

# BMS-LIKE ALGEBRAS: CANONICAL REALISATIONS AND BRST QUANTISATION

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ABSTRACT. We generalise BMS algebras in three dimensions by the introduction of an arbitrary real parameter  $\lambda$ , recovering the standard algebras (BMS, extended BMS and Weyl-BMS) for  $\lambda = -1$ . We exhibit a realisation of the (centreless) Weyl  $\lambda$ -BMS algebra in terms of the symplectic structure on the space of solutions of the massless Klein-Gordon equation in  $2 + 1$ , using the eigenstates of the spacetime momentum operator. The quadratic Casimir of the Lorentz algebra plays an essential rôle in the construction. The Weyl  $\lambda$ -BMS algebra admits a three-parameter family of central extensions, resulting in the (centrally extended) Weyl-BMS algebra, which we reformulate in terms of operator product expansions. We construct the BRST complex of a putative Weyl-BMS string and show that the BRST cohomology is isomorphic to the chiral ring of a topologically twisted  $N = 2$  superconformal field theory. We also comment on the obstructions to obtaining a conformal BMS Lie algebra – that is, one that includes in addition the special-conformal generators – and the need to consider a W-algebra. We then construct the quantum version of this W-algebra in terms of operator product expansions. We show that this W-algebra does not admit a BRST complex.

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## 1. INTRODUCTION AND SUMMARY OF THE RESULTS

The BMS algebra was introduced in [1, 2] as the asymptotic symmetry algebra of a four-dimensional flat spacetime at null infinity. The BMS algebra extends the Poincaré algebra by an infinite number of “super-translations”. Longhi and Materassi [3] found a canonical realisation of this algebra in terms of the natural symplectic structure of the Fourier modes of a free Klein–Gordon (KG) field in Minkowski spacetime. A comprehensive presentation of recent applications of BMS symmetries can be found in [4].

The BMS algebra was later extended to include an infinite number of “super-rotations” [5–7] (see also [8] for more recent developments). There also exists a canonical realisation of this extended BMS algebra in terms of a free massless KG field in three-dimensional Minkowski spacetime [9], expressed in terms of plane waves. The massless KG equation is conformally invariant, so it is natural to ask whether there is a way to further extend the algebra by adding (super-)dilatations and (super-)special-conformal transformations. The answer for dilatations is positive and results in the Weyl–BMS algebra [10–13], which further extends the extended BMS algebra by superdilatations. The answer for special-conformal

transformations seems to be negative; although there is a W-algebra which may be argued to extend the Weyl–BMS algebra [12].

One of the main aims of this paper is to generalise these algebras, in the case of  $2 + 1$  dimensions, by the introduction of an arbitrary real parameter  $\lambda$ , recovering the standard algebras (BMS, extended BMS and Weyl-BMS) for  $\lambda = -1$ . We call them (Weyl)  $\lambda$ -BMS algebras, but they should not be confused with the cosmological  $\Lambda$ -BMS algebras discussed, for instance, in [14].

The paper is divided into three main sections. In Section 2 we will exhibit a realisation of the (centreless) Weyl  $\lambda$ -BMS algebra in terms of the symplectic structure on the space of solutions of the three-dimensional massless Klein–Gordon equation. Deriving inspiration from the case of the BMS algebra (corresponding to  $\lambda = -1$ ), we consider in Section 2.1 the semidirect product of the Lorentz Lie algebra with the super-translations. The super-translations form an infinite-dimensional abelian Lie algebra spanned by the eigenfunctions of eigenvalue  $\lambda(\lambda - 1)$  of the quadratic Casimir  $C_2$  of the Lorentz Lie algebra, thought of as a second-order differential operator on the smooth functions on the lightcone (i.e., the massless mass shell).

The Lorentz Lie algebra is realised as vector fields on the lightcone and naturally act on the super-translations via the Lie derivative. In Section 2.2 we observe that not only the Lorentz Lie algebra, but indeed any vector field on the lightcone which commutes with  $C_2$  also acts on the super-translations. The additional such vector fields are the super-rotations and they form a Lie algebra isomorphic to the Witt algebra (i.e., the centreless Virasoro algebra). Together with the super-translations, one obtains the (extended)  $\lambda$ -BMS algebra. In Section 2.1.1 we determine the quadratic Casimirs of the  $\lambda$ -BMS algebra, which might be a result of independent interest.

The massless Klein–Gordon field is not only Poincaré invariant, but actually conformally invariant. In particular it is invariant under dilatations, whose generator acts on functions as a differential operator of degree at most one, of which the super-translations form an eigenspace. We then ask whether there are other differential operators of degree at most one which commute with  $C_2$  and which act on super-translations with the same eigenvalue as the dilatation. The answer is positive and we obtain in this way a family of such operators  $D_n(k_1, k_2, k_3)$  depending on an integer parameter ( $n$ ) and three real parameters ( $k_i$ ). The real parameters are fixed by demanding that for  $n = 0$  we should recover the dilatation in the conformal algebra. This results in generators  $D_n$ , which we call superdilatations. The resulting Lie algebra is the (centreless) Weyl  $\lambda$ -BMS Lie algebra.

A natural question is whether one can do the same with the special-conformal generators and extend them to a “conformal”  $\lambda$ -BMS algebra. We argue in Section 2.4 that no such extension exists as a Lie algebra. It is known, however, that there is an extension as a W-algebra [12], which is discussed in Section 4.

In Section 3 we show that the Weyl  $\lambda$ -BMS algebra admits a three-parameter family of central extensions, resulting in the (centrally extended) Weyl–BMS algebra. We then reformulate the centrally extended algebra in terms of operator product expansions. In that language we then proceed to construct the BRST complex of putative Weyl–BMS strings, showing that that it exists provided the central charges are chosen judiciously. We then show that the BRST cohomology is isomorphic to the chiral ring of a topologically twisted  $N = 2$  superconformal theory obtained by coupling the Weyl–BMS string to topological gravity in the form of a Koszul topological conformal field theory. This provides further evidence for a conjecture in [15, 16] that the BRST cohomology of every topological conformal field theory is isomorphic to the chiral ring of an  $N = 2$  superconformal field theory.

In Section 4 we return to the case of  $\lambda = -1$  and we first of all construct the fully quantum conformal BMS W-algebra of [12] in terms of operator product expansions and then report on calculations showing that, perhaps contrary to expectations, there is no BRST complex for this W-algebra.

The paper ends with a short Section 5 with conclusions and a look at future extensions of this work and two appendices with technical results used in Section 2.

## 2. CANONICAL REALISATIONS

In this section we will consider a canonical realisation of a generalisation of the Weyl-BMS algebra that depends on one parameter. We will see the crucial role of the Casimir of the Lorentz group. We will start by including only the super-translations, and their  $\lambda$ -depending generalisations, and then will proceed to include the super-rotations and the super-dilatations.

The Lagrangian density for a real massless scalar field in flat Minkowski space time<sup>1</sup> is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \quad (2.1)$$

The solution to the equation of motion, Klein–Gordon equation, in terms of the Fourier modes  $\mathbf{a}(\vec{k})$ ,

$$\phi(\mathbf{t}, \vec{x}) = \int d\vec{k} \left( \mathbf{a}(\vec{k})e^{i\mathbf{k}\cdot\mathbf{x}} + \bar{\mathbf{a}}(\vec{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right), \quad (2.2)$$

with  $\mathbf{x} = (\mathbf{t}, \vec{x})$ ,  $\mathbf{k}\cdot\mathbf{x} = -\omega\mathbf{t} + \vec{k}\cdot\vec{x}$ ,  $\omega = k^0 = \sqrt{\vec{k}^2}$ . The solution is expressed in terms of the plane waves that are eigenstates of the momentum operator and

$$d\vec{k} = \frac{d^2k}{\Omega(\vec{k})}, \quad \Omega(\vec{k}) = (2\pi)^2 2\omega, \quad (2.3)$$

and where the Fourier modes satisfy the non-zero Poisson brackets

$$\{\mathbf{a}(\vec{k}), \bar{\mathbf{a}}(\vec{q})\} = -i\Omega(\vec{k})\delta^2(\vec{k} - \vec{q}). \quad (2.4)$$

Notice that we parameterise the mass-shell manifold of the massless scalar-field,  $k^2 = 0$ , using  $\vec{k} \in \mathbb{R}^2$ . Alternatively, we could expand the solution in terms of other eigenfunctions, for example using the eigenfunctions of a boost generator, which would lead in a natural way to celestial holography (see, e.g., [17, 18]) expressing the four dimensional theory in terms of two-dimensional complex conformal field theory. In our case, in three dimensions, a chiral two-dimensional conformal field theory seems to appear naturally.

The conserved charges associated to the translation and Lorentz generators are

$$P^\mu = \int d\vec{k} \bar{\mathbf{a}}(\vec{k})k^\mu\mathbf{a}(\vec{k}), \quad \mu = 0, 1, 2, \quad (2.5)$$

$$M^{ij} = -i \int d\vec{k} \bar{\mathbf{a}}(\vec{k}) \left( k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) \mathbf{a}(\vec{k}), \quad j = 1, 2 \quad (2.6)$$

$$M^{0j} = tP^j - i \int d\vec{k} \bar{\mathbf{a}}(\vec{k})k^0 \frac{\partial}{\partial k^j} \mathbf{a}(\vec{k}), \quad j = 1, 2, \quad (2.7)$$

while the charge associated to dilatations is

$$D = -tP^0 + i \int d\vec{k} \bar{\mathbf{a}}(\vec{k}) \left( k^j \frac{\partial}{\partial k^j} + \frac{1}{2} \right) \mathbf{a}(\vec{k}). \quad (2.8)$$

From the above equations one can read, at  $t = 0$ , the differential operators for boosts, rotation and dilatation:

$$\hat{B}_j = i\omega \frac{\partial}{\partial k^j}, \quad j = 1, 2, \quad (2.9)$$

$$\hat{J} = -i \left( k^1 \frac{\partial}{\partial k^2} - k^2 \frac{\partial}{\partial k^1} \right), \quad (2.10)$$

$$\hat{D} = i \left( k^j \frac{\partial}{\partial k^j} + \frac{1}{2} \right), \quad (2.11)$$

with  $B_j = M_{0j}$ . As shown in Appendix B, the algebra of the Poisson brackets of the charges induced by the symplectic structure of the Fourier coefficients is the same, up to a  $-i$  factor, of that of the associated differential operators, and we will work with the later.

Using polar coordinates in the  $(k^1, k^2)$  plane,  $k^1 = r \cos \phi$ ,  $k^2 = r \sin \phi$ ,  $\omega = r$ , one has

$$\hat{B}_1 = ir \cos \phi \partial_r - i \sin \phi \partial_\phi, \quad (2.12)$$

$$\hat{B}_2 = ir \sin \phi \partial_r + i \cos \phi \partial_\phi, \quad (2.13)$$

$$\hat{J} = -i\partial_\phi, \quad (2.14)$$

$$\hat{D} = i \left( r\partial_r + \frac{1}{2} \right). \quad (2.15)$$

In terms of these operators, the quadratic Casimir of the Lorentz group in  $2+1$ ,  $\frac{1}{2}M^{\mu\nu}M_{\mu\nu}$ , is

$$\hat{C}_2 = -\hat{B}_1^2 - \hat{B}_2^2 + \hat{J}^2 = r^2\partial_r^2 + 2r\partial_r. \quad (2.16)$$

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<sup>1</sup>The signature of the Minkowski metric is  $(-++)$

In this realisation, there is a fundamental relation between the Casimir and the square of the dilatation generator,

$$\hat{C}_2 = -\hat{D}^2 - \left(\frac{1}{2}\right)^2. \quad (2.17)$$

**2.1. Super-translations and  $\lambda$ -BMS algebras.** Following [19], which generalises to arbitrary dimensions the results in [3], we can construct BMS-like algebras by considering the solutions of

$$-\hat{C}_2\xi = \alpha\xi, \quad (2.18)$$

where the minus sign is added in order to more easily connect with the standard notation of the representations of the Lorentz group. As shown in Appendix A, the solutions to (2.18) provide representations of the Lorentz algebra.

