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## NEW RESULTS SIMILAR TO LAGRANGE'S FOUR-SQUARE THEOREM

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ABSTRACT. In this paper we establish some new results similar to Lagrange's four-square theorem. For example, we prove that any integer  $n > 1$  can be written  $w(5w+1)/2+x(5x+1)/2+y(5y+1)/2+z(5z+1)/2$  with  $w, x, y, z \in \mathbb{Z}$ . Here we state two general results:

(1) If  $a$  and  $b$  are odd integers with  $a > 0, b > -a$  and  $\gcd(a, b) = 1$ , then all sufficient large integers can be written as

$$\frac{w(aw+b)}{2} + \frac{x(ax+b)}{2} + \frac{y(ay+b)}{2} + \frac{z(az+b)}{2}$$

with  $w, x, y, z$  nonnegative integers.

(2) If  $a$  and  $b$  are integers with  $a > 0, b > -a, 2 \mid a$  and  $\gcd(a, b) = 1$ , then all sufficient large integers can be written as

$$w(aw+b) + x(ax+b) + y(ay+b) + z(az+b)$$

with  $w, x, y, z$  nonnegative integers.

### 1. INTRODUCTION

Lagrange's four-square theorem states that any  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  can be written as  $w^2 + x^2 + y^2 + z^2$  with  $w, x, y, z \in \mathbb{Z}$ .

For any integer  $m \geq 3$ , the polygonal numbers of order  $m$  (or  $m$ -gonal numbers) are those nonnegative integers

$$p_m(n) = (m-2) \binom{n}{2} + n = \frac{n((m-2)n - (m-4))}{2} \quad (n = 0, 1, 2, \dots).$$

Those

$$\bar{p}_m(n) = p_m(-n) = \frac{n((m-2)n + (m-4))}{2} \quad (n = 0, 1, 2, \dots)$$

are called second kind  $m$ -gonal numbers, and those  $p_m(x)$  with  $x \in \mathbb{Z}$  are called generalized  $m$ -gonal numbers. Note that

$$p_3(x) = \frac{x(x+1)}{2}, \quad p_4(x) = x^2, \quad p_5(x) = \frac{x(3x-1)}{2} \quad \text{and} \quad p_8(x) = x(3x-2).$$

Fermat claimed that each  $n \in \mathbb{N}$  is a sum of  $m$   $m$ -gonal numbers; this was confirmed by Lagrange, Gauss and Cauchy for the cases  $m = 4, m = 3$ , and  $m \geq 5$ , respectively (cf. [3, 4]).

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In 2016 the author [6] proved that each  $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$  can be written a sum of four generalized octagonal numbers one of which is odd. This is quite similar to Lagrange's four-square theorem.

In [7, 8] the author investigated those tuples  $(a, b, c, d, e, f) \in \mathbb{Z}^6$  with  $a - b, c - d, e - f$  all even such that each  $n \in \mathbb{N}$  can be written as

$$\frac{x(ax+b)}{2} + \frac{y(cy+d)}{2} + \frac{z(ez+f)}{2}$$

with  $x, y, z$  integers or nonnegative integers. Some further results along this direction were proved in [?]. Note that if  $a > b > 0$  and  $a \equiv b \pmod{2}$ , then

$$\left\{ \frac{x(ax+b)}{2} : x \in \mathbb{Z} \right\} \subseteq \mathbb{N}.$$

If  $a > 0, b > -a$  and  $a \equiv b \pmod{2}$ , then

$$\left\{ \frac{x(ax+b)}{2} : x \in \mathbb{N} \right\} \subseteq \mathbb{N}.$$

Motivated by the above work, we establish some new results similar to Lagrange's four-square theorem.

**Theorem 1.1.** *Let  $a$  and  $b$  be positive odd integers with  $a > b$  and  $\gcd(a, b) = 1$ . Then any integer  $n > a^3/2 + ab$  can be written as*

$$\frac{w(aw+b)}{2} + \frac{x(ax+b)}{2} + \frac{y(ay+b)}{2} + \frac{z(az+b)}{2}$$

with  $w, x, y, z \in \mathbb{Z}$ .

For convenience, for  $a, b \in \mathbb{Z}$  with  $a \equiv b \pmod{2}$ , we set

$$S_{a,b} = \left\{ \frac{w(aw+b)}{2} + \frac{x(ax+b)}{2} + \frac{y(ay+b)}{2} + \frac{z(az+b)}{2} : w, x, y, z \in \mathbb{Z} \right\}.$$

Via Theorem 1.1 and some easy numerical computations, we get the following corollary.

