

A model for heat generation by acoustic waves in piezoelectric materials: Global large-data solutions

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Abstract

A model for the generation of heat due to mechanical losses during acoustic wave propagation in a solid is considered in a Kelvin-Voigt type framework. In contrast to previous studies on related thermoviscoelastic models, in line with recent experimental findings the present manuscript focuses on situations in which elastic parameters depend on temperature. Despite an apparent loss of mathematically favorable structural properties thereby encountered, in the framework of a suitably generalized concept of solvability a result on global existence of solutions is derived under mild assumptions which, in particular, do not involve any smallness condition on the initial data.

Key words: viscous wave equation, thermoviscoelasticity, generalized solvability

MSC 2020: 35D99 (primary), 35L05, 74F05, 74J10 (secondary)

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1 Introduction

As a mathematical subject, the theory of elasticity goes back to Cauchy and has again flourished since the mid-twentieth century (cf. eg. [60]). Thermoviscoelasticity with its strong nonlinear coupling between heat generation and the deformation of materials offers mathematical challenges at various levels. Classical one-dimensional existence results include local [43] or local and, under smallness conditions on initial data, global existence of classical solutions [58, 14, 28, 26] – although, on the other hand, for large data, C^2 solutions may blow up, [15, 32] – or global existence of weak solutions, [46, 11]. Also the long-time behavior of solutions has been studied, [46, 42, 5]. Only substantially later than [58, 14] were results for higher-dimensional systems obtained, e.g. [45, 56, 6] or [19] (the latter under assumptions of radial symmetry). Once again, there are nonlinearities that can cause solutions to lose C^2 regularity, [44]. Existence results concern local-in-time solutions [27, 8] or solvability under smallness conditions [45, 56].

Global weak solutions in three-dimensional domains were found in [37] for a system with irreversible mechanical deformations manifesting in an inelastic constitutive relation taking the form of an ODE. Without this inelastic effect, heat capacity growing with temperature has been leveraged in several constructions of certain solutions, [50, 6, 40, 20, 21], where this growth was relied on to deal with analytic difficulties stemming from the effects of temperature dilation. For a simplified system neglecting viscosity-driven transfer of mechanical energy into heat, a statement on global solvability has recently been derived in [12]. Further extended models feature fourth-order terms [65, 49, 64, 52, 34] (which may be beneficial in the derivation of a priori bounds) or additional 'internal variables' describing e.g. plasticity [51, 2, 3, 4, 38, 39, 53]; for related settings regarding adhesive contact see [48, 49].

Common to all these precedent studies on higher-dimensional cases seems a concentration on situations in which crucial system ingredients such as the elastic parameters and viscosities are constants; recent experimental observations concerned with the behavior of certain piezoceramics, however, have revealed some partially significant dependencies of these constituents on temperature. The present manuscript investigates a multi-dimensional model for thermoviscoelasticity capable of taking such effects into account, and intends to develop a basic theory of global solvability on the basis of a generalized solution concept that seems novel in this context; in particular, the design of this framework will be motivated by the ambition to cope with an apparent loss of some favorable structural properties going along with such modifications.

Before introducing the mathematical main results, let us consider the model with its physical background.

Application and modeling background. The generation of excess heat has a detrimental effect in many industrial and scientific applications. Not only are involuntary thermal emissions an indicator for a lack of efficiency, the increased temperature may also damage components. Among the materials that are particularly susceptible to temperature related damage are piezoelectric ceramics [55]. They are used as electromechanical transducers to generate and detect mechanical vibrations and acoustic waves in a variety of applications, ranging from microphones and loudspeakers to ultrasonic welding. The piezoelectric effect in these ceramics, which also show ferroelectric properties, is only present if the material is polarised. This intrinsic polarisation vanishes if a certain temperature threshold, the Curie temperature, is exceeded, rendering the piezoelectric ceramic inert. This is especially problematic in

high-power applications, such as ultrasonic bonding and welding, where piezoelectric ceramics are used as actors and are thus a primary source of heat [31, 66]. Additionally, the resonant behavior, which is crucial for the function of these devices, shows strong dependence on temperature [61]. Therefore, it is of utmost importance to accurately consider the heat generated by the acoustic waves in and around the piezoelectric material.

To account for mechanical losses during acoustic wave propagation, a suitable refinement of Hooke's law is required. Known as the apparently most basic description for the linear mechanical behavior of a solid, Hooke's law postulates the mechanical stress tensor $T = (T_{ij})_{i,j \in \{1,2,3\}} \in \mathbb{R}^{3 \times 3}$ to be related to the mechanical strain tensor $S = (S_{kl})_{k,l \in \{1,2,3\}} \in \mathbb{R}^{3 \times 3}$ through the fourth-rank elasticity tensor $C = (C_{ijkl})_{i,j,k,l \in \{1,2,3\}} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ [36] according to the tensor product

$$T = C : S, \quad (1.1)$$

that is, to the relation

$$T_{ij} = \sum_{k,l=1}^3 C_{ijkl} S_{kl}, \quad i, j \in \{1, 2, 3\},$$

(cf. also below for a more compact summary on notation used here); we note that both the stress and strain tensor are symmetric in the sense of satisfying $T_{ij} = T_{ji}$ and $S_{ij} = S_{ji}$ for $i, j \in \{1, 2, 3\}$. Now in further development of this, the Kelvin-Voigt model proposes to describe mechanical losses, and thus the generation of heat, by means of the modified relation ([22])

$$T = C : (S + \tau S_t), \quad (1.2)$$

where $\tau \in \mathbb{R}$ is the retardation time constant quantifying losses and S_t denotes the time derivative of the strain [33]; thus containing only one single additional constant compared to the above purely elastic model, this Kelvin-Voigt law is one of the fundamental viscoelastic material models. The differential equation for the displacement field $u = (u_i)_{i \in \{1,2,3\}} \in \mathbb{R}^3$ in a material described by the Kelvin-Voigt model can be derived from the Cauchy momentum equation [7]

$$u_{tt} = \frac{1}{\rho} \sum_{j=1}^3 \partial_j T_{.j} = \frac{1}{\rho} \operatorname{div} T,$$

and the definition of the strain via the symmetric gradient

$$S_{kl} = \frac{1}{2} (\partial_l u_k + \partial_k u_l) = (\nabla^s u)_{kl}, \quad k, l \in \{1, 2, 3\}. \quad (1.3)$$

With $\rho > 0$ denoting the density of the material, the resulting viscous wave equation

$$\rho u_{tt} = \operatorname{div} (C : \nabla^s u) + \tau \operatorname{div} (C : \nabla^s u_t) \quad (1.4)$$

does not only model the behavior of a viscoelastic solid described by the Kelvin-Voigt model but also arises in fluid acoustics from a linearized form of the Navier-Stokes equations [54]. For $\tau = 0$ and scalar quantities, (1.4) takes the form of a classical wave equation with phase velocity $c_{\text{ph}} = \sqrt{C/\rho}$. The

mixed third-order derivative term quantifies absorption for acoustic waves and is directly related to the viscosity when considering fluids [54]; this term thus describes a conversion of mechanical energy into thermal energy.

To quantify the energy conversion process, the work $P \in \mathbb{R}$ done on a solid can be determined by the scalar product of the mechanical stress T with the time derivative of the mechanical strain S_t [7] according to

$$P = \sum_{i,j=1}^3 T_{ij} S_{ij,t} = \langle T, S_t \rangle,$$

which in conjunction with (1.2) yields the identity

$$P = \langle C : S, S_t \rangle + \langle \tau C : S_t, S_t \rangle.$$

The first term in this expression describes reversible energy storage due to elastic deformations; for harmonic processes especially, it is easily shown that the time average of $\langle C : S, S_t \rangle$ is zero. The second term quantifies the conversion of mechanical work into thermal energy. It can be described in terms of the strain S or, using (1.3), in terms of the displacement u [59, 7],

$$Q = \langle \tau C : S_t, S_t \rangle = \langle \tau C : \nabla^s u_t, \nabla^s u_t \rangle.$$

The quantity $Q \in \mathbb{R}$ has the physical unit of an energy source density and can be inserted directly as a source for heat generation [7] in the parabolic equation

$$c\rho\Theta_t = \lambda\Delta\Theta + \langle \tau C : \nabla^s u_t, \nabla^s u_t \rangle, \quad (1.5)$$

for the temperature distribution Θ , where $c > 0$ and $\lambda > 0$ denote the heat capacity and the thermal conductivity of the material, respectively; in this sense, (1.5) thus couples thermal and mechanical effects. Now the main novelty to be considered in the present study stems from the observation that in some materials relevant to applications, temperature dependencies of the elastic parameters are not negligible. As an example, a parameter of the elasticity C of a piezoelectric material is shown in Fig. 1, indicating a near-linear increase over temperature within the considered range [18]. Thus led to considering situations when

$$C = C(\Theta),$$

we note that allowing for such types of dependencies further increases the complexity with respect to modelling the thermoelastic behavior of a component in general, and to thermal losses generated by acoustic waves in particular. In fact, it is to be expected that the overall thermal stability of a system may depend on a specific type of temperature dependence.

Specifying a class of initial-boundary value problems. Notation and main results. On supplementing (1.4) and (1.5) with prototypically simple boundary and initial conditions we arrive at the problem

$$\begin{cases} u_{tt} = \operatorname{div}(\gamma(\Theta) : \nabla^s u_t) + a \operatorname{div}(\gamma(\Theta) : \nabla^s u), & x \in \Omega, t > 0, \\ \Theta_t = D\Delta\Theta + \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle, & x \in \Omega, t > 0, \\ u = 0, \quad \Theta = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_{0t}(x), \quad \Theta(x, 0) = \Theta_0(x), & x \in \Omega, \end{cases} \quad (1.6)$$

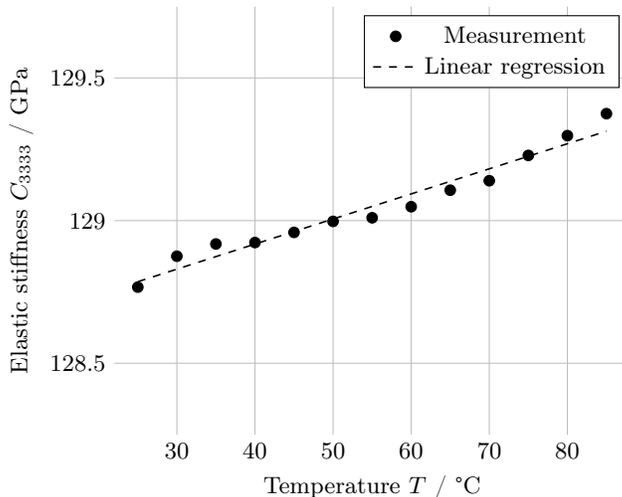


Figure 1: Measurement result for the temperature dependence of the elastic parameter C_{3333} of a piezoelectric ceramic, showing a near-linear relationship [18].

as the precise mathematical object to be subsequently considered; in comparison with the above, to prepare convenient notation throughout our analysis we have thus set $a = \frac{1}{\tau}$, $D = \frac{\lambda}{c\rho}$, $\gamma(\Theta) = \frac{\tau}{\rho}C(\Theta)$ and $\Gamma(\Theta) = \frac{\tau}{c\rho}C(\Theta)$, although we will not rely on any relation between γ and Γ . We note that (1.6) assumes temperature dependencies to be limited to the elasticity tensor, hence neglecting possible variations of the parameter a with respect to Θ . Apart from that, we underline that effects of thermoelasticity are not regarded within the scope of this study; in fact, including such mechanisms is well-known to bring about substantial challenges for the mathematical analysis already in cases when γ does not depend on Θ ([50], [6]). After all, at least in simple near-linear situations in which mechanical processes are purely harmonic, such mechanisms would lead to additional contributions which with respect to heat generation would involve production rates with vanishing temporal averages. In this sense, their neglect might be expected to be of minor impact in comparison to possible effects exerted by temperature dependencies in the key system constituent γ .

In order to formulate our results, and for further reference below, let us comment on the notation used throughout sequel. For matrices $Y = (Y_{ij})_{i,j \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n}$ and tensors $\beta = (\beta_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n \times n \times n}$, we write $\langle X, Y \rangle := \sum_{i,j=1}^n X_{ij}Y_{ij} \in \mathbb{R}$ and $X^t := (X^t)_{ij}$ with $(X^t)_{ij} := X_{ji}$ for $i, j \in \{1, \dots, n\}$, and define the matrix $\beta : X \in \mathbb{R}^{n \times n}$ by letting $(\beta : X)_{ij} := \sum_{k,l=1}^n \beta_{ijkl}X_{kl}$ for $i, j \in \{1, \dots, n\}$. Moreover, given vectors $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ we introduce a matrix $w \otimes z \in \mathbb{R}^{n \times n}$ by writing $(w \otimes z)_{ij} := w_i z_j$ for $i, j \in \{1, \dots, n\}$. Finally, for vector functions $\psi = (\psi_1, \dots, \psi_n) \in W^{1,1}(\Omega; \mathbb{R}^n)$ we let $\nabla \psi := (\partial_j \psi_i)_{i,j=1, \dots, n}$ and $\nabla^s \psi := \frac{1}{2}(\partial_j \psi_i + \partial_i \psi_j)_{i,j=1, \dots, n}$ denote its Jacobian and the associated symmetrized gradient, respectively.

Now in an attempt to undertake a first step toward an understanding of possible influences that temperature dependencies of the above type may have, this manuscript focuses on the issue of global solvability in (1.6). We again recall here that in the case when both γ and Γ are temperature-independent, then previous literature asserts global existence of so-called weak-renormalized solutions actually even in

slightly more complex variants of (1.6) in which, inter alia, some suitably mild thermoelastic effects can be admitted ([6]; cf. also [50] for a close relative addressing Neumann boundary conditions for Θ); we emphasize, however, that precedent studies in essential parts rely on a favorable energy structure which in the corresponding version of the simple system (1.6) with Θ -independent γ is formally expressed in the identity

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} |u_t|^2 + \frac{a}{2} \int_{\Omega} \langle \gamma : \nabla^s u, \nabla^s u \rangle \right\} = - \int_{\Omega} \langle \gamma : \nabla^s u_t, \nabla^s u_t \rangle. \quad (1.7)$$

Our main results now make sure that although this structure apparently breaks down when $\gamma = \gamma(\Theta)$, within a suitably generalized concept of solvability, to be discussed in more detail in Section 2, some global solution to (1.6) can be found under assumptions mild enough so as to be satisfied by any sufficiently smooth and bounded ingredients γ and Γ :

Theorem 1.1 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $a > 0$ and $D > 0$, and suppose that*

$$\begin{cases} \gamma = (\gamma_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap C^2([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \\ \Gamma = (\Gamma_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap C^2([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \end{cases} \quad \text{and} \quad (1.8)$$

are such that

$$\gamma_{ijkl}(\xi) = \gamma_{klij}(\xi) \quad \text{for all } \xi \geq 0 \text{ and } (i, j, k, l) \in \{1, \dots, n\}^4 \quad (1.9)$$

and

$$\gamma_{ijkl}(\xi) = \gamma_{jikl}(\xi) \quad \text{for all } \xi \geq 0 \text{ and } (i, j, k, l) \in \{1, \dots, n\}^4, \quad (1.10)$$

that

$$\Gamma_{ijkl}(\xi) = \Gamma_{klij}(\xi) \quad \text{for all } \xi \geq 0 \text{ and } (i, j, k, l) \in \{1, \dots, n\}^4, \quad (1.11)$$

and that

$$\langle \gamma(\xi) : X, X \rangle \geq K_\gamma |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n} \quad (1.12)$$

as well as

$$\langle \Gamma(\xi) : X, X \rangle \geq K_\Gamma |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n} \quad (1.13)$$

with some $K_\gamma > 0$ and $K_\Gamma > 0$. Then whenever

$$\begin{cases} u_0 \in W_0^{1,2}(\Omega; \mathbb{R}^n), \\ u_{0t} \in L^2(\Omega; \mathbb{R}^n) \quad \text{and} \\ \Theta_0 \in L^1(\Omega; \mathbb{R}) \text{ is nonnegative,} \end{cases} \quad (1.14)$$

there exist functions

$$\begin{cases} u \in L_{loc}^\infty([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)) \quad \text{and} \\ \Theta \in L_{loc}^\infty([0, \infty); L^1(\Omega; \mathbb{R})) \cap \bigcap_{q \in [1, \frac{n+2}{n})} L_{loc}^q(\bar{\Omega} \times [0, \infty); \mathbb{R}) \cap \bigcap_{r \in [1, \frac{n+2}{n+1})} L_{loc}^r([0, \infty); W_0^{1,r}(\Omega; \mathbb{R})) \end{cases} \quad (1.15)$$

such that

$$u_t \in L_{loc}^\infty([0, \infty); L^2(\Omega; \mathbb{R}^n)) \cap L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)), \quad (1.16)$$

and that (u, Θ) forms a global generalized solution of (1.6) in the sense of Definition 2.1 below.

We note that here the assumption in (1.10), as already required in previous literature ([50], [34], [6]) encodes the requirement from (1.1) that γ turns symmetric strains to symmetric stresses.

Main ideas. Due to the absence of an energy structure of the form in (1.7), a major challenge for any analysis of (1.6) seems to be linked to the essentially quadratic type of coupling therein: Already in the simple particular case when Γ coincides with the identity mapping on $\mathbb{R}^{n \times n}$, heat production occurs at a rate which is bounded from below by $|\nabla^s u_t|^2$. In the present setting, *a priori* knowledge for this quantity which is available without requirements on information about Θ seems to reduce to estimates of the form

$$\int_0^T \int_{\Omega} \langle \gamma(\Theta) : \nabla^s(u_t + au), \nabla^s(u_t + au) \rangle \leq C(T)$$

which reflect the dissipative action in the first equation from (1.6) in the framework of some associated zero-order energy inequality; even in the favorable setting of bounded and uniformly positive definite γ addressed in Theorem 1.1, this seems to merely yield L^2 bounds for ∇u_t – and hence, equivalently, for $\nabla^s u_t$ – in the sense of implying that

$$\left(\int_0^T \int_{\Omega} |\nabla u_t|^2 + \right) \int_0^T \int_{\Omega} |\nabla^s u_t|^2 \leq C(T)$$

for $T > 0$ (cf. Lemma 3.4). Accordingly, the heat source can apparently be controlled in the non-reflexive space $L^1(\Omega \times (0, T))$ for $T > 0$ only, which seems to go along with a lack of knowledge on suitable compactness properties in any meaningful approximation scheme, and hence seems to mark a crucial difference to the situation of Θ -independent γ addressed in [6] and [50], for instance.