The Casimir eigenvalue equation (2.18) in terms of polar coordinates is

$$r^2\partial_r^2\xi + 2r\partial_r\xi = -\alpha\xi, \quad (2.19)$$

Since this does not depend on the angular coordinate, the solutions will be of the form  $\xi_n(r, \phi) = f(r)e^{in\phi}$ , with  $n \in \mathbb{Z}$ . Looking for radial solutions of the form

$$f(r) = r^\beta \quad (2.20)$$

one finds

$$\beta = \frac{-1 \pm \sqrt{1-4\alpha}}{2} = \frac{1}{2}(-1 \pm (1-2\lambda)), \quad (2.21)$$

where we have defined

$$1-4\alpha = (1-2\lambda)^2. \quad (2.22)$$

with inverse relation

$$\alpha = -\lambda(\lambda-1). \quad (2.23)$$

One gets thus the two families of solutions, with  $\beta = -\lambda$  and  $\beta = \lambda-1$ . In order to get all possible values of  $\beta \in \mathbb{R}$ , it is enough to take  $\lambda \in \mathbb{R}$  and consider only the solutions  $\beta = -\lambda$ .<sup>2</sup>

The complete solution, with the  $S^1$  angular coordinate, will be

$$\omega_n(r, \phi) = r^{-\lambda}e^{in\phi}, \quad n \in \mathbb{Z}, \quad (2.24)$$

and then

$$\hat{C}_2\omega_n = \lambda(\lambda-1)\omega_n. \quad (2.25)$$

The action of the Lorentz and dilatation generators on  $\omega_n(r, \phi)$  is

$$\begin{aligned} \hat{B}_1\omega_n &= -\frac{i}{2}(n+\lambda)\omega_{n+1} + \frac{i}{2}(n-\lambda)\omega_{n-1}, \\ \hat{B}_2\omega_n &= -\frac{1}{2}(n+\lambda)\omega_{n+1} - \frac{1}{2}(n-\lambda)\omega_{n-1}, \\ \hat{J}\omega_n &= n\omega_n, \end{aligned} \quad (2.26)$$

These equations provide an infinite-dimensional realisation of the  $2+1$  Lorentz algebra in the space of the  $\{\omega_n\}_{n \in \mathbb{Z}}$ . Looking at the zeros of the coefficients appearing in the above equations, several cases, depending on the value of  $\lambda$ , can be considered:

- (1) If  $\lambda \notin \mathbb{Z}$ , then the coefficients in the above equations can never be zero and the representation is irreducible. This corresponds to the complementary series in the standard representation theory of  $SO(2,1)$  (see, e.g., [20]).
- (2) In  $\lambda = -N$ ,  $N \in \mathbb{N}$ , then one has a finite representation in the space  $\{\omega_n\}_{|n| \leq N}$ . In particular, for  $\lambda = -1$  one obtains the vector representation in the space of the  $\{\omega_{-1}, \omega_0, \omega_1\}$ .
- (3) If  $\lambda = 0$ , one obtains the trivial representation spanned by  $\{\omega_0\}$ .
- (4) If  $\lambda = N$ ,  $N \in \mathbb{N}$ , there appear two infinite-dimensional representations, spanned by  $\{\omega_n\}_{n \leq -N}$  and  $\{\omega_n\}_{n \geq N}$ , respectively, which correspond to the highest and lowest weight representations of the standard literature.

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<sup>2</sup>One can also consider complex values of  $\lambda$  provided that the real part is  $1/2$ , since in this case  $\alpha$  remains real. In the representation theory of  $SO(2,1)$  this corresponds to the unitary principal series representation (see, e.g., [20]). However, we will not pursue this possibility in this work.

Up to this point, we have only talked about representations of the Lorentz group, which are provided by the set of  $\omega_n$ . However, since the Lorentz operators are of first order (without zero order term) and the  $\omega_n$  are of zeroth order, the above expressions coincide with the commutators

$$[\hat{B}_1, \omega_n] = -\frac{i}{2}(n+\lambda)\omega_{n+1} + \frac{i}{2}(n-\lambda)\omega_{n-1}, \quad (2.27)$$

$$[\hat{B}_2, \omega_n] = -\frac{1}{2}(n+\lambda)\omega_{n+1} - \frac{1}{2}(n-\lambda)\omega_{n-1}, \quad (2.28)$$

$$[\hat{J}, \omega_n] = n\omega_n. \quad (2.29)$$

These commutators, together with those between  $\hat{B}_1$ ,  $\hat{B}_2$  and  $\hat{J}$ , and adding the trivial ones between zero order operators

$$[\omega_n, \omega_m] = 0 \quad (2.30)$$

provide a realisation of an infinite dimensional algebra which, for  $\lambda = -1$ , is the standard BMS algebra in 2+1, which contains the finite Poincaré algebra obtained by considering the subset  $\{\omega_{-1}, \omega_0, \omega_1\}$ . For general  $\lambda \in \mathbb{R}$  we obtain what we will call  $\lambda$ -BMS algebras.

As is the case for the representations of Lorentz, for  $\lambda = -N$ ,  $N \in \mathbb{N}$ , one can obtain finite subalgebras of dimension  $3 + (2N + 1)$ , with generators  $\hat{B}_1$ ,  $\hat{B}_2$ ,  $\hat{J}$  and  $\omega_{-N}, \dots, \omega_N$ , which we call  $\lambda$ -Poincaré.

Notice that  $\lambda = -1$ , which is the value that leads to the Poincaré subalgebra in  $d = 3$  spacetime, corresponds to  $\alpha = 2$  in equation (2.19), which is equal to  $d - 1$  for  $d = 3$ . As shown in Appendix A,  $d - 1$  is the eigenvalue in (2.19) that makes Poincaré appear as a subalgebra of  $\lambda$ -Poincaré for arbitrary spacetime dimension  $d$ .

It is customary to write the above algebras in terms of<sup>3</sup>  $L_0 = -J$  and the ladder operators  $L_1 = -iB_1 + B_2$ ,  $L_{-1} = iB_1 + B_2$ , which are explicitly given by

$$L_1 = ie^{i\phi}(\partial_\phi - ir\partial_r), \quad (2.31)$$

$$L_{-1} = ie^{-i\phi}(\partial_\phi + ir\partial_r), \quad (2.32)$$

$$L_0 = i\partial_\phi. \quad (2.33)$$

One has then the  $\lambda$ -algebra in the form

$$[L_1, L_{-1}] = 2L_0, \quad [L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad (2.34)$$

$$[L_1, \omega_n] = -(n+\lambda)\omega_{n+1}, \quad (2.35)$$

$$[L_{-1}, \omega_n] = -(n-\lambda)\omega_{n-1}, \quad (2.36)$$

$$[L_0, \omega_n] = -n\omega_n, \quad (2.37)$$

$$[\omega_n, \omega_m] = 0. \quad (2.38)$$

In terms of these operators, the Lorentz Casimir can be expressed as

$$C_2 = -L_1L_{-1} + L_0^2 + L_0. \quad (2.39)$$

It should be noticed that, since the Lorentz Casimir  $C_2$  is a second order operator, the commutator of  $C_2$  with  $\omega_n$  differs from the action of  $C_2$  on  $\omega_n$ , yielding a first order operator (with zeroth term) instead of a function:

$$\begin{aligned} [C_2, \omega_n] &= [r^2\partial_r^2 + 2r\partial_r, r^{-\lambda}e^{in\phi}] = \lambda(\lambda-1)r^{-\lambda}e^{in\phi} - 2\lambda r^{-\lambda+1}e^{in\phi}\partial_r \\ &= C_2\omega_n - 2\lambda\omega_n E, \end{aligned} \quad (2.40)$$

where  $E = r\partial_r$  is the Euler operator, which in Cartesian coordinates has the expression  $E = k_1\partial_{k_1} + k_2\partial_{k_2}$ .

**2.1.1. Quadratic Casimirs of the  $\lambda$ -BMS algebras.** Since the quadratic Casimir of the Lorentz algebra does not commute with the  $\omega_n$ , a general quadratic Casimir of the  $\lambda$ -BMS algebra can only involve the  $\omega_n$  generators, and will be of the form

$$C_2^\lambda = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} A_{nm} \omega_n \omega_m, \quad A_{nm} = A_{mn}. \quad (2.41)$$

Demanding  $[L_0, C_2^\lambda] = 0$  leads to that  $A_{nm}$  can be different from zero only if  $n + m = 0$ , so that

$$C_2^\lambda = \sum_{n \in \mathbb{Z}} A_n \omega_n \omega_{-n}, \quad A_n = A_{n(-n)}, \quad (2.42)$$

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<sup>3</sup>For the rest of this section we will suppress the  $\hat{\cdot}$  on the differential operators.

with  $A_n = A_{-n}$  due to the symmetry of the  $A_{nm}$ . Imposing  $[L_1, C_2^\lambda] = 0$ , re-arranging terms and using  $A_n = A_{-n}$ , one obtains the set of equations

$$(-\lambda - n)A_n + (-\lambda + n + 1)A_{n+1} = 0, \quad n = 0, 1, 2, \dots, \quad (2.43)$$

which are the same relations that are obtained imposing  $[L_{-1}, C_2^\lambda] = 0$ . For  $\lambda \notin \mathbb{Z}$ , the recurrence relations (2.43) can be solved for all the  $A_n$  in terms of  $A_0$ , and the Casimir has infinite terms. We will discuss (2.43) for  $\lambda \in \mathbb{Z}$ , separating the cases  $\lambda < 0$  and  $\lambda > 0$  (the case  $\lambda = 0$  is obviously trivial).

- If  $-\lambda = N \in \mathbb{N}$ , equations (2.43) become

$$(N - n)A_n + (N + n + 1)A_{n+1} = 0, \quad n = 0, 1, 2, \dots \quad (2.44)$$

which can be solved for  $A_1, A_2, \dots, A_N$  in terms of  $A_0$ , and one obtains

$$A_n = (-1)^n \frac{N(N-1) \cdots (N-n+1)}{(N+n)(N+n-1) \cdots (N+1)} A_0, \quad n = 1, 2, \dots, N. \quad (2.45)$$

However, the equation for  $n = N$  is just

$$0 \cdot A_N + (2N + 1)A_{N+1} = 0,$$

which implies  $A_{N+1} = 0$ , and subsequently also  $A_n = 0$  for all  $n > N$ . Thus, using that  $A_n = A_{-n}$  and taking  $A_0 = 1$ , the Casimir boils down to

$$C_2^\lambda = \omega_0^2 + 2 \sum_{n=1}^N (-1)^n \frac{N(N-1) \cdots (N-n+1)}{(N+n)(N+n-1) \cdots (N+1)} \omega_n \omega_{-n}, \quad (2.46)$$

which will also be the Casimir for the finite dimensional  $\lambda$ -Poincaré algebra.

