

Graphical models for topological groups: A case study on countable Stone spaces

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Abstract

By analogy with the Cayley graph of a group with respect to a finite generating set or the Cayley–Abels graph of a totally disconnected, locally compact group, we introduce countable connected graphs associated to Polish groups that we term Cayley–Abels–Rosendal graphs. A group admitting a Cayley–Abels–Rosendal graph acts on it continuously, coarsely metrically properly and cocompactly by isometries of the path metric. By an expansion of the Milnor–Schwarz lemma, it follows that the group is generated by a coarsely bounded set and the group equipped with (a suitably compatible alteration of) a word metric with respect to a coarsely bounded generating set and the graph are quasi-isometric. In other words, groups admitting Cayley–Abels–Rosendal graphs are topological analogues of the finitely generated groups. Our goal is to introduce this topological perspective to a geometric group theorist.

We apply these concepts to homeomorphism groups of countable Stone spaces. We completely characterize when these homeomorphism groups are coarsely bounded, when they are locally bounded (all of them are), and when they admit a Cayley–Abels–Rosendal graph, and if so produce a coarsely bounded generating set.

1 Introduction

A *Cayley–Abels graph* for a totally disconnected, locally compact group G is a connected, locally finite graph Γ on which G acts continuously, vertex transitively, and with compact stabilizers. Such a group G is then compactly generated and in fact quasi-isometric to Γ when equipped with a word metric with respect to a compact generating set. (This metric makes G discrete, but one can exchange it with a quasi-isometric metric additionally generating the topology on G .) Cayley–Abels graphs are the extension of the notion of a Cayley graph (with respect to a finite generating set) to locally compact groups.

We expand the notion further to the setting of Polish groups, introducing the notion of a *Cayley–Abels–Rosendal graph*. Briefly for experts, a Cayley–Abels–Rosendal graph for a group G is a connected, countable graph Γ on which G acts continuously, vertex transitively, with coarsely bounded stabilizers and finitely many orbits of edges. The group G is then generated by a coarsely bounded set, and with respect to the word metric associated to such a generating set becomes quasi-isometric to Γ .

Just as Cayley graphs are a powerful way to study finitely generated groups geometrically, so too do we believe that Cayley–Abels–Rosendal graphs are fundamental to the large-scale geometric study of Polish groups.

To illustrate their use, we construct Cayley–Abels–Rosendal graphs for a family of homeomorphism groups. A *Stone space* is compact, Hausdorff, and totally disconnected. Stone spaces are dual to Boolean algebras, and second countable Stone spaces show up as spaces of ends of, for example, surfaces or locally finite graphs. The group of homeomorphisms of a Stone space is a

non-Archimedean Polish group, and may profitably be thought of as a “0-dimensional” analogue of the mapping class group of a locally finite infinite graph or surface of infinite type.

Countable Stone spaces turn out to be classified by two parameters: a positive integer n and a countable ordinal α . Any such Stone space is homeomorphic to $\omega^\alpha \cdot n + 1$ in the order topology. Recall that ordinals are either *successor* ordinals, being of the form $\alpha = \beta + 1$, or they are not, in which case they are *limit* ordinals.

Theorem A. Let $E = \omega^\alpha \cdot n + 1$ be a countable Stone space. The group $\text{Homeo}(E)$ is always locally bounded and is

- coarsely bounded if and only if $n = 1$, and
- boundedly generated but not coarsely bounded if and only if $n > 1$ and α is a successor ordinal.

Moreover, if $\text{Homeo}(E)$ is boundedly generated, we compute its quasi-isometry type by constructing a Cayley–Abels–Rosendal graph for it.

Rosendal [Ros22], inspired by work of Roe [Roe03] has expanded the framework of coarse geometry from the realm of discrete or locally compact groups to the wider setting of Polish topological groups. It is also our intention in this paper to make the fundamentals of this theory more accessible to the mathematician familiar with the basic techniques of geometric group theory.

Here are the salient features of this work, intended for a geometric group theorist. First: a *Polish* topological space is one that is *separable*, having a countable dense set, and *completely metrizable*—that is, not just admitting a metric generating the topology, but one that is additionally complete.

Now, there are *a priori* many topologies one can put on a group G . As geometric group theorists, we are interested in the *metrizable* topologies, since G should be able to act on itself by isometries. In fact, this latter observation suggests that we are really interested in *left-invariant* (or right-invariant) *compatible metrics* on G ; that is, those metrics that generate the given topology, which is then *a fortiori* metrizable, and that are compatible with the group structure. As it turns out, by the Birkhoff–Kakutani metrization theorem [Bir36, Kak36], every metrizable topological group admits a compatible left-invariant metric, and moreover a topological group is metrizable if and only if it is Hausdorff and first countable (i.e. every point, or equivalently just the identity, has a countable neighborhood basis).

Even allowing that (metrizable) topological groups are interesting, it is not immediately clear from a geometric group theory point of view why Polish groups might be the “right” objects to consider. Indeed, an abstract group G may admit many metrizable topologies. Let us offer this suggestion: the *discrete* topology on a group G is Polish only when the required countable dense subset of G is G itself—so for Polish groups, the discrete groups are countable. As it turns out, for Polish groups, it is also true that the countable groups are discrete: a topological group G has an isolated point if and only if the identity is isolated if and only if it is discrete, but a nonempty complete metric space with no isolated points is uncountable.

Geometric group theorists remember that although one can define a “word metric” on a (discrete) group G as soon as one has a generating set for G , in order for two word metrics on a group G to yield quasi-isometric metric spaces, one needs in general for these generating sets to be *finite*. The reader familiar with locally compact groups may realize that *finite* may profitably be replaced with *compact*, but may wonder what one does without local compactness.