To overcome related difficulties, the existence theory to be developed below resorts to a notion of generalized solvability that substantially deviates from those introduced in some precedent studies in which certain renormalized solution concepts for the corresponding temperature distributions were introduced, but in which standard weak solvability is considered with respect to the displacement variable ([6], [9], [10]). Specifically, with regard to the crucial solution component Θ our concept will require validity of two *inequalities*, instead of fulfillment of one identity such as in standard weak solution concepts. In contrast to some cases of rather far relatives in the recent literature on fully parabolic problems ([30], [62]), the approach pursued here in this regard will need to appropriately cope with the wave type structure of the first equation in (1.6), and with thus fairly restricted options to make use of dissipative features. Specifically, the core of our analysis, to be foreshadowed in Definition 2.1 and executed in Lemma 7.4, will examine quantities of the form

$$\mathcal{F} := \frac{1}{2}|u_t|^2 + \frac{\kappa}{2}|\nabla u|^2 + \lambda\Theta, \quad \kappa > 0, \lambda > 0, \quad (1.17)$$

which at a pointwise level couple the temperature field to the constituents $|u_t|^2$ and $|\nabla u|^2$ of the fundamental energy structure in the wave part of (1.6). A key observation will reveal that when here the free parameters κ and λ are chosen suitably, certain integrated versions of \mathcal{F} will indeed satisfy a one-sided inequality which can be viewed as providing an upper bound for $\partial_t \mathcal{F}$ that is optimal in the sense of being satisfied as an identity along smooth trajectories (see (2.5), Proposition 2.2 and (7.31)).

At a technical level, key parts of these considerations will rely on appropriate exploitation of weak

lower semicontinuity of norms in L^2 spaces, to be adapted to settings in which differences between such expressions and integrals involving terms of the form

$$\langle \beta(z) : \nabla^s w, \nabla^s w \rangle \quad (1.18)$$

occur, comparable to those in (1.9) and (1.12) (Section 6). Apart from that, due to limited information on regularity of $\frac{\partial \Theta}{\partial \nu}$ on $\partial \Omega$ (cf. Lemma 3.3 and Lemma 5.1), spatial localization of the arguments related to \mathcal{F} seems in order (see (2.8)).

2 A concept of generalized solvability. Approximate solutions

The following describes the notion of generalized solvability that will form the target object of our subsequent considerations. While essentially standard requirements on natural weak solvability with regard to the first equation in (1.6) are imposed, with respect to the component Θ an associated one-sided inequality, (2.4), is combined with the localized energy dissipation feature (2.5):

Definition 2.1 *Let $\gamma \in L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ and $\Gamma \in L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$, let $a > 0$ and $D > 0$, and suppose that $u_0 \in W_0^{1,2}(\Omega; \mathbb{R}^n)$, $u_{0t} \in L^2(\Omega; \mathbb{R}^n)$ and $\Theta_0 \in L^1(\Omega; \mathbb{R})$. Then a pair (u, Θ) of functions*

$$\begin{cases} u \in L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)) & \text{and} \\ \Theta \in L_{loc}^1([0, \infty); W_0^{1,1}(\Omega; \mathbb{R})) \end{cases} \quad (2.1)$$

will be called a global generalized solution of (1.6) if

$$u_t \in L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)) \quad (2.2)$$

and $\Theta \geq 0$ a.e. in $\Omega \times (0, \infty)$, if

$$\begin{aligned} & \int_0^\infty \int_\Omega u \cdot \varphi_{tt} + \int_\Omega u_0 \cdot \varphi_t(\cdot, 0) - \int_\Omega u_{0t} \cdot \varphi(\cdot, 0) \\ &= - \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u_t, \nabla \varphi \rangle - a \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, \nabla \varphi \rangle \end{aligned} \quad (2.3)$$

for all $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^n)$, if

$$- \int_0^\infty \int_\Omega \Theta \widehat{\varphi}_t - \int_\Omega \Theta_0 \widehat{\varphi}(\cdot, 0) \geq D \int_0^\infty \int_\Omega \Theta \Delta \widehat{\varphi} + \int_0^\infty \int_\Omega \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle \widehat{\varphi} \quad (2.4)$$

for each nonnegative $\widehat{\varphi} \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R})$, and if there exist $\kappa > 0$, $\lambda > 0$ and $\mu > 0$ with the property that for any nonnegative $\psi \in C_0^\infty(\Omega; \mathbb{R})$ and arbitrary nonincreasing $\zeta \in C_0^\infty([0, \infty); \mathbb{R})$, the inequality

$$\int_0^\infty \int_\Omega \mathcal{D}^{(\kappa, \lambda, \mu, \zeta, \psi)} \leq \zeta(0) \int_\Omega \mathcal{F}_0^{(\kappa, \lambda, \psi)} + \int_0^\infty \int_\Omega \mathcal{R}^{(\lambda, \mu, \zeta, \psi)} \quad (2.5)$$

holds, where

$$\mathcal{F}_0^{(\kappa, \lambda, \psi)} \equiv \mathcal{F}_0^{(\kappa, \lambda, \psi)}(x) := \left(\frac{1}{2} |u_{0t}|^2 + \frac{\kappa}{2} |\nabla u_0|^2 + \lambda \Theta_0 \right) \psi, \quad x \in \Omega, \quad (2.6)$$

and

$$\begin{aligned}
\mathcal{D}^{(\kappa, \lambda, \mu, \zeta, \psi)} &\equiv \mathcal{D}^{(\kappa, \lambda, \mu, \zeta, \psi)}(x, t) \\
&:= \left\{ \langle \gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle + a \langle \gamma(\Theta) : \nabla^s u, \nabla^s u_t \rangle \right. \\
&\quad \left. - \kappa \langle \nabla u, \nabla u_t \rangle - \lambda \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle \right\} \zeta(t) e^{-\mu t} \psi \\
&\quad + \mathcal{F}^{(\kappa, \lambda, \psi)}(\mu \zeta(t) - \zeta_t(t)) e^{-\mu t}, \quad x \in \Omega, \quad t > 0,
\end{aligned} \tag{2.7}$$

with

$$\mathcal{F}^{(\kappa, \lambda, \psi)} \equiv \mathcal{F}^{(\kappa, \lambda, \psi)}(x, t) := \left(\frac{1}{2} |u_t|^2 + \frac{\kappa}{2} |\nabla u|^2 + \lambda \Theta \right) \psi, \quad x \in \Omega, \quad t > 0, \tag{2.8}$$

and where

$$\begin{aligned}
\mathcal{R}^{(\lambda, \mu, \zeta, \psi)} &\equiv \mathcal{R}^{(\lambda, \mu, \zeta, \psi)}(x, t) \\
&:= \left\{ \langle \gamma(\Theta) : \nabla^s u_t, u_t \otimes \nabla \psi \rangle + a \langle \gamma(\Theta) : \nabla^s u, u_t \otimes \nabla \psi \rangle + \lambda D \Theta \Delta \psi \right\} \zeta(t) e^{-\mu t}, \\
&\quad x \in \Omega, \quad t > 0.
\end{aligned} \tag{2.9}$$

Consistency of this concept with that of classical solvability is underlined by the following observation.

Proposition 2.2 *Let γ and Γ belong to $L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap C^1([0, \infty); \mathbb{R}^{n \times n \times n \times n})$, let $a > 0$ and $D > 0$, and let $u_0 \in C^0(\bar{\Omega}; \mathbb{R}^n)$, $u_{0t} \in C^0(\bar{\Omega}; \mathbb{R}^n)$, $\Theta_0 \in C^0(\bar{\Omega}; \mathbb{R})$ as well as*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, \infty); \mathbb{R}^n) \cap C^{2,1}(\Omega \times (0, \infty); \mathbb{R}^n) & \text{and} \\ \Theta \in C^0(\bar{\Omega} \times [0, \infty); \mathbb{R}) \cap C^{2,1}(\Omega \times (0, \infty); \mathbb{R}) \end{cases}$$

be such that

$$\begin{aligned}
u_t &\in C^0(\bar{\Omega} \times [0, \infty); \mathbb{R}^n) \cap C^{2,1}(\Omega \times (0, \infty); \mathbb{R}^n) \\
t &\mapsto \int_{\Omega} |\nabla u(\cdot, t)|^2 \quad \text{is continuous,} \\
u_{tt} &\in L^1_{loc}(\Omega \times [0, \infty)), \quad \Theta_t, \Delta \Theta \in L^1_{loc}(\Omega \times [0, \infty))
\end{aligned} \tag{2.10}$$

and that (u, Θ) forms a global generalized solution of (1.6) in the sense of Definition 2.1. Then (1.6) is actually satisfied in the classical sense.

PROOF. That $u = 0$ and $\Theta = 0$ on $\bar{\Omega} \times (0, \infty)$ immediately results from the functions belonging to $L^2_{loc}([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n))$ and $L^1_{loc}([0, \infty); W_0^{1,1}(\Omega; \mathbb{R}^n))$, respectively, and their continuity.

If we insert arbitrary $\varphi \in C_0^\infty(\Omega \times (0, \infty))$ in (2.3) and integrate by parts twice with respect to time and once to space (which is obviously possible, because all of the integrands φu_{tt} , $\varphi_t u_t$, $\varphi_{tt} u$, $\langle \gamma(\Theta) : \nabla^s u_t, \nabla \varphi \rangle + a \langle \gamma(\Theta) : \nabla^s u, \nabla \varphi \rangle$, $\text{div}(\gamma(\Theta) : \nabla^s (u_t + a u)) \varphi$ are continuous on $\text{supp } \varphi$), from the fundamental lemma of the calculus of variations (applicable, since u_{tt} and $\text{div}(\gamma(\Theta) : \nabla^s (u_t + a u))$ belong to $L^1_{loc}(\Omega \times (0, \infty))$), we obtain that the first equation of (1.6) is satisfied at every point in $\Omega \times (0, \infty)$. For $\psi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ inserting $\varphi(x, t) = \psi(x) \left(\frac{t}{\varepsilon} - \frac{2t^2}{\varepsilon^2} + \frac{t^3}{\varepsilon^3} \right) \chi_{[0, \varepsilon]}(t)$ or $\varphi(x, t) = \psi(x) \left(1 - \frac{2t^2}{\varepsilon^2} + \frac{t^4}{\varepsilon^4} \right) \chi_{[0, \varepsilon]}(t)$

in (2.3) and taking the limit $\varepsilon \searrow 0$ (using continuity of u and u_t at $t = 0$) shows that u fulfils the initial conditions.

From (2.10) and an analogous treatment of (2.4) we find that $\Theta_t \geq D\Delta\Theta + \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle$ in $\Omega \times (0, \infty)$ and $\Theta(0) \geq \Theta_0$ in Ω .

From a choice of $\zeta(t) := (1 - \frac{t}{\varepsilon})_+$ in (2.5) due to continuity of $t \mapsto \int_{\Omega} \mathcal{F}^{(\kappa, \lambda, \psi)}(\cdot, t)$,

$$\int_{\Omega} \left(\frac{1}{2} |u_t(\cdot, 0)|^2 + \frac{\kappa}{2} |\nabla u(\cdot, 0)|^2 + \lambda \Theta(\cdot, 0) \right) \psi \leq \int_{\Omega} \left(\frac{1}{2} |u_{0t}|^2 + \frac{\kappa}{2} |\nabla u_0|^2 + \lambda \Theta_0 \right) \psi$$

for each nonnegative $\psi \in C_0^\infty(\Omega)$ and hence $\Theta(0) \leq \Theta_0$ in Ω .

Accordingly, for any $\psi \in C_0^\infty(\Omega)$ and $\zeta \in C_0^\infty([0, \infty))$,

$$\begin{aligned} \int_0^\infty \int_{\Omega} \left(\frac{1}{2} |u_t|^2 + \frac{\kappa}{2} |\nabla u|^2 + \lambda \Theta \right) \psi (-\zeta e^{-\mu t})_t &= \\ \int_0^\infty \int_{\Omega} (u_t u_{tt} + \kappa \langle \nabla u, \nabla^s u_t \rangle + \lambda \Theta_t) \psi \zeta e^{-\mu t} + \zeta(0) \int_{\Omega} \left(|u_{0t}|^2 + \frac{\kappa}{2} |\nabla u_0|^2 + \lambda \Theta_0 \right) \psi & \end{aligned}$$

since $u_{tt}, \Theta_t \in L_{loc}^1(\Omega \times [0, \infty))$. If we insert the equality for u and integrate by parts with respect to space, many terms in (2.5) are cancelled and we see that

$$\int_0^\infty \int_{\Omega} (-\lambda \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle + \lambda \Theta_t) \zeta(t) e^{-\mu t} \psi \leq \int_0^\infty \int_{\Omega} \lambda D\Delta\Theta \psi \zeta(t) e^{-\mu t},$$

which finally due to $\Delta\Theta \in L_{loc}^1(\Omega \times [0, \infty))$ shows the remaining inequality $\Theta_t \leq D\Delta\Theta + \langle \Gamma(\Theta) : \nabla^s u_t, \nabla^s u_t \rangle$. □

Throughout the sequel, we shall consider Ω, a, D, γ and Γ as well as u_0, u_{0t} and Θ_0 to be fixed and such that the assumptions of Theorem 1.1 are met. We can then pick $(u_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset C_0^\infty(\Omega; \mathbb{R}^n)$, $(u_{0t\varepsilon})_{\varepsilon \in (0, 1)} \subset C_0^\infty(\Omega; \mathbb{R}^n)$ and $(\Theta_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset C_0^\infty(\Omega; \mathbb{R})$ in such a way that $\Theta_{0\varepsilon} \geq 0$ in Ω for all $\varepsilon \in (0, 1)$, and that

$$u_{0\varepsilon} \rightarrow u_0 \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^n), \quad u_{0t\varepsilon} \rightarrow u_{0t} \quad \text{in } L^2(\Omega; \mathbb{R}^n) \quad \text{and} \quad \Theta_{0\varepsilon} \rightarrow \Theta_0 \quad \text{in } L^1(\Omega; \mathbb{R}) \quad \text{as } \varepsilon \searrow 0. \quad (2.11)$$

In line with standard theory of local solvability in parabolic systems ([1]), this particularly ensures that for each $\varepsilon \in (0, 1)$, the regularized variant of (1.6) given by

$$\begin{cases} v_{\varepsilon t} = -\varepsilon \Delta^2 v_\varepsilon + \operatorname{div}(\gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon) + a \operatorname{div}(\gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon), & x \in \Omega, \quad t > 0, \\ u_{\varepsilon t} = \varepsilon \Delta u_\varepsilon + v_\varepsilon, & x \in \Omega, \quad t > 0, \\ \Theta_{\varepsilon t} = D\Delta\Theta_\varepsilon + \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle, & x \in \Omega, \quad t > 0, \\ v_\varepsilon = 0, \quad \Delta v_\varepsilon = 0, \quad u_\varepsilon = 0, \quad \Theta_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ v_\varepsilon(x, 0) = u_{0t\varepsilon}(x), \quad u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad \Theta_\varepsilon(x, 0) = \Theta_{0\varepsilon}(x), & x \in \Omega, \end{cases} \quad (2.12)$$

admits a local-in-time classical solution in the following sense:

Lemma 2.3 *Let $\varepsilon \in (0, 1)$. Then there exist $T_{max,\varepsilon} \in (0, \infty]$ as well as functions*

$$\begin{cases} v_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon}]; \mathbb{R}^n) \cap C^{4,1}(\bar{\Omega} \times (0, T_{max,\varepsilon}); \mathbb{R}^n), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon}]; \mathbb{R}^n) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon}); \mathbb{R}^n) \cap C^0([0, T_{max,\varepsilon}]; W_0^{1,2}(\Omega; \mathbb{R}^n)) \\ \Theta_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon}]; \mathbb{R}) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon}); \mathbb{R}) \end{cases} \quad \text{and}$$

such that $\Theta_\varepsilon \geq 0$ in $\bar{\Omega} \times [0, T_{max,\varepsilon})$, that $(v_\varepsilon, u_\varepsilon, \Theta_\varepsilon)$ solves (2.12) classically in $\Omega \times (0, T_{max,\varepsilon})$, and that

if $T_{max,\varepsilon} < \infty$, then for all $\eta > 0$,

$$\limsup_{t \nearrow T_{max,\varepsilon}} \left\{ \|v_\varepsilon(\cdot, t)\|_{W^{2+2\eta,\infty}(\Omega)} + \|u_\varepsilon(\cdot, t)\|_{W^{1+\eta,\infty}(\Omega)} + \|\Theta_\varepsilon(\cdot, t)\|_{W^{1+\eta,\infty}(\Omega)} \right\} = \infty \quad (2.13)$$

PROOF. For any $p > n$, we let $E_1 = \{(v, u, \Theta) \in W^{4,p}(\Omega) \times W^{2,p}(\Omega) \times W^{2,p}(\Omega) \mid v|_{\partial\Omega} = 0, \Delta v|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0, \Theta|_{\partial\Omega} = 0\}$ and $E_0 = (L^p(\Omega))^3$ and denote by $E_\gamma = (E_0, E_1)_\gamma$ the interpolation space of order $\gamma \in (0, 1)$, let $f \equiv 0$ and

$$A(v, u, \Theta) = \begin{pmatrix} \varepsilon \Delta^2 - \operatorname{div}(\gamma(\Theta) : \nabla^s \cdot) & -a \operatorname{div}(\gamma(\Theta) \nabla^s \cdot) \\ & -\varepsilon \Delta \\ -\langle \Gamma(\Theta) : \nabla^s v : \nabla^s \cdot \rangle & & -D\Delta \end{pmatrix} \in L(E_1, E_0), \quad (u, v, \Theta) \in E_\beta.$$

Since for $(v_1, u_1, \Theta_1), (v_2, u_2, \Theta_2) \in E_\beta$ and $(\tilde{v}, \tilde{u}, \tilde{\Theta}) \in E_1$,

$$\begin{aligned} & \|\operatorname{div}((\gamma(\Theta_1) - \gamma(\Theta_2)) \nabla^s \tilde{u})\|_{L^p(\Omega)} \\ & \leq \|((\gamma(\Theta_1) - \gamma(\Theta_2)) D^2 \tilde{u})\|_{L^p(\Omega)} + \|(\gamma'(\Theta_1) - \gamma'(\Theta_2)) \nabla \Theta_1 \nabla^s \tilde{u}\|_{L^p(\Omega)} + \|\gamma'(\Theta_2) (\nabla \Theta_1 - \nabla \Theta_2) \nabla^s \tilde{u}\|_{L^p(\Omega)} \\ & \leq (\|\gamma(\Theta_1) - \gamma(\Theta_2)\|_{L^\infty(\Omega)} + \|(\gamma'(\Theta_1) - \gamma'(\Theta_2)) \nabla \Theta_1\|_{L^p(\Omega)} + \|\gamma'(\Theta_2) (\nabla \Theta_1 - \nabla \Theta_2)\|_{L^p(\Omega)}) \|\tilde{u}\|_{W^{2,p}} \end{aligned}$$

and thus

$$\begin{aligned} & \|\operatorname{div}((\gamma(\Theta_1) - \gamma(\Theta_2)) \nabla^s \cdot)\|_{L(E_1, E_0)} \\ & \leq \|\gamma(\Theta_1) - \gamma(\Theta_2)\|_{L^\infty(\Omega)} + \|(\gamma'(\Theta_1) - \gamma'(\Theta_2)) \nabla \Theta_1\|_{L^p(\Omega)} + \|\gamma'(\Theta_2) (\nabla \Theta_1 - \nabla \Theta_2)\|_{L^p(\Omega)} \end{aligned}$$

and related estimates for the other terms show the required Lipschitz continuity of $A: E_\beta \rightarrow L(E_1, E_0)$ for $\beta = \frac{1}{2}$, we may employ the general existence result of [1, Thm. 12.1], obtaining a solution and deriving an extensibility criterion in E_δ for $\delta > \frac{1}{2}$ and thus, in consequence, (2.13), from [1, Thm. 12.5]. \square

3 Basic testing procedures. *A priori* estimates for v_ε and u_ε

A first testing procedure applied to (2.12) is designed here in such a way that not only the derivation of spatially global estimates is prepared (Lemma 3.4), but that later on also our analysis of (2.5) can be built on this (see Lemma 7.4).

