mathbb{R})_e^\infty$ the collection of even Schwartz class functions on $\mathbb{R}$ which are supported outside an interval containing 0. By the derivative estimates of $|\mathbf{c}(\lambda)|^{-2}$ (see (2.5)), we note that for $\kappa \in \mathscr{S}(\mathbb{R})_e^\infty$ the multiplier defined by,

$$\mathfrak{m}(\kappa)(\lambda) := \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^{n-1}} \kappa(\lambda) \, ,$$

is a bijection from $\mathscr{S}(\mathbb{R})_e^\infty$ onto itself (its inverse being the multiplier corresponding to $\frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}}$). We also define, $\mathscr{S}(\mathbb{R}^n)_o^\infty$ to be the collection of all radial Schwartz class functions h on $\mathbb{R}^n$ whose Euclidean Spherical Fourier transform $\mathscr{F}h$ belongs to $\mathscr{S}(\mathbb{R})_e^\infty$, that is, $\mathscr{S}(\mathbb{R}^n)_o^\infty := \mathscr{F}^{-1}\left(\mathscr{S}(\mathbb{R})_e^\infty\right)$. Now as $\mathscr{F} : \mathscr{S}(\mathbb{R}^n)_o \to \mathscr{S}(\mathbb{R})_e$ is a bijection, we get an induced map $\mathcal{M}$ obtained by conjugating $\mathfrak{m}$ with $\mathscr{F}$, which is a bijection from $\mathscr{S}(\mathbb{R}^n)_o^\infty$ onto itself:

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^n)_o^\infty & \xrightarrow{\mathcal{M}} & \mathscr{S}(\mathbb{R}^n)_o^\infty \\ \downarrow \mathscr{F} & & \downarrow \mathscr{F} \\ \mathscr{S}(\mathbb{R})_e^\infty & \xrightarrow{\mathfrak{m}} & \mathscr{S}(\mathbb{R})_e^\infty \end{array}$$

Thus for $h \in \mathscr{S}(\mathbb{R}^n)_o^\infty$, we get that

$$\mathscr{F}(\mathcal{M}h)(\lambda) = \frac{|\mathbf{c}(\lambda)|^{-2}}{\lambda^{n-1}} \mathscr{F}h(\lambda) \, ,$$

and similarly,

$$\mathscr{F}(\mathcal{M}^{-1}h)(\lambda) = \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}}\mathscr{F}h(\lambda)\,. \tag{4.7}$$

Lemma 4.3. *Under the hypothesis of Theorem 1.4, with* $\delta_2 \le \frac{2}{3C_6}\min\{\varepsilon, R_0\}$, *where* ε *is as in (4.4) and* R_0 *is as in Lemma 2.2, we have for all* $f \in \mathscr{S}(\mathbb{R}^n)_o$ *with* $Supp\,(\mathscr{F}(f)) \subset (1,\infty)$*,*

$$\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,,$$

where the implicit constant depends only on δ_1, δ_2 *and* n*.*

Proof. Let $\mathscr{S}^2(S)_o^\infty$ denote the collection of all radial L^2-Schwartz class functions on S whose Spherical Fourier transform belongs to $\mathscr{S}(\mathbb{R})_e^\infty$, that is, $\mathscr{S}^2(S)_o^\infty := \wedge^{-1}\left(\mathscr{S}(\mathbb{R})_e^\infty\right)$. Now as $\wedge : \mathscr{S}^2(S)_o \to \mathscr{S}(\mathbb{R})_e$ is a bijection, its restriction also defines a bijection from $\mathscr{S}^2(S)_o^\infty$ onto $\mathscr{S}(\mathbb{R})_e^\infty$. Then combining (4.6), (4.7) and the fact that $Supp\,(\mathscr{F}(f)) \subset (1,\infty)$, we define $g := \left(\mathscr{A}^{-1}\circ\mathcal{M}^{-1}\right)f$. The interaction of g with the Fourier transform is best understood via the following commutative diagram, where each arrow is a bijection:

$$\begin{array}{ccccc}
\mathscr{S}(\mathbb{R}^n)_o^\infty & \xrightarrow{\mathcal{M}^{-1}} & \mathscr{S}(\mathbb{R}^n)_o^\infty & \xrightarrow{\mathscr{A}^{-1}} & \mathscr{S}^2(S)_o^\infty \\
\downarrow \mathscr{F} & & \downarrow \mathscr{F} & \swarrow \wedge & \\
\mathscr{S}(\mathbb{R})_e^\infty & \xrightarrow{\mathfrak{m}^{-1}} & \mathscr{S}(\mathbb{R})_e^\infty & &
\end{array}$$