Moreover, the above framing of the “fundamental observation” of geometric group theory—i.e. the Milnor–Schwarz Lemma—actually sells it short: if one has a (metrically) properly discontinuous, cobounded action of a group G on a geodesic metric space X , one concludes that G is *in fact* finitely generated—with care, one can even extract a finite generating set—and G equipped with *any* word metric with respect to some finite generating set is quasi-isometric to the space X .

(For finitely generated groups, one can always consider for X a Cayley graph for G with respect to some finite generating set. This space X is then additionally *proper*, that is, closed balls are compact, so one can replace “metrically properly discontinuous” with “properly discontinuous” and “cobounded” with “cocompact” with no loss.)

As it turns out, the correct enlargement of finiteness or compactness is the notion of *coarse boundedness*, and that with this expanded notion, one still has a Milnor–Schwarz Lemma. This allows one to compute quasi-isometry types of those Polish groups that admit them. Moreover, as it turns out, the only coarsely bounded subsets of a discrete group are the finite ones, so this expanded field is really a conservative extension of the theory: for countable discrete groups, it continues to pick out the finitely generated ones.

2 Coarsely bounded subsets and the Milnor–Schwarz Lemma

A subset A of a topological group G is said to be *coarsely bounded in G* if whenever G acts continuously by isometries on a metric space X , some and hence every orbit of A is bounded in X .

Let us remark that it is clear that if $A \subset G$ is finite (or precompact), then it is coarsely bounded. The converse is also true for discrete (or locally compact) groups. If G is finitely generated, this is clear by examining the action of an infinite subset A on some Cayley graph of G with respect to a finite generating set.

Although the definition above is conceptually clear, it is functionally impossible to verify in practice. Fortunately, Rosendal gives us the following formulation. We give a proof for completeness.

Lemma 1 (Proposition 2.15 of [Ros22]). *Suppose that G is metrizable and that for every identity neighborhood U , there exists a countable subset C of G such that G is generated by $U \cup C$. (This holds for G Polish, for instance.)*

A subset A of G is coarsely bounded in G if and only if for every identity neighborhood $U \subset G$, there exists a finite subset $F \subset G$ and $N \in \mathbb{N}$ such that

$$A \subset (FU)^N = \{f_1 u_1 \dots f_\ell u_\ell : \ell \leq N, f_i \in F \text{ and } u_i \in U\}.$$

Proof. Suppose first that A satisfies Rosendal’s criterion for coarse boundedness, and let X be a metric space equipped with a continuous, isometric action of G and $x_0 \in X$ a basepoint. For each $\epsilon > 0$, the subset of G moving x_0 a distance at most ϵ is an open neighborhood of the identity in G : indeed by continuity, the map $G \times X \rightarrow X$ given by $(g, x) \mapsto g.x$ is continuous, and the set we are interested in is the preimage of the ball of radius ϵ centered at x_0 under the continuous map $g \mapsto (g, x_0) \mapsto g.x_0$. Call this identity neighborhood U . For each finite subset F of G and $k \in \mathbb{N}$, observe that the orbit of x_0 under $(FU)^N$, and hence A , is bounded. Therefore if A satisfies Rosendal’s criterion, it is coarsely bounded in G .

Next suppose that A is coarsely bounded in G . Since G is metrizable by assumption, we may take a left-invariant compatible metric on G . By compatibility, for any identity neighborhood U in G , we can find $\epsilon > 0$ such that U contains the ball of radius ϵ about the identity in G , B_ϵ . By assumption, G is generated by B_ϵ together with some countable set $\{x_1, x_2, \dots\}$. Let $F_n = \{x_1, \dots, x_n\}$ and consider the sets $V_n = (F_n B_\epsilon)^{2n}$. Observe that by the triangle inequality, we have that each V_n is bounded and their union is all of G . Since A is coarsely bounded in G , it is contained in some V_n and hence satisfies Rosendal’s criterion. \square

Suppose that G acts continuously by isometries on a metric space X . The space X is *geodesic* if between any two points of X there exists a rectifiable curve from one to the other whose length is equal to the distance between them—such a curve is a *geodesic*. G is *locally bounded* (some authors

have “locally CB”) if it has a coarsely bounded identity neighborhood and *boundedly generated* (some authors have “CB generated”) if it is generated by a coarsely bounded set.

Let us remark that *a priori* bounded generation appears to be somewhat orthogonal to local boundedness. This is not so for Polish groups, as we now show.

Lemma 2. *Suppose that G is a boundedly generated Polish group. Then G is locally bounded.*

First, some useful terminology. A subset S of a space X is *nowhere dense* if its closure has empty interior. A countable union of nowhere dense sets is said to be *meagre*, while a set containing a countable intersection of open and dense sets is said to be *residual*. A quick set-theoretical argument shows that the residual sets are exactly the *comeagre sets*, i.e. those whose complement is meagre. A set is *non-meagre* if it is not meagre. The *Baire category theorem* says that in a Polish space, residual sets are dense.

Observe that if N is nowhere dense, then $X - \bar{N}$ is open and dense. If some nonempty open set U of a Polish space X (or just a space satisfying the Baire category theorem) were meagre, say equal to $\bigcup_{n \in \mathbb{N}} N_n$ where each N_n is nowhere dense, we would have that the closed set $\overline{\bigcap_{n \in \mathbb{N}} (X - N_n)}$ is equal to the complement of U , in contradiction to the fact that it is residual and hence dense. Therefore in Baire spaces, open sets are non-meagre. It is also clear that a non-meagre closed set must have nonempty interior.

A subset $A \subset X$ has the *Baire property* if there exists an open set $U \subset X$ such that the symmetric difference $A \Delta U$ is meagre. Equivalently we may write A as the symmetric difference $A = U \Delta M$ where U is open and M is meagre. If X is Polish, we say that $A \subset X$ is *analytic* if it is the continuous image of some Polish space. Borel sets are analytic, and analytic subsets have the Baire property.

In fact we have a lemma.

