A new lower bound for the multicolor Ramsey number $r_k(K_{2,t+1})$

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Abstract

In this short note, we provide a new infinite family of $K_{2,t+1}$-free graphs for each prime power t. Using these graphs, we show that it is possible to partition the edges of K_n into parts, such that each part is isomorphic to our $K_{2,t+1}$-free graph. This yields an improved lower bound to the multicolor Ramsey number $r_k(K_{2,t+1})$ when k and t are powers of the same prime. For these values of k and t, our coloring implies that

$$tk^2 + 1 \le r_k(K_{2,t+1}) \le tk^2 + k + 2.$$

where the upper bound is due to Chung and Graham.

1 Introduction

Let $k \ge 1$ be an integer. The Ramsey number $r_k(F)$ denotes the smallest number of vertices n such that in any k-coloring of the complete graph K_n, there exists a monochromatic copy of F (i.e. F is a subgraph of one of the color classes). Ramsey theory has received much attention over the last century with a number of ground breaking results appearing the last few years [1, 11, 3, 13, 15]. We contribute to this area by improving the lower bound for $r_k(K_{2,t+1})$ for infinitely many k and t.

Chung and Graham [6] proved that

$$r_k(K_{2,t+1}) \le \begin{cases} k^2 + k + 1 & \text{if } t = 1 \\ tk^2 + k + 2 & \text{if } t > 1 \end{cases}$$

They conjectured that when $t >> S$, $r_k(K_{s,t}) = (t-1)k^s + o(k^s)$. Chung [5] in her dissertation had demonstrated that when $s = 2$,

$$\lim_{t\to\infty} \frac{r_k(K_{2,t})}{t} = k^2$$

implying the correct asymptotics up the constant term of t in the conjecture. However, it was not until Axenovich, Füredi, and Mubayi [1] that the conjecture was verified for $s = 2$. Their coloring relied on the construction of Füredi of a dense $K_{2,t}$-free graphs [7]. In particular, the result from [1] implies that given any integer t, and there are infinitely many k for which a graph of order tk^2 can be colored with $k + O(\sqrt{tk}\log(tk))$ colors such that each color class is $K_{2,t+1}$-free. For large k, this roughly implies the lower bound

$$tk^2 - c_t k^{3/2}\log(k) \leq r_k(K_{2,t+1}) \tag{1}$$

for some constant c_t dependent on t. The authors leverage a prime density argument to demonstrate that the leading term asymptotics are correct. In this paper, we give an explicit coloring which removes the lower order term in (1) entirely when k and t are any powers of the same prime.

For $s = 2$ and $t = 1$, incremental progress has been made by Chung [4], Irving [9], and most recently by Lazebnik and Woldar [11]. In [11], the authors gave an explicit coloring which showed that $r_k(K_{2,2}) \geq k^2 + 2$ when k is an odd prime power. Thus, when k is an odd prime power we have the bounds

$$k^2 + 2 \leq r_k(K_{2,2}) \leq k^2 + k + 1. \tag{2}$$

We will demonstrate that lower bound extends to the case when k is an even prime power as well.

Let $ex(n, F)$ denote the maximum number of edges in an F-free graph on n vertices. This function is called the Turán number of F. The Ramsey number $r_k(F)$ and Turán number are related in the following way: If $r_k(F) = n + 1$, then

$$ex(n, F) \geq \frac{1}{k}\binom{n}{2}.$$

Füredi's construction of $K_{2,t+1}$-free graphs gives the best-known lower bound for $ex(n, K_{2,t+1})$ [7]. Together with the best-known upper bound due to Hyltén-Cavallius [8] (which is a small improvement over the famous Kövári, Sós, and Turán theorem [10]), we have that

$$\frac{\sqrt{t}}{2}n^{3/2} - \frac{n}{2} < ex(n, K_{2,t+1}) < \frac{1}{2}n\sqrt{tn - t + 1/4} + \frac{n}{4}$$

for infinitely many n. The lower order term in the lower bound can be improved for specific n by performing a case by case analysis of Füredi's construction. A prime density argument implies that the leading term asymptotics in this construction are correct for all n. Until now, Füredi's construction is the only know construction capable of producing infinite families of $K_{2,t+1}$-free graphs with the right edge count and for infinitely many t.

