LIPCHITZ CURVE SELECTION AND ITS APPLICATION TO THAMRONGTHANYALAK'S OPEN PROBLEM

MASATO FUJITA

ABSTRACT. We solve an open problem posed in Thamrongthanyalak's paper on the definable Banach fixed point property. A Lipschitz curve selection is a key of our solution. In addition, we show a definable version of Caristi fixed point theorem.

1. INTRODUCTION

Throughout, $\mathcal{F} = (F, <, +, \cdot, 0, 1, ...)$ is a definably complete expansion of an ordered field. 'Definable' means 'definable with parameters'. We recall basic notions. \mathcal{F} is *definably complete* if every definable subset of F has a supremum and infimum in $F \cup \{\pm \infty\}$ [9]. \mathcal{F} is *locally o-minimal* if, for every $a \in F$ and every definable subset X of F, there exists an open interval I such that $a \in I$ and $X \cap I$ is a union of finitely many points and open intervals [12]. We call a locally o-minimal structure \mathcal{F} o-minimal when we can choose I = F [3]. An open core \mathcal{F}° of \mathcal{F} is the structure on F generated by open sets definable in \mathcal{F} [2, 5, 10].

In Thamrongthanyalak's paper [11], the Banach fixed point property (BFPP for short) is investigated. A definable set E has the *BFPP* if every definable contraction on E has a fixed point. Every nonempty definable closed set enjoys BFPP by [11, 1.4]. \mathcal{F} possesses the *BFPP* (resp. strong *BFPP*) if every locally closed definable set (resp. every definable set) having the BFPP is closed. Thamrongthanyalak showed that structures having o-minimal open core enjoy the strong BFPP and, if \mathcal{F} possesses the BFPP, \mathcal{F} has a locally o-minimal open core. The following question is posed in [11].

If \mathcal{F} is definably complete and possesses the strong BFPP, is it o-minimal?

The following theorem answers the above question in the negative because nono-minimal definably complete locally o-minimal expansion of an ordered field is already known [5, Example 3.11].

Theorem 1.1. A definably complete locally o-minimal expansion of an ordered field possesses the strong BFPP.

We prove a Lipschitz curve selection lemma in Section 2. Theorem 1.1 follows from the lemma in the same manner as [11, Theorem A]. A rough strategy of its proof is only given in the present paper.

Theorem 1.1 implies the 'only if' part of the following corollary. The 'if' part was already proved as [11, Theorem B].

²⁰²⁰ Mathematics Subject Classification. Primary 03C64; Secondary 54H25.

 $Key\ words\ and\ phrases.$ locally o-minimal structure; Banach fixed point theorem; Caristi fixed point theorem.

M. FUJITA

Corollary 1.2. A definably complete expansion of an ordered field has a locally *o*-minimal open core if and only if it possesses the BFPP.

The Caristi fixed point theorem is a generalization of the Banach fixed point theorem [1]. A metric space has the Caristi fixed point property if and only if it is complete [13]. We prove a similar equivalence holds in definably complete structures in Section 3.

Theorem 1.3 (Definable Caristi fixed point theorem). For a definable subset X of F^n , the following are equivalent:

- (1) X is closed.
- (2) For every definable lower semi-continuous function $f : X \to [0, \infty)$, there exists $x_0 \in X$ such that $S_f(x_0) = \{x_0\}$, where $S_f(x) := \{y \in X \mid ||x y|| \le f(x) f(y)\}$ for every $x \in X$.

2. Lipschitz curve selection

Thamrongthanyalak used Fischer's Λ^m -regular stratification [4] in his paper [11]. In locally o-minimal structures, such a stratification is not always available. For our purpose, a weaker substitute called Lipschitz curve selection (Lemma 2.2) is enough.

Let X and T be definable sets. The parameterized family $\{S_t \mid t \in T\}$ of definable subsets of X is called *definable* if the union $\bigcup_{t \in T} \{t\} \times S_t$ is definable. For a set X, a definable family C of subsets of X is called a *definable filtered collection* if, for any $B_1, B_2 \in C$, there exists $B_3 \in C$ with $B_3 \subseteq B_1 \cap B_2$. We say that X is *definably compact* if, for every definable filtered collection C of nonempty closed subsets of X, $\bigcap_{C \in C} C$ is non-empty.

Lemma 2.1. A definable subset X of F^n is bounded and closed if and only if it is definably compact.