Lemma 3 (Pettis’ Lemma, Theorem 1 of [Pet50]). *Suppose that A is a non-meagre analytic subset of a Polish topological group G . Then $A^{-1}A$ is an identity neighborhood.*

Proof. Since A has the property of Baire, we may write it as $W \Delta M$ for some open set W and some meagre set M . By continuity of the map $(g, h) \mapsto gh^{-1}$, every open neighborhood V of the identity contains an open neighborhood U such that $UU^{-1} \subset V$. Since arbitrary open sets are left-translates of open identity neighborhoods, we see that there exists $g \in G$ and an open identity neighborhood U such that $gUU^{-1} \subset W$. We claim that $U \subset A^{-1}A$. Indeed, take $h \in U$. Observe that $E_h = (G - M) \cap [(G - M)h^{-1}]$ is residual, being the intersection of two comeagre sets, so has nontrivial intersection with the open set gUh^{-1} . Take x in this intersection. We claim that x and xh are in A , whence $h \in A^{-1}$. Indeed, by the definition of E_h , both are in the complement of M , while by the containment $gUh^{-1} \subset gUU^{-1}$, we have that they are contained in W . \square

Proof of Lemma 2. Let S be a coarsely bounded set generating G ; we may assume that it is symmetric without loss. To say that S generates is equivalent to the observation that the union of the sequence S, S^2, \dots is all of G , hence the same is true of the sequence $\bar{S}, \bar{S}^2, \dots$. Observe that each S^k is coarsely bounded by Rosenthal’s criterion. By the definition of coarse boundedness, so are their closures, which have the property of Baire, being Borel. Since G is Polish it is non-meagre, so some \bar{S}^k must be non-meagre. Then $S^{2k} = (S^k)^{-1}S^k$ is a coarsely bounded identity neighborhood by Pettis’s lemma. \square

We come now to the Milnor–Schwarz Lemma. To state it, we need a couple definitions. Recall that in the setting of a topological group G acting (continuously) on a topological space X , the action is *proper* if the map $G \times X \rightarrow X \times X$ given by $(g, x) \mapsto (g.x, x)$ is a proper map; that is, preimages of compact sets are compact. When X is locally compact and Hausdorff, this is equivalent to the condition that for each compact set $K \subset X$, the set

$$\{g \in G : g.K \cap K \neq \emptyset\}$$

is compact in G . When G is discrete, the compact sets are finite and this latter condition is “proper discontinuity” of the action. When X is a (proper, i.e. closed metric balls are compact) metric space, we may instead consider closed metric balls B instead of compact sets and obtain the notion of a *metrically proper* action. Without the assumption that X is *proper*, these conditions are really different. If instead of compact, we require that the sets

$$\{g \in G : g.B \cap B \neq \emptyset\}$$

are coarsely bounded, we have the notion of a *coarsely metrically proper* group action.

A group action is *cocompact* (respectively *cobounded*) if there is a compact (respectively bounded) set whose translates cover the space.

Proposition 4 (Milnor–Schwarz Lemma). *Suppose that the Polish group G acts continuously, coarsely (metrically) properly and coboundedly by isometries on a geodesic metric space (X, d) . Then G is boundedly generated, and although the word metric with respect to a symmetric, analytic, coarsely bounded generating set need not be compatible with the topology on G (in fact, it will not be if G is not discrete), any such metric is quasi-isometric—via any orbit map—to (X, d) .*

Let us remark that in the proof we will need to use a metric which is not compatible with the topology on G . We will show that it is quasi-isometric to a metric that *is* compatible in Lemma 5 directly following the proof.

Proof. The proof is essentially the classical geometric group theory proof. Afficionados of the proof will recognize that we do not need the full strength of “geodesic.” We also do not need that G is Polish except in proving Lemma 5.

Let B be an open bounded set whose G -translates cover X , (say a sufficiently large metric ball) and consider the set $S \subset G$ of elements $g \in G$ such that $g.B \cap B \neq \emptyset$. Let us remark that by continuity this set is open: the sets $\{(g, x) : g.x \in B\}$ and $\{(g, x) : x \in B\}$ are open, and S is the image of their intersection under the (open) projection map from $G \times X$ to G . By coarse properness of the action, the set S is coarsely bounded.

Consider $g \in G$ and $x_0 \in B$. Let γ be a geodesic from x_0 to $g.x_0$. Since translates of B cover X , they certainly cover the image of γ , and by compactness of this image, finitely many suffice, say $B, g_1.B, \dots, g_k.B, g.B$, where we have ordered the g_i so that $g_i.B \cap g_{i+1}.B \neq \emptyset$, and we write $1 = g_0$ and $g = g_{k+1}$. By writing $s_i = g_i^{-1}g_{i+1}$, we see that each s_i for $0 \leq i \leq k$ belongs to S and that $g = s_0 \cdots s_k$. Therefore the set S is an open coarsely bounded generating set for G .

Next suppose that S is a symmetric, *analytic* coarsely bounded generating set for G , and consider the word metric on G with respect to S . Analyticity is necessary to prove in Lemma 5 below that this word metric is quasi-isometric to one with respect to an open, coarsely bounded generating set. Since S is coarsely bounded, we have that for any $x_0 \in X$, there exists M large such that for each $g \in S$, we have that $d(x_0, g.x_0) \leq M$. By the triangle inequality, if $g \in G$ is arbitrary, we have that $d(x_0, g.x_0)$ is bounded above by M times the word length of g with respect to S . Conversely, suppose that B is an open ball centered at x_0 of radius $2C$, where translates of the ball of radius C cover X . The collection S' of elements of G which fail to move B off of itself is coarsely bounded in G , hence by Lemma 5 to follow, we see that word lengths of elements in S' with respect to S are bounded by some constant M' . Write

$$k = \left\lceil \frac{d(x_0, g.x_0)}{C} \right\rceil.$$

We can find a sequence of $k + 1$ points $x_0, x_1, \dots, x_k = g.x_0$ on any geodesic from x_0 to $g.x_0$ such that the distance between successive points is at most C . Associated to these points we can find

$g_i \in G$ such that $x_i \in g_i.B$, from which we conclude that the word length of g with respect to S is at most

$$Mk + M \leq M \frac{d(x_0, g.x_0)}{C} + M,$$

showing that the word metric on G with respect to S is quasi-isometric to the metric on the orbit of x_0 , or equivalently by coboundedness, the metric on X . \square

To complete the proof, we need the following lemma. Analyticity is used in the proof in order to appeal to Pettis's lemma.

