MONOCHROMATIC CONFIGURATIONS INDUCED FROM THE SET OF SUMS OF TWO SQUARES

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Abstract. The set of sums of two squares plays an important role in the elementary number theory. In [18], Di Nasso investigated several infinite monochromatic patterns in integers considering operations induced from affine maps and asked whether one can find different sets of infinite monochromatic configurations in natural numbers with the structure induced from the set of sums of two squares. This article investigates Nasso's question and along the way proves several classical Ramsey-type theorems in this new setting.

1. Introduction

Finding monochromatic patterns for a colouring of certain algebraic structures has been a long-standing thrust in Ramsey's theory. Among all, there are a few ground-breaking results, such as the *van der Waerden theorem*, which deals with the existence of monochromatic arithmetic progression for any finite partition of natural numbers. In other words, the family of subsets of natural numbers given by arithmetic progression is *partition regular*. One notices that the van der Waerden theorem is a finitary Ramsey-type result that only uses the additive structure of $\mathbb{N}$. The infinitary version is widely known as the *Hindman's theorem*, and it states that for any finite colouring of $\mathbb{N}$, an injective sequence exists such that all finite sums are monochromatic. There are many ways to pass from the additive structure of $\mathbb{N}$ to a multiplicative structure, such as via the maps $n \mapsto p^n$ for any prime p. This change in structure suggests the multiplicative van der Waerden theorem, which states that for any finite partition of $\mathbb{N}$, a monochromatic geometric progression of arbitrary length exists.

The patterns that mix natural numbers' additive and multiplicative structures are much more difficult to proceed. The first instance in this direction is due to Bergelson [6]and Hindman [13]. They independently prove that the configuration $\{a, b, c, d\}$ with a+b = c.d is monochromatic. Later in [16], J. Moreira proved that the pattern $\{a, a + b, a.b\}$, and consequently, with J.M. Barrett and M. Lupini, Moreira showed the pattern {a, a+b, a+b+a.b} is monochromatic in [1]. In the area of mixed pattern, the monochromaticity of the configuration $\{a, b, a + b, a.b\}$ is still an open problem.

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Recently, D. Nasso developed a new idea to construct a new operation on $\mathbb{Z}$ involving both the additive and multiplicative structure in [18]. He considers a class of associative and commutative operations on the integers originating from the affine transformations. For specific cases, the multiplication for these operations gives the symmetric polynomials (these are combinations of old additive and multiplicative structures.) Then, using the algebra of Stone-Čech compactification of this new semigroup, Nasso produced a lot of new monochromatic patterns in integers. Furthermore, polynomial extension of some Ramsey theorems have been considered in [10] along the line of Nasso.

Thus, it is interesting to investigate the Ramsey-type properties of well-structures. Let $\mathbb{N}_0$ be the set of natural numbers along with zero. Consider the set $S(m,n) := \{x_1^m + \cdots + x_n^m : x_1, \cdots, x_m \in \mathbb{N}_0\}$. It is well known from the elementary number theory that $S(2,4)$ is $\mathbb{N}_0$, $S(2,3)$ is the set of the natural those are not of the form of $4^a(8b+7)$ for some $a, b \in \mathbb{N}_0$. This set is not multiplicative, whereas $S(2,4)$ is multiplicative but not interesting to us as it is the whole set. Therefore, this article focuses on $\Sigma = S(2,2)$, which is nothing but the set of all natural numbers such that the prime divisors of form $4k+3$ (if any) having even exponents. Monochromatic patterns induced from the set of all sum of two squares, Σ, were asked by D. Nasso in [18] and we answered this question in this article.

In this article, we investigate a new structure on natural numbers induced from the set of sums of two squares, Σ and with this new algebraic structure along the Stone-Čech compactification of it, we prove several classical results such as Hindman theorem, Deuber theorem, Brauer's theorem, Miliken-Taylor theorem, geo-arithmetic progression, and polynomial van der Waerden theorem.

2. A new structure and Stone-Čech compactification

The set $\Sigma = \{a^2 + b^2 : a, b \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$ of sums of two squares, is a commutative semigroup with respect to the usual multiplication of natural numbers since $(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (ac + bd)^2 \in \Sigma$. Now, one consider the ordering on Σ induced from $\mathbb{N}_0$ and write

$$\Sigma = \{s_0 < s_1 < s_2 < s_3 < s_4 < s_5 < s_6 < s_7 < \cdots\},$$

where $s_0 = 0$,$s_1 = 1$, $s_2 = 2$, $s_3 = 4$, $s_4 = 5$, $s_5 = 8$, $s_6 = 9$, $s_7 = 10$, $s_8 = 13$, $s_9 = 16$, $s_{10} = 17$, $s_{11} = 18$, $s_{12} = 20$, $s_{13} = 25$, $s_{14} = 26$, $s_{15} = 29$, $s_{16} = 32$ and so on.

