Exact solution for a class of quantum models of interacting bosons

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Quantum models of interacting bosons have wide range of applications, among them the propagation of optical modes in nonlinear media, such as the k-photon down conversion. Many of such models are related to nonlinear deformations of finite group algebras, thus, in this sense, they are exactly solvable. Whereas the advanced group-theoretic methods have been developed to study the eigenvalue spectrum of exactly solvable Hamiltonians, in quantum optics the prime interest is not the spectrum of the Hamiltonian, but the evolution of an initial state, such as the generation of optical signal modes by a strong pump mode propagating in a nonlinear medium. I propose a simple and general method of derivation of the solution to such a state evolution problem, applicable to a wide class of quantum models of interacting bosons. For the k-photon down conversion model and its generalizations, the solution to the state evolution problem is given in the form of an infinite series expansion in the powers of propagation time with the coefficients defined by a recursion relation with a single polynomial function, unique for each nonlinear model. As an application, I compare the exact solution to the parametric down conversion process with the semiclassical parametric approximation.

I. INTRODUCTION

Finding the exact solution to a physically relevant model is of high value even in the age of widespread computer simulations. Some crucial features of the solution, such as the asymptotic or qualitative behavior, could be only studied by the analytical approach. Widely applicable algebraic methods have been discovered in order to derive exactly solvable models and to obtain the exact solutions to the eigenvalue problem. The first of such methods was the famous Bethe's ansatz for the Heisenberg antiferromagnetic chain model [\[1](#page-9-0)], extended also to the Bose gas with the zero-range interaction potential [\[2](#page-9-1)] and to the one-dimensional Hubbard model [\[3](#page-9-2)]. Other algebraic methods have been developed in order to find and analyze exactly solvable models in Quantum Mechanics, e.g., the method of factorization of second-order differential operator (representing the quantum Hamiltonian) into a product of two first-order differential (ladder) operators [\[4\]](#page-9-3), the Darboux transformations [\[5](#page-9-4)], and the Quantum Inverse Scattering Method [\[6](#page-9-5)].

Exactly solvable models are abundant also in quantum optics. The effective potential method for the generating function approach has been found for the Dicke model [\[7\]](#page-9-6) and the spin systems [\[8\]](#page-9-7), the Bethe ansatz allows to derive the energy spectra of the three-boson model [\[9\]](#page-9-8). Moreover, it was found that some quantum Hamiltonians of the forth order in the boson creation and annihilation operators allow exact solutions for a part of the energy spectra subject to a hidden symmetry [\[10](#page-9-9)]. The latter models, which include the k-photon down conversion and multiple photon cascades, are the so-called quasi-exactly solvable models, allowing for analytical analysis of some part of the energy spectra [\[11](#page-9-10)]. With the help of the group-theoretic methods the Hamiltonian can sometimes be mapped onto the generators of a deformed Lie algebra, which allows one to derive the equations for the eigenvalues and eigenfunctions of a nonlinear quantum optical Hamiltonian, including the second-harmonic generation model [\[12](#page-9-11)[–14\]](#page-9-12). Such methods have been applied recently to find the energy spectra for a wide range of such nonlinear boson models [\[15,](#page-9-13) [16\]](#page-9-14).

In the ever growing field of research on exactly solvable models, the focus is usually on finding the spectrum of the Hamiltonian and the associated eigenstates. On the other hand, in quantum optics applications it is more important to solve the initial value problem, whereas the eigenstates of the full Hamiltonian do not correspond to the optical energy (given by the free propagation part of the Hamiltonian, see below). The most important example is the work-horse of all quantum optics – the twophoton parametric down conversion process in a nonlinear medium with the second-order nonlinearity [\[17,](#page-9-15) [18](#page-9-16)] (see also the reviews [\[19](#page-9-17)[–21](#page-9-18)]). The recent experimental realization of the long sought three-photon spontaneous parametric down-conversion [\[22](#page-9-19)] adds another integrable model to applications in quantum optics.

The analytical approaches developed previously for deriving the analytical results for the exactly solvable models, such as the deformed Lie algebras and the Quantum Inverse Scattering Method, are quite involved in their technical part, which may explain the fact that the same models in the physical literature, such as the k-photon down-conversion [\[23,](#page-9-20) [24\]](#page-9-21), have been studied analytically by other methods, such as the semiclassical or the WKBlike approximations [\[25](#page-9-22)[–28](#page-10-0)], or resorting to numerical simulations [\[29](#page-10-1)[–33](#page-10-2)] helped by the reductions based on the conservation laws. The theoretical approaches in physical literature continue to be based on various approximations, for instance, in order to go beyond the explicitly solvable quasi-classical, a.k.a., parametric approximation [\[34,](#page-10-3) [35\]](#page-10-4).

The purpose of the present work is propose a simple algebraic method of derivation of exact solution to the state evolution problem for a wide class of integrable models of interacting bosons. The method is used to derive the exact solution to the problem of generation of optical signal modes by a pump mode propagating in a nonlinear

medium for a wide class of quantum optical models, such as the k -photon down-conversion and the related models.

In section [II](#page-1-0) the class of quantum models of interacting bosons is described which can be studied by the unified approach of section [III.](#page-2-0) Section [III](#page-2-0) is the main section, where the unified algebraic approach is developed for derivation of the solution to the state evolution problem, where Theorem 1 and Corollary 1 give the solution the problem of state evolution relevant to the quantum optics applications. The derived solution is then verified by substitution to the Schrödinger equation in the Fock space, subsection [III A,](#page-5-0) whereas in subsections [III B,](#page-5-1) [III C,](#page-6-0) and [III D](#page-6-1) the important mathematical features of the solution are exposed and discussed. Section [IV](#page-7-0) discusses the application to generation of optical signal modes by a pump mode propagating in a nonlinear medium, the scaling transformation of the propagation time for a strong coherent pump mode is discussed, subsection [IV A,](#page-7-1) and comparison of the exact solution with the parametric approximation for the squeezed states generation is given, subsection [IV B.](#page-7-2) Section [V](#page-9-23) contains brief description of the results and open problems.

II. MODELS OF INTERACTING BOSONS IN QUANTUM OPTICS

In quantum optics the models of interacting bosons appear due to propagation of optical light modes in a nonlinear media, when the phase matching conditions are satisfied in the medium. One is usually interested in the generation of signal modes from the vacuum state by the strong pump $mode(s)$. In a lossless medium, by the energy conservation (i.e., the Manley–Rowe relations, see for more details Ref. [\[36\]](#page-10-5)) the total electromagnetic field energy is conserved during the propagation. The resulting model Hamiltonian can be partitioned, accordingly, into two terms, the free propagation term, \hat{H}_0 , which corresponds to the total electromagnetic energy and is quadratic in the boson creation and annihilation operators of the optical modes, and the interaction term, \hat{H}_1 , of higher-order in the boson operators, which describes the photon conversion between the optical modes propagating in the nonlinear medium. When the phase matching conditions are satisfied, the interaction part of the Hamiltonian preserves the total optical energy: $[\hat{H}_0, \hat{H}_1] = 0$.

The above class of the interaction models includes the k -photon down conversion processes [\[24\]](#page-9-21), such as the generation of the squeezed states of light by the parametric down conversion process in the second-order nonlinear medium (e.g., Refs. [\[20](#page-9-24), [21\]](#page-9-18)) and the recently experimentally achieved three-photon state generation [\[22\]](#page-9-19). In the simplest case, the k-photon down conversion process involves propagation of a single pump mode and of a single signal mode. It is described by the following two-mode Hamiltonian (see e.g., Ref. [\[24](#page-9-21)]) $\hat{H} = \hat{H}_0 + \tilde{H}_1$

$$
\hat{H}_0 = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \hbar \omega \hat{b}^\dagger \hat{b}, \quad \hat{H}_1 = \hbar \Omega \left\{ \hat{a}^\dagger \hat{b}^k + \hat{a} (\hat{b}^\dagger)^k \right\} (1)
$$

where \hat{a} and b are the annihilation boson operators for the pump mode and the signal, respectively, and $\hbar\Omega$ is the photon conversion strength specific to the nonlinear medium. The phase matching demands that $\omega_0 =$ $k\omega$, leading to the above discussed zero commutator $[\hat{H}_0, \hat{H}_1] = 0$. The Hilbert space of the two interacting modes can be partitioned into a direct sum of the invariant subspaces corresponding to the eigenvalues (the total electromagnetic energy) of \hat{H}_0 , $\mathcal{E} = \hbar \omega_0 N + \hbar \omega \ell$, where N and ℓ are integers $(\ell \leq k-1)$. The dimension of the invariant subspace $\mathcal{H}_{N,\ell}$ is $\dim \mathcal{H}_{N,\ell} = N+1$. In the context of the model in Eq. [\(1\)](#page-1-1), the initial value problem, which is of interest in quantum optics, is the conversion of

the optical pump mode (\hat{a}) into the signal mode (respectively, \hat{b}), where initially (in the running time variable) the signal mode is in the vacuum state (thus $\ell = 0$).

