Transformation representations of diagram monoids

Reinis Cirpons[1], James East[2], James D. Mitchell[1]

Abstract

We obtain formulae for the minimum transformation degrees of the most well-studied families of finite diagram monoids, including the partition, Brauer, Temperley–Lieb and Motzkin monoids. For example, the partition monoid $\mathcal{P}_n$ has degree $1 + \frac{B(n+2)-B(n+1)+B(n)}{2}$ for $n \geq 2$, where these are Bell numbers. The proofs involve constructing explicit faithful representations of the minimum degree, many of which can be realised as (partial) actions on projections.

Keywords: Diagram monoids, transformation representations, transformation degrees.
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Contents

[1]Mathematical Institute, School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife KY16 9SS, UK. *Emails:* rc234@st-andrews.ac.uk, jdm3@st-andrews.ac.uk.

[2]Centre for Research in Mathematics and Data Science, Western Sydney University, Locked Bag 1797, Penrith NSW 2751, Australia. *Email:* j.east@westernsydney.edu.au. Supported by ARC Future Fellowship FT190100632.

1 Introduction

The *minimum transformation degree* of a finite monoid M, denoted $\deg(M)$, is the least integer $n \geq 1$ such that M can be faithfully represented by transformations (self-maps) of a set of size n. This is well defined because of Cayley's Theorem, which says that M acts faithfully on itself by translations.

Much of the initial work on transformation degrees was undertaken by Easdown and Schein [10–12,42,43], but see also [24,44,45,47] for some earlier work on transformation representations. Some more recent studies include [6,15,36]. See also [13,14,20,26,29,32,41,48] for group-theoretic investigations, and [2,4,15,16,21,25,33–35] for connections to computational algebra.

The current article concerns the class of *diagram monoids*, which consist of various kinds of set partitions that are depicted and multiplied diagrammatically. These monoids, and related algebras and categories built from them, have origins and applications in many fields of mathematics and science [3,5,30,37,46].

The diagram monoids we consider here are all submonoids of the partition monoid $\mathcal{P}_n$. This monoid, which will be defined below, contains the full transformation monoid $\mathcal{T}_n$, as well as its *opposite monoid* $\mathcal{T}_n^{\text{op}}$. Margolis and Steinberg have recently shown [36] that $\deg(\mathcal{T}_n^{\text{op}}) = 2^n$, and this leads to the lower bound of $\deg(\mathcal{P}_n) \geq 2^n$. As far as we are aware, no upper bound for $\deg(\mathcal{P}_n)$ has been given in the literature, apart from $\deg(\mathcal{P}_n) \leq |\mathcal{P}_n|$, from Cayley's Theorem. However, since $\mathcal{P}_n$ is fundamental [19], it follows from [28, Theorem 5] that $\mathcal{P}_n$ can be faithfully represented in $\mathcal{T}_P \times \mathcal{T}_P^{\text{op}}$, where $P = P(\mathcal{P}_n)$ is the set of *projections* of $\mathcal{P}_n$ (again, see below for the precise definitions). Combining this with the result from [36] mentioned above, this leads to the upper bound of $\deg(\mathcal{P}_n) \leq |P| + 2^{|P|}$. For example, when $n = 3$ these bounds are:

$$8 \leq \deg(\mathcal{P}_3) \leq 4194\,326.$$

This upper bound is not an improvement on Cayley's Theorem, which gives $\deg(\mathcal{P}_3) \leq |\mathcal{P}_3| = 203$. As we will see, the exact value of $\deg(\mathcal{P}_3)$ turns out to be 22. One our main results here is that in fact $\deg(\mathcal{P}_n) = |Q| + 1$, where $Q \subseteq P$ is the set of projections of rank at most 2. A combinatorial argument then leads to the concise formula

$$\deg(\mathcal{P}_n) = 1 + \frac{B(n+2) - B(n+1) + B(n)}{2},$$

in terms of Bell numbers. The '1+' in the above formula can be removed to obtain the minimum *partial* transformation degree, $\deg'(\mathcal{P}_n)$.

We also obtain analogous results for several other important families of diagram monoids—namely the Brauer monoid $\mathcal{B}_n$, the partial Brauer monoid $\mathcal{PB}_n$, the planar partition monoid $\mathscr{P}\mathcal{P}_n$, the Motzkin monoid $\mathcal{M}_n$, and the Temperley–Lieb monoid $\mathcal{TL}_n$—in terms of equally-natural number sequences, such as Catalan and Motzkin numbers. These formulae (stated for *partial* transformation degrees) are summarised in Table 1; calculated values are given in Table 2.

The paper is organised as follows. We begin in Section 2 with preliminaries on semigroups and diagram monoids. Section 3 contains general results on transformation representations, and their connections to actions and right congruences. Sections 4 and 5 then apply this to the monoids $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{P}\mathcal{P}_n$, $\mathcal{M}_n$ and $\mathcal{TL}_n$. The main results here are Theorems 4.1 and 5.1, which show that the degree of each such monoid is $1 + |Q|$, for a suitable set Q of low-rank projections. The Brauer monoid $\mathcal{B}_n$ is treated in Section 6, where we must use rather different methods; see Theorem 6.1. En route to proving these theorems, we construct explicit faithful actions/representations of the stated degree; see Theorems 4.9, 5.4, 6.16 and 6.21. Since every such action contains a global fixed point, the minimum *partial* transformation degree is given by $\deg'(M) = \deg(M) - 1$. Finally, in Section 7 we give combinatorial formulae for $|Q|$ for each the monoids $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{P}\mathcal{P}_n$, $\mathcal{M}_n$ and $\mathcal{TL}_n$.

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Monoid M	Validity	Minimum partial transformation degree $\deg'(M)$
$\mathcal{P}_n$	$n \geq 2$	$\frac{B(n+2)-B(n+1)+B(n)}{2}$
$\mathcal{PB}_n$	$n \geq 2$	$\frac{I(n+2)}{2}$
$\mathcal{B}_n$	$n \geq 3$ odd	$\frac{n+1}{2} \cdot n!!$
	$n \geq 4$ even	$\frac{(n+4)(n+2)}{8} \cdot (n-1)!!$
$\mathscr{P}\mathcal{P}_n$	$n \geq 2$	$C(n+2) - 2C(n+1) + C(n)$
$\mathcal{M}_n$	$n \geq 2$	$M(n+2) - M(n+1)$
$\mathcal{TL}_n$	$n = 2k-1 \geq 3$	$C(k+1) - C(k)$
	$n = 2k \geq 4$	$C(k+2) - 2C(k+1) + C(k)$

Table 1. Formulae for the minimum partial transformation degree, $\deg'(M)$, for diagram monoids M, valid for the stated values of n. For each such M and n, the minimum transformation degree is equal to $\deg(M) = 1 + \deg'(M)$. Here $B(n)$, $I(n)$, $C(n)$ and $M(n)$ are the nth Bell, involution, Catalan and Motzkin numbers, and $m!! = m(m-2)(m-4)\cdots 1$ for odd m. See Theorems 4.1, 5.1 and 6.1, and Propositions 7.4–7.8.

n	0	1	2	3	4	5	6	7	8	9	10	OEIS
$\deg'(\mathcal{P}_n)$	1	1	6	21	83	363	1733	8942	49 484	291 871	1825 501	A087649
$\deg'(\mathcal{PB}_n)$	1	1	5	13	38	116	382	1310	4748	17 848	70 076	A001475
$\deg'(\mathcal{B}_n)$	1		2		18		150		1575		19 845	$\frac{1}{3}\times$A001194
		1		6		45		420		4725		A001879
$\deg'(\mathscr{P}\mathcal{P}_n)$	1	1	6	19	62	207	704	2431	8502	30 056	107 236	A026012
$\deg'(\mathcal{M}_n)$	1	1	5	12	30	76	196	512	1353	3610	9713	A002026
$\deg'(\mathcal{TL}_n)$	1		1		6		19		62		207	A026012
		1		3		9		28		90		A000245

Table 2. Calculated values of $\deg'(M)$ for diagram monoids M, and their corresponding sequence numbers on the OEIS [1]. Black entries are those for which the formulae in Table 1 hold. For these entries we also have $\deg(M) = 1 + \deg'(M)$.

2 Preliminaries

We begin by recalling the preliminary ideas and results we need concerning semigroups (Section 2.1), regular $*$-semigroups (Section 2.2) and diagram monoids (Section 2.3). For more basic background on semigroup theory, see for example [7, 27].

2.1 Semigroups

Let S be a semigroup, and let S^1 be the monoid completion of S. So $S^1 = S$ if S is a monoid, or else $S = S \cup \{1\}$, where 1 is a symbol not belonging to S, acting as an adjoined identity element. Green's $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ pre-orders and equivalences are defined, for $a, b \in S$, by

$$a \leq_{\mathscr{L}} b \iff a \in S^1 b, \qquad a \mathrel{\mathscr{L}} b \iff S^1 a = S^1 b,$$
$$a \leq_{\mathscr{R}} b \iff a \in b S^1, \qquad a \mathrel{\mathscr{R}} b \iff a S^1 = b S^1,$$
$$a \leq_{\mathscr{J}} b \iff a \in S^1 b S^1, \qquad a \mathrel{\mathscr{J}} b \iff S^1 a S^1 = S^1 b S^1.$$

So, for example, $a \mathrel{\mathscr{L}} b$ holds when either $a = b$, or else $a = sb$ and $b = ta$ for some $s, t \in S$; similar comments apply to $\mathscr{R}$ and $\mathscr{J}$. We also have the $\mathscr{H}$ and $\mathscr{D}$ relations, defined by $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$

and $\mathscr{D} = \mathscr{L} \vee \mathscr{R}$, where the latter is the join of $\mathscr{L}$ and $\mathscr{R}$ in the lattice of all equivalences of S. If S is finite, then $\mathscr{D} = \mathscr{J}$. The $\mathscr{R}$-class of an element $a \in S$ is denoted by R_a, and similarly for $\mathscr{L}$-classes, etc. The set $S/\mathscr{R}$ of all $\mathscr{R}$-classes is partially ordered by

$$R_a \leq R_b \iff a \leq_{\mathscr{R}} b \qquad \text{for } a, b \in S. \tag{2.1}$$

A semigroup is *stable* if

$$sa \mathrel{\mathscr{J}} a \iff sa \mathrel{\mathscr{L}} a \qquad \text{and} \qquad as \mathrel{\mathscr{J}} a \iff as \mathrel{\mathscr{R}} a \qquad \text{for all } a, s \in S. \tag{2.2}$$

Any finite semigroup is stable.

An equivalence relation σ on a semigroup S is a *right congruence* if it is *right-compatible*, meaning that

$$(a,b) \in \sigma \Rightarrow (as, bs) \in \sigma \qquad \text{for all } a, b, s \in S.$$

Left congruences are defined symmetrically. A *(two-sided) congruence* is an equivalence that is both a left and right congruence. For example, $\mathscr{L}$ is a right congruence, and $\mathscr{R}$ is a left congruence. The *trivial* and *universal* congruences are respectively denoted

$$\Delta_S = \{(a,a) : a \in S\} \qquad \text{and} \qquad \nabla_S = S \times S.$$

For a set of pairs $\Sigma \subseteq S \times S$, we write $\Sigma^\sharp$ for the (two-sided) congruence of S generated by Σ, i.e. the intersection of all congruences containing Σ. When $\Sigma = \{(a,b)\}$ for some $a, b \in S$, we write $(a,b)^\sharp = \Sigma^\sharp$; such a congruence is called *principal*.

A *right ideal* of a semigroup S is a subset $I \subseteq S$ such that $IS \subseteq I$. *Left ideals* and *(two-sided) ideals* are defined analogously. Any left, right or two-sided ideal is a union of $\mathscr{L}$-, $\mathscr{R}$- or $\mathscr{J}$-classes, respectively. If I is a right ideal, then we have the *Rees right congruence*

$$\mathcal{R}_I = \nabla_I \cup \Delta_S = \{(x,y) \in S \times S : x = y \text{ or } x, y \in I\}.$$

As special cases we have $\mathcal{R}_\varnothing = \Delta_S$ and $\mathcal{R}_S = \nabla_S$.

2.2 Regular $*$-semigroups

A *regular $*$-semigroup* is a semigroup S with an additional unary operation $S \to S : a \mapsto a^*$ satisfying the identities

$$a^{**} = a = aa^*a \qquad \text{and} \qquad (ab)^* = b^*a^* \qquad \text{for all } a, b \in S.$$

These were introduced by Nordahl and Scheiblich in [39], and the diagram monoids considered here are key examples. The set of *projections* of a regular $*$-semigroup S is denoted

$$P(S) = \{p \in S : p^2 = p = p^*\}.$$

It is well known (see for example [40]) that $P(S) = \{aa^* : a \in S\} = \{a^*a : a \in S\}$, and that

$$a \mathrel{\mathscr{L}} b \iff a^*a = b^*b \qquad \text{and} \qquad a \mathrel{\mathscr{R}} b \iff aa^* = bb^* \qquad \text{for all } a, b \in S. \tag{2.3}$$

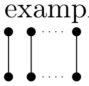
Since every element of S is $\mathscr{R}$-related to a unique projection (namely $a \mathrel{\mathscr{R}} aa^*$), with a similar statement for the $\mathscr{L}$ relation, it follows that any $\mathscr{D}$-class D of S contains $|D/\mathscr{L}| = |D/\mathscr{R}|$ projections, and that $|P(S)| = |S/\mathscr{L}| = |S/\mathscr{R}|$.

2.3 Diagram monoids

Fix a non-negative integer $n \geq 0$, and write $\mathbf{n} = \{1, \dots, n\}$ and $\mathbf{n}' = \{1', \dots, n'\}$. The *partition monoid* $\mathcal{P}_n$ is the set of all set partitions of $\mathbf{n} \cup \mathbf{n}'$. Such a partition is identified with any graph on vertex set $\mathbf{n} \cup \mathbf{n}'$ whose connected components are the blocks of the partition; the vertices from $\mathbf{n}$ and $\mathbf{n}'$ are drawn on an upper and lower row, respectively, ordered $1 < \cdots < n$ and $1' < \cdots < n'$.

The product of $\alpha, \beta \in \mathcal{P}_n$ is defined as follows. We first write $\mathbf{n}'' = \{1'', \dots, n''\}$, and we let:

- $\alpha^\vee$ be the graph on vertex set $\mathbf{n} \cup \mathbf{n}''$ obtained by changing each lower vertex x' of α to x'',
- $\beta^\wedge$ be the graph on vertex set $\mathbf{n}'' \cup \mathbf{n}'$ obtained by changing each upper vertex x of β to x'',
- $\Pi(\alpha, \beta)$ be the graph on vertex set $\mathbf{n} \cup \mathbf{n}'' \cup \mathbf{n}'$ whose edge set is the union of the edge sets of $\alpha^\vee$ and $\beta^\wedge$.

We call $\Pi(\alpha, \beta)$ the *product graph* of α and β, and we draw it with vertices $1'' < \cdots < n''$ in a new middle row. The product $\alpha\beta \in \mathcal{P}_n$ is then the unique partition of $\mathbf{n} \cup \mathbf{n}'$ such that $x, y \in \mathbf{n} \cup \mathbf{n}'$ belong to the same block of $\alpha\beta$ if and only if they belong to the same connected component of $\Pi(\alpha, \beta)$. An example calculation is given in Figure 1 in the case $n = 6$. The identity element of $\mathcal{P}_n$ is $\mathrm{id}_n =$ , and the group of units is (isomorphic to) the symmetric group $\mathcal{S}_n$.

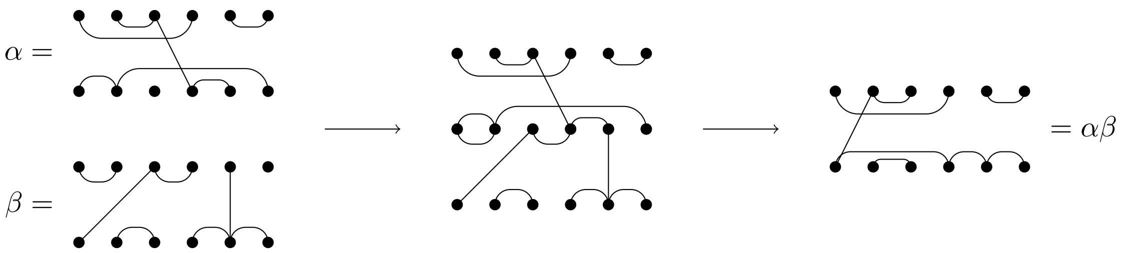


Figure 1. Multiplication of partitions $\alpha, \beta \in \mathcal{P}_6$, with the product graph $\Pi(\alpha, \beta)$ in the middle.

We now recall the definitions of the submonoids of $\mathcal{P}_n$ we will work with. The first two are:

- $\mathcal{PB}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size} \leq 2\}$, the *partial Brauer monoid*,
- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size } 2\}$, the *Brauer monoid*.

A partition from $\mathcal{P}_n$ is *planar* if it is represented by a graph whose edges are all contained within the rectangle spanned by the vertices, and have no crossings. For example, $\beta \in \mathcal{P}_6$ from Figure 1 is planar, but α is not. We then have the further three submonoids of $\mathcal{P}_n$:

- $\mathscr{P}\mathcal{P}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is planar}\}$, the *planar partition monoid*,
- $\mathcal{M}_n = \mathscr{P}\mathcal{P}_n \cap \mathcal{PB}_n$, the *Motzkin monoid*,
- $\mathcal{TL}_n = \mathscr{P}\mathcal{P}_n \cap \mathcal{B}_n$, the *Temperley–Lieb monoid* (sometimes called the *Jones monoid*).

The containments among these monoids, along with sample elements, are shown in Figure 2. It is well known (see for example [23,30]) that $\mathscr{P}\mathcal{P}_n \cong \mathcal{TL}_{2n}$, via an isomorphism $\alpha \mapsto \widetilde{\alpha}$ illustrated in Figure 3.

A non-empty subset $\varnothing \neq X \subseteq \mathbf{n} \cup \mathbf{n}'$ is called:

- an *upper non-transversal* if $X \subseteq \mathbf{n}$, i.e. if X contains only un-dashed vertices,
- a *lower non-transversal* if $X \subseteq \mathbf{n}'$, i.e. if X contains only dashed vertices, or
- a *transversal* otherwise, i.e. if X contains both dashed and un-dashed vertices.