Denote by $\Sigma/s = \{y \in S : y < s\}$, the set of all predecessors of s in Σ. Consider the function $g : S \to \mathbb{N}_0$ given by

$$g(s) = \text{card}(\Sigma/s).$$

One can easily verify that g is a bijective function with the inverse $f(n)$ is the n^{th} term s_n in Σ.

Definition 2.1. $m *_f n = g(f(m).f(n)) = \text{card}(\Sigma/s_m \cdot s_n)$.

Example 2.2. For example, $m = 2$ and $n = 5$, then $s_m = 2$ and $s_n = 8$. Now $s_m s_n = 2 \cdot 8 = 16 = s_9$. So, for $m = 2, n = 5$,we have $\text{card}(S/s_m \cdot s_n) = \text{card}(\{y \in S : y < s_m s_n\}) = \text{card}(\{y \in S : y < s_9\}) = 9$. Therefore, $2 *_f 5 = 9$.

Proposition 2.3. *$(\mathbb{N}_0, *_f)$ is a commutative semigroup and it contains identity element and $f : (\mathbb{N}_0, *_f) \to (\Sigma, .)$ is a semigroup homomorphism.*

Proof. The associativity and commutativity properties of multiplication on the set of sums of two squares Σ are inherited by the operation $*_f$, via the isomorphism f. The identity element in $(\mathbb{N}_0, *_f)$ is 1 as $1 *_f n = \text{card}(\Sigma/s_1.s_n) = \text{card}(\Sigma/s_n) = n$. Similarly, we can prove that $n *_f 1 = n$.

The second part evident from the definition of $*_f$. Hence the result follows. □

We write $x^{(0)} = 1$ and $x^{(n)} := \underbrace{x *_f x *_f \cdots *_f x}_{n \text{ times}}$ which in turn gives $\text{card}(\Sigma/s_x^n)$.

2.0.1. *Algebra in the Stone-Čech compactification.* We briefly recall the algebraic structure of the Stone-Čech compactification βS for a discrete semigroup $(S, \cdot)$. The elements of βS consist of the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$ let us set, $\overline{A} = \{p \in \beta S \mid A \in p\}$. Then the set $\{\overline{A} \mid A \subseteq S\}$ is a basis for a topology on βS. The operation $\cdot$ on S can be extended to the Stone-Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S \mid x^{-1} \cdot A \in q\} \in p$, where $x^{-1} \cdot A = \{y \in S \mid x \cdot y \in A\}$. This is a fundamental result due to Ellis, which states that in every compact right topological semigroup, there exists an idempotent element. Thus, idempotent ultrafilter p in $(\beta S, \cdot)$ such that $p \cdot p = p$ exists.

For any sequence $\langle x_n \rangle_{n=1}^{\infty}$ of elements in a semigroup $(S, \cdot)$, denote by $FP(\langle x_n \rangle_{n=1}^{\infty})$ the corresponding set of finite products:

$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_{n_1}.x_{n_2} \cdots .x_{n_k} : n_1 < n_2 < \cdots < n_k\}.$$

The relevance of idempotent ultrafilters in Ramsey Theory can be understood by *Galvin's theorem* which says that given a semigroup $(S, \cdot)$ and an idempotent ultrafilter $p = p \cdot p$ in the Stone-Čech compactification $(\beta S, \cdot)$. Then for every $A \in p$ there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that the set of finite products $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. To study the details about Stone-Čech compactification of any set and it's application on Ramsey theory, one can see [14], [21].

As we understood that $(\mathbb{N}_0, *_f)$ is a discrete commutative semigroup, then its Stone-Čech compactification $(\beta\mathbb{N}_0, *_f)$ responsible for a lot of Ramsey type results in $\mathbb{N}_0$ induced by this new operation $*_f$.

3. Study on Monochromatic Configurations

In this section, we write down new monochromatic configurations in the semigroup $(\mathbb{N}, *_f)$ that encodes information about the sum of two squares. We start with the following definition:

For any infinite sequence of natural numbers $\langle x_n \rangle_{n=1}^{\infty}$, the corresponding set of finite sums is denoted and defined by

$$FS(\langle x_n \rangle_{n=1}^{\infty}) = \{x_{n_1} + x_{n_2} + \cdots + x_{n_k} : n_1 < n_2 < \cdots < n_s\}.$$

A famous result in arithmetic Ramsey Theory shows the existence of infinite monochromatic patterns of finite sums, which was proved by Hindman stated as follows:

Theorem 3.1 (Hindman's finite sum theorem, [12]). *Let $r \geq 1$. For every r-coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist a color C_i and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C_i$.*

The above result is true if we consider finite products instead of finite sums. The same result is true even if we take any semigroup [14, Theorem 5.8]. In analogy with the set of finite sums, we can define the set of finite $*_f$-operations for the semigroup $(\mathbb{N}_0, *_f)$.