For example, in the quadratic nonlinear medium (in this case $k = 2$, assuming a strong pump and short propagation times [\[27](#page-10-6)], the usual approach is to consider the pump mode to be in a coherent state with a large amplitude α [\[19\]](#page-9-17)

$$
|\alpha\rangle \equiv e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{(\hat{a}^{\dagger})^N}{\sqrt{N!}} |Vac\rangle, \tag{2}
$$

where we denote the vacuum state for the (multimode) quantum optical models by $|Vac\rangle$, i.e., $\hat{a}|Vac\rangle = 0$. One then adopts the parametric approximation by replacing the boson operator \hat{a} by a scalar $\hat{a} \to \alpha$, which converts the interaction Hamiltonian \hat{H}_1 in Eq. [\(1\)](#page-1-1) for $k = 2$ into the semiclassical equivalent being of the second order in boson operators $\hat{b}, \hat{b}^{\dagger}$ and thus allowing an explicit solution to the initial value problem in the form of the so-called squeezed state (see, for more details, the recent reviews [\[20,](#page-9-24) [21\]](#page-9-18)). Such semiclassical approximation has proved to be very useful. However, quite recently, we have witnessed the revival of the interest in describing the 2-photon down conversion process beyond the simple parametric approximation by considering the full quantum model using numerical simulations and approximations [\[34,](#page-10-3) [35\]](#page-10-4). To that one can add the known fact that the semiclassical approach fails quite spectacularly for the higher-order $(k > 3)$ processes described by the model of Eq. [\(1\)](#page-1-1) (the norm of the quantum state evolved by the similar parametric approximation simply diverges in a finite time) [\[23\]](#page-9-20), requiring one to account for the quantum effects on the pump mode itself [\[31](#page-10-7), [32](#page-10-8)].

It turns out that the optical model Hamiltonians of Eq. [\(1\)](#page-1-1) have exact (and quite simple in form) solution to the evolution problem of conversion of the pump mode into the signal mode(s). Moreover, the exact solution to such an evolution problem is given in a unified single form applicable to a wide class of similar models. Such models have the same main feature: the ladder operators, which partition the Hilbert space into the invariant subspaces of finite dimensions.

Consider, for example, the class of models of Eq. [\(1\)](#page-1-1). Introducing the ladder operator for the k-photon down

conversion process by $\hat{A} \equiv \hat{a}^{\dagger} \hat{b}^k$, one can cast the inter-action Hamiltonian in Eq. [\(1\)](#page-1-1) as follows $\hat{H}_1 = \hat{A} + \hat{A}^{\dagger}$. This is one of the required prerequisites. Another one is the above discussed partition of the Hilbert space H into the invariant subspaces labelled by the values of the full set of commuting conserved quantities, which have discrete values (e.g. $\mathcal{I} = (N, \ell)$ in the model of Eq. [\(1\)](#page-1-1)): $\mathcal{H} = \sum_{\mathcal{I}} \mathcal{H}_{\mathcal{I}}$, where the dimension of each invariant subspace $\mathcal{H}_{\mathcal{I}}$ is finite. Such are all the higher-order boson interaction models, which preserve the equivalent of the optical energy given by the term \hat{H}_0 in the quantum Hamiltonian, e.g., the exactly integrable boson models whose energy spectra were considered in Refs. [\[12](#page-9-11)[–16\]](#page-9-14). The generalizations of the model in Eq. [\(1\)](#page-1-1), to which the approach of the next section is applicable, have the following ladder operator \hat{A} :

$$
\hat{A} = (\hat{a}^{\dagger})^m \prod_{s=1}^{S} \hat{b}_s^{k_s},\tag{3}
$$

for arbitrary integer parameters $m \geq 1$, $S \geq 1$ and $k_s \geq$ 0, where the mode \hat{a} is the pump mode, whereas the other modes are the signal modes, initially in a state annihilated by \hat{A} (the approach below trivially applies also to the initial population of the signal mode(s) below the energy conversion condition, e.g., for the model with $S = 1$ of Eq. [\(3\)](#page-2-1) with ℓ photons initially in the signal mode, when $\ell \leq k_1 - 1$).

III. SOLUTION TO THE EVOLUTION PROBLEM

Similar to the approaches described in the Introduction, the approach presented below is completely algebraic. Similarly to the factorization method [\[4](#page-9-3)] it is based on the usage of the ladder operators, but now appearing as partition (not as a product) in the quantum Hamiltonian for a many-body problem (and not a single-particle Schrödinger equation). Similarly to the group-theoretic methods, such as in Refs. [\[12](#page-9-11)[–16](#page-9-14)], the conservation laws are used to partition the Hilbert space, but in contrast, only an elementary algebra is utilized, no usage of the advanced machinery of the Lie algebras or the deformed Lie algebras is necessary. We will work in the interaction picture (as in section [II](#page-1-0) we consider the Hamiltonians having two parts, $\hat{H} = \hat{H}_0 + \hat{H}_1$, with $[\hat{H}_0, \hat{H}_1] = 0$).

As have been discussed in the previous section, the method is based on just two key assumptions, described in the items (i) and (ii) below.

(i). The Hilbert space $\mathcal H$ of a quantum system can be partitioned into a direct sum (infinite, in general) of some finite-dimensional invariant subspaces (with respect to the interaction Hamiltonian \hat{H}_1). Without loss of generality, let us label the invariant subspaces by a discrete quantity N having integer values $N \geq 0$: $\mathcal{H} = \sum^{\oplus} \mathcal{H}_N$, where \mathcal{H}_N is $(N + 1)$ -dimensional, i.e.,

$$
\mathcal{H}_N = \text{Span}\{|\Psi_0^{(N)}\rangle, |\Psi_1^{(N)}\rangle, \dots, |\Psi_N^{(N)}\rangle\}
$$
 (4)

(if some subspaces have the same dimension, one can introduce additional label to distinguish them). Below, we mainly work in just one of the invariant subspaces \mathcal{H}_N , thus we will suppress the index "N" identifying the subspace in order to have less cumbersome notations.

(*ii*). The interaction Hamiltonian \hat{H}_1 of the quantum system can be partitioned into two Hermitian-conjugated parts,

$$
\hat{H}_1 = \hat{A} + \hat{A}^\dagger,\tag{5}
$$

acting as the (up and down) ladder operators on the subspace \mathcal{H}_N , i.e., for $n = 0, \ldots, N$ only the nearestneighbor averages are non-zero: $\langle \Psi_n | \hat{A} | \Psi_{n+1} \rangle \neq 0$ and $\langle \Psi_{n+1} | \hat{A}^{\dagger} | \Psi_n \rangle \neq 0$. One can always adjust the overall phases of the basis states in \mathcal{H}_N such that the latter averages are real and, therefore, must coincide. Now let us introduce (in each \mathcal{H}_N) a real *n*-dependent number β_n such that

$$
\hat{A}|\Psi_{n+1}\rangle = \sqrt{\beta_n}|\Psi_n\rangle, \quad \hat{A}^\dagger|\Psi_n\rangle = \sqrt{\beta_n}|\Psi_{n+1}\rangle, \quad (6)
$$

where necessarily $\beta_N = 0$ since the finite-dimensional subspace \mathcal{H}_N has the dimension $N + 1$. Eq. [\(6\)](#page-2-2) is satisfied by the models of interacting bosons at the resonant energy conversion (a.k.a. the phase matching conditions in quantum optics).

Let us note for below, that the only quantity which characterizes the specific quantum model in the above described scheme is the parameter $\beta_n^{(N)}$ (which may have different values in different invariant subspaces \mathcal{H}_N). The latter accounts also for any additional features of the basis in each \mathcal{H}_N (for instance, when there are more than one subspace of dimension $N + 1$, as discussed in section [II,](#page-1-0) we have to introduce an additional index to distinguish them, see also section [IV](#page-7-0) below, for more details).