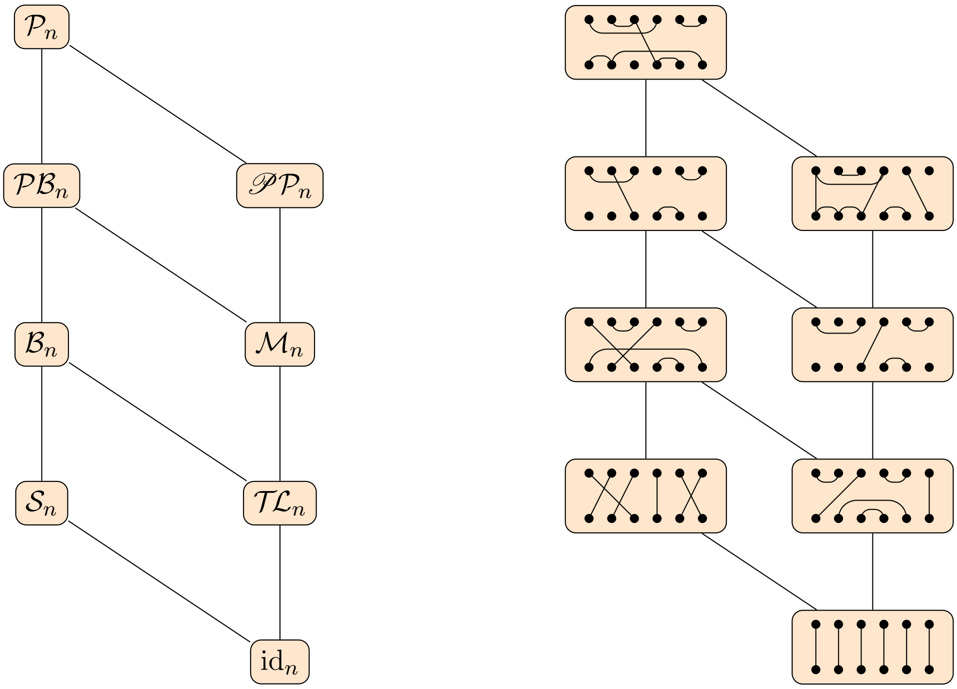


Figure 2. Submonoids of $\mathcal{P}_n$ (left) and representative elements from each submonoid (right).

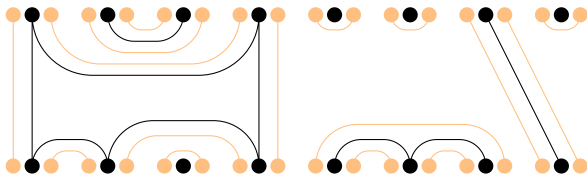


Figure 3. A planar partition $\alpha \in \mathscr{P}\mathcal{P}_8$ (black), with its corresponding Temperley–Lieb element $\widetilde{\alpha} \in \mathcal{TL}_{16}$ (orange), illustrating the isomorphism $\mathscr{P}\mathcal{P}_n \to \mathcal{TL}_{2n}$.

For $\alpha \in \mathcal{P}_n$ we define

- $\mathrm{dom}(\alpha) = \{x \in \mathbf{n} : x \text{ is contained in a transversal of } \alpha\}$, the *domain* of α,
- $\mathrm{codom}(\alpha) = \{x \in \mathbf{n} : x' \text{ is contained in a transversal of } \alpha\}$, the *codomain* of α,
- $\ker(\alpha) = \{(x,y) \in \mathbf{n} \times \mathbf{n} : x \text{ and } y \text{ are contained in the same block of } \alpha\}$, the *kernel* of α,
- $\mathrm{coker}(\alpha) = \{(x,y) \in \mathbf{n} \times \mathbf{n} : x' \text{ and } y' \text{ are contained in the same block of } \alpha\}$, the *cokernel* of α.

We also define $\mathrm{rank}(\alpha)$, the *rank* of α, to be the number of transversals of α. For example, with $\alpha \in \mathcal{P}_6$ as in Figure 1, we have (using an obvious notation for equivalences):

$$\begin{aligned} \mathrm{dom}(\alpha) &= \{2,3\}, & \ker(\alpha) &= (1,4 \mid 2,3 \mid 5,6), & & \\ \mathrm{codom}(\alpha) &= \{4,5\}, & \mathrm{coker}(\alpha) &= (1,2,6 \mid 3 \mid 4,5), & \mathrm{rank}(\alpha) &= 1. \end{aligned}$$

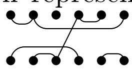
For $\alpha \in \mathcal{P}_n$ we use the tabular notation

$$\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{array} \right) \tag{2.4}$$

to indicate that α has transversals $A_i \cup B_i'$ $(i = 1, \ldots, r)$, upper non-transversals C_i $(i = 1, \ldots, s)$ and lower non-transversals D_i' $(i = 1, \ldots, t)$. Note that one or two of r, s, t could be 0 in (2.4), but not all three (unless $n = 0$). When we use this tabular notation, we always assume that the transversals are ordered so that $\min(A_1) < \cdots < \min(A_r)$. When $\alpha \in \mathscr{P}\mathcal{P}_n$ is planar, this in fact implies $A_1 < \cdots < A_r$ and $B_1 < \cdots < B_r$. (Here for subsets $X, Y \subseteq \mathbf{n}$ we write $X < Y$ to indicate that $x < y$ for all $x \in X$ and $y \in Y$.)

For $\alpha \in \mathcal{P}_n$ as in (2.4), we define

$$\alpha^* = \left(\begin{array}{c|c|c|c|c|c} B_1 & \cdots & B_r & D_1 & \cdots & D_t \\ A_1 & \cdots & A_r & C_1 & \cdots & C_s \end{array}\right),$$

which is the partition obtained by interchanging dashed and un-dashed elements of $\mathbf{n} \cup \mathbf{n}'$. Diagrammatically, α^* is obtained by reflecting any graph representing α in a horizontal axis. For example, with $\alpha \in \mathcal{P}_6$ from Figure 1 we have $\alpha^* =$ . It is easy to see that this gives $\mathcal{P}_n$ the structure of a regular $*$-monoid, i.e. that

$$\alpha^{**} = \alpha = \alpha\alpha^*\alpha \qquad \text{and} \qquad (\alpha\beta)^* = \beta^*\alpha^* \qquad \text{for all } \alpha, \beta \in \mathcal{P}_n.$$

The projections of $\mathcal{P}_n$, i.e. the elements $\varepsilon \in \mathcal{P}_n$ satisfying $\varepsilon^2 = \varepsilon = \varepsilon^*$, have the form

$$\varepsilon = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ A_1 & \cdots & A_r & C_1 & \cdots & C_s \end{array}\right).$$

Each of $\mathcal{PB}_n$, $\mathcal{B}_n$, $\mathscr{P}\mathcal{P}_n$, $\mathcal{M}_n$ and $\mathcal{TL}_n$ is a regular $*$-submonoid of $\mathcal{P}_n$. So too is $\mathcal{S}_n$, in which we have $\alpha^* = \alpha^{-1}$.

3 Transformation monoids and representations

For a set X, the *full transformation monoid* $\mathcal{T}_X$ consists of all self-maps of X under composition. For $f \in \mathcal{T}_X$ and $x \in X$ we write xf for the image of x under f, and compose transformations left to right. When $X = \{1, \ldots, n\}$ for a positive integer n, we write $\mathcal{T}_X = \mathcal{T}_n$.

A *transformation representation* of a semigroup S is a homomorphism $S \to \mathcal{T}_X$ for some set X. When the representation is injective we say it is *faithful*, and that S *embeds* in $\mathcal{T}_X$. Cayley's Theorem states that any semigroup S embeds in $\mathcal{T}_{S^1}$; the proof is the observation that S acts faithfully on S^1 by right translations [27, Theorem 1.1.2]. The *minimum transformation degree* of a finite semigroup S is defined to be

$$\deg(S) = \min\{n \geq 1 : S \text{ embeds in } \mathcal{T}_n\}.$$

Our main goal in this paper is to compute this degree parameter for several families of diagram monoids. In this section we establish some of the theoretical groundwork for doing this. We begin by recalling the connections between transformation representations and actions (Section 3.1), partial representations (Section 3.2) and right congruences (Section 3.3). In Section 3.4 we give a useful construction of right congruences from $\mathscr{R}$-classes. Finally, we show in Section 3.5 how to build families of (partial) actions on projections of regular $*$-semigroups.

3.1 Transformation representations and actions

In all that follows, it will be convenient to view transformation representations and degrees through the lens of actions. For more detailed background, we refer to [31].

Recall that a *(right) action* of a semigroup S on a set X is a map

$$\mu : X \times S \to X \qquad \text{for which} \qquad \mu(\mu(x, a), b) = \mu(x, ab) \qquad \text{for all } x \in X \text{ and } a, b \in S.$$

We say μ is a *monoid action* if S is a monoid with identity 1 and $\mu(x, 1) = x$ for all $x \in X$. The *degree* of the action μ is defined to be $|X|$. It is standard to abbreviate $\mu(x, a)$ to xa, in which case the defining property above says $(xa)b = x(ab)$. However, since we will often have to deal with several actions at once, we will only occasionally use such shorthand notation.

Given an action $\mu : X \times S \to X$, a subset $Y \subseteq X$ is called a *sub-act* if $\mu(y, a) \in Y$ for all $y \in Y$ and $a \in S$. In this case, μ restricts to an action $\mu\restriction_Y : Y \times S \to Y$. For an arbitrary subset Y of X, we define the sub-act

$$\langle Y\rangle_\mu = Y \cup \{\mu(y, a) : y \in Y,\ a \in S\},$$

which we call the *sub-act generated by* Y. (Of course '$Y \cup$ ' can be deleted if μ is a monoid action.) If $X = \langle x\rangle_\mu$ for some $x \in X$, we say that μ is *monogenic*.

Consider actions $\mu_1 : X_1 \times S \to X_1$ and $\mu_2 : X_2 \times S \to X_2$. We say these are *isomorphic*, and write $\mu_1 \cong \mu_2$, if there is a bijection $\xi : X_1 \to X_2$ such that

$$(\mu_1(x, a))\xi = \mu_2(x\xi, a) \qquad \text{for all } x \in X_1 \text{ and } a \in S.$$

We say sub-acts $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ are isomorphic, and write $Y_1 \cong Y_2$ if the restrictions are isomorphic: $\mu_1\restriction_{Y_1} \cong \mu_2\restriction_{Y_2}$.

Consider again actions $\mu_1 : X_1 \times S \to X_1$ and $\mu_2 : X_2 \times S \to X_2$, where this time we assume that X_1 and X_2 are disjoint. The (disjoint) union

$$\mu_1 \sqcup \mu_2 : (X_1 \sqcup X_2) \times S \to X_1 \sqcup X_2$$

is then an action. More generally, suppose we have isomorphic sub-acts $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$, as witnessed by the bijection $\xi : Y_1 \to Y_2$. We define $X = X_1 \sqcup_\xi X_2$ to be the quotient of $X_1 \sqcup X_2$ by the equivalence relation that equates y and $y\xi$ for each $y \in Y_1$. Writing $[x] \in X$ for the equivalence class of each $x \in X_1 \sqcup X_2$, we then have a well-defined action

$$\mu : X \times S \to X \qquad \text{given by} \qquad \mu([x], a) = \begin{cases} [\mu_1(x, a)] & \text{if } x \in X_1 \\ [\mu_2(x, a)] & \text{if } x \in X_2. \end{cases} \tag{3.1}$$

This is called the *push-out along* ξ, and is denoted by $\mu = \mu_1 \sqcup_\xi \mu_2$.

Given an action $\mu : X \times S \to X$, one can define a transformation representation

$$\phi_\mu : S \to \mathcal{T}_X : a \mapsto f_a, \qquad \text{where} \qquad xf_a = \mu(x, a) \qquad \text{for } a \in S \text{ and } x \in X.$$

Conversely, given a transformation representation $\phi : S \to \mathcal{T}_X : a \mapsto f_a$, one can define an action

$$\mu_\phi : X \times S \to X, \qquad \text{where} \qquad \mu_\phi(x, a) = xf_a \qquad \text{for } a \in S \text{ and } x \in X.$$

These constructions are mutually inverse.

The *kernel* of an action $\mu : X \times S \to X$, denoted $\ker(\mu)$, is defined to be the kernel of the corresponding representation $\phi_\mu : S \to \mathcal{T}_X$, so

$$\ker(\mu) = \ker(\phi_\mu) = \{(a, b) \in S \times S : \mu(x, a) = \mu(x, b) \text{ for all } x \in X\}.$$

We say the action μ is *faithful* if ϕ_μ is faithful, which is equivalent to $\ker(\mu) = \Delta_S$. Thus, we also have

$$\deg(S) = \min\{n \geq 1 : S \text{ has a faithful action of degree } n\}.$$

3.2 Partial actions and representations

The *partial transformation monoid* $\mathcal{PT}_X$ consists of all partial self-maps of X under (relational) composition. Let $-$ be a symbol not belonging to X, and write $X^- = X \cup \{-\}$. Then $\mathcal{PT}_X$ is isomorphic to the submonoid of $\mathcal{T}_{X^-}$ consisting of all transformations that fix $-$. In particular, we have embeddings $\mathcal{T}_n \hookrightarrow \mathcal{PT}_n \hookrightarrow \mathcal{T}_{n+1}$ for any n.

One can of course speak of partial transformation representations and degrees. Denote the *minimum partial transformation degree* of a finite semigroup S by

$$\deg'(S) = \min\{n \geq 1 : S \text{ embeds in } \mathcal{PT}_n\}.$$

It follows from the above-mentioned embeddings that $\deg'(S) \leq \deg(S) \leq \deg'(S) + 1$, and that $\deg'(S) = \deg(S) - 1$ holds precisely when there is a minimum-degree faithful transformation representation with a global fixed point. This will be the case for every representation/action we construct in Sections 4–6.

3.3 Actions and right congruences

One special family of actions (and hence transformation representations) come from right congruences, and these will be especially important in the current work. To describe them, fix some right congruence σ on a semigroup S. Write $[x]_\sigma$ for the σ-class of $x \in S$, and let $S/\sigma = \{[x]_\sigma : x \in S\}$ be the set of all such classes. We then have a well-defined action

$$\mu_\sigma : (S/\sigma) \times S \to S/\sigma \qquad \text{given by} \qquad \mu_\sigma([x]_\sigma, a) = [xa]_\sigma \qquad \text{for } a, x \in S. \tag{3.2}$$

We say an action μ is a *right congruence action* if $\mu \cong \mu_\sigma$ for some right congruence σ. The *minimum right congruence degree* of S is defined by

$$\deg_{\text{rc}}(S) = \min\{n \geq 1 : S \text{ has a faithful right congruence action of degree } n\}.$$

Of course we have $\deg(S) \leq \deg_{\text{rc}}(S)$.

The next result is a special case of [47, Proposition 1.2], formulated in a way that is convenient for our purposes. We provide a simple proof for convenience, and to keep the paper self contained.

Proposition 3.3. *Let σ be a right congruence of a monoid S, and let $\mu_\sigma : (S/\sigma) \times S \to S/\sigma$ be the action in (3.2). Then $\ker(\mu_\sigma)$ is the largest two-sided congruence of S contained in σ. Consequently, μ_σ is faithful if and only if σ contains no non-trivial two-sided congruences.*

Proof. We first show that $\ker(\mu_\sigma) \subseteq \sigma$. To do so, let $(a, b) \in \ker(\mu_\sigma)$. Then

$$[a]_\sigma = [1a]_\sigma = \mu_\sigma([1]_\sigma, a) = \mu_\sigma([1]_\sigma, b) = [1b]_\sigma = [b]_\sigma,$$

meaning that $(a, b) \in \sigma$.

We can complete the proof by showing that any two-sided congruence τ with $\tau \subseteq \sigma$ satisfies $\tau \subseteq \ker(\mu_\sigma)$. So fix some such τ, and let $(a, b) \in \tau$. We must show that $\mu_\sigma([x]_\sigma, a) = \mu_\sigma([x]_\sigma, b)$ for all $x \in S$, i.e. that $[xa]_\sigma = [xb]_\sigma$ for all x. But since τ is a congruence, we have $(xa, xb) \in \tau \subseteq \sigma$, so indeed $[xa]_\sigma = [xb]_\sigma$. □

Given an action $\mu : X \times S \to X$, and an element $x \in X$, it is easy to check that the relation

$$\sigma_x = \{(a, b) \in S \times S : \mu(x, a) = \mu(x, b)\} \tag{3.4}$$

is a right congruence of S. Note that $\ker(\mu) = \bigcap_{x \in X} \sigma_x$.

Lemma 3.5. *Any monogenic monoid action is a right congruence action.*

Proof. Let $\mu : X \times S \to X$ be a monogenic monoid action, say with $X = \langle z \rangle_\mu$. For each $x \in X$, fix some $b_x \in S$ such that $x = \mu(z, b_x)$. We will show that $\mu \cong \mu_\sigma$, where

$$\sigma = \sigma_z = \{(a, b) \in S \times S : \mu(z, a) = \mu(z, b)\}$$

is the right congruence from (3.4). For any $a \in S$ we have $[a]_\sigma = [b_x]_\sigma$, where $x = \mu(z,a)$, so it follows that the map $\xi : X \to S/\sigma : x \mapsto [b_x]_\sigma$ is surjective. It is injective as well, because $[b_x]_\sigma = [b_y]_\sigma$ implies $\mu(z,b_x) = \mu(z,b_y)$, i.e. $x = y$. It remains to check that

$$(\mu(x,a))\xi = \mu_\sigma(x\xi,a) \qquad \text{for all } a,x \in S,$$

i.e. that $[b_{\mu(x,a)}]_\sigma = \mu_\sigma([b_x]_\sigma,a)$ for all a,x. By definition, and the above observation, we have

$$\mu_\sigma([b_x]_\sigma,a) = [b_xa]_\sigma = [b_y]_\sigma, \qquad \text{where} \qquad y = \mu(z,b_xa) = \mu(\mu(z,b_x),a) = \mu(x,a),$$

as required. □

We say that an action $\mu : X \times S \to X$ *separates* a pair $(a,b) \in S \times S$ if $\mu(x,a) \neq \mu(x,b)$ for some $x \in X$. In this case we say that x *witnesses* the μ-separation of (a,b).

A two-sided congruence σ of S is *minimal* if $\sigma \neq \Delta_S$, and the only congruence properly contained in σ is Δ_S. Such a congruence is necessarily principal. If S is finite, then every non-trivial congruence contains a minimal congruence.

Lemma 3.6. *Let S be a finite monoid, and let $\Gamma \subseteq S \times S$ be such that $\{(a,b)^\sharp : (a,b) \in \Gamma\}$ consists of all the minimal two-sided congruences of S. Then a semigroup action $\mu : X \times S \to X$ is faithful if and only if it separates each pair from Γ.*

Proof. If a pair $(a,b) \in \Gamma$ was not separated by μ, then we would have $\mu(x,a) = \mu(x,b)$ for all $x \in X$, meaning that $(a,b) \in \ker(\mu)$. But then $\ker(\mu) \neq \Delta_S$, so μ is not faithful.

Conversely, suppose μ is not faithful, so that $\ker(\mu) \neq \Delta_S$. We can then fix some minimal congruence $\sigma \subseteq \ker(\mu)$, and by assumption we have $\sigma = (a,b)^\sharp$ for some $(a,b) \in \Gamma$. Since $(a,b) \in \sigma \subseteq \ker(\mu)$ we have $\mu(x,a) = \mu(x,b)$ for all $x \in X$, so that μ does not separate (a,b). □

We say an equivalence relation σ on a set A *separates* a subset $B \subseteq A$ if $(a,b) \notin \sigma$ for distinct $a,b \in B$.

Lemma 3.7. *Let S be a semigroup, and suppose $a,b \in S$ and $C \subseteq S$ are such that any right congruence on S separating $\{a,b\}$ also separates C. Also let $\mu : X \times S \to X$ be an action with $\mu(x,a) \neq \mu(x,b)$ for some $x \in X$. Then the map $C \to X : c \mapsto \mu(x,c)$ is injective.*

Proof. Let σ_x be the right congruence from (3.4). The assumption $\mu(x,a) \neq \mu(x,b)$ says that σ_x separates $\{a,b\}$, so it follows that σ_x separates C. But this says precisely that the stated map is injective. □

3.4 Right congruences from $\mathscr{R}$-classes

For some of our later applications, we show how to build congruences on a semigroup starting from a specified $\mathscr{R}$-class. These will have the form $\mathcal{R}_I \vee \mathscr{L}^U$ for a carefully chosen right ideal I and subsemigroup U of S. Here $\mathcal{R}_I$ is a Rees right congruence, and $\mathscr{L}^U$ denotes *Green's relative $\mathscr{L}$ relation*, defined for $a,b \in S$ by

$$a \mathrel{\mathscr{L}^U} b \;\Leftrightarrow\; U^1a = U^1b.$$

This is again a right congruence, and as special cases we have $\mathscr{L}^S = \mathscr{L}$ and $\mathscr{L}^\varnothing = \Delta_S$.

