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Penrose and super-Penrose energy extraction from a Reissner-Nordström black hole spacetime with a cosmological constant through the BSW mechanism: Full story

Duarte Feiteira,^{1,*} José P. S. Lemos,^{1,†} and Oleg B. Zaslavskii^{2,‡}

¹Centro de Astrofísica e Gravitação - CENTRA, Departamento de Física, Instituto Superior Técnico - IST, Universidade de Lisboa - UL, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal ²Department of Physics and Technology, Kharkov V. N. Karazin

National University, 4 Svoboda Square, 61022 Kharkov, Ukraine

The Penrose process, a process that transfers energy from a black hole to infinity, together with the BSW mechanism, a mechanism that uses collisions of ingoing particles at the event horizon of a black hole to locally produce large amounts of energy, is studied in a combined description for a d dimensional extremal Reissner-Nordström black hole spacetime with negative, zero, or positive cosmological constant, i.e., for an asymptotically anti-de Sitter (AdS), flat, or de Sitter (dS) spacetime. This blending of the Penrose process with BSW collisions is an instance of a collisional Penrose process. In an electrically charged extremal Reissner-Nordström black hole background, in the vicinity of the event horizon, several types of radial collisions between electrically charged particles can be considered, the most interesting one is between a critical particle, i.e., a particle that has its electric charge adjusted in a specific way to the other relevant parameters, and a usual particle, as it gives a divergent center of mass frame energy locally. A divergent center of mass frame energy at the point of collision is a favorable condition to extract energy from the black hole, but not sufficient, since, e.g., the product particles might go down the hole. So, to understand whether energy can be extracted or not in a Penrose process, we investigate in detail a collision between ingoing particles 1 and 2, from which particles 3 and 4 emerge, with the possibility that particle 3 can carry the energy extracted far out from the black hole horizon, i.e., there is a high Killing energy transported by particle 3. One finds that the mass, the energy, the electric charge, and the initial direction of motion of particle 3 can have different values, depending on the collision internal process itself. But, the different possible values of the the parameters of the emitted particle 3 lie within some range, and moreover the energy of particle 3 can, in some cases, be arbitrarily high but not infinite, characterizing a super-Penrose process. It is also shown that particle 4 has negative energy, as required in a Penrose process, living in its own electric ergosphere while it exists, i.e., before being engulfed by the event horizon. For zero cosmological constant we find that the results do not depend on the number of dimensions, but they do for negative and positive cosmological constant. The value of the cosmological constant also introduces differences in the lower bound for the energy extracted.

I. INTRODUCTION

Processes that can extract energy from a black hole are relevant on several counts. They have astrophysical import whereby matter or waves passing in the vicinity of an astrophysical black hole succeed in getting energy out of it. They have physical relevance since black holes can exist in all scales, in particular in microscales, and thus can be of use, in principle, as an additional power source in an advanced technology scheme. Whereas astrophysical black holes arise through gravitational collapse of large quantities of matter, micro black holes with very small radii, of the order of the neutron radius or even lower, can appear, for instance, through pair creation in strong field regimes, or by smashing matter fragments of exceedingly high energy against each other. Thus, the detail knowledge of all the possible mechanisms that can be employed to extract energy from a black hole of any size worth of pursuit.

Extraction of energy from a black hole started by the observation that the ergoregion of a rotating black hole contains states with negative energy with respect to infinity, and thus it is possible to somehow deposit energy far away at the expense of the black hole energy. This is the Penrose process [1, 2]. In the original process, it was proposed that a particle decays into two new particles inside the ergoregion, and one of the created particles carries an extra energy to infinity, with the energy source being the black hole angular momentum, this process being thus a decayment Penrose process. A static electrically charged black hole can allow as well for a Penrose process, the energy extracted coming now from the electric ergosphere of the black hole [3]. A general study of the electric Penrose

^{*} duartefeiteira@tecnico.ulisboa.pt

[†] joselemos@ist.utl.pt

 $^{^{\}ddagger}$ zaslav@ukr.net

process where electrically charged particles suffer a decayment in an electrically charged black hole background has been done in [4].

Another process that yields a possible extraction of energy from a black hole is the BSW mechanism [5]. The original BSW mechanism involves the collision at the horizon of an extremal rotating Kerr black hole of two ingoing test particles, one of the particles being critical, i.e., its angular momentum being critically tuned for the effect to occur, so that generation of unbounded center of mass energies would arise. Thus, we define BSW mechanism as a mechanism involving two particles moving inwards and colliding at the horizon of a black hole, or in the vicinity of it, where an unlimited center of mass energy is produced locally. The derivation of this mechanism motivated further study to understand which type of black holes can provide the effect and whether the emission of highly energetic or supermassive particles is possible after such a collision, and where the colliding particles can be of any type, including dark matter and electrically charged particles. In this way, it was shown that instead of an extremal black hole, a nonextremal black hole could be used if the parameters for the particle were properly adjusted at the point of the collision [6]. It was also proved in [7] that the BSW effect is generic, i.e., it is due to the general properties of a black hole horizon, and thus, many results that followed can be seen as particular cases of this feature.

As is typical in black hole physics, if some phenomenon exists for rotating black holes and neutral particles, it is quite natural that it also exists for electrically charged black holes and electrically charged particles, and consequently it was soon proposed that an extremal electrically charged Reissner-Nordström black hole, could also yield, by the collision of two ingoing electrically charged particles, with one of them critical, a divergent center of mass frame energy, in what is the electrically charged version of the BSW effect [8]. Several other important advancements on the BSW effect appeared as we now mention. By using the innermost stable circular orbit up to the extremal Kerr state it was confirmed in [9] that high center of mass energies can be created at the horizon and highly energetic particles are subsequently emitted at the point of collision. A geometric and general explanation, based on spacetime properties of systems composed of rotating black holes and matter systems, pictured the effect as attributes of null and timelike vectors in the vicinity of the future event horizon [10]. The creation of unbounded center of mass energies at the point of collision was exhibit to persist with neutral particles in Kerr black hole spacetimes in a cosmological constant setting [11]. The collision of two generic geodesic particles around a Kerr black hole was further explored to produce unlimited center of mass energies at the horizon in [12].

Clarifications and interesting applications of the BSW effect kept emerging in the literature. For instance, a kinematic explanation of the effect was developed in [13] in that a collision between a particle with velocity tending to the velocity of light and a particle with a velocity smaller than the velocity of light, both velocities seen in a locally nonrotating frame at the horizon, produce an unlimited center of mass energy. An anti-de Sitter (AdS) background for extremal electrically charged rotating cylindrical black holes was the scene to inspect the existence of the BSW effect [14]. For dirty rotating black holes, i.e., black holes with surrounding matter, it was found that in the collisions of particles near the horizon, the energy of the particles scales with a power of the inverse surface gravity of the black hole [15]. That ultrahigh energies at the center of mass scale with some power of the inverse surface gravity of the black hole was also uncovered by considering, near the horizon of a Reissner-Nordström black hole with negative cosmological constant, that one of the electrically charged colliding particles is critical and at rest, which is possible due to the electric repulsion on one hand, and the gravitational and the extra attraction from the negative cosmological constant on the other hand [16]. Additionally, it was realized in [17] that electrically charged particles in radial motion that collide at the horizon of an extremal Reissner-Nordström black hole.

The BSW effect has also been studied in higher and lower dimensions, and has been implemented in several possible scenarios. In higher-dimensional spacetimes, the effect has been tested specifically in extremal Myers-Perry black holes [18]. The analysis of collisions for higher-dimensional rotating black holes was revisited in [19] establishing that high center of mass energies can be produced. A review of the BSW effect with emphasis on the collision of particles in an extremal Kerr black hole was produced in [20]. The joint effect of rotation and electric charge was taken into account for the extremal Kerr-Newman black hole, and in particular, a noticeable BSW effect was found when simultaneously the angular momentum of the particle is very large and the black hole charge is very small [21]. Lower-dimensional spacetimes were the target of a work for the implementation of the effect, namely, the BTZ rotating extremal black hole which is a solution of 3-dimensional general relativity with negative cosmological constant [22]. An additional 3-dimensional rotating black hole, solution of a topologically massive gravity, acts, as expected, as a particle accelerator as discussed in [23]. Several types of classification for different BSW scenarios have been performed in [24–26].

One can combine the Penrose process and the BSW effect to get a collisional Penrose process. In the Penrose process one uses the negative states existing in the ergosphere to extract energy of the black hole which is then transferred to infinity. The first instance found for a physical Penrose process was through particle decay in the ergosphere, but one can think of other processes, a distinct one is indeed through particle collisions, such as a BSW collision. For collisions, the energy extracted in a Penrose process from a rotating black hole can be somewhat enhanced. A relevant development and clarification of the BSW effect demonstrated that although the center of mass energy grows without

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bound in a collision near or at an extremal Kerr horizon, the energy of the particles at infinity, i.e., the Killing energy, is finite and not large [27–29]. Still in the rotating black hole case, the relation between the energy in the center of mass frame of BSW colliding particles and the net Penrose energy extracted was uncovered in [30]. Improvements and refinements of the collisional Penrose process have appeared. For instance, particle collisions in the axis of symmetry of a Kerr-Newman black hole along with the possibility of energy extraction were described in [31]. For electrically charged colliding particles it was established that even in a nonextremal Reissner-Nordström black hole background, the emergent particles can carry arbitrarily large Killing energy to infinity, thus yielding a super-Penrose process of real energy extraction from a black hole [32], and confirming previous results for extremal electrically charged black hole backgrounds. This extraction of arbitrary large amounts of energy in particle collisions in electric charged black holes contrasts with the finite and moderate energy extraction from the ejected particles of a rotating black hole. In addition, a general and encompassing treatment of the BSW effect in the equatorial plane of Kerr-Newman black holes where unbounded energies taken by the particle debris to infinity can happen, yielding thus real energy extraction from the black hole, was reported in [33, 34].

One could continue to enumerate processes that extract energy from a black hole. In a collision of two particles, one ingoing the other outgoing, and so different from a BSW collision in which the two colliding particles are ingoing, it was shown that such a collisional Penrose process could have an energetic extraction enhancement by a factor close to 14 [35], see also [36]. In a extended version of the decayment Penrose process, one employs a mirror at some radius surrounding the black hole, to obtain multi processes and so more energy extraction, leading possibly to a black hole bomb, see [37] for the electrically charged case. Other processes, being somewhat different in character from the BSW effect or the Penrose process, can still extract energy from a collision of particles. These processes include particle collisions in black hole spacetimes with additional physical features, in spacetimes without black holes, and in other special spacetimes. One can also extract energy from black holes using waves in the superradiance phenomenon.

It is our aim to study the Penrose process in conjunction with the BSW effect, i.e., the collision Penrose process, of black hole energy extraction due to the collision of two electrically charged particles at the event horizon of an extremal Reissner-Nordström black hole in a background with a cosmological constant in generic d dimensions in a unified way. The cosmological constant will be allowed to have negative values, in which case the spacetime is asymptotically AdS, to have zero value, in which case the spacetime is asymptotically flat, and to have positive values, in which case the spacetime is asymptotically de Sitter (dS), in this latter case there is also the cosmological horizon, but it does not play any major role in our analysis. The number of dimensions obeys $d \ge 4$, and thus some of the results already obtained for the particular case d = 4 are recovered. One can list some of the reasons why these spacetimes are relevant to study. Reissner-Nordström black holes in isolation can in some instances keep their own electric charge, although when surrounded by some medium they are exposed to be completely discharged. Asymptotically AdS spacetimes possess many symmetries and are the base of some extended theories of gravity such as supergavity and string theory. Asymptotically flat spacetimes are the appropriate environment in the study of a sufficiently large neighborhood of any cosmic environment, such as the environment surrounding a black hole, since for it the asymptotic structure is approximately flat. Asymptotically dS spacetimes have implications in fundamental theories and can be used describe the universe at large. The interest in studying spacetimes with d generic dimensions comes from the possibility of understanding what is peculiar to d = 4 and what is generic, and from the fact that several possible suitable theories live consistently in spacetimes with higher dimensions, which makes physical effects in these dimensions worth pursuing.