**Corollary 1.1.** *For any integer  $n > 1$ , there are  $w, x, y, z \in \mathbb{Z}$  such that*

$$n = \frac{w(5w+1)}{2} + \frac{x(5x+1)}{2} + \frac{y(5y+1)}{2} + \frac{z(5z+1)}{2}.$$

Also,

$$S_{7,1} = \mathbb{N} \setminus \{1, 2, 5\}, \quad S_{7,3} = \mathbb{N} \setminus \{1, 3, 25\}, \quad S_{7,5} = \mathbb{N} \setminus \{5, 23\},$$

$$S_{9,1} = \mathbb{N} \setminus \{1, 2, 3, 6, 7, 11, 35, 37\}, \quad S_{9,5} = \mathbb{N} \setminus \{1, 3, 5, 10, 12, 31, 67\},$$

and

$$S_{9,7} = \mathbb{N} \setminus \{5, 6, 7, 15, 29, 65\}.$$

**Theorem 1.2.** *Let  $a, b \in \mathbb{Z}^+$  with  $a \not\equiv b \pmod{2}$  and  $\gcd(a, b) = 1$ .*

(i) *If  $2 \mid a, n \in \mathbb{N}$  and  $n \geq 2^{1+(-1)^n} a^3 + (3 + (-1)^n)ab$ , then there are  $w, x, y, z \in \mathbb{Z}$  such that*

$$n = w(aw+b) + x(ax+b) + y(ay+b) + z(az+b).$$

(ii) If  $2 \mid b$ , then any integer  $n \geq a^3 + 2ab$  with  $4 \nmid n$  can be written as

$$w(aw + b) + x(ax + b) + y(ay + b) + z(az + b)$$

with  $w, x, y, z \in \mathbb{Z}$ .

For  $a, b \in \mathbb{Z}$ , we define

$$T_{a,b} = S_{2a,2b} = \{w(aw + b) + x(ax + b) + y(ay + b) + z(az + b) : w, x, y, z \in \mathbb{Z}\}.$$

**Corollary 1.2.** *We have*

$$\begin{aligned} T_{4,1} &= \mathbb{N} \setminus \{1, 2, 4, 7, 30\}, \\ \{n \in \mathbb{N} : 4 \nmid n \text{ and } n \notin T_{5,2}\} &= \{1, 2, 5, 11, 18\}, \\ \{n \in \mathbb{N} : 4 \nmid n \text{ and } n \notin T_{5,4}\} &= \{5, 6, 7, 17\}, \\ \max(\mathbb{N} \setminus T_{6,1}) &= 168, \quad \max(\mathbb{N} \setminus T_{6,5}) = 182. \end{aligned}$$

*Remark 1.1.* Sun [6] proved that  $T_{3,2} = \mathbb{N}$ , and Meng and Sun [2, Corollary 1.1] showed that  $T_{4,3} = \{p_{10}(w) + p_{10}(x) + p_{10}(y) + p_{10}(z) : w, x, y, z \in \mathbb{Z}\}$  coincides with  $\mathbb{N} \setminus \{5, 6, 26\}$ . It seems that both  $\mathbb{N} \setminus T_{5,2}$  and  $\mathbb{N} \setminus T_{5,4}$  contain infinitely many multiples of 4.

**Theorem 1.3.** *Let  $a \in \mathbb{Z}^+$  and  $b \in \mathbb{Z}$  with  $b > -a$ ,  $2 \nmid ab$  and  $\gcd(a, b) = 1$ . Let  $n \in \mathbb{N}$  with*

$$n > 2a(7a(a-1)+1) + (7a-2)b + (2a(2a-1)+b)\sqrt{6(2a(a-1)+b)}. \quad (1.1)$$

*Then there are  $w, x, y, z \in \mathbb{N}$  such that*

$$n = \frac{w(aw + b)}{2} + \frac{x(ax + b)}{2} + \frac{y(ay + b)}{2} + \frac{z(az + b)}{2}.$$