Below two more operators will be used, additionally to the above introduced ladder operators in Eq. [\(6\)](#page-2-2). The state-number operator in \mathcal{H}_N , \hat{n} , which satisfies $\hat{n}|\Psi_n\rangle =$ $n|\Psi_n\rangle$, for all $n=0,\ldots,N$, and the commutator of the ladder operators $\hat{B} \equiv [\hat{A}, \hat{A}^{\dagger}]$. From Eq. [\(6\)](#page-2-2) we see that the latter is a scalar function of \hat{n} :

$$
\hat{B} \equiv [\hat{A}, \hat{A}^{\dagger}] = B(\hat{n}), \quad B(n) \equiv \beta_n - \beta_{n-1}.
$$
 (7)

The ladder operators \hat{A} , \hat{A}^{\dagger} have the following commutation relations with the state-number operator

$$
[\hat{n}, \hat{A}^{\dagger}] = \hat{A}^{\dagger}, \quad [\hat{n}, \hat{A}] = -\hat{A}, \tag{8}
$$

reminiscent of the usual boson creation and annihilation operators, in which case \hat{n} would be the number of bosons (the difference is that \hat{B} is not a scalar, but is a function of \hat{n}). In this interpretation Eq. [\(7\)](#page-2-3) introduces an arbitrary (nonlinear in \hat{n}) deformation of the usual bosonoperator algebra. For any scalar function $F(x)$ from Eq. [\(8\)](#page-2-4) we get

$$
F(\hat{n})\hat{A}^{\dagger} = \hat{A}^{\dagger}F(\hat{n}+1). \tag{9}
$$

Using Eq. [\(9\)](#page-2-5) and the definition of $B(n)$ in Eq. [\(7\)](#page-2-3) we find

$$
\hat{A}(\hat{A}^{\dagger})^m = (\hat{A}^{\dagger})^m \hat{A} + \sum_{l=0}^{m-1} (\hat{A}^{\dagger})^l B(\hat{n}) (\hat{A}^{\dagger})^{m-1-l} \n= (\hat{A}^{\dagger})^m \hat{A} + (\hat{A}^{\dagger})^{m-1} \sum_{s=0}^{m-1} B(\hat{n} + s).
$$
\n(10)

In view of applications to the pump mode conversion to the signal modes in nonlinear optical media, we will focus below on the initial value problem for the quantum evolution with the initial state being annihilated by one of the two ladder operators, here chosen to be $A[38]$ $A[38]$ $A[38]$, i.e., the state whose evolution we have to find reads

$$
|\psi(0)\rangle = \sum_{N=0}^{\infty} c_N |\Psi_0^{(N)}\rangle.
$$
 (11)

Consider the quantum state at positive times, in the interaction picture, $|\psi(\tau)\rangle = e^{-i\tau \hat{H}_1} |\psi(0)\rangle$ (we rescale the propagation time to a dimensionless parameter τ) with the initial state of Eq. [\(11\)](#page-3-0). We will work in a single subspace \mathcal{H}_N thus our goal is to find the evolution of the state $|\Psi_0^{(N)}\rangle$. It proves convenient to expand such an evolved state with the help of the ladder operators (omitting the index "N" of the subspace \mathcal{H}_N)

$$
e^{-i\tau(\hat{A}+\hat{A}^{\dagger})}|\Psi_0\rangle = \sum_{n=0}^{N} \gamma_n(\tau)(-i\hat{A}^{\dagger})^n |\Psi_0\rangle, \tag{12}
$$

where $(-i)^n$ is judiciously introduced in order to have real coefficients γ_n . By using Eq. [\(6\)](#page-2-2) we can relate γ_n and the normalized quantum amplitudes $\psi_n(\tau)$ of the expansion of the state in Eq. [\(12\)](#page-3-1) over the basis in Eq. [\(4\)](#page-2-6) in \mathcal{H}_N :

$$
\psi_n \equiv \langle \Psi_n | \psi(\tau) \rangle = (-i)^n \gamma_n \sqrt{\prod_{\ell=0}^{n-1} \beta_\ell}.
$$
 (13)

The solution to the state evolution problem in Eq. [\(12\)](#page-3-1) amounts to finding the general expression for $|m\rangle \equiv (\hat{A} + \hat{A}^{\dagger})^m |\Psi_0\rangle$ for an arbitrary $m \geq 0$, which appears in the expansion of the evolution operator $e^{-i\tau(\hat{A}+\hat{A}^{\dagger})}$. Let us consider first few such states $|m\rangle$ starting from $|0\rangle \equiv |\Psi_0\rangle$. By application of $\hat{A} + \hat{A}^{\dagger}$ to the state $|m-1\rangle$, utilizing Eqs. [\(7\)](#page-2-3) and [\(10\)](#page-3-2) and the fact that $\hat{A}|\Psi_0\rangle = 0$, we have:

$$
|1\rangle = (\hat{A}^{\dagger})^{1} |\Psi_{0}\rangle,
$$

\n
$$
|2\rangle = \left\{ (\hat{A}^{\dagger})^{2} + (\hat{A}^{\dagger})^{0} \beta_{0} \right\} |\Psi_{0}\rangle,
$$

\n
$$
|3\rangle = \left\{ (\hat{A}^{\dagger})^{3} + (\hat{A}^{\dagger})^{1} \sum_{s=0}^{1} \beta_{s} \right\} |\Psi_{0}\rangle,
$$

\n
$$
|4\rangle = \left\{ (\hat{A}^{\dagger})^{4} + (\hat{A}^{\dagger})^{2} \sum_{s=0}^{2} \beta_{s} + (\hat{A}^{\dagger})^{0} \beta_{0} \sum_{s=0}^{1} \beta_{s} \right\} |\Psi_{0}\rangle,
$$

\n
$$
|5\rangle = \left\{ (\hat{A}^{\dagger})^{5} + (\hat{A}^{\dagger})^{3} \sum_{s=0}^{3} \beta_{s} + (\hat{A}^{\dagger})^{1} \sum_{s_{1}=0}^{1} \beta_{s_{1}} \sum_{s_{2}=0}^{s_{1}+1} \beta_{s_{2}} \right\} |\Psi_{0}\rangle,
$$

\n
$$
|6\rangle = \left\{ (\hat{A}^{\dagger})^{6} + (\hat{A}^{\dagger})^{4} \sum_{s=0}^{4} \beta_{s} + (\hat{A}^{\dagger})^{2} \sum_{s_{1}=0}^{2} \beta_{s_{1}} \sum_{s_{2}=0}^{s_{1}+1} \beta_{s_{2}}
$$

\n
$$
+ (\hat{A}^{\dagger})^{0} \beta_{0} \sum_{s_{2}=0}^{s_{1}+1} \beta_{s_{2}} \sum_{s_{3}=0}^{s_{2}+1} \beta_{s_{3}} \right\} |\Psi_{0}\rangle.
$$

Observe that in the above sequence the respective powers of the ladder operator in the expansion of the state $|m\rangle$ are given by $(\hat{A}^{\dagger})^{m-2l}$ with $0 \leq l \leq \left[\frac{m}{2}\right]$ (here the bracket [. . .] denotes the integer part). The nested sums come from two sources: (i) from the multiplication of the lower power term $(\hat{A}^{\dagger})^{(m-1)-2l}$ by \hat{A}^{\dagger} and (ii) from commutation of \hat{A} with a higher power term $(\hat{A}^{\dagger})^{(m-1)-2(l-1)}$ in the expression for $|m-1\rangle$. Taking these facts together suggest the following.

Theorem 1 The quantum state $|m\rangle \equiv (\hat{A} + \hat{A}^{\dagger})^m |\Psi_0\rangle$ reads

$$
|m\rangle = \left\{ \sum_{l=0}^{\left[\frac{m}{2}\right]} (\hat{A}^{\dagger})^{m-2l} \prod_{j=1}^{l} \sum_{s_j=0}^{s_{j-1}+1} \beta_{s_j} \right\} |\Psi_0\rangle, \qquad (14)
$$

where for $l = 0$ the empty product is equal to 1, while for $l \geq 1$ the sum over s_1 has the upper limit $s_0 + 1$ with $s_0 \equiv m - 2l - 1.$

Proof. The theorem can be proven by induction. For $m = 1$ it is trivial. Assuming it to be valid for m, we proceed by multiplying the expression for $|m\rangle$ in Eq. [\(14\)](#page-3-3) by $\hat{A} + \hat{A}^{\dagger}$, commuting \hat{A} with the respective power of \hat{A}^{\dagger} using Eq. [\(10\)](#page-3-2) (separating the term with $l = 0$) with the result

$$
|m+1\rangle = \left\{ (\hat{A}^{\dagger})^{m+1} + (\hat{A}^{\dagger})^{m-1} \beta_{m-1} + \sum_{l=1}^{\left[\frac{m}{2}\right]} (\hat{A}^{\dagger})^{m+1-2l} \prod_{j=1}^{l} \sum_{s_j=0}^{s_{j-1}+1} \beta_{s_j} + \sum_{l=1}^{\left[\frac{m-1}{2}\right]} (\hat{A}^{\dagger})^{m-1-2l} \beta_{m-1-2l} \prod_{j=1}^{l} \sum_{s_j=0}^{s_{j-1}+1} \beta_{s_j} \right\} | \Psi_0 \rangle, \tag{15}
$$

where the first two terms come from the product $(A +$ \hat{A}^{\dagger}) $(\hat{A}^{\dagger})^m$, the third from multiplication of the sum with $1 \leq l \leq \left[\frac{m}{2}\right]$ from the left by \hat{A}^{\dagger} and the last term from multiplication on the left by \hat{A} with the account that the remaining power of \hat{A}^{\dagger} is non-negative, hence the upper limit $l \leq \left[\frac{m-1}{2}\right]$ in the last sum. Below we consider the two cases of even and odd m separately.