Proof. See [8, Proposition 3.10], which proves the lemma when the structure is o-minimal; the same proof works when the structure is definably complete. \Box

We introduce the notations used in the proof of Lemma 2.2. Let \mathbb{M}_n be the set of $n \times n$ matrices with entries in F. Set $\mathbb{H}_{n,d} := \{A \in \mathbb{M}_n \mid {}^tA = A, A^2 = A, \operatorname{tr}(A) = d\}$ and $\mathbb{H}_n = \bigcup_{d=0}^n \mathbb{H}_{n,d}$. For every linear subspace H of F^n of dimension d, we can find $A \in \mathbb{H}_{n,d}$ such that $H = AF^n$ and A is the linear projection of F^n onto H. The algebraic set $\mathbb{H}_{n,d}$ is bounded and closed in F^{n^2} . By Lemma 2.1, $\mathbb{H}_{n,d}$ is definably compact. Let ||A|| be the Euclidean norm of a matrix $A \in \mathbb{M}_n$ under the natural identification of \mathbb{M}_n with F^{n^2} . We define ||v|| for $v \in F^n$ similarly, and $||A||_{\operatorname{op}} := \sup_{||v||=1} ||Av||$. The function $\delta : \mathbb{H}_n \times \mathbb{H}_n \to F$ is given by $\delta(A, B) = ||B^{\perp}A||_{\operatorname{op}}$, where $B^{\perp} = I_n - B$ and I_n is the identity matrix of size $n \times n$.

Lemma 2.2 (Lipchitz definable curve selection). Suppose that \mathcal{F} is locally ominimal. Let X be a definable subset of F^n and $a \in \partial X$, where ∂X is the frontier of X. Then there exists a definable injective Lipschitz continuous map $\gamma : [0,d] \to F^n$ such that $\gamma(0) = a$ and $\gamma((0,d]) \subseteq X$.

Proof. We assume that a is the origin of F^n for simplicity. By [5, Lemma 5.16], there exist d' > 0 and a definable continuous map $f : [0, d') \to X$ such that f(0) = a and $f(t) \in X$ for t > 0. We may assume that f is injective by [5, Theorem 5.1].

The definable set M := f((0, d')) is decomposed into finitely many definable C^1 submanifolds using [5, Theorem 5.11] in the same manner as [5, Theorem 5.6]. For some e > 0, f((0, e)) coincides with one of them by local o-minimality. By setting d' = e, we may assume that M is a definable C^1 submanifold of F^n . We have dim M = 1 by [7, Proposition 2.8(6)].

Fix a sufficiently small $\varepsilon > 0$. Let $\tau : M \to \mathbb{H}_{n,1}$ be the definable continuous map sending $x \in M$ to the matrix which represents the projection onto the tangent space of M at x. Since $\mathbb{H}_{n,1}$ is definably compact, [6, Theorem 4.5] yields $A := \lim_{t\to 0} \tau(f(t))$. By linear change of coordinates, we may assume that $AF^n = F \times \{0\}^{n-1}$. Let $\overline{\tau} : M \cup \{a\} \to \mathbb{H}_{n,1}$ be the extension of τ given by $\overline{\tau}(a) = A$. The map $\overline{\tau} \circ f$ is continuous. Therefore, we may assume that $\|\tau(f(t)) - A\| < \varepsilon$ for 0 < t < d'by taking a smaller d' if necessary. In addition, the tangent space $\tau(f(t))F^n$ of Mat f(t) is not orthogonal to $F \times \{0\}^{n-1}$ because ε is sufficiently small.

Let $\pi: F^n \to F$ be the coordinate projection onto the first coordinate. For every subset S of F^n and $u \in F$, we denote $S \cap \pi^{-1}(u)$ by S_u . We show that M_u is closed and discrete for every $u \in \pi(M)$. For contradiction, assume that dim $M_u > 0$ for some $u \in \pi(M)$. We have dim $f^{-1}(M_u) > 0$ by [7, Proposition 2.8(6)]. A nonempty open interval I is contained in $f^{-1}(M_u)$. The tangent space of M at $f(t), t \in I$, is orthogonal to the first coordinate axis, which is absurd. We have shown that dim $M_u = 0$ for every $u \in \pi(M)$. This implies that M_u is discrete and closed by [7, Proposition 2.8(1)]. If $0 \in \pi(M)$, the set $f^{-1}(M \cap \pi^{-1}(0))$ is discrete and closed by [7, Proposition 2.8(1),(6)]. Therefore, we may assume that $0 \notin \pi(M)$ by taking a smaller d' > 0.