Lemma 5 (Lemma 2.51 of [Ros22]). *Suppose that d is a word metric on a Polish group G with respect to some symmetric, analytic coarsely bounded generating set S . There exists a left-invariant compatible metric ∂ on G such that (G, ∂) is quasi-isometric to (G, d) .*

Proof. First, observe that we may suppose that S is open. Indeed, if it is not already, observe that because G is Polish and boundedly generated, it is locally bounded by Lemma 2, so we may take some coarsely bounded, symmetric open identity neighborhood U . By Rosendal's criterion, there exists a finite (we may assume symmetric) set F such that $S \subset (FU)^k$. But this says precisely that S has bounded word lengths with respect to the open generating set FU . Conversely, FU is coarsely bounded, so if we can show that some S^k contains an identity neighborhood, we can conclude that FU has bounded word lengths with respect to S , and thus that the corresponding word lengths are bi-Lipschitz equivalent.

This is where we need analyticity of S : the point is that we yet again have $G = \bigcup_{k \in \mathbb{N}} S^k$, so some S^k is nonmeagre and has the property of Baire, so by Pettis's lemma, since S^k is symmetric, S^{2k} is an identity neighborhood. Using an open identity neighborhood $V \subset S^{2k}$ and Rosendal's criterion for coarse boundedness of FU , we see that FU has bounded word lengths with respect to S , so their word metrics are bi-Lipschitz equivalent.

Supposing that S is an open identity neighborhood, which is a symmetric and coarsely bounded generating set, write $\|g\|$ for the word norm on g with respect to S . Let d' be any compatible left-invariant metric on G with corresponding norm $|g| = d'(1, g)$. Because S is coarsely bounded, we have some constant M such that for each $h \in S$, we have $|h| \leq M$. Define a metric ∂ on G by the rule that

$$\partial(g, h) = \inf\{|s_1| + \dots + |s_k| : h^{-1}g = s_1 \dots s_k, s_i \in S\}.$$

First, it is not hard to argue that this really is a left-invariant metric on G , since d' is one. Since ∂ agrees with d' on the open set S by the triangle inequality and otherwise satisfies $\partial \geq d'$, we see that ∂ is compatible with the topology on G . Notice that we have $\partial(g, h) \leq M\|h^{-1}g\|$ by construction. On the other hand, because S is open, it contains the d' ball of some radius $2\epsilon > 0$ about the identity. Since $\partial(g, h)$ is defined as an infimum, we may take some actual sequence s_1, \dots, s_k from S such that $h^{-1}g = s_1 \dots s_k$ and such that $|s_1| + \dots + |s_k| \leq \partial(g, h) + 1$. Indeed, among such sequences, choose k to be the smallest possible. Now, notice that if $s_i s_{i+1}$ were in S , we could do better on choosing k , so in fact they do not belong to S . It follows that $|s_i s_{i+1}| \geq 2\epsilon$, whence by the triangle inequality at least one of $|s_i|$ or $|s_{i+1}|$ must be at least ϵ . Now, we have $1 + \partial(g, h) \geq |s_1| + \dots + |s_k| \geq \lfloor \frac{k}{2} \rfloor \cdot \epsilon$, and the word length $\|g^{-1}h\| \leq k$ by assumption, so

$$\frac{\epsilon\|g^{-1}h\|}{2} - 1 - \frac{\epsilon}{2} \leq \partial(g, h),$$

from which we conclude that the identity map from (G, d) to (G, ∂) is a quasi-isometry. \square

By the by, there is also a "finite generation" condition underlying the notion of bounded generation, as we show in the next lemma. Recall that for a countable discrete group G , being finitely generated is equivalent to the condition that whenever G may be written as the union of a chain of subgroups $G_1 \leq G_2 \leq \dots$, this chain terminates in G after finitely many steps.

Lemma 6 (cf. Theorem 2.30 of [Ros22]). *Suppose that G is locally bounded and Polish. Then G is boundedly generated if and only if it is not the union of a countably infinite chain of proper open subgroups.*

Proof. If G is boundedly generated, say by S , let $U = G_1$ be our first open subgroup. By Rosendal's criterion, there exists a finite set F and k such that $S \subset (FU)^k$. But F , and hence $(FU)^k$ and G , is therefore contained in some G_n , so the chain terminates in G after finitely many steps.

Conversely, since G is locally bounded and Polish, there exists an open, coarsely bounded set U and a countable collection $\{x_1, x_2, \dots\}$ of elements such that $G = \langle U, x_1, x_2, \dots \rangle$. But then by considering the open subgroups $G_n = \langle U, x_1, \dots, x_n \rangle$, we see that finitely many of the x_i suffice. Since the set $U \cup \{x_1, \dots, x_k\}$ is clearly coarsely bounded for any k by Rosendal's criterion, we see that G is boundedly generated. \square

3 Cayley–Abels–Rosendal graphs

A connected, countable simplicial graph Γ is a *Cayley–Abels–Rosendal graph* for a topological group G if G admits a continuous, vertex-transitive action on Γ with finitely many orbits of edges and coarsely bounded (necessarily open) vertex stabilizers.