Thus $g \in \mathscr{S}^2(S)_o^\infty$ with

$$\widehat{g}(\lambda) = \mathscr{F}\left(\mathcal{M}^{-1}f\right)(\lambda) = \frac{\lambda^{n-1}}{|\mathbf{c}(\lambda)|^{-2}}\mathscr{F}f(\lambda)\,. \tag{4.8}$$

So in particular, $Supp(\widehat{g}) \subset (1,\infty)$.

We also note that using the hypothesis, a computation similar to Lemma 3.1 implies that $\|\gamma_s(t)\| < \min\{\varepsilon, R_0\}$, for all $s \in [0,\delta_2)$ and $t \in [-T,T]$. Hence by (4.4), we see that

$$d\left(e, \left(\mathcal{E}^*\gamma\right)_s(t)\right) = \|\gamma_s(t)\| < \min\{\varepsilon, R_0\}\,. \tag{4.9}$$

Then for $s \in (\delta_1,\delta_2)$, using Lemmata 4.2 and 3.1, the local growth asymptotics of the density function of S (2.2), the relationship between $\widehat{g}$ and $\mathscr{F}f$ (4.8) and the isometric relation (4.9), we obtain

$$\begin{aligned}
|T_{\mathcal{E}^*\gamma,4}\, g(s)| &\asymp \left|\int_1^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\, d\left(e,\left(\mathcal{E}^*\gamma\right)_s(t(s))\right)\right) e^{it(s)\lambda^2}\,\widehat{g}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\, d\lambda\right| \\
&= \left|\int_1^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\,\|\gamma_s(t(s))\|\right) e^{it(s)\lambda^2}\,\mathscr{F}f(\lambda)\,\lambda^{n-1}\, d\lambda\right| \\
&= \left|\tilde{T}_\gamma f(s)\right|\,.
\end{aligned}$$

Thus again by the local growth asymptotics of the density function of S, it follows that

$$\left(\int_{\delta_1}^{\delta_2} |T_{\mathcal{E}^*\gamma,4}\, g(s)|^2 A(s)\, ds\right)^{1/2} \asymp \left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} . \tag{4.10}$$

Using (4.8) and the large frequency asymptotics of $|\mathbf{c}(\lambda)|^{-2}$ (2.4), we also compare the Homogeneous Sobolev norms of g and f:

$$\begin{aligned} \|g\|_{\dot{H}^{1/4}(S)} &= \left(\int_1^\infty \lambda^{1/2}\, |\widehat{g}(\lambda)|^2\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\right)^{1/2} \\ &= \left(\int_1^\infty \lambda^{1/2}\, |\mathscr{F} f(\lambda)|^2\, \frac{\lambda^{2(n-1)}}{|\mathbf{c}(\lambda)|^{-4}}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\right)^{1/2} \\ &\asymp \left(\int_1^\infty \lambda^{1/2}\, |\mathscr{F} f(\lambda)|^2\, \lambda^{n-1}\, d\lambda\right)^{1/2} \\ &= \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)} . \end{aligned} \tag{4.11}$$

Then by Lemma 4.2, Corollary 3.2, (4.10) and (4.11), the result follows. □

We now complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $f \in \mathscr{S}(\mathbb{R}^n)_o$. Then $\mathscr{F} f \in \mathscr{S}(\mathbb{R})_e$. Now let us choose an auxiliary non-negative even function $\psi \in C_c^\infty(\mathbb{R})$ such that $Supp(\psi) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}\xi) = 1 \, , \; \xi \neq 0 \, .$$

We set

$$\psi_1(\xi) := \sum_{k=-\infty}^{0} \psi(2^{-k}\xi) \, , \;\text{ and } \psi_2(\xi) := \sum_{k=1}^{\infty} \psi(2^{-k}\xi) \, .$$