Throughout this section, let R be a fixed $\mathscr{R}$-class of a semigroup S. Define the sets

$$T = \{a \in S : aR \subseteq R\}, \qquad K = \{a \in S : R \leq R_a\} \qquad \text{and} \qquad I = S \setminus K = \{a \in S : R \not\leq R_a\}, \tag{3.8}$$

where $\leq$ is the ordering on $\mathscr{R}$-classes in (2.1). It is easy to check that T is a (possibly empty) subsemigroup of S, and that I is a right ideal.

Lemma 3.9. *If z is an arbitrary element of R, then*

(i) $K = \{a \in S : z \leq_{\mathscr{R}} a\}$,

(ii) $T = \{a \in S : az \mathrel{\mathscr{R}} z\}$,

(iii) $T = \{a \in S : az = z\}$ *if S is stable and R is $\mathscr{H}$-trivial.*

Proof. (i). Since $R = R_z$, this follows immediately from the definitions.

(ii). Let $a \in S$. If $aR \subseteq R$, then $az \in aR \subseteq R = R_z$, which says $az \mathrel{\mathscr{R}} z$.

Conversely, suppose $az \mathrel{\mathscr{R}} z$, and let $x \in R$ be arbitrary, so that $x \mathrel{\mathscr{R}} z$. Since $\mathscr{R}$ is a left congruence it follows that $ax \mathrel{\mathscr{R}} az \mathrel{\mathscr{R}} z$, which says $ax \in R$, showing that $aR \subseteq R$.

(iii). Suppose S is stable and R is $\mathscr{L}$-trivial. Given part (ii), it is enough to show that $az \mathrel{\mathscr{R}} z \Rightarrow az = z$. So suppose $az \mathrel{\mathscr{R}} z$. Since $\mathscr{R} \subseteq \mathscr{J}$ we have $az \mathrel{\mathscr{J}} z$, and it follows from stability (see (2.2)) that $az \mathrel{\mathscr{L}} z$, and then from $\mathscr{H}$-triviality that $az = z$. ☐

In what follows, we typically use parts (i) and (ii) of Lemma 3.9 without explicit reference.

Lemma 3.10. *For any subsemigroup $U \subseteq T$, we have $\mathscr{L}^U \subseteq \nabla_I \cup \nabla_K$, and consequently $\mathscr{L}^U = \mathscr{L}^U{\restriction}_I \cup \mathscr{L}^U{\restriction}_K$.*

Proof. Let $(a, b) \in \mathscr{L}^U$. By symmetry, it suffices to show that $a \in K \Rightarrow b \in K$. So suppose $a \in K$. Also let $z \in R$, so that $z \leq_{\mathscr{R}} a$, which gives $z = as$ for some $s \in S^1$. Since $(a, b) \in \mathscr{L}^U$ we have $b = ua$ for some $u \in U^1$. Since $U \subseteq T$ we have $uz \mathrel{\mathscr{R}} z$. It follows that $z \mathrel{\mathscr{R}} uz = uas = bs \leq_{\mathscr{R}} b$, which gives $z \leq_{\mathscr{R}} b$, i.e. $b \in K$. ☐

With R, T, K and I as above, and for any subsemigroup $U \subseteq T$, we define the right congruence

$$\sigma = \mathcal{R}_I \vee \mathscr{L}^U,$$

where here $\mathcal{R}_I = \nabla_I \cup \Delta_S = \nabla_I \cup \Delta_K$ is the Rees right congruence, and where $\vee$ denotes the join in the lattice of equivalences of S. It follows from Lemma 3.10 that in fact

$$\sigma = \nabla_I \cup \mathscr{L}^U{\restriction}_K = \big\{(a, b) : a, b \in I \text{ or } [a, b \in K \text{ and } U^1 a = U^1 b]\big\}. \tag{3.11}$$

In particular, we have $S/\sigma = \{I\} \cup (K/\mathscr{L}^U)$.

3.5 Regular ∗-semigroups and (partial) actions on projections

We now show how the projections of a regular ∗-semigroup can be used to define (partial) actions, and hence transformation representations.

Let S be a regular ∗-semigroup, and let $P = P(S) = \{p \in S : p^2 = p = p^*\}$ be its set of projections. For $p \in P$ and $a \in S$ we write

$$p^a = a^* pa = (pa)^* pa \in P.$$

Since $(p^a)^b = p^{ab}$ for all $p \in P$ and $a, b \in S$, this defines an action

$$P \times S \to P : (p, a) \mapsto p^a. \tag{3.12}$$

Now suppose $Q \subseteq P$ is a set of projections that is closed under the action (3.12), meaning that $p^a \in Q$ for all $p \in Q$ and $a \in S$. Also suppose $\preceq$ is a left-compatible pre-order on S containing $\leq_{\mathscr{R}}$, meaning that:

- $\preceq$ is reflexive and transitive,
- $a \preceq b \Rightarrow sa \preceq sb$ for all $a, b, s \in S$, and

- $as \preceq a$ for all $a, s \in S$.

Let $\approx \ = \ \preceq \cap \succeq$ be the equivalence on S induced by $\preceq$, and note that $\approx$ is a left congruence containing $\mathscr{R}$. Let $-$ be a symbol not belonging to P, let $Q^- = Q \cup \{-\}$, and define

$$\mu : Q^- \times S \to Q^- \qquad \text{by} \qquad \mu(p,a) = \begin{cases} p^a & \text{if } p \in Q \text{ and } p \approx pa \\ - & \text{otherwise.} \end{cases} \tag{3.13}$$

So in particular $\mu(-, a) = -$ for all $a \in S$.

Proposition 3.14. *If S is a regular $*$-semigroup, and if Q, $\preceq$ and $\approx$ are as above, then (3.13) determines an action $\mu : Q^- \times S \to Q^-$. If S is a monoid, then the action is monoidal.*

Proof. For the first assertion (the second is clear), we must show that

$$\mu(p, ab) = \mu(\mu(p,a), b) \qquad \text{for all } p \in Q^- \text{ and } a, b \in S.$$

This is clear if $p = -$, so for the rest of the proof we assume that $p \in Q$. Given that (3.12) determines an action of S on Q, it is in fact enough to show that

$$\mu(p, ab) \in Q \qquad \Leftrightarrow \qquad \mu(\mu(p,a), b) \in Q.$$

Following the definitions, this amounts to showing that

$$p \approx pab \qquad \Leftrightarrow \qquad [p \approx pa \text{ and } p^a \approx p^a b].$$

For the forward implication, suppose $p \approx pab$. Since $\leq_{\mathscr{R}} \subseteq \preceq$, we then have

$$p \approx pab \preceq pa \preceq p,$$

so that $p \approx pa \approx pab$. From $pa \approx pab$ we have $a^*pa \approx a^*pab$ (as $\approx$ is left-compatible), i.e. $p^a \approx p^a b$.

Conversely, suppose $p \approx pa$ and $p^a \approx p^a b$. From the latter (and left-compatibility) we have $pa \cdot p^a \approx pa \cdot p^a b$. But

$$pa \cdot p^a = pa \cdot a^*pa = pa \cdot (pa)^* \cdot pa = pa,$$

so the previous conclusion says $pa \approx pab$. Combined with $p \approx pa$, it follows that $p \approx pab$. □

Remark 3.15. (i) One obvious choice for the pre-order $\preceq$ is $\leq_{\mathscr{R}}$ itself, in which case $\approx$ is $\mathscr{R}$. Also taking the obvious $Q = P$, this leads to the action

$$P^- \times S \to P^- \qquad \text{given by} \qquad (p,a) \mapsto \begin{cases} p^a & \text{if } p \mathrel{\mathscr{R}} pa \\ - & \text{otherwise,} \end{cases}$$

which exists for an arbitrary regular $*$-semigroup S.

(ii) Another obvious choice for $\preceq$ is ∇_S, in which case $\approx$ is also equal to ∇_S. Again taking $Q = P$, and in this case keeping in mind that $p \approx pa$ for all $p \in P$ and $a \in S$, the action (3.13) essentially reduces to (3.12), with the symbol $-$ acting as an adjoined fixed point.

Remark 3.16. (Partial) actions on projections will be used in the next two sections to construct minimum-degree faithful representations/actions for many of our diagram monoids. However, it is worth noting that the degree of a general regular $*$-semigroup need not have anything to do with the number of its projections. For example, a billion-by-billion rectangular band has a billion projections, but degree 64 [6].

4 The partition, partial Brauer, planar partition and Motzkin monoids

With the theoretical results of Section 3 now established, we now come to our main task: the calculation of the minimum transformation degrees of our diagram monoids.

Throughout this section, we let M denote any of the partition monoid $\mathcal{P}_n$, the partial Brauer monoid $\mathcal{PB}_n$, the planar partition monoid $\mathscr{PP}_n$ or the Motzkin monoid $\mathcal{M}_n$. The Temperley–Lieb monoid $\mathcal{TL}_n$ will be treated in Section 5, and the Brauer monoid $\mathcal{B}_n$ in Section 6. Our goal here is to prove the following:

Theorem 4.1. *If $n \geq 2$, and if M is any of $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{PP}_n$ or $\mathcal{M}_n$, then*

$$\deg'(M) = |Q| \qquad \text{and} \qquad \deg(M) = \deg_{\mathrm{rc}}(M) = 1 + |Q|,$$

where $Q = \{\varepsilon \in P(M) : \operatorname{rank}(\varepsilon) \leq 2\}$. Formulae for $|Q|$ can be found in Propositions 7.4–7.7.

To prove the theorem, we show that $1 + |Q|$ is an upper bound for $\deg_{\mathrm{rc}}(M)$ in Section 4.1 (see Theorem 4.9), and a lower bound for $\deg(M)$ in Section 4.2 (see Proposition 4.14). In fact, Theorem 4.9 gives an explicit right congruence action of degree $1 + |Q|$. Since this action has a global fixed point it will follow (as explained in Section 3.2) that $\deg'(M) = \deg(M) - 1$.

The proof of the theorem requires some understanding of the (two-sided) congruences of M (being $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{PP}_n$ or $\mathcal{M}_n$). These were classified in [19, Theorems 5.4, 6.1 and 7.3]. We do not need the full classification here, but we do need to know that M has precisely three *minimal* (non-trivial) congruences:

$$\begin{aligned}
\lambda &= \Delta_M \cup \{(\alpha, \beta) \in M \times M : \operatorname{rank}(\alpha) = \operatorname{rank}(\beta) = 0,\ \alpha \mathrel{\mathscr{L}} \beta\}, \\
\rho &= \Delta_M \cup \{(\alpha, \beta) \in M \times M : \operatorname{rank}(\alpha) = \operatorname{rank}(\beta) = 0,\ \alpha \mathrel{\mathscr{R}} \beta\}, \\
\eta &= \Delta_M \cup \{(\alpha, \beta) \in M \times M : \operatorname{rank}(\alpha), \operatorname{rank}(\beta) \leq 1,\ \widehat{\alpha} = \widehat{\beta}\}.
\end{aligned} \tag{4.2}$$

(These were denoted λ_0, ρ_0 and μ_1 in [19].) The congruence η involves the mapping

$$\mathcal{P}_n \to \mathcal{P}_n : \alpha = \begin{pmatrix} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{pmatrix} \mapsto \widehat{\alpha} = \begin{pmatrix} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ B_1 & \cdots & B_r & D_1 & \cdots & D_t \end{pmatrix}.$$

Since the above congruences are minimal, they are generated by any non-trivial pair they contain. Thus, for example, we have

$$\lambda = (\zeta, \alpha)^\sharp, \qquad \rho = (\zeta, \beta)^\sharp \qquad \text{and} \qquad \eta = (\zeta, \gamma)^\sharp, \tag{4.3}$$

in terms of the partitions

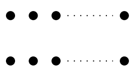
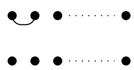
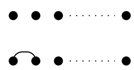
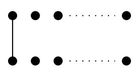
$\zeta =$ , $\alpha =$ , $\beta =$  and $\gamma =$ . (4.4)

Figure 4 shows a Hasse diagram of the congruence lattice $\operatorname{Cong}(M)$, which is the set of all congruences of M, ordered by inclusion. For more detailed diagrams see [19, Figures 5 and 6].

4.1 Upper bound

Recall that M denotes any of $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{PP}_n$ or $\mathcal{M}_n$. For the rest of this section we write

$$P = P(M) = \{\varepsilon \in M : \varepsilon^2 = \varepsilon = \varepsilon^*\} \qquad \text{and} \qquad P_r = \{\varepsilon \in P : \operatorname{rank}(\varepsilon) = r\} \qquad \text{for } 0 \leq r \leq n.$$

We also define

$$Q = Q(M) = P_0 \cup P_1 \cup P_2 = \{\varepsilon \in P : \operatorname{rank}(\varepsilon) \leq 2\}. \tag{4.5}$$

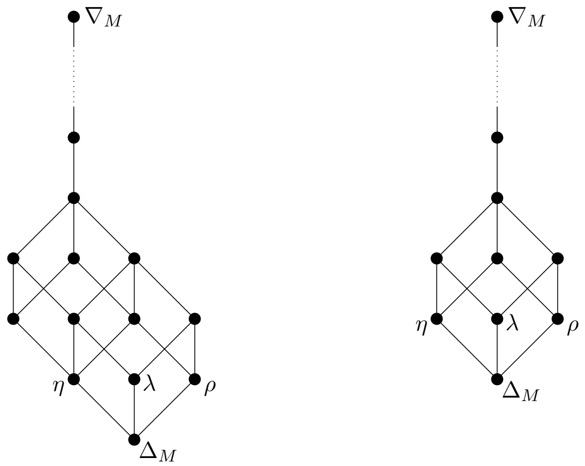


Figure 4. The congruence lattice $\mathrm{Cong}(M)$, for $M = \mathcal{P}_n$ or $\mathcal{PB}_n$ (left), and for $M = \mathscr{P}\mathcal{P}_n$ or $\mathcal{M}_n$ (right).

Since $\mathrm{rank}(\varepsilon^\alpha) = \mathrm{rank}(\alpha^*\varepsilon\alpha) \leq \mathrm{rank}(\varepsilon)$ for all $\varepsilon \in P$ and $\alpha \in M$, it is clear that Q is closed under the action (3.12).

We now define the relation $\preceq$ on M by

$$\alpha \preceq \beta \ \Leftrightarrow\ \ker(\alpha) \supseteq \ker(\beta) \qquad \text{for } \alpha, \beta \in M. \tag{4.6}$$

This relation played a crucial role in [18], in connection with so-called *Ehresmann structures* on $\mathcal{P}_n$. Of particular relevance to the current situation, it was shown in the proof of [18, Lemma 4.6] that $\preceq$ is left-compatible, and it is clearly transitive. We obtain $\leq_{\mathscr{R}} \subseteq \preceq$ from the identity $\ker(\alpha\beta) \supseteq \ker(\alpha)$. It then follows from Proposition 3.14 that we have an action

$$\mu : Q^- \times M \to Q^- \qquad \text{given by} \qquad \mu(\varepsilon, \alpha) = \begin{cases} \varepsilon^\alpha & \text{if } \varepsilon \in Q \text{ and } \ker(\varepsilon) = \ker(\varepsilon\alpha) \\ - & \text{otherwise.} \end{cases} \tag{4.7}$$

During this section, it will be convenient in some circumstances to omit singleton blocks in the tabular notation for partitions. For example, we write $\alpha = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & & & \\ B_1 & \cdots & B_r & D_1 & \cdots & D_s \end{array}\right)$ to indicate that the upper non-transversals of α are all singletons. As concrete examples, the partitions α, β and γ in (4.4) can be denoted as $\alpha = \left(\begin{smallmatrix} 1,2 \\ \ \end{smallmatrix}\right)$, $\beta = \left(\begin{smallmatrix} \ \\ 1,2 \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$.

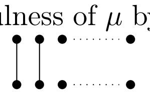
In the remainder of this section, an important role will be played by the map

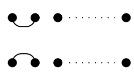
$$P \to M : \varepsilon = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_r & C_1 & \cdots & C_s \\ A_1 & \cdots & A_r & C_1 & \cdots & C_s \end{array}\right) \mapsto \bar{\varepsilon} = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & & & \\ A_1 & \cdots & A_r & C_1 & \cdots & C_s \end{array}\right). \tag{4.8}$$

Here as usual we assume that $\min(A_1) < \cdots < \min(A_r)$, which ensures that $\bar{\varepsilon}$ is planar whenever ε is. Keeping in mind the convention regarding singleton blocks, we note that $\ker(\bar{\varepsilon}) = \Delta_{\mathbf{n}}$ for all $\varepsilon \in P$. Because of the identity $\varepsilon = \bar{\varepsilon}^*\bar{\varepsilon}$, the map $\varepsilon \mapsto \bar{\varepsilon}$ is injective.

Theorem 4.9. *Let $n \geq 2$, let M be any of $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{P}\mathcal{P}_n$ or $\mathcal{M}_n$, and let $\mu : Q^- \times M \to Q^-$ be the action in (4.7). Then μ is faithful and monogenic, and consequently*

$$\deg(M) \leq \deg_{\mathrm{rc}}(M) \leq 1 + |Q|.$$

Proof. As explained above, the minimal congruences of M are $(\zeta, \alpha)^\sharp$, $(\zeta, \beta)^\sharp$ and $(\zeta, \gamma)^\sharp$, where $\zeta, \alpha, \beta, \gamma \in M$ are as in (4.4). Thus, by Lemma 3.6 we can establish faithfulness of μ by showing that it separates each of (ζ, α), (ζ, β) and (ζ, γ). For this, we define $\pi =$  $\in Q$, and one can check that

$$\mu(\pi, \zeta) = \zeta, \qquad \mu(\pi, \alpha) = -, \qquad \mu(\pi, \beta) =$$  $$\qquad \text{and} \qquad \mu(\pi, \gamma) = \gamma,$$

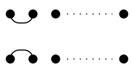
which are all distinct.