Note: Our definition follows more closely the definition of a Cayley–Abels graph for a totally disconnected locally compact group, rather than that of a Cayley graph. While groups act vertex-transitively on their Cayley graphs, as commonly defined, there is no assumption on the finiteness of the generating set, nor continuity of the action. We also direct the reader to [Ros22, Section 6.2] in which Rosendal carries out a similar discussion in the context of non-Archimedean Polish groups and automorphisms of countable first-order structures.

We begin this section with some generalities on continuous actions of groups on graphs. Suppose that a group G acts on a graph Γ continuously. This induces a representation from G into the group of bijections of the vertex set $V\Gamma$. This latter group is a topological group with the *permutation topology*, where pointwise stabilizers of finite sets are basic open neighborhoods of the identity, so this representation will be continuous if and only if the stabilizer of a vertex is open in G . Such a permutation actually arises as a graph automorphism just when it preserves adjacency.

When Γ is simplicial, adjacency is a (symmetric) *relation* on $V\Gamma$, that is, a subset of $V\Gamma \times V\Gamma$, so a bijection of $V\Gamma$ corresponds (uniquely) to a graph automorphism just when it preserves this relation in the diagonal action on $V\Gamma \times V\Gamma$; it is not hard to see that the topology on $\text{Aut}(\Gamma)$ is precisely the subspace topology; so if a continuous representation from a group G into the group of bijections of $V\Gamma$ preserves the adjacency relation, then G acts continuously on Γ , and we see that this happens if and only if vertex stabilizers are open in G .

Now when Γ is a simplicial graph and G acts continuously with one orbit of vertices, write V for the stabilizer of the vertex v , and notice that there exists an element $k \in G$ sending the oriented edge $e = (g.v, h.v)$ to the oriented edge $e' = (g'.v, h'.v)$ precisely when the pair (kgV, khV) is equal to the pair $(g'V, h'V)$, or put another way, when the *double coset* $Vg^{-1}hV$ is equal to the double coset $Vg'^{-1}h'V$. So when Γ is simplicial, every orbit of oriented edges corresponds (uniquely) to a double coset of V in G .

Let A be a set of representatives for the double cosets corresponding to oriented edges in Γ . We may and will assume that A is symmetric, so that $AV = VA$, since V and A are symmetric. Now, notice that if v is the vertex of Γ with stabilizer V , the vertices $a.v$ are adjacent to v . In fact, we have $VA = VAV$. The containment $VA \subset VAV$ is obvious since V is a subgroup. For any element $g \in VAV$, notice that $(v, g.v)$ is an oriented edge of Γ , since it corresponds to the double coset VgV , which is represented by some $a \in A$, since $g \in VAV$. In other words, the set of elements sending v to an adjacent vertex is precisely VAV . But it is also VA , since the action of G on $V\Gamma$ corresponds to the action of G on G/V .

Conversely, given an open subgroup $V \leq G$ and a set of double cosets in $V \backslash G / V$ represented by the set A chosen to be symmetric and meet V trivially, we construct a simplicial graph $\Gamma = \Gamma(V, A)$. Its vertex set is G/V and two vertices (cosets) gV and hV are declared adjacent when the double coset $Vg^{-1}hV$ has a representative in A . The action of G on $\Gamma(V, A)$ is continuous, vertex transitive, and A is in one-to-one correspondence with the set of oriented edge orbits.

Now, it is not hard to see that the graph Γ will be connected precisely when $AV = VA = VAV$ is an open generating set for G . Notice that any continuous vertex-transitive action of a group G on a connected graph Γ becomes cobounded as soon as we associate to Γ the geodesic path metric that assigns length 1 to each edge. If such an action is to be coarsely metrically proper, it must be the case that the set $VAV = AV = VA$ associated to Γ as above is coarsely bounded in G , since it fails to move the closed 1-neighborhood of v off of itself. But since Γ is connected, the coarsely bounded open set VAV generates G . What's more, if G is additionally Polish, Rosendal's criterion applies, and we may write VAV as being contained in $(FV)^k$ for some finite set $F \leq G$. A little elbow grease shows that we may take F to be symmetric, meet V trivially, and actually be contained in A , implying that the action of G on Γ must actually have finitely many orbits of edges.

From here, the rest of the work proving the following proposition should be clear.

Proposition 7. *Suppose that G admits a Cayley–Abels–Rosendal graph. Then G is boundedly generated and by the Milnor–Schwarz Lemma is quasi-isometric to any Cayley–Abels–Rosendal graph for G . If G is boundedly generated and Polish, for any open, coarsely bounded subgroup V of G (supposing it exists), there exists a Cayley–Abels–Rosendal graph for G with V as the stabilizer of a vertex.*

Not every topological group G admits proper open subgroups, and even those that do may not have ones which are sufficiently small to apply Proposition 7. One family of groups that do are the *non-Archimedean groups*, those having a neighborhood basis of the identity given by open subgroups. The Polish non-Archimedean groups turn out to be the closed subgroups of the symmetric group on a countable set, a rich and interesting family of groups to which Proposition 7 may readily be applied.

Now, suppose that G is locally bounded and Polish but not boundedly generated. Then it follows that if V is a coarsely bounded, open subgroup of G , the foregoing also proves the following dichotomy for vertex-transitive actions of G on graphs Γ with V as the stabilizer of a vertex: either Γ has infinitely many orbits of edges, or Γ is disconnected.

4 Examples: Homeomorphism Groups of Countable Stone Spaces

In this final section, we apply the machinery described above to give a full classification of when homeomorphism groups of countable Stone spaces admit Cayley–Abels–Rosendal graphs. This will allow us to obtain a full classification of when these groups are (1) coarsely bounded, (2) locally bounded, and/or (3) boundedly generated.

Definition 8. A *Stone space* is a topological space that is compact, Hausdorff, and totally disconnected.