Then both ψ_1 and ψ_2 are even non-negative C^∞ functions with $Supp(\psi_1) \subset (-2,2)$, $Supp(\psi_2) \subset \mathbb{R} \setminus (-1,1)$ and $\psi_1 + \psi_2 \equiv 1$. Then we consider,

$$\mathscr{F} f = (\mathscr{F} f)\, \psi_1 + (\mathscr{F} f)\, \psi_2 \, .$$

As $(\mathscr{F} f)\, \psi_j$ for $j = 1, 2$, are even Schwartz class functions on $\mathbb{R}$, by the Schwartz isomorphism theorem, there exist $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)_o$ such that $\mathscr{F} f_j = (\mathscr{F} f)\, \psi_j$ for $j = 1, 2$. Then by Lemmata 4.1 and 4.3, we get the desired estimate on the linearized maximal function,

$$
\begin{aligned}
&\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \\
\leq\ &\left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f_1(s)\right|^2 s^{n-1}\, ds\right)^{1/2} + \left(\int_{\delta_1}^{\delta_2} \left|\tilde{T}_\gamma f_2(s)\right|^2 s^{n-1}\, ds\right)^{1/2} \\
\lesssim\ &\left(\int_{0}^{2} \lambda^{1/2} \left|\mathscr{F} f_1(\lambda)\right|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} + \left(\int_{1}^{\infty} \lambda^{1/2} \left|\mathscr{F} f_2(\lambda)\right|^2 \lambda^{n-1}\, d\lambda\right)^{1/2} \\
\leq\ & 2\, \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\, .
\end{aligned}
$$

□

Proof of Theorem 1.6. By standard arguments, in order to prove the result on almost everywhere pointwise convergence, it suffices to prove that given any $0 < r_1 < r_2 < \infty$, there exists some fixed $T > 0$, such that one has for all $f \in \mathscr{S}(\mathbb{R}^n)_o$, the maximal estimate,

$$
\left(\int_{r_1}^{r_2} \left(\sup_{t\in[-T,T]} \left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \lesssim \|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\, , \tag{4.12}
$$

where the implicit constant can possibly depend only on r_1, r_2 and n. We first note that by compactness of $[0, r_2]$, there exists $0 < T < (r_1/2C_7)^2$, such that

$$
\|\gamma_s(t)\| = s + C_7 t^{1/2}\, ,
$$

for all $s \in [0, r_2)$ and $t \in [-T, T]$. Then for all $s \in [0, r_2)$ and $t, t' \in [-T, T]$,

$$
|\|\gamma_s(t)\| - \|\gamma_s(t')\|| = C_7 |t^{1/2} - t'^{1/2}| \leq C_7 |t - t'|^{1/2}\, .
$$

Also for all $s, s' \in [0, r_2)$ and $t \in [-T, T]$,

$$
|\|\gamma_s(t)\| - \|\gamma_{s'}(t)\|| = |s - s'|\, .
$$

Therefore, $\gamma_\cdot(\cdot)$ satisfies $(\mathscr{H}_3)$ for $\alpha = 1/2$ and $(\mathscr{H}_4)$ for $s, s' \in [0, r_2)$ and $t, t' \in [-T, T]$, for some fixed $0 < T < (r_1/2C_7)^2$.

Let $\delta \in (0, r_2)$ be sufficiently small and set $\eta = r_2/\delta$ and $\delta' = r_1/\eta$. For $f \in \mathscr{S}(\mathbb{R}^n)_o$, we consider

$$
f_\eta(x) := f(\eta x)\, .
$$

Now the solutions of the Schrödinger equation with initial data f and f_η are related as follows: for $s \in (r_1, r_2)$ and $t \in [-T, T]$,

$$\begin{aligned}
\tilde{S}_t f(\|\gamma_s(t)\|) &= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda\|\gamma_s(t)\|)\, e^{it\lambda^2}\, \widehat{f}(\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \eta^n \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}(\lambda\|\gamma_s(t)\|)\, e^{it\lambda^2}\, \widehat{f}_\eta(\eta\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\frac{\|\gamma_s(t)\|}{\eta}\right) e^{i\left(\frac{t}{\eta^2}\right)\lambda^2}\, \widehat{f}_\eta(\lambda)\, \lambda^{n-1}\, d\lambda\,.
\end{aligned}$$