Lemma 3.1 *Let $\psi \in C^2(\bar{\Omega}; \mathbb{R})$ and $\varepsilon \in (0, 1)$. Then*

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} (|v_{\varepsilon}|^2)_t \psi + \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi + \varepsilon \int_{\Omega} |\Delta v_{\varepsilon}|^2 \psi \\
&= -a \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s u_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi \\
&\quad - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, v_{\varepsilon} \otimes \nabla \psi \rangle - a \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s u_{\varepsilon}, v_{\varepsilon} \otimes \nabla \psi \rangle \\
&\quad - 2\varepsilon \int_{\Omega} (\nabla v_{\varepsilon} \cdot \nabla \psi) \cdot \Delta v_{\varepsilon} - \varepsilon \int_{\Omega} (v_{\varepsilon} \cdot \Delta v_{\varepsilon}) \Delta \psi \quad \text{for all } t \in (0, T_{max, \varepsilon}). \tag{3.1}
\end{aligned}$$

PROOF. According to the first equation in (2.12),

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} (|v_{\varepsilon}|^2)_t \psi &= \int_{\Omega} v_{\varepsilon} \cdot \operatorname{div} (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}) \psi + a \int_{\Omega} v_{\varepsilon} \cdot \operatorname{div} (\gamma(\Theta_{\varepsilon}) : \nabla^s u_{\varepsilon}) \psi \\
&\quad - \varepsilon \int_{\Omega} (v_{\varepsilon} \cdot \Delta^2 v_{\varepsilon}) \psi \quad \text{for all } t \in (0, T_{max, \varepsilon}), \tag{3.2}
\end{aligned}$$

where thanks to the boundary conditions $v_{\varepsilon} = 0$ and $\Delta v_{\varepsilon} = 0$ on $\partial\Omega \times (0, T_{max, \varepsilon})$, two integrations by parts show that for all $t \in (0, T_{max, \varepsilon})$,

$$\begin{aligned}
-\varepsilon \int_{\Omega} (v_{\varepsilon} \cdot \Delta^2 v_{\varepsilon}) \psi &= -\varepsilon \int_{\Omega} \Delta v_{\varepsilon} \cdot \Delta (v_{\varepsilon} \cdot \psi) \\
&= -\varepsilon \int_{\Omega} |\Delta v_{\varepsilon}|^2 \psi - 2\varepsilon \int_{\Omega} (\nabla v_{\varepsilon} \cdot \nabla \psi) \cdot \Delta v_{\varepsilon} - \varepsilon \int_{\Omega} (v_{\varepsilon} \cdot \Delta v_{\varepsilon}) \Delta \psi. \tag{3.3}
\end{aligned}$$

Again since $v_{\varepsilon} \equiv (v_{\varepsilon 1}, \dots, v_{\varepsilon n}) = 0$ on $\partial\Omega \times (0, T_{max, \varepsilon})$, by another integration by parts we find that

$$\begin{aligned}
\int_{\Omega} v_{\varepsilon} \cdot \operatorname{div} (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}) \psi &= \sum_{i=1}^n \int_{\Omega} \psi v_{\varepsilon i} \left(\operatorname{div} (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}) \right)_i \\
&= \sum_{i,j=1}^n \int_{\Omega} \psi v_{\varepsilon i} \partial_j (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon})_{ij} \\
&= - \sum_{i,j=1}^n \int_{\Omega} \partial_j (\psi v_{\varepsilon i}) (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon})_{ij} \\
&= - \sum_{i,j=1}^n \int_{\Omega} \psi \partial_j v_{\varepsilon i} (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon})_{ij} - \sum_{i,j=1}^n \int_{\Omega} \partial_j \psi v_{\varepsilon i} (\gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon})_{ij} \\
&= - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla v_{\varepsilon} \rangle \psi - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, v_{\varepsilon} \otimes \nabla \psi \rangle \\
&= - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, v_{\varepsilon} \otimes \nabla \psi \rangle \tag{3.4}
\end{aligned}$$

for all $t \in (0, T_{max, \varepsilon})$, because our assumption (1.10) guarantees that

$$\langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, (\nabla v_{\varepsilon})^t \rangle = \sum_{i,j,k,l=1}^n \gamma_{ijkl}(\Theta_{\varepsilon}) (\partial_l v_{\varepsilon k} + \partial_k v_{\varepsilon l}) \partial_i v_{\varepsilon j}$$

$$\begin{aligned}
&= \sum_{i,j,k,l=1}^n \gamma_{jikl}(\Theta_\varepsilon)(\partial_l v_{\varepsilon k} + \partial_k v_{\varepsilon l}) \partial_i v_{\varepsilon j} \\
&= \sum_{i',j',k,l=1}^n \gamma_{i'j'kl}(\Theta_\varepsilon)(\partial_l v_{\varepsilon k} + \partial_k v_{\varepsilon l}) \partial_{j'} v_{\varepsilon i'} \\
&= \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla v_\varepsilon \rangle
\end{aligned}$$

and hence

$$\langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla v_\varepsilon \rangle = \frac{1}{2} \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla v_\varepsilon \rangle + \frac{1}{2} \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, (\nabla v_\varepsilon)^t \rangle = \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle$$

in $\Omega \times (0, T_{max,\varepsilon})$. As, similarly,

$$a \int_{\Omega} v_\varepsilon \cdot \operatorname{div} (\gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon) \psi = -a \int_{\Omega} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle - a \int_{\Omega} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle$$

for all $t \in (0, T_{max,\varepsilon})$, combining (3.2) with (3.3) and (3.4) leads to (3.1). \square

A second basic feature of (2.12) is rather evident.

Lemma 3.2 *If $\psi \in C^1(\bar{\Omega}; \mathbb{R})$ and $\varepsilon \in (0, 1)$, then*

$$\frac{1}{2} \int_{\Omega} (|\nabla u_\varepsilon|^2)_t \psi + \varepsilon \int_{\Omega} |\Delta u_\varepsilon|^2 \psi = \int_{\Omega} \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \psi - \varepsilon \int_{\Omega} (\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon \quad (3.5)$$

for all $t \in (0, T_{max,\varepsilon})$.

PROOF. From the second equation in (2.12), we obtain the identity

$$\frac{1}{2} \int_{\Omega} (|\nabla u_\varepsilon|^2)_t \psi = \int_{\Omega} \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \psi + \varepsilon \int_{\Omega} \langle \psi \nabla u_\varepsilon, \nabla \Delta u_\varepsilon \rangle \quad \text{for all } t \in (0, T_{max,\varepsilon}). \quad (3.6)$$

Since the boundary condition $u_\varepsilon = (u_{\varepsilon 1}, \dots, u_{\varepsilon n}) = 0$ on $\partial\Omega \times (0, T_{max,\varepsilon})$ implies that also $u_{\varepsilon t}|_{\partial\Omega \times (0, T_{max,\varepsilon})} = 0$, and since thus $\Delta u_\varepsilon = \frac{u_{\varepsilon t} - v_\varepsilon}{\varepsilon} = 0$ on $\partial\Omega \times (0, T_{max,\varepsilon})$, we may here integrate by parts without encountering nonzero boundary integrals, thereby confirming that

$$\begin{aligned}
\varepsilon \int_{\Omega} \langle \psi \nabla u_\varepsilon, \nabla \Delta u_\varepsilon \rangle &= \varepsilon \sum_{i,j=1}^n \int_{\Omega} \psi \partial_j u_{\varepsilon i} \partial_j \Delta u_{\varepsilon i} \\
&= -\varepsilon \sum_{i,j=1}^n \int_{\Omega} \psi \partial_{jj} u_{\varepsilon i} \Delta u_{\varepsilon i} - \varepsilon \sum_{i,j=1}^n \int_{\Omega} \partial_j \psi \partial_j u_{\varepsilon i} \Delta u_{\varepsilon i} \\
&= -\varepsilon \int_{\Omega} |\Delta u_\varepsilon|^2 \psi - \varepsilon \int_{\Omega} (\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon \quad \text{for all } t \in (0, T_{max,\varepsilon}).
\end{aligned}$$

Therefore, (3.6) is equivalent to (3.5). \square

We furthermore record another simple property of solutions to (2.12).

Lemma 3.3 *Whenever $\psi \in C^2(\bar{\Omega}; \mathbb{R})$ and $\varepsilon \in (0, 1)$, we have*

$$\int_{\Omega} \Theta_{\varepsilon t} \psi = D \int_{\Omega} \Theta_{\varepsilon} \Delta \psi + D \int_{\partial\Omega} \frac{\partial \Theta_{\varepsilon}}{\partial \nu} \psi + \int_{\Omega} \langle \Gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi \quad \text{for all } t \in (0, T_{max, \varepsilon}). \quad (3.7)$$

PROOF. From (2.12) we obtain that

$$\int_{\Omega} \Theta_{\varepsilon t} \psi = D \int_{\Omega} \Delta \Theta_{\varepsilon} \psi + \int_{\Omega} \langle \Gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi \quad \text{for all } t \in (0, T_{max, \varepsilon}),$$

and since two integrations by parts relying on the identity $\Theta_{\varepsilon}|_{\partial\Omega \times (0, T_{max, \varepsilon})} = 0$ show that

$$\begin{aligned} D \int_{\Omega} \Delta \Theta_{\varepsilon} \psi &= -D \int_{\Omega} \nabla \Theta_{\varepsilon} \cdot \nabla \psi + D \int_{\partial\Omega} \frac{\partial \Theta_{\varepsilon}}{\partial \nu} \psi \\ &= D \int_{\Omega} \Theta_{\varepsilon} \Delta \psi + D \int_{\partial\Omega} \frac{\partial \Theta_{\varepsilon}}{\partial \nu} \psi \quad \text{for all } t \in (0, T_{max, \varepsilon}), \end{aligned}$$

this yields (3.7). □

Choosing $\psi \equiv 1$ in (3.1) and (3.5) lets a linear combination of the resulting identities become an essentially straightforward energy analysis of the wave subsystem of (2.12), leading to fairly natural results as follows.

Lemma 3.4 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\int_{\Omega} |v_{\varepsilon}(\cdot, t)|^2 \leq C(T) \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1) \quad (3.8)$$

and

$$\int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2 \leq C(T) \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1), \quad (3.9)$$

that

$$\int_0^t \int_{\Omega} |\nabla v_{\varepsilon}|^2 \leq C(T) \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1) \quad (3.10)$$

and that

$$\varepsilon \int_0^t \int_{\Omega} |\Delta v_{\varepsilon}|^2 \leq C(T) \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1) \quad (3.11)$$

as well as

$$\varepsilon \int_0^t \int_{\Omega} |\Delta u_{\varepsilon}|^2 \leq C(T) \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1). \quad (3.12)$$

PROOF. On choosing $\psi \equiv 1$ therein, from Lemma 3.1 and Lemma 3.2 we infer that

$$y_{\varepsilon}(t) := \int_{\Omega} |v_{\varepsilon}(\cdot, t)|^2 + \int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^2, \quad t \in [0, T_{max, \varepsilon}], \varepsilon \in (0, 1), \quad (3.13)$$

satisfies

$$y'_{\varepsilon}(t) + 2 \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle + 2\varepsilon \int_{\Omega} |\Delta v_{\varepsilon}|^2 + 2\varepsilon \int_{\Omega} |\Delta u_{\varepsilon}|^2$$

$$= -2a \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s u_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle + 2 \int_{\Omega} \langle \nabla u_{\varepsilon}, \nabla v_{\varepsilon} \rangle \quad \text{for all } t \in (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1). \quad (3.14)$$

Here, a combination of (1.12) with Korn's inequality (see e.g. [35, 29]) shows that with some $c_1 > 0$ we have

$$2 \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \geq 2K_{\gamma} \int_{\Omega} |\nabla^s v_{\varepsilon}|^2 \geq c_1 \int_{\Omega} |\nabla v_{\varepsilon}|^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1),$$

while the boundedness of γ on $[0, \infty)$ ensures the existence of $c_2 > 0$ such that thanks to Young's inequality,

$$\begin{aligned} -2a \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s u_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle &\leq c_2 \int_{\Omega} |\nabla u_{\varepsilon}| |\nabla v_{\varepsilon}| \\ &\leq \frac{c_1}{4} \int_{\Omega} |\nabla v_{\varepsilon}|^2 + \frac{c_2^2}{c_1} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1). \end{aligned}$$

As moreover, again by Young's inequality,

$$2 \int_{\Omega} \langle \nabla u_{\varepsilon}, \nabla v_{\varepsilon} \rangle \leq \frac{c_1}{4} \int_{\Omega} |\nabla v_{\varepsilon}|^2 + \frac{4}{c_1} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1), \quad (3.15)$$

from (3.14) and (3.13) we conclude that

$$y'_{\varepsilon}(t) + h_{\varepsilon}(t) \leq c_3 y_{\varepsilon}(t) \quad \text{for all } t \in (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1),$$

where $c_3 := \frac{c_2^2}{c_1} + \frac{4}{c_1}$, and where

$$h_{\varepsilon}(t) := \frac{c_1}{2} \int_{\Omega} |\nabla v_{\varepsilon}(\cdot, t)|^2 + 2\varepsilon \int_{\Omega} |\Delta v_{\varepsilon}(\cdot, t)|^2 + 2\varepsilon \int_{\Omega} |\Delta u_{\varepsilon}(\cdot, t)|^2, \quad t \in (0, T_{max, \varepsilon}), \varepsilon \in (0, 1).$$

By nonnegativity of h_{ε} for $\varepsilon \in (0, 1)$, using Gronwall's inequality we firstly infer from (3.15) that

$$y_{\varepsilon}(t) \leq c_4 := \left\{ \sup_{\varepsilon \in (0, 1)} y_{\varepsilon}(0) \right\} e^{c_3 T} \quad \text{for all } t \in [0, T] \cap [0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1), \quad (3.16)$$

with c_4 being finite due to (2.11). This directly yields (3.8) and (3.9), whereas (3.10), (3.11) and (3.12) result from an integration in (3.15), which in view of (3.16) shows that

$$\int_0^t h_{\varepsilon}(s) ds \leq y_{\varepsilon}(0) + c_3 \int_0^t y_{\varepsilon}(s) ds \leq c_4 + c_3 c_4 T \quad \text{for all } t \in (0, T) \cap (0, T_{max, \varepsilon}) \text{ and } \varepsilon \in (0, 1),$$

namely. □

4 Global existence in the regularized problems

For fixed $\varepsilon \in (0, 1)$, due to the fourth-order parabolic regularization in the first equation from (2.12) the moderate information on L^2 -boundedness of v_{ε} in (3.8) is already sufficient to ensure bounds for this quantity with respect to the norm in any of the spaces $W^{s,p}(\Omega; \mathbb{R}^n)$ with $s < 3$ and $p < \infty$; indeed:

Lemma 4.1 *Let $p \geq 2$, and let A denote the realization of Δ^2 under the boundary conditions $(\cdot)|_{\partial\Omega} = 0$ and $\Delta(\cdot)|_{\partial\Omega} = 0$ in $L^p(\Omega; \mathbb{R}^n)$, with domain given by $D(A) := \{\psi \in W^{4,p}(\Omega; \mathbb{R}^n) \mid \psi = \Delta\psi = 0 \text{ on } \partial\Omega\}$. Then whenever $\varepsilon \in (0, 1)$ is such that $T_{max,\varepsilon} < \infty$, for each $\alpha \in (\frac{1}{4}, \frac{3}{4})$ there exists $C(\alpha, \varepsilon) > 0$ such that the corresponding fractional power satisfies*

$$\|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C(\alpha, \varepsilon) \quad \text{for all } t \in (\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}). \quad (4.1)$$

PROOF. We first note that according to known smoothing properties of the analytic semigroup $(e^{-t\varepsilon A})_{t \geq 0}$ ([17]) and a duality argument in the flavor of [25, Lemma 2.1], one can find $\alpha' = \alpha'(\alpha) \in (\frac{1}{4}, \frac{3}{4})$, $c_1 = c_1(\alpha, \varepsilon) > 0$ and $c_2 = c_2(\alpha, \varepsilon) > 0$ such that

$$\|A^\alpha e^{-t\varepsilon A} \operatorname{div} \psi\|_{L^p(\Omega)} \leq c_1 t^{-\alpha' - \frac{1}{4}} \|\psi\|_{L^p(\Omega)} \quad \text{for all } t > 0 \text{ and each } \psi \in C^1(\bar{\Omega}; \mathbb{R}^n) \text{ fulfilling } \psi|_{\partial\Omega} = 0,$$

and that

$$\|A^\alpha e^{-t\varepsilon A} \psi\|_{L^p(\Omega)} \leq c_2 \|\psi\|_{W^{4,p}(\Omega)} \quad \text{for all } t > 0 \text{ and any } \psi \in D(A).$$