When X is a second countable Stone space, the group $\text{Homeo}(X)$ equipped with the compact–open topology turns out to be non-Archimedean and Polish. Two of the present authors [BA24] exhibit $\text{Homeo}(X)$ as the automorphism group of a countable graph, whose vertices are (certain) partitions of X into two clopen (closed and open) sets U and V . The stabilizer of this vertex is the subgroup of $\text{Homeo}(X)$ either preserving U and V each setwise, or exchanging them. Since the

intersection of clopen sets is clopen, a basis for the compact–open topology on $\text{Homeo}(X)$ then is given by $U_{\mathcal{P}}$, where \mathcal{P} is a finite partition $\mathcal{P} = P_0 \sqcup P_1 \sqcup \dots \sqcup P_n$ of X into clopen sets P_i , and $f \in \text{Homeo}(X)$ belongs to $U_{\mathcal{P}}$ when it permutes the P_i .

4.1 Countable Stone Spaces

For the remainder of this paper we will be concerned with *countable* Stone spaces. See Figure 1 for some examples of countable Stone spaces. These form a particularly nice class of Stone spaces as they are exactly classified by a pair (α, n) where α is a countable ordinal and $n \in \mathbb{N}$. In fact, a consequence of this is that any countable Stone space is exactly homeomorphic to the countable ordinal $\omega^\alpha \cdot n + 1$ equipped with the order topology. This was first proven by Mazurkiewicz and Sierpiński [MS20], but for the sake of completeness we provide a proof here.

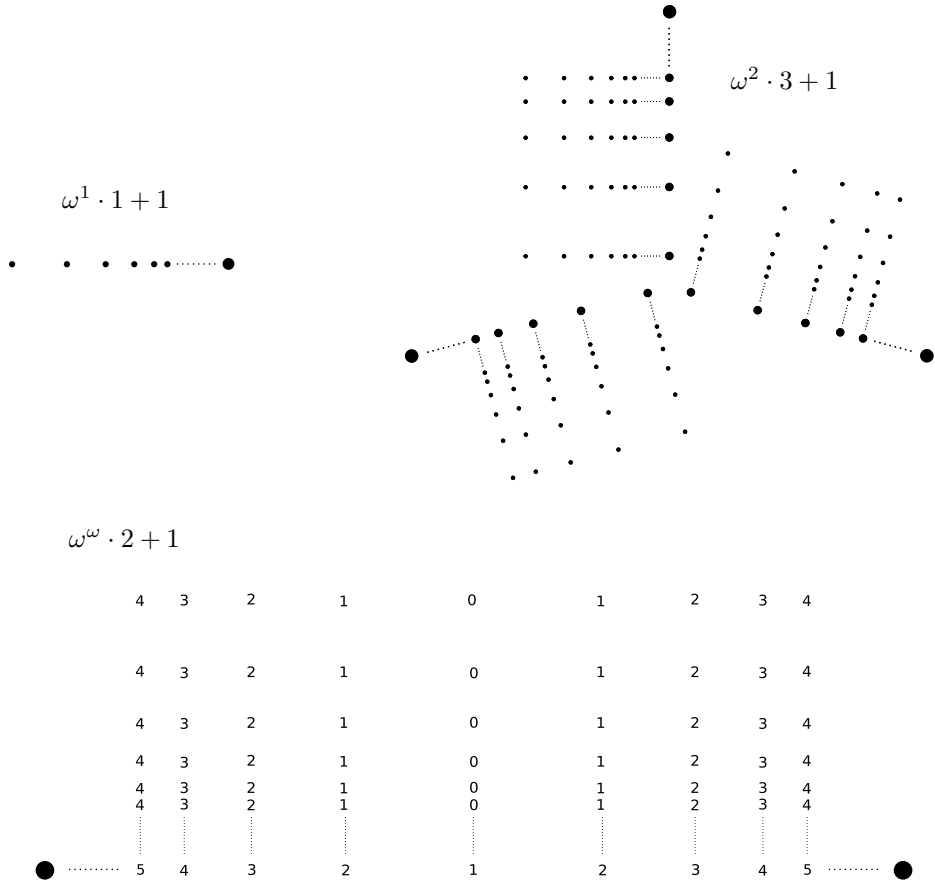


Figure 1: Three examples of countable Stone spaces as subspaces of the plane. The top two figures are examples of successor ordinals and the bottom is a limit ordinal. In the bottom figure, the number $n \in \mathbb{Z}_{\geq 0}$ is used to represent a small copy of $\omega^n + 1$. We will make use of this convention throughout this document.

First we need to introduce a type of “derivative” map on topological spaces. We refer the reader to [Kec95, Section 6.C] and [Mil11] for a more thorough treatment of what follows.

Definition 9. The **derived set** of a topological space X is the set of all accumulation points of X . We denote the derived set of X by X' . For an ordinal α we define the **α -th Cantor-Bendixson**

derivative, X^α , of X recursively as

- $X^0 = X$,
- $X^{\alpha+1} = (X^\alpha)'$, and
- $X^\lambda = \bigcap_{\alpha < \lambda} X^\alpha$ if λ is a limit ordinal.

Next we check that for a second countable space, the Cantor-Bendixson derivatives eventually stabilize at some *countable* ordinal.

Theorem 10. [Kec95, Theorem 6.9] *If a topological space X is second countable, then there exists some countable ordinal ρ such that $X^\rho = X^{\rho+1}$.*

Proof. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable open basis for X . Given any closed set $F \subset X$, set $N(F) = \{n : U_n \cap F \neq \emptyset\}$. Since $X \setminus F = \bigcup_{n \notin N(F)} U_n$, the map $F \mapsto N(F)$ is injective into the power set of the natural numbers. Also note that if $F_1 \subseteq F_2$ are two closed subsets, then $N(F_1) \subseteq N(F_2)$.