In this paper, we generalize the approach of Lazebnik and Woldar [11] to give two main results. The first is a new construction of $K_{2,t+1}$-free graphs which is yields an equal or slightly improved lower bound for $ex(n, K_{2,t+1})$ over Füredi's construction when t is a prime power.

Theorem 1.1. *Let q and t be powers of the same prime with $t < q$. If $n = q^2/t$, then*

$$\frac{\sqrt{t}}{2} n^{3/2} - \frac{\sqrt{tn}}{2} \leq ex(n, K_{2,t+1}).$$

Second, we use this construction to give a coloring that gives a result similar in spirit to (2). Just like in [11], a nice property of our coloring is all the color classes are all isomorphic to our construction which yields Theorem 1.1. Such decompositions of K_n are exceedingly rare, and algebraically defined graphs seem to be the only known class of graphs which seem to be able to consistently produce such decompositions [12].

Theorem 1.2. *Let t and k be powers of the same prime, then*

$$tk^2 + 1 \leq r_k(K_{2,t+1}).$$

An observation will also lead us to extending the result in [11] to even prime powers.

Theorem 1.3. *Let $k = 2^e$. Then*

$$k^2 + 2 \leq r_k(C_4).$$

2 A $K_{2,t+1}$-free graph

Let p be a prime and $q = p^e$, and $\mathbb{F}_q$ be a finite field of order q. Given a positive integer $d < e$, let $f(x)$ be an $\mathbb{F}_p$-linear polynomial (i.e. $f(x + y) = f(x) + f(y)$) in $\mathbb{F}_q[x]$ with p^d roots. We note that such a polynomial always exists as it can be thought of as a linear transformation of rank $e - d$ on the vector space $\mathbb{F}_p^e$. Denote the range of $f(x)$ by the set $R_f = \{f(x) : x \in \mathbb{F}_q\}$. Since $f(x)$ is $\mathbb{F}_p$-linear, we know that R_f is closed under addition and that $|R_f| = p^{e-d}$.

Define the graph Γ_f with vertex set $V = \mathbb{F}_q \times R_f$. Two distinct vertices $(v_1, v_2), (w_1, w_2) \in V$ are adjacent if and only if

$$v_2 + w_2 = f(v_1 w_1). \tag{3}$$

Let $t = p^d$ and note that $|V| = q|R_f| = q^2/t$.

Claim: Γ_f is $K_{2,t+1}$-free.

Proof. Observe that given any vertex $v = (v_1, v_2)$, and any $\alpha \in \mathbb{F}_q$, $\alpha \neq v_1$, the vertex (v_1, v_2) has a unique neighbor with first coordinate α. In particular, using (3), we have that v is adjacent to $(\alpha, f(\alpha v_1) - v_2)$.

Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$ be two vertices in Γ, we will show that they cannot have more than t common neighbors. If $v_1 = w_1$, then v and w share no common neighbors because if they did, it would imply that this common neighbor is adjacent to two distinct vertices with same first coordinate, a contradiction. So we suppose that $v_1 \neq w_1$.

Let $x = (x_1, x_2) \in V$ be a common neighbor of both v and w. Then the equations

$$x_2 + v_2 = f(x_1 v_1) \tag{4}$$
$$x_2 + w_2 = f(x_1 w_1) \tag{5}$$

must be satisfied by (x_1, x_2). Subtracting equation (5) from equation (4), we obtain that

$$v_2 - w_2 = f(x_1(v_1 - w_1)). \tag{6}$$

Thus if x is adjacent to v and w, it's first coordinate must be a solution to (6). Since $v_1 \neq w_1$, then we know that the polynomial on the right hand side is not 0. Let $a = v_1 - w_1$ and $b = v_2 - w_2$, then x_1 satisfies

$$f(ax_1) = b$$

Since $b \in R_f$, then we know that there exists at least one solution. On the other hand, $f(x)$ is $\mathbb{F}_p$-linear over $\mathbb{F}_q$ and has $t = p^d$ roots and so must in-fact be a t-to-1 mapping. Therefore, there are exactly t different values of x_1 which satisfy (6), and so v and w have at most t common neighbors. Thus Γ_f is $K_{2,t+1}$-free. □