Given Lemma 3.5, it remains to check that μ is monogenic, and for this we claim that $Q^- = \langle \pi \rangle_\mu$, where $\pi \in Q$ is as above. We have already observed that $- = \mu(\pi, \alpha)$, and for $\varepsilon \in Q$ it is easy to check that $\varepsilon = \mu(\pi, \bar{\varepsilon})$, where $\bar{\varepsilon}$ is as in (4.8). $\square$

Remark 4.10. If instead we took $\preceq \, = \, \leq_{\mathscr{R}}$ and $Q = P$, we would obtain the alternative action $P^- \times M \to P^-$, as in Remark 3.15(i), which we will here denote by μ'. It turns out that μ' is not faithful for any of the diagram monoids in Theorem 4.9. Indeed, for any $\alpha \in M$ with $\operatorname{rank}(\alpha) = 0$, and for $\varepsilon \in Q^-$, one can check that

$$\mu'(\varepsilon, \alpha) = \begin{cases} \alpha^*\alpha & \text{if } \varepsilon \in P_0 \\ - & \text{otherwise.} \end{cases}$$

Combining this with (2.3), it follows that $(\alpha, \beta) \in \ker(\mu')$ if $\alpha \mathrel{\mathscr{L}} \beta$. This is to say that $\lambda \subseteq \ker(\mu')$, where λ is the congruence of M in (4.2). On the other hand, with $\beta, \gamma, \zeta \in M$ as in (4.4), we have

$$\mu'(\zeta, \zeta) = \zeta, \qquad \mu'(\zeta, \beta) = $$  $$, \qquad \mu'(\gamma, \zeta) = - \qquad \text{and} \qquad \mu'(\gamma, \gamma) = \gamma,$$

which shows that neither (ζ, β) nor (ζ, γ) belongs to $\ker(\mu')$. Since $(\zeta, \beta) \in \rho$ and $(\zeta, \gamma) \in \eta$, it follows that $\rho \not\subseteq \ker(\mu')$ and $\eta \not\subseteq \ker(\mu')$. Thus, keeping the structure of the congruence lattice $\operatorname{Cong}(M)$ in mind (see Figure 4), it follows that in fact $\ker(\mu') = \lambda$.

4.2 Lower bound

We continue to fix the monoid M, being one of $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{PP}_n$ or $\mathcal{M}_n$, with $n \geq 2$. Our goal now is to show that $1 + |Q|$ is a lower bound for $\deg(M)$, thus completing the proof of Theorem 4.1.

In what follows, we fix the elements $\zeta, \alpha, \beta, \gamma \in M$ from (4.4), as well as the map $P = P(M) \to M : \varepsilon \mapsto \bar{\varepsilon}$ from (4.8). For $X \subseteq P$ we write $\bar{X} = \{\bar{\varepsilon} : \varepsilon \in X\}$, noting that $|\bar{X}| = |X|$, as $\varepsilon \mapsto \bar{\varepsilon}$ is injective.

Lemma 4.11. *Let σ be a right congruence of M.*

(i) *If σ separates $\{\zeta, \alpha\}$, then it separates $\bar{P}_2$.*

(ii) *If σ separates $\{\zeta, \beta\}$, then it separates $\bar{P}_0$.*

(iii) *If σ separates $\{\zeta, \gamma\}$, then it separates $\bar{P}_1$.*

Proof. (i). Aiming to prove the contrapositive, suppose $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_2$, and write

$$\bar{\varepsilon}_1 = \left(\begin{smallmatrix} 1 \\ A \end{smallmatrix} \middle| \begin{smallmatrix} 2 \\ B \end{smallmatrix} \middle| \begin{smallmatrix} \\ C_1 \end{smallmatrix} \middle| \begin{smallmatrix} \\ \cdots \end{smallmatrix} \middle| \begin{smallmatrix} \\ C_s \end{smallmatrix} \right) \quad \text{and} \quad \bar{\varepsilon}_2 = \left(\begin{smallmatrix} 1 \\ D \end{smallmatrix} \middle| \begin{smallmatrix} 2 \\ E \end{smallmatrix} \middle| \begin{smallmatrix} \\ F_1 \end{smallmatrix} \middle| \begin{smallmatrix} \\ \cdots \end{smallmatrix} \middle| \begin{smallmatrix} \\ F_t \end{smallmatrix} \right). \tag{4.12}$$

We must show that $(\zeta, \alpha) \in \sigma$.

Case 1. Suppose first that $A \cup B \neq D \cup E$. Without loss of generality, we can fix some $a \in A$ and $b \in B$ such that at least one of a, b does not belong to $D \cup E$. Then with $\theta = \left(\begin{smallmatrix} a, b \\ \end{smallmatrix} \right) \in M$, we have $(\alpha, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$.

Case 2. Next suppose $A \cup B = D \cup E$, but $A \neq D$ (and $B \neq E$). (Since $\min(A) < \min(B)$ and $\min(D) < \min(E)$, it is impossible to have $A = E$ and $B = D$.) Without loss of generality, we can fix some $a \in A$ and $b \in B$ such that $a, b \in D$ or $a, b \in E$. Then with $\theta = \left(\begin{smallmatrix} a, b \\ \end{smallmatrix} \right) \in M$, we again have $(\alpha, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$.

Case 3. Finally, suppose $A = D$ and $B = E$. Since $\varepsilon_1 \neq \varepsilon_2$, we can assume without loss of generality that there exist $x, y \in C_1$ such that $x \in F_1$ and $y \in F_2$. Also let $a = \min(A)$ and

$b = \min(B)$, noting that $a < b$. Then with $\theta = \left(\begin{smallmatrix} a,x & b,y \\ & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a,y & b,x \\ & \end{smallmatrix}\right)$, whichever is planar (and hence belongs to M), we again have $(\alpha, \zeta) = (\bar\varepsilon_1\theta, \bar\varepsilon_2\theta) \in \sigma$.

(ii). Suppose $(\bar\varepsilon_1, \bar\varepsilon_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_0$; this time we must show that $(\zeta, \beta) \in \sigma$. Without loss of generality, we can fix some $(x, y) \in \operatorname{coker}(\varepsilon_1) \setminus \operatorname{coker}(\varepsilon_2)$, say with $x < y$. Then with $\theta = \left(\begin{smallmatrix} x & y \\ 1 & 2 \end{smallmatrix}\right) \in M$ we have $(\beta, \zeta) = (\bar\varepsilon_1\theta, \bar\varepsilon_2\theta) \in \sigma$.

(iii). Suppose $(\bar\varepsilon_1, \bar\varepsilon_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_1$, and write

$$\bar\varepsilon_1 = \left(\begin{smallmatrix} 1 & & & \\ A & B_1 & \cdots & B_s \end{smallmatrix}\right) \quad \text{and} \quad \bar\varepsilon_2 = \left(\begin{smallmatrix} 1 & & & \\ C & D_1 & \cdots & D_t \end{smallmatrix}\right). \tag{4.13}$$

Case 1. Suppose first that $A \neq C$. Without loss of generality, we can fix some $a \in A \setminus C$. Then with $\theta = \binom{a}{1} \in M$ we have $(\gamma, \zeta) = (\bar\varepsilon_1\theta, \bar\varepsilon_2\theta) \in \sigma$.

Case 2. Now suppose $A = C$, and fix some $a \in A$. Without loss of generality, we can also fix some $(x, y) \in \operatorname{coker}(\varepsilon_1) \setminus \operatorname{coker}(\varepsilon_2)$. Then with $\theta = \left(\begin{smallmatrix} x & a,y \\ 1 & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} y & a,x \\ 1 & \end{smallmatrix}\right)$, whichever is planar, we have $(\gamma, \zeta) = (\bar\varepsilon_1\theta, \bar\varepsilon_2\theta) \in \sigma$. □

Here then is the last part of the proof of Theorem 4.1:

Proposition 4.14. *If $n \geq 2$, and if M is any of $\mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{P}\mathcal{P}_n$ or $\mathcal{M}_n$, then*

$$\deg(M) \geq 1 + |Q|.$$

Proof. Let $\mu : X \times M \to X$ be a faithful action. We prove the result by showing that $|X| \geq 1 + |Q|$. Throughout the proof it will be convenient to write $x\delta = \mu(x, \delta)$ for $x \in X$ and $\delta \in M$.

Since μ is faithful, it follows from Lemma 3.6 and (4.3) that we can fix elements $x_0, x_1, x_2 \in X$ for which

$$x_0\zeta \neq x_0\beta, \qquad x_1\zeta \neq x_1\gamma \qquad \text{and} \qquad x_2\zeta \neq x_2\alpha. \tag{4.15}$$

We note that x_0, x_1, x_2 need not be distinct. We also define the sets

$$Y_i = x_i\bar P_i = \{x_i\bar\varepsilon : \varepsilon \in P_i\} \qquad \text{for } i = 0, 1, 2.$$

By Lemmas 3.7 and 4.11 the maps $P_i \to Y_i : \varepsilon \mapsto x_i\bar\varepsilon$ are bijections, and hence

$$|Y_i| = |P_i| \qquad \text{for } i = 0, 1, 2. \tag{4.16}$$

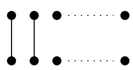
Our strategy for obtaining $|X| \geq 1 + |Q|$ involves showing that $|Y_0 \cup Y_1 \cup Y_2| = |Q|$, and that $X \setminus (Y_0 \cup Y_1 \cup Y_2)$ is non-empty.

We begin by claiming that

$$(Y_1 \cup Y_2) \cap \{x\zeta : x \in X\} = \varnothing. \tag{4.17}$$

To prove this, suppose to the contrary that $x_i\bar\varepsilon = x\zeta$ for some $\varepsilon \in P_i$ and $x \in X$, where $i = 1$ or 2. It then follows that

$$x\zeta = x\zeta\bar\varepsilon^* = x_i\bar\varepsilon\bar\varepsilon^* = \begin{cases} x_1\gamma & \text{if } i = 1 \\ x_2\pi & \text{if } i = 2, \end{cases}$$

where again $\pi =$ . The $i = 1$ and $i = 2$ cases lead respectively to

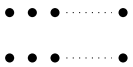
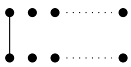
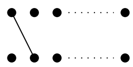
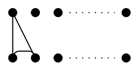
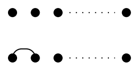
$$x_1\zeta = x_1\gamma\zeta = x\zeta\zeta = x\zeta = x_1\gamma \qquad \text{or} \qquad x_2\zeta = x_2\pi\zeta = x\zeta\zeta = x\zeta = x\zeta\alpha = x_2\pi\alpha = x_2\alpha,$$

both contradicting (4.15).

Now that we have proved (4.17), our next claim is that

$$Y_1 \cap Y_2 = \varnothing. \tag{4.18}$$

To prove this, suppose to the contrary that $x_1\bar{\varepsilon}_1 = x_2\bar{\varepsilon}_2$ for some $\varepsilon_1 \in P_1$ and $\varepsilon_2 \in P_2$. By considering the form of $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$, one can check that $\bar{\varepsilon}_1\bar{\varepsilon}_2^*$ is equal to one of

, , ,  or .

In any case, it follows that $\bar{\varepsilon}_1\bar{\varepsilon}_2^*\alpha = \zeta$. On the other hand, we have $\bar{\varepsilon}_2\bar{\varepsilon}_2^*\alpha = \pi\alpha = \alpha$. Combining these, we obtain

$$x_1\zeta = (x_1\bar{\varepsilon}_1)\bar{\varepsilon}_2^*\alpha = (x_2\bar{\varepsilon}_2)\bar{\varepsilon}_2^*\alpha = x_2\alpha.$$

From $\bar{\varepsilon}_1\zeta = \bar{\varepsilon}_2\zeta = \zeta$, we also have

$$x_1\zeta = (x_1\bar{\varepsilon}_1)\zeta = (x_2\bar{\varepsilon}_2)\zeta = x_2\zeta.$$

Combining the last two conclusions, it follows that $x_2\alpha = x_2\zeta$, which contradicts (4.15).

Now that we have proved (4.18), our next claim is that

$$(Y_1 \cup Y_2) \cap Y_0 = \varnothing. \tag{4.19}$$

To prove this, suppose to the contrary that $x_i\bar{\varepsilon}_i = x_0\bar{\varepsilon}_0$ for some $\varepsilon_0 \in P_0$ and $\varepsilon_i \in P_i$, where $i = 1$ or 2. We then have $x_i\bar{\varepsilon}_i\bar{\varepsilon}_i^* = x_0\bar{\varepsilon}_0\bar{\varepsilon}_i^*$. Now,

$$\bar{\varepsilon}_i\bar{\varepsilon}_i^* = \gamma \text{ or } \pi \text{ (for } i = 1 \text{ or } 2\text{, respectively)} \qquad \text{and} \qquad \bar{\varepsilon}_0\bar{\varepsilon}_i^* = \zeta \text{ or } \beta,$$

but we note that $\bar{\varepsilon}_0\bar{\varepsilon}_i^* = \beta$ is only possible when $i = 2$. We see then that one of the following holds:

$$x_1\gamma = x_0\zeta, \qquad x_2\pi = x_0\zeta \qquad \text{or} \qquad x_2\pi = x_0\beta.$$

Keeping in mind $\gamma = \bar{\gamma}$ and $\pi = \bar{\pi}$, the first two options contradict (4.17), so suppose instead that $x_2\pi = x_0\beta$. As noted above, this case arises when $i = 2$ and $\bar{\varepsilon}_0\bar{\varepsilon}_2^* = \beta$, and so our original assumption was that $x_2\bar{\varepsilon}_2 = x_0\bar{\varepsilon}_0$. Putting this all together, we have

$$x_2\zeta = x_2\pi\zeta = x_0\beta\zeta = x_0\zeta = (x_0\bar{\varepsilon}_0)\bar{\varepsilon}_0^* = (x_2\bar{\varepsilon}_2)\bar{\varepsilon}_0^* = x_2(\bar{\varepsilon}_0\bar{\varepsilon}_2^*)^* = x_2\beta^* = x_2\alpha,$$

which contradicts (4.15). This completes the proof of (4.19).

Combining (4.18) and (4.19), we see that Y_0, Y_1 and Y_2 are pairwise disjoint. Given (4.16), it follows that the subset $Y_0 \cup Y_1 \cup Y_2$ of X has size $|P_0| + |P_1| + |P_2| = |Q|$. Thus, it remains only to show that

$$X \setminus (Y_0 \cup Y_1 \cup Y_2) \neq \varnothing. \tag{4.20}$$

This is certainly true if either of $x_1\zeta$ or $x_2\zeta$ does not belong to $Y_0 \cup Y_1 \cup Y_2$. Given (4.17), the only alternative is that $x_1\zeta$ and $x_2\zeta$ both belong to Y_0, so we now assume that this is the case. Thus, for $i = 1, 2$ we have $x_i\zeta = x_0\bar{\varepsilon}_i$ for some $\varepsilon_i \in P_0$, and then $x_i\zeta = x_i\zeta\zeta = x_0\bar{\varepsilon}_i\zeta = x_0\zeta$, so in fact

$$x_0\zeta = x_1\zeta = x_2\zeta. \tag{4.21}$$

We can complete the proof of (4.20), and hence of the proposition, by showing that

$$x_2\alpha \notin Y_0 \cup Y_1 \cup Y_2.$$

To do so, suppose to the contrary that $x_2\alpha \in Y_i$ for some $i = 0, 1, 2$, so that $x_2\alpha = x_i\bar{\varepsilon}$ for some $\varepsilon \in P_i$. Keeping (4.21) in mind, we then have

$$x_2\alpha = x_2\alpha\zeta = x_i\bar{\varepsilon}\zeta = x_i\zeta = x_2\zeta,$$

contradicting (4.15). As noted above, this completes the proof. □

5 The Temperley–Lieb monoid

In this section we consider the Temperley–Lieb monoid $\mathcal{TL}_n$, our main result being the following:

Theorem 5.1. *For $n \geq 3$ we have*

$$\deg'(\mathcal{TL}_n) = |Q| \qquad \textit{and} \qquad \deg(\mathcal{TL}_n) = \deg_{\mathrm{rc}}(\mathcal{TL}_n) = 1 + |Q|,$$

where

$$Q = \begin{cases} P_0 \cup P_2 \cup P_4 & \textit{if } n \textit{ is even} \\ P_1 \cup P_3 & \textit{if } n \textit{ is odd.} \end{cases}$$

A formula for $|Q|$ can be found in Proposition 7.8.

Here as usual $P_r = \{\varepsilon \in P(\mathcal{TL}_n) : \mathrm{rank}(\varepsilon) = r\}$ for $0 \leq r \leq n$, but we note that this set is non-empty precisely when $r \equiv n \pmod 2$. The even case of Theorem 5.1 in fact follows from Theorem 4.1, and the previously-mentioned isomorphism $\mathscr{P}\mathcal{P}_n \cong \mathcal{TL}_{2n}$, which maps $P_r(\mathscr{P}\mathcal{P}_n) \to P_{2r}(\mathcal{TL}_{2n})$; see Figure 3.

We are therefore left to deal with the odd case of Theorem 5.1. For this it is again convenient to work with an isomorphic copy of $\mathcal{TL}_{2n-1}$. Specifically, it was explained in [23, Section 1] that $\mathcal{TL}_{2n-1}$ is isomorphic to the monoid

$$M = \{\alpha \in \mathscr{P}\mathcal{P}_n : 1 \text{ and } 1' \text{ belong to the same block of } \alpha\}.$$

Indeed, the isomorphism $\mathscr{P}\mathcal{P}_n \to \mathcal{TL}_{2n}$ maps an element $\alpha \in M$ to a Temperley–Lieb diagram $\widetilde{\alpha} \in \mathcal{TL}_{2n}$ with transversal $\{1, 1'\}$, the set of which is clearly isomorphic to $\mathcal{TL}_{2n-1}$; again see Figure 3.

We fix the above monoid $M (\cong \mathcal{TL}_{2n-1})$ for the remainder of the section. We also write

$$P = P(M) \quad \text{and} \quad Q = P_1 \cup P_2, \quad \text{where} \quad P_r = \{\varepsilon \in P : \mathrm{rank}(\varepsilon) = r\} \qquad \text{for } 1 \leq r \leq n.$$

(Because of its defining property, M contains no partitions of rank 0.) We therefore wish to prove the following:

Theorem 5.2. *For $n \geq 2$ we have*

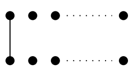
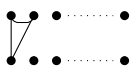
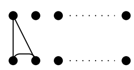
$$\deg'(M) = |Q| \qquad \textit{and} \qquad \deg(M) = \deg_{\mathrm{rc}}(M) = 1 + |Q|, \qquad \textit{where} \qquad Q = P_1 \cup P_2.$$

Our strategy for proving this is similar to that for Theorem 4.1. We will therefore be somewhat briefer.

As before, we need to understand the minimal congruences of M. Since $M \cong \mathcal{TL}_{2n-1}$, it follows from [19, Theorem 9.1] that there are two of these, which we again denote by

$$\lambda = \Delta_M \cup \{(\alpha, \beta) \in M \times M : \mathrm{rank}(\alpha) = \mathrm{rank}(\beta) = 1,\ \alpha \mathrel{\mathscr{L}} \beta\},$$
$$\rho = \Delta_M \cup \{(\alpha, \beta) \in M \times M : \mathrm{rank}(\alpha) = \mathrm{rank}(\beta) = 1,\ \alpha \mathrel{\mathscr{R}} \beta\}.$$

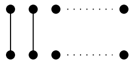
We also have $\lambda = (\zeta, \alpha)^\sharp$ and $\rho = (\zeta, \beta)^\sharp$, where

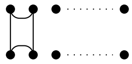
$\zeta =$  , $\alpha =$  and $\beta =$  . (5.3)

We still have the pre-order $\preceq$ on M, as in (4.6), and its associated equivalence $\approx\ =\ \preceq \cap \succeq$, leading to the representation $\mu : Q^- \times M \to Q^-$, as in (4.7). In the proofs that follow we make use of the fact that the map $\varepsilon \mapsto \bar{\varepsilon}$ from (4.8) still maps P into M.