Then writing $t_0 := \frac{1}{2}T_{max,\varepsilon}$ and using a Duhamel representation associated with the first equation in (2.12), we see that for all $t \in (t_0, T_{max,\varepsilon})$,

$$\begin{aligned} \|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} &= \left\| A^\alpha e^{-(t-t_0)\varepsilon A} v_\varepsilon(\cdot, t_0) + \int_{t_0}^t A^\alpha e^{-(t-s)\varepsilon A} \operatorname{div} \{ \gamma(\Theta_\varepsilon) : \nabla^s (v_\varepsilon + au_\varepsilon) \}(\cdot, s) ds \right\|_{L^p(\Omega)} \\ &\leq \|A^\alpha e^{-(t-t_0)\varepsilon A} v_\varepsilon(\cdot, t_0)\|_{L^p(\Omega)} \\ &\quad + c_1 \int_{t_0}^t (t-s)^{-\alpha' - \frac{1}{4}} \|\gamma(\Theta_\varepsilon) : \nabla^s (v_\varepsilon + au_\varepsilon)\|_{L^p(\Omega)}(\cdot, s) ds, \end{aligned}$$

so that since $v_\varepsilon(\cdot, t_0) \in D(A)$ by Lemma 2.3, and since γ is bounded on $[0, \infty)$, with some $c_3 = c_3(\alpha, \varepsilon) > 0$ we have

$$\|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq c_3 + c_3 \int_{t_0}^t (t-s)^{-\alpha' - \frac{1}{4}} \{ \|v_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} + \|u_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} \} ds \quad (4.2)$$

for all $t \in (t_0, T_{max,\varepsilon})$. Since a standard interpolation property involving fractional powers of A ([17]) provides $\vartheta_1 = \vartheta_1(\alpha) \in (0, 1)$ and $c_4 = c_4(\alpha) > 0$ fulfilling

$$\|v_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} \leq c_4 \|A^\alpha v_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^{\vartheta_1} \|v_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^{1-\vartheta_1} \quad \text{for all } s \in (t_0, T_{max,\varepsilon}),$$

and since the Gagliardo-Nirenberg inequality together with (3.8) shows that with some $\vartheta_2 \in (0, 1)$, $c_5 > 0$ and $c_6 > 0$ we have

$$\|v_\varepsilon(\cdot, s)\|_{L^p(\Omega)} \leq c_5 \|v_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)}^{\vartheta_2} \|v_\varepsilon(\cdot, s)\|_{L^2(\Omega)}^{1-\vartheta_2} \leq c_6 \|v_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)}^{\vartheta_2} \quad \text{for all } s \in (t_0, T_{max,\varepsilon}),$$

we readily infer the existence of $\vartheta_3 = \vartheta_3(\alpha) \in (0, 1)$ and $c_7 = c_7(\alpha, \varepsilon) > 0$ such that if for $T \in (t_0, T_{max,\varepsilon})$ we let

$$M(T) := \sup_{t \in (t_0, T)} \|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)},$$

then

$$\|v_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} \leq c_7 \|A^\alpha v_\varepsilon(\cdot, s)\|_{L^p(\Omega)}^{\vartheta_3} \leq c_7 M^{\vartheta_3}(T) \quad \text{for all } s \in (t_0, T) \text{ and } T \in (t_0, T_{max,\varepsilon}). \quad (4.3)$$

As $\alpha' + \frac{1}{4} < 1$, (4.2) therefore implies the existence of $c_8 = c_8(\alpha, \varepsilon) > 0$ such that

$$\begin{aligned} \|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} &\leq c_3 + c_3 c_7 M^{\vartheta_3}(T) \int_{t_0}^t (t-s)^{-\alpha' - \frac{1}{4}} ds + c_3 \int_{t_0}^t (t-s)^{-\alpha' - \frac{1}{4}} \|u_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} ds \\ &\leq c_8(1 + M^{\vartheta_3}(T)) + c_3 \int_{t_0}^t (t-s)^{-\alpha' - \frac{1}{4}} \|u_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} ds \end{aligned} \quad (4.4)$$

for all $t \in (t_0, T)$ and $T \in (t_0, T_{max,\varepsilon})$, and in order to appropriately estimate the rightmost summand herein, we similarly employ regularity features of the Dirichlet heat semigroup $(e^{-t\varepsilon A_2})_{t \geq 0}$, where A_2 realizes $-\Delta$ in $L^p(\Omega; \mathbb{R}^n)$ in the domain $D(A_2) := \{\psi \in W^{2,p}(\Omega; \mathbb{R}^n) \mid \psi = 0 \text{ on } \partial\Omega\}$, and thereby obtain $c_9 = c_9(\alpha, \varepsilon) > 0$ such that

$$\begin{aligned} \|u_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} &= \left\| e^{-(t-t_0)\varepsilon A_2} u_\varepsilon(\cdot, t_0) + \int_{t_0}^t e^{-(t-s)\varepsilon A_2} v_\varepsilon(\cdot, \sigma) d\sigma \right\|_{W^{1,p}(\Omega)} \\ &\leq c_9 \|u_\varepsilon(\cdot, t_0)\|_{W^{1,p}(\Omega)} + c_9 \int_{t_0}^t \|v_\varepsilon(\cdot, \sigma)\|_{W^{1,p}(\Omega)} d\sigma \quad \text{for all } t \in (t_0, T_{max,\varepsilon}). \end{aligned}$$

Using that $u_\varepsilon(\cdot, t_0) \in C^2(\bar{\Omega}; \mathbb{R}^n)$, we may thus once more draw on (4.3) to find $c_{10} = c_{10}(\alpha, \varepsilon) > 0$ such that

$$\|u_\varepsilon(\cdot, s)\|_{W^{1,p}(\Omega)} \leq c_{10} + c_{10} M^{\vartheta_3}(T) \quad \text{for all } s \in (t_0, T) \text{ and } T \in (t_0, T_{max,\varepsilon}),$$

whence again relying on the inequality $\alpha' + \frac{1}{4} < 1$, from (4.4) we obtain $c_{11} = c_{11}(\alpha, \varepsilon) > 0$ such that

$$\|A^\alpha v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq c_{11} + c_{11} M^{\vartheta_3}(T) \quad \text{for all } t \in (t_0, T) \text{ and } T \in (t_0, T_{max,\varepsilon}).$$

Thus,

$$M(T) \leq c_{11} + c_{11} M^{\vartheta_3}(T) \quad \text{for all } T \in (t_0, T_{max,\varepsilon}),$$

which thanks to the inequality $\vartheta_3 < 1$ ensures that, indeed, $\sup_{t \in (t_0, T_{max,\varepsilon})} M(T) < \infty$. \square

This implies that, in fact, for any such ε the second alternative in the extensibility criterion in (2.13) cannot occur:

Lemma 4.2 *For each $\varepsilon \in (0, 1)$, the solution of (2.12) from Lemma 2.3 is global in time; that is, we have $T_{max,\varepsilon} = \infty$.*

PROOF. If $T_{max,\varepsilon}$ was finite for some $\varepsilon \in (0, 1)$, then fixing $\eta \in (0, \frac{1}{2})$ we could choose an arbitrary $\alpha \in (\frac{1}{2}, \frac{3}{4})$ and any $p \geq 2$ such that $4\alpha - \frac{n}{p} > 2 + 2\eta$, and that thus the fractional powers in Lemma 4.1 have the property that $D(A^\alpha) \hookrightarrow W^{2+2\eta, \infty}(\Omega; \mathbb{R}^n)$ ([23, Thm. 1.6.1]).

An application of (4.1) would therefore yield $c_1 > 0$ such that, again with $t_0 := \frac{1}{2}T_{max}$,

$$\|v_\varepsilon(\cdot, t)\|_{W^{2+2\eta, \infty}(\Omega)} \leq c_1 \quad \text{for all } t \in (t_0, T_{max,\varepsilon}), \quad (4.5)$$

which by boundedness of Γ on $[0, \infty)$ particularly implies that we could find $c_2 > 0$ such that $h_\varepsilon := \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle$ would satisfy

$$|h_\varepsilon(x, t)| \leq c_2 \quad \text{for all } x \in \Omega \text{ and } t \in (t_0, T_{max, \varepsilon}). \quad (4.6)$$

Therefore, if we pick $\beta \in (\frac{1}{2}, 1)$ and $q > 1$ large such that $2\beta - \frac{n}{q} > 1 + \eta$, then letting B represent the Dirichlet Laplacian $-\Delta$ in $L^q(\Omega)$ and noting that $D(B^\beta) \hookrightarrow W^{1+\eta, \infty}(\Omega)$ ([23, Thm. 1.6.1]), we could employ standard heat semigroup estimates to obtain $c_3 > 0$, $c_4 > 0$ and $c_5 > 0$ fulfilling

$$\begin{aligned} \|\Theta_\varepsilon(\cdot, t)\|_{W^{1+\eta, \infty}(\Omega)} &\leq c_3 \|B^\beta \Theta_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \\ &= c_3 \left\| B^\beta e^{-(t-t_0)B} \Theta_\varepsilon(\cdot, t_0) + \int_{t_0}^t B^\beta e^{-(t-s)B} h_\varepsilon(\cdot, s) ds \right\|_{L^q(\Omega)} \\ &\leq c_4 \|B^\beta \Theta_\varepsilon(\cdot, t_0)\|_{L^q(\Omega)} + c_4 \int_{t_0}^t (t-s)^{-\beta} \|h_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} ds \\ &\leq c_4 \|B^\beta \Theta_\varepsilon(\cdot, t_0)\|_{L^q(\Omega)} + c_2 c_4 \int_{t_0}^t (t-s)^{-\beta} ds \\ &\leq c_5 \quad \text{for all } t \in (t_0, T_{max, \varepsilon}). \end{aligned} \quad (4.7)$$

As a combination of (4.5) with the second equation in (2.12) would similarly provide $c_6 > 0$ satisfying

$$\|u_\varepsilon(\cdot, t)\|_{W^{1+\eta, \infty}(\Omega)} \leq c_6 \quad \text{for all } t \in (t_0, T_{max, \varepsilon}),$$

the boundedness properties in (4.5) and (4.7) would contradict (2.13), however. \square

5 Regularity properties of Θ_ε

Returning to the problem of identifying ε -independent properties of solutions to (2.12), in this section we focus on the key quantity Θ_ε for which Lemma 3.3, also applied to $\psi \equiv 1$ here, entails the following immediate consequence.

Lemma 5.1 *For each $T > 0$ there exists $C(T) > 0$ such that*

$$\int_{\Omega} \Theta_\varepsilon(\cdot, t) \leq C(T) \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1) \quad (5.1)$$

and

$$\int_0^T \int_{\partial\Omega} \left| \frac{\partial \Theta_\varepsilon}{\partial \nu} \right| \leq C(T) \quad \text{for all } \varepsilon \in (0, 1). \quad (5.2)$$

PROOF. Upon applying Lemma 3.3 to $\psi \equiv 1$, we see that

$$\frac{d}{dt} \int_{\Omega} \Theta_\varepsilon - D \int_{\partial\Omega} \frac{\partial \Theta_\varepsilon}{\partial \nu} = \int_{\Omega} \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \quad (5.3)$$

where thanks to the boundedness of Γ on $[0, \infty)$, with some $c_1 > 0$ we have

$$\int_{\Omega} \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \leq c_1 \int_{\Omega} |\nabla v_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).$$

Moreover, combining the nonnegativity of Θ_ε with the identity $\Theta_\varepsilon|_{\partial\Omega \times (0, \infty)} = 0$ for $\varepsilon \in (0, 1)$, we particularly find that $\frac{\partial\Theta_\varepsilon}{\partial\nu} \leq 0$ on $\partial\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$, so that integrating (5.3) shows that

$$\int_{\Omega} \Theta_\varepsilon(\cdot, t) + D \int_0^t \int_{\partial\Omega} \left| \frac{\partial\Theta_\varepsilon}{\partial\nu} \right| \leq \int_{\Omega} \Theta_{0\varepsilon} + c_1 \int_0^t \int_{\Omega} |\nabla v_\varepsilon|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).$$

In view of (2.11) and (3.10), this implies both (5.1) and (5.2). \square

Based on (5.2), we can appropriately control boundary integrals appearing in another testing procedure which is independent from that in Lemma 3.3 and, unlike the latter, capable of providing some spatially global information about temperature gradients.

Lemma 5.2 *Let $p \in (0, 1)$. Then for all $T > 0$ there exists $C(p, T) > 0$ such that*

$$\int_0^T \int_{\Omega} (\Theta_\varepsilon + 1)^{p-2} |\nabla\Theta_\varepsilon|^2 \leq C(p, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (5.4)$$

PROOF. Once more explicitly using the third equation in (2.12), we obtain that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} (\Theta_\varepsilon + 1)^p &= \int_{\Omega} (\Theta_\varepsilon + 1)^{p-1} \left\{ D\Delta\Theta_\varepsilon + \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \right\} \\ &\geq D \int_{\Omega} (\Theta_\varepsilon + 1)^{p-1} \Delta\Theta_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \end{aligned}$$

because $\langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \geq 0$ in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$ due to (1.13). Since an integration by parts shows that for all $t > 0$ and $\varepsilon \in (0, 1)$, thanks to the identity $(\Theta_\varepsilon + 1)^{p-1}|_{\partial\Omega} = 1$ we have

$$\begin{aligned} D \int_{\Omega} (\Theta_\varepsilon + 1)^{p-1} \Delta\Theta_\varepsilon &= (1-p)D \int_{\Omega} (\Theta_\varepsilon + 1)^{p-2} |\nabla\Theta_\varepsilon|^2 + D \int_{\partial\Omega} (\Theta_\varepsilon + 1)^{p-1} \frac{\partial\Theta_\varepsilon}{\partial\nu} \\ &\geq (1-p)D \int_{\Omega} (\Theta_\varepsilon + 1)^{p-2} |\nabla\Theta_\varepsilon|^2 - D \int_{\partial\Omega} \left| \frac{\partial\Theta_\varepsilon}{\partial\nu} \right|, \end{aligned}$$

this entails that

$$\begin{aligned} (1-p)D \int_0^T \int_{\Omega} (\Theta_\varepsilon + 1)^{p-2} |\nabla\Theta_\varepsilon|^2 &\leq \frac{1}{p} \int_{\Omega} (\Theta_\varepsilon(\cdot, T) + 1)^p - \frac{1}{p} \int_{\Omega} (\Theta_{0\varepsilon} + 1)^p + D \int_0^T \int_{\partial\Omega} \left| \frac{\partial\Theta_\varepsilon}{\partial\nu} \right| \\ &\leq \frac{1}{p} \int_{\Omega} (\Theta_\varepsilon(\cdot, T) + 1)^p + D \int_0^T \int_{\partial\Omega} \left| \frac{\partial\Theta_\varepsilon}{\partial\nu} \right| \quad (5.5) \end{aligned}$$

for all $T > 0$ and $\varepsilon \in (0, 1)$. Using Young's inequality in estimating

$$\frac{1}{p} \int_{\Omega} (\Theta_\varepsilon(\cdot, T) + 1)^p \leq \frac{1}{p} \int_{\Omega} (\Theta_\varepsilon(\cdot, T) + 1) + \frac{|\Omega|}{p} \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1),$$

in light of (5.1) and (5.2) we infer (5.4) from (5.5). \square

Through straightforward Gagliardo-Nirenberg interpolation, this implies bounds for Θ_ε in some reflexive Lebesgue spaces.

Lemma 5.3 *Let $q \in (1, \frac{n+2}{n})$. Then for each $T > 0$ there exists $C(q, T) > 0$ such that*

$$\int_0^T \int_{\Omega} (\Theta_{\varepsilon} + 1)^q \leq C(q, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (5.6)$$

PROOF. By boundedness of Ω , we may assume without loss of generality that $q > \frac{2}{n}$, and that thus our assumption implies that $p \equiv p(q) := q - \frac{2}{n}$ satisfies $p \in (0, 1)$. An application of the Gagliardo-Nirenberg inequality then yields $c_1 = c_1(q) > 0$ such that

$$\int_{\Omega} |\psi|^{\frac{2q}{p}} \leq c_1 \|\nabla \psi\|_{L^2(\Omega)}^{\frac{2q}{p}\vartheta} \|\psi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2q}{p}(1-\vartheta)} + c_1 \|\psi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2q}{p}} \quad \text{for all } \psi \in W^{1,2}(\Omega; \mathbb{R}), \quad (5.7)$$

where by definition of p , the number $\vartheta \equiv \vartheta(q) := (\frac{np}{2} - \frac{np}{2q}) / (1 - \frac{n}{2} + \frac{np}{2})$ satisfies

$$\frac{2q}{p}\vartheta = \frac{nq - n}{1 - \frac{n}{2} + \frac{np}{2}} = \frac{nq - n}{1 - \frac{n}{2} + \frac{n}{2}(q - \frac{2}{n})} = \frac{nq - n}{-\frac{n}{2} + \frac{nq}{2}} = 2.$$

Therefore, (5.7) implies that

$$\begin{aligned} \int_{\Omega} (\Theta_{\varepsilon} + 1)^q &= \int_{\Omega} \left\{ (\Theta_{\varepsilon} + 1)^{\frac{p}{2}} \right\}^{\frac{2q}{p}} \\ &\leq c_1 \left\{ \int_{\Omega} |\nabla (\Theta_{\varepsilon} + 1)^{\frac{p}{2}}|^2 \right\} \left\{ \int_{\Omega} (\Theta_{\varepsilon} + 1) \right\}^{\frac{2}{n}} + c_1 \left\{ \int_{\Omega} (\Theta_{\varepsilon} + 1) \right\}^q \\ &\leq c_1 c_2^{\frac{2}{n}} \int_{\Omega} |\nabla (\Theta_{\varepsilon} + 1)^{\frac{p}{2}}|^2 + c_1 c_2^q \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1), \end{aligned} \quad (5.8)$$

where $c_2 \equiv c_2(T) := \sup_{\varepsilon \in (0,1)} \sup_{t \in (0,T)} \int_{\Omega} (\Theta_{\varepsilon}(\cdot, t) + 1)$ is finite due to Lemma 5.1. Since $|\nabla (\Theta_{\varepsilon} + 1)^{\frac{p}{2}}|^2 = \frac{p^2}{4} (\Theta_{\varepsilon} + 1)^{p-2} |\nabla \Theta_{\varepsilon}|^2$ for all $\varepsilon \in (0, 1)$, integrating (5.8) over $t \in (0, T)$ and using Lemma 5.2 we arrive at (5.6). \square

One further standard interpolation step yields bounds for $\nabla \Theta_{\varepsilon}$ which in contrast to those in Lemma 5.2 do no longer contain weight functions.