For $m = 2p + 1$ $(p > 1$, as the case of $m = 1$ is trivial) we have in both sums in Eq. [\(15\)](#page-3-4) the upper limit to be $l \leq p$, whereas for $m + 1 = 2(p + 1)$ the upper limit by Eq. [\(14\)](#page-3-3) must be $l \leq p+1$. We perform the following two operations on the terms in Eq. [\(15\)](#page-3-4).

 (i) We combine the second term on the right hand side of Eq. [\(15\)](#page-3-4) and the term with $l = 1$ in the first sum to obtain the term with $l = 1$ required by Eq. [\(14\)](#page-3-3) for $m+1,$

$$
(\hat{A}^{\dagger})^{m-1} \sum_{s_1=0}^{m-1} \beta_{s_1}, \tag{16}
$$

since we must have $s_0 = (m + 1) - 2l - 1 = m - 2$, for $l = 1$, thus indeed $s_1 \leq s_0 + 1$ in Eq. [\(16\)](#page-4-0), as required.

 (ii) To combine the rest of the first sum (i.e., with $2 \leq l \leq p$) with the last sum in Eq. [\(15\)](#page-3-4), we introduce a new index $l' \equiv l - 1$, instead of l, into the last sum, with the new index satisfying $2 \leq l' \leq p+1 = \left[\frac{m+1}{2}\right]$, i.e., as required by Eq. [\(14\)](#page-3-3). The two terms read

$$
\sum_{l=2}^{p} (\hat{A}^{\dagger})^{m+1-2l} \prod_{j=1}^{l} \sum_{s_j=0}^{s_{j-1}+1} \beta_{s_j}
$$

+
$$
\sum_{l=2}^{p+1} (\hat{A}^{\dagger})^{m+1-2l'} \beta_{m+1-2l'} \prod_{j=2}^{l'} \sum_{s'_j=0}^{s'_{j-1}+1} \beta_{s'_j}, \quad (17)
$$

where, in accordance with the product over $2 \leq j \leq l'$, we have shifted the indices in β 's as follows: $s_j \rightarrow s'_{j+1}$. In the second term in Eq. (17) the summation over s'_2 has the upper limit $s'_1 + 1 \equiv s_0 + 1$, with $s_0 = m - 2l - 1 = (m + 1)$ 1) $-2l'$ i.e., exactly the index in the extra (the first from the left to the right) β -factor. Precisely this value of s_1 is missing in the first term on the right hand side of Eq. [\(17\)](#page-4-1) in order to reproduce the sum over $s_1 \leq (m+1) - 2l$ for $l \leq p$, as suggested by Eq. [\(14\)](#page-3-3) applied to $m+1$, whereas for $l = p + 1$ the respective sum contains just $\beta_{m+1-2l'}$ coming from the second term in Eq. [\(17\)](#page-4-1). Hence, the two terms in Eq. [\(17\)](#page-4-1) indeed combine to reproduce the respective sum over $2 \leq l \leq \left[\frac{m+1}{2}\right]$, to coincide with Eq. (14) for $m + 1$.

Consider now an even $m = 2p$. In this case we have $1 \leq l \leq p$ in Eq. [\(14\)](#page-3-3) both for m and for $m + 1$. We proceed in a similar way as in the considered above case of $m = 2p + 1$.

(i) The first step is exactly the same as for $m = 2p +$ 1 above with the same result as in Eq. [\(16\)](#page-4-0). Similar arguments show that the result coincides with the term with $l = 1$ as required by Eq. [\(14\)](#page-3-3) for $m + 1 = 2p + 1$.

(ii) We perform exactly the same operations, as in the above considered case of $m = 2p + 1$, on the last sum to combine it with the rest of the first sum (with $2 < l <$ $p)$ on the right hand side of Eq. [\(15\)](#page-3-4). Now, however, $\lfloor \frac{m-1}{2} \rfloor = p-1$, thus the upper limits in the two sums (in the second for the new index $l' = l - 1$) coincide:

$$
\sum_{l=2}^{p} (\hat{A}^{\dagger})^{m+1-2l} \prod_{j=1}^{l} \sum_{s_j=0}^{s_{j-1}+1} \beta_{s_j}
$$

+
$$
\sum_{l=2}^{p} (\hat{A}^{\dagger})^{m+1-2l'} \beta_{m+1-2l'} \prod_{j=2}^{l'} \sum_{s'_j=0}^{s'_{j-1}+1} \beta_{s'_j}.
$$
 (18)

Now we have to verify that we have a new $s_0 = (m+1) - 2l - 1$ as the upper limit for the sum over s_1 in the two terms, where in the second term we have only one addend for this sum, i.e., the first (from the left to the right) β -factor with the index $m + 1 - 2l'$. The latter is the β -addend to the sum over s_1 with the highest index, as required by Eq. [\(14\)](#page-3-3), since we have to have $s_1 \leq (m+1) - 2l$. Similar arguments, as in the odd m case, convince that we have the upper limit for s'_2 in the second term aligned with the index of the first β -factor as required by Eq. [\(14\)](#page-3-3). This concludes the proof of the Theorem. Q.E.D.

The mth term in the power series expansion of the evolution operator, given in Theorem 1, allows to find the coefficients $\gamma_n(t)$ in Eq. [\(12\)](#page-3-1) in the form of a power series in t. Let us introduce a concise notation for the nested sums of the β -factors, which have appeared on the right hand side of Eq. [\(14\)](#page-3-3). We set $g_n^{(0)} \equiv 1$ and for $l \ge 1$ and $0 \leq n \leq N$

$$
g_n^{(l)} \equiv \sum_{s_1=0}^n \beta_{s_1} \sum_{s_2=0}^{s_1+1} \beta_{s_2} \dots \sum_{s_l=0}^{s_{l-1}+1} \beta_{s_l}.
$$
 (19)

Both the factor β_n defined in Eq. [\(6\)](#page-2-2) and, therefore, $g_n^{(l)}$ are, in general, different in each invariant subspace \mathcal{H}_N (for instance, in the applications below, they depend on the index "N"). A curious feature of $g_n^{(l)}$ is that, by the condition that $\dim(\mathcal{H}_N) = N + 1$, as above discussed, requires $\beta_N = 0$, thus $g_N^{(l)} = g_N^{(l)}$ $\sum_{N=1}^{(i)}$ in each invariant subspace \mathcal{H}_N . Note also the recursive relation:

$$
g_n^{(l)} = \sum_{s=0}^n \beta_s g_{s+1}^{(l-1)}, \quad g_n^{(0)} = 1.
$$
 (20)

With the above notations, we get the following corollary to Theorem 1.

Corollary 1 The evolution of the initial state of Eq. [\(11\)](#page-3-0) projected onto invariant subspace \mathcal{H}_N reads (omitting the index " N " for simplicity)

$$
e^{-i\tau(\hat{A}+\hat{A}^{\dagger})}|\Psi_0\rangle = \sum_{n=0}^{N} \gamma_n(\tau)(-i\hat{A}^{\dagger})^n |\Psi_0\rangle,
$$

$$
\gamma_n(\tau) = \sum_{l=0}^{\infty} \frac{(-1)^l \tau^{n+2l}}{(n+2l)!} g_n^{(l)}.
$$
 (21)

In other words, the amplitude $\gamma_n(\tau)$ is a power series expansion in time, where the coefficients are recursively defined by Eq. (20) .

Proof. The proof of Eq. [\(21\)](#page-5-2) amounts to substitution of the result of Theorem 1 into the power series expansion of the evolution exponent and exchange of the order of the summations with introduction of a new index of summation, $n \equiv m - 2l$ (treating the even and odd values separately, since the even/odd values of index n correspond to the even/odd values of index m :

$$
e^{-i\tau(\hat{A}+\hat{A}^{\dagger})}|\Psi_0\rangle = \sum_{m=0}^{\infty} \frac{(-i\tau)^m}{m!} \sum_{l=0}^{[\frac{m}{2}]} g_{m-2l}^{(l)}(\hat{A}^{\dagger})^m |\Psi_0\rangle
$$

=
$$
\sum_{n=0}^{N} \left(\sum_{l=0}^{\infty} \frac{(-1)^l \tau^{n+2l}}{(n+2l)!} g_n^{(l)} \right) (-i\hat{A}^{\dagger})^n |\Psi_0\rangle,
$$

where the sum in the brackets is precisely the expression given in Eq. (21) . Q.E.D.