Theorem 5.4 (cf. Theorem 4.9). *For $n \geq 2$, the action $\mu : Q^- \times M \to Q^-$ is faithful and monogenic, and consequently*

$$\deg(M) \leq \deg_{\mathrm{rc}}(M) \leq 1 + |Q|.$$

Proof. Taking $\pi =$  $\in Q$, faithfulness of μ follows from the fact that

$$\mu(\pi, \zeta) = \zeta, \qquad \mu(\pi, \alpha) = - \qquad \text{and} \qquad \mu(\pi, \beta) =$$ 

are distinct. Monogenicity follows from the fact that $\varepsilon = \mu(\pi, \bar{\varepsilon})$ for all $\varepsilon \in Q$. □

Lemma 5.5 (cf. Lemma 4.11). *Let σ be a right congruence of M.*

(i) *If σ separates $\{\zeta, \alpha\}$, then it separates $\bar{P}_2$.*

(ii) *If σ separates $\{\zeta, \beta\}$, then it separates $\bar{P}_1$.*

Proof. (i). Suppose $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_2$, and write $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ as in (4.12), noting that $1 \in A < B$ and $1 \in D < E$. We must show that $(\zeta, \alpha) \in \sigma$.

Case 1. Suppose first that $A \neq D$. Without loss of generality, fix some $a \in A \setminus D$, and let $b \in B$ be arbitrary, noting that $1 < a < b$. It follows that $\theta = \left(\begin{array}{c|c} 1 & a, b \\ 1 & \end{array}\right) \in M$. We then have $(\alpha, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$. (Note that by the form of $\bar{\varepsilon}_2$ and θ, the product $\bar{\varepsilon}_2\theta$ could only be equal to α or ζ. It could only be equal to α if the edge $\{a, b\}$ of θ connected D and E. Since $a \notin D$, this could only be the case if $a \in E$ and $b \in D$, but this is impossible since $a < b$ and $D < E$.)

Case 2. Next suppose $A = D$ but $B \neq E$, and without loss of generality, fix $b \in B \setminus E$. Then with $\theta = \binom{1, b}{1} \in M$, we have $(\alpha, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$.

Case 3. Finally, suppose $A = D$ and $B = E$, and without loss of generality fix some $(x, y) \in \operatorname{coker}(\varepsilon_1) \setminus \operatorname{coker}(\varepsilon_2)$. We assume that $x < y$, and we also let $b \in B$ be arbitrary. By planarity of ε_1, we either have $1 < x < y < b$ or $1 < b < x < y$. We then define $\theta = \left(\begin{array}{c|c} 1, x & b, y \\ 1 & \end{array}\right)$ or $\left(\begin{array}{c|c} 1, y & b, x \\ 1 & \end{array}\right)$, respectively, and we have $\theta \in M$, and $(\alpha, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$.

(ii). Suppose $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_1$, and write $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ as in (4.13).

Case 1. Suppose first that $A \neq C$, and without loss of generality fix some $a \in A \setminus C$, noting that $a \neq 1$. Then with $\theta = \left(\begin{array}{c|c} 1 & a \\ 1 & 2 \end{array}\right) \in M$ we have $(\beta, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$.

Case 2. Now suppose $A = C$, and without loss of generality fix some $(x, y) \in \operatorname{coker}(\varepsilon_1) \setminus \operatorname{coker}(\varepsilon_2)$ with $x < y$. Then with $\theta = \left(\begin{array}{c|c} 1, x & y \\ 1 & 2 \end{array}\right) \in M$ we have $(\beta, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$. □

The next result completes the proof of Theorem 5.2, and hence of Theorem 5.1.

Proposition 5.6 (cf. Proposition 4.14). *If $n \geq 2$, then* $\deg(M) \geq 1 + |Q|$.

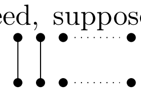
Proof. Let $\mu : X \times M \to X$ be a faithful action, denoted $\mu(x, \delta) = x\delta$. We must show that $|X| \geq 1 + |Q|$. Since μ is faithful, we can fix elements $x_1, x_2 \in X$ such that

$$x_1\zeta \neq x_1\beta \qquad \text{and} \qquad x_2\zeta \neq x_2\alpha. \tag{5.7}$$

We define the sets $Y_i = x_i\bar{P}_i = \{x_i\bar{\varepsilon} : \varepsilon \in P_i\}$ for $i = 1, 2$, noting that $|Y_i| = |P_i|$ by Lemmas 3.7 and 5.5.

We first claim that

$$Y_2 \cap \{x\zeta : x \in X\} = \varnothing. \tag{5.8}$$

Indeed, suppose to the contrary that $x_2\bar{\varepsilon} = x\zeta$ for some $\varepsilon \in P_2$ and $x \in X$. Again writing $\pi =$ , we have $x_2\pi = x_2\bar{\varepsilon}\bar{\varepsilon}^* = x\zeta\bar{\varepsilon}^* = x\zeta$. It follows that

$$x_2\zeta = x_2\pi\zeta = x\zeta\zeta = x\zeta = x\zeta\alpha = x_2\pi\alpha = x_2\alpha,$$

contradicting (5.7).

Next we claim that

$$Y_1 \cap Y_2 = \varnothing. \tag{5.9}$$

Indeed, suppose to the contrary that $x_1\bar{\varepsilon}_1 = x_2\bar{\varepsilon}_2$ for some $\varepsilon_1 \in P_1$ and $\varepsilon_2 \in P_2$. Then

$$x_2\pi = x_2\bar{\varepsilon}_2\bar{\varepsilon}_2^* = x_1\bar{\varepsilon}_1\bar{\varepsilon}_2^*.$$

Noting that $\bar{\varepsilon}_1\bar{\varepsilon}_2^* = \zeta$ or β, it follows that $x_2\pi = x_1\zeta$ or $x_2\pi = x_1\beta$. The first contradicts (5.8), so suppose instead that $x_2\pi = x_1\beta$. This occurs when $\bar{\varepsilon}_1\bar{\varepsilon}_2^* = \beta$, from which it follows that $\bar{\varepsilon}_2\bar{\varepsilon}_1^* = \beta^* = \alpha$. We then calculate

$$x_1\zeta = x_1\bar{\varepsilon}_1\bar{\varepsilon}_1^* = x_2\bar{\varepsilon}_2\bar{\varepsilon}_1^* = x_2\alpha \qquad \text{and} \qquad x_1\zeta = x_1\beta\zeta = x_2\pi\zeta = x_2\zeta,$$

so that $x_2\alpha = x_2\zeta$, contradicting (5.7).

Now that we have proved (5.9), it follows that $|Y_1 \cup Y_2| = |Y_1| + |Y_2| = |P_1| + |P_2| = |Q|$, so it remains to show that

$$X \setminus (Y_1 \cup Y_2) \neq \varnothing. \tag{5.10}$$

This is certainly true if $x_2\zeta \notin Y_1 \cup Y_2$, so suppose this is not the case. It follows from (5.8) that $x_2\zeta \in Y_1$, so we have $x_2\zeta = x_1\bar{\varepsilon}$ for some $\varepsilon \in P_1$. We then have $x_2\zeta = x_2\zeta\zeta = x_1\bar{\varepsilon}\zeta = x_1\zeta$. We will complete the proof of (5.10), and hence of the proposition, by showing that

$$x_2\alpha \notin Y_1 \cup Y_2.$$

To do so, suppose to the contrary that $x_2\alpha = x_i\bar{\varepsilon}_i$ for some $\varepsilon_i \in P_i$, where $i = 1$ or 2. Then

$$x_2\alpha = x_2\alpha\zeta = x_i\bar{\varepsilon}_i\zeta = x_i\zeta = x_2\zeta,$$

using $x_1\zeta = x_2\zeta$ in the last step. This again contradicts (5.7). □

6 The Brauer monoid

We now come to our last diagram monoid, the Brauer monoid $\mathcal{B}_n$. Our main result here is the following, stated in terms of the projection sets $P_r = \{\varepsilon \in P(\mathcal{B}_n) : \text{rank}(\varepsilon) = r\}$:

Theorem 6.1. *For $n \geq 3$ we have*

$$\deg'(\mathcal{B}_n) = \begin{cases} |P_1| + 3|P_3| & \textit{if } n \textit{ is odd} \\ |P_0| + 2|P_2| + 3|P_4| & \textit{if } n \textit{ is even,} \end{cases} \qquad \textit{and} \qquad \deg(\mathcal{B}_n) = 1 + \deg'(\mathcal{B}_n).$$

If n is odd, then $\deg(\mathcal{B}_n) = \deg_{\text{rc}}(\mathcal{B}_n)$.

It was shown in [17, Theorem 8.4] that $|P_r| = \binom{n}{r}(n-r-1)!!$. Here as usual for a positive odd integer m we define the double factorial $m!! = m(m-2)(m-4)\cdots 1$, and by convention $(-1)!! = 1$. One can then easily check that Theorem 6.1 leads to the explicit formulae

$$\deg'(\mathcal{B}_n) = \begin{cases} \frac{n+1}{2} \cdot n!! & \text{if } n \geq 3 \text{ is odd} \\ \frac{(n+4)(n+2)}{8} \cdot (n-1)!! & \text{if } n \geq 4 \text{ is even.} \end{cases} \tag{6.2}$$

Remark 6.3. As we will see, the methods we use here are necessarily rather different from those of previous sections. For one thing, as indicated by the theorem itself, we generally have the strict inequality $\deg(\mathcal{B}_n) < \deg_{\text{rc}}(\mathcal{B}_n)$ for even $n \geq 4$. For example, it follows from the theorem that $\deg(\mathcal{B}_4) = 19$, and from GAP computations [22, 38] that $\deg_{\text{rc}}(\mathcal{B}_4) = 22$. Another dissimilarity is that $\deg'(\mathcal{B}_n)$ is not simply a sum of $|P_r|$ parameters, but rather a (non-trivial) linear combination. The reason for this will become clear as we progress.

Our strategy of proving Theorem 6.1 is as follows. After gathering the required preliminaries in Section 6.1, we build a right congruence σ in Section 6.2, using the construction from Section 3.4. We then use this to show that the claimed value for $\deg(\mathcal{B}_n)$ is an upper bound in Sections 6.3 and 6.4 for odd and even n, respectively; see Theorems 6.16 and 6.21. The even case is more involved, and the faithful action of the desired degree is constructed as a push-out of two smaller actions (only one of which involves σ). Finally, we show that the claimed value of $\deg(\mathcal{B}_n)$ is a lower bound in Sections 6.5 and 6.6, again split by parity; see Propositions 6.23 and 6.28. As usual, the minimum-degree faithful actions we construct have global fixed points, so it follows that $\deg'(\mathcal{B}_n) = \deg(\mathcal{B}_n) - 1$.

6.1 Preliminaries

In what follows, we will need the characterisation of Green's relations on $\mathcal{B}_n$ from [8, Proposition 3.1]. For $\alpha, \beta \in \mathcal{B}_n$ we have

$$\begin{aligned} \alpha \leq_{\mathscr{L}} \beta &\Leftrightarrow \operatorname{coker}(\alpha) \supseteq \operatorname{coker}(\beta), & \alpha \mathscr{L} \beta &\Leftrightarrow \operatorname{coker}(\alpha) = \operatorname{coker}(\beta), \\ \alpha \leq_{\mathscr{R}} \beta &\Leftrightarrow \ker(\alpha) \supseteq \ker(\beta), & \alpha \mathscr{R} \beta &\Leftrightarrow \ker(\alpha) = \ker(\beta), \\ \alpha \leq_{\mathscr{J}} \beta &\Leftrightarrow \operatorname{rank}(\alpha) \leq \operatorname{rank}(\beta), & \alpha \mathscr{J} \beta &\Leftrightarrow \operatorname{rank}(\alpha) = \operatorname{rank}(\beta), \end{aligned}$$

and of course $\mathscr{D} = \mathscr{J}$ as $\mathcal{B}_n$ is finite. Keeping in mind that the rank of a Brauer partition has the same parity as n, it follows that the $\mathscr{D}$-classes and non-empty ideals of $\mathcal{B}_n$ are the sets

$$D_r = \{\alpha \in \mathcal{B}_n : \operatorname{rank}(\alpha) = r\} \qquad \text{and} \qquad I_r = \{\alpha \in \mathcal{B}_n : \operatorname{rank}(\alpha) \leq r\}$$

for $0 \leq r \leq n$ with $r \equiv n \pmod 2$. Note that $I_r = D_r \cup D_{r-2} \cup D_{r-4} \cup \cdots$.

Remark 6.4. It is also worth noting that $\leq_{\mathscr{R}}$ is precisely the pre-order $\preceq$ used in Section 4; see (4.6). As in Remark 4.10, it follows that the resulting action of $\mathcal{B}_n$ on $P^- = P \cup \{-\}$ is not faithful, hence the need for the different approach we take here.

Throughout this section we write

$$P = P(\mathcal{B}_n) = \{\varepsilon \in \mathcal{B}_n : \varepsilon^2 = \varepsilon = \varepsilon^*\} \qquad \text{and} \qquad P_r = P_r(\mathcal{B}_n) = P \cap D_r \qquad \text{for } 0 \leq r \leq n,$$

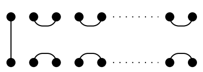
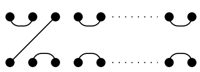
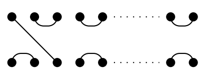
noting that P_r is non-empty precisely when $r \equiv n \pmod 2$. We also write

$$p_r = |P_r| = \tbinom{n}{r}(n-r-1)!!.$$

As before, we will need to understand the minimal congruences of $\mathcal{B}_n$, as described in [19, Section 8]. For odd n, there are two minimal congruences, which we will denote by

$$\lambda = \Delta_{\mathcal{B}_n} \cup \mathscr{L}{\restriction}_{D_1} \qquad \text{and} \qquad \rho = \Delta_{\mathcal{B}_n} \cup \mathscr{R}{\restriction}_{D_1},$$

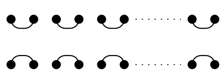
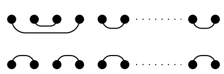
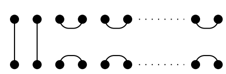
and we have $\lambda = (\zeta, \alpha)^\sharp$ and $\rho = (\zeta, \beta)^\sharp$, where

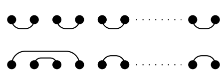
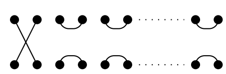
$\zeta =$  , $\alpha =$  and $\beta =$  . (6.5)

For even n there are three minimal congruences, which we will denote by

$$\lambda = \Delta_{\mathcal{B}_n} \cup \mathscr{L}{\restriction}_{D_0}, \qquad \rho = \Delta_{\mathcal{B}_n} \cup \mathscr{R}{\restriction}_{D_0} \qquad \text{and} \qquad \eta = \Delta_{\mathcal{B}_n} \cup \mathscr{H}{\restriction}_{D_2},$$

and we have $\lambda = (\zeta, \alpha)^\sharp$, $\rho = (\zeta, \beta)^\sharp$ and $\eta = (\gamma, \delta)^\sharp$, where

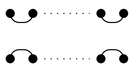
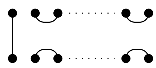
$\zeta =$  , $\alpha =$  , $\gamma =$  ,

$\beta =$  , $\delta =$  . (6.6)

6.2 A right congruence

We will soon have to split into cases according to the parity of n. However, a certain right congruence σ that is parity-independent will play an important role, and we introduce this here. This will have the form $\sigma = \mathcal{R}_I \vee \mathscr{L}^U = \nabla_I \cup \mathscr{L}^U{\restriction}_K$, as in Section 3.4; see (3.11).

Throughout this section we assume that $n \geq 3$, and we fix $k = \lfloor n/2 \rfloor$, so that $n = 2k$ or $2k+1$. We also fix the partition

$\zeta =$  or $\zeta =$  .

We write $\kappa = \ker(\zeta) = \operatorname{coker}(\zeta)$, and we denote the non-trivial κ-classes by

$$Z_1 < \cdots < Z_k, \qquad \text{so} \qquad Z_i = \begin{cases} \{2i, 2i+1\} & \text{if } n \text{ is odd} \\ \{2i-1, 2i\} & \text{if } n \text{ is even.} \end{cases}$$

When n is even, these are *all* the κ-classes, but when n is odd, the last κ-class is $\{1\}$. Define the $\mathscr{R}$-class $R = R_\zeta$, and let T, K and $I = \mathcal{B}_n \setminus K$ be as in (3.8). We then have

$$R = \{\alpha \in \mathcal{B}_n : \ker(\alpha) = \kappa\}, \quad K = \{\alpha \in \mathcal{B}_n : \ker(\alpha) \subseteq \kappa\} \quad \text{and} \quad I = \{\alpha \in \mathcal{B}_n : \ker(\alpha) \not\subseteq \kappa\}.$$

Note that the condition $\ker(\alpha) \subseteq \kappa$ says that the non-trivial $\ker(\alpha)$-classes are among the Z_i. Since $\mathcal{B}_n$ is stable (as it is finite), and since D_ζ (being D_0 or D_1) is $\mathscr{H}$-trivial, it follows from Lemma 3.9(iii) that

$$T = \{\alpha \in \mathcal{B}_n : \alpha\zeta = \zeta\}.$$

This set is not easy to describe concisely, but we will instead work with its sub(semi)group

$$U = T \cap \mathcal{S}_n = \{\alpha \in \mathcal{S}_n : \alpha\zeta = \zeta\} = \{\alpha \in \mathcal{S}_n : (\forall i \in \mathbf{k})(\exists j \in \mathbf{k})\ Z_i\alpha = Z_j\}.$$

This subgroup of $\mathcal{S}_n$ is the stabiliser of the partition $\mathbf{n}/\kappa$, and is (isomorphic to) the wreath product $C_2 \wr \mathcal{S}_k$. In later arguments (in which we will need to vary the underlying parameter n), we will also denote this group by $U = \mathcal{G}_n$, and we note that $|\mathcal{G}_n| = 2^k k!$.

In what follows, it will be convenient in some circumstances to omit some upper and/or lower non-transversals when using the tabular notation for Brauer partitions, although we always list *all* transversals. Specifically, if we write $\alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_r & A_1 & \cdots & A_s \\ b_1 & \cdots & b_r & B_1 & \cdots & B_t \end{array}\right)$, and if the unlisted upper vertices are $c_1 < \cdots < c_u$ (there must be an even number of these), then we assume the remaining upper blocks of α are $\{c_1, c_2\}, \{c_3, c_4\}, \ldots, \{c_{u-1}, c_u\}$. A similar statement holds for unlisted lower vertices/blocks. Thus, for example, the elements α and β in (6.5) can be denoted as $\alpha = \binom{3}{1}$ and $\beta = \binom{1}{3}$, while α in (6.6) can be written as either $\alpha = \left(\begin{array}{c} 1,4 \\ \ \end{array}\right)$ or $\left(\begin{array}{c} 2,3 \\ \ \end{array}\right)$.