For instance, for  $\lambda = -1$  one obtains the well-known Poincaré quadratic Casimir

$$C_2^{\lambda=-1} = \omega_0^2 - \omega_1 \omega_{-1}, \quad (2.47)$$

while for  $\lambda = -2$  one gets

$$C_2^{\lambda=-2} = \omega_0^2 - \frac{4}{3} \omega_1 \omega_{-1} + \frac{1}{3} \omega_2 \omega_{-2}. \quad (2.48)$$

- If  $\lambda = N \in \mathbb{N}$  the recurrence relation is

$$(-N - n)A_n + (-N + n + 1)A_{n+1} = 0, \quad n = 0, 1, 2, \dots \quad (2.49)$$

For  $n = N - 1$  one obtains

$$(-2N + 1)A_{N-1} + 0 \cdot A_N = 0, \quad (2.50)$$

so that  $A_{N-1} = 0$ , which then forces  $A_{N-2} = \dots = A_1 = A_0 = 0$ , and the resulting Casimir has infinite terms

$$\tilde{C}_2^\lambda = 2 \sum_{n=N}^{\infty} A_n \omega_n \omega_{-n}, \quad (2.51)$$

with all the  $A_n$  computed in terms of  $A_N$  using the recurrence relation (2.49), which can be solved for the generating function  $A(z) = \sum_{n=0}^{\infty} A_{n+N} z^n$  as follows:

$$A(z) = \frac{A_N}{(1-z)^{2N}}. \quad (2.52)$$

**2.2. Super-rotations and extended  $\lambda$ -BMS algebras.** In the same way that super-translations are obtained by computing all the solutions of the Lorentz Casimir  $C_2$  eigenvalue equation, one may try to obtain generalisations of the Lorentz generators by computing all the first order differential operators that commute with  $C_2$ .

Using polar coordinates in the massless mass-shell manifold we look for an operator

$$L(r, \phi) = a(r, \phi) \partial_r + b(r, \phi) \partial_\phi \quad (2.53)$$

such that  $[L, C_2] = 0$ , with  $C_2$  the second order differential operator in (2.16). One has

$$\begin{aligned} [L, C_2] &= (2ar - 2r^2 \partial_r a) \partial_r^2 \\ &\quad + (2a - 2r \partial_r a - 2r \partial_r b - r^2 \partial_r^2 a) \partial_r \\ &\quad - 2r^2 \partial_r b \partial_r \partial_\phi - r^2 \partial_r^2 b \partial_\phi. \end{aligned} \quad (2.54)$$

The cancellation of the term in  $\partial_r \partial_\phi$  forces  $\partial_r b = 0$ , so that  $b = b(\phi)$ . This also cancels the term in  $\partial_\phi$ , and demanding that the two remaining terms are zero leads to  $\partial_r^2 a = 0$  and to

$$r \partial_r a = a, \quad (2.55)$$

with general solution  $\alpha(r, \phi) = rc(\phi)$ , which also satisfies  $\partial_r^2 \alpha = 0$ . Hence, the most general first-order differential operator commuting with the Lorentz Casimir is

$$L = rc(\phi)\partial_r + b(\phi)\partial_\phi. \quad (2.56)$$

Besides commuting with the Casimir, the differential operators associated to the Lorentz generators are also divergenceless. Since the Lorentz-invariant volume form on the massless mass-shell manifold is proportional to  $dr \wedge d\phi$ , the divergence of a field in polar coordinates is just the sum of the partial derivatives (see also appendix B in [9]), and we have

$$0 = \text{div } L = c(\phi) + \partial_\phi b(\phi), \quad (2.57)$$

from which  $c(\phi) = -b'(\phi)$ . The first order differential operators that share all the relevant properties with the Lorentz generators are thus

$$L = -rb'(\phi)\partial_r + b(\phi)\partial_\phi. \quad (2.58)$$

Since  $\phi \in S_1$ , we can expand in Fourier series and obtain an infinite set of operators  $L_n$  indexed by  $n \in \mathbb{Z}$ . Writing  $b(\phi) = ie^{in\phi}$ , the resulting operators, called super-rotations [5], are

$$L_n = ie^{in\phi}(\partial_\phi - inr\partial_r), \quad n \in \mathbb{Z}, \quad (2.59)$$

which coincide with the standard Lorentz generators for  $n = -1, 0, 1$ .

Adding the super-rotations one gets the extended  $\lambda$ -BMS algebras

$$[L_n, L_m] = (n - m)L_{n+m}, \quad (2.60)$$

$$[L_n, \omega_m] = -(m + \lambda n)\omega_{n+m}, \quad (2.61)$$

$$[\omega_n, \omega_m] = 0, \quad (2.62)$$

with  $n, m \in \mathbb{Z}$ . This algebra is the semi-direct sum of the Witt algebra with its tensor density modules (see section 3). Following the notation of [21], we refer to it as  $\mathfrak{g}_\lambda$ . It is a special case of the  $W(\mathfrak{a}, \mathfrak{b})$  algebras [22], where  $\mathfrak{a} \in \mathbb{Z}$  and  $\mathfrak{b} = \lambda$ . Setting  $\lambda = -1$  recovers the centreless BMS algebra, whose most general deformation is shown to be  $W(\mathfrak{a}, \mathfrak{b})^4$  (even when central extensions are included) [25].

From (2.61) one also obtains

$$L_n \omega_m = -(m + \lambda n)\omega_{n+m}, \quad (2.63)$$

which provides, for each  $\lambda \in \mathbb{R}$ , a representation of the Witt algebra.

Notice that, using (2.60) and the Lorentz Casimir in the form (2.16)

$$\begin{aligned} [L_n, C_2] &= -(n-1)L_{n+1}L_{-1} - (n+1)L_1L_{n-1} + nL_nL_0 + nL_0L_n + nL_n \\ &= -(n-1)L_{n+1}L_{-1} - (n+1)L_{n-1}L_1 + 2nL_nL_0 - 2L_n. \end{aligned} \quad (2.64)$$

By construction of the  $L_n$  this must be zero, and the following identity must hold:

$$-(n-1)L_{n+1}L_{-1} - (n+1)L_{n-1}L_1 + 2nL_nL_0 - 2L_n = 0, \quad n \in \mathbb{Z}. \quad (2.65)$$

This can be checked directly by using the explicit form (2.59).<sup>5</sup>

**2.3. Super-dilatations and the centreless Weyl  $\lambda$ -BMS algebras.** Consider a general differential operator of order up to one

$$\mathbb{D} = a\partial_r + b\partial_\phi + c. \quad (2.66)$$

Demanding that the  $\omega_n$  provide a representation of  $\mathbb{D}$ ,

$$\mathbb{D}\omega_n = \alpha\omega_m, \quad (2.67)$$

with  $\alpha$  possibly depending on  $m, n$ , one immediately gets

$$a = k_1 r e^{i(m-n)\phi}, \quad b = k_2 e^{i(m-n)\phi}, \quad c = k_3 e^{i(m-n)\phi}, \quad (2.68)$$

with  $k_i$  constants, and then

$$\alpha = -k_1 \lambda + ik_2 n + k_3. \quad (2.69)$$

One has then the operators

$$D_{m-n} = e^{i(m-n)\phi}(k_1 r \partial_r + k_2 \partial_\phi + k_3), \quad (2.70)$$

<sup>4</sup>We refer the reader to [23, 24] for some appearances of  $W(0, -2)$  in physics.

<sup>5</sup>As a vector in the universal enveloping algebra  $\mathfrak{U}$  of the Witt algebra,  $(n-1)L_{n+1}L_{-1} + (n+1)L_{n-1}L_1 - 2nL_nL_0 + 2L_n \neq 0$ , but of course it is in the kernel of the algebra homomorphism from  $\mathfrak{U}$  to differential operators on the punctured plane acting on smooth functions. The kernel of this homomorphism has been calculated in [26, Theorem 1.2] and it is a principal two-sided ideal of  $\mathfrak{U}$  generated by  $Z_1 := \frac{1}{2}(L_2L_{-1} - L_1L_0 - L_1)$ . A quick calculation shows that  $[L_{-1}, Z_1] = C_2$  and therefore  $[L_n, C_2] = [L_n, [L_{-1}, Z_1]] = L_nL_{-1}Z_1 - L_nZ_1L_{-1} - L_{-1}Z_1L_n + Z_1L_{-1}L_n$  indeed belongs to the two-sided ideal  $\mathfrak{U}Z_1\mathfrak{U}$ .



or, setting  $p = m - n$ ,

$$D_p = e^{ip\phi} (k_1 r \partial_r + k_2 \partial_\phi + k_3), \quad (2.71)$$

which yields

$$D_p \omega_n = (-k_1 \lambda + ik_2 n + k_3) \omega_{n+p}. \quad (2.72)$$

Demanding  $D_0 = D$  fixes  $k_1 = 1$ ,  $k_2 = 0$  and  $k_3 = 1/2$ , and one gets

$$D_n = e^{in\phi} \left( r \partial_r + \frac{1}{2} \right), \quad n \in \mathbb{Z}, \quad (2.73)$$

which, for  $n = 0$  gives  $D_0 = -iD$ , with  $D$  the standard dilatation operator defined in (2.15). Up to an overall constant factor, these agree with the super-dilatation operators introduced in [11].

Adding the commutators of  $D_n$  with the super-translations and super-rotations one obtains the centreless Weyl  $\lambda$ -BMS algebra:

$$[L_n, L_m] = (n - m)L_{n+m}, \quad (2.74)$$

$$[L_n, P_m] = -(m + \lambda n)P_{n+m}, \quad (2.75)$$

$$[P_n, P_m] = 0, \quad (2.76)$$

$$[L_n, D_m] = -mD_{m+n}, \quad (2.77)$$

$$[D_n, P_m] = -\lambda P_{m+n}, \quad (2.78)$$

$$[D_n, D_m] = 0, \quad (2.79)$$

where we have nominally replaced the super-translation functions  $\omega_n$  with the tensor densities<sup>6</sup>  $P_n$  since, as shown by (2.78), the  $\omega_n$  do not transform as true functions under super-dilatations, unless  $\lambda = 0$ .

Notice that (in the case  $\lambda \neq 0$ ) by redefining  $D_n \rightarrow -D_n/\lambda$  we can set the coefficient of the right-hand side of (2.78) to unity without changing the other commutation relations. We note that the algebra is not centrally extended in this realisation. We will see in Section 3 that this algebra admits a three-parameter family of central extensions (see also [13] for the case of  $\lambda = -1$ ).

**2.4. BMS-like super-special-conformal generators.** The conformal algebra adds generators  $D$  (dilations) and  $K^\mu$  (special-conformal transformations) to the Poincaré algebra. As shown in [11], the corresponding differential operators<sup>7</sup> in  $2 + 1$  spacetime acting on the mass-shell manifold of a massless scalar field are, besides  $D$ ,

$$K^0 = \omega \frac{\partial^2}{\partial k_i \partial k_i}, \quad (2.80)$$

$$K^j = \left( k_j \frac{\partial}{\partial k_i} - 2k_i \frac{\partial}{\partial k_j} \right) \frac{\partial}{\partial k_i} - \frac{\partial}{\partial k_i}, \quad i = 1, 2, \quad (2.81)$$

and the commutators involving them are, besides those of the Poincaré algebra,

$$[D, K^\mu] = iK^\mu, \quad (2.82)$$

$$[K^\mu, M^{\nu\sigma}] = -i(\eta^{\mu\sigma} K^\nu - \eta^{\mu\nu} K^\sigma), \quad (2.83)$$

$$[K^\mu, P^\nu] = -2i(\eta^{\mu\nu} D + M^{\mu\nu}), \quad (2.84)$$

$$[K^\mu, K^\nu] = 0, \quad (2.85)$$

for  $\mu, \nu, \sigma = 0, 1, 2$ .