A. Direct verification by substitution

One can easily verify the solution by substitution to the evolution equation for the amplitudes $\gamma_n(\tau)$. Expanding the Schrödinger equation for $|\Psi(\tau)\rangle \equiv e^{-i\tau(\hat{A}+\hat{A}^{\dagger})}|\Psi_0\rangle$ in the basis $(-i\hat{A}^{\dagger})^n|\Psi_0\rangle$, using the definition of $\gamma_n(\tau)$ in Eq. (12) and the identity in Eq. (10) we get

$$
\frac{d\gamma_n}{d\tau} = \gamma_{n-1} - \beta_n \gamma_{n+1}, \quad \gamma_{-1} \equiv \gamma_{N+1} \equiv 0. \tag{22}
$$

To verify that our amplitudes $\gamma_n(\tau)$ from Eq. [\(21\)](#page-5-2) satisfy Eq. [\(22\)](#page-5-3) we will use the following relation of the coefficients $g_n^{(l)}$ for all $l \geq 1$

$$
g_n^{(l)} = g_{n-1}^{(l)} + \beta_n g_{n+1}^{(l-1)},\tag{23}
$$

which evidently follows from Eq. [\(20\)](#page-4-2). Differentiating γ_n of Eq. [\(21\)](#page-5-2) for $n \geq 1$ and using Eq. [\(23\)](#page-5-4) we obtain (separating the term with $l = 0$)

$$
\frac{d\gamma_n}{d\tau} = \frac{\tau^{n-1}}{(n-1)!} + \sum_{l=1}^{\infty} \frac{(-1)^l \tau^{n-1+2l}}{(n-1+2l)!} g_{n-1}^{(l)}
$$

+
$$
\beta_n \sum_{l=1}^{\infty} \frac{(-1)^l \tau^{n-1+2l}}{(n-1+2l)!} g_{n+1}^{(l-1)}
$$

=
$$
\sum_{l=0}^{\infty} \frac{(-1)^l \tau^{n-1+2l}}{(n-1+2l)!} g_{n-1}^{(l)} - \beta_n \sum_{l'=0}^{\infty} \frac{(-1)^l \tau^{n+1+2l'}}{(n+1+2l')!} g_{n+1}^{(l')}
$$

=
$$
\gamma_n - \beta_n \gamma_{n+1},
$$

$$
\frac{d\gamma_0}{d\tau} = \beta_0 \sum_{l=1}^{\infty} \frac{(-1)^l \tau^{2l-1}}{(2l-1)!} g_1^{(l-1)}
$$

$$
= -\beta_0 \sum_{l'=0}^{\infty} \frac{(-1)^{l'} \tau^{2l'+1}}{(2l'+1)!} g_1^{(l')} = -\beta_0 \gamma_1,
$$

by the same procedure.

Finally, one could have solved Eq. [\(22\)](#page-5-3) by assuming a series expansion in the powers of τ , where the twodimensional linear recursive equation has to be resolved for the coefficient of the *n*th amplitude γ_n at the *mth* power of τ .

B. Amplitude $\gamma_n(\tau)$ is a holomorphic function of τ

The solution in Eq. [\(21\)](#page-5-2) applies for all finite propagation times τ , i.e., the infinite power series, which defines $\gamma_n(\tau)$, converges. Indeed, as follows from Eq. [\(13\)](#page-3-5), the state normalization condition requires that

$$
|\gamma_n(\tau)| \le \left(\prod_{\ell=0}^{n-1} \beta_\ell\right)^{-\frac{1}{2}},\tag{24}
$$

hence, the power series in τ in Eq. [\(21\)](#page-5-2) must converge irrespectively of the values of the β -factors. This can be shown by using the evident bound $g_n^{(l)} \leq \left(\sum_{s=0}^{N-1} \beta_s\right)^l$ $(g_{N-1}^{(1)})^l$ (recall, that $\beta_N = 0$). The absolute convergence (y_{N-1}) (recall, that $p_N = 0$). The absolute convergence follows, since the power series giving $\gamma_n(\tau)$ is bounded by the uniformly convergent series with the infinite radius of convergence:

$$
|\gamma_n(\tau)| \le \sum_{l=0}^{\infty} \frac{|\tau|^{n+2l}}{(n+2l)!} \left(\sum_{s=0}^{N-1} \beta_s\right)^l < \infty, \quad \forall |\tau| < \infty. \tag{25}
$$

Therefore, $\gamma_n(z)$ is a holomorphic function in the complex plane of $z \in \mathbb{C}$ and bounded on the real line $\tau = \Re(z)$.

Let us also give the upper limit on how many terms $l(\epsilon)$ in the power series expansion one has to keep in order to obtain an approximation, to a given error $\epsilon \ll 1$, to the exact quantum amplitudes $\gamma_n(\tau)$. Setting T_{l+1} T_l $= \epsilon,$ where T_l and T_{l+1} are two consecutive terms in the power series in Eq. [\(25\)](#page-5-5) we get an estimate on the index of the first discarded term in the series for $\gamma_n(t)$ (the maximum is for $n = 0$)

$$
\bar{l}(\epsilon) \lesssim \frac{\tau}{\sqrt{\epsilon}} \sqrt{\sum_{s=0}^{N-1} \beta_s}.
$$
 (26)

Holomorphic functions which are bounded on the real line include most of the known elementary and special functions. Hence, at least in some special cases, the quantum amplitude $\gamma_n(\tau)$ must be reducible to the compositions of known, special or even elementary, holomorphic functions, which are bounded on the real line. However, finding such a combination, when it occurs, is a nontrivial problem.

The beam-splitter example illustrates the above point. Consider the quadratic Hamiltonian describing the unitary linear four-port interferometer (called also the beam-splitter), which is obtained in Eq. [\(1\)](#page-1-1) when $k = 1$. This case admits an explicit solution by a combination of powers of trigonometric functions, which can be obtained by direct integration (in the Heisenberg picture) or by the group methods. Observing that the interaction Hamilto-nian of Eq. [\(1\)](#page-1-1) with $k = 1$ has the following commutators (we use below $\hbar = \Omega = 1$)

$$
[\hat{H}_1, \hat{a}^\dagger] = \hat{b}^\dagger, \quad [\hat{H}_1, \hat{b}^\dagger] = \hat{a}^\dagger,\tag{27}
$$

we get the evolved operator \hat{a}^{\dagger} (in the Heisenberg picture) as follows

$$
e^{-i\tau \hat{H}_1} \hat{a}^\dagger e^{i\tau \hat{H}_1} = \hat{a}^\dagger \cos \tau - i\hat{b}^\dagger \sin \tau.
$$
 (28)

In each subspace \mathcal{H}_N the basis introduced in section [II](#page-1-0) is given in this case by the Fock states $|N-n,n\rangle \equiv \frac{(\hat{a}^{\dagger})^{N-n}(\hat{b}^{\dagger})^n}{\sqrt{(N-n)!n!}}|0,0\rangle$ (where $|0,0\rangle = |Vac\rangle$)

and, in the nomenclature of section [II,](#page-1-0) the state $|\Psi_0^{(N)}\rangle$ annihilated by the ladder operator $\hat{A} \equiv \hat{a}^{\dagger} \hat{b}$ is the Fock state $|N, 0\rangle$. In the interaction picture, Eq. [\(28\)](#page-6-2) leads to

$$
e^{-i\tau \hat{H}_1} |N,0\rangle = \frac{1}{\sqrt{N!}} \left[\hat{a}^\dagger \cos \tau - i \hat{b}^\dagger \sin \tau \right]^N |0,0\rangle
$$

$$
= \cos^N \tau \sum_{n=0}^N {N \choose n}^{\frac{1}{2}} \left[-i \tan \tau \right]^n |N-n,n\rangle. \tag{29}
$$

The coefficients γ_n appearing in the equivalent expansion of the state in Eq. [\(29\)](#page-6-3), given by Eq. [\(12\)](#page-3-1) of section [II,](#page-1-0) can be obtained also from Eq. [\(29\)](#page-6-3) which gives the respective quantum amplitudes, as in Eq. [\(13\)](#page-3-5). Observing that the corresponding parameter $\beta_{\ell} = (N - \ell)(\ell + 1)$ gives $\prod_{\ell=0}^{n-1} \beta_{\ell} = \binom{N}{n}$, we get the identities for the rescaled coefficients $n! \gamma_n$

$$
\tau^n \sum_{p=0}^{\infty} \frac{(-\tau^2)^p}{(n+1)\dots(n+2p)} g_n^{(p)} = \cos^{N-n} \tau \sin^n \tau, \quad (30)
$$

for all $n = 0, 1, 2, \ldots, N$. Eq. [\(30\)](#page-6-4) can be verified by the Taylor expansion at $t = 0$ of the function on the right hand side.