To determine the exact number of edges in Γ_f we need to count the number of vertices of degree q and degree $q-1$. Note that the beginning of the proof of the claim above implies that each vertex has degree at least $q-1$. To account for the remaining edges, note that adjacency relation (3) implies that some vertices (v_1, v_2) are adjacent to distinct vertices with same first coordinate, (v_1, v_3). In particular, we have (v_1, v_2) is adjacent to (v_1, v_3) if and only if

$$v_2 + v_3 = f(v_1^2).$$

Such vertices have degree q. A vertex has degree $(q-1)$ if the equation above only holds when $v_2 = v_3$. So we count the number of solutions in V to the equation.

$$2v_2 = f(v_1^2)$$

- If q is even, then (v_1, v_2) is a solution of $0 = f(v_1^2)$ if and only if v_1^2 is a root of $f(x)$. Since x^2 is a permutation of $\mathbb{F}_q$ then we know that there are t such values of v_1. The second coordinate v_2 belongs to R_f, so there are $t|R_f| = q$ solutions.

- If q is odd, then we have $2v_2 = f(v_1^2)$. For each $v_1 \in \mathbb{F}_q$, there exists a unique v_2 which solves the equation. Thus there are q solutions to this equation as well.

Consequently, Γ has q vertices of degree $(q-1)$ and $(q^2/t) - q$ vertices of degree q. Let $n = |V| = q^2/t$, then the number of edges in Γ_f is

$$\frac{1}{2}\left(q\left(\frac{q^2}{t} - q\right) + (q-1)q\right) = \frac{q^3}{2t} - \frac{q}{2} = \frac{\sqrt{t}}{2}n^{3/2} - \frac{\sqrt{tn}}{2}$$

This finishes the proof of Theorem 1.1.

3 A lower bound for $r_k(K_{2,t+1})$

Let p be a prime, $q = p^e$, and $f(x) \in \mathbb{F}_q[x]$ be an $\mathbb{F}_p$-linear polynomial with $t = p^d$ roots (for some $d < e$). We now demonstrate how the graphs Γ_f can be used to partition the edges of K_n, when $n = q^2/t$, into $k = |R_f|$ color classes, such that each class is $K_{2,t+1}$-free.

Denote the k colors by the elements of R_f and label the vertices of K_n by the set $V = \mathbb{F}_q \times R_f$. Assign the color $\alpha \in R_f$ to the edge between (v_1, v_2) and (w_1, w_2) if

$$v_2 + w_2 + \alpha = f(v_1 w_1).$$

Clearly, no edge can be assigned two different colors in this way. Furthermore, if (v_1, v_2) and (w_1, w_2) are distinct vertices, then setting

$$\alpha = f(v_1 w_1) - v_2 - w_2,$$

demonstrates that this edge belongs to the color class α and so each edge receives a color. Denote by $\Gamma_{f,\alpha}$ the subgraph of K_n defined by the color class α. Observe that $\Gamma_{f,0} = \Gamma_f$.

Lemma 3.1. $\Gamma_{f,\alpha} \cong \Gamma_f$.

Proof. Here we will provide an isomorphism depending on if q is odd or q is even. It is straight forward to verify the isomorphism.