Lemma 5.4 *If $r \in (1, \frac{n+2}{n+1})$, then given any $T > 0$ one can find $C(r, T) > 0$ such that*

$$\int_0^T \int_{\Omega} |\nabla \Theta_{\varepsilon}|^r \leq C(r, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (5.9)$$

PROOF. Since $r < \frac{n+2}{n+1}$ and hence $(3n+2)r - 2(n+2) < nr$, we can fix $p = p(r) \in (0, 1)$ suitably close to 1 such that

$$p > \frac{(3n+2)r - 2(n+2)}{nr}.$$

As thus $2 - p < \frac{n+2}{n} \cdot \frac{2-r}{r}$, letting $q \equiv q(r) := \frac{(2-p)r}{2-r}$ defines a number $q \in (1, \frac{n+2}{n})$, and since Young's inequality says that according to this definition of q we have

$$\int_0^T \int_{\Omega} |\nabla \Theta_{\varepsilon}|^r = \int_0^T \int_{\Omega} \left\{ (\Theta_{\varepsilon} + 1)^{p-2} |\nabla \Theta_{\varepsilon}|^2 \right\}^{\frac{r}{2}} (\Theta_{\varepsilon} + 1)^{\frac{(2-p)r}{2}}$$

$$\leq \int_0^T \int_{\Omega} (\Theta_{\varepsilon} + 1)^{p-2} |\nabla \Theta_{\varepsilon}|^2 + \int_0^T \int_{\Omega} (\Theta_{\varepsilon} + 1)^q \quad \text{for all } T > 0 \text{ and } \varepsilon \in (0, 1),$$

the claim is a consequence of Lemma 5.2 and Lemma 5.3. \square

In a natural manner, Lemma 5.3 and Lemma 3.4 imply some information on regularity of $\Theta_{\varepsilon t}$ that will be used to extract pointwise a.e. convergent subsequences by means of an Aubin-Lions lemma (Lemma 7.3).

Lemma 5.5 *Let $s > \frac{n+2}{2}$. Then for all $T > 0$ there exists $C(s, T) > 0$ such that*

$$\int_0^T \|\Theta_{\varepsilon t}(\cdot, t)\|_{(W_0^{2,s}(\Omega; \mathbb{R}))^*} dt \leq C(s, T) \quad \text{for all } \varepsilon \in (0, 1). \quad (5.10)$$

PROOF. For fixed $\psi \in C_0^\infty(\Omega; \mathbb{R})$, again due to the boundedness of Γ on $[0, \infty)$ the identity in (3.7) shows that with some $c_1 > 0$,

$$\begin{aligned} \left| \int_{\Omega} \Theta_{\varepsilon t} \psi \right| &= \left| D \int_{\Omega} \Theta_{\varepsilon} \Delta \psi + \int_{\Omega} \langle \Gamma(\Theta_{\varepsilon}) : \nabla^s v_{\varepsilon}, \nabla^s v_{\varepsilon} \rangle \psi \right| \\ &\leq D \int_{\Omega} \Theta_{\varepsilon} |\Delta \psi| + c_1 \int_{\Omega} |\nabla v_{\varepsilon}|^2 |\psi| \\ &\leq D \|\Theta_{\varepsilon}\|_{L^{\frac{s}{s-1}}(\Omega)} \|\Delta \psi\|_{L^s(\Omega)} + c_1 \|\nabla v_{\varepsilon}\|_{L^2(\Omega)}^2 \|\psi\|_{L^\infty(\Omega)} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \end{aligned}$$

Since the inequality $s > \frac{n+2}{2} > \frac{n}{2}$ particularly ensures that $W^{2,s}(\Omega; \mathbb{R})$ is continuously embedded into $L^\infty(\Omega; \mathbb{R})$, by definition of the norm in $(W_0^{2,s}(\Omega; \mathbb{R}))^*$ this entails the existence of $c_2 = c_2(s) > 0$ fulfilling

$$\|\Theta_{\varepsilon t}\|_{(W_0^{2,s}(\Omega; \mathbb{R}))^*} \leq D \|\Theta_{\varepsilon}\|_{L^{\frac{s}{s-1}}(\Omega)} + c_1 c_2 \|\nabla v_{\varepsilon}\|_{L^2(\Omega)}^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1).$$

Young's inequality thus implies that for all $T > 0$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned} \int_0^T \|\Theta_{\varepsilon t}(\cdot, t)\|_{(W_0^{2,s}(\Omega; \mathbb{R}))^*} dt &\leq D \int_0^T \|\Theta_{\varepsilon}(\cdot, t)\|_{L^{\frac{s}{s-1}}(\Omega)} dt + c_1 c_2 \int_0^T \|\nabla v_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)}^2 dt \\ &\leq D \int_0^T \int_{\Omega} \Theta_{\varepsilon}^{\frac{s}{s-1}} + DT + c_1 c_2 \int_0^T \int_{\Omega} |\nabla v_{\varepsilon}|^2, \end{aligned}$$

so that the claim results upon employing (3.8) and Lemma 5.3, and observing that $\frac{s}{s-1} = (1 - \frac{1}{s})^{-1} < (1 - \frac{2}{n+2})^{-1} = \frac{n+2}{n}$ thanks to our hypothesis. \square

6 Exploiting weak lower semicontinuity of L^2 norms

This section collects some consequences of lower semicontinuity of norms in L^2 spaces with respect to weak convergence, arranged here in such a way that expressions of the form in (1.18) can adequately be coped with in several different particular situations arising below.

Let us first record an essentially well-known consequence of Lebesgue's theorem for products of a.e. convergent and L^2 -weakly convergent sequences. This will be used not only in the subsequent Lemma 6.3, but later on also in our extraction of convergent subsequences of $((v_{\varepsilon}, u_{\varepsilon}, \Theta_{\varepsilon}))_{\varepsilon \in (0,1)}$ (Lemma 7.3) and in the derivation of (2.5) (Lemma 7.4).

Lemma 6.1 *Let $N \geq 1$ and $G \subset \mathbb{R}^N$ be measurable with $|G| < \infty$, and suppose that $(f_j)_{j \in \mathbb{N}} \subset L^\infty(G)$, $(g_j)_{j \in \mathbb{N}} \subset L^2(G)$, $f \in L^\infty(G)$ and $g \in L^2(G)$ are such that*

$$\sup_{j \in \mathbb{N}} \|f_j\|_{L^\infty(G)} < \infty, \quad (6.1)$$

and that as $j \rightarrow \infty$ we have

$$f_j \rightarrow f \quad \text{a.e. in } G \quad (6.2)$$

and

$$g_j \rightarrow g \quad \text{in } L^2(G). \quad (6.3)$$

Then

$$f_j g_j \rightarrow f g \quad \text{in } L^2(G) \quad \text{as } j \rightarrow \infty. \quad (6.4)$$

PROOF. If the claim was false, then since (6.1) and (6.3) particularly assert that $(f_j g_j)_{j \in \mathbb{N}}$ is bounded in $L^2(G)$, we could find a subsequence $(j_k)_{k \in \mathbb{N}}$ along which $f_{j_k} g_{j_k} \rightarrow h$ would hold in $L^2(G)$ as $k \rightarrow \infty$ with some $h \in L^2(G)$ satisfying $|\{h \neq fg\}| > 0$. As $|G|$ is finite, this would especially imply that $f_{j_k} g_{j_k} \rightarrow h$ in $L^1(G)$ as $k \rightarrow \infty$. But uniform boundedness of $(f_j)_{j \in \mathbb{N}}$ according to (6.1) together with (6.2) shows that $f_j \rightarrow f$ in $L^2(G)$ by Lebesgue's dominated convergence theorem. Accordingly, $f_j g_j \rightarrow g$ in $L^1(G)$ due to (6.3), and hence we could infer that $h = fg$, which is absurd. \square

Lemma 6.2 *Let $B = (B_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ be such that*

$$B_{ijkl}(\xi) = B_{klij}(\xi) \quad \text{for all } \xi \geq 0 \text{ and } (i, j, k, l) \in \{1, \dots, n\}^4, \quad (6.5)$$

and that there exists $K_B > 0$ such that

$$\langle B(\xi) : X, X \rangle \geq K_B |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n}. \quad (6.6)$$

Then there exists $\sqrt{B} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ such that

$$\langle B(\xi) : W, W \rangle = \langle \sqrt{B(\xi)} : W, \sqrt{B(\xi)} : W \rangle \quad \text{for all } \xi \geq 0 \text{ and } W \in \mathbb{R}^{n \times n}. \quad (6.7)$$

PROOF. For every $\xi \in [0, \infty)$ interpreting $B(\xi)$ as (by (6.5)) symmetric and (by (6.6)) positive definite $n^2 \times n^2$ matrix, we take $\sqrt{B}(\xi)$ to be its unique positive definite square root [24, Cor. 1.30], which in particular satisfies (6.7). Then $\xi \mapsto \sqrt{B}(\xi)$ is continuous by continuity of B and [24, Thm. 6.12]. Boundedness of \sqrt{B} follows from $B \in L^\infty([0, \infty); \mathbb{R}^{(n \times n) \times (n \times n)})$ and is most easily seen in the spectral norm, as $\|\sqrt{B}(\xi)\|_\sigma = \sqrt{\|B(\xi)\|_\sigma}$. \square

For bounded and uniformly positive definite symmetric tensor-valued functions, we can use Lemma 6.1 together with the lower semicontinuity property under consideration to obtain the following basic property.

Lemma 6.3 *Let $B = (B_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ satisfy (6.5) and let there be $K_B > 0$ such that (6.6) holds. Then whenever $T > 0$ as well as $\varphi \in L^\infty(\Omega \times (0, T); \mathbb{R})$, $(W_j)_{j \in \mathbb{N}} \subset L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ and the measurable functions $z_j : \Omega \times (0, T) \rightarrow \mathbb{R}$, $j \in \mathbb{N}$, are such that*

$$\varphi \geq 0 \quad \text{a.e. in } \Omega \times (0, T) \quad (6.8)$$

and

$$z_j \geq 0 \quad \text{a.e. in } \Omega \times (0, T) \quad \text{for all } j \in \mathbb{N}, \quad (6.9)$$

and that as $j \rightarrow \infty$ we have

$$W_j \rightharpoonup W \quad \text{in } L^2(\Omega \times (0, T); \mathbb{R}^{n \times n}) \quad (6.10)$$

and

$$z_j \rightarrow z \quad \text{a.e. in } \Omega \times (0, T) \quad (6.11)$$

with some $W \in L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ and some measurable $z : \Omega \times (0, T) \rightarrow \mathbb{R}$, it follows that

$$\int_0^T \int_{\Omega} \langle B(z) : W, W \rangle \varphi \leq \liminf_{j \rightarrow \infty} \int_0^T \int_{\Omega} \langle B(z_j) : W_j, W_j \rangle \varphi. \quad (6.12)$$

PROOF. According to (6.5), (6.6) and Lemma 6.2, there exists $\sqrt{B} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ such that (6.7) holds.

Thus, if for $j \in \mathbb{N}$ we let

$$\rho_j := \sqrt{\varphi} \sqrt{B(z_j)} : W_j, \quad (6.13)$$

then we obtain a sequence $(\rho_j)_{j \in \mathbb{N}} \subset L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ which due to Lemma 6.1 and (6.10) satisfies

$$\rho_j \rightharpoonup \rho := \sqrt{\varphi} \sqrt{B(z)} : W \quad \text{in } L^2(\Omega \times (0, T); \mathbb{R}^{n \times n}) \quad \text{as } j \rightarrow \infty, \quad (6.14)$$

because $\sup_{j \in \mathbb{N}} \|\sqrt{B(z_j)}\|_{L^\infty(\Omega \times (0, T))} < \infty$ and $\sqrt{B(z_j)} \rightarrow \sqrt{B(z)}$ a.e. in $\Omega \times (0, T)$ by boundedness and continuity of \sqrt{B} , and by (6.11). Thanks to the lower semicontinuity of the norm in $L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ with respect to weak convergence, from (6.14) we infer that, in line with (6.7),

$$\int_0^T \int_{\Omega} \langle B(z) : W, W \rangle \varphi = \int_0^T \int_{\Omega} |\rho|^2 \leq \liminf_{j \rightarrow \infty} \int_0^T \int_{\Omega} |\rho_j|^2 = \liminf_{j \rightarrow \infty} \int_0^T \int_{\Omega} \langle B(z_j) : W_j, W_j \rangle \varphi,$$

as intended. \square

Of crucial importance for our reasoning is now the following consequence of Lemma 6.3.

Lemma 6.4 *Let $\beta = (\beta_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ be such that*

$$\beta_{ijkl}(\xi) = \beta_{klij}(\xi) \quad \text{for all } \xi \geq 0 \text{ and } (i, j, k, l) \in \{1, \dots, n\}^4, \quad (6.15)$$

Then there exists $\eta_1 = \eta_1(\beta) > 0$ with the property that if $\eta \in (0, \eta_1)$ and $T > 0$, if $\varphi \in L^\infty(\Omega \times (0, T); \mathbb{R})$ is nonnegative, and if $(w_j)_{j \in \mathbb{N}} \subset L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n))$, $(z_j)_{j \in \mathbb{N}} \subset L^1(\Omega \times (0, T); \mathbb{R})$, $w \in L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n))$, and $z \in L^1(\Omega \times (0, T); \mathbb{R})$ are such that

$$z_j \geq 0 \quad \text{a.e. in } \Omega \times (0, T) \quad \text{for all } j \in \mathbb{N} \quad \text{and} \quad z_j \rightarrow z \quad \text{a.e. in } \Omega \times (0, T) \quad \text{as } j \rightarrow \infty, \quad (6.16)$$

and that

$$w_j \rightharpoonup w \quad \text{in } L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n)) \quad \text{as } j \rightarrow \infty, \quad (6.17)$$

then

$$\begin{aligned} & \int_0^T \int_{\Omega} |\nabla w|^2 \varphi - \eta \int_0^T \int_{\Omega} \langle \beta(z) : \nabla^s w, \nabla^s w \rangle \varphi \\ & \leq \liminf_{j \rightarrow \infty} \left\{ \int_0^T \int_{\Omega} |\nabla w_j|^2 \varphi - \eta \int_0^T \int_{\Omega} \langle \beta(z_j) : \nabla^s w_j, \nabla^s w_j \rangle \varphi \right\}. \end{aligned} \quad (6.18)$$

PROOF. We let $(\delta_{ij})_{i,j \in \{1, \dots, n\}}$ denote the Kronecker delta on $\{1, \dots, n\}^2$, and define

$$\widehat{\beta}_{ijkl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (i, j, k, l) \in \{1, \dots, n\}^4, \quad (6.19)$$

observing that then

$$(\widehat{\beta} : X)_{ij} = \frac{1}{2} \sum_{k,l=1}^n \delta_{ik}\delta_{jl}X_{kl} + \frac{1}{2} \sum_{k,l=1}^n \delta_{il}\delta_{jk}X_{kl} = \frac{1}{2}X_{ij} + \frac{1}{2}X_{ji} \quad \text{for all } (i, j) \in \{1, \dots, n\}^2$$

and hence

$$\widehat{\beta} : X = \frac{1}{2}(X + X^t)$$

for each $X = (X_{ij})_{i,j \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n}$. As clearly

$$\widehat{\beta}_{ijkl} = \widehat{\beta}_{klij} \quad \text{for all } (i, j, k, l) \in \{1, \dots, n\}^4, \quad (6.20)$$

by straightforward linear algebra we thus obtain that

$$\begin{aligned} \langle \beta(\xi) : \left\{ \frac{1}{2}(X + X^t) \right\}, \frac{1}{2}(X + X^t) \rangle &= \langle \beta(\xi) : (\widehat{\beta} : X), \widehat{\beta} : X \rangle \\ &= \langle \widehat{\beta} : \{ \beta(\xi) : (\widehat{\beta} : X) \}, X \rangle \\ &= \langle \widehat{B}(\xi) : X, X \rangle \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n} \end{aligned} \quad (6.21)$$

with

$$\widehat{B}(\xi) := \widehat{\beta} \odot (\beta(\xi) \odot \widehat{\beta}), \quad \xi \geq 0, \quad (6.22)$$

where for $\beta^{(\iota)} = (\beta_{ijkl}^{(\iota)})_{i,j,k,l \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n \times n \times n}$, $\iota \in \{1, 2\}$, we have set

$$(\beta^{(1)} \odot \beta^{(2)})_{ijkl} := \sum_{m,m'=1}^n \beta_{ijmm'}^{(1)} \beta_{mm'kl}^{(2)}, \quad (i, j, k, l) \in \{1, \dots, n\}^4.$$

Now due to the assumed boundeness of β and (6.19), from (6.22) we infer that with some $c_1 = c_1(\beta) > 0$ we have

$$\langle \widehat{B}(\xi) : X, X \rangle \leq c_1 |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n}, \quad (6.23)$$

and fixing any $\eta_1 = \eta_1(\beta) \in (0, \frac{1}{c_1})$ and assuming that $\eta \in (0, \eta_1)$, we thereupon let

$$B(\xi) := I - \eta \widehat{B}(\xi), \quad \xi \geq 0, \quad (6.24)$$

where $I := (\delta_{ik}\delta_{jl})_{i,j,k,l \in \{1, \dots, n\}}$ represents the identity mapping on $\mathbb{R}^{n \times n}$. Again thanks to (6.20), the hypothesis in (6.15) ensures that B has the symmetry property in (6.5), while (6.24) along with (6.23) guarantees that

$$\langle B(\xi) : X, X \rangle = |X|^2 - \eta \langle \widehat{B}(\xi) : X, X \rangle \geq c_2 |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n},$$

with $c_2 \equiv c_2(\beta) := 1 - \eta_1 c_1$ being positive due to our selection of η_1 . The lemma thereby becomes a consequence of (6.17) and Lemma 6.3, according to which, namely, in line with (6.24) and (6.22) it follows that

$$\begin{aligned}
& \int_0^T \int_{\Omega} |\nabla w|^2 \varphi - \eta \int_0^T \int_{\Omega} \langle \beta(z) : \nabla^s w, \nabla^s w \rangle \varphi \\
&= \int_0^T \int_{\Omega} \langle B(z) : \nabla w, \nabla w \rangle \varphi \\
&\leq \liminf_{j \rightarrow \infty} \int_0^T \int_{\Omega} \langle B(z_j) : \nabla w_j, \nabla w_j \rangle \varphi \\
&= \liminf_{j \rightarrow \infty} \left\{ \int_0^T \int_{\Omega} |\nabla w_j|^2 \varphi - \eta \int_0^T \int_{\Omega} \langle \beta(z_j) : \nabla^s w_j, \nabla^s w_j \rangle \varphi \right\},
\end{aligned}$$

as claimed. \square

In preparation for a relative of Lemma 6.4 involving differences with ordering opposite to those in (6.18), let us state the following observation which essentially relies on vanishing boundary values.