If we assume, to the contrary, that X^ρ does not stabilize for any countable ordinal ρ , then by considering the transfinite sequence $(X^\rho)_{\rho < \omega_1}$, where ω_1 is the smallest uncountable ordinal, one obtains a monotonic transfinite sequence $(N(X^\rho))_{\rho < \omega_1}$ of uncountably many subsets of \mathbb{N} , a contradiction. \square

Definition 11. Let X be a second countable topological space. The **Cantor-Bendixson rank** of X is the smallest countable ordinal ρ such that $X^\rho = X^{\rho+1}$.

Let X be a countable Stone space and ρ_X be the Cantor-Bendixson rank of X . Now, $X^{\rho_X} = \emptyset$ as X^{ρ_X} must be perfect and countable. Also, as X is compact and hence has the finite intersection property we also have that $\rho_X = \alpha_X + 1$ for some countable ordinal α_X . Following a standard abuse of notation, in this case we will refer to α_X as the Cantor-Bendixson rank of X . For example, in fig. 1, the Cantor-Bendixson ranks are 1, 2, and ω . We set $n_X = |X^{\alpha_X}| \in \mathbb{N}$. We often refer to the n_X points of X^{α_X} as the set of **maximal points** of X . We say that the **characteristic pair** of X is the pair (α_X, n_X) . We say that a point $x \in X$ has **rank** α if $x \in X^\alpha \setminus X^{\alpha+1}$. Note that every point $x \in X$ of rank α has a clopen neighborhood $U_x \subset X$ with characteristic pair exactly $(\alpha, 1)$. This follows from the fact that x having rank α implies that x is not a limit point of X^α .

Finally, we are ready to state the classification theorem of countable Stone spaces.

Theorem 12. [MS20, Théorème 1] *Two countable Stone spaces, X and Y , are homeomorphic if and only if they have the same characteristic pair, i.e. $(\alpha_X, n_X) = (\alpha_Y, n_Y)$. In particular, any countable Stone space X is homeomorphic to the countable ordinal $\omega^\alpha \cdot n + 1$ equipped with the order topology.*

Proof. If X and Y are homeomorphic, then $X^\rho \cong Y^\rho$ for any countable ordinal ρ . In particular, we must have that $\alpha_X = \alpha_Y$ and $X^{\alpha_X} \cong Y^{\alpha_Y}$ so that $n_X = n_Y$ as well.

We will prove the reverse implication by transfinite induction on α . For the base case, if $\alpha = 0$, then X and Y are simply finite sets and hence homeomorphic if and only if they are of the same cardinality. Now, let X and Y be two countable Stone spaces with characteristic pair $(\alpha_0, 1)$. Our induction hypothesis will be that for any $\alpha < \alpha_0$, two spaces with characteristic pair $(\alpha, 1)$ are homeomorphic.

Let $x_0 \in X$ and $y_0 \in Y$ be the maximal points of X and Y , respectively. Let $\{U_n\}_{n \in \mathbb{N}}$ and $\{V_n\}_{n \in \mathbb{N}}$ be sequences of clopen subsets such that $\bigcap U_n = \{x_0\}$ and $\bigcap V_n = \{y_0\}$. Without loss of generality we assume that $U_1 = X$ and $V_1 = Y$. Now set

$$\begin{aligned} A_n &:= U_n \setminus U_{n+1}, \text{ and} \\ B_n &:= V_n \setminus V_{n+1}. \end{aligned}$$

Note that each A_n and B_n is clopen. Let (α_n, a_n) and (β_n, b_n) be the sequences of characteristic pairs of $\{A_n\}$ and $\{B_n\}$, respectively. Up to replacing a given A_n or B_n with a finite disjoint union of the A_n 's or B_n 's, respectively, we may assume without loss of generality that for each n , either $\alpha_n < \beta_n$ or $a_n < b_n$, and either $\beta_n < \alpha_{n+1}$ or $b_n < b_{n+1}$.

We now use the induction hypothesis to construct a homeomorphism $f : X \rightarrow Y$. We first define Note that since one of β_1 or b_1 is larger than α_1 or a_1 , Y contains a clopen subset, B'_1 , with characteristic pair (α_1, a_1) . Thus, by the induction hypothesis, there exists a homeomorphism $f_1 : A_1 \rightarrow B'_1$. Similarly, there exists an $A'_2 \subset A_2$ that is homeomorphic to $B_2 \setminus B'_2$. This allows us to define a homeomorphism $g_1 : B_1 \setminus B'_1 \rightarrow A'_2$. We continue this “back-and-forth” process recursively to obtain, for all n , two homeomorphisms

$$\begin{aligned} f_n &: A_n \setminus A'_n \rightarrow B'_n \\ g_n &: B_n \setminus B'_n \rightarrow A'_{n+1}. \end{aligned}$$

Note that the collection of maps of the form f_n and g_n^{-1} have disjoint support and the union of their supports is exactly $X \setminus \{x\}$. Additionally, the union of the images is exactly $Y \setminus \{y\}$. Thus we define the map

$$f(x) := \begin{cases} y_0 & \text{if } x = x_0 \\ f_n(x) & \text{if } x \in A_n \setminus A'_n \\ g_n^{-1}(x) & \text{if } x \in A'_{n+1} \end{cases}$$

where we take $A'_1 = \emptyset$. Note that since each of the f_n and g_n are bijections, f is also a bijection.

To finish we only need to verify that f is continuous. As X is compact, Hausdorff, and second countable it is metrizable and hence it suffices to check sequential continuity. Let $x_i \rightarrow x$ be a convergent sequence in X . The argument now splits into two cases.

If x has rank strictly less than α_0 , then x is contained in one of the A_n . Therefore, throwing out finitely many terms of the sequence, we have that f restricted to the sequence (x_i) is exactly equal to one of f_n or g_n^{-1} restricted to (x_i) . However, now we must have that $f(x_i) \rightarrow x_i$ by the continuity of each of f_n and g_n^{-1} .