$$\begin{aligned}\mathfrak{FP}_f(\langle x_n \rangle_{n=1}^{\infty}) &= \{x_{n_1} *_f x_{n_2} *_f \cdots *_f x_{n_k} : n_1 < n_2 < \cdots < n_k\} \\ &= \{\text{card}(\Sigma / s_{x_{n_1}} s_{x_{n_2}} \cdots s_{x_{n_k}}) : n_1 < n_2 < \cdots < n_k\}.\end{aligned}$$

Theorem 3.2. *For any $r \geq 1$ and any finite r-coloring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a color C_i and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\mathfrak{FP}_f(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C_i$, that is,*

$$\{\text{card}(\Sigma / s_{x_{n_1}} s_{x_{n_2}} \cdots s_{x_{n_k}}) : n_1 < n_2 < \cdots < n_k\} \subseteq C_i,$$

Proof. The proof is similar to the ultrafilter proof of Hindman Theorem [14, Theorem 5.8] where the associative operation is $*_f$. □

A fundamental result in arithmetic Ramsey Theory is the Van der Waerden Theorem (1927). Next year, this theorem was strengthened by Brauer, who proved that one can also have the common difference as well as the elements of the progression in the same color.

Theorem 3.3 (van der Waerden Theorem, [20]). *For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $L \in \mathbb{N}$ there exists a monochromatic arithmetic progression of length L; that is, there exist a color C_i and elements $a, b \in \mathbb{N}$ such that $a, a+b, a+2b, \cdots, a+Lb \in C_i$.*

Theorem 3.4 (Brauer's Theorem, [9]). *For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $k \in \mathbb{N}$ there exists a monochromatic arithmetic progression of length k; that is, there exist a color C_i and elements $a, b \in \mathbb{N}$ such that $\{a, b, a+b, a+2b, \cdots, a+kb\} \subseteq C_i$.*

We have the analogues version of Brauer's Theorem in our context which is the following:

Theorem 3.5. *For every finite coloring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$ and for every $k \in \mathbb{N}$, there exist a color C_i and elements $x, y \in \mathbb{N}$ such that*

$$\{x, z, \mathrm{card}(\Sigma/s_x^j s_z) : j = 1, 2, \cdots, k\} \subseteq C_i$$

where Σ is the set of sums of two squares.

After few decades later, Deuber gave a result about generalized partition regularity of homogeneous systems of linear Diophantine equations: in particular, he showed the partition regularity of the so called (m, p, c)-sets.

Theorem 3.6 (Deuber Theorem, [11])**.** *For every $m, p, c \in \mathbb{N}$ and for every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exists a monochromatic (m, p, c)-set; that is, there exist a color C_i and elements $a_0, a_1, \cdots, a_m \in C_i$ such that $ca_j + \sum_{s=0}^{j-1} n_s a_s \in C_i$ for every $j \in \{1, 2, \cdots, m\}$ and for all $n_0, n_1, \cdots, n_{j-1} \in \{-p, \cdots, p\}$.*

We have the analogue of Deuber Theorem in our context which is following:

Theorem 3.7. *Let $m, p, r \in \mathbb{N}$. For every r-coloring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$, there exists a color C_i and elements $x_0, x_1, \cdots, x_m \in C_i$ such that*

$$\{\mathrm{card}(\Sigma/s_{x_0}^{n_0} s_{x_1}^{n_1} \cdots s_{x_{j-1}}^{n_{j-1}} s_{x_j}) : n_0, n_1, \cdots, n_{j-1} \in \{0, 1, \cdots, p\} \text{ and } j = 1, 2, \cdots, m\} \subseteq C_i$$

where Σ is the set of sums of two squares.

Proof. To prove this theorem, we will use a generalisation of Deuber's Theorem for commutative semirings, which was recently proved by V. Bergelson, J.H. Johnson, and J. Moreira in [8, Corollary 3.7]. It states as follows:

Let $(S, *)$ be a commutative semigroup, and for $j = 1, 2, \cdots, m$, let $\mathfrak{F}_j$ be a finite set of endomorphisms $\psi : S^j \to S$. Then for every r-coloring $S = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a color C_i and elements $x_0, x_1, \cdots, x_m$ different from identity, such that $x_0 \in C_i$ and $\psi(x_0, x_1, \cdots, x_{j-1}) * x_j \in C_i$ for every $j = 1, 2, \cdots, m$ and for every $\psi \in \mathfrak{F}_j$.