**Corollary 1.3.** (i) *We have*

$$\begin{aligned} &\left\{ \frac{w(3w-1)}{2} + \frac{x(3x-1)}{2} + \frac{y(3y-1)}{2} + \frac{z(3z-1)}{2} : w, x, y, z \in \mathbb{N} \right\} \\ &= \mathbb{N} \setminus \{9, 21, 31, 43, 55, 89\} \end{aligned}$$

and

$$\begin{aligned} &\left\{ \frac{w(3w+1)}{2} + \frac{x(3x+1)}{2} + \frac{y(3y+1)}{2} + \frac{z(3z+1)}{2} : w, x, y, z \in \mathbb{N} \right\} \\ &= \mathbb{N} \setminus \{1, 3, 5, 10, 12, 20, 25, 27, 38, 53, 65, 153, 165\}. \end{aligned}$$

(ii) *Any integer  $n \geq 884$  can be written as*

$$\frac{w(5w-1)}{2} + \frac{x(5x-1)}{2} + \frac{y(5y-1)}{2} + \frac{z(5z-1)}{2}$$

*with  $w, x, y, z \in \mathbb{N}$ , and any integer  $n \geq 776$  can be written as*

$$\frac{w(5w+1)}{2} + \frac{x(5x+1)}{2} + \frac{y(5y+1)}{2} + \frac{z(5z+1)}{2}$$

*with  $w, x, y, z \in \mathbb{N}$ . Also, any integer  $n \geq 1004$  can be written as*

$$\frac{w(5w-3)}{2} + \frac{x(5x-3)}{2} + \frac{y(5y-3)}{2} + \frac{z(5z-3)}{2}$$

with  $w, x, y, z \in \mathbb{N}$ , and any integer  $n \geq 786$  can be written as

$$\frac{w(5w+3)}{2} + \frac{x(5x+3)}{2} + \frac{y(5y+3)}{2} + \frac{z(5z+3)}{2}$$

with  $w, x, y, z \in \mathbb{N}$ .

**Theorem 1.4.** Let  $a \in \mathbb{Z}^+$  and  $b \in \mathbb{Z}$  with  $b > -a$ ,  $2 \mid a$  and  $\gcd(a, b) = 1$ . If  $n$  is an odd integer greater than

$$4a(7a(a-1)+1) + 2(7a-2)b + 2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)},$$

or  $n$  is an even integer with  $n/4$  greater than

$$28a^3 - 14a^2 + a + (7a-1)b + (2a(4a-1)+b)\sqrt{3(2a(2a-1)+b)},$$

then we can write  $n$  as

$$w(aw+b) + x(ax+b) + y(ay+b) + z(az+b)$$

with  $w, x, y, z \in \mathbb{N}$ .

**Corollary 1.4.** Any integer  $n \geq 2283$  can be written as  $w(4w+1) + x(4x+1) + y(4y+1) + z(4z+1)$  with  $w, x, y, z \in \mathbb{N}$ .

We are going to prove Theorems 1.1-1.2 and Theorems 1.3-1.4 in Sections 2 and 3 respectively. In Section 5, we pose some open conjectures.

## 2. PROOFS OF THEOREMS 1.1-1.2

We need the following lemma motivated by [2, Lemma 2.4] which extends Cauchy's lemma (cf. [4, p. 31, Lemma 1.12]).

**Lemma 2.1.** Let  $c, d \in \mathbb{N}$  with  $4 \nmid c$ ,  $c \equiv d \pmod{2}$  and  $4c > d^2$ . Then there are  $s, t, u, v \in \mathbb{Z}$  such that

$$s^2 + t^2 + u^2 + v^2 = c \text{ and } s + t + u + v = d. \quad (2.1)$$

*Proof.* We distinguish two cases.

*Case 1.*  $2 \nmid cd$ .

As  $4c - d^2 \equiv 3 \pmod{8}$ , by the Gauss-Legendre theorem on sums of three squares (cf. [4, Section 1.5]), there are odd integers  $x, y, z$  such that  $4c - d^2 = x^2 + y^2 + z^2$ . Since  $z$  or  $-z$  is congruent to the odd number  $d + x + y$ , without loss of generality we may assume that  $d + x + y + z \equiv 0 \pmod{4}$ . Then the four numbers

$$s = \frac{d+x+y+z}{4}, t = \frac{d+x-y-z}{4}, u = \frac{d-x+y-z}{4}, v = \frac{d-x-y+z}{4} \quad (2.2)$$

are integers and they satisfy (2.1).