One would like to find a way to go in the inverse direction in Eq. [\(30\)](#page-6-4), i.e., determine which combination of elementary and/or special functions the power series expansion on the left hand side in Eq. [\(21\)](#page-5-2) represents, and if such a combination exists at all. Te latter seems to be a hard problem in general for the recursively defined coefficients $g_n^{(l)}$ in Eq. [\(19\)](#page-4-3).

D. Matrix form of the power series

In the absence of a procedure to relate the infinite power series in Eq. [\(21\)](#page-5-2) to the combinations of the elementary and special functions, one can perform numerical computation of the expansion coefficients using the recursive definition in Eq. [\(20\)](#page-4-2). It turns out that this computation can be done in parallel for all $0 \leq n \leq N$ by using a matrix realization of the recursion for the vector-column of the coefficients: $\mathbf{g}^{(l)} \equiv (g_0^{(l)}, \ldots, g_N^{(l)})^T$, where " T " denotes the transposition. Indeed, the recursion in Eq. [\(20\)](#page-4-2) can be cast also in the matrix form with a $(N + 1)$ -dimensional matrix **B** of a very special class. Introducing the $(N+1)$ -dimensional vector of ones $\mathbf{1} \equiv (1, \ldots, 1)^T$ (coinciding with the values of $g_n^{(0)}$) we have for $p > 1$:

$$
\mathbf{g}^{(p)} = \mathbf{B}^p \cdot \mathbf{1}, \quad \mathbf{B}_{nl} \equiv \begin{cases} \beta_{l-1}, & 0 \le l \le \min(n+1, N) \\ 0, & n+2 \le l \le N. \end{cases} \tag{31}
$$

Observe that the $(N+1)$ -dimensional matrix **B** is almost lower-triangular, save the first diagonal above the main being non-zero, thus it belongs to the class of matrices known as the Hessenberg matrices.

More properties of the matrix \bf{B} can be most easily discussed by considering an expilicit example. Setting $N = 4$ we get the following lower Hessenberg matrix

$$
\mathbf{B} = \begin{pmatrix} 0 & \beta_0 & 0 & 0 & 0 \\ 0 & \beta_0 & \beta_1 & 0 & 0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & 0 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix},\tag{32}
$$

where one can note that the last two rows are identical (due to the definition of $g_n^{(l)}$ in Eq. [\(19\)](#page-4-3) and the fact that $\beta_N = 0$). In fact, we have rank(**B**) = N. Indeed, the matrix \bf{B} has a simple LU-decomposition, e.g., for the above example with $N = 4$ we get

$$
\mathbf{B} = \mathbf{L} \mathbf{U} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta_0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$
(33)

One can now write the infinite power series defining $\gamma_n(\tau)$ of Eq. [\(21\)](#page-5-2) as the following series in powers of **B**:

$$
\gamma_n(\tau) = \frac{\tau^n}{n!} \sum_{p=0}^{\infty} \frac{(-\tau^2)^p}{(n+1)\dots(n+2p)} (\mathbf{B}^p \cdot \mathbf{1})_n. \tag{34}
$$

Eq. [\(34\)](#page-6-5) can be used for numerical calculations of the power series coefficients via rapid algorithms of computing powers of the Hessenberg matrices [\[37\]](#page-10-10).

IV. APPLICATIONS IN QUANTUM OPTICS

To apply the theory of section [III](#page-2-0) all one has to do it to partition the (interaction) Hamiltonian into a sum of a ladder operator and its Hermitian conjugate, $\hat{H}_1 = \hat{A} +$ \hat{A}^{\dagger} and within the basis states in each invariant subspace of the Hilbert space identify the state $|\Psi_0\rangle$ annihilated by the ladder operator: $\hat{A}|\Psi_0\rangle = 0$.

Consider now application of the theory of section [III](#page-2-0) to the models of Eq. [\(3\)](#page-2-1) of section [II.](#page-1-0) In the simplest two-mode case $(S = 1)$ with $\hat{A} = (\hat{a}^{\dagger})^m \hat{b}^k$ we have the invariant subspaces labelled by the composite index $\mathcal{I} \equiv$ (M, ℓ) , where $M \geq 0$ and $0 \leq \ell \leq k-1$. In each subspace $\mathcal{H}_{\mathcal{I}}$ the basis states are Fock states of the two modes:

$$
|\Psi_n^{(\mathcal{I})}\rangle \equiv |M - mn, kn + \ell\rangle, \quad 0 \le n \le N \equiv \left[\frac{M}{m}\right],
$$

$$
|M - mn, kn + \ell\rangle \equiv \frac{(\hat{a}^\dagger)^{M - mn}(\hat{b}^\dagger)^{kn + \ell}}{\sqrt{(M - mn)!(kn + \ell)!}}|Vac\rangle,
$$

(35)

where $\left[\ldots\right]$ is the integer part. The corresponding parameter $\beta_n^{(\mathcal{I})}$ in each invariant subspace $\mathcal{H}_{\mathcal{I}}$ identified above can be found from Eqs. [\(6\)](#page-2-2) and [\(35\)](#page-7-3). We have

$$
\beta_n^{(\mathcal{I})} = \left[\prod_{i=0}^{m-1} (M - mn - i) \right] \prod_{j=1}^k (kn + \ell + j). \tag{36}
$$

Observe that, by the definition of N in Eq. [\(35\)](#page-7-3), there is such a $0 \le q \le m-1$ that $M = Nm + q$. Therefore, as dictated by the finite dimension of $\mathcal{H}_{\mathcal{I}}$, in each such subspace we obtain $\beta_N^{(\mathcal{I})} = 0$ from Eq. [\(36\)](#page-7-4), since there is a zero factor in the first product.

In the general multi-mode case of Eq. [\(3\)](#page-2-1) with \hat{A} = $(\hat{a}^\dagger)^m \prod^S$ $\prod_{s=1}^{\infty} \hat{b}_{s}^{k_s}$ we have the invariant subspaces labelled by the composite index $\mathcal{J} \equiv (M, \ell_1, \ldots, \ell_S)$, where $M \geq 0$ and $0 \leq \ell_s \leq k_s - 1$ with the corresponding Fock states being the basis, similar as in Eq. [\(35\)](#page-7-3). In a similar way, we get the following β -parameter

$$
\beta_n^{(\mathcal{J})} = \left[\prod_{i=0}^{m-1} (M - mn - i) \right] \prod_{s=1}^{S} \prod_{j=1}^{k_s} (k_s n + \ell_s + j). \tag{37}
$$

A. Rescaling propagation time by subspace size

The series in Eq. [\(21\)](#page-5-2) solves the evolution problem in each subspace \mathcal{H}_N of the Hilbert space, with the initial state being $|\Psi_0^{(N)}\rangle \equiv \frac{(\hat{a}^{\dagger})^N}{\sqrt{N!}}|Vac\rangle$. The (dimensionless) propagation time τ and the size of the respective Hilbert subspace \mathcal{H}_N can be combined by a scaling transformation for the class of models in Eq. [\(3\)](#page-2-1). Consider the simplest two-mode case with β -parameter of Eq. [\(36\)](#page-7-4). We can scale out the factor $M^m \sim (mN)^m$, i.e., introduce the rescaled β-coefficient

$$
\bar{\beta}_n \equiv \left[\prod_{i=0}^{m-1} \left(1 - \frac{mn - i}{M} \right) \right] \prod_{j=1}^k (kn + \ell + j) \tag{38}
$$

and rescale the propagation time in each invariant subspace \mathcal{H}_N as follows $\bar{\tau}^{(N)} \equiv \sqrt{M^m \tau} \sim \sqrt{(mN)^m \tau}$. The rescaling leaves invariant the quantum amplitudes $\psi_n^{(N)}$ given by Eqs. [\(13\)](#page-3-5)-[\(21\)](#page-5-2), namely, the function $\psi_n^{(N)}(\bar{\tau}^{(N)})$ is given by the same power series expansion of Eqs. [\(13\)](#page-3-5) and [\(21\)](#page-5-2) in the rescaled time $\bar{\tau}^{(N)}$ with the coefficients given by the rescaled β -parameter of Eq. [\(38\)](#page-7-5).