Equations (2.83) and (2.85) show, as is well known, that the conformal algebra contains a second realisation of the Poincaré algebra, given by the Lorentz generators and the special-conformal ones.

One might wonder whether it is possible to repeat the construction of the BMS-like algebras using the special-conformal generators instead of the translation ones. A proposal in this direction was discussed in [11, 27], generalising the special-conformal generators by making them dependent on an arbitrary integer index, but the resulting algebra was in fact a  $W$ -algebra, rather than a Lie algebra [12]. In this context,  $W$ -algebras are studied in Section 4 from the point of view of conformal field theory.

One can also study whether it is possible to write down some master equation similar to (2.18) but for the special conformal transformations. However, since in the canonical formalism these are second order differential operators, one is led to consider the commutator with the Lorentz Casimir instead of

<sup>6</sup>See Section 3.

<sup>7</sup>We work here in Cartesian coordinates since the expression of the special-conformal operators in polar coordinates is not particularly simple.



the action of the Casimir on these operators. One can see that, if  $K$  is any of the 3 special conformal generators in  $2+1$ , one has

$$[C_2, K] = -2KE, \quad (2.86)$$

with  $E$  the Euler operator. It is possible to obtain other solutions to this equation, in the form of second-order differential operators, but in the end one finds out that the resulting set of generators  $K_n$ , containing  $K^0$  and the complex combinations of  $K^1$  and  $K^2$  for  $n = 0, \pm 1$ , do not commute, except for  $n = 0, \pm 1$ , and hence do not qualify as super-special conformal transformations.

A separate but related question is whether it is possible to consider a generalised equation  $[C_2, K] = \lambda KE$ , with  $\lambda$  arbitrary and with the restriction of  $K$  being a second-order differential operator. A detailed study of the resulting set of differential equations shows that there are solutions only for  $\lambda = -2$  and  $\lambda = 0$ , and hence one does not have the liberty of adding the parameter  $\lambda$  that appears in the case of super-translations.

### 3. THE BRST COMPLEX OF THE WEYL $\lambda$ -BMS ALGEBRA

We will now depart from the mode algebra in equations (2.74)–(2.79) and explore its possible central extensions, resulting in the (centrally extended) Weyl  $\lambda$ -BMS algebra. We will then reformulate the centrally extended algebra in terms of operator product expansions. We will then discuss the construction of Weyl  $\lambda$ -BMS strings by constructing the BRST complex and showing that the BRST cohomology is isomorphic to the chiral ring of a topologically twisted  $N=2$  superconformal field theory.

**3.1. The (centrally extended) Weyl  $\lambda$ -BMS algebra.** Firstly, we reformulate the centreless Weyl  $\lambda$ -BMS algebra, determine the possible central extensions and rewrite the centrally extended algebra in terms of operator product expansions.

**3.1.1. The centreless Weyl  $\lambda$ -BMS algebra.** It is convenient to re-interpret the centreless Weyl  $\lambda$ -BMS algebra in terms of natural objects associated to the punctured complex plane.<sup>8</sup> To that end, we let  $A = \mathbb{C}[z, z^{-1}]$  denote the associative algebra of Laurent polynomials in a complex variable  $z$ . The Lie algebra  $W = \text{Der}_{\mathbb{C}} A$  of derivations is the **Witt algebra**, which is isomorphic to the Lie algebra of polynomial vector fields on the circle. Let  $\lambda \in \mathbb{Z}$  and let  $I(\lambda)$  denote the one-dimensional  $A$ -module spanned by  $(dz)^\lambda$ ; that is, a typical vector in  $I(\lambda)$  is of the form  $f(z)(dz)^\lambda$  with  $f(z) \in A$ . The tacit understanding is that if  $\lambda < 0$ ,  $(dz)^\lambda = \left(\frac{d}{dz}\right)^{-\lambda}$ . We introduce bases  $D_n := z^n$ ,  $L_n := -z^{n+1}\partial$ , where  $\partial f(z) = \frac{df}{dz}$  and  $P_n := z^{n-\lambda}(dz)^\lambda$ , for  $n \in \mathbb{Z}$ , for the  $A$ -modules  $A$ ,  $W$  and  $I(\lambda)$ , respectively.

We define a Lie algebra structure on the vector space  $W \oplus A \oplus I(\lambda)$  as follows:

- $W$  is a Lie algebra under the Lie bracket of vector fields:

$$[L_n, L_m] = (n - m)L_{n+m}; \quad (3.1)$$

- $A$  is an abelian Lie algebra, is acted on by  $W$  as derivations:

$$[L_n, D_m] = -mD_{n+m}; \quad (3.2)$$

- $I(\lambda)$  is an abelian Lie algebra and is acted on by  $W$  via the Lie derivative:

$$[L_n, P_m] = -(m + \lambda n)P_{n+m} \quad (3.3)$$

and by  $A$  via the module action:

$$[D_m, P_n] = P_{m+n}. \quad (3.4)$$

We could rescale  $D_m$  by any nonzero  $\mu$  and arrive at  $[D_m, P_n] = \mu P_{m+n}$ , without altering any of the other Lie brackets. In particular, choosing  $\mu = -\lambda$  we would arrive at the Lie bracket (2.78). As mentioned earlier, we will take  $\mu = 1$  in what follows.

The Lie algebra generated by  $\{L_n, D_n, P_n\}_{n \in \mathbb{Z}}$  given by the above brackets is the double semi-direct product of Lie algebras  $\mathfrak{w}_\lambda = (W \ltimes A) \ltimes I(\lambda)$  and it is clearly isomorphic to the centreless Weyl  $\lambda$ -BMS algebra in Section 2: see equations (2.74)–(2.79). The Lie subalgebra  $W \ltimes A$  is often called the **Heisenberg–Virasoro algebra** [28–31], and corresponds to the algebra of differential operators of degree at most 1. For  $\lambda = -1$ ,  $\mathfrak{w}_{-1}$  agrees with the Weyl BMS Poisson algebra of [11]: see equations (4.16)–(4.21) in that paper.

<sup>8</sup>The coordinates  $(r, \phi)$  used for the lightcone in Section 2 are reminiscent of polar coordinates for the punctured complex plane. There is however no change of variables relating  $(r, \phi)$  to the complex coordinate  $z$  used in this section. One way to see this is to consider the functions annihilated by the Witt generators  $L_n$ . For the realisation in this section, it is clear that any anti-holomorphic function – i.e., any function of  $\bar{z}$  – is annihilated by  $L_n = -z^{n+1}\partial$ , whereas the operator  $L_n$  given in equation (2.59) annihilates functions of  $re^{in\phi}$ , which clearly depends on  $n$ .

3.1.2. *Central extensions.* Our first order of business is to determine the possible central extensions (up to equivalence). They are classified by the second cohomology of  $\mathfrak{w}_\lambda$  with values in the trivial representation. We make use of the following lemma to simplify our computation.

**Lemma 1.** *Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{h} \subset \mathfrak{g}$  an ideal such that  $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$ . Then the canonical surjective homomorphism  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$  induces an injective linear map  $\pi^*: H^2(\mathfrak{g}/\mathfrak{h}) \rightarrow H^2(\mathfrak{g})$  in cohomology.*

*Proof.* Recall<sup>9</sup> that the space of  $\mathfrak{p}$ -cochains of any Lie algebra  $\mathfrak{g}$  with values in the base field (here  $\mathbb{R}$ ) viewed as a trivial representation is given by  $C^p(\mathfrak{g}) = \text{Hom}(\wedge^p \mathfrak{g}, \mathbb{R})$ . We are particularly interested in the first few terms:

$$C^1(\mathfrak{g}) \xrightarrow{d} C^2(\mathfrak{g}) \xrightarrow{d} C^3(\mathfrak{g}) \quad (3.5)$$

where for  $\beta \in C^1(\mathfrak{g})$  and  $\varphi \in C^2(\mathfrak{g})$ , their differentials are given by

$$d\beta(X, Y) = -\beta([X, Y]) \quad \text{and} \quad d\varphi(X, Y, Z) = -\varphi([X, Y], Z) + \varphi([X, Z], Y) - \varphi([Y, Z], X). \quad (3.6)$$

If  $\mathfrak{h} \subset \mathfrak{g}$  is an ideal, the Chevalley–Eilenberg complex for the quotient Lie algebra  $\mathfrak{g}/\mathfrak{h}$  can be understood as a subcomplex of  $C^*(\mathfrak{g})$ . Namely, any  $\varphi \in C^p(\mathfrak{g}/\mathfrak{h})$  may be extended uniquely to a  $\varphi \in C^p(\mathfrak{g})$  such that  $\varphi(X, \dots) = 0$  for all  $X \in \mathfrak{h}$ . Such cochains are preserved by the Chevalley–Eilenberg differential precisely because  $\mathfrak{h}$  is an ideal and hence define a subcomplex.

Now suppose that a cocycle  $\varphi \in C^2(\mathfrak{g}/\mathfrak{h})$  is a coboundary in  $C^2(\mathfrak{g})$ ; that is,  $\varphi = d\beta$  for some  $\beta \in C^1(\mathfrak{g})$ ; that is,  $\varphi(X, Y) = -\beta([X, Y])$ . We claim that  $\beta \in C^1(\mathfrak{g}/\mathfrak{h})$ . Indeed, suppose that  $Z \in \mathfrak{h}$ . By hypothesis, there exist  $X_i \in \mathfrak{h}$  and  $Y_i \in \mathfrak{g}$  such that  $Z = \sum_i [X_i, Y_i]$ . Hence

$$\beta(Z) = \sum_i \beta([X_i, Y_i]) = -\sum_i \varphi(X_i, Y_i) = 0, \quad (3.7)$$

since  $X_i \in \mathfrak{h}$  and  $\varphi \in C^2(\mathfrak{g}/\mathfrak{h})$ . In other words, if  $\pi^*([\varphi]) = 0 \in H^2(\mathfrak{g})$ , then  $[\varphi] = 0 \in H^2(\mathfrak{g}/\mathfrak{h})$ .  $\square$

**Remark.** Lemma 1 requires  $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$ , as can easily be seen when  $\mathfrak{h}$  is the centre of  $\mathfrak{g}$ .

**Proposition 2.** *The Lie algebra  $\mathfrak{w}_\lambda = (W \ltimes A) \ltimes I(\lambda)$  admits a 3-dimensional universal central extension generated by the 2-cocycles*

$$\begin{aligned} \gamma_{LL}(L_n, L_m) &= \frac{1}{12}n(n^2 - 1)\delta_{m+n}^0 \\ \gamma_{LD}(L_n, D_m) &= \frac{1}{6}n(n - 1)\delta_{m+n}^0 \\ \gamma_{DD}(D_n, D_m) &= n\delta_{m+n}^0. \end{aligned} \quad (3.8)$$

*Proof.* Consider the Lie algebra  $\mathfrak{g}$  generated by  $\{L_n, P_n, D_n, I_n\}_{n \in \mathbb{Z}}$ , with (nonzero) Lie brackets

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{m+n} & [L_n, D_m] &= -mD_{m+n} \\ [L_n, P_m] &= -(m + \lambda n)P_{m+n} & [L_n, I_m] &= (n - m)L_{m+n} \\ [D_m, P_n] &= P_{m+n} & [D_m, I_n] &= -I_{m+n}. \end{aligned} \quad (3.9)$$

For  $\lambda = -1$ , this is known as the planar galilean conformal algebra (GCA). It was shown by Gao, Liu and Pei that the second cohomology group of the planar GCA (with values in the trivial module  $\mathbb{C}$ ) is 3-dimensional [33]. It is easy to check that this statement holds true for any  $\lambda \in \mathbb{Z}$ .