The coming arguments will also use the mapping

$$\mathcal{B}_n \to K : \alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_r & A_1 & \cdots & A_s \\ b_1 & \cdots & b_r & B_1 & \cdots & B_s \end{array}\right) \mapsto \bar{\alpha} = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & & & \\ b_1 & \cdots & b_r & B_1 & \cdots & B_s \end{array}\right). \tag{6.7}$$

We assume that *all* the blocks of α have been listed here, and as usual we have $a_1 < \cdots < a_r$. The upper non-transversals of $\bar{\alpha}$ have been omitted, however, as per the above convention, and these are in fact $Z_{k-s+1}, \ldots, Z_k$. Note also that $\bar{\alpha} = \theta\alpha$, where $\theta \in \mathcal{S}_n$ is any permutation for which:

$$i\theta = a_i \text{ for each } i \in \mathbf{r}, \qquad \text{and} \qquad Z_{k-s+i}\theta = A_i \text{ for each } i \in \mathbf{s}. \tag{6.8}$$

Lemma 6.9. *For any $\alpha \in K$ we have $\alpha \mathrel{\mathscr{L}^U} \bar{\alpha}$.*

Proof. Let $\alpha \in K$. Since $\ker(\alpha) \subseteq \kappa$, we can write $\alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_r & Z_{i_1} & \cdots & Z_{i_s} \\ b_1 & \cdots & b_r & B_1 & \cdots & B_s \end{array}\right)$. Also writing $\mathbf{k} \setminus \{i_1, \ldots, i_s\} = \{j_1 < \cdots < j_t\}$, we note that:

- $Z_{j_1} = \{a_1, a_2\}$, $Z_{j_2} = \{a_3, a_4\}$, and so on, when n is even, or
- $Z_{j_1} = \{a_2, a_3\}$, $Z_{j_2} = \{a_4, a_5\}$, and so on, when n is odd, in which case we also have $a_1 = 1$.

It follows that any permutation θ as in (6.8) belongs to U, and $\bar{\alpha} = \theta\alpha$ gives the claim. □

We now let

$$\sigma = \mathcal{R}_I \vee \mathscr{L}^U = \nabla_I \cup \mathscr{L}^U{\restriction}_K = \big\{(\alpha, \beta) : \alpha, \beta \in I \text{ or } [\alpha, \beta \in K \text{ and } U\alpha = U\beta]\big\} \tag{6.10}$$

be the right congruence of $\mathcal{B}_n$ from (3.11). Of course I is a σ-class. By definition, and since $\mathscr{L}^U \subseteq \mathscr{L}$, every other σ-class is contained in $L \cap K$ for some $\mathscr{L}$-class L of $\mathcal{B}_n$. It follows from Lemma 3.10 that every such σ-class is also an $\mathscr{L}^U$-class. The next result tells us how many such classes are contained in $L \cap K$. For the proof, we define a map

$$\phi : \mathcal{B}_n \to \mathcal{S}_n \cup \mathcal{S}_{n-2} \cup \mathcal{S}_{n-4} \cup \cdots ,$$

as follows. Let $\alpha \in \mathcal{B}_n$, and write $\operatorname{dom}(\alpha) = \{a_1 < \cdots < a_r\}$ and $\operatorname{codom}(\alpha) = \{b_1 < \cdots < b_r\}$. Then we define $\phi(\alpha)$ to be the permutation $f \in \mathcal{S}_r$ for which α contains the transversal $\{a_i, b'_{if}\}$ for each i.

Lemma 6.11. *Let L be an $\mathscr{L}$-class of $\mathcal{B}_n$, and let the common rank of its elements be r. Then*

$$|(L \cap K)/\sigma| = |(L \cap K)/\mathscr{L}^U| = r!/|\mathcal{G}_r|.$$

Consequently,

$$|(D_r \cap K)/\sigma| = |(D_r \cap K)/\mathscr{L}^U| = p_r r!/|\mathcal{G}_r|.$$

Proof. Since D_r contains p_r $\mathscr{L}$-classes, it suffices to prove the first claim. We do this by showing that

$$\alpha \mathrel{\sigma} \beta \qquad \Leftrightarrow \qquad \phi(\alpha)\phi(\beta)^{-1} \in \mathcal{G}_r \qquad \text{for all } \alpha, \beta \in L \cap K.$$

So let $\alpha, \beta \in L \cap K$, write $f = \phi(\alpha)$ and $g = \phi(\beta)$, let $\operatorname{codom}(\alpha) = \operatorname{codom}(\beta) = \{b_1 < \cdots < b_r\}$, and let (all) the non-trivial $\operatorname{coker}(\alpha) = \operatorname{coker}(\beta)$-classes be $B_1, \ldots, B_s$. We then have

$$\bar{\alpha} = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & & & \\ b_{1f} & \cdots & b_{rf} & B_1 & \cdots & B_s \end{array}\right) \qquad \text{and} \qquad \bar{\beta} = \left(\begin{array}{c|c|c|c|c|c} 1 & \cdots & r & & & \\ b_{1g} & \cdots & b_{rg} & B_1 & \cdots & B_s \end{array}\right). \tag{6.12}$$

Since $\alpha, \beta \in K$, it follows from Lemma 6.9 (and the definition of σ) that

$$\alpha \mathrel{\sigma} \beta \;\Leftrightarrow\; \alpha \mathrel{\mathscr{L}^U} \beta \;\Leftrightarrow\; \bar{\alpha} \mathrel{\mathscr{L}^U} \bar{\beta} \;\Leftrightarrow\; \bar{\alpha} \in U\bar{\beta}.$$

So it remains to show that

$$\bar{\alpha} \in U\bar{\beta} \qquad \Leftrightarrow \qquad fg^{-1} \in \mathcal{G}_r. \tag{6.13}$$

For the forwards implication, suppose $\bar{\alpha} = \theta\bar{\beta}$ for some $\theta \in U$. Examining (6.12), we see that θ fixes $\mathbf{r} = \{1, \ldots, r\}$ set-wise, and so $\vartheta = \theta{\restriction}_{\mathbf{r}} \in \mathcal{S}_r$. Moreover, it follows from $\theta \in U = \mathcal{G}_n$ that $\vartheta \in \mathcal{G}_r$. Now, for every $i \in \mathbf{r}$, the product $\theta\bar{\beta}$ contains the transversal $\{i, b'_{i\vartheta g}\}$. Keeping in mind that $\theta\bar{\beta} = \bar{\alpha}$, and comparing with (6.12), it follows that $f = \vartheta g$, i.e. $fg^{-1} = \vartheta \in \mathcal{G}_r$.

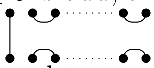
Conversely, if $fg^{-1} \in \mathcal{G}_r$, then $\bar{\alpha} = \theta\bar{\beta}$, where $\theta \in U$ is such that:

$$i\theta = ifg^{-1} \text{ for } 1 \leq i \leq r \qquad \text{and} \qquad j\theta = j \text{ for } r+1 \leq j \leq n.$$

This completes the proof of (6.13), and hence of the lemma. □

Remark 6.14. Using $|\mathcal{G}_r| = 2^s s!$, where $s = \lfloor r/2 \rfloor$, one can show that $r!/|\mathcal{G}_r| = r!!$ or $(r-1)!!$ for r odd or even, respectively.

6.3 Upper bound – odd case

We are now ready to construct faithful actions of the degree specified in Theorem 6.1. Here we consider the case that $n \geq 3$ is odd, and the even case will be treated in Section 6.4. We continue to fix the partition $\zeta =$ , and the right congruence σ from (6.10). Using square brackets to denote σ-classes, let

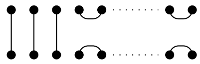
$$\Omega = \{[\alpha]_\sigma : \alpha \in I_3\}, \qquad \text{where} \qquad I_3 = D_1 \cup D_3.$$

Since I_3 is an ideal of $\mathcal{B}_n$, the action of $\mathcal{B}_n$ on $\mathcal{B}_n/\sigma$ restricts to an action

$$\mu : \Omega \times \mathcal{B}_n \to \Omega \qquad \text{given by} \qquad \mu([\alpha]_\sigma, \beta) = [\alpha]_\sigma \beta = [\alpha\beta]_\sigma. \tag{6.15}$$

Theorem 6.16. *For odd $n \geq 3$, the action $\mu : \Omega \times \mathcal{B}_n \to \Omega$ is faithful and monogenic, and consequently*

$$\deg(\mathcal{B}_n) \leq \deg_{\mathrm{rc}}(\mathcal{B}_n) \leq 1 + 3p_3 + p_1.$$

Proof. As explained in Section 6.1, the minimal congruences of $\mathcal{B}_n$ are $(\zeta, \alpha)^\sharp$ and $(\zeta, \beta)^\sharp$, where α and β are as in (6.5). Thus, by Lemma 3.6, we can establish faithfulness by showing that μ separates (ζ, α) and (ζ, β). For this we let $\pi =$ , and we note that

$$[\pi]_\sigma \zeta = [\zeta]_\sigma, \qquad [\pi]_\sigma \alpha = [\alpha]_\sigma = I \qquad \text{and} \qquad [\pi]_\sigma \beta = [\beta]_\sigma.$$

These are distinct because $\zeta, \beta \in K$, and $[\zeta]_\sigma = U\zeta = \{\zeta\}$.

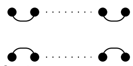
For monogenicity, we claim that $\Omega = \langle [\pi]_\sigma \rangle_\mu$. For this, we have seen that $I = [\pi]_\sigma \alpha$. Any other σ-class has the form $[\delta]_\sigma$ for some $\delta \in I_3 \cap K$, and by Lemma 6.9 we have

$$[\delta]_\sigma = [\bar{\delta}]_\sigma = [\pi\bar{\delta}]_\sigma = [\pi]_\sigma \bar{\delta}.$$

Finally, using Lemma 6.11, we have

$$\deg_{\mathrm{rc}}(\mathcal{B}_n) \leq |\Omega| = 1 + |(D_3 \cap K)/\sigma| + |(D_1 \cap K)/\sigma| = 1 + 3p_3 + p_1. \qquad \square$$

6.4 Upper bound – even case

Now we consider the case that $n = 2k \geq 4$ is even. We fix the partition $\zeta =$ , and the right congruence σ from (6.10). Unlike the odd case, minimum-degree faithful actions of $\mathcal{B}_n$ are not always monogenic; cf. Remark 6.3. Consequently, we will define such an action by first defining two separate actions μ_1 and μ_2, and then taking an appropriate push-out $\mu_1 \sqcup_\xi \mu_2$.

In all that follows, an important role will be played by the set

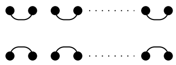
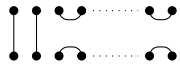
$$J = I_2 \cap K = I_2 \setminus I = \{\alpha \in \mathcal{B}_n : \mathrm{rank}(\alpha) \leq 2,\ \ker(\alpha) \subseteq \kappa\}.$$

Lemma 6.17. *J is a right ideal of $\mathcal{B}_n$.*

Proof. Let $\alpha \in J$ and $\beta \in \mathcal{B}_n$; we must show that $\alpha\beta \in J$, which amounts to showing that $\alpha\beta \in K$, as we certainly have $\alpha\beta \in I_2$. Now $\alpha\beta$ contains all the upper blocks of α, and these are either

$$Z_1, \ldots, Z_k \qquad \text{or} \qquad Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_k \ \text{ for some } i \in \mathbf{k}.$$

In the first case, the upper blocks of $\alpha\beta$ are of course $Z_1, \ldots, Z_k$. In the second case, the upper blocks of $\alpha\beta$ are the same as for α if $\mathrm{rank}(\alpha\beta) = 2$, or else $Z_1, \ldots, Z_k$ if $\mathrm{rank}(\alpha\beta) = 0$. In all cases, it follows that $\alpha\beta \in K$. $\square$

Now, in addition to $\zeta =$ , we also define $\gamma =$ . It follows from the proof of Lemma 6.17 that

$$\Omega_1 = R_\gamma \cup R_\zeta = \{\alpha \in \mathcal{B}_n : \alpha \mathrel{\mathscr{R}} \gamma \text{ or } \alpha \mathrel{\mathscr{R}} \zeta\} \tag{6.18}$$

is a right ideal of $\mathcal{B}_n$. Consequently, we have an action

$$\mu_1 : \Omega_1 \times \mathcal{B}_n \to \Omega_1 \qquad \text{given by} \qquad \mu_1(\alpha, \beta) = \alpha\beta.$$

To define the second action, we need to work a little harder. We first define a map

$$\mathcal{B}_n \to I_0 = D_0 : \alpha = \left(\begin{array}{c|c|c|c|c|c} a_1 & \cdots & a_r & A_1 & \cdots & A_s \\ b_1 & \cdots & b_r & B_1 & \cdots & B_s \end{array}\right) \mapsto \widehat{\alpha} = \left(\begin{array}{c|c|c|c|c|c} a_1, a_2 & \cdots & a_{r-1}, a_r & A_1 & \cdots & A_s \\ b_1, b_2 & \cdots & b_{r-1}, b_r & B_1 & \cdots & B_s \end{array}\right).$$

In particular, note that $\widehat{\alpha} = \alpha$ for $\alpha \in D_0$. By [19, Theorem 8.4], the relation

$$\chi = \Delta_{\mathcal{B}_n} \cup \{(\alpha, \beta) \in I_2 \times I_2 : \widehat{\alpha} = \widehat{\beta}\}$$

is a (two-sided) congruence on $\mathcal{B}_n$. We now define the relation

$$\tau = \nabla_I \sqcup \chi{\restriction}_J \sqcup \mathscr{L}^U{\restriction}_{K\setminus J},$$

and we note that $K \setminus J = K \cap (D_4 \cup D_6 \cup \cdots \cup D_n)$.

Lemma 6.19. *τ is a right congruence on $\mathcal{B}_n$.*

Proof. Since $\mathcal{B}_n = I \sqcup K$ and $K = J \sqcup (K \setminus J)$, τ is the union of equivalences on disjoint sets, and is therefore an equivalence itself. For right-compatibility, let $(\alpha, \beta) \in \tau$ and $\theta \in \mathcal{B}_n$; we must show that $(\alpha\theta, \beta\theta) \in \tau$. This is clear if $(\alpha, \beta) \in \nabla_I$.

Next suppose $(\alpha, \beta) \in \chi{\restriction}_J$. Since χ is a congruence we have $(\alpha\theta, \beta\theta) \in \chi$, and since J is a right ideal we have $\alpha\theta, \beta\theta \in J$. Together these give $(\alpha\theta, \beta\theta) \in \chi{\restriction}_J \subseteq \tau$.

Finally suppose $(\alpha, \beta) \in \mathscr{L}^U{\restriction}_{K\setminus J}$. Since $\mathscr{L}^U{\restriction}_{K\setminus J} \subseteq \mathscr{L}^U{\restriction}_K \subseteq \sigma$, it follows that $(\alpha\theta, \beta\theta)$ belongs to σ, and hence to either ∇_I or $\mathscr{L}^U{\restriction}_K$. In the first case we are done, so suppose $(\alpha\theta, \beta\theta) \in \mathscr{L}^U{\restriction}_K$. Since $\mathscr{L}^U \subseteq \mathscr{D}$ we have $\operatorname{rank}(\alpha\theta) = \operatorname{rank}(\beta\theta) = r$, say.

- If $r \geq 4$ then $(\alpha\theta, \beta\theta) \in \mathscr{L}^U{\restriction}_{K\setminus J} \subseteq \tau$.
- If $r = 0$ then $\alpha\theta$ and $\beta\theta$ belong to $D_0 \cap K = R$, and are $\mathscr{L}^U$- and hence $\mathscr{L}$-related. Since D_0 is $\mathscr{H}$-trivial, it follows that $\alpha\theta = \beta\theta$ in this case.
- Finally, if $r = 2$, then from $\alpha\theta, \beta\theta \in D_2 \cap K (\subseteq J)$ and $(\alpha\theta, \beta\theta) \in \mathscr{L}^U \subseteq \mathscr{L}$, we can write

  $$\alpha\theta = \left(\begin{array}{c|c|c|c|c} 2i-1 & 2i & & & \\ a & b & C_1 & \cdots & C_{k-1} \end{array}\right) \qquad \text{and} \qquad \beta\theta = \left(\begin{array}{c|c|c|c|c} 2j-1 & 2j & & & \\ a & b & C_1 & \cdots & C_{k-1} \end{array}\right) \qquad \text{for some } 1 \leq i, j \leq k.$$

  We then have $\widehat{\alpha\theta} = \left(\begin{array}{c|c|c|c} & & & \\ a, b & C_1 & \cdots & C_{k-1} \end{array}\right) = \widehat{\beta\theta}$, so that $(\alpha\theta, \beta\theta) \in \chi{\restriction}_J \subseteq \tau$. $\square$

Using square brackets for τ-classes, let

$$\Omega_2 = \big\{[\alpha]_\tau : \alpha \in I_4\big\}, \qquad \text{where} \qquad I_4 = D_0 \cup D_2 \cup D_4.$$

Since I_4 is an ideal, the action of $\mathcal{B}_n$ on $\mathcal{B}_n/\tau$ restricts to an action

$$\mu_2 : \Omega_2 \times \mathcal{B}_n \to \Omega_2 \qquad \text{given by} \qquad \mu_2([\alpha]_\tau, \beta) = [\alpha\beta]_\tau.$$

It follows from the definition of τ that

$$\Omega_2 = \{I\} \cup (D_4 \cap K)/\mathscr{L}^U \cup \Omega_3, \qquad \text{where} \qquad \Omega_3 = \big\{[\alpha]_\tau : \alpha \in J\big\} = \big\{[\alpha]_\tau : \alpha \in R_\zeta\big\}, \tag{6.20}$$

and that each τ-class from Ω_3 is in fact a χ-class. Since $\widehat{\alpha} = \alpha$ for all $\alpha \in R_\zeta$, the map

$$\xi : R_\zeta \to \Omega_3 : \alpha \mapsto [\alpha]_\tau$$

is a bijection. Since R_ζ is a right ideal of $\mathcal{B}_n$, it follows that R_ζ and Ω_3 are sub-acts of Ω_1 and Ω_2, respectively, and it is easy to see that ξ is in fact an isomorphism of these sub-acts. As in (3.1), we can therefore form the push-out

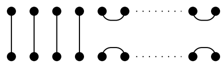
$$\mu = \mu_1 \sqcup_\xi \mu_2 : \Omega \times \mathcal{B}_n \to \Omega, \qquad \text{where} \qquad \Omega = \Omega_1 \sqcup_\xi \Omega_2.$$

Theorem 6.21. *For even $n \geq 4$, the action $\mu : \Omega \times \mathcal{B}_n \to \Omega$ is faithful, and consequently*

$$\deg(\mathcal{B}_n) \leq 1 + 3p_4 + 2p_2 + p_0.$$

Proof. As explained in Section 6.1, the minimal congruences of $\mathcal{B}_n$ are $(\zeta, \alpha)^\sharp$, $(\zeta, \beta)^\sharp$ and $(\gamma, \delta)^\sharp$, where these partitions are as in (6.6). Thus, by Lemma 3.6, we can establish faithfulness by showing that μ separates (ζ, α), (ζ, β) and (γ, δ). Since $\mu = \mu_1 \sqcup_\xi \mu_2$, we can do this by showing that μ_1 separates (ζ, β) and (γ, δ), while μ_2 separates (ζ, α). For the former we have

$$\mu_1(\zeta, \zeta) = \zeta, \qquad \mu_1(\zeta, \beta) = \beta, \qquad \mu_1(\gamma, \gamma) = \gamma \qquad \text{and} \qquad \mu_1(\gamma, \delta) = \delta,$$

which are all distinct. For the latter, and with $\pi =$ , we have

$$\mu_2([\pi]_\tau, \zeta) = [\zeta]_\tau \qquad \text{and} \qquad \mu_2([\pi]_\tau, \alpha) = [\alpha]_\tau = I.$$

It now follows that

$$\deg(\mathcal{B}_n) \leq |\Omega| = |\Omega_1 \sqcup_\xi \Omega_2| = |\Omega_1| + |\Omega_2| - |\Omega_3|.$$

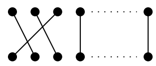
Combining (6.18) and (6.20) with Lemma 6.11 we have

$$|\Omega_1| = |R_\gamma| + |R_\zeta| = 2p_2 + p_0 \qquad \text{and} \qquad |\Omega_2| = 1 + 3p_4 + |\Omega_3|.$$

It follows that $|\Omega| = 1 + 3p_4 + 2p_2 + p_0$. □

6.5 Lower bound – odd case

We now turn to the task of showing that the claimed value for $\deg(\mathcal{B}_n)$ in Theorem 6.1 is a lower bound, again treating the odd and even cases in separate sections.