The work is organized as follows. In Sec. II, the d-dimensional Reissner-Nordström black hole spacetime in a cosmological constant background, nonextremal and extremal, is introduced along with the important horizon radii. The equations describing electrically charged particle motion in such a spacetime are presented and the electric ergosphere is defined. In Sec. III, the definitions and necessary conditions for energy extraction from particle collisions are given, in particular, we present the definition of critical, near-critical, and usual particles, we make an analysis of the energy of the BSW collisions at the center of mass frame, give the prerequisites for the particles to reach the horizon, and show the only type of collision that yields an unbounded center of mass energy is the collision between a critical and a usual particle. In Sec. IV, we examine in detail the energy extraction from a collisional Penrose process. For that, we uncover the BSW collision between a critical particle that goes into the black hole and a usual particle that also goes in, from which emerges a near critical particle that goes out and a usual particle with negative energy that goes in, with the energy of the emitted particle, i.e., the possible energy extracted in the Penrose process, being established in terms of lower and upper bounds. It is shown that super-Penrose processes can occur. We add a discussion on the dependence of the process on the cosmological constant and on the spacetime dimension d and make further comments. In Sec. V, we conclude. In the Appendix we derive the center of mass energy expression from the equations obtained for the energy of the emitted particle, and uncover some details of the energetics of the particle with negative energy that falls in after the collision.

II. LINE ELEMENT, BLACK HOLE HORIZON RADII, EQUATIONS OF MOTION FOR THE PARTICLES, AND THE ELECTRIC ERGOSPHERE

A. Line element

In this work, it is provided a general analysis of the collisional Penrose process, i.e., the combination of BSW effect with the Penrose process of extraction of energy, in a Reissner-Nordström black hole background, with a negative, zero, and positive cosmological constant in d dimensions with $d \ge 4$. Thus, we consider the Reissner-Nordström-Tangherlini, or simply Reissner-Nordström, line element for the interval s. Generically, the spacetime line element is written as $ds^2 = g_{ab}dx^a dx^b$, where g_{ab} is the metric and the dx^a are the coordinate components of the infinitesimal displacements, with a, b running over the time and spatial coordinates. In usual spherical coordinates $x^a = (t, r, \theta_1, \dots, \theta_{d-2})$, one can write $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega_{d-2}$, where g_{tt} and g_{rr} are the time-time and the radius-radius components of the metric, respectively, which in general depend on t and r, and $d\Omega^2_{d-2}$ is the line element on a d-2 sphere, $d\Omega_{d-2} = d\theta_1^2 + \sin \theta_1^2 d\theta_2^2 + \dots + \prod_{i=2}^{d-3} \sin \theta_i^2 d\theta_{d-2}^2$. Then, the electrovacuum Einstein-Maxwell field equations yield that g_{tt} and g_{rr} only depend on r, and indeed the line element has the d-dimensional Reissner-Nordström form given by

$$ds^{2} = -f(r) dt^{2} + \frac{dr^{2}}{f(r)} + r^{2} d\Omega_{d-2}, \qquad (1)$$

with the metric potential f(r) being given by

$$f(r) = 1 - \frac{2\mu M}{r^{d-3}} + \frac{\chi Q^2}{r^{2(d-3)}} - k \frac{r^2}{l^2},$$
(2)

where M is the mass of the black hole, Q is its electric charge, $l^2 = \frac{3}{|\Lambda|}$ is the length scale related with the absolute value of the cosmological constant Λ , k = -1, 0, 1 for spacetimes with negative, zero or positive cosmological constant, respectively, and μ and χ are defined as $\mu = \frac{8\pi}{(d-2)\Omega_{d-2}}, \chi = \frac{8\pi}{(d-2)\Omega_{d-2}}$, and $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ is the area of the (d-2)-dimensional unit sphere with Γ being the gamma function. The time coordinate range is $-\infty < t < \infty$, the radial coordinate range is $r_+ < r < \infty$, with r_+ being the black hole event horizon, and the angular coordinate ranges are $0 \le \theta_i \le \pi$ for i = 1, ..., d-3, and $0 \le \theta_{d-2} < 2\pi$. The electric potential φ of the spacetime is

$$\varphi = \frac{Q}{\left(d-3\right)r^{d-3}}\,,\tag{3}$$

i.e., a Coulomb electric potential.

B. The black hole horizon radius r_+ and the extremal black hole: f(r) and its factorization

A nonextremal black hole has two characteristic radii, the event horizon radius r_+ and the Cauchy horizon radius r_- . The black hole horizon radius r_+ is one of the solutions of the equation $f(r_+) = 0$, where f(r) is given in Eq. (2). Thus, in general $r_+ = r_+ (M, Q, l, d)$. There is another possible radius for which f(r) = 0. It is the Cauchy horizon radius r_- obeying $r_- \leq r_+$, with $r_- = r_- (M, Q, l, d)$. Thus, in general one has

$$r_{+} = r_{+} (M, Q, l, d) , \qquad r_{-} = r_{-} (M, Q, l, d) .$$
 (4)

In terms of r_+ and r_- , the function f(r) of Eq. (2) can be written as

$$f(r) = \frac{1}{r^2} (r - r_+) (r - r_-) g(r), \qquad (5)$$

where $g(r) \equiv \frac{p_{2(d-4+k^2)}(r;k)}{r^{2(d-4)}}$, and $p_{2(d-4+k^2)}(r;k)$ is a polynomial function of r, of degree $2(d-4+k^2)$, whose coefficients can depend on k.

An extremal black hole has one characteristic radius, the event horizon radius has now the same value as the the Cauchy horizon radius $r_{+} = r_{-}$. For the BSW mechanism, the most interesting black hole, the one from which large amounts of energy can be extracted from the particle collision, is the extremal black hole. For an extremal black hole,

there are two conditions, the former one $f(r_+) = 0$, and a new one $\frac{df}{dr}(r_+) = 0$, and they are such the black hole horizon r_+ and the Cauchy horizon r_- indeed coincide, i.e.,

$$r_{+}(M,Q,l,d) = r_{-}(M,Q,l,d) .$$
(6)

From the two conditions on f(r) one finds that $r_{+}^{d-3}\left(1-\frac{d-2}{d-3}k\frac{r_{+}^{2}}{l^{2}}\right) = \mu M$. For k = 0, one readily obtains $r_{+}^{d-3} = \mu M$. Using again the two conditions on f(r) one gets that r_{+} of an extremal black hole obeys $r_{+} = \sqrt{-kl^{2}R(M,Q,d) + l^{2}\sqrt{R^{2}(M,Q,d) + \left(\frac{d-3}{d-2}\right)^{2}\left(\frac{\mu^{2}M^{2}}{\chi Q^{2}} - 1\right)}}$, where we have used the abbreviation $R(M,Q,d) \equiv \frac{1}{2}\frac{\mu^{2}M^{2}}{\chi Q^{2}}\left(\frac{d-3}{d-2}\right)^{2}\left[\frac{2(d-2)}{d-3}\left(1-\frac{\chi Q^{2}}{\mu^{2}M^{2}}\right)-1\right]$. Note that we still have $r_{+} = r_{+}(M,Q,l,d)$, and we stick to the horizon radius notation r_{+} , knowing that this is the extremal case $r_{+} = r_{-}$. Moreover, since $r_{+} = r_{-}$, and each radius is a root of f(r), in the extremal case, one has that f(r) of Eq. (2) has a double route, and so it assumes the form

$$f(r) = \frac{1}{r^2} \left(r - r_+ \right)^2 g(r) , \qquad (7)$$

where $g(r) \equiv \frac{p_{2(d-4+k^2)}(r;k)}{r^{2(d-4)}}$, and $p_{2(d-4+k^2)}(r;k)$ is a polynomial function of r, of degree $2(d-4+k^2)$, whose coefficients can depend on k. Notice that, with this factorization, the electric potential defined in Eq. (3) can be written as $\varphi = \frac{Q}{(d-3)r_+^{d-3}} \left(1 - \sqrt{\frac{f(r)}{g(r)}}\right)^{d-3}$.

For k = -1 or k = 0, in even dimensions g(r) has no real zeros, and in odd dimensions it has one negative zero and the other zeros are not real. For k = 1, i.e., for positive cosmological constant, there is yet another zero of f(r). It is given by the cosmological horizon radius r_c , with $r_c = r_c (M, Q, l, d)$ and $r_- \leq r_+ \leq r_c$, so it is the largest horizon radius. We will be interested in the black hole r_+ radius for the collision processes, so r_c will not appear in our developments. In this case of k = 1, in fact, one can work with the variables $\{M, Q, l, d\}$, or with the variables $\{r_+, r_-, r_c, d\}$, or with a combination of the two. The most natural combination of variables in the problem at hand is $\{r_+, Q, l, d\}$. In the case of k = 0 and k = 1 there is no cosmological horizon and one can work with the natural combination of variables in the problem at hand, i.e., $\{r_+, Q, l, d\}$.

C. Equations of motion for particles and the electric ergosphere

1. Equations of motion for particles

Neutral particles follow geodesics in a Reissner-Nodtsröm spacetime, but electrically charged particles do not. To obtain the equations of motion for a charged particle it is useful to resort to the Lagrangian of the particle and its Euler-Lagrange equations of motion. Consider a particle with mass m and specific charge \tilde{e} , moving in a d-dimensional Reissner-Nodtsröm spacetime. The Lagrangian L for the particle can then be written as $L = \frac{1}{2}g_{ab}u^a u^b - A_a u^a$, where $u^a = \frac{dx^a}{d\tau}$ is the four-velocity of the particle, τ is its proper time defined through $d\tau^2 = -ds^2$, and A_a is the electromagnetic four-potential. For the Coulomb interaction of the particle with the black hole spacetime one has $A_a = \varphi \delta_a^t$, and assuming pure radial motion, i.e., $u^a = u^t \delta_t^a + u^r \delta_r^a$, the Lagrangian L for the particle is then given by $L = \frac{1}{2} \left(-f(r) \dot{t}^2 + \frac{\dot{r}^2}{f(r)} - \frac{2\tilde{e}Q}{(d-3)r^{d-3}}\dot{t}\right)$, where a dot means a derivative relative to the particle's proper time τ , and we have used $g_{tt} = -f$, $g_{rr} = \frac{1}{f}$, $\varphi = \frac{Q}{(d-3)r^{d-3}}$, $u^t = \dot{t}$, and $u^r = \dot{r}$. From the Euler-Lagrange equations for L one obtains that the equation for the coordinate t is $\frac{d}{d\tau} \left(\frac{\partial L}{\partial t}\right) = 0$, i.e., $\frac{d}{d\tau} \left(f\dot{t} + \frac{\tilde{e}Q}{(d-3)r^{d-3}}\right) = 0$, which can be integrated to $\dot{t} = \frac{E^- \frac{eQ}{(d-3)r^{d-3}}}{mf}$, where E is the energy of the particle, a conserved quantity, and the particle's total electric charge has been defined as $e = m\tilde{e}$. For pure radial motion, one finds that the first integral for the radial coordinate is $\dot{r}^2 = \frac{\left(E^- \frac{eQ}{(d-3)r^{d-3}}\right)}{mf} - f$. Defining the four-momentum as $p^a \equiv mu^a$, the time component of the four-momentum of the particle is $p^t \equiv m\dot{t}$ and the radial component is $p^r \equiv m\dot{r}$. One then finds that the equations of motion for the form

$$p^t \equiv m\dot{t} = \frac{X}{f}, \qquad p^r \equiv m\dot{r} = \varepsilon Z,$$
(8)

$$X(r) = E - \frac{eQ}{(d-3)r^{d-3}}, \qquad Z(r) = \sqrt{X^2 - m^2 f(r)}, \qquad (9)$$

with f = f(r) being given in Eq. (7), and $\varepsilon = \pm 1$ defining the direction of the particle's motion, -1 for inward radial motion and +1 for outward motion. The forward in time condition, $\dot{t} > 0$ implies that X > 0 outside the horizon. In what follows, it will always be assumed that the electric charge of the black hole is positive, Q > 0, without loss of generality.