Let N := f([0, d'/2]). Observe that N is definably compact by [9, Proposition 1.10] and Lemma 2.1. Observe that $a \in N$ and $0 \in \pi(N)$. We show $0 \in cl(\pi(N) \setminus \{0\}) = cl(\pi(N \setminus \{a\}))$, where $cl(\cdot)$ denotes the closure in F. Assume for contradiction that $0 \notin cl(\pi(N) \setminus \{0\})$. By local o-minimality, 0 is isolated in $\pi(N)$. We have $\pi(N) = \{0\}$ because $\pi(N)$ is definably connected by [9, Corollary 1.5]. This deduces that dim $M \cap \pi^{-1}(0) \ge \dim N \cap \pi^{-1}(0) = \dim N = 1$, which is a contradiction. We have shown that $0 \in cl(\pi(N) \setminus \{0\})$. This deduces, by local o-minimality, that a closed interval one of whose endpoints is 0 is contained in $\pi(N)$. After a linear transformation, we may assume that the closed interval [0, d] is contained in $\pi(N)$ for some d > 0.

By [5, Lemma 5.15], we can find a definable map $\gamma : [0, d] \to N$ such that $\pi(\gamma(u)) = u$ for $0 \le u \le d$. We may assume that $\gamma|_{(0,d)}$ is of class \mathcal{C}^1 and $\gamma|_{(0,d)}$ is continuous by [5, Theore 5.11] by taking a smaller d > 0. γ is continuous at 0. In fact, by [6, Theorem 4.5], $\lim_{t\to 0} \gamma(t)$ exists in the definably compact set N. The continuity of π implies $\pi(\lim_{t\to 0} \gamma(t)) = \lim_{t\to 0} \pi(\gamma(t)) = 0$. This implies that $\lim_{t\to 0} \gamma(t) = a = \gamma(0)$ because $N \cap \pi_1^{-1}(0)$ is the singleton $\{a\}$.

We prove $||J_{\gamma}(t)|| < 1/\sqrt{1-\varepsilon^2}$, where J_{γ} is the Jacobian matrix of γ . Fix 0 < t < d. Let $V = \tau(\gamma(t))$ and $\omega : VF^n \to F$ be the linear bijection defined by $w(v) = \pi(v)$. Let e = (1) be the unit vector in F. We have $J_{\gamma}(t) = \omega^{-1}(e)$. By the definition of M and [4, Proposition 3.1(c)], we have $\delta(V, A) \leq ||V - A|| < \varepsilon$. This inequality and the inequality $\delta(V, A) + \delta(V, A^{\perp}) \geq 1$ imply $||AV||_{\text{op}} = \delta(V, A^{\perp}) > \sqrt{1-\varepsilon^2}$. Let w be a unit vector in VF^n . By the definitions of V and A, we have $||AV||_{\text{op}} = ||\omega(w)||$. Therefore, $||J_{\gamma}(t)|| = ||\omega^{-1}(e)|| = 1/||AV||_{\text{op}} < 1/\sqrt{1-\varepsilon^2}$.

Put $L = \sqrt{n/(1-\varepsilon^2)}$. We show that γ is *L*-Lipchitz continuous. To show this, fix $0 \leq b_1 < b_2 \leq d$. Let $\gamma_i(t)$ be the *i*th coordinate of $\gamma(t)$. The mean

M. FUJITA

value theorem is deduced only from the extreme value theorem. Therefore, the mean value theorem holds in \mathcal{F} by [9, p.1786]. Apply it to γ_i , then we have $|\gamma_i(b_2) - \gamma_i(b_1)| \leq \sup_{b_1 < t < b_2} |\gamma'_i(t)|(b_2 - b_1) \leq ||J_{\gamma}||(b_2 - b_1) < (b_2 - b_1)/\sqrt{1 - \varepsilon^2}$. This deduces $||\gamma(b_2) - \gamma(b_1)|| \leq L(b_2 - b_1)$, which means that γ is *L*-Lipchitz continuous.

Proof of Theorem 1.1. For every non-closed definable subset E of F^n , we have only to construct a definable contraction on E that has no fixed point. Let $a \in \partial E$. By Lemma 2.2, we can pick an L-Lipschitz definable map $\gamma : [0, d] \to E \cup \{a\}$ for some L > 0 such that $\gamma(0) = a$ and $\gamma((0, d]) \subseteq E$. Let $H : E \to E$ be the definable map defined by $H(x) = \gamma(\min(d, ||x - a||)/2L)$. We can show that H is a definable contraction on E having no fixed point in the same manner as the proof of [11, Theorem A]. We omit the details.

3. Caristi fixed point theorem

We prove Theorem 1.3 in this section.