Here we assume $n = 2k + 1 \geq 3$ is odd, and we fix $\zeta, \alpha, \beta \in \mathcal{B}_n$, as in (6.5). For a subset $X \subseteq P = P(\mathcal{B}_n)$ we write $\bar{X} = \{\bar{\varepsilon} : \varepsilon \in X\}$, where here $\bar{\varepsilon}$ is as in (6.7). We will also use cycle notation for permutations from $\mathcal{S}_n \subseteq \mathcal{B}_n$, so that for example $(1, 2, 3) =$ . We fix the cyclic subgroup

$$\mathcal{C}_3 = \langle (1, 2, 3) \rangle = \{\mathrm{id}_n, (1, 2, 3), (1, 3, 2)\} \leq \mathcal{S}_n.$$

Lemma 6.22. *Let σ be a right congruence of $\mathcal{B}_n$, where $n \geq 3$ is odd.*

(i) *If σ separates $\{\zeta, \alpha\}$, then it separates $\mathcal{C}_3\bar{P}_3 = \{\xi\bar{\varepsilon} : \xi \in \mathcal{C}_3, \ \varepsilon \in P_3\}$.*

(ii) *If σ separates $\{\zeta, \beta\}$, then it separates $\bar{P}_1$.*

Proof. (i). As usual, we prove the contrapositive. So suppose $(\xi_1\bar{\varepsilon}_1, \xi_2\bar{\varepsilon}_2) \in \sigma$ for some $\xi_1, \xi_2 \in \mathcal{C}_3$ and $\varepsilon_1, \varepsilon_2 \in P_3$, with $\xi_1\bar{\varepsilon}_1 \neq \xi_2\bar{\varepsilon}_2$; we must show that $(\zeta, \alpha) \in \sigma$. Write

$$\xi_1\bar{\varepsilon}_1 = \left(\begin{smallmatrix} 1 & 2 & 3 & & \\ a_1 & a_2 & a_3 & A_1 & \cdots & A_{k-1} \end{smallmatrix} \right) \qquad \text{and} \qquad \xi_2\bar{\varepsilon}_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & & \\ b_1 & b_2 & b_3 & B_1 & \cdots & B_{k-1} \end{smallmatrix} \right),$$

and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. We now split into cases; in each we define an element $\theta \in \mathcal{B}_n$ for which $\{\xi_1\bar{\varepsilon}_1\theta, \xi_2\bar{\varepsilon}_2\theta\} = \{\zeta, \alpha\}$.

Case 1. Suppose first that $A \neq B$. If $a_1 \notin \{b_1, b_2\}$ or $a_3 \notin \{b_2, b_3\}$, then we take $\theta = \left(\begin{smallmatrix} a_1 & b_1, b_2 \\ 1 & \end{smallmatrix}\right)$ or $\theta = \left(\begin{smallmatrix} a_3 & b_2, b_3 \\ 1 & \end{smallmatrix}\right)$, respectively. The cases in which $b_1 \notin \{a_1, a_2\}$ or $b_3 \notin \{a_2, a_3\}$ are analogous. This leave us to consider the case in which

$$a_1 \in \{b_1, b_2\}, \qquad a_3 \in \{b_2, b_3\}, \qquad b_1 \in \{a_1, a_2\} \qquad \text{and} \qquad b_3 \in \{a_2, a_3\}.$$

Since $A \neq B$, this implies that $a_2 \notin B$ and $b_2 \notin A$, so in fact $a_1 = b_1$ and $a_3 = b_3$. We can then assume without loss of generality that $B_1 = \{a_2, x\}$ for some $x \in \mathbf{n}$, and we take $\theta = \left(\begin{smallmatrix} x & a_1, a_2 \\ 1 & \end{smallmatrix}\right)$.

Case 2. Next suppose $a_i = b_i$ for $i = 1, 2, 3$. Since $\xi_1\bar{\varepsilon}_1 \neq \xi_2\bar{\varepsilon}_2$, we can assume without loss of generality that $A_1 = \{u, v\}$, $B_1 = \{u, x\}$ and $B_2 = \{v, y\}$ for distinct $u, v, x, y \in \mathbf{n}$. We then take $\theta = \left(\begin{smallmatrix} x & a_1, u & a_2, v \\ 1 & & \end{smallmatrix}\right)$.

Case 3. Up to symmetry, the final case to consider is where $b_1 = a_2$, $b_2 = a_3$ and $b_3 = a_1$, and we then take $\theta = \left(\begin{smallmatrix} a_1 & a_2, a_3 \\ 1 & \end{smallmatrix}\right)$.

(ii). Suppose $(\bar{\varepsilon}_1, \bar{\varepsilon}_2) \in \sigma$ for distinct $\varepsilon_1, \varepsilon_2 \in P_1$. Since then $\operatorname{coker}(\varepsilon_1) \neq \operatorname{coker}(\varepsilon_2)$, there exist distinct $u, v, w \in \mathbf{n}$ such that $(u, v) \in \operatorname{coker}(\varepsilon_1)$ and $(v, w) \in \operatorname{coker}(\varepsilon_2)$. Then with $\theta = \left(\begin{smallmatrix} u & v & w \\ 1 & 2 & 3 \end{smallmatrix}\right)$ we have $(\beta, \zeta) = (\bar{\varepsilon}_1\theta, \bar{\varepsilon}_2\theta) \in \sigma$. □

We can now complete the proof of Theorem 6.1 in the odd case:

Proposition 6.23. *If $n \geq 3$ is odd, then* $\deg(\mathcal{B}_n) \geq 3p_3 + p_1 + 1$.

Proof. Let $\mu : X \times \mathcal{B}_n \to X$ be a faithful action, denoted $\mu(x, \delta) = x\delta$. We must show that $|X| \geq 3p_3 + p_1 + 1$. Since μ is faithful, and since the minimal congruences of $\mathcal{B}_n$ are generated by the pairs (ζ, α) and (ζ, β), we can fix elements $x_1, x_3 \in X$ such that

$$x_1\zeta \neq x_1\beta \qquad \text{and} \qquad x_3\zeta \neq x_3\alpha. \tag{6.24}$$

We define the sets

$$Y_1 = x_1\bar{P}_1 = \{x_1\bar{\varepsilon} : \varepsilon \in P_1\} \qquad \text{and} \qquad Y_3 = x_3\mathcal{C}_3\bar{P}_3 = \{x_3\xi\bar{\varepsilon} : \xi \in \mathcal{C}_3,\ \varepsilon \in P_3\},$$

noting that $|Y_1| = p_1$ and $|Y_3| = |\mathcal{C}_3\bar{P}_3| = 3p_3$ by Lemmas 3.7 and 6.22. Thus, we can show that $|X| \geq 3p_3 + p_1 + 1$ by showing that

$$Y_1 \cap Y_3 = \varnothing \qquad \text{and} \qquad X \setminus (Y_1 \cup Y_3) \neq \varnothing. \tag{6.25}$$

To prove the first assertion, suppose to the contrary that $x_1\bar{\varepsilon}_1 = x_3\xi\bar{\varepsilon}_3$ for some $\varepsilon_i \in P_i$ and $\xi \in \mathcal{C}_3$. Then with $\theta_1 = \bar{\varepsilon}_3^*\xi^{-1}\alpha$ and $\theta_2 = \bar{\varepsilon}_3^*\xi^{-1}\zeta$, we have

$$\xi\bar{\varepsilon}_3\theta_1 = \alpha \qquad \text{and} \qquad \bar{\varepsilon}_1\theta_1 = \bar{\varepsilon}_1\theta_2 = \xi\bar{\varepsilon}_3\theta_2 = \zeta.$$

It follows that

$$x_3\alpha = (x_3\xi\bar{\varepsilon}_3)\theta_1 = (x_1\bar{\varepsilon}_1)\theta_1 = x_1\zeta = (x_1\bar{\varepsilon}_1)\theta_2 = (x_3\xi\bar{\varepsilon}_3)\theta_2 = x_3\zeta,$$

contradicting (6.24).

This leaves us to prove the second assertion in (6.25). This is certainly true if $x_3\zeta \notin Y_1 \cup Y_3$, so suppose otherwise.

If $x_3\zeta \in Y_3$, say with $x_3\zeta = x_3\xi\bar{\varepsilon}$ for $\xi \in \mathcal{C}_3$ and $\varepsilon \in P_3$, then with $\theta = \bar{\varepsilon}^*\xi^{-1}\alpha$ we have $\zeta\theta = \zeta$ and $\xi\bar{\varepsilon}\theta = \alpha$, and this gives $x_3\zeta = (x_3\zeta)\theta = (x_3\xi\bar{\varepsilon})\theta = x_3\alpha$, again contradicting (6.24).

So we must instead have $x_3\zeta \in Y_1$, say with $x_3\zeta = x_1\bar{\varepsilon}$ for $\varepsilon \in P_1$, and it follows that $x_3\zeta = x_3\zeta\zeta = x_1\bar{\varepsilon}\zeta = x_1\zeta$. We complete the proof by showing that $x_3\alpha \notin Y_1 \cup Y_3$. Indeed, if $x_3\alpha \in Y_1$, say with $x_3\alpha = x_1\bar{\varepsilon}_1$ for $\varepsilon_1 \in P_1$, then

$$x_3\alpha = x_3\alpha\zeta = x_1\bar{\varepsilon}_1\zeta = x_1\zeta = x_3\zeta,$$

contradicting (6.24). But if $x_3\alpha \in Y_3$, say with $x_3\alpha = x_3\xi\bar{\varepsilon}_3$ for $\xi \in \mathcal{C}_3$ and $\varepsilon_3 \in P_3$, then

$$x_3\alpha = x_3\alpha \cdot \bar{\varepsilon}_3^*\xi^{-1}\zeta = x_3\xi\bar{\varepsilon}_3 \cdot \bar{\varepsilon}_3^*\xi^{-1}\zeta = x_3\zeta,$$

contradicting (6.24). □

6.6 Lower bound – even case

We now assume $n = 2k \geq 4$ is even, and we fix $\zeta, \alpha, \beta, \gamma, \delta \in \mathcal{B}_n$, as in (6.6). In addition to the subgroup $\mathcal{C}_3 = \langle (1,2,3) \rangle = \{\mathrm{id}_n, (1,2,3), (1,3,2)\}$ of $\mathcal{S}_n$, we also fix $\mathcal{C}_2 = \langle (1,2) \rangle = \{\mathrm{id}_n, (1,2)\}$.

Lemma 6.26. *Let σ be a right congruence of $\mathcal{B}_n$, where $n \geq 4$ is even.*

(i) *If σ separates $\{\zeta, \alpha\}$, then it separates $\mathcal{C}_3\bar{P}_4$.*

(ii) *If σ separates $\{\zeta, \beta\}$, then it separates $\bar{P}_0$.*

(iii) *If σ separates $\{\gamma, \delta\}$, then it separates $\mathcal{C}_2\bar{P}_2$.*

Proof. (i). Suppose $(\xi_1\bar{\varepsilon}_1, \xi_2\bar{\varepsilon}_2) \in \sigma$ for some $\xi_1, \xi_2 \in \mathcal{C}_3$ and $\varepsilon_1, \varepsilon_2 \in P_4$, with $\xi_1\bar{\varepsilon}_1 \neq \xi_2\bar{\varepsilon}_2$, and write

$$\xi_1\bar{\varepsilon}_1 = \left(\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & & & \\ a_1 & a_2 & a_3 & a_4 & A_1 & \cdots & A_{k-2} \end{array}\right) \qquad \text{and} \qquad \xi_2\bar{\varepsilon}_2 = \left(\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & & & \\ b_1 & b_2 & b_3 & b_4 & B_1 & \cdots & B_{k-2} \end{array}\right).$$

We must show that $(\zeta, \alpha) \in \sigma$. We now consider various cases; in each we define an element $\theta \in \mathcal{B}_n$ for which $\{\xi_1\bar{\varepsilon}_1\theta, \xi_2\bar{\varepsilon}_2\theta\} = \{\zeta, \alpha\}$.

Case 1. First, if

(a) $\{a_1, a_2\} \cap \{b_1, b_4\} = \varnothing$,

(b) $\{a_1, a_2\} \cap \{b_2, b_3\} = \varnothing$,

(c) $\{a_3, a_4\} \cap \{b_1, b_4\} = \varnothing$, or

(d) $\{a_3, a_4\} \cap \{b_2, b_3\} = \varnothing$,

then we take (a) $\theta = \left(\begin{smallmatrix} a_1, a_2 | b_1, b_4 \\ \ \end{smallmatrix}\right)$, (b) $\theta = \left(\begin{smallmatrix} a_1, a_2 | b_2, b_3 \\ \ \end{smallmatrix}\right)$, (c) $\theta = \left(\begin{smallmatrix} a_3, a_4 | b_1, b_4 \\ \ \end{smallmatrix}\right)$, or (d) $\theta = \left(\begin{smallmatrix} a_3, a_4 | b_2, b_3 \\ \ \end{smallmatrix}\right)$, respectively.

Case 2. Now suppose all four of

$$\{a_1, a_2\} \cap \{b_1, b_4\}, \qquad \{a_1, a_2\} \cap \{b_2, b_3\}, \qquad \{a_3, a_4\} \cap \{b_1, b_4\} \qquad \text{and} \qquad \{a_3, a_4\} \cap \{b_2, b_3\}$$

are non-empty, noting then that that this forces $\{a_1, a_2, a_3, a_4\} = \{b_1, b_2, b_3, b_4\}$. In particular, this says that $\mathrm{codom}(\varepsilon_1) = \mathrm{codom}(\varepsilon_2)$, and it follows that $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ have the same transversals. Since $\xi_1, \xi_2 \in \mathcal{C}_3$, it follows that the ordered tuple (b_1, b_2, b_3, b_4) is equal to either

(a) (a_1, a_2, a_3, a_4),

(b) (a_2, a_3, a_1, a_4), or

(c) (a_3, a_1, a_2, a_4).

In cases (b) and (c) we take $\theta = \left(\begin{smallmatrix} a_1, a_4 | a_2, a_3 \\ \ \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a_1, a_2 | a_3, a_4 \\ \ \end{smallmatrix}\right)$, respectively. So, finally consider case (a), in which we have $a_i = b_i$ for $i = 1, 2, 3, 4$. Since $\xi_1\bar{\varepsilon}_1 \neq \xi_2\bar{\varepsilon}_2$, we can assume without loss of generality that $A_1 = \{u, v\}$, $B_1 = \{u, x\}$ and $B_2 = \{v, y\}$ for distinct $u, v, x, y \in \mathbf{n}$, and we take $\theta = \left(\begin{smallmatrix} a_1, u | a_2, v | a_3, y | a_4, x \\ \ \end{smallmatrix}\right)$.

(ii). Suppose $(\bar\varepsilon_1,\bar\varepsilon_2)\in\sigma$ for distinct $\varepsilon_1,\varepsilon_2\in P_0$, and fix

$$(u,v)\in\operatorname{coker}(\varepsilon_1)\qquad\text{and}\qquad(u,x),(v,y)\in\operatorname{coker}(\varepsilon_2),\qquad\text{where}\qquad u,v,x,y\in\mathbf{n}\text{ are distinct.}\tag{6.27}$$

Then with $\theta=\left(\begin{smallmatrix}x&u&v&y\\1&2&3&4\end{smallmatrix}\right)$ we have $(\beta,\zeta)=(\bar\varepsilon_1\theta,\bar\varepsilon_2\theta)\in\sigma$.

(iii). Suppose $(\xi_1\bar\varepsilon_1,\xi_2\bar\varepsilon_2)\in\sigma$ for some $\xi_1,\xi_2\in\mathcal{C}_2$ and $\varepsilon_1,\varepsilon_2\in P_2$, with $\xi_1\bar\varepsilon_1\neq\xi_2\bar\varepsilon_2$, and write

$$\xi_1\bar\varepsilon_1=\left(\begin{smallmatrix}1&2&&&\\a_1&a_2&A_1&\cdots&A_{k-1}\end{smallmatrix}\right)\qquad\text{and}\qquad\xi_2\bar\varepsilon_2=\left(\begin{smallmatrix}1&2&&&\\b_1&b_2&B_1&\cdots&B_{k-1}\end{smallmatrix}\right).$$

If $a_1\neq b_1$ or $a_2\neq b_2$, then with $\theta=\left(\begin{smallmatrix}a_1&b_1\\1&2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}a_2&b_2\\2&1\end{smallmatrix}\right)$, respectively, we have $(\gamma,\delta)=(\xi_1\bar\varepsilon_1\theta,\xi_2\bar\varepsilon_2\theta)\in\sigma$. Now suppose $a_1=b_1$ and $a_2=b_2$. Since $\xi_1\bar\varepsilon_1\neq\xi_2\bar\varepsilon_2$, we can fix $u,v,x,y\in\mathbf{n}$ as in (6.27), and with $\theta=\left(\begin{smallmatrix}v&x&a_1,u&a_2,y\\1&2&&\end{smallmatrix}\right)$ we have $(\gamma,\delta)=(\xi_1\bar\varepsilon_1\theta,\xi_2\bar\varepsilon_2\theta)\in\sigma$. □

Here then is the final piece of the proof of Theorem 6.1.

Proposition 6.28. *If $n\geq4$ is even, then $\deg(\mathcal{B}_n)\geq3p_4+2p_2+p_0+1$.*

Proof. Let $\mu:X\times\mathcal{B}_n\to X$ be a faithful action. We must show that $|X|\geq3p_4+2p_2+p_0+1$. Since μ is faithful, and since the minimal congruences of $\mathcal{B}_n$ are generated by the pairs (ζ,α), (ζ,β) and (γ,δ), we can fix elements $x_0,x_2,x_4\in X$ such that

$$x_0\zeta\neq x_0\beta,\qquad x_2\gamma\neq x_2\delta\qquad\text{and}\qquad x_4\zeta\neq x_4\alpha,\tag{6.29}$$

where we again use shorthand notation for the action. We define the sets

$$Y_0=x_0\bar P_0,\qquad Y_2=x_2\mathcal{C}_2\bar P_2\qquad\text{and}\qquad Y_4=x_4\mathcal{C}_3\bar P_4,$$

noting that $|Y_0|=p_0$, $|Y_2|=2p_2$ and $|Y_4|=3p_4$, by Lemmas 3.7 and 6.26. Thus, we can show that $|X|\geq3p_4+2p_2+p_0+1$ by showing that

$$Y_0,\ Y_2\text{ and }Y_4\text{ are pairwise disjoint}\qquad\text{and}\qquad X\setminus(Y_0\cup Y_2\cup Y_4)\neq\varnothing.\tag{6.30}$$

Before we do this, we first claim that

$$(Y_2\cup Y_4)\cap\{x\zeta:x\in X\}=\varnothing.\tag{6.31}$$

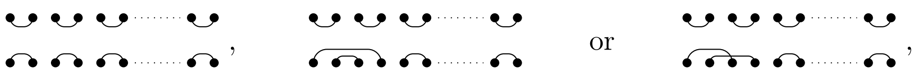
To prove this, suppose to the contrary that $x\zeta\in Y_2\cup Y_4$ for some $x\in X$.