The statement of the theorem is a generalisation of Theorem 3.6 for $c = 1$ case. We will apply Bergelson-Johnson-Moreira's result with $(S, *) = (\mathbb{N}_0, *_f)$. For every j-tuple $\bar{n} = (n_0, n_1, \cdots, n_{j-1}) \in (\mathbb{N}_0)^j$, let

$$\psi_{\bar{n}} : (x_0, x_1, \cdots, x_{j-1}) \mapsto {x_0}^{(n_0)} *_f {x_1}^{(n_1)} *_f \cdots *_f {x_j}^{(n_j)}.$$

Since $(\mathbb{N}_0, *_f)$ is a commutative semigroup, then $\psi_{\bar{n}} : (\mathbb{N}_0, *_f)^j \to (\mathbb{N}_0, *_f)$ is a semigroup homomorphism. Let

$$\mathfrak{F}_j = \{\psi_{\bar{n}} : \mathbb{N}^j \to \mathbb{N} : \bar{n} = (n_0, n_1, \cdots, n_{j-1}) \in \{0, 1, 2, \cdots, p\}^j\}$$

be the sets of homomorphisms for $j = 1, 2, \cdots, m$. Then by Bergelson-Johnson-Moreira Theorem, for every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a color C_i and elements $x_0, x_1, \cdots, x_m$ different from the identity such that:

(1) $x_0 \in C_i$

(2) $\psi_{\bar{n}}(x_0, x_1, \cdots, x_{j-1}) *_f x_j \in C_i$, for every $j = 1, 2, \cdots, m$ and for all $\bar{n} \in \{0, 1, 2, \cdots, p\}^j$.

Finally, from the second point, we have

$$\psi_{\bar{n}}(x_0, x_1, \cdots, x_{j-1}) *_f x_j \in C_i$$

which implies that ${x_0}^{(n_0)} *_f {x_1}^{(n_1)} *_f \cdots *_f {x_j}^{(n_j)} *_f x_j \in C_i$.

Therefore, for all $(n_0, n_1, \cdots, n_{j-1}) \in \{0, 1, \cdots, p\}^j$ and $j = 1, 2, \cdots, m$ we obtain

$$\text{card}(\Sigma / s_{x_0}^{n_0} s_{x_1}^{n_1} \cdots s_{x_{j-1}}^{n_{j-1}} s_{x_j}) \in C_i.$$

Hence, the result follows. □

Proof. (Proof of Brauer's Theorem 3.5) The proof of Brauer's theorem relies on the Bergelson-Johnson-Moreira Theorem that we discussed in the proof of Theorem 3.7 for $(S, *) = (\mathbb{N}_0, *_f)$. For $j = 0, 1, 2, \cdots, k+1$, let $\psi_j : (\mathbb{N}, *_f) \to (\mathbb{N}, *_f)$ be the endomorphism defined by $\psi_j(x) = x^{(j)}$. Then for every r-coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, there exist a color C_i and elements x, y different from 1 such that

$$\{x, \psi_0(x) *_f y, \psi_1(x) *_f y, \psi_2(x) *_f y, \cdots, \psi_{k+1}(x) *_f y\} \subseteq C_i. \tag{3.1}$$

Note that $\psi_0(x) *_f y = x^{(0)} *_f y = 1 *_f y = y$ and $\phi_j(x) *_f y = \text{card}(\Sigma / s_x^j s_y)$. Now if we let, $z := x *_f y$, then from (3.1) we obtain

$$\{x, z, x *_f z, x^{(2)} *_f z, x^{(3)} *_f z, \cdots, x^{(k)} *_f z\} \subseteq C_i.$$

Which in turn gives

$$\{x, z, \text{card}(\Sigma / s_x^j s_z) : j = 1, 2, \cdots, k\} \subseteq C_i.$$

Hence the result follows. □

4. Milliken-Taylor theorem

Milliken-Taylor Theorem simultaneously generalises the Hindman finite sums theorem and Ramsey's Theorem. It has often been utilized in the literature, including various powerful generalisations of Szemeredi's Theorem on arithmetic progressions.

To state the Milliken-Taylor Theorem, one needs to introduce some notations. The set of all finite non-empty subsets of $\mathbb{N}$ is denoted by $\mathcal{P}_f(\mathbb{N})$ and for any natural number m, we denote $[\mathbb{N}]^m = \{A \subseteq \mathbb{N} : |A| = m\}$, the family of all subsets of $\mathbb{N}$ of cardinality m and for any $F, G \in \mathcal{P}_f(\mathbb{N})$ we consider the ordering given by $F < G$ if and only if $\max F < \min G$.

Theorem 4.1 (Milliken-Taylor Theorem, [17] [19]). *For $r \geq 1$ and every r-coloring $[\mathbb{N}]^m = C_1 \cup C_2 \cup \cdots \cup C_r$ there exists an injective sequence $(x_n)_{n=1}^{\infty}$ of natural numbers and a color C_i such that*

$$\{(x_{F_1}, \cdots, x_{F_m}) : F_1 < \cdots < F_m\} \subseteq C_i,$$

where for $F = \{n_1 < n_2 < \cdots < n_k\} \in \mathcal{P}_f(\mathbb{N})$ we denoted $x_F = x_{n_1} + x_{n_2} + \cdots + x_{n_k}$.