- If q is odd, define $\Phi_\alpha : V \to V$ by

  $$\Phi_\alpha((v_1, v_2)) = (v_1, v_2 - 2^{-1}\alpha)$$

  Then Φ_α is an isomorphism from Γ_f to $\Gamma_{f,\alpha}$. In particular, we observe that $\Phi_\alpha((v_1, v_2))$ is adjacent to $\Phi_\alpha((w_1, w_2))$ in $\Gamma_{f,\alpha}$ if and only if

  $$(v_2 - 2^{-1}\alpha) + (w_2 - 2^{-1}\alpha) + \alpha = f(v_1 w_1) \iff v_2 + w_2 = f(v_1 w_1).$$

  The last equality holds true if and only if (v_1, v_2) is adjacent to (w_1, w_2) in Γ_f.

- If q is even, then x^2 is a permutation of $\mathbb{F}_q$. Thus, for each $\alpha \in R_f$, there exists a solution to $f(x^2) = \alpha$. Let β be such a solution and define $\Phi_\beta : V \to V$ by

  $$\Phi_\beta((v_1, v_2)) = (v_1 + \beta, v_2 + f(\beta v_1)).$$

  Then Φ_β is an isomorphism from Γ_f to $\Gamma_{f,\alpha}$. A similar calculation as in the odd case verifies the isomorphism.

□

Consequently, each of the k color classes is $K_{2,t+1}$-free. Note that the only constraint on the relationship between k and t is that $q = kt$. Thus, given any two powers of the same prime k and t, and setting $q = kt$, the above implies Theorem 1.2.

4 A lower bound for $r_k(C_4)$ when $k = 2^e$

The coloring we provide which yields Theorem 1.2 is very similar to the one given in [11] to lower bound $r_k(C_4)$. Note that one key difference is that our coloring works when $k = 2^e$, while Lazenik and Woldar's coloring is only given for k and odd prime power.

We wish to quickly demonstrate that the coloring in [11] does extend naturally to the even prime power case. Just as in Lemma 3.1, we had to provide different isomorphisms depending on whether q was even or odd, so too needs to happen with the coloring in [11].

Let q be a power of two and let $V = \mathbb{F}_q^2$ and label the vertex set of K_{q^2} by the elements of V. Denote the colors classes by the elements of $\mathbb{F}_q$, and we color an edge with color $\alpha \in \mathbb{F}_q$ if and only if

$$v_2 + w_2 + \alpha = v_1 w_1. \tag{7}$$

Observe that no edge can be assigned to distinct colors. Finally, each edge is assigned a color. In particular, the edge between vertices (v_1, v_2) and (w_1, w_2) is assigned the color

$$\alpha = v_2 + w_2 + v_1 w_1.$$

When $\alpha = 0$, it is a well-known fact that the graph produced with the equation (7) is C_4-free. The missing piece is the isomorphism, between color classes. Let $\Gamma_{q,\alpha}$ be the subgraph of K_n defined by the color class α. Let β be such that $\beta^2 = \alpha$. Note then that the function

$$\Phi_\beta((v_1, v_2)) = (v_1 + \beta, v_2 + v_1\beta)$$

is an isomorphism from $\Gamma_{q,0}$ to $\Gamma_{q,\alpha}$ by following the same type of calculations as were performed in the proof of Lemma 3.1. The same argument as in [11] implies one can add one more vertex and color the edges so that each color class remains $K_{2,q+1}$-free. In particular, we add one vertex x and may assign the color α to all edges between x and those of the form (α, x). Before the addition of the vertex x, no color class had a vertex which is adjacent to two vertices of the form (α, x) because the neighbors of any vertex in $\Gamma_{q,\alpha}$ must all have distinct first coordinates. Therefore, each color class remains $K_{2,q+1}$-free when x is added and its incident edges are colored in the right way. Thus, when $k = q = 2^e$, we obtain the same bound as in [11] for these values of k as well,

$$k^2 + 2 \leq r_k(C_4).$$

5 Concluding Remarks

All the arguments in this paper can be repeated to obtain a bipartite version of all given results. It is not hard to see that our construction for Theorem 1.1 is maximal. In fact, it is not hard to see that any two vertices at distance two are contained in a $K_{2,t}$. The addition of any other single edge will create a $K_{2,t+1}$. The trick used to add an extra vertex as was done in [11] is not possible for our construction. We conjecture that the lower bound given in Theorem 1.1 is the true upper bound for $r_k(K_{2,t+1})$.