Lemma 3.1. Let X be a definable, bounded and closed subset of F^n and $f: X \to [0, \infty)$ be a definable lower semi-continuous function. Then, $\inf f(X) \in f(X)$.

Proof. Set T := f(X). For every $t \in T$, put $C_t := \{x \in X \mid f(x) \leq t\}$. C_t is a closed subset of X because f is lower semi-continuous. Consider the family $\mathcal{C} = \{C_t \mid t \in T\}$, which is a definable filtered collection. By Lemma 2.1, there exists $x_0 \in \bigcap_{t \in T} C_t$. We have $f(x_0) \leq t$ for every $t \in T$. This implies that $f(x_0) = \inf T$.

Proof of Theorem 1.3. We denote $S_f(x)$ by S(x) in the proof.

 $(1) \Rightarrow (2)$: We first show that $S(y) \subseteq S(x)$ whenever $y \in S(x)$. We pick an arbitrary $z \in S(y)$. We have $||z-x|| \le ||y-z|| + ||x-y|| \le f(z) - f(y) + f(y) - f(x) = f(z) - f(x)$. This means that $z \in S(x)$.

We next reduce to the case where X is bounded in F^n . Take an arbitrary point $x' \in X$. Put X' = S(x'). The definable closed set X' is bounded because, for every element $z \in X'$, we have $||z - x'|| \leq f(x')$. For every $x \in X'$, we have $S_f(x) \subseteq S(x') = X'$. Therefore, we may assume that X is bounded by replacing X with X'.

Since X is closed and bounded, we can find $x_0 \in X$ such that $f(x_0) = \inf f(X)$ by Lemma 3.1. We have $f(y) \ge f(x_0)$ for every $y \in X$. This implies that $S(x_0) = \{x_0\}$.

 $(2) \Rightarrow (1)$: Assume that X is not closed. Let p be a point in the frontier of X. We define the definable function $f: X \to [0, \infty)$ by f(x) = 2||x - p||. f is continuous, and it is also lower semi-continuous. We show that $S(x) \neq \{x\}$ for every $x \in X$. We fix $x \in X$ to show this relation. Let y be an arbitrary point in X. Triangle inequality implies $\frac{1}{2}(f(x) + f(y)) \geq ||x - y||$. Therefore, we have $f(x) - f(y) \geq ||x - y|| + \frac{1}{2}(f(x) - 3f(y))$. If we choose $y_0 \in X$ sufficiently close to p, we have $3f(y_0) < f(x)$. We obtain $y_0 \in S(x)$ and $S(x) \neq \{x\}$.

References

J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc., 215 (1976), 241-251.

^[2] A. Dolich, C. Miller and C. Steinhorn, Structure having o-minimal open core, Trans. Amer. Math. Soc., 362 (2010), 1371-1411.

- [3] L. van den Dries, Tame topology and o-minimal structures, London Mathematical Society Lecture Note Series, Vol. 248. Cambridge University Press, Cambridge, 1998.
- [4] A. Fischer, O-minimal Λ^m -regular stratification, Ann. Pure Appl. Logic, **147** (2007), 101–112.
- [5] A. Fornasiero, Locally o-minimal structures and structures with locally o-minimal open core, Ann. Pure Appl. Logic, 164 (2013), 211–229.
- [6] M. Fujita, Definable compactness in definably complete locally o-minimal structures, Fund. Math., 267 (2024) 129–156.
- [7] M. Fujita, T. Kawakami and W. Komine, Tameness of definably complete locally o-minimal structures and definable bounded multiplication, Math. Logic Quart., 68 (2022), 496-515.
- [8] W. Johnson, Interpretable sets in dense o-minimal structures, J. Symbolic Logic, 83 (2018), 1477–1500.
- [9] C. Miller, Expansions of dense linear orders with the intermediate value property, J. Symbolic Logic, 66 (2001), 1783-1790.
- [10] C. Miller and P. Speissegger, Expansions of the real line by open sets: o-minimality and open cores, Fund. Math., 162 (1999), 193-208.
- [11] A. Thamrongthanyalak, Expansions of real closed fields with the Banach fixed point property, Math. Logic Quart., 70 (2024), 197-204.
- [12] C. Toffalori and K. Vozoris, Notes on local o-minimality, Math. Logic Quart., 55 (2009), 617-632.
- [13] J. W. Weston, A characterization of metric completeness, Proc. Amer. Math. Soc., 64 (1977), 186-188.

Department of Liberal Arts, Japan Coast Guard Academy, 5-1 Wakaba-cho, Kure, Hiroshima 737-8512, Japan

Email address: fujita.masato.p34@kyoto-u.jp