Now for $s \in (r_1, r_2)$ and $t \in [-T, T]$, we have

$$\frac{\|\gamma_s(t)\|}{\eta} = \frac{s}{\eta} + C_7\frac{t^{1/2}}{\eta} = \frac{s}{\eta} + C_7\left(\frac{t}{\eta^2}\right)^{1/2} = \|\gamma_{s/\eta}(t/\eta^2)\|\,.$$

Therefore, for $s \in (r_1, r_2)$ and $t \in [-T, T]$,

$$\begin{aligned}
\tilde{S}_t f(\|\gamma_s(t)\|) &= \int_0^\infty \mathscr{J}_{\frac{n-2}{2}}\left(\lambda\|\gamma_{s/\eta}(t/\eta^2)\|\right) e^{i\left(\frac{t}{\eta^2}\right)\lambda^2}\, \widehat{f}_\eta(\lambda)\, \lambda^{n-1}\, d\lambda \\
&= \tilde{S}_{t/\eta^2} f_\eta(\|\gamma_{s/\eta}(t/\eta^2)\|)\,.
\end{aligned}$$

Then as $T < (r_1/2C_7)^2$, we have

$$\left(\frac{T}{\eta^2}\right)^{1/2} < \frac{\delta'}{2C_7}\,,$$

and hence by Theorem 1.4,

$$\begin{aligned}
&\left(\int_{r_1}^{r_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \left(\int_{r_1}^{r_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_{t/\eta^2} f_\eta(\|\gamma_{s/\eta}(t/\eta^2)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \left(\int_{r_1}^{r_2}\left(\sup_{t\in\left[-\frac{T}{\eta^2},\frac{T}{\eta^2}\right]}\left|\tilde{S}_t f_\eta(\|\gamma_{s/\eta}(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
=\;& \eta^{n/2}\left(\int_{\delta'}^{\delta}\left(\sup_{t\in\left[-\frac{T}{\eta^2},\frac{T}{\eta^2}\right]}\left|\tilde{S}_t f_\eta(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\, ds\right)^{1/2} \\
\lesssim\;& \eta^{n/2}\,\|f_\eta\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.
\end{aligned}$$

We next note that

$$\begin{aligned}\|f_\eta\|_{\dot{H}^{1/4}(\mathbb{R}^n)} &= \left(\int_0^\infty \lambda^{1/2}\left|\widehat{f_\eta}(\lambda)\right|^2 \lambda^{n-1}\,d\lambda\right)^{1/2}\\ &= \eta^{\left(\frac{1}{4}-\frac{n}{2}\right)}\left(\int_0^\infty \lambda^{1/2}\left|\widehat{f}(\lambda)\right|^2 \lambda^{n-1}\,d\lambda\right)^{1/2}\\ &= \eta^{\left(\frac{1}{4}-\frac{n}{2}\right)}\|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,,\end{aligned}$$

which we plug in the above to obtain

$$\left(\int_{r_1}^{r_2}\left(\sup_{t\in[-T,T]}\left|\tilde{S}_t f(\|\gamma_s(t)\|)\right|\right)^2 s^{n-1}\,ds\right)^{1/2} \lesssim \eta^{\frac{1}{4}}\|f\|_{\dot{H}^{1/4}(\mathbb{R}^n)}\,.$$

Thus we obtain (4.12), which completes the proof. □

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References

An91. Anker, J-P. *The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan.* J. Funct. Anal. Volume 96, Issue 2, 1991, 331-349,

ADY96. Anker, J-P., Damek, E. and Yacoub, C. *Spherical analysis on harmonic AN groups.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, 23 (1996), no. 4, 643-679.

AP14. Anker, J-P. and Pierfelice, V. *Wave and Klein-Gordon equations on hyperbolic spaces.* Anal. PDE, 2014, 7(4), pp.953-995.

APV15. Anker, J-P., Pierfelice, V. and Vallarino, M. *The wave equation on Damek-Ricci spaces*, Ann. Mat. Pura Appl. (4) 194 (2015), no. 3, 731-758.

As95. Astengo, F. *A class of L^p convolutors on harmonic extensions of H-type groups.* J. Lie Theory 5 (1995), no. 2, 147-164.