Lemma 6.5 *Let $w \in W_0^{1,2}(\Omega; \mathbb{R}^n)$ and $\psi \in C^1(\bar{\Omega}; \mathbb{R})$. Then*

$$\int_{\Omega} |\nabla^s w|^2 \psi = \frac{1}{2} \int_{\Omega} |\nabla w|^2 \psi + \frac{1}{2} \int_{\Omega} |\operatorname{div} w|^2 \psi + \frac{1}{2} \int_{\Omega} (\operatorname{div} w)(w \cdot \nabla \psi) - \frac{1}{2} \int_{\Omega} w \cdot (\nabla w \cdot \nabla \psi). \quad (6.25)$$

PROOF. For $w = (w_1, \dots, w_n) \in C_0^\infty(\Omega; \mathbb{R}^n)$ and $\psi \in C^1(\bar{\Omega}; \mathbb{R})$, in the identity

$$\begin{aligned}
\int_{\Omega} |\nabla^s w|^2 \psi &= \frac{1}{4} \sum_{i,j=1}^n \int_{\Omega} (\partial_j w_i + \partial_i w_j)^2 \psi \\
&= \frac{1}{4} \left\{ \sum_{i,j=1}^n \int_{\Omega} (\partial_j w_i)^2 \psi + \sum_{i,j=1}^n \int_{\Omega} (\partial_i w_j)^2 \psi + 2 \sum_{i,j=1}^n \int_{\Omega} \partial_j w_i \partial_i w_j \psi \right\} \\
&= \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} (\partial_j w_i)^2 \psi + \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \partial_i w_j \partial_j w_i \psi \\
&= \frac{1}{2} \int_{\Omega} |\nabla w|^2 \psi + \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \partial_i w_j \partial_j w_i \psi
\end{aligned}$$

we may twice integrate by parts to see that

$$\begin{aligned}
\frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \partial_i w_j \partial_j w_i \psi &= -\frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} w_j \partial_{ij} w_i \psi - \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} w_j \partial_j w_i \partial_i \psi \\
&= \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} \partial_j w_j \partial_i w_i \psi + \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} w_j \partial_i w_i \partial_j \psi - \frac{1}{2} \sum_{i,j=1}^n \int_{\Omega} w_j \partial_j w_i \partial_i \psi \\
&= \frac{1}{2} \int_{\Omega} |\operatorname{div} w|^2 \psi + \frac{1}{2} \int_{\Omega} (\operatorname{div} w)(w \cdot \nabla \psi) - \frac{1}{2} \int_{\Omega} w \cdot (\nabla w \cdot \nabla \psi),
\end{aligned}$$

so that (6.25) immediately follows by means of a completion argument. \square

In fact, making appropriate use of this we can complement Lemma 6.4 by means of a different argument as follows.

Lemma 6.6 *Suppose that $\beta = (\beta_{ijkl})_{i,j,k,l \in \{1, \dots, n\}} \in C^0([0, \infty); \mathbb{R}^{n \times n \times n \times n}) \cap L^\infty([0, \infty); \mathbb{R}^{n \times n \times n \times n})$ satisfies (6.15) and*

$$\langle \beta(\xi) : X, X \rangle \geq K_\beta |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n} \quad (6.26)$$

with some $K_\beta > 0$. Then there exists $\eta_2 = \eta_2(\beta) > 0$ such that assuming that $T > 0$, that $0 \leq \varphi \in L^\infty(\Omega \times (0, T); \mathbb{R})$, that $(w_j)_{j \in \mathbb{N}} \subset L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n))$, $(z_j)_{j \in \mathbb{N}} \subset L^1(\Omega \times (0, T); \mathbb{R})$, $w \in L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n))$, and $z \in L^1(\Omega \times (0, T); \mathbb{R})$ satisfy (6.16), (6.17) as well as

$$w_j \rightarrow w \quad \text{in } L^2(\Omega \times (0, T); \mathbb{R}^n) \quad \text{as } j \rightarrow \infty, \quad (6.27)$$

one can conclude that

$$\begin{aligned} & \int_0^T \int_\Omega \langle \beta(z) : \nabla^s w, \nabla^s w \rangle \varphi - \eta \int_0^T \int_\Omega |\nabla w|^2 \varphi \\ & \leq \liminf_{j \rightarrow \infty} \left\{ \int_0^T \int_\Omega \langle \beta(z_j) : \nabla^s w_j, \nabla^s w_j \rangle \varphi - \eta \int_0^T \int_\Omega |\nabla w_j|^2 \varphi \right\} \end{aligned} \quad (6.28)$$

for all $\eta \in (0, \eta_2)$.

PROOF. We fix any $\eta_2 = \eta_2(\beta) \in (0, \frac{1}{2}K_\beta)$, and for $\eta \in (0, \eta_2)$ we let

$$B(\xi) := \beta(\xi) - 2\eta I, \quad \xi \geq 0,$$

where again $I := (\delta_{ik}\delta_{jl})_{i,j,k,l \in \{1, \dots, n\}} \in \mathbb{R}^{n \times n \times n \times n}$. Then B satisfies (6.5) due to (6.15), whereas the requirement in (6.26) guarantees that writing $c_1 \equiv c_1(\beta) := K_\beta - 2\eta_2 > 0$ we have

$$\langle B(\xi) : X, X \rangle = \langle \beta(\xi) : X, X \rangle - 2\eta |X|^2 \geq c_1 |X|^2 \quad \text{for all } \xi \geq 0 \text{ and } X \in \mathbb{R}^{n \times n}.$$

Since $\nabla^s w_j \rightharpoonup \nabla^s w$ in $L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ as $j \rightarrow \infty$, we may therefore employ Lemma 6.3 to infer that

$$\begin{aligned} & \int_0^T \int_\Omega \langle \beta(z) : \nabla^s w, \nabla^s w \rangle \varphi - 2\eta \int_0^T \int_\Omega |\nabla^s w|^2 \varphi \\ & = \int_0^T \int_\Omega \langle B(z) : \nabla^s w, \nabla^s w \rangle \varphi \\ & \leq \liminf_{j \rightarrow \infty} \int_0^T \int_\Omega \langle B(z_j) : \nabla^s w_j, \nabla^s w_j \rangle \varphi \\ & = \liminf_{j \rightarrow \infty} \left\{ \int_0^T \int_\Omega \langle \beta(z_j) : \nabla^s w_j, \nabla^s w_j \rangle \varphi - 2\eta \int_0^T \int_\Omega |\nabla^s w_j|^2 \varphi \right\}. \end{aligned} \quad (6.29)$$

To appropriately complement this, we now utilize Lemma 6.5 to confirm that for all $j \in \mathbb{N}$,

$$2\eta \int_0^T \int_\Omega |\nabla^s w_j|^2 \varphi - \eta \int_0^T \int_\Omega |\nabla w_j|^2 \varphi$$

$$= \eta \int_0^T \int_{\Omega} |\operatorname{div} w_j|^2 \varphi + \eta \int_0^T \int_{\Omega} (\operatorname{div} w_j)(w_j \cdot \nabla \varphi) - \eta \int_0^T \int_{\Omega} w_j \cdot (\nabla w_j \cdot \nabla \varphi), \quad (6.30)$$

where as a consequence of (6.17) and the strong convergence feature in (6.27),

$$\eta \int_0^T \int_{\Omega} (\operatorname{div} w_j)(w_j \cdot \nabla \varphi) \rightarrow \eta \int_0^T \int_{\Omega} (\operatorname{div} w)(w \cdot \nabla \varphi) \quad (6.31)$$

and

$$-\eta \int_0^T \int_{\Omega} w_j \cdot (\nabla w_j \cdot \nabla \varphi) \rightarrow -\eta \int_0^T \int_{\Omega} w \cdot (\nabla w \cdot \nabla \varphi) \quad (6.32)$$

as $j \rightarrow \infty$. Since by lower semicontinuity of the norm in $L^2(\Omega \times (0, T); \mathbb{R})$ we furthermore readily obtain from (6.17) that

$$\eta \int_0^T \int_{\Omega} |\operatorname{div} w|^2 \varphi \leq \liminf_{j \rightarrow \infty} \left\{ \eta \int_0^T \int_{\Omega} |\operatorname{div} w_j|^2 \varphi \right\},$$

once more relying on Lemma 6.5 we may combine (6.30) with (6.31) and (6.32) to see that

$$\begin{aligned} & 2\eta \int_0^T \int_{\Omega} |\nabla^s w|^2 \varphi - \eta \int_0^T \int_{\Omega} |\nabla w|^2 \varphi \\ &= \eta \int_0^T \int_{\Omega} |\operatorname{div} w|^2 \varphi + \eta \int_0^T \int_{\Omega} (\operatorname{div} w)(w \cdot \nabla \varphi) - \eta \int_0^T \int_{\Omega} w \cdot (\nabla w \cdot \nabla \varphi) \\ &\leq \liminf_{j \rightarrow \infty} \left\{ \eta \int_0^T \int_{\Omega} |\operatorname{div} w_j|^2 \varphi + \eta \int_0^T \int_{\Omega} (\operatorname{div} w_j)(w_j \cdot \nabla \varphi) - \eta \int_0^T \int_{\Omega} w_j \cdot (\nabla w_j \cdot \nabla \varphi) \right\} \\ &= \liminf_{j \rightarrow \infty} \left\{ 2\eta \int_0^T \int_{\Omega} |\nabla^s w_j|^2 \varphi - \eta \int_0^T \int_{\Omega} |\nabla w_j|^2 \varphi \right\}. \end{aligned}$$

In conjunction with (6.29), this establishes (6.28) due to the basic fact that $\liminf_{j \rightarrow \infty} (\omega_j + \widehat{\omega}_j) \geq \liminf_{j \rightarrow \infty} \omega_j + \liminf_{j \rightarrow \infty} \widehat{\omega}_j$ for bounded sequences $(\omega_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ and $(\widehat{\omega}_j)_{j \in \mathbb{N}} \subset \mathbb{R}$. \square

7 Passing to the limit. Proof of Theorem 1.1

Final preliminaries for our limit passage in (2.12) derive the following information on regularity of temporal derivatives from Lemma 3.4.

Lemma 7.1 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\int_0^T \|v_{\varepsilon t}(\cdot, t)\|_{(W_0^{2,2}(\Omega; \mathbb{R}^n))^*}^2 dt \leq C(T) \quad \text{for all } \varepsilon \in (0, 1). \quad (7.1)$$

PROOF. Given $\psi \in C_0^\infty(\Omega; \mathbb{R}^n)$ such that $\|\psi\|_{W_0^{2,2}(\Omega)} \leq 1$, we integrate by parts in the first equation from (2.12) to see that due to the Cauchy-Schwarz inequality,

$$\left| \int_{\Omega} v_{\varepsilon t} \cdot \psi \right| = \left| -\varepsilon \int_{\Omega} \Delta^2 v_{\varepsilon} \cdot \psi + \int_{\Omega} \operatorname{div} (\gamma(\Theta_{\varepsilon}) : \nabla^s (v_{\varepsilon} + a u_{\varepsilon})) \cdot \psi \right|$$

$$\begin{aligned}
&= \left| -\varepsilon \int_{\Omega} \Delta v_{\varepsilon} \cdot \Delta \psi - \int_{\Omega} \langle \gamma(\Theta_{\varepsilon}) : \nabla^s(v_{\varepsilon} + au_{\varepsilon}), \nabla \psi \rangle \right| \\
&\leq \varepsilon^{\frac{1}{2}} \left\{ \varepsilon \int_{\Omega} |\Delta v_{\varepsilon}|^2 \right\}^{\frac{1}{2}} \|\Delta \psi\|_{L^2(\Omega)} + \left\{ \int_{\Omega} |\gamma(\Theta_{\varepsilon}) : \nabla^s(v_{\varepsilon} + au_{\varepsilon})|^2 \right\}^{\frac{1}{2}} \|\nabla \psi\|_{L^2(\Omega)}
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. According to the boundedness of γ and the definition of the norm in $(W_0^{2,2}(\Omega; \mathbb{R}^n))^*$, this implies the existence of $c_1 > 0$ such that

$$\|v_{\varepsilon t}(\cdot, t)\|_{(W_0^{2,2}(\Omega; \mathbb{R}^n))^*}^2 \leq c_1 \varepsilon \int_{\Omega} |\Delta v_{\varepsilon}|^2 + c_1 \int_{\Omega} |\nabla v_{\varepsilon}|^2 + c_1 \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

which in view of (3.11), (3.10) and (3.9) entails (7.1) upon an integration in time. \square

A similar statement holds for the derivative $u_{\varepsilon t}$.

Lemma 7.2 *Let $T > 0$. Then there exists $C(T) > 0$ such that*

$$\|u_{\varepsilon t}\|_{L^\infty((0,T);(W_0^{2,2}(\Omega; \mathbb{R}^n))^*)} \leq C(T) \quad \text{for all } \varepsilon \in (0, 1).$$

PROOF. Given $\psi \in C_0^\infty(\Omega; \mathbb{R}^n)$ such that $\|\psi\|_{W_0^{2,2}(\Omega)} \leq 1$, we conclude from the second equation of (2.12), integration by parts and Hölder's inequality that

$$\left| \int_{\Omega} u_{\varepsilon t}(\cdot, t) \psi \right| \leq \varepsilon \int_{\Omega} |u_{\varepsilon}(\cdot, t)| |\Delta \psi| + \int_{\Omega} v_{\varepsilon} |\psi| \leq \varepsilon \|u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} + \|v_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)},$$

which is bounded according to (3.9) combined with Poincaré's inequality and (3.8). \square

With these preparations at hand, by means of a straightforward subsequence extraction, followed by applications of the Vitali convergence theorem, Fatou's lemma, Lemma 6.1 and especially Lemma 6.3, we can now proceed to the construction of a limit pair (u, Θ) which satisfies (2.3) and (2.4).

Lemma 7.3 *There exists $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ such that $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$, and that with some*

$$\begin{cases} v \in L_{loc}^\infty([0, \infty); L^2(\Omega; \mathbb{R}^n)) \cap L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)), \\ u \in L_{loc}^\infty([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)) \quad \text{and} \\ \Theta \in L_{loc}^\infty([0, \infty); L^1(\Omega; \mathbb{R})) \cap \bigcap_{q \in [1, \frac{n+2}{n}]} L_{loc}^q(\bar{\Omega} \times [0, \infty); \mathbb{R}) \cap \bigcap_{r \in [1, \frac{n+2}{n+1}]} L_{loc}^r([0, \infty); W_0^{1,r}(\Omega; \mathbb{R})) \end{cases} \quad (7.2)$$

fulfilling $\Theta \geq 0$ a.e. in $\Omega \times (0, \infty)$, we have

$$v_{\varepsilon} \rightharpoonup v \quad \text{in } L_{loc}^2(\bar{\Omega} \times [0, \infty); \mathbb{R}^n) \text{ and a.e. in } \Omega \times (0, \infty), \quad (7.3)$$

$$v_{\varepsilon} \rightharpoonup v \quad \text{in } L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)), \quad (7.4)$$

$$u_{\varepsilon} \rightarrow u \quad \text{in } L_{loc}^2(\bar{\Omega} \times [0, \infty); \mathbb{R}^n), \quad (7.5)$$

$$u_{\varepsilon} \rightharpoonup u \quad \text{in } L_{loc}^2([0, \infty); W_0^{1,2}(\Omega; \mathbb{R}^n)) \quad \text{and} \quad (7.6)$$

$$\Theta_{\varepsilon} \rightarrow \Theta \quad \text{in } L_{loc}^1(\bar{\Omega} \times [0, \infty); \mathbb{R}) \text{ and a.e. in } \Omega \times (0, \infty), \quad (7.7)$$

as $\varepsilon = \varepsilon_j \searrow 0$. These limit functions have the properties that

$$u_t = v \quad \text{a.e. in } \Omega \times (0, \infty), \quad (7.8)$$

that (2.3) holds for each $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^n)$, and that (2.4) is satisfied for any nonnegative $\hat{\varphi} \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R})$.

PROOF. From Lemma 3.4 and Lemma 7.1, we know that for each $T > 0$,

$$(v_\varepsilon)_{\varepsilon \in (0,1)} \text{ is bounded in } L^\infty((0, T); L^2(\Omega; \mathbb{R}^n)) \text{ and in } L^2((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n)),$$

that

$$(v_{\varepsilon t})_{\varepsilon \in (0,1)} \text{ is bounded in } L^2((0, T); (W_0^{2,2}(\Omega; \mathbb{R}^n))^*),$$

that

$$(u_\varepsilon)_{\varepsilon \in (0,1)} \text{ is bounded in } L^\infty((0, T); W_0^{1,2}(\Omega; \mathbb{R}^n)),$$

and that

$$(u_{\varepsilon t})_{\varepsilon \in (0,1)} \text{ is bounded in } L^\infty((0, T); (W_0^{2,2}(\Omega; \mathbb{R}^n))^*),$$

while Lemma 5.1 in conjunction with Lemma 5.3, Lemma 5.4 and Lemma 5.5 says that whenever $T > 0$, $q \in [1, \frac{n+2}{n})$ and $r \in [1, \frac{n+2}{n+1})$,

$$(\Theta_\varepsilon)_{\varepsilon \in (0,1)} \text{ is bounded in } L^\infty((0, T); L^1(\Omega; \mathbb{R})), \text{ in } L^q(\Omega \times (0, T); \mathbb{R}) \text{ and in } L^r((0, T); W_0^{1,r}(\Omega; \mathbb{R})),$$

and that for each $s > \frac{n+2}{2}$,

$$(\Theta_{\varepsilon t})_{\varepsilon \in (0,1)} \text{ is bounded in } L^1((0, T); (W_0^{2,s}(\Omega; \mathbb{R}))^*).$$

A straightforward extraction procedure involving three applications of an Aubin-Lions lemma ([57]) combined with Vitali's convergence theorem and Fatou's lemma therefore yields $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ such that $\varepsilon_j \searrow 0$ as $j \rightarrow \infty$, and that with some functions v, u and Θ fulfilling (7.2) as well as $\Theta \geq 0$ a.e. in $\Omega \times (0, \infty)$, the convergence properties in (7.3)-(7.7) hold.