*Case 2.*  $2 \parallel c$  and  $2 \mid d$ .

Write  $c = 2c_0$  and  $d = 2d_0$  with  $c_0, d_0 \in \mathbb{Z}$ . Then  $2c_0 - d_0^2 \equiv 1, 2 \pmod{4}$ . By the Gauss-Legendre theorem, we have  $2c_0 - d_0^2 = x_0^2 + y_0^2 + z_0^2$  for some  $x_0, y_0, z_0 \in \mathbb{Z}$ . Without loss of generality, we may assume that  $x_0 \equiv d_0 \pmod{2}$  and  $y_0 \equiv z_0 \pmod{2}$ . Set  $x = 2x_0$ ,  $y = 2y_0$  and  $z = 2z_0$ . Then

$4c - d^2 = 4(2c_0 - d_0^2) = x^2 + y^2 + z^2$ , and the numbers  $s, t, u, v$  given by (2.2) are integers satisfying (2.1).

In view of the above, we have completed the proof of Lemma 2.1.

It is easy to see that our following theorem implies Theorems 1.1-1.2.

**Theorem 2.1.** *Let  $a, b \in \mathbb{Z}^+$  with  $a > b$  and  $\gcd(a, b) = 1$ .*

(i) *Let  $n \in \mathbb{N}$  with  $n \geq a^3 + 2ab$ . If  $a(n - b)$  or  $b(n - a)$  is odd, then we can write  $n$  as*

$$w(aw + b) + x(ax + b) + y(ay + b) + z(az + b)$$

with  $w, x, y, z \in \mathbb{Z}$ .

(ii) *Suppose that  $a \not\equiv b \pmod{2}$ . Let  $n$  be an integer such that  $n \equiv 0 \pmod{2}$  and  $n \geq 4a(a^2 + b)$  if  $2 \mid a$ , and  $n \equiv 2 \pmod{4}$  and  $n \geq a^3 + 2ab$  if  $2 \nmid a$ . Then we can write  $n$  as*

$$w(aw + b) + x(ax + b) + y(ay + b) + z(az + b)$$

with  $w, x, y, z \in \mathbb{Z}$ .

**Proof of Theorem 2.1(i).** As  $n \geq a^3 + 2ab$ , we have  $an + b^2 \geq (a^2 + b)^2$  and hence  $\sqrt{an + b^2} - b \geq a^2$ . In view of the first assertion of Lemma 2.1, it suffices to find positive odd integers  $c$  and  $d$  with  $c > d^2/4$  such that  $n = ac + bd$ .

*Case 1.*  $2 \nmid a(n - b)$ .

In this case, we have  $\gcd(a, 2b) = 1$ . Thus, for certain  $r \in \{1, \dots, 2b\}$  the integer

$$c = \left\lfloor \frac{an + 2b^2 - 2b\sqrt{an + b^2}}{a^2} \right\rfloor + r$$

satisfies the congruence  $n - ac \equiv b \pmod{2b}$ . Hence  $n - ac = bd$  for an odd integer  $d$ . As  $bd \equiv b \not\equiv n \pmod{2}$ , we also have  $2 \nmid c$ . Note that  $c > 0$  since

$$(an + 2b^2)^2 - 4b^2(an + b^2) = a^2n^2 \geq 0.$$

Note that  $d = (n - ac)/b \in \mathbb{N}$  because

$$c \leq \frac{an + 2b^2 - 2b\sqrt{an + b^2}}{a^2} + 2b = \frac{n}{a} + \frac{2b}{a^2}(a^2 + b - \sqrt{an + b^2}) \leq \frac{n}{a}.$$

Since  $a^2c > an + 2b^2 - 2b\sqrt{an + b^2} = (\sqrt{an + b^2} - b)^2$ , we have

$$(a\sqrt{c} + b)^2 > an + b^2$$

and hence  $a^2c + 2ab\sqrt{c} > an$ . Therefore  $bd = n - ac < 2b\sqrt{c}$  and hence  $d^2 < 4c$ .

*Case 2.*  $2 \nmid b(n - a)$ .