Let  $\mathfrak{h}$  denote the abelian ideal in  $\mathfrak{g}$  spanned by  $\{I_n\}_{n \in \mathbb{Z}}$ . Since  $\mathfrak{w}_\lambda = \mathfrak{g}/\mathfrak{h}$  and  $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$ , we can make use of Lemma 1 to deduce that the canonical surjective homomorphism  $\tilde{\pi}: \mathfrak{g} \rightarrow \mathfrak{w}_\lambda$  induces an injective map in cohomology  $\tilde{\pi}_*: H^2(\mathfrak{w}_\lambda) \rightarrow H^2(\mathfrak{g})$ . This implies that  $\dim H^2(\mathfrak{w}_\lambda) \leq 3$ .

Now let  $\mathfrak{h}$  denote the abelian ideal of  $\mathfrak{w}_\lambda$  spanned by  $\{P_n\}_{n \in \mathbb{Z}}$ . Indeed,  $[\mathfrak{w}_\lambda, \mathfrak{h}] = \mathfrak{h}$ , and we have a surjective Lie algebra homomorphism  $\pi: \mathfrak{w}_\lambda \rightarrow \mathfrak{w}_\lambda/\mathfrak{h} =: \mathfrak{g}_0$  (recall that  $\mathfrak{g}_\lambda$  is given by eqs. (2.60) to (2.62)). Once again, using Lemma 1, the injectivity of the induced map  $\pi^*: H^2(\mathfrak{g}_0) \rightarrow H^2(\mathfrak{w}_\lambda)$  implies that  $\dim H^2(\mathfrak{w}_\lambda) \geq \dim H^2(\mathfrak{g}_0)$ . Since  $\dim H^2(\mathfrak{g}_0) = 3$  (as shown by Arbarello, De Concini, Kac and Procesi [29], see also [22]), we arrive at the nested inequality  $3 \leq \dim H^2(\mathfrak{w}_\lambda) \leq 3$ . Thus,  $\dim H^2(\mathfrak{w}_\lambda) = 3$  indeed. We may now obtain the explicit form of the representative 2-cocycles by pulling back the three representative 2-cocycles on  $\mathfrak{g}_0$  [29]

$$\begin{aligned} \gamma_1(L_n, L_m) &= \frac{1}{12}n(n^2 - 1)\delta_{m+n}^0 \\ \gamma_2(L_n, D_m) &= \frac{1}{6}n(n - 1)\delta_{m+n}^0 \\ \gamma_3(D_n, D_m) &= n\delta_{m+n}^0 \end{aligned} \quad (3.10)$$

<sup>9</sup>We refer the reader to [32] for a review of Lie algebra cohomology.

by  $\pi^*$  to get  $\gamma_{LL} = \pi^*(\gamma_1)$ ,  $\gamma_{LD} = \pi^*(\gamma_2)$  and  $\gamma_{DD} = \pi^*(\gamma_3)$  as given by (3.8).<sup>10</sup>  $\square$

**Remark.** One might have expected that for the values of  $\lambda$  for which the  $\lambda$ -BMS algebra admits additional central extensions (namely,  $\lambda = -1, 0, 1$ ) so would the Weyl  $\lambda$ -BMS algebra. This however is not the case. For example, for  $\lambda = -1$ , corresponding to the BMS algebra, the BMS algebra admits a central extension  $c_P$  in the bracket  $[L_m, P_n]$ . This however has to vanish in the Weyl BMS algebra: indeed, the zero mode  $D_0$  of the super-dilatations acts diagonally with  $[D_0, L_m] = 0$  and  $[D_0, P_m] = P_m$  and hence by Jacobi  $[D_0, [L_m, P_n]] = [L_m, P_n]$  and hence there can be no central terms in  $[L_m, P_n]$ .

**Definition 3.** The universal central extension  $\widehat{\mathfrak{w}}_\lambda$  of  $\mathfrak{w}_\lambda$  is called the **Weyl  $\lambda$ -BMS algebra**. For  $\lambda = -1$ , this is the three-dimensional version of the Weyl BMS algebra in [13].

3.1.3. *The Weyl  $\lambda$ -BMS algebra in terms of OPEs.* Next we reformulate the Lie bracket of the Weyl  $\lambda$ -BMS algebra in terms of operator product expansions for the fields

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad D(z) = \sum_{n \in \mathbb{Z}} D_n z^{-n-1} \quad \text{and} \quad P(z) = \sum_{n \in \mathbb{Z}} P_n z^{-n-(1-\lambda)}. \quad (3.11)$$

The operator product algebra is given by

$$\begin{aligned} T(z)T(w) &= \frac{\frac{1}{2}c_L}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} \\ T(z)D(w) &= \frac{c_{TD}}{(z-w)^3} + \frac{D(w)}{(z-w)^2} + \frac{\partial D(w)}{z-w} + \text{reg} \\ T(z)P(w) &= \frac{(1-\lambda)P(w)}{(z-w)^2} + \frac{\partial P(w)}{z-w} + \text{reg} \\ D(z)P(w) &= \frac{P(w)}{z-w} + \text{reg} \\ D(z)D(w) &= \frac{c_D}{(z-w)^2} + \text{reg} \\ P(z)P(w) &= \text{reg}, \end{aligned} \quad (3.12)$$

where  $c_L$ ,  $c_D$  and  $c_{TD}$  are the three central charges. One can check that the above operator product expansions are associative. This and many of the calculations below have been performed using the Mathematica package `OPEdefs` written by Kris Thielemans [34–36].

For further reference, we remind the reader that the operator product expansion can be reformulated in terms of a sequence of bilinear products indexed by the integers:

$$A(z)B(w) = \sum_{n \ll \infty} \frac{[A, B]_n(w)}{(z-w)^n}, \quad (3.13)$$

where the sum is over all  $n$  less than some positive integer. The commutativity and associativity of the operator product expansion translate into axioms for the brackets  $[-, -]_n$  which are reminiscent to those satisfied by the bracket in a Lie algebra. All the operator product algebras in this paper are conformal, so that there is always a field  $T(z)$  satisfying the operator product expansion of the Virasoro algebra with some central charge (as in the first operator product expansion in equation (3.12)). We assume that all other fields have a well-defined conformal weight, so that for any field  $\Phi(z)$ ,  $[T, \Phi]_2 = h\Phi$ , where  $h$  is the conformal weight. If  $A(z)$  and  $B(z)$  have conformal weights  $h_A$  and  $h_B$ , respectively, we expand them in modes according to

$$A(z) = \sum_n A_n z^{-n-h_A} \quad \text{and} \quad B(z) = \sum_n B_n z^{-n-h_B} \quad (3.14)$$

and the Lie algebra of modes can be read off from the singular part of the operator product expansion via the formula

$$[A_n, B_m] = \sum_{\ell \geq 1} \binom{n+h_A-1}{\ell-1} ([A, B]_\ell)_{m+n}, \quad (3.15)$$

with  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  the usual binomial coefficient.

<sup>10</sup>Likewise,  $\widehat{\gamma}_1 = \widehat{\pi}^* \circ \pi^*(\gamma_1)$ ,  $\widehat{\gamma}_2 = \widehat{\pi}^* \circ \pi^*(\gamma_2)$  and  $\widehat{\gamma}_3 = \widehat{\pi}^* \circ \pi^*(\gamma_3)$  agree with the expressions for the three representative two-cocycles on  $\mathfrak{g}$  [33].

**3.2. The BRST complex of the Weyl  $\lambda$ -BMS algebra.** We now construct the BRST complex for the Weyl  $\lambda$ -BMS algebra above. Its cohomology, which is the semi-infinite cohomology of the  $\lambda$ -BMS algebra relative to the centre and with values in an admissible representation, can be interpreted as the spectrum of (a chiral sector of) a putative Weyl  $\lambda$ -BMS string, as was first observed for the bosonic string in [37].

**3.2.1. The ghosts BC systems.** The ghosts are described by fermionic BC systems  $(b_i, c_i)$  for  $i = 1, 2, 3$  with conformal weights  $(2, -1)$ ,  $(1, 0)$  and  $(1 - \lambda, \lambda)$ , respectively. Their operator product expansions are the standard ones as in

$$b_i(z)c_j(w) = \frac{\delta_{ij}}{z-w} + \text{reg}, \quad b_i(z)b_j(w) = \text{reg} \quad \text{and} \quad c_i(z)c_j(w) = \text{reg}. \quad (3.16)$$

The Virasoro element is given by

$$T^{\text{gh}} = -2(b_1\partial c_1) - (\partial b_1 c_1) - (b_2\partial c_2) - (1 - \lambda)(b_3\partial c_3) + \lambda(\partial b_3 c_3), \quad (3.17)$$

where parentheses indicate the normal-ordered product, which associates to the left so that  $(ABC) := (A(BC))$ , et cetera. We define the ghost number as usual by declaring  $b_i$  to have ghost number  $-1$  and  $c_i$  ghost number  $+1$ .

**3.2.2. The ghost Weyl  $\lambda$ -BMS algebra.** There is an embedding of the Weyl  $\lambda$ -BMS algebra in the operator product algebra of the ghost BC systems. The Virasoro element is given by  $T^{\text{gh}}$  in equation (3.17), which results in a central charge

$$c_L^{\text{gh}} = -6(5 - 2\lambda + 2\lambda^2). \quad (3.18)$$

One can find expressions for  $D^{\text{gh}}$  and  $P^{\text{gh}}$  after some trial and error<sup>11</sup> and this leads to

$$D^{\text{gh}} = (b_3 c_3) + \partial(c_1 b_2) \quad (3.19)$$

which results in central charges

$$c_{TD}^{\text{gh}} = 1 - 2\lambda \quad \text{and} \quad c_D^{\text{gh}} = 1. \quad (3.20)$$

Finally we find

$$P^{\text{gh}} = (c_1 \partial b_3) + (c_2 b_3) + (1 - \lambda)(\partial c_1 b_3). \quad (3.21)$$

One finds that  $T^{\text{gh}}$ ,  $D^{\text{gh}}$  and  $P^{\text{gh}}$  obey the Weyl  $\lambda$ -BMS algebra with the above values for the central charges. It should be mentioned that this is not the unique embedding of the Weyl  $\lambda$ -BMS algebra in the operator product algebra of the ghosts, but it is the one induced by the BRST differential to be introduced presently.

**3.2.3. The BRST current.** The BRST current  $J$  is a conformal weight 1 and ghost number 1 field of the form

$$J = (c_1 T) + (c_2 D) + (c_3 P) + \dots \quad (3.22)$$

where  $T$ ,  $D$  and  $P$  are a representation of the Weyl  $\lambda$ -BMS algebra with opposite central charges:

$$c_L = -c_L^{\text{gh}} = 6(5 - 2\lambda + 2\lambda^2), \quad c_{TD} = -c_{TD}^{\text{gh}} = 2\lambda - 1 \quad \text{and} \quad c_D = -c_D^{\text{gh}} = -1. \quad (3.23)$$

The fundamental property of  $J$  is that its zero mode  $d$  is a differential:  $d^2 = 0$ , where the action of  $d$  is given by the first order pole of the operator product expansion with  $J$ ; that is,  $d = [J, -]_1$ . By a result of Füsün Akman [38],  $d$  is a differential if and only if  $T^{\text{tot}} := db_1$ ,  $D^{\text{tot}} := db_2$  and  $P^{\text{tot}} := db_3$  obey the centreless Weyl  $\lambda$ -BMS algebra. The BRST current is of course only defined up to the addition of a total derivative, since that does not change its zero mode.