One can easily observe that for $m = 1$, we have Hindman's Theorem, and when all F_i contains a single element, one has the Ramsey Theorem.

Milliken-Taylor Theorem also has an analogue version in our context. To prove our result, the main tool is the notion of tensor products of ultrafilters.

Definition 4.2. Let $k \in \mathbb{N}$ and for $i \in \{1, 2, \cdots, k\}$, let S_i be a semigroup and let $p_i \in \beta S_i$. We define $\bigotimes_{i=1}^{k} p_i \in \beta(\times_{i=1}^{k} S_i)$ inductively as follows:

(1) $\bigotimes_{i=1}^{1} p_i = p_1$.

(2) Given $k \in \mathbb{N}$ and $A \subseteq \times_{i=1}^{k+1} S_i$, $A \in \bigotimes_{i=1}^{k+1} p_i$ if and only if

$$\{(x_1, x_2, \cdots, x_k) \in \times_{i=1}^{k} S_i : \{x_{k+1} \in S_{k+1} : (x_1, x_2, \cdots, x_{k+1}) \in A\} \in p_{k+1}\} \in \bigotimes_{i=1}^{k} p_i.$$

One can easily verify that $\bigotimes_{i=1}^{k} p_i$ is an ultrafilter on $\times_{i=1}^{k} S_i$. We will use the general version of Milliken-Taylor Theorem, which is characterised by the sets contained in the tensor products of idempotent ultrafilters, as follows:

Theorem 4.3 (Bergelson-Hindman-Williams, [7])**.** *Let S be a semigroup, let $m \in \mathbb{N}$, and let $A \subseteq \times_{i=1}^{m} S$. The following statements are equivalent:*

(a) *There is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that*

$$\{(x_{F_1}, \cdots, x_{F_m}) : F_1 < \cdots < F_m\} \subseteq A.$$

(b) *There is an idempotent $p \in \beta S$ such that $A \in \bigotimes_{i=1}^{m} p$*

Suppose $\phi : X \to Y$ is any function, It induces a map $\phi_* : \beta X \to \beta Y$ defined by

$$\phi_*(p) := \{B \subseteq Y : \phi^{-1}(B) \in p\}, \text{ for } p \in \beta X.$$

Notice that $A \in p \Rightarrow \phi(A) \in \phi_*(p)$, but not conversely.

Corollary 4.4. Let $(S, \cdot)$ be a semigroup, and $m \in \mathbb{N}$. Let $\phi : S^m \to S$ be any function, and let $B \subseteq S$. Then the following statements are equivalent:

(a) There is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that

$$\{\phi(x_{F_1}, \cdots, x_{F_m}) : F_1 < \cdots < F_m\} \subseteq B.$$

(b) $B \in \phi_*(\bigotimes_{i=1}^{m} p)$, where p is an idempotent in $(\beta S, \cdot)$

Proof. In Bergelson-Hindman-Williams theorem, if we consider $A := \phi^{-1}(B) \in \bigotimes_{i=1}^{m} p$, then we have the required result. For details see [18, Corollary 5.4]. □

Theorem 4.5. *For $r \geq 1$ and $\phi : (\mathbb{N}_0)^m \to \mathbb{N}_0$ be any map. Then for every r-coloring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$ there exists a sequence $(x_n)_{n=1}^{\infty}$ of natural numbers and a color C_i such that*

$$\{\phi(\text{card}(\Sigma/F_1), \cdots, \text{card}(\Sigma/F_m)) : F_1 < \cdots < F_m\} \subseteq C_i,$$

where $F_j = \{n_{j_1} < \cdots < n_{j_{k_j}}\} \in \mathcal{P}_f(\mathbb{N})$ for all $j = 1, \cdots, m$ and $\Sigma/F_j = \Sigma/s_{x_{n_{j_1}}} s_{x_{n_{j_2}}} \cdots s_{x_{n_{j_{k_j}}}}$ where Σ is the set of sums of two squares.

Proof. In Corollary 4.4, we consider $(S,\cdot) = (\mathbb{N}_0, *_f)$. Pick an idempotent ultrafilter $p \in (\beta\mathbb{N}_0, *_f)$. Thus $\bigotimes_{i=1}^m p$ is an ultrafilter on $(\mathbb{N}_0)^m$. For the given map $\phi : (\mathbb{N}_0)^m \to \mathbb{N}_0$, we set $q = \phi_*(\bigotimes_{i=1}^m p)$. Given a finite coloring $\mathbb{N}_0 = C_1 \cup C_2 \cup \cdots \cup C_r$, let C_i be the color such that $C_i \in q$, and by Corollary 4.4, there is a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}_0$ such that

$$\{\phi(x_{F_1}, \cdots, x_{F_m}) | F_1 < \cdots < F_m\} \subseteq C_i$$

holds for all $F_i \in \mathcal{P}_f(\mathbb{N})$, $i = 1, 2, \cdots, m$ and $F_1 < \cdots < F_m$.