Now according to the second equation in (2.12), for each $\phi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^n)$ and any $\varepsilon \in (0, 1)$ we have

$$-\int_0^\infty \int_\Omega u_\varepsilon \cdot \phi_t - \int_\Omega u_\varepsilon(\cdot, 0) \cdot \phi(\cdot, 0) = \varepsilon \int_0^\infty \int_\Omega \Delta u_\varepsilon \cdot \phi + \int_0^\infty \int_\Omega v_\varepsilon \cdot \phi, \quad (7.9)$$

so that since $\varepsilon \Delta u_\varepsilon \rightarrow 0$ in $L_{loc}^2(\overline{\Omega} \times [0, \infty); \mathbb{R}^n)$ as $\varepsilon \searrow 0$ by (3.12), from (7.3), (2.11) and (7.6) we obtain on letting $\varepsilon = \varepsilon_j \searrow 0$ in (7.9) that

$$-\int_0^\infty \int_\Omega u \cdot \phi_t - \int_\Omega u_0 \cdot \phi(\cdot, 0) = \int_0^\infty \int_\Omega v \cdot \phi, \quad (7.10)$$

and that thus, in particular, (7.8) holds.

Next, for a verification of (2.3) we fix $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^n)$ and integrate by parts in the first equation in (2.12) to see that

$$\begin{aligned} -\int_0^\infty \int_\Omega v_\varepsilon \cdot \varphi_t - \int_\Omega v_{0\varepsilon} \cdot \varphi(\cdot, 0) &= -\varepsilon \int_0^\infty \int_\Omega \Delta v_\varepsilon \cdot \Delta \varphi \\ &\quad - \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla \varphi \rangle - a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla \varphi \rangle, \quad \text{for all } \varepsilon \in (0, 1), \end{aligned} \quad (7.11)$$

where by (7.3) and (2.11),

$$-\int_0^\infty \int_\Omega v_\varepsilon \cdot \varphi_t - \int_\Omega v_{0\varepsilon} \cdot \varphi(\cdot, 0) \rightarrow -\int_0^\infty \int_\Omega v \cdot \varphi_t - \int_\Omega u_{0t} \cdot \varphi(\cdot, 0) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \quad (7.12)$$

and where

$$-\varepsilon \int_0^\infty \int_\Omega \Delta v_\varepsilon \cdot \Delta \varphi \rightarrow 0 \quad \text{as } \varepsilon \searrow 0, \quad (7.13)$$

because (3.11) ensures that also $\varepsilon \Delta v_\varepsilon \rightarrow 0$ in $L^2_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^n)$ as $\varepsilon \searrow 0$. Moreover, using that $(\gamma(\Theta_\varepsilon))_{\varepsilon \in (0,1)}$ is bounded in $L^\infty(\Omega \times (0, \infty); \mathbb{R}^{n \times n \times n \times n})$ with $\gamma(\Theta_\varepsilon) \rightarrow \gamma(\Theta)$ a.e. in $\Omega \times (0, \infty)$ as $\varepsilon = \varepsilon_j \searrow 0$ by (7.7) and the continuity of γ , from the L^2 convergence properties of $(\nabla^s v_{\varepsilon_j})_{j \in \mathbb{N}}$ and $(\nabla^s u_{\varepsilon_j})_{j \in \mathbb{N}}$ entailed by (7.4) and (7.6) we infer through Lemma 6.1 that $\gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon \rightharpoonup \gamma(\Theta) : \nabla^s v$ and $\gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon \rightharpoonup \gamma(\Theta) : \nabla^s u$ in $L^2_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^{n \times n})$ as $\varepsilon = \varepsilon_j \searrow 0$. Therefore, (7.11)-(7.13) imply that

$$-\int_0^\infty \int_\Omega v \cdot \varphi_t - \int_\Omega u_{0t} \cdot \varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s v, \nabla \varphi \rangle - a \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, \nabla \varphi \rangle,$$

whence (2.3) follows upon observing that $\nabla^s v = \nabla^s u_t$ by (7.8), and that according to (7.10) when applied to $\phi := \varphi_t$,

$$-\int_0^\infty \int_\Omega v \cdot \varphi_t = \int_0^\infty \int_\Omega u \cdot \varphi_{tt} + \int_\Omega u_0 \cdot \varphi_t(\cdot, 0).$$

Finally, for arbitrary nonnegative $\widehat{\varphi} \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R})$ and $\varepsilon \in (0, 1)$ we may use the third equation in (2.12) to see that

$$\int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \widehat{\varphi} = -\int_0^\infty \int_\Omega \Theta_\varepsilon \widehat{\varphi}_t - \int_\Omega \Theta_{0\varepsilon} \widehat{\varphi}(\cdot, 0) - D \int_0^\infty \int_\Omega \Theta_\varepsilon \Delta \widehat{\varphi}, \quad (7.14)$$

and note that here, due to (7.7) and (2.11),

$$\begin{aligned} & -\int_0^\infty \int_\Omega \Theta_\varepsilon \widehat{\varphi}_t - \int_\Omega \Theta_{0\varepsilon} \widehat{\varphi}(\cdot, 0) - D \int_0^\infty \int_\Omega \Theta_\varepsilon \Delta \widehat{\varphi} \\ & \rightarrow -\int_0^\infty \int_\Omega \Theta \widehat{\varphi}_t - \int_\Omega \Theta_0 \widehat{\varphi}(\cdot, 0) - D \int_0^\infty \int_\Omega \Theta \Delta \widehat{\varphi} \end{aligned} \quad (7.15)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since in view of (1.11), (1.13), (7.7) and (7.4) we may employ Lemma 6.3 to infer thanks to the boundedness of $\text{supp } \widehat{\varphi}$ that

$$\int_0^\infty \int_\Omega \langle \Gamma(\Theta) : \nabla^s v, \nabla^s v \rangle \widehat{\varphi} \leq \liminf_{\varepsilon = \varepsilon_j \searrow 0} \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \widehat{\varphi},$$

from (7.14) and (7.15) we obtain that indeed the inequality in (2.4) holds, whereby the proof is completed. \square

The most crucial step in our analysis, however, can be found in the following argument which now makes full use of the results from Section 6 to derive (2.5) after an appropriate rearrangement of the key contributions to the corresponding dissipation rate (see (2.7)) as essentially quadratic expressions (cf. (7.30)).

Lemma 7.4 *Let u and Θ be as in Lemma 7.3. Then there exist $\kappa > 0, \lambda > 0$ and $\mu > 0$ such that (2.5) is valid for each nonnegative $\psi \in C_0^\infty(\Omega; \mathbb{R})$ and any nonincreasing $\zeta \in C_0^\infty([0, \infty); \mathbb{R})$.*

PROOF. We let $K_\gamma > 0$ be as in (1.12), and taking $\eta_1(\cdot)$ and $\eta_2(\cdot)$ from Lemma 6.4 and Lemma 6.6, we fix $\kappa > 0$ in such a way that

$$\frac{\kappa}{a} < \eta_2(\gamma). \quad (7.16)$$

We thereupon choose $\lambda > 0$ small enough fulfilling

$$\frac{\kappa}{a} \eta_1(\Gamma) > \lambda \quad (7.17)$$

and pick some $\mu > 0$ suitably large such that

$$\frac{\kappa(a + 2\mu)}{4} \eta_1(\gamma) > \frac{a^2}{4}. \quad (7.18)$$

To verify that then (2.5) holds for any nonnegative $\psi \in C_0^\infty(\Omega; \mathbb{R})$ and any $\zeta \in C_0^\infty([0, \infty); \mathbb{R})$ with $\zeta_t \leq 0$, letting

$$\mathcal{F}_\varepsilon \equiv \mathcal{F}_\varepsilon^{(\kappa, \lambda, \psi)}(x, t) := \left(\frac{1}{2} |v_\varepsilon|^2 + \frac{\kappa}{2} |\nabla u_\varepsilon|^2 + \lambda \Theta_\varepsilon \right) \psi \quad (7.19)$$

we go back to Lemma 3.1, Lemma 3.2 and Lemma 3.3 and thereby see that since $\text{supp } \zeta$ is bounded and $\psi|_{\partial\Omega} = 0$,

$$\begin{aligned} -\zeta(0) \int_\Omega \mathcal{F}_\varepsilon(\cdot, 0) &= \int_0^\infty \int_\Omega \partial_t \{ \mathcal{F}_\varepsilon \zeta(t) e^{-\mu t} \} \\ &= \int_0^\infty \int_\Omega \left\{ \frac{1}{2} (|v_\varepsilon|^2)_t \psi + \frac{\kappa}{2} (|\nabla u_\varepsilon|^2)_t \psi + \lambda \Theta_{\varepsilon t} \psi \right\} \zeta(t) e^{-\mu t} \\ &\quad + \int_0^\infty \int_\Omega \left\{ \frac{1}{2} |v_\varepsilon|^2 \psi + \frac{\kappa}{2} |\nabla u_\varepsilon|^2 \psi + \lambda \Theta_\varepsilon \psi \right\} \{ \zeta_t(t) - \mu \zeta(t) \} e^{-\mu t} \\ &= \int_0^\infty \int_\Omega \left\{ -\langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \psi - \varepsilon |\Delta v_\varepsilon|^2 \psi \right. \\ &\quad \left. - a \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \psi + \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \right. \\ &\quad \left. + a \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \right. \\ &\quad \left. - 2\varepsilon (\nabla v_\varepsilon \cdot \nabla \psi) \Delta v_\varepsilon - \varepsilon (v_\varepsilon \cdot \Delta v_\varepsilon) \Delta \psi \right\} \zeta(t) e^{-\mu t} \\ &\quad + \kappa \int_0^\infty \int_\Omega \left\{ -\varepsilon |\Delta u_\varepsilon|^2 \psi + \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \psi - \varepsilon (\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon \right\} \zeta(t) e^{-\mu t} \\ &\quad + \lambda \int_0^\infty \int_\Omega \left\{ D\Theta_\varepsilon \Delta \psi + \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \psi \right\} \zeta(t) e^{-\mu t} \\ &\quad + \int_0^\infty \int_\Omega \left\{ \frac{1}{2} |v_\varepsilon|^2 \psi + \frac{\kappa}{2} |\nabla u_\varepsilon|^2 \psi + \lambda \Theta_\varepsilon \psi \right\} \{ \zeta_t(t) - \mu \zeta(t) \} e^{-\mu t} \\ &= - \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi - a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\ &\quad - \frac{\kappa \mu}{2} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi + \kappa \int_0^\infty \int_\Omega \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \end{aligned}$$

$$\begin{aligned}
& +\lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\
& -J^{(1)}(\varepsilon) + J^{(2)}(\varepsilon) \quad \text{for all } \varepsilon \in (0, 1),
\end{aligned} \tag{7.20}$$

where

$$\begin{aligned}
J^{(1)}(\varepsilon) & := \int_0^\infty \int_\Omega \left(\frac{1}{2} |v_\varepsilon|^2 + \lambda \Theta_\varepsilon \right) (\mu \zeta(t) - \zeta_t(t)) e^{-\mu t} \psi - \frac{\kappa}{2} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi \\
& + \varepsilon \int_0^\infty \int_\Omega |\Delta v_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi + \kappa \varepsilon \int_0^\infty \int_\Omega |\Delta u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi, \quad \varepsilon \in (0, 1),
\end{aligned} \tag{7.21}$$

and

$$\begin{aligned}
J^{(2)}(\varepsilon) & := \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} + a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \\
& + \lambda D \int_0^\infty \int_\Omega \Theta_\varepsilon \zeta(t) e^{-\mu t} \Delta \psi \\
& - 2\varepsilon \int_0^\infty \int_\Omega \{ (\nabla v_\varepsilon \cdot \nabla \psi) \cdot \Delta v_\varepsilon \} \zeta(t) e^{-\mu t} - \varepsilon \int_0^\infty \int_\Omega (v_\varepsilon \cdot \Delta v_\varepsilon) \zeta(t) e^{-\mu t} \Delta \psi \\
& - \kappa \varepsilon \int_0^\infty \int_\Omega \{ (\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon \} \zeta(t) e^{-\mu t}, \quad \varepsilon \in (0, 1).
\end{aligned} \tag{7.22}$$

Here since Lemma 6.1 along with (7.4), (7.7) and the boundedness of γ again ensures that with $(\varepsilon_j)_{j \in \mathbb{N}}$ taken from Lemma 7.3 we have $\gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon \rightharpoonup \gamma(\Theta) : \nabla^s v$ in $L_{loc}^2(\bar{\Omega} \times [0, \infty); \mathbb{R}^{n \times n})$ as $\varepsilon = \varepsilon_j \searrow 0$, the strong convergence property in (7.3) implies that

$$\int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \rightarrow \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s v, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{7.23}$$

because $T := \sup\{t > 0 \mid \zeta(t) \neq 0\}$ is finite. Similarly, (7.6), (7.7) and (7.3) guarantee that

$$a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, v_\varepsilon \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \rightarrow a \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \tag{7.24}$$

while (7.7) and the compactness of $\text{supp } \Delta \psi \subset \Omega$ entail that

$$\lambda D \int_0^\infty \int_\Omega \Theta_\varepsilon \zeta(t) e^{-\mu t} \Delta \psi \rightarrow \lambda D \int_0^\infty \int_\Omega \Theta \zeta(t) e^{-\mu t} \Delta \psi \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{7.25}$$

To see that the last three summands vanish in the limit $\varepsilon \searrow 0$, we abbreviate $c_1 := \|\zeta\|_{L^\infty((0, \infty))}$, $c_2 := \|\nabla \psi\|_{L^\infty(\Omega)}$ and $c_3 := \|\Delta \psi\|_{L^\infty(\Omega)}$, and use the Cauchy-Schwarz inequality in estimating

$$\begin{aligned}
\left| -2\varepsilon \int_0^\infty \int_\Omega \{ (\nabla v_\varepsilon \cdot \nabla \psi) \cdot \Delta v_\varepsilon \} \zeta(t) e^{-\mu t} \right| & \leq 2c_1 c_2 \varepsilon \int_0^T \int_\Omega |\nabla v_\varepsilon| |\Delta v_\varepsilon| \\
& \leq 2c_1 c_2 \varepsilon^{\frac{1}{2}} \left\{ \varepsilon \int_0^T \int_\Omega |\Delta v_\varepsilon|^2 \right\}^{\frac{1}{2}} \left\{ \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned} \left| -\varepsilon \int_0^\infty \int_\Omega (v_\varepsilon \cdot \Delta v_\varepsilon) \zeta(t) e^{-\mu t} \Delta \psi \right| &\leq c_1 c_3 \varepsilon \int_0^T \int_\Omega |v_\varepsilon| |\Delta v_\varepsilon| \\ &\leq c_1 c_3 \varepsilon^{\frac{1}{2}} \left\{ \varepsilon \int_0^T \int_\Omega |\Delta v_\varepsilon|^2 \right\}^{\frac{1}{2}} \left\{ \int_0^T \int_\Omega |v_\varepsilon|^2 \right\}^{\frac{1}{2}} \end{aligned}$$

as well as

$$\begin{aligned} \left| -\kappa \varepsilon \int_0^\infty \int_\Omega \{(\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon\} \zeta(t) e^{-\mu t} \right| &\leq c_1 c_2 \kappa \varepsilon \int_0^T \int_\Omega |\nabla u_\varepsilon| |\Delta u_\varepsilon| \\ &\leq c_1 c_2 \kappa \varepsilon^{\frac{1}{2}} \left\{ \varepsilon \int_0^T \int_\Omega |\Delta u_\varepsilon|^2 \right\}^{\frac{1}{2}} \left\{ \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \right\}^{\frac{1}{2}} \end{aligned}$$

for all $\varepsilon \in (0, 1)$. In consequence, from (3.11), (3.10), (3.8), (3.12) and (3.9) we thus infer that, indeed,

$$\begin{aligned} -2\varepsilon \int_0^\infty \int_\Omega \{(\nabla v_\varepsilon \cdot \nabla \psi) \cdot \Delta v_\varepsilon\} \zeta(t) e^{-\mu t} - \varepsilon \int_0^\infty \int_\Omega (v_\varepsilon \cdot \Delta v_\varepsilon) \zeta(t) e^{-\mu t} \Delta \psi \\ - \kappa \varepsilon \int_0^\infty \int_\Omega \{(\nabla u_\varepsilon \cdot \nabla \psi) \cdot \Delta u_\varepsilon\} \zeta(t) e^{-\mu t} \rightarrow 0 \quad \text{as } \varepsilon \searrow 0, \end{aligned}$$

whence (7.22)-(7.25) imply that

$$\begin{aligned} J^{(2)}(\varepsilon) &\rightarrow \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s v, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} + a \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \\ &\quad + \lambda D \int_0^\infty \int_\Omega \Theta \zeta(t) e^{-\mu t} \Delta \psi \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \end{aligned} \quad (7.26)$$

Next addressing (7.21), we may rely on the fact that both ζ and ψ are nonnegative to simply estimate

$$\varepsilon \int_0^\infty \int_\Omega |\Delta v_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi + \kappa \varepsilon \int_0^\infty \int_\Omega |\Delta u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi \geq 0 \quad \text{for all } \varepsilon \in (0, 1), \quad (7.27)$$

and use the pointwise approximation features contained in (7.3) and (7.7) to see that since also $-\zeta_t$ is nonnegative, Fatou's lemma implies that

$$\int_0^\infty \int_\Omega \left(\frac{1}{2} |v|^2 + \lambda \Theta \right) (\mu \zeta(t) - \zeta_t(t)) e^{-\mu t} \psi \leq \liminf_{\varepsilon = \varepsilon_j \searrow 0} \int_0^\infty \int_\Omega \left(\frac{1}{2} |v_\varepsilon|^2 + \lambda \Theta_\varepsilon \right) (\mu \zeta(t) - \zeta_t(t)) e^{-\mu t} \psi. \quad (7.28)$$

Since in view of (7.6) the lower semicontinuity of the norm in $L^2(\Omega \times (0, T); \mathbb{R}^{n \times n})$ ensures that

$$\begin{aligned} -\frac{\kappa}{2} \int_0^\infty \int_\Omega |\nabla u|^2 \zeta_t(t) e^{-\mu t} \psi &= \frac{\kappa}{2} \int_0^T \int_\Omega \left| \sqrt{|\zeta_t(t)| e^{-\mu t} \psi} \nabla u \right|^2 \\ &\leq \frac{\kappa}{2} \liminf_{\varepsilon = \varepsilon_j \searrow 0} \int_0^T \int_\Omega \left| \sqrt{|\zeta_t(t)| e^{-\mu t} \psi} \nabla u_\varepsilon \right|^2 \\ &= \liminf_{\varepsilon = \varepsilon_j \searrow 0} \left\{ -\frac{\kappa}{2} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta_t(t) e^{-\mu t} \psi \right\}, \end{aligned}$$

a combination of (7.27) with (7.28) shows that

$$\begin{aligned} & \int_0^\infty \int_\Omega \left(\frac{1}{2} |v|^2 + \lambda \Theta \right) (\mu \zeta(t) - \zeta_t(t)) e^{-\mu t} \psi - \frac{\kappa}{2} \int_0^\infty \int_\Omega |\nabla u|^2 \zeta_t(t) e^{-\mu t} \psi \\ & \leq \liminf_{\varepsilon = \varepsilon_j \searrow 0} J^{(1)}(\varepsilon). \end{aligned} \quad (7.29)$$