2. The electric ergosphere

Given the forward in time condition, $\dot{t} > 0$, one has X > 0 for any radii outside the horizon. From Eq. (9) this implies $E - \frac{eQ}{(d-3)r^{d-3}} > 0$. Given a black hole electric charge Q, with Q positive as assumed, it is clear from the latter expression that one can have negative energy states E for the particle, as long as its electric charge e is negative and the position of the particle r is sufficiently small. Writing for these negative energy states E = -|E| and e = -|e|, we have that indeed these states exist when the radial position r of the particle obeys

$$r_{+} \le r < r_{\rm ergo}, \qquad r_{\rm ergo}^{d-3} = \frac{|e|Q}{(d-3)|E|}, \qquad E < 0, \ e < 0.$$
 (10)

The region $r_+ \leq r < r_{ergo}$ is an ergosphere that comes from the existence of electric charge, and is called electric ergosphere or generalized ergosphere.

III. DEFINITIONS AND NECESSARY CONDITIONS FOR ENERGY PRODUCTION FROM BSW PARTICLE COLLISIONS: DEFINITION OF CRITICAL PARTICLES AND ANALYSIS OF THE ENERGY AT THE CENTER OF MASS FRAME

A. Definition of critical particles and energy at the center of mass frame

1. Critical, near-critical, and usual particles

We nominate each particle as particle *i*, in general i = 1, 2, 3, 4. Each particle has attributes like its energy E_i , its mass m_i , its electric charge e_i , and so on. In what follows, definitions of critical, near-critical, and usual particles will be important. Thus, we establish here definitions for the different particles, see [8].

The horizon radius r_+ obeys the equation $f(r_+) = 0$ as we have seen. If in addition $X(r_+)$ in Eq. (9) is also zero, $X(r_+) = 0$, then p^t is undetermined a priori and interesting things can happen. When for a given particle *i* one has $X_i(r_+) = 0$, then for a value E_i of the energy of the particle there corresponds a definite value of the electric charge, the critical electric charge e_{ic} given by

$$e_{ic} = \frac{r_+^{d-3} \left(d-3\right)}{Q} E_i. \tag{11}$$

A particle *i* is then defined as critical if its electric charge e_i is equal to the critical electric charge, i.e.,

$$e_i = e_{ic}$$
, critical particle. (12)

A particle i is defined as near critical if its electric charge e_i is almost equal to the critical electric charge, i.e.,

$$e_i = e_{ic} (1+\delta)$$
, near critical particle, (13)

with $|\delta| \ll 1$ and δ positive or negative. If $\delta = 0$, one recovers the definition of a critical particle. A particle *i* is defined as usual if its electric charge e_i differs from the critical and near-critical electric charges, i.e.,

$$e_i \neq e_{ic} \left(1+\delta\right), \qquad \text{usual particle}, \tag{14}$$

i.e., the electric charge of the particle is significantly different from the critical electric charge. These definitions are important.

2. Particle collision and energy at the center of mass frame

To study the energy generated from the BSW effect, a collision between two ingoing particles is assumed to occur in a *d* dimensional extremal Reissner-Nordström black hole spacetime with horizon radius r_+ , electric charge Q, and cosmological constant $k\Lambda$. The mass of particle *i* is denoted as m_i , and the electric charge of particle *i* is denoted as e_i , so that, before the collision, particles i = 1, 2 have masses and charges m_1 and e_1 , and m_2 and e_2 , respectively, and after the collision, particles i = 3, 4 have masses and charges m_3 and e_3 , and m_4 and e_4 , respectively, assuming, as we do, that two particles come out of the collision.

To have a grasp on the collision process and to understand the necessary conditions for energy extraction, we start by making a simplifying assumption, the incoming masses are equal $m_1 = m_2$, although the electric charges e_1 and e_2 are different. So we put $m \equiv m_1 = m_2$. An important quantity is the energy of the center of mass $E_{\rm CM}$. To calculate an expression for it, note that the total four-momentum vector p^a is $p^a = p_1^a + p_2^a$, where p_1^a is the momentum of particle 1 and p_2^a is the momentum of particle 2. In the center of mass the total three-momentum is zero, so the four-momentum is only composed of the center of mass energy in the local frame and we can write in the center of mass $\bar{p}^l = E_{\rm CM} \delta_t^l$. But $p^a = \bar{p}^l e_l^a$ where e_l^a is the local tetrad. So, $p^2 = g_{ab} p^a p^b = g_{ab} \bar{p}^l \bar{p}^m e_l^a e_m^b = \bar{p}^l \bar{p}^m \eta_{lm} = E_{\rm CM}^2 \delta_t^l \delta_t^m \eta_{lm} = E_{\rm CM}^2 \eta_{lt}^b = -E_{\rm CM}^2$, where g_{ab} is here the Reissner-Nordström metric and η_{lm} is the Minkowski metric. On the other hand $p^2 = g_{ab} p^a p^b = g_{ab} (p_1^a + p_2^a) (p_1^b + p_2^b) = g_{ab} p_1^a p_1^b + g_{ab} p_2^a p_2^b + 2g_{ab} p_1^a p_2^b$. Now $p_1^a = mu_1^a$ and $p_2^a = mu_2^a$, so $g_{ab} p_1^a p_1^b = -m^2$, $g_{ab} p_2^a p_2^b = -m^2$, and $2g_{ab} p_1^a p_2^b = 2m^2 g_{ab} u_1^a u_2^b$. Since at r, p^2 is an invariant, one gets $-(E_{\rm CM}^2) = -2m^2 + 2m^2 g_{ab} u_1^a u_2^b$, i.e., $\frac{E_{\rm CM}^2}{2m^2} = 1 - g_{ab} u_1^a u_2^b$. For a pure radial collision one has $u_1^a = u_1^t \delta_t^a + u_1^r \delta_r^a$ and $u_2^a = u_2^t \delta_t^a + u_2^r \delta_r^a$, so that $\frac{E_{\rm CM}^2}{2m^2} = 1 - g_{rr} u_1^r u_2^r$. Since $u_1^t = \dot{t}_1$, $u_1^r = \dot{r}_1$, $u_2^t = \dot{t}_2$, $u_2^r = \dot{r}_2$, $g_{tt} = -f$, and $u_2^a = u_2^t \delta_t^a + u_2^r \delta_r^a$, so that $\frac{E_{\rm CM}^2}{2m^2} = 1 - g_{ab} u_1^a u_2^b$. For a pure radial collision one has $u_1^a = u_1^t \delta_t^a + u_1^r \delta_r^a$ an

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{X_1(r)X_2(r) - Z_1(r)Z_2(r)}{m^2 f(r)},\tag{15}$$

where X_i and Z_i , i = 1, 2, are the quantities defined in Eq. (9) evaluated for particle i = 1, 2 at the point of collision r, the value $\varepsilon_i = -1$, i = 1, 2, was used since we assume a BSW collision and so the two particles move inwards, and f is given in Eq. (7). $E_{\rm CM}$ which clearly is a function of r, $E_{\rm CM}(r)$, is to be computed at some radius, possibly near the black hole horizon r_+ . Note that after the collision, particles 3 and 4 have the same $E_{\rm CM}$ as before the collision.

Three types of collision between the incoming particles can happen. They are the collision between two critical particles, the collision between a critical particle and a usual particle, and the collision between a near-critical particle and a usual particle. For each type of collision, the energy at the center of mass frame can be computed and the conditions which particles must obey to allow the occurrence of the collision at the horizon can be established.

B. The three types of collisions: Estimates for the produced energy

1. Collision between two critical particles

Here we study a collision between two ingoing critical particles in the vicinity of the extremal black hole event horizon, at some radius r which is near or at r_+ . So we assume that particle 1 is exactly critical and particle 2 is exactly critical. Then particle 1 has $e_1 = e_{1c}$, see Eq. (12), and from Eq. (9) we have $X_1 = E_1 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2$ and $Z_1 = \sqrt{E_1^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f}$. Particle 1 can only move for radii for which Z_1 is well defined, in the sense that the argument of the square root has to be positive, $E_1^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f > 0$. Therefore, particle 1 can only move for radii for which Z_1 is compared that the term $-m^2 f(r)$ approaches zero faster than $E_1^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2$ when $r \to r_+$. Particle 2 is critical, it has $e_2 = e_{2c}$, see Eq. (12), and from Eq. (9) we have $X_2 = E_2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2$ and $Z_2 = \sqrt{E_2^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f}$. Particle 2 can as well only move for radii for which Z_2 is well defined, in the sense that the argument of the square root has to be positive, $E_2^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f > 0$. Therefore, particle 2 can only $Z_2 = \sqrt{E_2^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f}$. Particle 2 can as well only move for radii for which Z_2 is well defined, in the sense that the argument of the square root has to be positive, $E_2^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f > 0$. Therefore, particle 2 can only reach the horizon provided that the term $-m^2 f(r)$ approaches zero faster than $E_2^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2$ when $r \to r_+$. To find if particles 1 and 2 can reach the black hole event horizon, one has to study how the metric potential f(r)

approaches zero when $r \to r_+$. This will depend on the factorization of this function f(r). The case in which f goes faster to zero when the horizon r_+ is approached is the extremal case with the factorization obtained in Eq. (7), since $1 - \frac{r_+}{r} \le 1 - \left(\frac{r_+}{r}\right)^{d-3}$. The factorization for nonextremal black holes, Eq. (5), shows that the approach is not fast

enough. The generic expression for Z_1 in the extremal case is then $Z_1 = \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right) \sqrt{E_1^2 - m^2 g(r) \frac{\left(1 - \frac{r_+}{r}\right)^2}{\left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2}}$.