The above scaling transformation can be useful in the quantum optical applications, when the pump mode is the output state of a strong laser. A strong semiclassical coherent pump state, such as the coherent state of Eq. [\(2\)](#page-1-2) with $\alpha \gg 1$, can be expanded over Fock states within the relatively small interval about the average $|N/\langle N\rangle - 1| = \mathcal{O}\left(\alpha^{-1}\right)$, where $\langle N \rangle = \alpha^2$, since the Poisson distribution $\mathcal{P}_N(\alpha) = e^{-\alpha^2} \frac{\alpha^{2N}}{N!}$ $\frac{N!}{N!}$ of the relative weights of the subspaces \mathcal{H}_N is small outside the above interval. This makes the scaling of the propagation time in the invariant subspaces \mathcal{H}_N with significant part of the state norm uniform over all such \mathcal{H}_N to the relative error $\mathcal{O}\left(\alpha^{-1}\right)$.

Let us illustrate the above point using as an example the "generalized squeezing" process, introduced in Ref. [\[24\]](#page-9-21), which correspond to the class of models in Eq. [\(3\)](#page-2-1) with $S = m = 1$ and an arbitrary $k \geq 2$. The corresponding rescaled time is $\bar{\tau} = \sqrt{N\tau}$. The maximum propagation time of the parametric approximation for such models was found in Ref. [\[31\]](#page-10-7): $\tau_c \sim \frac{1}{\alpha}$, when the pump mode is in a strong coherent state $\alpha = \sqrt{\langle N \rangle}$. This translates to our rescaled time $\bar{\tau}_c \sim 1$ for the significant Fock state components of the coherent state.

B. Comparing the exact solution with the Gaussian squeezed state

Consider now the most important case of the secondorder nonlinearity $(k = 2$ in terms of Eq. [\(1\)](#page-1-1) of section [II\)](#page-1-0), which models the spontaneous down conversion process $[17, 18]$ $[17, 18]$ (see also the reviews $[19-21]$).

First of all, let us briefly remind the standard parametric approximation for the strong coherent pump, i.e., in the pump in the state of Eq. [\(2\)](#page-1-2) of section [II](#page-1-0) with $\alpha \gg 1$ (here $\alpha > 0$). In this approximation one assumes that during the evolution the pump remains in the same coherent state and replaces the boson operators of the pump mode by the scalar parameter: $\hat{a} \to \alpha$ and $\hat{a}^{\dagger} \to \alpha$, hence the name of the approximation. The approximate

Hamiltonian, obtained from that of Eq. [\(1\)](#page-1-1) by the above procedure,

$$
\hat{H}_1^{(\alpha)} \equiv \hat{H}_1 \left[\begin{array}{c} \hat{a} \to \alpha \\ \hat{a}^\dagger \to \alpha \end{array} \right] = \hbar \Omega \alpha (\hat{b}^2 + \hat{b}^{\dagger 2}) \tag{39}
$$

is mapped in this case to the generators of the $SU(1, 1)$ group: $K_{-} = \frac{1}{2}\hat{b}^{2}$, $K_{+} = K_{-}^{\dagger}$, and $K_{3} = \frac{1}{2}(\hat{n} + \frac{1}{2})$, with $\hat{n} \equiv \hat{b}^{\dagger} \hat{b}$. The commutators $[K_-, K_+] = 2K_3$ and $[K_3, K_+] = K_+$ allow one to express the unitary evolution exponent as follows

$$
e^{-ir(K_{-}+K_{+})} = e^{-iu(r)K_{+}}e^{-v(r)K_{3}}e^{-iu(r)K_{-}},
$$

$$
r \equiv 2\alpha\Omega t, u = \tanh r, v = 2\ln(\cosh r).
$$
 (40)

Application of the expression on the right hand side of Eq. [\(40\)](#page-8-0) to the vacuum state in the signal mode results in the standard parametric approximation (the squeezed state [\[19](#page-9-17)])

$$
|Sr\rangle \equiv e^{-ir(K_{-}+K_{+})}|0\rangle
$$
\n
$$
= \sqrt{\text{sechr}} \sum_{n=0}^{\infty} {2n \choose n}^{\frac{1}{2}} \left(\frac{-i \tanh r}{2}\right)^{n} |2n\rangle,
$$
\n(41)

where we have denoted by $|2n\rangle$ the Fock state with $2n$ photons.

To compare the above described parametric approximation with the exact solution, let us map the joint state of the pump-signal system onto the invariant subspaces $\mathcal{H}_N \equiv \text{Span}\{|N-n, 2n\rangle, 0 \leq n \leq N\},\$ with $|N-n, 2n\rangle$ denoting the Fock state with $N - n$ photons in the pump mode and $2n$ in the signal mode. Expanding the coherent state over the Fock states $|M\rangle = \frac{(\hat{a}^{\dagger})^M}{\sqrt{M!}}|0\rangle$, as in Eq. [\(2\)](#page-1-2), and rearranging the two summations by introducing a new index $N \equiv M + n$, where *n* is from Eq. [\(41\)](#page-8-1), we get

$$
|\alpha\rangle|Sr\rangle = e^{-\frac{\alpha^2}{2}} \sum_{M=0}^{\infty} \frac{\alpha^M}{\sqrt{M!}} |M\rangle|Sr\rangle
$$

=
$$
e^{-\frac{\alpha^2}{2}} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} \sum_{n=0}^N \widetilde{\psi}_n^{(N)} |N-n, 2n\rangle, \qquad (42)
$$

where the quantum amplitude $\widetilde{\psi}_n^{(N)}$, a parametric approximation to the exact amplitude $\psi_n^{(N)}$ of Eqs. [\(12\)](#page-3-1)-[\(13\)](#page-3-5) of section [III,](#page-2-0) is given as follows

$$
\widetilde{\psi}_n^{(N)} = \left[(N)_n \operatorname{sechr} \binom{2n}{n} \right]^{\frac{1}{2}} \left(\frac{-i \tanh r}{2\alpha} \right)^n, \qquad (43)
$$

where $(N)_n = N(N-1)...(N-n+1)$.

The parametric approximation to the coefficient $\widetilde{\gamma}_n^{(N)}$ of the exact solution Eq. [\(21\)](#page-5-2), for the original quantum Hamiltonian $\hat{H}_1 = \hbar \Omega (\hat{a}^\dagger \hat{b}^2 + \hat{a} \hat{b}^\dagger{}^2)$, can be derived from the relation in Eq. [\(13\)](#page-3-5). Taking into account that in this case $m = 1$ and $q = 0$ $(M = N)$ in Eq. [\(36\)](#page-7-4), giving

$$
\beta_n^{(N)} = (N - n)(2n + 1)(2n + 2), \prod_{l=0}^{n-1} \beta_l^{(N)} = (N)_n (2n)!,
$$
\n(44)

we obtain from Eqs. [\(13\)](#page-3-5) and [\(43\)](#page-8-2) the respective coefficients in the parametric approximation (as functions of the dimensionless time $\tau = \Omega t = \frac{r}{2\alpha}$)

$$
\widetilde{\gamma}_n = \sqrt{\text{sechr}} \frac{\left(\frac{\tanh r}{2\alpha}\right)^n}{n!}.
$$
\n(45)

Observe also the independence of the parametric approximation $\tilde{\gamma}_n$ from the subspace dimension parameter "N". In comparison, the exact coefficient does depend on N. This feature is due to the fact that the parametric approximation is applied for $\alpha \gg 1$, thus only the subspaces \mathcal{H}_n with N satisfying $|N - \alpha^2| = \mathcal{O}(\alpha)$ contribute significantly to the result.

Let us now compare the parametric approximation of Eq. [\(45\)](#page-8-3), applicable for $\alpha \gg 1$, to the exact result in Eq. [\(21\)](#page-5-2) of section [III.](#page-2-0) By the standard Taylor series expansions of the involved elementary functions in Eq. [\(45\)](#page-8-3) we obtain, up to an error on the order $\mathcal{O}(n^2r^4)$,

$$
\widetilde{\gamma}_n = \frac{\tau^n}{n!} \left(1 - \left[\frac{n}{3} + \frac{1}{4} \right] r^2 + \mathcal{O}(n^2 r^4) \right). \tag{46}
$$

At the same time Eq. [\(21\)](#page-5-2) gives the following exact series up to the same order of error

$$
\gamma_n^{(N)} = \frac{\tau^n}{n!} \left(1 - \left[\frac{n}{3} + \frac{1}{4} - \frac{n(n+1)}{4N} \right] \frac{Nr^2}{\alpha^2} + \mathcal{O}(n^2 r^4) \right). \tag{47}
$$

Comparing the expansions in Eqs. (46) and (47) we conclude that the parametric approximation has the following relative error (in the significant interval about $\langle N \rangle = \alpha^2$, see the discussion above)

$$
\frac{\widetilde{\gamma}_n - \gamma_n^{(N)}}{\gamma_n^{(N)}} = \mathcal{O}\left(\frac{n+1}{\alpha}\right),\tag{48}
$$

thus it approximates quite well the quantum amplitudes in Eq. [\(42\)](#page-8-6) for $n \ll \alpha = \sqrt{\langle N \rangle}$.