Henceforth directing our attention toward the crucial first five summands on the right of (7.20), we create quadratic expression therein by observing that thanks to the symmetry property (1.9),

$$\begin{aligned} \langle \gamma(\Theta_\varepsilon) : \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon), \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon) \rangle &= \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle + \frac{a}{2} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \\ & \quad + \frac{a}{2} \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s u_\varepsilon \rangle + \frac{a^2}{4} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s u_\varepsilon \rangle \\ &= \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle + a \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \\ & \quad + \frac{a^2}{4} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s u_\varepsilon \rangle \end{aligned}$$

and hence

$$\begin{aligned} & \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle + a \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \\ & = \langle \gamma(\Theta_\varepsilon) : \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon), \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon) \rangle - \frac{a^2}{4} \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s u_\varepsilon \rangle \end{aligned}$$

in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$, and that similarly,

$$-\kappa \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle = -\frac{\kappa}{a} |\nabla(v_\varepsilon + \frac{a}{2}u_\varepsilon)|^2 + \frac{\kappa}{a} |\nabla v_\varepsilon|^2 + \frac{\kappa a}{4} |\nabla u_\varepsilon|^2$$

in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$. Therefore, indeed,

$$\begin{aligned} & \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi + a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\ & + \frac{\kappa \mu}{2} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \kappa \int_0^\infty \int_\Omega \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\ & - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\ & = \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon), \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon) \rangle \zeta(t) e^{-\mu t} \psi - \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla(v_\varepsilon + \frac{a}{2}u_\varepsilon)|^2 \zeta(t) e^{-\mu t} \psi \\ & \quad + \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \\ & \quad + \frac{\kappa(a + 2\mu)}{4} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \frac{a^2}{4} \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s u_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \end{aligned} \quad (7.30)$$

for all $\varepsilon \in (0, 1)$. Now (7.16) together with (1.12) and (7.3)-(7.7) enables us to see upon an application of Lemma 6.6 to $\beta := \gamma$, to $w_j := v_\varepsilon + \frac{a}{2}u_\varepsilon$ and $z_j := \Theta_\varepsilon$ for $\varepsilon = \varepsilon_j$, and to $\varphi(x, t) := \zeta(t) e^{-\mu t} \psi(x)$ for $(x, t) \in \bar{\Omega} \times [0, T]$ that

$$\int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s(v + \frac{a}{2}u), \nabla^s(v + \frac{a}{2}u) \rangle \zeta(t) e^{-\mu t} \psi - \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla(v + \frac{a}{2}u)|^2 \zeta(t) e^{-\mu t} \psi$$

$$\leq \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon), \nabla^s(v_\varepsilon + \frac{a}{2}u_\varepsilon) \rangle \zeta(t) e^{-\mu t} \psi \right. \\ \left. - \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla(v_\varepsilon + \frac{a}{2}u_\varepsilon)|^2 \zeta(t) e^{-\mu t} \psi \right\},$$

while similarly from (7.17) and Lemma 6.4 we obtain that

$$\frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla v|^2 \zeta(t) e^{-\mu t} \psi - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta) : \nabla^s v, \nabla^s v \rangle \zeta(t) e^{-\mu t} \psi \\ \leq \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \right\},$$

and while (7.18) in view of Lemma 6.4 ensures that

$$\frac{\kappa(a+2\mu)}{4} \int_0^\infty \int_\Omega |\nabla u|^2 \zeta(t) e^{-\mu t} \psi - \frac{a^2}{4} \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, \nabla^s u \rangle \zeta(t) e^{-\mu t} \psi \\ \leq \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \frac{\kappa(a+2\mu)}{4} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \frac{a^2}{4} \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s u_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \right\}.$$

Since clearly

$$\zeta(0) \int_\Omega \mathcal{F}_\varepsilon(\cdot, 0) = \zeta(0) \int_\Omega \left(\frac{1}{2} |v_{0\varepsilon}|^2 + \frac{\kappa}{2} |\nabla u_{0\varepsilon}|^2 + \lambda \Theta_{0\varepsilon} \right) \psi \\ \rightarrow \zeta(0) \int_\Omega \left(\frac{1}{2} |u_{0t}|^2 + \frac{\kappa}{2} |\nabla u_0|^2 + \lambda \Theta_0 \right) \psi \quad \text{as } \varepsilon \searrow 0$$

by (2.11), in conjunction with (7.26) and (7.29) this shows that (7.20) and (7.30) imply the inequality

$$\int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s(v + \frac{a}{2}u) \nabla^s(v + \frac{a}{2}u) \rangle \zeta(t) e^{-\mu t} \psi - \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla(v + \frac{a}{2}u)|^2 \zeta(t) e^{-\mu t} \psi \\ + \frac{\kappa}{a} \int_0^\infty \int_\Omega |\nabla v|^2 \zeta(t) e^{-\mu t} \psi - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta) : \nabla^s v, \nabla^s v \rangle \zeta(t) e^{-\mu t} \psi \\ + \frac{\kappa(a+2\mu)}{4} \int_0^\infty \int_\Omega |\nabla u|^2 \zeta(t) e^{-\mu t} \psi - \frac{a^2}{4} \int_0^\infty \int_\Omega \langle \gamma(\Theta) : \nabla^s u, \nabla^s u \rangle \zeta(t) e^{-\mu t} \psi \\ + \int_0^\infty \int_\Omega \left(\frac{1}{2} |v|^2 + \lambda \Theta \right) (\mu \zeta(t) - \zeta_t(t)) e^{-\mu t} \psi - \frac{\kappa}{2} \int_0^\infty \int_\Omega |\nabla u|^2 \zeta_t(t) e^{-\mu t} \psi \\ \leq \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi + a \int_0^\infty \int_\Omega \langle \gamma(\Theta_\varepsilon) : \nabla^s u_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \right. \\ \left. + \frac{\kappa\mu}{2} \int_0^\infty \int_\Omega |\nabla u_\varepsilon|^2 \zeta(t) e^{-\mu t} \psi - \kappa \int_0^\infty \int_\Omega \langle \nabla u_\varepsilon, \nabla v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \right. \\ \left. - \lambda \int_0^\infty \int_\Omega \langle \Gamma(\Theta_\varepsilon) : \nabla^s v_\varepsilon, \nabla^s v_\varepsilon \rangle \zeta(t) e^{-\mu t} \psi \right. \\ \left. + J^{(1)}(\varepsilon) \right\}$$

$$\begin{aligned}
&= \liminf_{\varepsilon=\varepsilon_j \searrow 0} \left\{ \zeta(0) \int_{\Omega} \mathcal{F}_{\varepsilon}(\cdot, 0) + J^{(2)}(\varepsilon) \right\} \\
&= \zeta(0) \int_{\Omega} \left(\frac{1}{2} |u_{0t}|^2 + \frac{\kappa}{2} |\nabla u_0|^2 + \lambda \Theta_0 \right) \psi \\
&\quad + \int_0^{\infty} \int_{\Omega} \langle \gamma(\Theta) : \nabla^s v, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} + a \int_0^{\infty} \int_{\Omega} \langle \gamma(\Theta) : \nabla^s u, v \otimes \nabla \psi \rangle \zeta(t) e^{-\mu t} \\
&\quad + \lambda D \int_0^{\infty} \int_{\Omega} \Theta \zeta(t) e^{-\mu t} \Delta \psi.
\end{aligned} \tag{7.31}$$

In line with (2.6), (2.7), (2.8) and (2.9), rewriting the left-hand side herein in the style of (7.30) leads to (2.5), because $u_t = v$ a.e. in $\Omega \times (0, \infty)$ by (7.8). \square

The proof of our main result has thereby actually been completed:

PROOF of Theorem 1.1. Taking (u, Θ) from Lemma 7.3, we only need to combine the latter with Lemma 7.4. \square

8 Numerical experiments

This section intends to conduct some numerical experiments to illustrate some possible influences of temperature-dependent material parameters on the results of acoustic processes. Specifically, the coupled thermal and mechanical behavior of a one-dimensional acoustic resonator is considered using a finite-difference time-domain (FDTD) method [63, 13], and to facilitate a connection to corresponding literature we shall here return to the original variables by considering the evolution system

$$\begin{aligned}
\rho u_{tt} &= \tau C(\Theta) u_{xxt} + C(\Theta) u_{xx}, \\
c\rho \Theta_t &= \lambda \Theta_{xx} + \tau C(\Theta) u_{xt}^2,
\end{aligned} \tag{8.32}$$

in which all considered quantities are scalar.

The setup roughly approximates an actively cooled ($\Theta = 0$ at the boundaries), mechanically clamped ($u = u_t = 0$ at the boundaries) layer that is driven to oscillate at its resonance frequency. The viscous wave equation and the heat equation in (8.32) are solved with parameter values listed in Table 1. The elasticity C of the material is assumed to be temperature dependent, using two different models. To explore temperature dependent behavior that is akin to the observations recorded in Figure 1, a power-law in temperature is used,

$$C = C_0 \cdot (1 + k\Theta^p), \tag{8.33}$$

neglecting here the boundedness requirements made in Theorem 1.1. For the analysis of convergent behavior, an exponential law is presupposed,

$$C = C_0 \cdot (\alpha + (1 - \alpha)e^{-b\Theta}), \tag{8.34}$$

which yields $C = C_0$ for $\Theta = 0$ and converges to $C = \alpha \cdot C_0$ in the limit of large positive values of Θ .

The thickness of the resonator is 1 mm, resulting in a resonance frequency (at $\Theta = 0$) of 2 MHz. A continuous sinusoidal signal at this frequency, implemented as a time-dependent Dirichlet boundary

Table 1: Material parameters for the numerical study. Values approximate the mechanical and thermal properties of a piezoelectric ceramic [16, 41].

| Parameter | Value | Unit |
|-----------|-------|-----------------------------------|
| C_0 | 124.8 | GPa |
| ρ | 7800 | kg m^{-3} |
| τ | 1 | ns |
| c | 350 | $\text{J K}^{-1} \text{ kg}^{-1}$ |
| λ | 1.1 | $\text{W m}^{-1} \text{ K}^{-1}$ |

condition for the velocity u_t at the centre of the resonator, is applied to excite the system. Due to the high mechanical frequency and comparatively slow thermal processes, the simulation requires large number of steps ($5 \cdot 10^5$) in time-domain to show significant effects. The spatial and temporal resolution are chosen to over-satisfy the conditions of stability for the solution of wave equations [47] by a factor of 2.5 ($\Delta x / \Delta t = 2.5 \cdot c_{\text{ph}}$).

To analyse the mechanical and thermal behavior of the resonator, the temperature and the velocity u_t are observed. For a clearer depiction, the normalized mean temperature and the envelope of the velocity signal are shown. The results presented in Figure 2 constitute an archetypical behavior observed for many parameter value tuples (k, p) and (α, b) : Initially, the velocity of the oscillation increases rapidly because of resonance. The mean temperature of the material initially shows superlinear increase due to the mechanical losses, causing the elasticity to change and thus a shift in the resonance frequency of the oscillator. Because the excitation remains at 2 MHz, the system is no longer excited in resonance, leading to a reduced oscillation velocity after a short period of beating. The trend of the mean temperature increase thus changes to a sublinear behavior. The overall behavior of the mean temperature and the velocity envelope is similar for large ranges of k and p of the power law as well as α and b for the exponential expression, with only quantitative variations, e.g. when the velocity envelope begins to decrease. However, there are a number of configurations, especially for large values of k , where the simulation destabilises and the temperature field values overflow. Similar behavior can be observed when using the convergent, exponential expression to model the temperature dependence, however, overflow can only be brought about if the increase in elasticity occurs sudden and steep at the start of the simulation, e.g. both parameters α and b need to have high values. It is still to be shown if this behavior results from numerical problems with the finite-difference scheme or is inherent to the system of equations.

For further analysis, the spatial temperature distributions for different values of k and p for the power law and α and b convergent material behavior are examined. The simulation results for the time-dependent temperature distribution for four different configurations are visualized in Figure 3. It is immediately visible that qualitatively different results may arise, depending on the chosen parameter values. Due to the mechanical boundary conditions, the temperature increase is expected to be maximal at the boundary of the system ($x = 0 \text{ mm}$ and $x = 1 \text{ mm}$). However, the thermal boundary conditions force the temperature to be zero at the same boundaries, resulting in the distributions shown in Figures 3a, 3d or 3f, with local maxima close to but not at the spatial boundaries. Aside from these results, qualitatively different temperature distributions (Figures 3b and 3e) are also observed. In these cases, the temperature field forms a number of distinct, small hot spots along the spatial axis. The number and periodicity of these hot spots depends on the choice of the parameters. There are also

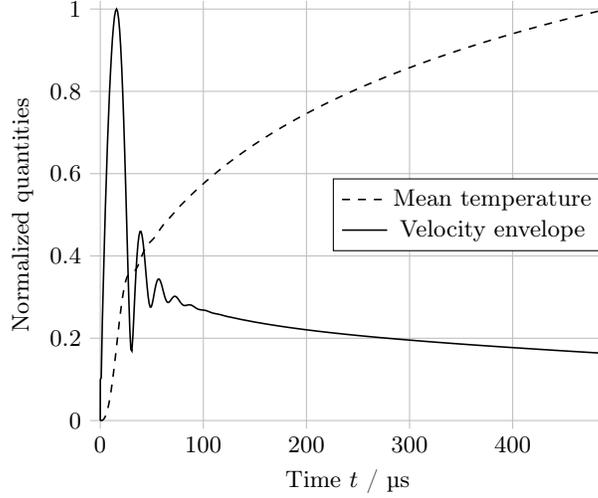


Figure 2: Typical result for the mean temperature and for the envelope of the mechanical oscillation from a coupled thermo-acoustic simulations of a one-dimensional resonator. Parameters for the temperature dependence power-law of the elasticity are $k = 10^7$ and $p = 1$.

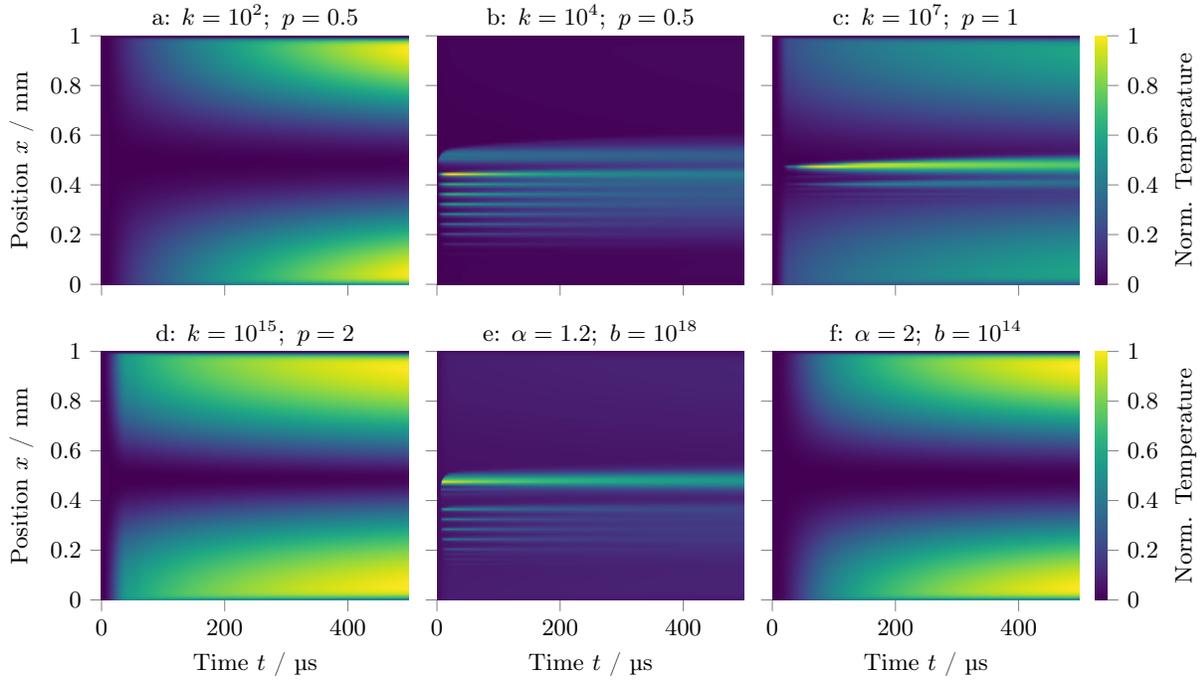


Figure 3: Results for the thermal field for coupled thermo-acoustic simulations of a one-dimensional resonator with different values and different models for the dependence of the elasticity on temperature (a to d: power law, (8.33); e and f: convergent exponential expression, (8.34)).

observations, which show a superposition of the expected behavior and hot spots (Figure 3c). Hot spots are observed primarily for larger values of k when applying the power law, which is obvious when comparing Figures 3a and 3b. Increasing k further (e.g. to 10^5) causes the simulation to destabilise. Additionally, observations show that the temperature distribution will not show hot spots for values of $p \leq 2$ for the power law. For the convergent behavior, hot spots occur primarily for large values of b , e.g. when the initial increase in elasticity C is large. If the observed, qualitative differences in the spatial temperature distribution arise from effects of imminent numerical instability or if they also exist in physical systems is a subject to be explored in future research. If it is found that the observed effects (formation of hot spots and unexpected rapid temperature increases) can occur in physical systems, even ideal cooling systems for acoustic resonators, such as high power piezoelectric actuators, may not be sufficient to keep such a system under stable operating conditions. Because the cause for this behavior is an unfavourable temperature-dependence of the material, it may rule out certain material classes for an application in these systems. Even if the observed artefacts are caused by numerical effect, further study of the coupled thermal and mechanical equation system is necessary to develop predictably stable simulation environments in the future.

Acknowledgement. MW and LC acknowledge support of the *Deutsche Forschungsgemeinschaft* (No. 444955436).

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