Since $\lim_{r \to r_+} \frac{\left(1 - \frac{r_+}{r}\right)^2}{\left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2} = \frac{1}{(d-3)^2}$, Z_1 near the horizon is given by $Z_1 = \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)\sqrt{E_1^2 - \frac{m^2}{(d-3)^2}g(r)}$. Note that by definition the function g(r) is always perpendent the horizon gives the part of f(r) that we have $r = \frac{1}{(d-3)^2}g(r)$.

that, by definition, the function g(r) is always nonzero at the horizon, since the part of f(r) that vanishes at the horizon was already factorized. Therefore, Z_1 is well defined and the horizon is reachable by a critical particle provided that $E_1^2 > \frac{m^2}{(d-3)^2} g(r_+)$. The same can be said for Z_2 , so that $Z_2 = \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right) \sqrt{E_2^2 - \frac{m^2}{(d-3)^2} g(r)}$, therefore, Z_2 is well defined and the horizon is reachable by a critical particle provided that $E_2^2 > \frac{m^2}{(d-3)^2} g(r_+)$. Thus in brief, we can write

$$X_{1}(r) = E_{1}\left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right), \qquad Z_{1} = \left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right)\sqrt{E_{1}^{2} - \frac{m^{2}}{(d-3)^{2}}g(r)},$$

$$X_{2}(r) = E_{2}\left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right), \qquad Z_{2} = \left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right)\sqrt{E_{2}^{2} - \frac{m^{2}}{(d-3)^{2}}g(r)},$$
(16)

Substituting now X_1 , Z_1 , X_2 , and Z_2 of Eq. (16) in the expression for the energy at the center of mass frame for a collision, Eq. (15), when the collision is between a critical and another critical particle at the extremal horizon r_+ , one gets

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{(d-3)^2}{m^2 g(r_+)} \left[E_1 E_2 - \sqrt{E_1^2 - \frac{m^2}{(d-3)^2} g(r_+)} \sqrt{E_2^2 - \frac{m^2}{(d-3)^2} g(r_+)} \right].$$
(17)

Note that, as displayed in Eq. (17), $E_{\rm CM}(r)$ does not depend on r in zero order in $r - r_+$. So, $E_{\rm CM}$ remains finite at the horizon,

$$E_{\rm CM}(r_+) = \text{finite}, \quad \text{particle 1 critical } e_1 = e_{1c}, \quad \text{particle 2 critical } e_2 = e_{2c}.$$
 (18)

Therefore, there is no great gain. When two critical particles collide near or at the horizon there is no great amount or even no energy generation.

2. Collision between a critical and a usual particle

Here we study a collision between an ingoing critical particle and an ingoing usual particle in the vicinity of the extremal black hole event horizon, at some radius r which is near or at r_+ . So, we assume that particle 1 is exactly critical and particle 2 is usual. Then particle 1 has $e_1 = e_{1c}$, see Eq. (12), and from Eq. (9) we have $X_1 = E_1 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)$ and $Z_1 = \sqrt{E_1^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f}$. Since particle 1 is critical, the calculations previously made hold here, and we do not repeat them, noting that f has to be the extremal black hole function in order to have a possible gain. Particle 2 is usual, it has $e_2 \neq e_{2c} (1 + \delta)$, see Eq. (14), and one simply has that $X_2 = E_2 - \frac{e_2 Q}{(d-3)r^{d-3}}$ and $Z_2 = \sqrt{\left(E_2 - \frac{e_2 Q}{(d-3)r^{d-3}}\right)^2 - m^2 f}$. To find if particle 2 can reach the black hole event horizon, one has to study how the metric potential f(r) approaches zero when $r \to r_+$. Due to the forward in time condition, one has that $X_2 > 0$ for all possible values of the radial coordinate. This means that, when $r \to r_+$, the term $\left(E_2 - \frac{e_2 Q}{(d-3)r^{d-3}}\right)^2$ remains positive, while the term $-m^2 f(r) \to 0$. Therefore, particle 2 can always reach the horizon. Thus in brief, we can write

$$X_{1}(r) = E_{1}\left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right), \qquad Z_{1} = \left(1 - \left(\frac{r_{+}}{r}\right)^{d-3}\right)\sqrt{E_{1}^{2} - \frac{m^{2}}{(d-3)^{2}}},$$

$$X_{2}(r) = E_{2} - \frac{e_{2}Q}{(d-3)r^{d-3}}, \qquad Z_{2}(r) = \sqrt{\left(E_{2} - \frac{e_{2}Q}{(d-3)r^{d-3}}\right)^{2} - m^{2}f(r)},$$
(19)

Substituting now X_1 , Z_1 , X_2 , and Z_2 of Eq. (19) in the expression for the energy at the center of mass frame for a collision, Eq. (15), when the collision is between a critical and a usual particle one gets the expression,

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{E_2 - \frac{e_2 Q}{(d-3)r_+^{d-3}}}{m^2 g(r_+)} \left[E_1 - \sqrt{E_1^2 - \frac{m^2}{(d-3)^2} g(r_+)} \right] \frac{d-3}{1 - \frac{r_+}{r}},$$
(20)

for a collision happening at a radius r near r_+ . When $r \to r_+$, the factor $\frac{1}{1-\frac{r_+}{r}} \to \infty$. Thus, the energy at the center of mass frame explicitly diverges when the collision is at the horizon. Therefore, a collision between a critical and a usual particle in the vicinity of the event horizon leads to a divergent energy at the center of mass frame

$$E_{\rm CM}(r_+) = \infty$$
, particle 1 critical $e_1 = e_{1c}$, particle 2 usual. (21)

Therefore, here one can have a great gain. When two particles collide near or at the horizon, one of them is critical and the other is usual, there is the possibility of a great amount of energy generation.

3. Collision between a near-critical and a usual particle

We now consider a collision between an ingoing near-critical particle, which for definiteness is chosen to be particle 1, and an ingoing usual particle, which is particle 2. The energy at the center of mass frame for collisions happening at the black hole event horizon, Eq. (15), is exactly $\frac{E_{CM}^2}{2m^2} = 1 + \frac{1}{2} \left(\frac{e_{2c} - e_2}{e_{1c} - e_1} + \frac{e_{1c} - e_1}{e_{2c} - e_2} \right)$. Now, particle 1 being near critical has an electric charge given by $e_1 = e_{1c} (1 + \delta)$, with $|\delta| \ll 1$, and δ has to be negative or zero, $\delta \leq 0$, since to arrive at the horizon the electric charge of the particle has to be less or equal to the critical charge due to the forward in time condition, i.e., $e_1 \leq e_{1c}$. So, $\frac{e_1 - e_{1c}}{e_{1c}} = \delta$. Thus, the energy at center of mass frame scales as the inverse of the square root of $|\delta|$. Indeed,

$$\frac{E_{\rm CM}^2(r)}{2m^2} = \frac{1}{2} \frac{e_{2c} - e_2}{e_{1c} - e_{1c}} + 1, \qquad (22)$$

plus $\mathcal{O}\left(\frac{e_1-e_{1c}}{e_{1c}}\right) = \mathcal{O}(\delta)$ for a collision at r_+ . Therefore, since the denominator is equal to $e_{1c}|\delta|$, the energy at the center of mass frame can be made as large as one wants. In particular, if $\delta = 0$, i.e., if particle 1 is exactly critical, the energy seems to diverge. However, to know whether this divergence is possible or not, one has to assume from the very beginning that particle 1 is exactly critical, in order to find how the horizon can be reached by such a particle. This is what we have done in the previous case, and no need to take this case into account.

C. Picking the only available interesting type of collision

The energy at the center of mass frame was computed for three different types of collisions between two ingoing particles with equal masses and different electric charges. The types of collision are between two critical particles, between a critical and a usual particle, and between a near-critical and a usual particle. For the first type, a collision between two critical particles, it was verified whether these particles were able to reach the black hole event horizon, and it was found that this is only the case if one assumed an extremal black hole, and it was further found that there is no divergence in the energy at the center of mass frame. For the second type, a collision between a critical particle and a usual particle, it was verified whether these particles were able to reach the black hole event horizon, and it was found that this is only the case if one assumed an extremal black hole, and it was further found that there is divergence in the energy at the center of mass frame. For the second type, a collision between a critical particle and a usual particle, it was verified whether these particles were able to reach the black hole event horizon, and it was found that this is only the case if one assumed an extremal black hole, and it was further found that there is divergence in the energy at the center of mass frame when the collision occurs exactly at the black hole event horizon. For the third type, a collision between a near critical particle and a usual particle, it was concluded that the energy at the center of mass frame when the collision occurs exactly at the black hole event horizon. For the third type, a collision between a near critical particle and a usual particle, it was concluded that the energy at the center of mass frame can be as large as one wants. However, this quantity is always finite, diverging only in the limit in which particle 1 is exactly critical, so one is back in second type.

In brief, the only interesting type of collision, the one that yields a divergingly high energy generation, is the second one, i.e., a collision between a critical and a usual particle in an extremal black hole background. However, even when the energy at the center of mass frame is divergingly high, it can happen that energy extraction is not possible, for instance, the particles that come out of the collision all enter into the black hole. Thus, one is advised to analyze carefully the collision process and consider the general case in which the particles have different masses. Therefore, for the situations in which the energy at the center of mass frame is divergent, we analyze a collision between ingoing particles i = 1, 2 with masses m_1 and m_2 that can be different in general, originating two final particles, i = 3, 4, with masses m_3 and m_4 , also different in general. For definiteness, it is assumed that particle 3 is outgoing in the end of the process, while particle 4 is ingoing and falls into the black hole. Thus, the energy extracted essentially equals the energy of the emitted particle, in this case the energy of particle 3. We want to establish bounds on the energy of the emitted particle, finding under which conditions energy extraction through a Penrose process is possible using the BSW effect. Thus, we aim to get a better measure, better than simply unbounded energy at the center of mass frame, of extracted energy from the black hole due to the BSW effect when combined with the Penrose process in the collisional Penrose process.

IV. PENROSE AND SUPER-PENROSE ENERGY EXTRACTION FROM A COLLISION BETWEEN A CRITICAL AND A USUAL PARTICLE: FULL STORY

A. Conservation laws and energy of critical and usual particles

Up to now, it has been shown that, under certain conditions, a BSW collision between two particles in the vicinity of the horizon of an extremal black hole can lead to a divergence of the energy at the center of mass frame. This happens for a collision between particle 1, a critical particle, and particle 2, a usual particle. However, this divergence is not a sufficient condition to guarantee that there is some outgoing particle with some net energy, let alone an unbounded energy. In this section, we find the energy and the mass of a particle which results from a collision near the black horizon and is sent to larger radii, i.e., we examine the collisional Penrose process in which the collisions are of BSW type. For spacetimes with zero or positive cosmological constant, i.e., asymptotically flat or asymptotically dS spacetimes, the escaping particle can reach infinity, for spacetimes with negative cosmological constant, i.e., asymptotically AdS spacetimes, this escaping particle only reaches infinity if it is massless.

We assume a collision between the initial ingoing particles i = 1, 2 and now consider the more general case in which the particles can have different masses, i.e., particle 1 has mass m_1 and electric charge e_1 and particle 2 has mass m_2 and electric charge e_2 . We further assume that two particles i = 3, 4 issue after the collision, with mass m_3 and electric charge e_3 , and mass m_4 and electric charge e_4 . We consider that energy, radial momentum, and electric charge are conserved in the collision process, which implies the following three conservation laws,

$$X_1 + X_2 = X_3 + X_4, (23)$$

$$\varepsilon_1 Z_1 + \varepsilon_2 Z_2 = \varepsilon_3 Z_3 + \varepsilon_4 Z_4, \tag{24}$$

$$e_1 + e_2 = e_3 + e_4, \tag{25}$$

respectively.