Conjecture 5.1. *Let k and t be any integers, then*

$$r_k(K_{2,t+1}) \leq \begin{cases} k^2+2 & \text{if } t=1 \\ tk^2+1 & \text{if } t>1 \end{cases}.$$

It is known that this conjecture holds when $t = 1$ and $k = 2, 3, 4$ [14]. In fact, for these values of k, we have that $r_k(K_{2,2}) = k^2 + 2$, which now implies that the coloring of Lazebnik and Woldar produces extremal examples for the corresponding Ramsey number. The first open case is when $k = 5$, where the bounds are $27 \leq r_5(C_4) \leq 29$.

The method used here for constructing $K_{2,t+1}$-free graphs can likely be extended to $K_{3,t}$-free graphs if only we had an algebraically defined $K_{3,3}$-free graph to work with as a base. To date, we have only one infinite family of $K_{3,3}$-free graphs with asymptotically the right number of edges which was constructed by Brown [2]. It is natural to ask if a new $K_{3,3}$-free family can be constructed from algebraically defined graphs.

Problem: Find an infinite family algebraically defined $K_{3,3}$-free graphs of order n with approximately $n^{1+2/3}$ edges.

References

[1] M. Axenovich, Z. Füredi, and D. Mubayi, "On Generalized Ramsey Theory: The Bipartite Case", *Journal of Combinatorial Theory, Series B*, 79 (1) (2000), 66–86.

[2] W. G. Brown, "On graphs that do not contain a Thomsen graph", *Canadian Mathematics Bulletin*, 9 (1966), 281–285.

[3] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe, "An exponential improvement for diagonal Ramsey numbers", `https://arxiv.org/abs/2303.09521`, (2023):

[4] F. R. K. Chung, "On triangular and cyclic Ramsey numbers with k colors", in *Graphs and Combinatorics* (R. Bari and F. Harari, Eds.), Lecture Notes in Mathematics, 406 (1974), 236–242.

[5] F. R. K. Chung, "Ramsey Numbers in Multi-Colors", Dissertation, University of Pennsylvania, (1974).

[6] F. R. K. Chung and R. L. Graham, "On multicolor Ramsey numbers for complete bipartite graphs", *Journal of Combinatorial Theory, Series B*, 18 (1975), 164–169.

[7] Z. Füredi, “New Asymptotics for Bipartite Turán Numbers”, *Journal of Combinatorial Theory, Series A*, 75 (1996), 141–144.

[8] C. Hyltén-Cavallius, “On a combinatorial problem”, Colloquim in Mathematics, 6 (1958), 59–65.

[9] R. W. Irving, “Generalized Ramsey numbers for small graphs”, *Discrete Mathematics*, 9, 251–264 (1974).

[10] T. Kövári, V. T. Sós, and P. Turán, “On a problem of K. Zarankiewicz”, *Colloqium in Mathematics*, 3 (1954), 50–57.

[11] F. Lazebnik and A. Woldar, “New Lower Bounds on the Multicolor Ramsey Numbers $r_k(C_4)$”, *Journal of Combinatorial Theory, Series B*, 79 (2000), 172–176.

[12] F. Lazebnik and A. Woldar, “General properties of some families of graphs defined by systems of equations”, *Journal of Graph Theory*, 38 (2001), 65-86.

[13] S. Mattheus and J. Verstraete, “The asymptotics of $r(4,t)$”, *Annals of Mathematics*, 199 (2), (2024): 919-941.

[14] S. P. Radziszowski, “Small Ramsey Numbers”, *Electronic Journal of Combinatorics*,

[15] Y. Widgerson, “Upper bounds on diagonal Ramsey numbers [after Campos, Griffiths, Morris, and Sahasrabudhe]”, `https://arxiv.org/abs/2411.09321`, (2024).