BX98. Bloom, W.R. and Xu, Z. *Fourier transforms of Schwartz functions on Chébli-Trimèche Hypergroups.* Monatsh. Math. 125, 89-109 (1998).

Bo16. Bourgain, J. *A note on the Schrödinger maximal function.* J. Anal. Math. 130 (2016), 393-396.

CM23. Cao, Z. and Miao, C. *Sharp pointwise convergence on the Schrödinger operator along one class of curves.* Bull. Sci. Math. 184 (2023) 103254.

Ca80. Carleson, L. *Some analytic problems related to statistical mechanics, Euclidean harmonic analysis.* Lecture Notes in Math. 779, Springer, Berlin, 1980, 5-45.

CLV12. Cho, C., Lee, S. and Vargas, A. *Problems on pointwise convergence of solutions to the Schrödinger equation.* J. Fourier Anal. Appl. 18 (2012), 972-994.

Co83. Cowling, M. *Pointwise behavior of solutions to Schrödinger equations, harmonic analysis.* Lecture Notes in Math. 992. Springer, Berlin, 1983, 83-90.

CDKR98. Cowling, M., Dooley, A., Korányi, A. and Ricci, F. *An approach to symmetric spaces of rank one via groups of Heisenberg type.* J. Geom. Anal. 8 (1998), no. 2, pp. 199–237.

DK82. Dahlberg, B.E.J. and Kenig, C.E. *A note on the almost everywhere behavior of solutions of the Schrödinger equation.* Lecture Notes in Math. 908. Springer-Verlag, Berlin, 1982, 205-208.

De24. Dewan, U. *Regularity and pointwise convergence of solutions of the Schrödinger operator with radial initial data on Damek-Ricci spaces.* arXiv:2407.13736 .

DR24. Dewan, U. and Ray, S.K. *Mapping properties of the local Schrödinger maximal function with radial initial data on Damek-Ricci spaces.* arXiv:2411.04084 .

DN17. Ding, Y. and Niu, Y. *Weighted maximal estimates along curve associated with dispersive equations.* Anal. Appl. vol. 15, no. 2, pp. 225-240 (2017).

DGL17. Du, X., Guth, L. and Li, X. *A sharp Schrödinger maximal estimate in* $\mathbb{R}^2$. Ann. Math. 2017, 186, 607-640.

DZ19. Du, X. and Zhang, R. *Sharp* L^2 *estimates of the Schrödinger maximal function in higher dimensions.* Ann. Math. 2019, 189, 837-861.

Gr09. Grafakos, L. *Modern Fourier Analysis.* Graduate Texts in Mathematics, 2nd ed, Springer Science+Business Media, LLC 2009.

Mi24. Minguillón, J. *A note on almost everywhere convergence along tangential curves to the Schrödinger equation initial datum.* J. Geom. Anal. (2024) 34:333.

RS09. Ray, S.K. and Sarkar, R.P. *Fourier and Radon transform on harmonic NA groups.* Trans. Amer. Math. Soc. 361 (2009), no. 8, 4269–4297.

SS89. Sjögren, P. and Sjölin, P. *Convergence properties for the time dependent Schrödinger equation.* Ann. Acad. Sci. Fenn. Math. 14 (1989), pp. 13-25.

Sj87. Sjölin, P. *Regularity of solutions to the Schrödinger equation.* Duke Math J. 55(1987), 699-715.

Sj97. Sjölin, P. L^p *maximal estimates for solutions to the Schrödinger equation.* Math. Scand., vol. 81, no. 1 (1997), pp. 35-68.

ST78. Stanton, R. J. and Tomas, P. A. *Expansions for spherical functions on noncompact symmetric spaces.* Acta Math. 140 (1978), no. 3-4, pp. 251-276.

St56. Stein, E.M. *Interpolation of Linear Operators.* Trans. Amer. Math. Soc. vol. 83, no. 2 (1956), 482–492.

SW90. Stein, E.M. and Weiss, G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, Princeton, New Jersey, Sixth printing, 1990.

Ve88. Vega, L. *Schrödinger equations: pointwise convergence to the initial data.* Proc. Amer. Math. Soc. 102(1988), 874-878.

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