We further assume that particle 1 is critical and goes into the black hole so that $\varepsilon_1 = -1$, particle 2 is usual and also goes into the black hole so that $\varepsilon_2 = -1$, and particle 4 which is one of the particles that comes out of the collision is usual and goes into the black hole so that $\varepsilon_4 = -1$. The black hole is extremal since we have seen that extremal black holes are the ones that can yield large amounts of energy. We want to find the properties of particle 3 that emerges from the collision and is outgoing, see Fig. 1. For critical particles, the quantity X evaluated at the horizon, $X(r_+)$, vanishes, while for usual particles one has that $X(r_+) \neq 0$. Therefore, for particle 1, $X_1(r_+) = 0$ and its energy can be related with its electric charge as $E_1 = \frac{e_1Q}{(d-3)r_+^{d-3}}$, where Eq. (9) was used. For particle 2, which is usual, one uses the forward in time condition, t > 0, to find a lower bound for the energy. From Eqs. (8) and (9) this condition implies that $X_2(r_+) > 0$ and, therefore, $E_2 > \frac{e_2Q}{(d-3)r_+^{d-3}}$ and the same applies for particle 4, $E_4 > \frac{e_4Q}{(d-3)r_+^{d-3}}$. In brief, we can write for particles 1, 2, and 4 that $X_1 = E_1 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)$, $Z_1 = \sqrt{E_1^2 \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right)^2 - m^2 f}$, $X_2 = E_2 - \frac{e_2Q}{(d-3)r_+^{d-3}}$, $Z_2 = \sqrt{\left(E_2 - \frac{e_2Q}{(d-3)r_+^{d-3}}\right)^2 - m^2 f}$, $X_4 = E_4 - \frac{e_4Q}{(d-3)r_+^{d-3}}$, and $Z_4 = \sqrt{\left(E_4 - \frac{e_4Q}{(d-3)r_+^{d-3}}\right)^2 - m^2 f}$. Since particle 1 is assumed critical, and therefore $X_1(r_+) = 0$ and so $e_1 = \frac{(d-3)r_+^{d-3}}{Q} E_1$, we can evaluate X(r) and Z(r) of Eq. (9) near r_+ , i.e., expand it in $r - r_+$, or what amounts to the same thing in $\sqrt{f(r)}$. Doing the expansion, one finds $X_1(r) = E_1\sqrt{f(r)}\frac{d-3}{\sqrt{g(r)}}$ and $Z_1(r) = E_1\sqrt{f(r)}\sqrt{\frac{(d-3)^2}{g(r)} - \frac{m_1^2}{E_1^2}}$, plus $\mathcal{O}(f)$ in both equations. Particle 2 and

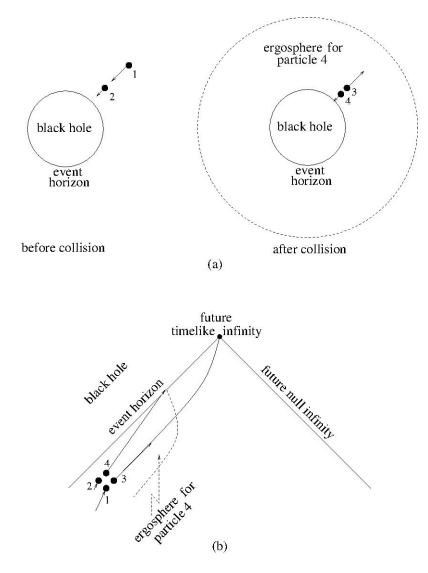


FIG. 1. (a) Schematic picture for an electrically charged BSW collision. The electrically charged particle 1 collides with the electrically charged particle 2 outside the event horizon of a black hole at the center. Particle 3 emerges from the collision escaping to infinity taking with it energy in a Penrose process, while particle 4, which is inside its own electric ergosphere, falls through the event horizon. On the left a snapshot before the collision. On the right a snapshot after the collision. Note that the electric ergosphere only materializes after particle 4 with negative energy is created in the collision. (b) Carter-Penrose diagram for the collisional Penrose process in a spacetime with zero cosmological constant. After particles 1 and 2 suffer a BSW collision, particles 3 and 4 emerge, with the former escaping to future timelike infinity through a Penrose process and the latter, being in its own electric ergosphere, falling into the black hole. Particle 3 can carry arbitrarily large energies to infinity characterizing a super-Penrose process.

one finds $X_2(r) = X_2(r_+) + \frac{e_2 Q}{r_+^{d-3}} \sqrt{\frac{f(r)}{g(r)}}, Z_2(r) = X_2(r), X_4(r) = X_4(r_+) + \frac{e_4 Q}{r_+^{d-3}} \sqrt{\frac{f(r)}{g(r)}}, Z_4(r) = X_4(r),$ plus $\mathcal{O}(f)$ in the four equations. Collecting these results gives

$$X_{1}(r) = E_{1} \frac{d-3}{\sqrt{g(r)}} \sqrt{f(r)}, \qquad Z_{1}(r) = E_{1} \sqrt{\frac{(d-3)^{2}}{g(r)} - \frac{m_{1}^{2}}{E_{1}^{2}}} \sqrt{f(r)}, \qquad Z_{2}(r) = X_{2}(r) + \frac{e_{2}Q}{r_{+}^{d-3}\sqrt{g(r)}} \sqrt{f(r)}, \qquad Z_{2}(r) = X_{2}(r), \qquad (26)$$

$$X_{4}(r) = X_{4}(r_{+}) + \frac{e_{4}Q}{r_{+}^{d-3}\sqrt{g(r)}} \sqrt{f(r)}, \qquad Z_{4}(r) = X_{4}(r),$$

plus $\mathcal{O}(f)$ in all equations, and where $X_2(r_+) = E_2 - \frac{e_2}{(d-3)} \frac{Q}{r_+^{d-3}}$ and $X_4(r_+) = E_4 - \frac{e_4}{(d-3)} \frac{Q}{r_+^{d-3}}$.

Thus, let us summarize what we have up to now. The situation in analysis in this section corresponds to a collision at some radius r near the extremal horizon radius r_+ between a critical and a usual particle, since this is the case for which the energy at the center of mass frame diverges. Therefore, particle 1 is critical and particle 2 is usual. We also assume that particle 4 that comes out of the collision is usual. Moreover, since we are dealing with the BSW effect, we assume that particle 1 goes in, particle 2 goes in, and further assume that particle 4 goes in. We consider that the other final particle, particle 3, escapes somehow after the collision. Several situations can happen, namely, particle 3 goes immediately out so that at the collision one has $\varepsilon_3 = +1$, or particle 3 goes in first, so that at the collision one has $\varepsilon_3 = -1$, and then reverses direction and goes out, or particle 3 goes in always and $\varepsilon_3 = -1$. Now we calculate the properties of the final outgoing emitted particle, particle 3.

B. Energy of the outgoing emitted particle

Particle 3, the one emitted, needs a very special treatment for the calculation of its quantities. The emitted particle, emerging from the particle collision out of an extremal black hole, can only move for radii for which the square root in the definition of Z, $Z = \sqrt{X^2 - m^2 f}$, see Eq. (9), is well defined. This means that particle motion can only happen for radii satisfying $X \ge m\sqrt{f}$, with $X = E - \frac{eQ}{(d-3)r^{d-3}}$, see Eq. (9). Thus, $X \ge m\sqrt{f}$, translates into $e \le \frac{(d-3)r^{d-3}}{Q} \left(E - m\sqrt{f}\right)$. From the definition of f of an extremal black hole, see Eq. (7), we have $r^{d-3} = \frac{r_+^{d-3}}{(1-\sqrt{\frac{f}{g}})^{d-3}}$, so that $e \le \frac{(d-3)r_+^{d-3}}{Q} \frac{E - m\sqrt{f(r)}}{(1-\sqrt{\frac{f(r)}{g(r)}})^{d-3}}$, i.e., $e \le \frac{(d-3)r_+^{d-3}E}{Q} \frac{1-\frac{m}{E}\sqrt{f(r)}}{(1-\sqrt{\frac{f(r)}{g(r)}})^{d-3}}$. We identified critical charge e_c as $e_c = \frac{(d-3)r_+^{d-3}E}{Q}$, see Eq.(11). So, one has $e \le e_c \frac{1-\frac{m}{E}\sqrt{f(r)}}{(1-\sqrt{\frac{f(r)}{g(r)}})^{d-3}}$. Now for each particle i, the function on the right hand side of this

inequality acts as a potential barrier, so we define the potential e_{i0} for particle i as $e_{i0}(r) \equiv e_{ic} \frac{1 - \frac{m_i}{E_i} \sqrt{f(r)}}{(1 - \sqrt{\frac{f(r)}{\sigma(r)}})^{d-3}}$, with

 $e_{i0}(r_+) = e_{ic}$. For particle 3, i = 3, we have $e_{30}(r) \equiv e_{3c} \frac{1 - \frac{m_3}{E_3}\sqrt{f(r)}}{(1 - \sqrt{\frac{f(r)}{f(r)}})^{d-3}}$ as the potential that particle 3 feels. Clearly

 $e_{30}(r)$ establishes where motion is allowed, and since $e_3 \leq e_{30}(r)$ the allowed motion depends on the particle's charge. Since, we are assuming Q > 0, we assume that $e_3 > 0$, so that the criticality condition $X_3(r_+) = 0$ can be satisfied, bearing in mind that these assumptions are without loss of generality. The collisions of interest occur near the horizon where f(r) is small, so $e_{30}(r)$ can be expanded in $\sqrt{f(r)}$ as

$$e_{30}(r) \equiv e_{3c} \left[1 + \left(\frac{d-3}{\sqrt{g(r)}} - \frac{m_3}{E_3} \right) \sqrt{f(r)} \right],$$
(27)

plus $\mathcal{O}(f)$ terms and with $e_{30}(r_+) = e_{3c}$. This $e_{30}(r)$ is the potential that particle 3 feels at this order of approximation, and the condition for particle 3 motion is then that the electric charge e_3 obeys

$$e_3 \le e_{30}(r)$$
 (28)

with $e_{30}(r)$ given in Eq. (27). Note that for particle 3 resulting from a collision in the vicinity of the black hole horizon to be able to escape outwards, one has to have $m_3 < \frac{d-3}{\sqrt{g(r)}}E_3$, such that the term proportional to \sqrt{f} in Eq. (27) is positive at least close to the horizon. Otherwise, $e_{30}(r)$ decreases with increasing r, and there would be a radius where the condition Eq. (28) would be violated and motion would not be possible. Note that $m_3 < \frac{d-3}{\sqrt{g(r)}}E_3$ can be written as $E_3 > \frac{\sqrt{g(r)}}{d-3}m_3$ so that one can define a lower energy $E_{3l} \equiv \frac{\sqrt{g(r)}}{d-3}m_3$, such that $E_3 > E_{3l}$. There are four cases of interest in which collisions can happen in the vicinity of the horizon. All of them have

There are four cases of interest in which collisions can happen in the vicinity of the horizon. All of them have $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3$. We now enumerate the four cases of this collisional Penrose process. **1.** $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3$, $e_3 \leq e_{3c}$, and $\varepsilon_3 = +1$. The collision happens in the allowed region, just near the horizon, where $e_3 \leq e_{3c} \leq e_{30}(r)$. After the collision, particle 3 moves to larger radii since $\varepsilon_3 = +1$. Particle 3 is near critical or critical, since in this case the collision can happen at the horizon itself. **2.** $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3$, $e_{3c} < e_3 < e_{30}(r)$, and $\varepsilon_3 = +1$. The collision happens in the allowed region, with the condition $e_3 < e_{30}(r)$ meaning that it happens at some radius just outside the horizon radius r_+ , i.e., particle 3 is near-critical. Since $\varepsilon_3 = +1$, particle 3 moves outwardly immediately after the collision.

3. $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3$, $e_3 \leq e_{3c} \leq e_{30}(r)$, and $\varepsilon_3 = -1$. The collision happens in the allowed region, just near the horizon, where $e_3 \leq e_{3c} \leq e_{30}(r)$. Particle 3 is near critical or critical, since in this case the collision can happen at the horizon itself. Since $\varepsilon_3 = -1$, the particle moves first toward the horizon, reaches a turning point before it, and moves back in the outward direction. This turning point is found from Eq. (28). After knowing the electric charge e_3 of the emitted particle 3, one finds from Eq. (28) the radius r at which $e_3 = e_{30}(r)$, with $e_{30}(r)$ given in Eq. (27), with this r being the turning point. **4.** $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3$, $e_{3c} < e_3 < e_{30}(r)$, and $\varepsilon_3 = -1$. The collision happens in the allowed region, with the condition $e_3 < e_{30}(r)$ meaning that it happens at some radius r just outside the horizon radius r_+ , i.e., particle 3 is near-critical. Since $\varepsilon_3 = -1$, the particle moves first toward the horizon, reaches a turning point before it, and moves back in the outward direction.