For each $j \in \{1, \cdots, m\}$, let $F_j = \{n_{j_1} < n_{j_2} < \cdots < n_{j_{k_j}}\}$, then one computes

$$x_{F_j} = x_{n_{j_1}} *_f x_{n_{j_2}} *_f \cdots *_f x_{n_{j_{k_j}}} = \mathrm{card}(\Sigma / s_{x_{n_{j_1}}} s_{x_{n_{j_2}}} \cdots s_{x_{n_{j_{k_j}}}}) = \mathrm{card}(\Sigma/F_j).$$

Thus, we derive the following by substituting these values in the conclusion of the Corollary 4.4.

$$\{\phi(\mathrm{card}(\Sigma/F_1), \cdots, \mathrm{card}(\Sigma/F_m)) : F_1 < F_2 < \cdots < F_m\} \subseteq C_i.$$

Hence, the result follows. □

5. Application of Hales-Jewett theorem

Let $\mathcal{A}$ be a nonempty, finite set of *alphabet* and $v \notin \mathcal{A}$ be a symbol, call it a *variable*. A located word in the alphabet $\mathcal{A}$ is finitely supported function $b : \mathrm{Dom}(b) \to \mathcal{A}$ where $\mathrm{Dom}(b)$ is a (possibly empty) finite subset of $\mathbb{N}_0$. Similarly, a *located variable word* in the alphabet $\mathcal{A}$ and variable v is a finitely supported function $b : \mathrm{Dom}(b) \to \mathcal{A} \cup \{v\}$ whose range contains v, where $\mathrm{Dom}(b)$ is a finite subset of $\mathbb{N}_0$. Let $L(\mathcal{A})$ be the set of *located words* in $\mathcal{A}$ and let $L(\mathcal{A}v)$ be the set of located variable words in $\mathcal{A}$ and the variable v. Then $S := L(\mathcal{A}) \cup L(\mathcal{A}v)$ has a natural partial semigroup operation [1], obtained by letting $b_0 + b_1$ be defined whenever the domains of b_0 and b_1 are disjoint. In such a case, $b_0 + b_1$ is just $b_0 \cup b_1$.

Theorem 5.1 (Hales-Jewett Theorem)**.** *Let $L(\mathcal{A})$ be finitely coloured. Then there exist $\alpha \in L(\mathcal{A})$ and $\gamma \in \mathcal{P}_f(\mathbb{N})$ such that $\mathrm{Dom}(\alpha) \cap \gamma = \emptyset$ and $\{\alpha \cup \gamma \times \{s\} : s \in \mathcal{A}\}$ is monochromatic.*

In 2008, M. Beiglböck extended the Hales–Jewett theorem which is stronger than the above Theorem. He involves partition regular families of $\mathbb{N}$. A *partition regular family* $\mathcal{F}$ of $\mathbb{N}$ is a subset of $\mathcal{P}_f(\mathbb{N})$ such that for any partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exists an $i \in \{1, \cdots, r\}$ such that $C_i \in \mathcal{F}$.

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[1](Partial semigroup) A *partial semigroup* is a triple $(S, X \subseteq S \times S, *)$ of a set S, a subset $X \subseteq S \times S$ and an operation $*$ defined on X satisfying

$$(x * y) * z = x * (y * z) \qquad \forall x, y, z \in G$$

in the sense that if either side is defined, so is the other and they are equal.

Theorem 5.2. [2, Theorem 3] *Let $\mathcal{F}$ be a partition regular family of finite subsets of $\mathbb{N}$ which contains no singletons, and let $\mathcal{A}$ be a finite alphabet set. For any finite coloring of $L(\mathcal{A})$, there exist $\alpha \in L(\mathcal{A})$, $\gamma \in \mathcal{P}_f(\mathbb{N})$ and $F \in \mathcal{F}$ such that* $\mathrm{Dom}(\alpha), \gamma, F$ *are pairwise disjoint sets and*

$$\{\alpha \cup (\gamma \cup \{t\} \times \{s\}) : s \in \mathcal{A}, t \in F\}$$

is monochromatic.

5.1. **Geo-arithmetic progression.** Vitaly Bergelson proved for any finite coloring of $\mathbb{Z}$, there exists a monochromatic geo-arithmetic progression of arbitrary length which can be considered as the combined extension of additive and multiplicative van der Waerden's Theorem. He proved this property by using the ergodic theory [5]. Later M. Beiglböck, V. Bergelson, N. Hindman and D. Strauss proved this result by using the algebra of Stone-Čech compactification in [3].