Since  $\gcd(2a, b) = 1$ , for certain  $r \in \{1, \dots, 2a\}$  the number

$$d := \left\lfloor \frac{2}{a}(\sqrt{an + b^2} - b) \right\rfloor - r \geq 0$$

satisfies the congruence  $n - bd \equiv a \pmod{2a}$ . Write  $n - bd = ac$  with  $c$  an odd integer. As  $ac \equiv a \not\equiv n \pmod{2}$ , we must have  $2 \nmid d$ . Since  $d < 2(\sqrt{an + b^2} - b)/a$ , we have  $4an + 4b^2 > (ad + 2b)^2$  and hence  $ac + bd = n > ad^2/4 + bd$ . Therefore  $c > d^2/4$ .

In view of the above, we have completed our proof of Theorem 2.1(i).  $\square$

**Proof of Theorem 2.1(ii).** In view of Lemma 2.1, it suffices to find  $c, d \in \mathbb{N}$  with  $2 \parallel c$ ,  $2 \mid d$ ,  $4c > d^2$  and  $n = ac + bd$ .

We distinguish two cases.

*Case 1.*  $2 \mid a$  and  $2 \nmid b$ .

In this case,  $n \geq 4a(a^2 + b)$  and hence  $\sqrt{an + b^2} \geq 2a^2 + b$ . As  $\gcd(2a, b) = 1$ , for certain  $r \in \{1, \dots, 2a\}$  the number

$$d_0 := \left\lfloor \frac{\sqrt{an + b^2} - b}{a} \right\rfloor - r \geq 0$$

satisfies the congruence  $bd_0 \equiv n/2 - a \pmod{2a}$ . Write  $n/2 - bd_0 = ac_0$  for  $c_0$  an odd integer. Note that  $c = 2c_0 \equiv 2 \pmod{4}$  and  $d = 2d_0 \equiv 0 \pmod{2}$ . As  $d_0 < (\sqrt{an + b^2} - b)/a$ , we have

$$an > (ad_0 + b)^2 - b^2 = a^2d_0^2 + 2abd_0$$

and hence

$$ac + bd = n > ad_0^2 + 2bd_0 = a \frac{d^2}{4} + bd.$$

Thus  $4c > d^2$ .

*Case 2.*  $2 \nmid a$  and  $2 \mid b$ .

In this case,  $n \equiv 2 \pmod{4}$ . As  $(an + 2b^2)^2 > 4b^2(an + b^2)$ , we have  $an + 2b^2 > 2b\sqrt{an + b^2}$ . As  $\gcd(a, b) = 1$ , for certain  $r \in \{1, \dots, b\}$  the number

$$c_0 := \left\lfloor \frac{an + 2b^2 - 2b\sqrt{an + b^2}}{2a^2} \right\rfloor + r > 0$$

satisfies the congruence  $ac_0 \equiv n/2 \pmod{b}$ . Note that  $c_0 \equiv ac_0 \equiv n/2 \equiv 1 \pmod{2}$ . Write  $n/2 = ac_0 + bd_0$  with  $d_0 \in \mathbb{Z}$ . Then  $c = 2c_0 \equiv 2 \pmod{4}$  and  $d = 2d_0 \equiv 0 \pmod{2}$ . As  $n \geq a(a^2 + 2b)$ , we have  $an + b^2 \geq (a^2 + b)^2$  and hence

$$c_0 \leq \frac{an + 2b^2 - 2b\sqrt{an + b^2}}{2a^2} + b \leq \frac{an + 2b^2 - 2b(a^2 + b)}{2a^2} + b = \frac{n}{2a} = c_0 + \frac{bd_0}{a}.$$

So  $d = 2d_0 \geq 0$ . Since

$$c = 2c_0 > 2 \times \frac{an + 2b^2 - 2b\sqrt{an + b^2}}{2a^2} = \frac{(\sqrt{an + b^2} - b)^2}{a^2},$$

we have  $an + b^2 < (a\sqrt{c} + b)^2 = a^2c + 2ab\sqrt{c} + b^2$  and hence  $ac + bd = n < ac + 2b\sqrt{c}$ . So  $d < 2\sqrt{c}$  and hence  $4c > d^2$ .

In view of the above, we have completed the proof of Theorem 2.1(ii).  $\square$

## 3. PROOFS OF THEOREMS 1.3-1.4

The following lemma is essentially [2, Lemma 2.4].