To distinguish systematically between cases 1., 2., 3., and 4., a parameter δ is defined in such a way that

$$e_3 = e_{3c} \left(1 + \delta \right), \tag{29}$$

where $\delta < 0$ in cases 1. and 4., and $\delta \ge 0$ in cases 2. and 3. For cases 1., 2., 3., and 4. the collisions occur at r near the horizon r_+ , and, therefore, X_3 and Z_3 can be expanded for small values of $r - r_+$ which can then be substituted by $\sqrt{f(r)}$. Thus, δ is very small, and comparing Eq. (13) with Eq. (29) we see that particle 3, the emitted particle, is indeed a near critical particle. For near critical particles, δ should be controlled and one way to do it is to expand δ in a series in \sqrt{f} which is near zero for r near r_+ ,

$$\delta = \frac{\Delta e_3}{e_3} \sqrt{f} \,, \tag{30}$$

plus $\mathcal{O}(f)$ and where $\frac{\Delta e_3}{e_3}$ is some constant value, not infinitesimal, see [17] for d = 4 and k = 0. The quantity $\frac{\Delta e_3}{e_3}$ is to be determined from the conservation equations as a function of geometrical quantities and particle quantities. There is also an upper bound for $\frac{\Delta e_3}{e_3}$, imposed by the condition $e_3 < e_{30}(r)$, which from Eqs. (27) and (29) reads $\frac{\Delta e_3}{e_3} < \frac{d-3}{\sqrt{g(r)}} - \frac{m_3}{E_3}$. Now, since particle 3 is near critical, X_3 and Z_3 can be expanded in $\sqrt{f(r)}$ as

$$X_{3} = E_{3} \left(\frac{d-3}{\sqrt{g(r)}} - \frac{\Delta e_{3}}{e_{3}} \right) \sqrt{f(r)}, \qquad Z_{3} = E_{3} \sqrt{\left(\frac{d-3}{\sqrt{g(r)}} - \frac{\Delta e_{3}}{e_{3}} \right)^{2} - \frac{m_{3}^{2}}{E_{3}^{2}} \sqrt{f(r)}, \tag{31}$$

plus $\mathcal{O}(f)$ terms. For $\delta < 0$ one has $\frac{\Delta e_3}{e_3} < 0$ and all is fine in Eq. (31). For $\delta > 0$ one gets from Eq. (31) the bound $\frac{\Delta e_3}{e_3} < \frac{d-3}{\sqrt{g(r)}}$ which is weaker than the one just found above.

From the momentum conservation relation, Eq. (24), i.e., $\varepsilon_1 Z_1 + \varepsilon_2 Z_2 = \varepsilon_3 Z_3 + \varepsilon_4 Z_4$, together with electric charge and energy conservation, Eqs. (25) and (23), respectively, assuming that particle 1 is critical and particles 2 and 4 are usual, see Eq. (26), and $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = -1$, one finds upon using Eq. (31) that

$$\frac{d-3}{\sqrt{g(r_{+})}} \left[E_1 - \sqrt{E_1^2 - m_1^2 \frac{g(r_{+})}{(d-3)^2}} \right] + E_3 \left(\frac{\Delta e_3}{e_3} - \frac{d-3}{\sqrt{g(r_{+})}} \right) = \varepsilon_3 E_3 \sqrt{\left(\frac{d-3}{\sqrt{g(r_{+})}} - \frac{\Delta e_3}{e_3} \right)^2 - \frac{m_3^2}{E_3^2}}, \quad (32)$$

valid in order \sqrt{f} . From this equation, the expression for the energy at the center of mass obtained in Eq. (20) can be recovered, considering that all the particles involved in this process have the same mass m, see the Appendix for the derivation. One can solve Eq. (32) for $\frac{\Delta e_3}{e_3}$. Before that, the first term that appears in Eq. (32) is an important quantity as we are about to find, and so we define

$$\Delta E_1 \equiv \frac{d-3}{\sqrt{g(r_+)}} \left[E_1 - \sqrt{E_1^2 - m_1^2 \frac{g(r_+)}{(d-3)^2}} \right].$$
(33)

Then, after solving Eq. (32), Δe_3 is the expression $\Delta e_3 = \frac{1}{E_3} \frac{d-3}{\sqrt{g(r_+)}} (E_3 - \frac{m_3^2 + (\Delta E_1)^2}{2\Delta E_1} \frac{\sqrt{g(r_+)}}{d-3}) e_3$. Clearly, the quantity $\frac{m_3^2 + (\Delta E_1)^2}{2\Delta E_1} \frac{\sqrt{g(r_+)}}{d-3}$ can be defined as an energy, and we define E_{3b} , with b for bound, as $E_{3b} \equiv \frac{m_3^2 + (\Delta E_1)^2}{2\Delta E_1} \frac{\sqrt{g(r_+)}}{d-3}$. Upon using ΔE_1 of Eq. (33), the expression for E_{3b} can be put in the form

$$E_{3b} = \frac{1}{2} \frac{m_3^2 + m_1^2}{m_1^2} E_1 + \frac{1}{2} \frac{m_3^2 - m_1^2}{m_1^2} \sqrt{E_1^2 - m_1^2 \frac{g(r_+)}{(d-3)^2}}.$$
(34)

Note that any information about particle 2 has disappeared from the formulas above, but in fact it has been deposited hiddenly in the quantities pertaining to particle 3. Now, from Eqs. (32)-(34) we have that

$$\Delta e_3 = \frac{d-3}{\sqrt{g(r_+)}} \left(1 - \frac{E_{3b}}{E_3}\right) e_3.$$
(35)

From Eq. (35), we deduce that if $E_3 \leq E_{3b}$, then $\frac{\Delta e_3}{e_3} \leq 0$ and from Eq. (30) one has $\delta \leq 0$, and then from Eq. (29) one has $e_3 \leq e_{3c}$. Then, for this case one can write $e_3 \leq e_{3c} \leq e_{30}(r)$. If $E_3 > E_{3b}$, then $\frac{\Delta e_3}{e_3} > 0$ and from Eq. (30) one has $\delta > 0$, and then from Eq. (29) one has $e_3 > e_{3c}$. Since $e_3 < e_{30}(r)$ for sure, one is within the case $e_{3c} < e_3 < e_{30}(r)$. A particular delicate case is when m_1 is small, even zero, $m_1 = 0$, as can be seen from Eq. (34), since then E_{3b} is very large, even infinite, and one has to decide whether E_3 is equal or lower than E_{3b} or E_3 can be higher than E_{3b} , so that $\frac{\Delta e_3}{e_3} \leq 0$ and there is no turning point or $\frac{\Delta e_3}{e_3} > 0$ and there is a turning point, respectively. We now show that to have a turning point one must have $m_1 > 0$, i.e., m_1 cannot be a massless particle. For that, we expand Eq. (35) for small $\frac{m_1}{E_1}$ to find that for $\frac{\Delta e_3}{e_3}$ to be positive then one has $\frac{m_3^2}{E_3} < \frac{m_1^2}{E_1}$. To have a turning point one has to have $Z_3 \neq X_3$ which from Eq. (31) means that $\frac{m_3}{E_3}$ is not zero, more precisely $\frac{m_3}{E_3} > 0$, therefore also $\frac{m_3}{\sqrt{E_3}} > 0$. Thus, we conclude that one has $m_1 > \sqrt{\frac{E_1}{E_3}} m_3$ for having a turning point. Since the right hand side is never zero, we have $m_1 > 0$ mandatorily, in order to have a turning point in the case m_1 small. So for this case $m_1 = 0$ is excluded, i.e., m_1 can be very small but not zero. In the case that $m_1 \leq \sqrt{\frac{E_1}{E_3}} m_3$ then $\frac{\Delta e_3}{e_3} \leq 0$ and there is no turning point, in particular $m_1 = 0$ has no turning point. There is still a further important equation that we have to explicitly give. Indeed, Eq. (32) together with Eq. (35) gives

$$\varepsilon_3 |(\Delta E_1)^2 - m_3^2| = (\Delta E_1)^2 - m_3^2.$$
(36)

If $\Delta E_1 > m_3$ we deduce from Eq. (36) that $\varepsilon_3 = +1$, particle 3 goes out from the point of collision. If $\Delta E_1 < m_3$ we deduce from Eq. (36) that $\varepsilon_3 = -1$, particle 3 goes in from the point of collision, in particular when m_1 is sufficiently small and so ΔE_1 is small, which comprises the case $m_1 = 0$, particle 3 goes in. If $\Delta E_1 = m_3$, Eq. (36) is an identity.

We have still to analyze the energy of particle 4, since to have energy extraction and in order that the BSW collision gives rise to a Penrose process, this energy has to be negative. Since the electric charge Q of the black hole is the same for all four particles, the point r of collision is the same for all four particles, and there is conservation of electric charge, i.e., $e_1 + e_2 - e_3 - e_4 = 0$, see Eq. (25), we have that the conservation equation $X_1 + X_2 = X_3 + X_4$, see Eq. (23), implies that at the point of collision, $E_1 + E_2 = E_3 + E_4$, i.e., $E_4 = E_1 + E_2 - E_3$. For energy extraction, one necessarily has $E_3 > E_1 + E_2$. Thus, when there exists extraction one finds $E_4 < 0$ necessarily, and so $e_4 < 0$. Therefore, when $t_{ergo 4} = e_{erg 2} = e_{erg 2} + e_{erg 2} = e_{erg 2} + e_{erg 2} = e_{erg 2} + e_{erg$

From the analysis made, namely, from the conditions that follow from Eqs. (35) and (36) and the discussions after them, the four cases considered above can be spelled out in detail, with the calculated bounds for the energy and mass of particle 3, the emitted particle. The four cases mentioned of the collisional Penrose process we are studying can be now enumerated in full detail as follows.

- 1. OUT⁻: $0 \le m_3 < \Delta E_1$ and $m_3 \frac{\sqrt{g(r_+)}}{d-3} \le E_3 \le E_{3b}$. Thus, $e_3 \le e_{3c} \le e_{30}(r)$ and $\varepsilon_3 = +1$, the particle goes directly out after the collision. This case, yields no energy extraction since E_3 is less than E_1 . This happens because, as $m_3 < \Delta E_1$ one has $E_{3b} < \Delta E_1 \frac{\sqrt{g(r_+)}}{d-3}$, which is lower than E_1 , from Eq. (33).
- 2. OUT^+ : $0 \le m_3 < \Delta E_1$ and $E_{3b} \le E_3 < \infty$. Thus, $e_{3c} < e_3 < e_{30}(r)$ and $\varepsilon_3 = +1$, the particle goes directly out after the collision. This case, can yield a Penrose process with energy extraction and it is not restricted, but there is an upper bound in m_3 , i.e., not any mass can be emitted. Moreover, clearly, when m_1 is very small then E_{3b} can be arbitrarily large but not infinite, and the energy extracted in particle 3, E_3 , can be arbitrarily large but not infinite as super-Penrose process. When there is energy extraction one has $E_4 < 0$ and there is an electric ergosphere.
- 3. IN⁻: $\Delta E_1 < m_3 < \infty$ and $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3 \leq E_{3b}$. Thus, $e_3 \leq e_{3c} \leq e_{30}(r)$ and $\varepsilon_3 = -1$, the particle goes in immediately after the collision and then continues the motion entering down the black hole. We have seen that for the case $m_1 \leq \sqrt{\frac{E_1}{E_3}} m_3$ there is no turning point for the ingoing particle. So, small masses m_1 which yield high E_{3b} have no turning points. There is no extraction of energy at all. In particular, for $m_1 = 0$, which falls within this case, one obtains $E_{3b} = \infty$, and therefore $E_3 = \infty$, but all this energy goes down the black hole.