Theorem 5.3 (Geo-arithmetic progression)**.** *If $n, r \in \mathbb{N}$, and $\mathbb{Z}$ is r-colored, then there exist a, b, and $d \in \mathbb{N}$ such that the set $\{a(b + id)^j : 0 \le i, j \le n\}$ is monochromatic.*

In [2], M. Beiglböck proved that the extension of the Hales-Jewett theorem is strong enough to yield Theorem 5.3. In this article, we will use this variant of the Hales-Jewett Theorem 5.2 to prove our geo-arithmetic structure .

Theorem 5.4. *If $k, r \in \mathbb{N}$, and $\mathbb{N}$ is r-colored, then there exist $a, b, d \in \mathbb{N}$ and $\gamma \in \mathcal{P}_f(\mathbb{N})$ such that the set $\{\mathrm{card}(\Sigma/s_b(\prod_{t\in\gamma} s_t s_{a+id})^j) : i, j = 0, 1, 2, \cdots, k\}$ is monochromatic, where Σ is the set of sums of two squares.*

Proof. Assume that $\mathbb{N}$ is finitely colored. Fix $k \in \mathbb{N}$, let $\mathcal{F} = \{\{a, a + d, \cdots, a + kd\} : a, d \in \mathbb{N}\}$ be the set of all $(k + 1)$-term arithmetic progressions, put $\mathcal{A} = \{0, 1, \cdots, k\}$ and define

$$h : L(\mathcal{A}) \to \mathbb{N} \text{ by } h(\alpha) = \underset{t\in \mathrm{Dom}(\alpha)}{*_f} t^{(\alpha(t))}.$$

We color each $\alpha \in L(\mathcal{A})$ with the color of $h(\alpha)$ and choose α, γ and $F = \{a, a + d, \cdots, a + kd\}$.

Then using the Theorem 5.2 we obtain that the set

$$\{h(\alpha \cup (\gamma \cup \{a + id\}) \times \{j\}) : i, j \in \{0, 1, \cdots, k\}$$

is monochromatic.

$$\begin{aligned} h(\alpha \cup (\gamma \cup \{a + id\}) \times \{j\}) &= (\underset{t\in \mathrm{Dom}(\alpha)}{*_f} t^{(\alpha(t))}) *_f (\underset{t\in\gamma}{*_f} t^{(j)}) *_f ((a + id)^{(j)}) \\ &= \mathrm{card}(\Sigma/(\prod_{t\in \mathrm{Dom}(\alpha)} s_t^{\alpha(t)})(\prod_{t\in\gamma} s_t s_{a+id})^j). \end{aligned}$$

Hence, we obtain that the following set

$$\{\mathrm{card}(\Sigma/(s_b(\prod_{t\in\gamma} s_t s_{a+id})^j) : i, j \in \{0, 1, \cdots, k\}\}$$

is monochromatic, where $s_b = \prod_{t\in \mathrm{Dom}(\alpha)} s_t^{\alpha(t)}$. □

5.2. **Polynomial van der Waerden's theorem.** Polynomial extension of van der Waerden's theorem relies on the polynomial version of the Hales-Jewett theorem. In 1988, V. Bergelson and A. Liebman proved the polynomial extension of the Hales-Jewett Theorem by introducing and developing the apparatus of set-polynomials (polynomials whose coefficients are finite sets) and applying the methods of topological dynamics in [4]. Later, M. Walters gave short and purely combinatorial proofs of those results in [22]. Let us start with the statement of the polynomial Hales-Jewett theorem with some relevant notations.

For fixed numbers q, N, d, let us consider the set $X(q,N,d) = \prod_{i=1}^d [q]^{N^i}$, where $[q] = \{1, \cdots q\}$. An element $x \in X(q,N,d)$ is of the form $(\vec{b}_1, \cdots, \vec{b}_d)$, where $\vec{b}_i : [N]^i \to [q]$. For $a = (\vec{a}_1, \cdots, \vec{a}_d)$, $\gamma \subseteq [N]$ and $(x_1, \cdots x_d) \in [q]^d$, define an element $x = a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d$ as follows:

If $x = (\vec{b}_1, \cdots, \vec{b}_d)$, then

$$\vec{b}_j((i_1, \cdots i_j)) = \begin{cases} x_j, & \text{if } (i_1, \cdots i_j) \in \gamma^j \\ \vec{a}_j((i_1, \cdots i_j)), & \text{otherwise.} \end{cases}$$

Theorem 5.5. *(PHJ Theorem) For any $q, k, d \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $X(q,N,d)$ is k-colored, there exist $a \in X(q,N,d)$ and $\gamma \subseteq [N]$ such that the set*

$$\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1, \cdots x_d) \in [q]^d\}$$

is monochromatic.