If the parametric approximation fails to represent a significant part of the norm of the quantum state of the signal (i.e., the higher-order amplitudes $\psi_n^{(N)}$ for $n \sim \sqrt{\langle N \rangle}$, then it fails to approximate the exact solution for such propagation times. Indeed, the normalization of the joint state in Eq. [\(42\)](#page-8-6), as well as the average number of photons in the signal mode $\langle \hat{n} \rangle$, both depend on the behavior of the quantum amplitudes $\widetilde{\psi}_n^{(N)}$ (and, hence $\tilde{\gamma}_n$) for large $n \geq \sqrt{N} \sim \alpha$. Therefore, the parametric approximation may fail for some squeezing parameters $r \geq r_c$, due to the approximation error in Eq. [\(48\)](#page-8-7). The detailed analysis of the applicability conditions of the parametric approximation to the spontaneous down conversion process, in view of its importance for the quantum technology, will be considered elsewhere.

V. CONCLUDING REMARKS

In the present work the exact solution was derived to a wide class of nonlinear models describing interacting bosons, with the immediate application to the state evolution problem as it is posed in the realm of quantum optics, where a pump mode propagating in a nonlinear optical media satisfying the phase matching conditions generates the signal mode(s) which are initially in the vacuum state. The solution to the state evolution problem is given as a series expansion in powers of the propagation time, where a single function, polynomial in the signal mode(s) populations, represents the specific quantum model. The results have immediate application to the nonlinear models in quantum optics, such as the kphoton down conversion process, important for the development of quantum technology.

A number of open problems is left for the future work.

- [1] H. Bethe, On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain, Z. Phys. 71, 205 (1931).
- [2] E. H. Lieb and W. Liniger, Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State, Phys. Rev. 130, 1605 (1963).
- [3] E. H. Lieb and F. Y. Wu, Absence of Mott Transition in an Exact Solution of the Short-Range, One-Band Model in One Dimension, Phys. Rev. Lett. 20, 1445 (1968).
- [4] L. Infeld and T. E. Hull, The Factorization Method, Rev. Mod. Phys. 23, 21 (1953).
- [5] V. G. Bagrov and B. F. Samsonov, Darboux transformation, factorization, and supersymmetry in one-dimensional quantum mechanics, Theor. Math. Phys. 104, 1051 (1995).
- [6] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions (Cambridge University Press, 1993).
- [7] O. B. Zaslavskii, Effective potential for spin—boson systems and quasi-exactly solvable problems, Phys. Lett. A 149, 365 (1990).
- [8] V. V. Ulyanov and O. B. Zaslavskii, New methods in the theory of quantum spin systems, Phys. Rep. 216, 179 (1992).
- [9] V. A. Andreev and O. A. Ivanova, The dynamics of threeboson interaction and algebraic Bethe ansatz, Phys. Lett. A. 171, 145 (1992).
- [10] S. N. Dolya and O. B. Zaslavskii, Quasi-exactly solvable quartic Bose Hamiltonians, J. Phys. A: Math. Gen. 34, 5955 (2001).
- $[11]$ G. Álvarez, F. Finkel, A. González-López, and M. A. Rodríguez, Quasi-exactly solvable models in nonlinear optics, J. Phys. A: Math. Gen. 35, 8705 (2002).
- [12] V. P. Karassiov and A. B.Klimov, An algebraic approach for solving evolution problems in some nonlinear quantum models, Phys. Lett. A 191, 117 (1994).
- [13] V. P. Karassiov, Symmetry approach to reveal hidden coherent structures in Quantum Optics. General outlook and examples, J. Russian Laser Research, 21, 370 (2000).

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- [14] V. P. Karassiov, A. A. Gusev, and S. I. Vinitsky, Polynomial Lie algebra methods in solving the second-harmonic generation model: some exact and approximate calculations, Phys. Lett. A 295, 247 (2002).
- [15] Y.-H. Lee, W.-Li Yang, and Y.-Zh. Zhang, Polynomial algebras and exact solutions of general quantum nonlinear optical models I: two-mode boson systems, J. Phys. A: Math. Theor. 43, 185204 (2010).
- [16] Y.-H. Lee, W.-Li Yang, and Y.-Zh. Zhang, Polynomial algebras and exact solutions of general quantum nonlinear optical models: II. Multi-mode boson systems, J. Phys. A: Math. Theor. 43, 375211 (2010).
- [17] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, Generation of Squeezed States by Parametric Down Conversion, Phys. Rev. Lett. 57, 2520 (1986).
- [18] R. E. Slusher, P. Grangier, A. LaPorta, B. Yurke, and M. J. Potasek, Pulsed Squeezed Light, Phys. Rev. Lett. 59, 2566 (1987).
- [19] R. Loudon and P. L. Knight, Squeezed light, J. Mod. Opt. 34, 709 (1987).
- [20] C. Couteau, Spontaneous parametric down-conversion, Contemporary Physics, 59, 291 (2018).
- [21] C. Zhang, Y.-F. Huang, B.-H. Liu, C.-F. Li, and G.-C. Guo, Spontaneous Parametric Down-Conversion Sources for Multiphoton Experiments, Adv. Quantum Technol. 4, 2000132 (2021).
- [22] W. S. Chang, C. Sabín, P. Forn-Díaz, F. Quijandría, A. M. Vadiraj, I. Nsanzineza, G. Johansson, and C. M. Wilson, Observation of Three-Photon Spontaneous Parametric Down-Conversion in a Superconducting Parametric Cavity, Phys. Rev. X 10, 011011 (2020).
- [23] R. A. Fisher, M. M. Nieto, and V. D. Sandberg, Impossibility of naively generalizing squeezed coherent states, Phys. Rev. A 29, 1107 (1984).
- [24] S. L. Braunstein and R. I. McLachlan, Generalized squeezing, Phys. Rev. A 35, 1659 (1987).
- [25] G. Scharf, Time Evolution of a Quantum Mechanical Maser Model, Annals of Physics, 83, 71 (1974).
- [26] M. Hillery and M.S. Zubary, Path-integral approach to the quantum theory of the degenerate parametric ampli-

fier, Phys. Rev. A 29, 1275 (1984).

- [27] G. Scharf and D. F. Walls, Effect of pump quantization on squeezing in parametric amplifier, Opt. Comm. 50, 245 (1984).
- [28] D. D. Crouch and S. L. Braunstein, Limitations to squeezing in a parametric amplifier due to pump quantum fluctuations, Phys. Rev. A 38, 4696 (1988).
- [29] M. D. Reid and P. D. Drummond, Quantum Correlations of Phase in Nondegenerate Parametric Oscillation, Phys. Rev. Lett. 60, 2731 (1988).
- [30] G. Drobný and I. Jex, Quantum properties of field modes in trilinear optical processes, Phys. Rev. A 46, 499 (1992).
- [31] V. Buzek and G. Drobný, Signal-pump entanglement in quantum , Phys. Rev. A 47, 1237 (1993).
- [32] G. Drobný and V. Buzek, Fundamental limit on energy transfer in k -photon down-conversion, Phys. Rev. A 50 , 3492 (1994).
- [33] M. Hillery, D. Yu, and J. Bergou, Effect of the pump

state on the dynamics of the parametric amplifier, Phys. Rev. A 52, 3209 (1995)

- [34] W. Xing and T. C. Ralph, Pump depletion in optical parametric amplification, Phys. Rev. A 107, 023712 (2023).
- [35] K. Chinni and N. Quesada, Beyond the parametric approximation: Pump depletion, entanglement, and squeezing in macroscopic down-conversion, Phys. Rev. A 110, 013712 (2024).
- [36] R. W. Boyd, Nonlinear Optics (Academic Press, Elsevier 2008).
- [37] C. P. Huang, Computing Powers of Arbitrary Hessenberg Matrices, Linear Algebra Appl. 21, 123 (1978).
- [38] Note that one can apply the theory to study the evolution of the states $|\Psi_{N+1}^{(N)}\rangle$ annihilated by \hat{A}^{\dagger} by changing the state enumeration $n' \equiv N - n$ and $\beta'_n \equiv \beta_{N-n-1}$ in Eq. [\(6\)](#page-2-2).