4. IN⁺: $\Delta E_1 < m_3 < \infty$ and $E_{3b} \leq E_3 < \infty$. Thus, $e_{3c} < e_3 < e_{30}(r)$ and $\varepsilon_3 = -1$, the particle goes in immediately after the collision and then reverses the motion at some radius nearer the horizon to move outward from then on. This case, can yield a Penrose process with energy extraction, it is not restricted and there is a lower bound in m_3 . Moreover, clearly, when m_1 is small then E_{3b} can be arbitrarily large but not infinite, and the energy extracted in particle 3, E_3 , can be arbitrarily large but no infinite as well, characterizing thus a super-Penrose process. There is also the possibility that m_3 be very large in this case, allowing thus for the emission of superheavy particles. When there is energy extraction one has $E_4 < 0$ and there is an electric ergosphere.

To have a hand on these results and what one gets out of them, let us suppose that the quantities given, the initial inputs, are $m_1, E_1, e_1, \varepsilon_1, m_2, E_2, e_2, \varepsilon_2$, and the point r of the collision of particles 1 and 2. This point of collision, to be of interest, is near the horizon radius r_{+} and can, in principle, be directly found from kinematic expressions for particles 1 and 2. It is also assumed that particle 4 travels inward so that ε_4 is known, plus that particle 4 is a usual particle. Then, m_3 , E_3 , e_3 , and ε_3 can have different values, depending on the collision internal process itself. One has that ε_3 can be either +1 or -1. For each collision, one could consider m_3 , E_3 , and e_3 as free parameters, but this cannot be, since for the given inputs above, m_3 , E_3 , and e_3 are entangled quantities, as can be seen from Eqs. (29), (30), and (35). Let us then consider m_3 and E_3 as free parameters. Thus, m_3 emitted can have some range of values and E_3 emitted is also within some range, but within these ranges m_3 and E_3 can be any, they are going to depend on the very details of the collision internal process. Given m_3 and E_3 , the electric charge e_3 is then fixed, with its allowed values being also within some range. Now, in the case particle 3 moves always inward then it is hard to measure m_3 and E_3 , but this is anyway irrelevant since it is the case IN⁻ and there is no energy extraction. In the case particle 3 moves outward in one stage or the other, which is the interesting case, we need a detector to measure m_3 and E_3 , with e_3 being then calculated from Eqs. (29) and (30). From the measured value of m_3 and the initial inputs, we find ε_3 , making use of Eq. (36), and so we can discern between the cases OUT, be it OUT⁻ or OUT⁺, that has $\varepsilon_3 = +1$, and IN^+ that has $\varepsilon_3 = -1$, this latter case yielding directly energy extraction. To distinguish between OUT⁻ and OUT^+ we have to calculate E_{3b} through Eq. (34). Given m_1 and E_1 , and measuring m_3 , one calculates E_{3b} . Then, if the measured E_3 is in the range $m_3 \frac{\sqrt{g(r_+)}}{d-3} \leq E_3 \leq E_{3b}$ one is in the case OUT⁻, and there is no energy extraction. If the measured E_3 is in the range $E_{3b} \leq E_3 < \infty$ one is in the case OUT⁺, and there is energy extraction. Moreover, we can find from the conservation laws, Eqs. (23)-(25), the other physical quantities of particle 4, i.e., m_4 , E_4 , and e_4 , given the inputs and the measured values of m_3 and E_3 . Note that since there are many quantities, namely, m_1 , $E_1, e_1, \varepsilon_1, m_2, E_2, e_2, \varepsilon_2, m_3, E_3, e_3, \varepsilon_3, m_4, E_4, e_4, \varepsilon_4$, and the point of collision r, what is initial input, what is measured, and what is deduced is a matter of choice. Above, we have given what we think is an interesting example of a collisional Penrose process in which the collisions are of BSW type. But a great number of other examples might be given. For instance, if in the example above, the detector is also able to measure the electric charge e_3 , then the point of collision does not need to be an initial input, it can be found a posteriori through Eqs. (30) and (35).

C. The dependence on the cosmological constant through k and on the dimension d and further comments

1. The dependence on the cosmological constant through k and on the dimension d

The bounds for the energy extracted, i.e., the energy of particle 3, depend on the factor $\frac{\sqrt{g(r_+)}}{d-3}$, see Eq. (34). Using the factorization of f(r) obtained in Eq. (7), it follows that $\frac{g(r_+)}{(d-3)^2} = \frac{1}{2} \frac{r_+^2}{(d-3)^2} \frac{d^2 f}{dr^2}(r_+)$. The second derivative of f(r) can be computed from Eq. (2), leading to $\frac{1}{2} \frac{d^2 f}{dr^2}(r_+) = -(d-3)(d-2)\frac{\mu}{r_+^{d-1}} + (d-3)(2d-5)\frac{\chi Q^2}{r_+^{2(d-2)}} - \frac{k}{l^2}$. Using this result in the previous equation, it follows that $\frac{g(r_+)}{(d-3)^2} = -\frac{d-2}{d-3} \frac{\mu M}{r_+^{d-3}} + \frac{2d-5}{d-3} \frac{\chi Q^2}{r_+^{2(d-3)}} - k \frac{1}{(d-3)^2} \frac{r_+^2}{l^2}$. Imposing the condition for an extremal black hole, Eq. (6), the following result can finally be obtained

$$\frac{g(r_{+})}{(d-3)^2} = 1 - k \frac{(d-2)(d-1)}{(d-3)^2} \frac{r_{+}^2}{l^2}.$$
(37)

From Eq. (37) we can discuss the dependence of the results on the spacetime dimension d and also on the cosmological constant Λ , i.e., the cosmological length l.

Firstly, we analyze the dependence on d. For k = -1, one has that $\frac{g(r_+)}{(d-3)^2}$ increases in the presence of the cosmological constant and depends on d. For k = 0, $\frac{g(r_+)}{(d-3)^2} = 1$, the bounds do not depend on the number of dimensions d, and the

bounds are the same obtained for an asymptotically flat Reissner-Nordström black hole spacetime in d = 4 [17]. For k = 1, the factor $\frac{g(r_+)}{(d-3)^2}$ decreases due to the presence of the cosmological constant, noting also that $\frac{(d-2)(d-1)}{(d-3)^2} \frac{r_+^2}{l^2} < 1$, since the r_+ we are considering is the black hole horizon, not the cosmological one, and the bound depends on d. Of course, one could redefine the cosmological constant, i.e., the cosmological length l, to include the d-dependent factors, but surely in such a case other quantities, not necessarily related to this problem, that depend on the pure cosmological constant would then depend on the dimension d through the inverse of the factor $\frac{(d-2)(d-1)}{(d-3)^2}$.

Secondly, we analyze the dependence on the cosmological constant Λ , i.e., the cosmological length l. The bound E_{3b} for the energy of the emitted particle can be interpreted as function of $\frac{\sqrt{g(r_+)}}{d-3}$ given by $E_{3b} = \frac{1}{2} \frac{m_3^2 + m_1^2}{m_1^2} E_1 + \frac{1}{2} \frac{m_2^2 + m_1^2}{m_1^2} E_1$ $\frac{1}{2}\frac{m_3^2-m_1^2}{m_1^2}\sqrt{E_1^2-m_1^2\frac{g(r_+)}{(d-3)^2}}$, see Eq. (34). Therefore, for the cases for which one has a collisional Penrose process and one can have energy extraction, i.e., OUT^+ and IN^+ , one has to consider different possibilities. First, for k = 0 the lower bound E_{3b} is independent of the cosmological constant, directly from Eq. (37), together with Eq. (34), and of d. Thus, we can compare the cases of negative cosmological constant, i.e., k = -1, with the cases of positive cosmological constant, i.e., k = 1, for fixed $\frac{r_+}{l}$, i.e., fixed horizon radius relative to the cosmological radius, which means compare $\frac{r_+}{l}$ in the various situations. If $m_3 < m_1$, the lower bound E_{3b} for the energy of the emitted particle is greater for negative cosmological constant, i.e., k = -1, than for positive cosmological constant, i.e., k = +1. If $m_3 = m_1$, the lower bound E_{3b} for the energy of the emitted particle is equal for negative cosmological constant, i.e., k = -1, and for positive cosmological constant, i.e., k = +1, which means that E_{3b} does not depend on the cosmological constant. If $m_3 > m_1$, the lower bound E_{3b} for the energy of the emitted particle is smaller for negative cosmological constant, i.e., k = -1, than for positive cosmological constant, i.e., k = +1. An interpretation can be tried. Define $\Delta m \equiv m_3 - m_1$. One can associate a net lengthscale λ by $\lambda = \frac{1}{\Delta m}$, with λ pointing inward, i.e., to decreasing radii, if $\Delta m < 0$, and λ pointing outward, i.e., to increasing radii, if $\Delta m > 0$. In this convention one also has that the AdS lengthscale lpoints inward, and the dS lengthscale l points outward. Thus, since for $\Delta m < 0$, λ net points inward, l AdS points inward and l dS points outward, one needs higher energy, higher E_{3b} , in AdS in relation to dS to have the particle going out. Also, since for $\Delta m > 0$, λ net points outward, l AdS points inward and l dS points outward, a particle that can go out, goes out with lower energy, lower E_{3b} , in AdS in relation to dS. For $\Delta m = 0$ one has that λ net is infinite and does not point in any direction, and so there is no dependence on the cosmological constant. This is the best we can offer as an interpretation.

In brief, different conclusions can be obtained depending on the parameter k, i.e., on the sign of the cosmological constant. For k = 0, i.e., for an asymptotically flat Reissner-Nordström black hole spacetime, the bounds for the energy extracted in the collisional Penrose processes considered do not depend on the number of dimensions. For $k = \pm 1$, these bounds depend on the number of dimensions, on the sign of k, and on the masses m_1 and m_3 . These bounds can be larger or lower than for k = 0 depending on the sign of k and on whether m_1 is lower or greater than m_3 .

We have considered the possible scenarios for the collisional Penrose process from a BSW collision, of colliding electrically charged particles in a Reissner-Nordström background with cosmological constant and in d dimensions. One could think of considering of doing a classification of the decayment Penrose process, instead of the collisional Penrose process, to find the possible scenarios in the generic background considered here. This decayment Penrose process of electrically charged particle classification has been done in a Reissner-Nordström background with zero cosmological constant in d = 4 dimensions [4]. A detailed comparison of these two types of Penrose processes, the collisional and the decayment, is certainly of great interest.

2. Further comments

In the collision processes that we have studied, an essential role is played by critical or near-critical particles in the vicinity of the black hole horizon. Such particles can be obtained by fine-tuning their parameters in their initial state, i.e., in the state previous to the collision. One might wonder how does such type of particles come into being in a plausible manner. We can think of two ways. One is through an experiment where the physicist prepares the particles in the necessary fine tuned manner. The other is to have have a cluster of a myriad of particles with all sorts of energy in the vicinity of the black hole such that statistically some of them are indeed near critical.