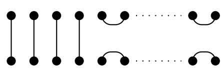
First consider the case that $x\zeta\in Y_2$, and write $x\zeta=x_2\xi\bar\varepsilon$ where $\xi\in\mathcal{C}_2$ and $\varepsilon\in P_2$. Then

$$x\zeta=x\zeta\bar\varepsilon^*\xi^{-1}=x_2\xi\bar\varepsilon\bar\varepsilon^*\xi^{-1}=x_2\gamma,$$

and we then have $x_2\delta=x_2\gamma\delta=x\zeta\delta=x\zeta=x_2\gamma$, contradicting (6.29).

Now suppose instead that $x\zeta\in Y_4$, and write $x\zeta=x_4\xi\bar\varepsilon$ where $\xi\in\mathcal{C}_3$ and $\varepsilon\in P_4$. This time we first note that $\zeta\bar\varepsilon^*\xi^{-1}$ is equal to one of



each of which has the form $\zeta\xi_0$ for some $\xi_0\in\mathcal{C}_3$. Then with $\pi=$ , it follows that

$$x\zeta\xi_0=x\zeta\bar\varepsilon^*\xi^{-1}=x_4\xi\bar\varepsilon\bar\varepsilon^*\xi^{-1}=x_4\pi.$$

But then $x_4\zeta=x_4\pi\zeta=x\zeta\xi_0\zeta=x\zeta=x\zeta\xi_0\alpha=x_4\pi\alpha=x_4\alpha$, contradicting (6.29).

Now that we have proved (6.31), our next claim is that

$$Y_2\cap Y_4=\varnothing.\tag{6.32}$$

To prove this, suppose to the contrary that $x_2\xi_2\bar\varepsilon_2 = x_4\xi_4\bar\varepsilon_4$ for some $\xi_2 \in \mathcal{C}_2$, $\xi_4 \in \mathcal{C}_3$, $\varepsilon_2 \in P_2$ and $\varepsilon_4 \in P_4$. Then

$$x_2\zeta = x_2\xi_2\bar\varepsilon_2\bar\varepsilon_4^*\xi_4^{-1}\alpha = x_4\xi_4\bar\varepsilon_4\bar\varepsilon_4^*\xi_4^{-1}\alpha = x_4\pi\alpha = x_4\alpha.$$

The previous calculation is valid with α replaced by ζ, leading to $x_2\zeta = x_4\zeta$. Combining these, it follows that $x_4\alpha = x_2\zeta = x_4\zeta$, contradicting (6.29).

Now that we have proved (6.32), our next claim is that

$$Y_0 \cap (Y_2 \cup Y_4) = \varnothing. \tag{6.33}$$

To prove this, suppose to the contrary that $x_0\bar\varepsilon_0 = x_i\xi_i\bar\varepsilon_i$, where $i = 2$ or 4, and where $\varepsilon_i \in P_i$ and $\xi_i \in \mathcal{C}_2$ or $\mathcal{C}_3$ as appropriate. We then have $x_0\bar\varepsilon_0\bar\varepsilon_i^*\xi_i^{-1} = x_i\xi_i\bar\varepsilon_i\bar\varepsilon_i^*\xi_i^{-1}$, and we note that

$$\xi_i\bar\varepsilon_i\bar\varepsilon_i^*\xi_i^{-1} = \begin{cases} \gamma & \text{if } i = 2 \\ \pi & \text{if } i = 4, \end{cases} \qquad \text{and} \qquad \bar\varepsilon_0\varepsilon_i^*\xi_i^{-1} = \zeta,\ \beta \text{ or } \beta\delta,$$

but that $\bar\varepsilon_0\varepsilon_i^*\xi_i^{-1} = \beta$ or $\beta\delta$ is only possible when $i = 4$. It follows that either

$$x_2\gamma = x_0\zeta, \qquad x_4\pi = x_0\zeta, \qquad x_4\pi = x_0\beta \qquad \text{or} \qquad x_4\pi = x_0\beta\delta.$$

Keeping in mind that $\gamma = \bar\gamma$ and $\pi = \bar\pi$, the first two options contradict (6.31). We now investigate the remaining two cases. First, if $x_4\pi = x_0\beta\delta$, then

$$x_4\alpha = x_4\pi\alpha = x_0\beta\delta\alpha = x_0\beta\delta\zeta = x_4\pi\zeta = x_4\zeta,$$

contradicting (6.29). So now suppose $x_4\pi = x_0\beta$. As noted above, this arises when $i = 4$ (so $x_0\bar\varepsilon_0 = x_4\xi_4\bar\varepsilon_4$) and $\bar\varepsilon_0\bar\varepsilon_4^*\xi_4^{-1} = \beta$. Here we calculate

$$x_4\zeta = x_4\pi\zeta = x_0\beta\zeta = x_0\zeta = x_0\bar\varepsilon_0\bar\varepsilon_0^* = x_4\xi_4\bar\varepsilon_4\bar\varepsilon_0^* = x_4(\bar\varepsilon_0\bar\varepsilon_4^*\xi_4^{-1})^* = x_4\beta^* = x_4\alpha,$$

again contradicting (6.29).

Now that we have proved (6.32) and (6.33), it remains to prove the second assertion in (6.30). This is clearly the case if either $x_2\zeta$ or $x_4\zeta$ does not belong to $Y_0 \cup Y_2 \cup Y_4$. Given (6.31), the only other possibility is that $x_2\zeta, x_4\zeta \in Y_0$, and we now assume this is the case. So for $i = 2, 4$ we have $x_i\zeta = x_0\bar\varepsilon_i$ for some $\varepsilon_i \in P_0$. It then follows that $x_i\zeta = x_i\zeta\zeta = x_0\bar\varepsilon_i\zeta = x_0\zeta$, so in fact

$$x_4\zeta = x_2\zeta = x_0\zeta.$$

We complete the proof by showing that $x_4\alpha \notin Y_0 \cup Y_2 \cup Y_4$. To do so, suppose to the contrary that $x_4\alpha = x_i\xi\bar\varepsilon$ for some $i \in \{0, 2, 4\}$, and some $\varepsilon \in P_i$ and $\xi \in \mathcal{C}_2 \cup \mathcal{C}_3$ as appropriate. We then note that $\xi\bar\varepsilon\bar\varepsilon^*\xi^{-1} = \zeta$, γ or π; in any case it follows that $\xi\bar\varepsilon\bar\varepsilon^*\xi^{-1}\zeta = \zeta$. But then

$$x_4\alpha = x_4\alpha\bar\varepsilon^*\xi^{-1}\zeta = x_i\xi\bar\varepsilon\bar\varepsilon^*\xi^{-1}\zeta = x_i\zeta = x_4\zeta,$$

contradicting (6.29). □

7 Combinatorics

In Section 6 we obtained an explicit formula for the transformation degree of the Brauer monoid $\mathcal{B}_n$; see (6.2). For the other diagram monoids $M = \mathcal{P}_n$, $\mathcal{PB}_n$, $\mathscr{P}\mathcal{P}_n$, $\mathcal{M}_n$ and $\mathcal{TL}_n$, we showed in Sections 4 and 5 that

$$\deg(M) = \deg_{\mathrm{rc}}(M) = 1 + |Q| \qquad \text{and} \qquad \deg'(M) = |Q|$$

for $n \geq 2$ or 3 (as appropriate), where

$$Q = Q(M) = \begin{cases} P_0 \cup P_1 \cup P_2 & \text{if } M = \mathcal{P}_n,\ \mathcal{PB}_n,\ \mathscr{P}\mathcal{P}_n \text{ or } \mathcal{M}_n \\ P_0 \cup P_2 \cup P_4 & \text{if } M = \mathcal{TL}_n \text{ for even } n \\ P_1 \cup P_3 & \text{if } M = \mathcal{TL}_n \text{ for odd } n. \end{cases}$$

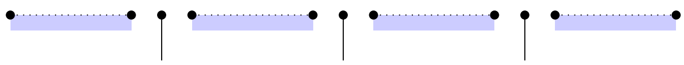
In this final section we obtain explicit formulae for $|Q|$. These are valid for arbitrary $n \geq 0$, but we note that some of the subsets P_r involved in the above unions are empty for very small n.

Formulae for the sizes of $P_r = P_r(M) = \{\varepsilon \in P : \text{rank}(\varepsilon) = r\}$ are known (see for example [9, Proposition 4.6]), and these could of course be added to obtain expressions for $|Q|$. However, since the relevant values of r are very small, a direct combinatorial analysis is possible, and allows us to express $|Q|$ in terms of very fundamental number sequences, such as Bell, Catalan and Motzkin numbers.

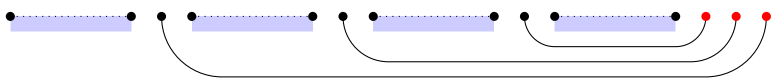
Our strategy relies on the existence, for any $0 \leq r \leq n$, of an injective mapping

$$P_r(\mathcal{P}_n) \to P_0(\mathcal{P}_{n+r}) : \varepsilon = \binom{A_1 \big| \cdots \big| A_r \big| C_1 \big| \cdots \big| C_s}{A_1 \big| \cdots \big| A_r \big| C_1 \big| \cdots \big| C_s} \mapsto \varepsilon^+ = \binom{A_1 \cup \{n+r\} \big| \cdots \big| A_r \cup \{n+1\} \big| C_1 \big| \cdots \big| C_s}{A_1 \cup \{n+r\} \big| \cdots \big| A_r \cup \{n+1\} \big| C_1 \big| \cdots \big| C_s}. \tag{7.1}$$

Recall that we always assume $\min(A_1) < \cdots < \min(A_r)$ when using this tabular notation. It follows that the map $\varepsilon \mapsto \varepsilon^+$ preserves planarity; it also maps (partial) Brauer diagrams to (partial) Brauer diagrams. As an example, consider a typical projection $\varepsilon \in P_3(\mathcal{TL}_n)$:

$\varepsilon =$ 

Here we have only drawn the top half of ε, as the bottom is just the mirror image. We then have

$\varepsilon^+ =$ 

(7.2)

To simplify expressions in what follows, we will write

$$p_r(M) = |P_r(M)| \qquad \text{for any diagram monoid } M, \text{ and any integer } r \geq 0.$$

We therefore have

$$|Q(M)| = p_0(M) + p_1(M) + p_2(M) \qquad \text{if } M \text{ is one of } \mathcal{P}_n,\ \mathcal{PB}_n,\ \mathscr{P}\mathcal{P}_n \text{ or } \mathcal{M}_n.$$

Analogous statements hold for $M = \mathcal{TL}_n$. Our ultimate goal, however, is to express each $|Q(M)|$ in terms of the relevant p_0 parameters, which themselves have the following simple forms:

Proposition 7.3 (see [9, Proposition 4.6]). *For $n \geq 0$, we have*

(i) $p_0(\mathcal{P}_n) = B(n)$, *the nth Bell number,*

(ii) $p_0(\mathcal{PB}_n) = I(n)$, *the nth involution number,*

(iii) $p_0(\mathscr{P}\mathcal{P}_n) = C(n)$ *the nth Catalan number,*

(iv) $p_0(\mathcal{M}_n) = M(n)$, *the nth Motzkin number,*

(v) $p_0(\mathcal{TL}_n) = C(n/2)$ *for even n.* □

The ubiquity of the numbers occurring in Proposition 7.3 is evidenced by their extremely low sequence numbers as A000085, A000108, A000110 and A001006 on the OEIS [1]. Note that $I(n)$ is the number of involutions (i.e. self-inverse permutations) of $\mathbf{n}$. These are given by the recurrence

$$I(0) = I(1) = 1 \qquad \text{and} \qquad I(n) = I(n-1) + (n-1)I(n-2) \qquad \text{for } n \geq 2.$$

We now proceed to give formulae for $|Q(M)|$, starting with $M = \mathcal{P}_n$.

Proposition 7.4. *For $n \geq 0$ we have* $|Q(\mathcal{P}_n)| = \dfrac{B(n+2) - B(n+1) + B(n)}{2}$.

Proof. Since $|Q(\mathcal{P}_n)| = p_0(\mathcal{P}_n) + p_1(\mathcal{P}_n) + p_2(\mathcal{P}_n)$, we can prove the result by showing that:

(i) $p_0(\mathcal{P}_n) = B(n)$,

(ii) $p_1(\mathcal{P}_n) = B(n+1) - B(n)$,

(iii) $p_2(\mathcal{P}_n) = \frac{B(n+2)-3B(n+1)+B(n)}{2}$.

(i). This is part of Proposition 7.3.

(ii). The image of the map $P_1(\mathcal{P}_n) \to P_0(\mathcal{P}_{n+1}) : \varepsilon \mapsto \varepsilon^+$ consists of all projections from $P_0(\mathcal{P}_{n+1})$ not containing the block $\{n+1\}$. Since there are $p_0(\mathcal{P}_n)$ such 'offending' projections, we have

$$p_1(\mathcal{P}_n) = p_0(\mathcal{P}_{n+1}) - p_0(\mathcal{P}_n) = B(n+1) - B(n).$$

(iii). This time we consider the map $P_2(\mathcal{P}_n) \to P_0(\mathcal{P}_{n+2}) : \varepsilon \mapsto \varepsilon^+$, and calculate the size of its image, via the following steps.

(a) First we note that $P_0(\mathcal{P}_{n+2})$ contains $B(n+2)$ projections.

(b) From $B(n+2)$ we subtract $B(n+1)$, corresponding to the projections for which $n+1$ and $n+2$ belong to the same block. (These projections are not in the image of the $\varepsilon \mapsto \varepsilon^+$ map.)

(c) We then subtract a further $B(n+1)$, for the projections containing the block $\{n+1\}$.

(d) We then subtract $B(n+1)$ again, for those containing the block $\{n+2\}$.

(e) We must now add $B(n)$, for the projections containing *both* blocks $\{n+1\}$ and $\{n+2\}$.

At this point we are left with a set Σ of $B(n+2) - 3B(n+1) + B(n)$ projections, but we must halve this total, as we have double-counted the image of the $\varepsilon \mapsto \varepsilon^+$ map. Specifically, for $\varepsilon = \binom{A_1 | A_2 | C_1 | \cdots | C_s}{A_1 | A_2 | C_1 | \cdots | C_s} \in P_2(\mathcal{P}_n)$, the set Σ contains both

$$\varepsilon^+ = \binom{A_1 \cup \{n+2\} | A_2 \cup \{n+1\} | C_1 | \cdots | C_s}{A_1 \cup \{n+2\} | A_2 \cup \{n+1\} | C_1 | \cdots | C_s} \quad \text{and} \quad \binom{A_1 \cup \{n+1\} | A_2 \cup \{n+2\} | C_1 | \cdots | C_s}{A_1 \cup \{n+1\} | A_2 \cup \{n+2\} | C_1 | \cdots | C_s}. \qquad \square$$

Proposition 7.5. *For $n \geq 0$ we have* $|Q(\mathcal{PB}_n)| = \dfrac{I(n+2)}{2}$.

Proof. This time we show that:

$$p_0(\mathcal{PB}_n) = I(n), \qquad p_1(\mathcal{PB}_n) = I(n+1) - I(n) \qquad \text{and} \qquad p_2(\mathcal{PB}_n) = \tfrac{I(n+2)-2I(n+1)}{2}.$$

The first two items are treated identically to the proof of Proposition 7.4. The third is *almost* identical. We follow steps (a)–(e), with $I(k)$ in place of $B(k)$, with the only significant difference being in step (b). Here we subtract $I(n)$, rather than $I(n+1)$, as there are $I(n)$ projections in $P_0(\mathcal{PB}_{n+2})$ for which $n+1$ and $n+2$ belong to the same block (as blocks have size ≤ 2). $\square$

Proposition 7.6. *For $n \geq 0$ we have* $|Q(\mathscr{P}\mathcal{P}_n)| = C(n+2) - 2C(n+1) + C(n)$.

Proof. This time the result follows from:

$$p_0(\mathscr{P}\mathcal{P}_n) = C(n), \quad p_1(\mathscr{P}\mathcal{P}_n) = C(n+1) - C(n) \quad \text{and} \quad p_2(\mathscr{P}\mathcal{P}_n) = C(n+2) - 3C(n+1) + C(n).$$

The proof is essentially the same as that of Proposition 7.4, except that we do not need to divide by 2 in the third calculation, as planarity ensures there is no double-counting. $\square$

Proposition 7.7. *For $n \geq 0$ we have $|Q(\mathcal{M}_n)| = M(n+2) - M(n+1)$.*

Proof. This time we have

$$p_0(\mathcal{M}_n) = M(n), \quad p_1(\mathcal{M}_n) = M(n+1) - M(n) \quad \text{and} \quad p_2(\mathcal{M}_n) = M(n+2) - 2M(n+1),$$

as in Proposition 7.5, but with no double-counting. □

Proposition 7.8. *For $n \geq 0$ we have*

$$|Q(\mathcal{TL}_n)| = \begin{cases} C(k+1) - C(k) & \text{if } n = 2k-1 \text{ is odd} \\ C(k+2) - 2C(k+1) + C(k) & \text{if } n = 2k \text{ is even.} \end{cases}$$

Proof. Given Proposition 7.6 and the isomorphism $\mathcal{TL}_{2k} \cong \mathscr{PP}_k$, we need only consider the case that $n = 2k-1$ is odd. Here we have $|Q(\mathcal{TL}_n)| = p_1(\mathcal{TL}_n) + p_3(\mathcal{TL}_n)$, and we claim that

$$p_1(\mathcal{TL}_n) = C(k) \qquad \text{and} \qquad p_3(\mathcal{TL}_n) = C(k+1) - 2C(k).$$

Indeed, the first holds because the map $P_1(\mathcal{TL}_n) \to P_0(\mathcal{TL}_{n+1}) : \varepsilon \mapsto \varepsilon^+$ is a bijection. (No Temperley–Lieb partition contains the block $\{n+1\}$.)

For the second, we note that the image of the map $P_3(\mathcal{TL}_n) \to P_0(\mathcal{TL}_{n+3}) : \varepsilon \mapsto \varepsilon^+$ consists of all projections from $P_0(\mathcal{TL}_{n+3})$ not containing the blocks $\{n+1, n+2\}$ or $\{n+2, n+3\}$. (See (7.2), and note that no Temperley–Lieb partition contains the block $\{n+1, n+3\}$.) We have $p_0(\mathcal{TL}_{n+3}) = C(k+1)$, and there are $p_0(\mathcal{TL}_{n+1}) = C(k)$ of each of the above two types of 'offending' projections, with no overlap between them. □

The values of $|Q(M)|$ in Propositions 7.4–7.8 appear as sequences A000245, A001475, A002026, A026012 and A087649 on the OEIS [1].

Remark 7.9. Denoting the *forward difference* of a sequence $s(n)$ by $\partial s(n) = s(n+1) - s(n)$, we see that

$$|Q(\mathscr{PP}_n)| = \partial^2 C(n), \quad |Q(\mathcal{M}_n)| = \partial M(n+1) \quad \text{and} \quad |Q(\mathcal{TL}_n)| = \begin{cases} \partial C(k) & \text{for } n = 2k-1 \\ \partial^2 C(k) & \text{for } n = 2k. \end{cases}$$

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