Some experimentation results in the following expression for the BRST current:

$$J = (c_1 T) + (c_2 D) + (c_3 P) + (b_1 c_1 \partial c_1) + (b_2 c_1 \partial c_2) + (b_3 c_1 \partial c_3) - \lambda(b_3 c_3 \partial c_1) + (c_2 b_3 c_3), \quad (3.24)$$

which up to a total derivative takes the more usual form

$$J' = (c_1 T) + (c_2 D) + (c_3 P) + \frac{1}{2}(c_1 T^{\text{gh}}) + \frac{1}{2}(c_2 D^{\text{gh}}) + \frac{1}{2}(c_3 P^{\text{gh}}). \quad (3.25)$$

One checks that  $[J', J']_1$  is indeed a total derivative and that  $T^{\text{tot}} = [J', b_1]_1 = T + T^{\text{gh}}$ ,  $D^{\text{tot}} = [J', b_2]_1 = D + D^{\text{gh}}$  and  $P^{\text{tot}} = [J', b_3]_1 = P + P^{\text{gh}}$  indeed give a representation of the centreless Weyl  $\lambda$ -BMS algebra.

The form of the BRST current (3.25) indicates that the fields  $T^{\text{gh}}$ ,  $D^{\text{gh}}$  and  $P^{\text{gh}}$  are indeed generating functionals formed from the semi-infinite wedge (i.e., fermionic Fock) representation of the Weyl  $\lambda$ -BMS algebra.

<sup>11</sup>Alternatively, one can follow the prescription given in [21, Section 3] for  $\mathfrak{g}_\lambda$  and apply it to  $\mathfrak{w}_\lambda$ .

**3.3. Quasi-isomorphism with a twisted  $N=2$  superconformal algebra.** The BRST cohomology is not just a graded vector space, but admits a richer algebraic structure, first formalised in the context of the bosonic string in [39] based on initial observations of the BRST cohomology of noncritical bosonic strings in [40]. The relevant structure is that of a Batalin–Vilkovisky (BV) algebra: a special kind of Gerstenhaber algebra where the bracket measures the failure of a second order operator (here the Virasoro antighost zero mode) being a derivation over the normal ordered product. We refer to [39] for the relevant facts and definitions.

A paradigmatic example of BV algebra is provided by the chiral ring of a topologically twisted  $N = 2$  superconformal algebra [41, 42]. The twisting gives the two supercharges  $G^\pm$ , originally of conformal weight  $\frac{3}{2}$ , conformal weights 1 and 2. The zero mode of the supercharge with conformal weight 1 plays the rôle of the BRST differential and the zero mode of the supercharge with conformal weight 2 plays the rôle of the Virasoro antighost zero mode.

As the case of the bosonic string theory shows, not every topological conformal field theory is of this form. Indeed, this is typical of most string theories, with the exception of the  $N = 2$  string itself [43, 44]. Nevertheless, exploiting the embedding [45] of the  $N = 1$  NSR string into the  $N = 2$  string, it was shown in [46], that the BRST cohomology of the  $N = 1$  string is isomorphic to the chiral ring of an  $N = 2$  superconformal field theory. This then suggested how to prove the same result for other string theories [47], resulting in a natural conjecture that the BRST cohomology of any string theory and, more generally, of any two-dimensional topological conformal field theory is isomorphic to the chiral ring of some twisted  $N = 2$  superconformal field theory [15, 16].

In this section we will test this conjecture (and show that it holds) for the BRST cohomology of the Weyl  $\lambda$ -BMS algebra.

To do this we will embed into the tensor product of the BRST complex of the Weyl  $\lambda$ -BMS algebra with a Koszul topological conformal algebra a topologically twisted  $N=2$  superconformal algebra. This embedding will turn out to be quasi-isomorphic, in that the BRST cohomology of the Weyl  $\lambda$ -BMS algebra will be isomorphic (as a BV algebra) to the chiral ring of the  $N=2$  superconformal algebra. This gives further evidence to the conjecture in [47, 15, 16].

We start by modifying the BRST current by a total derivative in order to simplify its operator product. Let us define

$$\begin{aligned}\mathbb{G}_W^+ &:= J + \partial((c_1 b_2 c_2) + (1 + \lambda)(c_1 b_3 c_3) + c_2 + \frac{1}{2}(7 - 4\lambda)\partial c_1) \\ \mathbb{G}_W^- &:= b_1 \\ \mathbb{J}_W &:= -(b_1 c_1) - (b_2 c_2) - (b_3 c_3) \\ \mathbb{T}_W &:= \mathbb{T}^{\text{tot}}.\end{aligned}\tag{3.26}$$

The resulting operator product expansions are

$$\begin{aligned}\mathbb{G}_W^+(z)\mathbb{G}_W^+(w) &= \frac{(\lambda - 1)\partial(c_1 \partial^2 c_1)(w)}{z - w} + \text{reg} \\ \mathbb{G}_W^+(z)\mathbb{G}_W^-(w) &= \frac{(7 - 4\lambda)}{(z - w)^3} + \frac{\mathbb{J}_W(w)}{(z - w)^2} + \frac{\mathbb{T}_W(w)}{z - w} + \text{reg} \\ \mathbb{G}_W^-(z)\mathbb{G}_W^-(w) &= \text{reg} \\ \mathbb{T}_W(z)\mathbb{J}_W(w) &= \frac{(2\lambda - 5)}{(z - w)^3} + \frac{\mathbb{J}_W(w)}{(z - w)^2} + \frac{\partial\mathbb{J}_W(w)}{z - w} + \text{reg} \\ \mathbb{T}_W(z)\mathbb{G}_W^+(w) &= \frac{6(\lambda - 1)c_1(w)}{(z - w)^4} + \frac{2(\lambda - 1)\partial c_1(w)}{(z - w)^3} + \frac{\mathbb{G}_W^+(w)}{(z - w)^2} + \frac{\partial\mathbb{G}_W^+(w)}{z - w} + \text{reg} \\ \mathbb{T}_W(z)\mathbb{G}_W^-(w) &= \frac{2\mathbb{G}_W^-(w)}{(z - w)^2} + \frac{\partial\mathbb{G}_W^-(w)}{z - w} + \text{reg} \\ \mathbb{J}_W(z)\mathbb{J}_W(w) &= \frac{3}{(z - w)^2} \\ \mathbb{J}_W(z)\mathbb{G}_W^+(w) &= \frac{6(1 - \lambda)c_1(w)}{(z - w)^3} + \frac{4(1 - \lambda)\partial c_1(w)}{(z - w)^2} + \frac{\mathbb{G}_W^+(w)}{z - w} + \text{reg} \\ \mathbb{J}_W(z)\mathbb{G}_W^-(w) &= \frac{-\mathbb{G}_W^-(w)}{z - w} + \text{reg},\end{aligned}\tag{3.27}$$

which are clearly not those of a topologically twisted  $N = 2$  superconformal algebra. For example,  $\mathbb{G}_W^+$  does not have regular operator product expansion with itself.

3.3.1. *The Koszul topological conformal algebra.* We now tensor with a Koszul topological conformal algebra consisting of one fermionic BC system  $(b, c)$  and one bosonic BC system  $(\beta, \gamma)$ , both of conformal weights  $(1 - \mu, \mu)$ , with basic operator product expansions:

$$b(z)c(w) = \frac{1}{z-w} + \text{reg} \quad \text{and} \quad \beta(z)\gamma(w) = \frac{1}{z-w} + \text{reg}. \quad (3.28)$$

The Koszul topological conformal algebra embeds a twisted N=2 superconformal algebra given by the following fields:

$$\begin{aligned} \mathbb{G}_K^+ &:= (b\gamma) \\ \mathbb{G}_K^- &:= (1 - \mu)(\partial c\beta) - \mu(c\partial\beta) \\ \mathbb{J}_K &:= \mu(bc) + (1 - \mu)(\beta\gamma) \\ \mathbb{T}_K &:= (1 - \mu)(\beta\partial\gamma) - \mu(\partial\beta\gamma) - (1 - \mu)(b\partial c) + \mu(\partial bc). \end{aligned} \quad (3.29)$$

These fields obey a twisted N=2 superconformal algebra on the nose:

$$\begin{aligned} \mathbb{G}_K^\pm(z)\mathbb{G}_K^\pm(w) &= \text{reg} \\ \mathbb{G}_K^+(z)\mathbb{G}_K^-(w) &= \frac{(2\mu - 1)}{(z-w)^3} + \frac{\mathbb{J}_K(w)}{(z-w)^2} + \frac{\mathbb{T}_K(w)}{z-w} + \text{reg} \\ \mathbb{J}_K(z)\mathbb{G}_K^\pm(w) &= \frac{\pm\mathbb{G}_K^\pm(w)}{z-w} + \text{reg}, \end{aligned} \quad (3.30)$$

from which the remaining operator product expansions follow by associativity, as shown independently in [48] and [49]:

$$\begin{aligned} \mathbb{J}_K(z)\mathbb{J}_K(w) &= \frac{(2\mu - 1)}{(z-w)^2} \\ \mathbb{T}_K(z)\mathbb{J}_K(w) &= \frac{(1 - 2\mu)}{(z-w)^3} + \frac{\mathbb{J}_K(w)}{(z-w)^2} + \frac{\partial\mathbb{J}_K(w)}{z-w} + \text{reg} \\ \mathbb{T}_K(z)\mathbb{G}_K^+(w) &= \frac{\mathbb{G}_K^+(w)}{(z-w)^2} + \frac{\partial\mathbb{G}_K^+(w)}{z-w} + \text{reg} \\ \mathbb{T}_K(z)\mathbb{G}_K^-(w) &= \frac{2\mathbb{G}_K^-(w)}{(z-w)^2} + \frac{\partial\mathbb{G}_K^-(w)}{z-w} + \text{reg} \\ \mathbb{T}_K(z)\mathbb{T}_K(w) &= \frac{2\mathbb{T}_K(w)}{(z-w)^2} + \frac{\partial\mathbb{T}_K(w)}{z-w} + \text{reg}. \end{aligned} \quad (3.31)$$

3.3.2. *A twisted N=2 superconformal algebra.* Now let

$$X := (1 - \lambda)((\partial c_1 c_1 c \beta) + (c_1 \beta \gamma) - (c_1 bc) - \partial c_1) \quad (3.32)$$

and define the following fields

$$\begin{aligned} \mathbb{G}^+ &:= \mathbb{G}_W^+ + \mathbb{G}_K^+ + \partial X \\ \mathbb{G}^- &:= \mathbb{G}_W^- + \mathbb{G}_K^-. \end{aligned} \quad (3.33)$$

It follows by calculation that

$$\begin{aligned} \mathbb{G}^\pm(z)\mathbb{G}^\pm(w) &= \text{reg} \\ \mathbb{G}^+(z)\mathbb{G}^-(w) &= \frac{2(2 + \mu - \lambda)}{(z-w)^3} + \frac{\mathbb{J}(w)}{(z-w)^2} + \frac{\mathbb{T}(w)}{z-w} + \text{reg}, \end{aligned} \quad (3.34)$$

which defines  $\mathbb{J}$  and  $\mathbb{T}$ :

$$\begin{aligned} \mathbb{J} &= \mathbb{J}_W + \mathbb{J}_K + (1 - \lambda)((bc) - (\beta\gamma) + \partial(c_1 c \beta)) \\ \mathbb{T} &= \mathbb{T}_W + \mathbb{T}_K. \end{aligned} \quad (3.35)$$