Another question that can be raised is how the collisions in the vicinity of an extremal black hole horizon can give rise to measurable physical observables. In our study, we have been interested in the understanding of the energetics of the possible processes. We have discussed not only the local center of mass energies $E_{\rm cm}$ in the collisions, which is indeed the original BSW effect, but have gone beyond it and discussed the Killing energies E that can be achieved in the collision. We have found that there are situations in which the energy E is larger than the initial energy characterizing a collisional Penrose process. In addition, we have found that there are situations that not only the center of mass energies $E_{\rm cm}$ are arbitrarily high but also the Killing energies are arbitrarily large. We have thus displayed examples of super-Penrose processes. If particles have arbitrarily large Killing energies it means they can reach far distances, although impediments to it can arise. For instance, the particles ejected can lose their energy through scattering processes or even fall down the black hole. Thus, independently of whether or not the ejected particles reach far distances, the issue of the measuring of physical observables by an asymptotic observer, is worth pursuing, although it has not been touched by us. Nevertheless, we can make now some comments related to the electrically charged case which is the one we have studied. On could think of direct and indirect detections. Direct detections are indeed possible here. Since the outgoing charged particle can have arbitrarily large energy, it can be detected directly at infinity by a charge counter. One thing we can say for sure in this direct detection case is that if the outgoing particle is directly detected then it is automatically a nearly fine tuned particle, as a usual outgoing particle cannot escape from a black hole horizon. On the other hand, indirect detections can happen from various radiative processes involving the outgoing particle interacting electromagnetically with other particles. One possible indirect observational signature could be provided by inverse Compton scattering, whereby the outgoing highly energetically particle loses energy when it encounters a photon, which photon can then be detected at infinity in the gamma ray range. Another possible observational signature is through bremsstrahlung, where the outgoing highly energetic particle is deflected by some other electric charged particle, loses energy and radiates photons. Bremsstrahlung has a characteristic spectrum, a continuous spectrum peaking at higher and higher frequencies as the deceleration of the particle increases. It would be of interest to find how these phenomena are affected for processes with or without electric charge, rotation, or cosmological constant.

A remark is in order. Electrically charged black holes of microscopic or macroscopic size are prone to be discharged by the surrounding medium. On the other hand, if the black holes are isolated, i.e., with no surrounding medium, quantum effects come into play. Due to vacuum polarization near the event horizon of the black hole, electrically charged isolated black holes are susceptible of discharging themselves sooner or later. Indeed, an electrically charged black hole favors the absorption of particles with opposite charge, losing therefore its charge. One can think of ways of bypassing this issue. One can take the charge to be a topological charge so that there are no particles to radiate. Or else, one can admit that the lightest charged particle of the theory is massive enough that cannot be created, and thus cannot be absorbed by the black hole. For instance, a magnetically charged black hole can lose its charge only by creating magnetic monopoles, which in principle are sufficiently massive such that the probability of creation is highly suppressed and stabilizes the discharging of the black hole. An alternative to this scheme would be to posit a central charge that arises in the algebra of supergravity theories. Interesting to note that for isolated black holes with sufficient high electric charge, the emission of charged massive particles is exponentially suppressed due to the Schwinger effect. Such black holes have to be large and massive, which implies a low electric field at the event horizon, quenching pair creation of massive particles. If in the electromagnetic theory in use, there are no massless charged particles, as is our case, then the black hole cannot create and emit the lightest possible massive particle. These large isolated black holes emit thus only neutral particles via Hawking radiation and tend to an extremal state, which are the ones that interest us here. Thus, the effects we have been studying in this work could arise in isolated electric charged black holes where suppression effects for quantum particle creation can occur to stabilize the electric charge of the black holes, be they microscopic or macroscopic. The astrophysical black holes that so far have been observed are macroscopic objects, which being surrounding by matter, would discharge quickly, and so for them, rotational effects are the important ones. We have not dealt with angular momentum effects of a rotating black hole, nevertheless, the effects we have found for electrically charged black holes may serve as a guide to black holes with angular momentum charge.

V. CONCLUSIONS

We have analyzed the BSW mechanism and the corresponding energy extraction in a collisional Penrose process, for a d-dimensional extremal black hole spacetime with horizon radius r_+ , electric charge Q, and cosmological constant $k\Lambda$, with k = -1, 0, +1, i.e., the spacetime can be asymptotically AdS, flat, or dS, respectively. By identifying that the relevant collision process, namely, the one that yields divergingly high center of mass energies in the collision of two ingoing particles, is the collision between a critical and a usual particle, we set on to study this process in all detail. In this, we have put bounds on the mass, energy, and electric charge of the particle emitted and have classified the cases in which a net large energy extraction can be obtained from the extremal Reissner-Nordström black hole. In some cases one finds that a super-Penrose process is possible, where an arbitarily large Killing energy can be carried by the outgoing particle. We have also shown that the created particle that falls down the hole has negative energy and is surrounded by an electric ergosphere allowing thus the existence of a Penrose process. Examples for the particle physical quantities that are given a priori, what might be measured in the collision process, and which information is obtained a posteriori can be made concrete, as was exemplified in one case for particles 1, 2, 3, and 4. We have shown that the bounds on the energy of the emitted particle do not depend on the dimension d for zero cosmological constant, i.e., for asymptotically flat black hole spacetimes. On the other hand, the bounds do depend on d for nonzero cosmological constant, i.e., for negative cosmological constant or asymptotically AdS spacetimes, and for positive cosmological constant or asymptotically dS spacetimes. This dependence can be seen from the expressions for $\frac{g(r_+)}{(d-3)^2}$ and E_{3b} above, and turns out to be a weak dependence, from a factor 6 when d = 4 to a factor 1 when d is infinite. We have also shown that the bounds for the energy extracted for each fixed dimension, are different depending on whether the cosmological constant is negative, zero, or positive.

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Appendix: Calculation supporting the results of the main text

1. Getting back the center of mass energy expression

In Sec. IV, we considered that the four particles have different masses between themselves and then obtained equations for the energy of the emitted particle. Now, we show that the expressions obtained in Sec. IV lead to the center of mass energy expression in the particular case that all masses are equal found in Eq. (20) of Sec. III.

We start by writing the energy at the center of mass frame in terms of the relevant quantities of particles 3 and 4, assuming $m \equiv m_1 = m_2 = m_3 = m_3$. This gives

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{X_3(r)X_4(r) + \varepsilon_3 Z_3(r)Z_4(r)}{m^2 f(r)}, \qquad (A.1)$$

where we have put $\varepsilon_4 = -1$ as was assumed, and since particle 3 can move in or out, we have left ε_3 unspecified, it can be -1 or +1. Then, from the expressions for $X_3(r)$, $X_4(r)$, $Z_3(r)$, and $Z_4(r)$ given in Eqs. (26) and (31), one finds

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{X_4(r_+)}{\sqrt{f(r)}m^2} \left[E_3\left(\frac{d-3}{\sqrt{g(r_+)}} - \frac{\Delta e_3}{e_3}\right) + \varepsilon_3 E_3 \sqrt{\left(\frac{d-3}{\sqrt{g(r_+)}} - \frac{\Delta e_3}{e_3}\right)^2 - \frac{m^2}{E_3^2}} \right].$$
 (A.2)

Since $X_1(r_+) = X_3(r_+) = 0$, we have from energy conservation, Eq. (23), that $X_4(r_+) = X_2(r_+) = E_2 - \frac{e_2Q}{(d-3)r_+^{d-3}}$, and thus

$$X_4(r_+) = E_2 - \frac{e_2 Q}{(d-3) r_+^{d-3}}.$$
(A.3)

From Eq. (32) one finds that in the case all masses are equal, the following expression holds

$$\frac{d-3}{\sqrt{g(r_+)}} \left[E_1 - \sqrt{E_1^2 - m^2 \frac{g(r_+)}{(d-3)^2}} \right] + E_3 \left(\frac{\Delta e_3}{e_3} - \frac{d-3}{\sqrt{g(r_+)}} \right) = \varepsilon_3 E_3 \sqrt{\left(\frac{d-3}{\sqrt{g(r_+)}} - \frac{\Delta e_3}{e_3} \right)^2 - \frac{m^2}{E_3^2}}.$$
 (A.4)

Substituting Eqs. (A.3) and (A.4) into Eq. (A.2) yields

$$\frac{E_{\rm CM}^2(r)}{2m^2} = 1 + \frac{E_2 - \frac{e_2 Q}{(d-3)r_+^{d-3}}}{m^2 g\left(r_+\right)} \left[E_1 - \sqrt{E_1^2 - \frac{m^2}{(d-3)^2}g\left(r_+\right)} \right] \frac{d-3}{1 - \frac{r_+}{r}},\tag{A.5}$$

which is precisely the expression given in Eq. (20).

2. The ergosphere of particle 4

In Sec. IV we stated that particle 4 has negative energy when there is energy extraction carried out by particle 3. Here we give the details supporting the statement. To have energy extraction, it is necessary that particle 3 has energy greater than the initial energy, the energy just before the collision of the two initial particles, i.e.,

$$E_3 > E_1 + E_2.$$
 (A.6)

At the point of collision the equality $E_1 + E_2 = E_3 + E_4$ holds. This equality comes from Eq. (23), $X_1 + X_2 = X_3 + X_4$ together with Eq. (9), if we use that the electric charge Q of the black hole is the same for all four particles, the point r of collision is the same for all four particles, and there is conservation of electric charge, i.e., $e_1 + e_2 - e_3 - e_4 = 0$, see Eq. (25). Thus, from $E_4 = E_1 + E_2 - E_3$ and the necessity of Eq. (A.6), one finds

$$\Xi_4 < 0, \tag{A.7}$$

if there is energy extraction in the collision process. From the forward in time condition $\dot{t} > 0$ for processes outside the horizon applied to particle 4, one has $X_4 > 0$, see Eq. (8), and so one finds $E_4 - \frac{e_4Q}{(d-3)r^{d-3}} > 0$, where r is the radius of the point of collision see Eq. (9). For energy extraction Eq. (A.7) holds, so one can write $-|E_4| - \frac{e_4Q}{(d-3)r^{d-3}} > 0$. Assuming without loss of generality Q > 0 as we do, for the latter equation to be true one necessarily has

$$e_4 < 0, \tag{A.8}$$

when there is energy extraction. Thus, $X_4 > 0$ can be written as $-|E_4| + \frac{|e_4|Q}{(d-3)r^{d-3}} > 0$. This equation defines a region within which the inequality is valid, namely

$$r_{+} \le r < r_{\mathrm{ergo}\,4}, \qquad \qquad r_{\mathrm{ergo}\,4}^{d-3} = \frac{|e_4|Q}{(d-3)\,|E_4|}, \tag{A.9}$$

which is the electric ergosphere for particle 4, and is the region where the collision has to take place in order that energy extraction can occur.

It is also interesting to note the following. For the two cases in which energy extraction can occur, i.e., IN^+ and OUT^+ , see Sec. IV, the electric charge of particle 3 obeys the equation $e_3 > e_{3c}$, where from Eq. (11) one has $e_{3c} = \frac{r_+^{d-3}(d-3)}{Q} E_3$, and its energy obeys $E_3 > E_{3b}$. Combining these two equations one finds $e_3 > \frac{r_+^{d-3}}{Q} E_{3b}$. Since we found that E_{3b} can be as large as one wants, e_3 can also assume arbitrarily large values. Using charge conservation, Eq. (25), one concludes that when energy extraction occurs, the electric charge of particle 4 becomes negative, $e_4 < 0$, because the energy of the emitted particle grows, as $e_4 = e_1 + e_2 - e_3$ and e_3 becomes dominant. Then, to have energy extraction $E_4 < 0$, and so $-|E_4| + \frac{|e_4|Q}{(d-3)r^{d-3}} > 0$, and there can be energy extraction if the collision occurs within the electric ergosphere defined in Eq. (A.9).

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