**Lemma 3.1.** *Let  $c, d \in \mathbb{N}$  with  $c \equiv d \pmod{2}$  and  $4 \nmid c$ . If*

$$4c > d^2 \text{ and } 3c < d^2 + 2d + 4,$$

*then there are  $w, x, y, z \in \mathbb{N}$  such that*

$$w^2 + x^2 + y^2 + z^2 = c \text{ and } w + x + y + z = d. \quad (3.1)$$

*Remark 3.1.* Lemma 3.1 in the case  $2 \nmid cd$  is Cauchy's lemma used to prove Cauchy's polynomial number theorem. In fact, when  $3c < d^2 + 2d + 4$  it is easy to show that the four numbers in (2.2) are greater than  $-1$  (cf. [4, p. 31] or the proof of [2, Lemma 2.4]).

**Lemma 3.2.** *Suppose that  $ac + bd = n \geq 4a/3$  with  $a \in \mathbb{Z}^+$ ,  $b, c, d \in \mathbb{Z}$  and  $b > -a$ . Then*

$$d > 0 \text{ and } \frac{d^2}{4} < c < \frac{d^2 + 2d + 4}{3} \quad (3.2)$$

*if and only if*

$$\frac{\sqrt{3a(n-a) + 3ab + \frac{9}{4}b^2} - a - 3b/2}{a} < d < \frac{2}{a}(\sqrt{an+b} - b). \quad (3.3)$$

*Proof.* Clearly,

$$3a(n-a) + 3ab + \frac{9}{4}b^2 \geq a^2 + 3ab + \frac{9}{4}b^2 = \left(a + \frac{3b}{2}\right)^2 \geq 0.$$

Thus  $d > 0$  if (3.3) holds.

Now assume  $d > 0$ . Note that

$$n > a\frac{d^2}{4} + bd \iff 4an > (ad + 2b)^2 - 4b^2 \iff d < \frac{2}{a}(\sqrt{an+b^2} - b).$$

As

$$ad + a + \frac{3b}{2} > ad + a - \frac{3a}{2} = a\left(d - \frac{1}{2}\right) > 0,$$

we have

$$\begin{aligned} n &< \frac{a}{3}(d^2 + 2d + 4) + bd \\ \iff 3an &< a^2d^2 + ad(2a + 3b) + 4a^2 = \left(ad + a + \frac{3b}{2}\right)^2 + 3a^2 - 3ab - \frac{9a^2}{b} \\ \iff \sqrt{3an - 3a^2 + 3ab + \frac{9}{4}b^2} &< ad + a + \frac{3b}{2} \\ \iff d &> \frac{\sqrt{3a(n-a) + 3ab + 9b^2/4} - a - 3b/2}{a}. \end{aligned}$$

As  $n = ac + bd$ , by the above,  $d^2/4 < c < (d^2 + 2d + 4)/3$  if and only if (3.3) holds.

In view of the above, we have completed the proof.  $\square$

Clearly, Theorem 1.3 follows from the following theorem.

**Theorem 3.1.** *Let  $a \in \mathbb{Z}^+$  and  $b \in \mathbb{Z}$  with  $b > -a$  and  $\gcd(a, b) = 1$ . Let  $n \in \mathbb{N}$  with*

$$n > 4a(7a(a-1)+1)+2(7a-2)b+2(2a(2a-1)+b)\sqrt{6(2a(a-1)+b)}. \quad (3.4)$$

*Suppose that both  $n - a$  and  $b$  are odd. Then we can write  $n$  as*

$$w(aw + b) + x(ax + b) + y(ay + b) + z(az + b)$$

*with  $w, x, y, z \in \mathbb{N}$ .*

*Proof.* Note that

$$2a(2a - 1) + b > 2a(a - 1) + b \geq 2a(a - 1) - a + 1 = a(2a - 3) + 1 \geq 0.$$

If  $a > 1$ , then by (3.4) we have

$$\frac{n}{2} > 2(7a^2 - 7a + 1) + (7a - 2)b > 2(7a^2 - 7a + 1) - a(7a - 2) = a(7a - 12) + 2 > a.$$

When  $a = 1$ , we have  $b \geq 1$  and hence

$$\frac{n}{2} > 2(7a^2 - 7a + 1) + (7a - 2)b \geq 14a^2 - 14a + 2 + 7a - 2 = 14a^2 - 7a > a.$$

Thus  $n > 2a > 4a/3$ .