Another calculation shows that

$$\mathbb{J}(z)\mathbb{G}^\pm(w) = \frac{\pm\mathbb{G}^\pm(w)}{z-w} + \text{reg}, \quad (3.36)$$

from which the other operator product expansions of the twisted N=2 superconformal algebra follow by associativity:

$$\begin{aligned}
\mathbb{J}(z)\mathbb{J}(w) &= \frac{2(2 + \mu - \lambda)}{(z - w)^2} + \text{reg} \\
\mathbb{T}(z)\mathbb{J}(w) &= \frac{-2(2 + \mu - \lambda)}{(z - w)^3} + \frac{\mathbb{J}(w)}{(z - w)^2} + \frac{\partial\mathbb{J}(w)}{z - w} + \text{reg} \\
\mathbb{T}(z)\mathbb{G}^+(w) &= \frac{\mathbb{G}^+(w)}{(z - w)^2} + \frac{\partial\mathbb{G}^+(w)}{z - w} + \text{reg} \\
\mathbb{T}(z)\mathbb{G}^-(w) &= \frac{2\mathbb{G}^-(w)}{(z - w)^2} + \frac{\partial\mathbb{G}^-(w)}{z - w} + \text{reg} \\
\mathbb{T}(z)\mathbb{T}(w) &= \frac{2\mathbb{T}(w)}{(z - w)^2} + \frac{\partial\mathbb{T}(w)}{z - w} + \text{reg}.
\end{aligned} \tag{3.37}$$

Notice that by choosing  $\mu$  appropriately, we can bring the central charges to any desired values, e.g., if  $\mu = \lambda - 2$ , then all central terms vanish. Notice that since  $\mathbb{G}^+$  differs from  $\mathbb{G}_W^+ + \mathbb{G}_K^+$  by a total derivative, the N=2 differential  $d_{N=2} = [\mathbb{G}^+, -]_1$  is the sum of the Weyl  $\lambda$ -BMS and Koszul differentials and hence we may apply the Künneth theorem to deduce that the chiral ring of the N=2 superconformal algebra is isomorphic to the graded tensor product of the cohomology of the Weyl  $\lambda$ -BMS differential  $d$  and the cohomology of the Koszul differential  $d_K = [\mathbb{G}_K^+, -]_1$ . Since (once we choose a picture for the  $\beta\gamma$  system) the latter cohomology is trivial except in degree 0 and isomorphic to  $\mathbb{C}$  there, we obtain that the chiral ring is isomorphic to the BRST cohomology of the Weyl  $\lambda$ -BMS algebra.

It is also the case that the isomorphism is one of BV algebras. Indeed, that the BRST cohomology of the Weyl  $\lambda$ -BMS algebra admits the structure of a BV algebra follows from results in [47, 50], which guarantee this is the case simply because the Virasoro antighost is a conformal primary with weight 2. In the BRST cohomology of the Weyl  $\lambda$ -BMS algebra, the BV differential is given by the zero mode of the Virasoro antighost ( $\mathbb{G}_W^-$ ), whereas in the case of the topologically N = 2 superconformal algebra it is given by the zero mode of  $\mathbb{G}^- = \mathbb{G}_W^- + \mathbb{G}_K^-$ , which acts the same way in cohomology, since  $\mathbb{G}_K^-$  acts trivially on the Koszul cohomology.

#### 4. THE CONFORMAL BMS W-ALGEBRA

Consider again the Weyl BMS algebra, which is the Weyl  $\lambda$ -BMS algebra with  $\lambda = -1$ . There is no way to extend it to a Lie algebra by the addition of a field  $K(z)$  of conformal weight 2 in such a way that the operator expansion  $K(z)P(w)$  is nonzero, but as shown in [12], such an extension exists as a W-algebra. In this section we show that contrary to many of the W-algebras which have been studied in this context, this one does not admit a BRST complex and hence there is no natural notion of W-strings for it.

**4.1. The W-algebra.** We consider the VOA generated by fields  $T, D, K, P$  with the following operator product expansions:

$$\begin{aligned}
T(z)T(w) &= \frac{\frac{1}{2}c_L}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w} + \text{reg} \\
T(z)D(w) &= \frac{D(w)}{(z - w)^2} + \frac{\partial D(w)}{z - w} + \text{reg} \\
T(z)P(w) &= \frac{2P(w)}{(z - w)^2} + \frac{\partial P(w)}{z - w} + \text{reg} \\
T(z)K(w) &= \frac{2K(w)}{(z - w)^2} + \frac{\partial K(w)}{z - w} + \text{reg} \\
D(z)D(w) &= \frac{c_D}{(z - w)^2} + \text{reg} \\
D(z)K(w) &= -\frac{K(w)}{z - w} + \text{reg} \\
D(z)P(w) &= \frac{P(w)}{z - w} + \text{reg} \\
K(z)K(w) &= \text{reg} \\
P(z)P(w) &= \text{reg},
\end{aligned} \tag{4.1}$$



and the operator product expansion  $K(z)P(w)$  has the following singular terms

$$K(z)P(w) = \sum_{\ell=1}^4 \frac{[K, P]_{\ell}(w)}{(z-w)^{\ell}} \quad (4.2)$$

with

$$\begin{aligned} [K, P]_4 &= \frac{3(c_D - 1)c_D^2}{1 + c_D} \\ [K, P]_3 &= \frac{3(1 - c_D)c_D}{1 + c_D} D \\ [K, P]_2 &= -c_D T + \frac{3(1 - c_D)c_D}{2(1 + c_D)} \partial D + \frac{2c_D - 1}{1 + c_D} D^2 \\ [K, P]_1 &= -\frac{1}{2}c_D \partial T - \frac{1 + c_D^2}{2(1 + c_D)} \partial^2 D + \frac{2c_D - 1}{1 + c_D} D \partial D + TD - \frac{1}{1 + c_D} D^3. \end{aligned} \quad (4.3)$$

Associativity of the operator product forbids any other central charges and even forces the Virasoro central charge to be

$$c_L = \frac{-2(6c_D^2 - 8c_D + 1)}{1 + c_D}. \quad (4.4)$$

The structure of the above operator product algebra can be understood a bit better if we define the following primary fields:

$$\begin{aligned} \Phi_2 &= \frac{2c_D - 1}{1 + c_D} \left( D^2 - \frac{2c_D}{c_L} T \right) \\ \Phi_3 &= -\frac{1}{1 + c_D} D^3 + \frac{1}{3 - 2c_D} (TD - \frac{1}{2} \partial^2 D), \end{aligned} \quad (4.5)$$

of conformal weights 2 and 3, respectively. Then the operator product expansion  $K(z)P(w)$  simply contains the conformal families of the identity,  $D$ ,  $\Phi_2$  and  $\Phi_3$  with coefficients

$$K(z)P(w) = \frac{3c_D^2(c_D - 1)}{1 + c_D} \frac{[\mathbb{1}](w)}{(z-w)^4} + \frac{3c_D(1 - c_D)}{1 + c_D} \frac{[D](w)}{(z-w)^3} + \frac{[\Phi_2](w)}{(z-w)^2} + \frac{[\Phi_3](w)}{z-w} + \text{reg}, \quad (4.6)$$

where for a primary field  $\phi$ , the notation  $[\phi]$  stands for its conformal family. Explicitly, in the above equation and up to regular terms, we have

$$\begin{aligned} [\mathbb{1}](w) &= \mathbb{1} + \frac{4}{c_L}(z-w)^2 T(w) + \frac{2}{c_L}(z-w)^3 \partial T(w) \\ [D](w) &= D(w) + \frac{1}{2}(z-w) \partial D(w) + \frac{2(1 + c_D)}{3c_D(3 - 2c_D)}(z-w)^2 \left( (TD)(w) - \frac{2c_D^2 - c_D + 2}{4(1 + c_D)} \partial^2 D(w) \right) \\ [\Phi_2](w) &= \Phi_2(w) + \frac{1}{2}(z-w) \partial \Phi_2(w) \\ [\Phi_3](w) &= \Phi_3(w). \end{aligned} \quad (4.7)$$

**4.2. Non-existence of BRST complex.** The proof of non-existence of the BRST complex for the above W-algebra is computational, but we will give some details setting up the calculation and then explain the result.

We have four quasiprimary fields in the W-algebra:  $T, D, P, K$  of conformal weights  $2, 1, 2, 2$ , respectively. We introduce fermionic ghost systems  $(b_i, c_i)$  for  $i = 1, 2, 3, 4$  of weights  $(2, -1)$  for  $i = 1, 3, 4$  and weights  $(1, 0)$  for  $i = 2$ . As usual we assign ghost numbers 1 to the  $c_i$  and  $-1$  to the  $b_i$ . The putative BRST current has ghost number 1 and conformal weight 1 and takes the form

$$J = c_1 T + c_2 D + c_3 P + c_4 K + \dots \quad (4.8)$$

where  $\dots$  refers to any terms of with one or more antighosts  $b_i$ . Our methodology is naive. We write the most general  $J$  of the above form and of conformal weight 1 and ghost number 1 and demand that  $d^2 = 0$ , with  $d := [J, -]_1$  its zero mode. The calculations have been performed in Mathematica on a 2020 MacBook Pro laptop with a 2.3 GHz Quad-Core Intel Core i7 processor and 16Gb of RAM, using the package `OPeDefs` (version 3.1 beta 4) written by Kris Thielemans [34–36]. A notebook is available upon request.

Table 1 lists the ingredients out of which we may write the terms in  $J$  of the form  $BC^2$ , along with their conformal weights. This and the following table is to be supplemented with the following table of

the fields of low conformal weight made out of the generators of the original W-algebra:

$\omega_X$	fields
0	$\mathbb{1}$
1	D
2	T, P, K, $D^2$ , $\partial D$ .

All the  $BC^2$  terms are obtained by picking one term from each table and ensuring that the sum of the conformal weights  $\omega_B + \omega_{C^2} + \omega_X = 1$ . It is easy to see that the possible triples  $(\omega_B, \omega_{C^2}, \omega_X)$  of conformal weights are  $(1, -2, 2)_{15}$ ,  $(1, -1, 1)_{12}$ ,  $(1, 0, 0)_{18}$ ,  $(2, -2, 1)_{12}$ ,  $(2, -1, 0)_{48}$  and  $(3, -2, 0)_{12}$ , where the subscript is the multiplicity. This means there are 117 such terms, which despite being easy to enumerate, we will refrain from doing so here.

TABLE 1. Ingredients of  $BC^2$  terms with their conformal weights

$\omega_B$	fields	$\omega_{C^2}$	fields
1	$b_2$	-2	$c_1 c_3, c_1 c_4, c_3 c_4$
2	$b_1, \partial b_2, b_3, b_4$	-1	$c_1 \partial c_3, \partial c_1 c_3, c_1 \partial c_4, \partial c_1 c_4, c_3 \partial c_4, \partial c_3 c_4,$ $c_1 \partial c_1, c_3 \partial c_3, c_4 \partial c_4, c_1 c_2, c_2 c_3, c_2 c_4$
3	$\partial b_1, \partial^2 b_2, \partial b_3, \partial b_4$	0	$c_1 \partial^2 c_3, \partial c_1 \partial c_3, \partial^2 c_1 c_3, c_1 \partial^2 c_4, \partial c_1 \partial c_4,$ $\partial^2 c_1 c_4, c_3 \partial^2 c_4, \partial c_3 \partial c_4, \partial^2 c_3 c_4, c_1 \partial^2 c_1$ $c_3 \partial^2 c_3, c_4 \partial^2 c_4, \partial c_1 c_2, c_1 \partial c_2, \partial c_2 c_3,$ $c_2 \partial c_3, \partial c_2 c_4, c_2 \partial c_4$

Table 2 lists the ingredients in terms of the form  $B^2 C^3$  along with their conformal weights. Again all terms are obtained by picking one term from each table and ensuring that the sum of the conformal weights  $\omega_{B^2} + \omega_{C^3} + \omega_X = 1$ . It is again easy to see that the possible triples  $(\omega_{B^2}, \omega_{C^3}, \omega_X)$  of conformal weights are  $(4, -3, 0)_{10}$ ,  $(3, -3, 1)_4$  and  $(3, -2, 0)_{48}$  for a total of 62, which we will also refrain from listing. There are no terms with three (or more) antighosts because the conformal weight of any term of the form  $B^3 C^4 X$  is bounded below by 2 and this only increases with terms with higher number of antighosts. In total there are 179 possible terms of ghost number 1 and conformal weight 1 we could add to the BRST current.