We'll use Theorem 5.5 to get our version of polynomial van der Waerden Theorem which is the following:

Theorem 5.6. *Let $d, k, \ell \in \mathbb{N}$ and $\{F_1, \cdots F_\ell\} \subset \mathcal{P}_f(\mathbb{N})$ with $F_i = \{a_{i1}, \cdots, a_{id}\}$ for all $i \in \{1, \cdots, \ell\}$. Then for any k-coloring of $\mathbb{N}$, there exist $b, c \in \mathbb{N}$ such that the set $\{\mathrm{card}(\Sigma / s_b s_{a_{i1}}^c \cdots s_{a_{id}}^{c^d}) : i = 1, \cdots, \ell\}$ is monochromatic, where Σ is the set of sums of two squares.*

Proof. Let us consider $q, k, d \in \mathbb{N}$ as in the statement of the polynomial Hales-Jewett theorem. Then using that theorem we get a natural number $N = N(q,k,d)$. Suppose $\chi : (\mathbb{N}, *_f) \to [k]$ a k-coloring of $\mathbb{N}$ and the canonical map $m : X(q,N,d) \to (\mathbb{N}, *_f)$ given by $m((\vec{b}_1, \cdots, \vec{b}_d)) = \mathop{*_f}\limits_{j=1}^{d} (\mathop{*_f}\limits_{(i_1,\cdots i_j)\in[N]^j} \vec{b}_j((i_1, \cdots i_j)))$. Then composite $\chi \circ m$ is a k-coloring of $X(q,N,d)$. Using the polynomial Hales-Jewett theorem, we derive that $\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1, \cdots x_d) \in [q]^d\}$ is monochromatic. Therefore the image $m(\{a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d : (x_1, \cdots x_d) \in [q]^d\})$ is monochromatic for the coloring χ of

$\mathbb{N}$. Note that

$$\begin{aligned} m(a \oplus x_1\gamma \oplus \cdots \oplus x_d\gamma^d) &= \mathop{*_f}_{j=1}^{d} \Big(\mathop{*_f}_{(i_1,\cdots i_j)\in[N]^j} \vec{b}_j((i_1,\cdots i_j))\Big) \\ &= \Big(\mathop{*_f}_{j=1}^{d} \Big(\mathop{*_f}_{(i_1,\cdots i_j)\in\gamma^j} \vec{b}_j((i_1,\cdots i_j))\Big) *_f \Big(\mathop{*_f}_{j=1}^{d} \Big(\mathop{*_f}_{(i_1,\cdots i_j)\in[N]^j\setminus\gamma^j} \vec{b}_j((i_1,\cdots i_j))\Big) \\ &= x_1^{(c)} *_f \cdots *_f x_d^{(c^d)} *_f b \\ &= \mathrm{card}(S/s_b s_{x_1}^{c} \cdots s_{x_d}^{c^d}). \end{aligned}$$

Here c is the cardinality of γ and $b = \mathop{*_f}_{j=1}^{d} \Big(\mathop{*_f}_{(i_1,\cdots i_j)\in[N]^j\setminus\gamma^j} \vec{a}_j((i_1,\cdots i_j))\Big)$. □

Remark 5.7. Let $(P,<)$ be a countably infinite poset. Then there is a canonical bijection

$$\phi : P \to \mathbb{N} \text{ given by } \phi(x) = \mathrm{card}\{y \in P : y < x\},$$

with an inverse ψ given by $\psi(n) = (n+1)^{st}$ smallest element of P. Since ψ does not preserve the multiplication of $\mathbb{N}$, hence if P is a countable multiplicative subgroup of $\mathbb{N}$, then the pullback ψ is not a semigroup homomorphism with respect to the usual multiplication. Thus we get a new operation on $\mathbb{N}$ via $m *_\psi n = \phi(\psi(m).\psi(n))$ (as we did before for the case of Σ). Note that $(\mathcal{P}_f(\mathbb{N}),<)$ is a countable ordered poset and for the map $\sigma : \mathcal{P}_f(\mathbb{N}) \to \mathbb{N}$ given by $\sigma(F) = \Sigma_{n\in F}2^n$. This map is not of the above form although it induces operations on $\mathcal{P}_f(\mathbb{N})$.

Remark 5.8. The set $\Sigma = \{a^2+b^2 : a, b \in \mathbb{N}_0\}$ is symmetric with respect to a, b, therefore there is canonical bijection between the set $T := \{(a,b) \in \mathbb{N}_0^2 : a \le b\}$ and Σ. Then the map $\psi : T \to \mathbb{N}$ given by $\psi(m, 2n) = (n+1)^2 - m$ and $\psi(m, 2n+1) = (n+1)(n+2) - m$ given a bijection which induces a new operation on T. Hence, one may write a lot of different monochromatic configurations in $\mathbb{N}$.

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