In view of Lemma 3.1, it suffices to find positive odd integers  $c$  and  $d$  such that

$$\frac{d^2}{4} < c < \frac{d^2 + 2d + 4}{3} \text{ and } n = ac + bd.$$

Motivated by Lemma 3.2, we consider the interval

$$I = \left( \frac{\sqrt{3a(n-a) + 3ab + 9b^2/4} - a - 3b/2}{a}, \frac{2}{a} \left( \sqrt{an + b^2} - b \right) \right). \quad (3.5)$$

Clearly the length of  $I$  is greater than  $2a$  if and only if

$$2\sqrt{an + b^2} - a(2a - 1) - \frac{b}{2} > \sqrt{3an - 3a^2 + 3ab + \frac{9}{4}b^2}. \quad (3.6)$$

As  $2a(2a - 1) + b > 0$ , we have

$$\begin{aligned} 2\sqrt{an + b^2} &\geq a(2a - 1) + \frac{b}{2} \\ \iff 4(an + b^2) &\geq a^2(2a - 1)^2 + (2a - 1)ab + \frac{b^2}{4}. \end{aligned}$$

If  $b \geq a(2a - 1)$ , then

$$4b^2 \geq a^2(2a - 1)^2 + a(2a - 1)b + 2b^2 > a^2(2a - 1)^2 + (2a - 1)ab + \frac{b^2}{4}.$$

When  $b < a(2a - 1)$ , we have  $a > 1$  and

$$\frac{n}{2} > 2a(7a^2 - 7a + 1) + (7a - 2)b$$



$$\begin{aligned}
&> 2a(7a^2 - 7a + 1) - a(7a - 2) = a(14a^2 - 21a + 4) \\
&> a \frac{(2a - 1)^2}{4}
\end{aligned}$$

and hence

$$4an > 2a^2(2a - 1)^2 > a^2(2a - 1)^2 + a(2a - 1)b.$$

Thus the left-hand side of (3.6) is nonnegative. Let  $t = \sqrt{an + b^2}$ . Then

(3.6) holds

$$\begin{aligned}
&\iff \left(2t - a(2a - 1) - \frac{b}{2}\right)^2 > 3an - 3a^2 + 3ab + \frac{9}{4}b^2 \\
&\iff 4t^2 - 2t(2a(2a - 1) + b) + \left(a(2a - 1) + \frac{b}{2}\right)^2 > 3an - 3a^2 + 3ab + \frac{9}{4}b^2 \\
&\iff (t - (2a(2a - 1) + b))^2 > 6a^2(2a(a - 1) + b),
\end{aligned}$$

and

$$\begin{aligned}
&\left(2a(2a - 1) + b + \sqrt{6a^2(2a(a - 1) + b)}\right)^2 \\
&\quad - 2(2a(2a - 1) + b)\sqrt{6a^2(2a(a - 1) + b)} \\
&= (2a(2a - 1) + b)^2 + 6a^2(2a(a - 1) + b) \\
&= 4a^2(7a^2 - 7a + 1) + 14a^2b - 4ab + b^2 \\
&< b^2 + a(n - 2(2a(2a - 1) + b))\sqrt{6(2a(a - 1) + b)}
\end{aligned}$$

by (3.4). Thus

$$t = \sqrt{an + b^2} > 2a(2a - 1) + b + \sqrt{6a^2(2a(a - 1) + b)}$$

and hence (3.6) does hold.

As the length of the interval  $I$  is greater than  $2a$ , there is an integer  $d \in I$  with  $n - bd \equiv a \pmod{2a}$ . Write  $n - bd = ac$  with  $c$  an odd integer. As  $ac \equiv a \not\equiv n \pmod{2}$ , we also have  $2 \nmid d$ . As  $d \in I$ , we have (3.2) by Lemma 3.2. So, with the aid of Lemma 2.1, we have the desired result.  $\square$

**Theorem 3.2.** *Let  $a \in \mathbb{Z}^+$  and  $b \in \mathbb{Z}$  with  $2 \mid a$ ,  $\gcd(a, b) = 1$  and  $b > -a$ . Let  $n \in \mathbb{N}$  with  $2 \mid n$  and*

$$\frac{n}{4} > 28a^3 - 14a^2 + a + (7a - 1)b + (2a(4a - 1) + b)\sqrt{3(2a(2a - 1) + b)}. \quad (3.7)$$