TABLE 2. Ingredients of  $B^2 C^3$  terms with their conformal weights

$\omega_{B^2}$	fields	$\omega_{C^3}$	fields
3	$b_1 b_2, b_2 b_3, b_2 b_4, b_2 \partial b_2$	-3	$c_1 c_3 c_4$
4	$b_1 b_3, b_1 b_4, b_3 b_4, b_2 \partial^2 b_2,$ $\partial b_1 b_2, b_1 \partial b_2, \partial b_2 b_3,$ $b_2 \partial b_3, \partial b_2 b_4, b_2 \partial b_4$	-2	$c_1 c_2 c_3, c_1 c_2 c_4, c_2 c_3 c_4, c_1 \partial c_1 c_3,$ $c_1 \partial c_1 c_4, c_1 c_3 \partial c_3, c_1 c_4 \partial c_4, c_3 \partial c_3 c_4,$ $c_3 c_4 \partial c_4, \partial c_1 c_3 c_4, c_1 \partial c_3 c_4, c_1 c_3 \partial c_4$

Given the most general J, depending on 179 parameters, we calculate the first-order pole  $[J, J]_1$  in the operator product expansion of the putative BRST current with itself. The equations are then  $[[J, J]_1, \phi]_1 = 0$  for  $\phi$  one of the generating fields of the VOA:  $b_i, c_i, T, D, P$  and  $K$ . This results in 3288 equations, which admit no solutions. The basic reason is the following. There are two central charges in the W-algebra: the Virasoro central charge  $c_L$  and that of the field D, denoted  $c_D$ . The existence of the BRST complex requires both of them to be critical, by which we mean that the values of  $c_L$  and  $c_D$  should cancel the ones of the ghost representation. However these central charges are not independent: associativity of the operator product expansion relates them:

$$c_L = \frac{-2(1 - 8c_D + 6c_D^2)}{1 + c_D}. \quad (4.9)$$

The critical values are  $c_L = 80$  and  $c_D = -2$ , which do not satisfy the above equation. Indeed, when  $c_D = -2$ , one finds that  $c_L = 82$ . (It is intriguing that the excess is small and integral.)

This result is perhaps surprising given our experience with other W-algebras. Deformable W-algebras, those which exist for generic values of the Virasoro central charge  $c_L$ , are typically constructed via Drinfel'd–Sokolov reduction [51, 52]. The starting point of this reduction is the affine Kac–Moody algebra

associated to the vacuum-preserving subalgebra  $\mathfrak{g}$  (a contraction  $c_L \rightarrow \infty$  of the algebra of the vacuum-preserving modes) [53]. The vacuum-preserving Virasoro modes  $L_{\pm 1}, L_0$  define an  $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$  subalgebra of  $\mathfrak{g}$  and these are in bijective correspondence with adjoint orbits of nilpotent elements in  $\mathfrak{g}$ , at least for  $\mathfrak{g}$  semisimple, which is typically the case. The best known W-algebras are associated to the principal nilpotent orbits, those with the smallest stabiliser, namely  $\mathfrak{so}(2, 1)$  itself. Perhaps the best known such example is the  $W_3$  algebra [54], whose BRST differential was constructed in [55] and whose cohomology was studied in detail in [56]. Those W-algebras always admit a BRST complex and indeed a reasonable notion of semi-infinite cohomology [57], as is the case for Lie algebras [58]. We also have constructions of BRST complexes for W-algebras associated with minimal nilpotent orbits, as in [59], which constructs a W-algebra out of the minimal nilpotent orbit of  $\mathfrak{sl}(3, \mathbb{R})$ ; although we are not aware of any general result for the existence of a semi-infinite cohomology theory for the minimal orbits. The conformal BMS W-algebra is one of four W-algebras which can be obtained by Drinfel'd–Sokolov reduction of the three-dimensional conformal algebra  $\mathfrak{so}(3, 2)$  [60], but it is not the W-algebra associated to the principal nilpotent orbit (whose BRST differential was constructed in [61]), nor indeed to the minimal nilpotent orbit (with stabiliser  $\mathfrak{so}(2, 2) \cong \mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$ ), but to an intermediate nilpotent orbit with stabiliser  $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(1, 1)$ . (There is another intermediate orbit with stabiliser  $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(2)$  which can also be shown to lack a BRST complex.) There are, to our knowledge, no theorems about the existence of a semi-infinite differential for such W-algebras and this example suggests that perhaps we should not expect them to exist.

## 5. CONCLUSIONS AND OUTLOOK

We have constructed the Weyl  $\lambda$ -BMS algebra in three dimensions: an extension of the three-dimensional Lorentz algebra  $\mathfrak{so}(2, 1)$  by super-translations, super-rotations and super-dilatations, which agrees for  $\lambda = -1$  with the Weyl–BMS algebra in the literature. We construct this algebra out of the Fourier modes of a free massless Klein–Gordon field, a construction in which the quadratic Casimir of  $\mathfrak{so}(2, 1)$  plays a crucial rôle. We then reformulate the Weyl  $\lambda$ -BMS algebra in terms of operator product expansions and show that it admits a three-parameter family of central extensions. We construct the BRST complex for critical values of the three central charges and show that the BRST cohomology is isomorphic to the chiral ring of a topologically twisted  $N = 2$  superconformal field theory. Returning to the classical case of  $\lambda = -1$ , we argue that there is no “conformal” BMS Lie algebra obtained by further extending the Weyl–BMS Lie algebra by super special-conformal transformations. There exists a classical conformal BMS W-algebra which we fully quantise in the language of operator product expansions and argue that the resulting W-algebra does not admit a BRST complex.

Some of these results can be extended to define a Weyl  $\lambda$ -BMS Lie superalgebra and a fully quantum conformal BMS W-superalgebra and we will report on this in future work.

An interesting question we do not know the answer to is whether the conformal BMS W-algebra admits a canonical realisation.

Another interesting question is whether there exist string sigma models with gauge algebra given by the Weyl–BMS Lie algebra. It is worth remarking that the flat ambitwistor string [62] gives a realisation of the Weyl–BMS Lie algebra. Indeed, if we let  $(X^\mu, \Pi_\mu)$  be  $d + 1$  bosonic BC systems with conformal weights  $(0, 1)$  which describe the flat ambitwistor string, then

$$T = -\partial X^\mu \Pi_\mu, \quad D = \frac{1}{2} X^\mu \Pi_\mu \quad \text{and} \quad P = \frac{1}{2} \eta^{\mu\nu} \Pi_\mu \Pi_\nu \quad (5.1)$$

provide a realisation of the Weyl–BMS Lie algebra with central charges  $c_L = 2(d + 1)$ ,  $c_D = -\frac{1}{4}(d + 1)$  and  $c_{TD} = -\frac{1}{2}(d + 1)$ . The critical values of the central charges of a Weyl–BMS string are given by  $c_L = 54$ ,  $c_D = -\frac{27}{4}$  and  $c_{TD} = -\frac{27}{2}$ . Even if we were to rescale  $D$ , we would not find the critical values for a putative Weyl–BMS string.

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#### APPENDIX A. CASIMIR EIGENVALUE CORRESPONDING TO POINCARÉ

The reason why the solutions to (2.18) provide representations of the Lorentz algebra is the following. [19] Assume that the  $\{\omega_\ell\}$  provide a complete set of solutions to

$$-C_2\omega_\ell = \alpha\omega_\ell, \quad (\text{A.1})$$

where, in general dimension,  $\ell$  is a multi-index. Acting with a Lorentz generator  $M_{\mu\nu}$  on (A.1) one has

$$-M_{\mu\nu}C_2\omega_\ell = \alpha M_{\mu\nu}\omega_\ell, \quad (\text{A.2})$$

and, because  $M_{\mu\nu}$  and  $C_2$  commute,

$$-C_2M_{\mu\nu}\omega_\ell = \alpha M_{\mu\nu}\omega_\ell, \quad (\text{A.3})$$

which means that  $M_{\mu\nu}\omega_\ell$  is also a solution of (A.1). Now, since the  $\{\omega_\ell\}$  are all the solutions to the eigenvalue equation (2.18), one has necessarily that

$$M_{\mu\nu}\omega_\ell = \sum_{\ell'} a_{\ell';\mu\nu}^{\ell'} \omega_{\ell'}, \quad (\text{A.4})$$

which indicate that indeed the  $\omega_\ell$  provide a representation of the  $M_{\mu\nu}$  via the matrices  $K_{\mu\nu}$  with elements

$$(K_{\mu\nu})_{\ell}^{\ell'} = a_{\ell';\mu\nu}^{\ell'},$$

for each  $\mu, \nu$ .

That the eigenvalue  $-\alpha = d - 1$  in (2.19) corresponds to an algebra which contains Poincaré can be proved in general in any dimension by considering the algebra commutator

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu), \quad (\text{A.5})$$

which yields<sup>12</sup>

$$[M^{\mu\nu}M_{\mu\nu}, P_\rho] = 4iP^\mu M_{\mu\rho} + 2(d-1)P_\rho, \quad (\text{A.6})$$

and hence, if  $C_2 = 1/2M^{\mu\nu}M_{\mu\nu}$ ,

$$[C_2, P_\mu] = 2iP^\nu M_{\nu\mu} + (d-1)P_\mu. \quad (\text{A.7})$$

If we represent the generators in terms of differential operators (not necessarily acting on the mass-shell hyperboloid),  $C_2$  is a second-order operator and the first term on the right-hand side of (A.7) will be a pure first-order one, without a zeroth-order contribution. From this it can be read that

$$\hat{C}_2P_\mu = (d-1)P_\mu, \quad (\text{A.8})$$

as stated.

#### APPENDIX B. ALGEBRA OF CONSERVED CHARGES

Consider two conserved charges, computed at  $t = 0$ , in terms of the corresponding differential operators acting on the Fourier coefficients of the scalar field,

$$\begin{aligned} P &= \int d\vec{k} \tilde{a}(\vec{k}) \hat{P}a(\vec{k}), \\ Q &= \int d\vec{k} \tilde{a}(\vec{k}) \hat{Q}a(\vec{k}). \end{aligned} \quad (\text{B.1})$$

Using the Poisson brackets of the Fourier modes, it can be shown, without resorting to any integration by parts, and hence without having to consider boundary contributions, that

$$\{P, Q\} = -i \int d\vec{k} \tilde{a}(\vec{k}) [\hat{P}, \hat{Q}]a(\vec{k}), \quad (\text{B.2})$$

which shows that if the algebra of the differential operators does not exhibit any central extension, neither does the Poisson algebra of the charges.

<sup>12</sup>For the interpretation that follows, it is important to obtain an expression with the  $M$  to the right of the  $P$ . One can also get  $[M^{\mu\nu}M_{\mu\nu}, P_\rho] = 4iM_{\mu\rho}P^\mu - 2(d-1)P_\rho$ .

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