Then there are  $w, x, y, z \in \mathbb{N}$  such that

$$n = w(aw + b) + x(ax + b) + y(ay + b) + z(az + b).$$

*Proof.* As  $b > -a$ , we have

$$\frac{n}{4} > 28a^3 - 14a^2 + a - a(7a - 1) = 8a^3 + a(20a^2 - 21a + 2) > 8a^3.$$

For the interval  $I$  given by (3.5), its length is greater than  $4a$  if and only if

$$2\sqrt{an + b^2} - a(4a - 1) - \frac{b}{2} > \sqrt{3a(n - a) + 3ab + \frac{9}{4}b^2}. \quad (3.8)$$

As  $2a(4a - 1) + b > 0$ , we have

$$\begin{aligned} 2\sqrt{an + b^2} &\geq a(4a - 1) + \frac{b}{2} \\ \iff 4(an + b^2) &\geq a^2(4a - 1)^2 + (4a - 1)ab + \frac{b^2}{4}. \end{aligned}$$

If  $b \geq a(4a - 1)$ , then

$$4b^2 \geq a^2(4a - 1)^2 + a(4a - 1)b + 2b^2 > a^2(4a - 1)^2 + (4a - 1)ab + \frac{b^2}{4}.$$

When  $b < a(4a - 1)$ , we have

$$4an > 4a \times 8a^3 > 2a^2(4a - 1)^2 > a^2(4a - 1)^2 + a(4a - 1)b.$$

Thus the left-hand side of (3.8) is nonnegative. Let  $t = \sqrt{an + b^2}$ . Then

(3.8) holds

$$\begin{aligned} \iff \left(2t - a(4a - 1) - \frac{b}{2}\right)^2 &> 3a(n - a) + 3ab + \frac{9}{4}b^2 \\ \iff 4t^2 - 2t(2a(4a - 1) + b) + \left(a(4a - 1) + \frac{b}{2}\right)^2 &> 3a(n - a) + 3ab + \frac{9}{4}b^2 \\ \iff (t - (2a(4a - 1) + b))^2 &> 12a^2(2a(2a - 1) + b), \end{aligned}$$

and

$$\begin{aligned} &\left(2a(4a - 1) + b + \sqrt{12a^2(2a(2a - 1) + b)}\right)^2 \\ &\quad - 4a(2a(4a - 1) + b)\sqrt{3(2a(2a - 1) + b)} \\ &= (2a(4a - 1) + b)^2 + 12a^2(2a(2a - 1) + b) \\ &= b^2 + 28a^2(4a^2 - 2a + b) + 4a^2 - 4ab \\ &< b^2 + an - 4a(2a(4a - 1) + b)\sqrt{3(2a(2a - 1) + b)} \end{aligned}$$

by (3.7). Thus

$$t = \sqrt{an + b^2} > 2a(4a - 1) + b + \sqrt{12a^2(2a(2a - 1) + b)}$$

and hence (3.8) holds. So the length of the interval  $I$  is greater than  $4a$ .

As  $\gcd(b, 4a) = 1$ , there is an integer  $d \in I$  with  $n - bd \equiv 2a \pmod{4a}$ . Write  $n - bd = ac$  with  $c \equiv 2 \pmod{4}$ . As both  $a$  and  $n$  are even but  $b$  is odd, we must have  $2 \mid d$ . As  $d \in I$ , we have (3.2) by Lemma 3.2. Now, by Lemma 3.1 there are  $w, x, y, z \in \mathbb{N}$  satisfying (3.1) Hence

$$n = ac + bd = w(aw + b) + x(ax + b) + y(ay + b) + z(az + b).$$

This concludes our proof.  $\square$

**Proof of Theorem 1.4.** As  $2 \mid a$  and  $2 \nmid b$ , combining Theorems 3.1 and ?? we immediately obtain the desired result.  $\square$

#### 4. SOME CONJECTURES

Motivated by Corollary 1.1, we pose the following conjectures.

**Conjecture 4.1.** *Any integer  $n > 1$  can be written as  $x(5x + 1) + y(5y + 1)/2 + z(5z + 1)/2$  with  $x, y, z \in \mathbb{Z}$ .*

**Conjecture 4.2.** *Any integer  $n > 51$  can be written as*

$$\frac{w(5w - 1)}{2} + \frac{x(5x - 1)}{2} + \frac{y(5y + 1)}{2} + \frac{z(5z + 1)}{2}$$

*with  $w, x, y, z \in \mathbb{N}$ .*

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