Next we suppose that $x = x_0$. In this case the fact that $f(x_i) \rightarrow f(x) = y$ follows because we chose the collections $\{U_n\}$ and $\{V_n\}$ such that $\bigcap U_n = \{x\}$ and $\bigcap V_n = \{y\}$. We thus conclude that f is a continuous bijection and hence a homeomorphism.

By transfinite induction, we now have that any two countable Stone spaces with characteristic pairs equal to $(\alpha, 1)$, for any countable ordinal α , are homeomorphic. To conclude the result for arbitrary n we simply note that a space with characteristic pair (α, n) can be partitioned into exactly n clopen subsets each with characteristic pair $(\alpha, 1)$.

The final statement follows from the fact that the countable ordinal $\omega^\alpha \cdot n + 1$ equipped with the order topology is exactly a countable Stone space with Cantor-Bendixson rank α and n maximal points. \square

Following this theorem, from now on we will always write $\omega^\alpha \cdot n + 1$ to refer to the unique (up to homeomorphism) countable Stone space with Cantor-Bendixson rank α and n maximal points. Given a point x in a countable Stone space X we say that x has **Cantor-Bendixson rank** α if $x \in X^\alpha$ but $x \notin X^{\alpha+1}$. Note that by the classification theorem this then implies that there exists a small clopen neighborhood about x homeomorphic to $\omega^\alpha + 1$. We no longer need to take derived sets, so from here on the prime symbol $'$ is used to decorate notation.

Our goal is to now prove the classification theorem stated in the introduction, recalled below for convenience.

Theorem A. Let $\omega^\alpha \cdot n + 1$ be a countable Stone space. The group $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is always locally bounded and is

- coarsely bounded if and only if $n = 1$, and
- boundedly generated but not coarsely bounded if and only $n > 1$ and α is a successor ordinal.

We break this into several steps. The first is to verify that when $n = 1$, the homeomorphism group is coarsely bounded. Next we will see that this immediately implies that all of these groups are locally bounded. We will then build Cayley-Abels-Rosendal graphs when α is a successor ordinal, thus proving bounded generation of $\text{Homeo}(\omega^\alpha \cdot n + 1)$. Finally, we will consider the limit ordinal case.

4.2 Coarsely Bounded and Locally Bounded

We begin with the case that $n = 1$.

Proposition 13. *For any open set $U \subset \text{Homeo}(\omega^\alpha + 1)$, there exists an open subgroup $V \subset U$ and an involution $f \notin V$ such that the graph $\Gamma(V, A)$ as defined in Section 3 associated to V and $A = \{f\}$ has diameter at most two (so is connected) and admits a vertex-transitive and edge-transitive action of $\text{Homeo}(\omega^\alpha + 1)$.*

Now suppose that $\text{Homeo}(\omega^\alpha + 1)$ acts continuously on a metric space X . Take for U in the statement above the open set comprising those elements failing to move a point $x \in X$ a distance more than $\epsilon > 0$, and construct the graph $\Gamma(V, A)$ with vertex stabilizer associated to V and A . Choosing an element $g \in G$ representing each coset of V defines a map $\Phi: V\Gamma \rightarrow X$ as $gV \mapsto g.x$. Write δ for $d(x, f.x)$, where f is the involution in the statement of the proposition. The map Φ is Lipschitz: supposing that $d(\Phi(gV), \Phi(hV)) \geq n \cdot (\epsilon + \delta)$, notice that the length of a shortest path in Γ from gV to hV must therefore have length at least n , since otherwise we could derive a contradiction from the triangle inequality because $g^{-1}h$ would have word length smaller than n in AV .

This argument, together with the statement of the proposition, has the following corollary

Corollary 14. *The group $\text{Homeo}(\omega^\alpha + 1)$ is coarsely bounded.*

Proof. The foregoing argument proves that every orbit in every continuous action of $\text{Homeo}(\omega^\alpha + 1)$ by isometries on a metric space is coarse-Lipschitz-dominated by an action on a connected graph as in the statement of Proposition 13. Since every such graph has diameter at most two, we conclude. \square

Proof of Proposition 13. By our description of the compact-open topology on $\text{Homeo}(\omega^\alpha + 1)$, beginning with an identity neighborhood U , we can find a finite partition $\mathcal{P} = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ of $\omega^\alpha + 1$ such that U contains $V = U_{\mathcal{P}}$.

Now, the graph Γ we will construct has as vertices those partitions \mathcal{P}' of X such that we may write $\mathcal{P}' = P'_0 \sqcup P'_1 \sqcup \cdots \sqcup P'_n$, where each P_i is homeomorphic to P'_i . Notice that as $\omega^\alpha + 1$ has a unique maximal point, one of the clopen sets, say P_0 contains this maximal point and thus is actually homeomorphic to $\omega^\alpha + 1$. Declare two such vertices to be adjacent when $P_1 \sqcup \cdots \sqcup P_n \subset P'_0$, from which it follows that $P'_1 \sqcup \cdots \sqcup P'_n \subset P_0$. In this case, notice that P_i and P'_i are disjoint provided $i > 0$, and there is an obvious involution f exchanging P_i and P'_i and acting as the identity on the complement, $P_0 \cap P'_0$. It is not hard to show that the graph Γ so described is exactly the graph associated to $V = U_{\mathcal{P}}$ and $A = \{f\}$.

So consider vertices P and P' . If $P_1 \sqcup \cdots \sqcup P_n$ is disjoint from $P'_1 \sqcup \cdots \sqcup P'_n$, then P and P' are adjacent. Otherwise, consider $P_0 \cap P'_0$. It is a clopen set containing the maximal point of $\omega^\alpha + 1$, so homeomorphic to $\omega^\alpha + 1$, and thus contains a clopen set homeomorphic to $P_1 \sqcup \cdots \sqcup P_n$. Partition this set accordingly as $P''_1 \sqcup \cdots \sqcup P''_n$, and let P''_0 be the complement. By construction, we see that this partition P'' corresponds to a vertex of Γ which is adjacent to both P and P' . \square

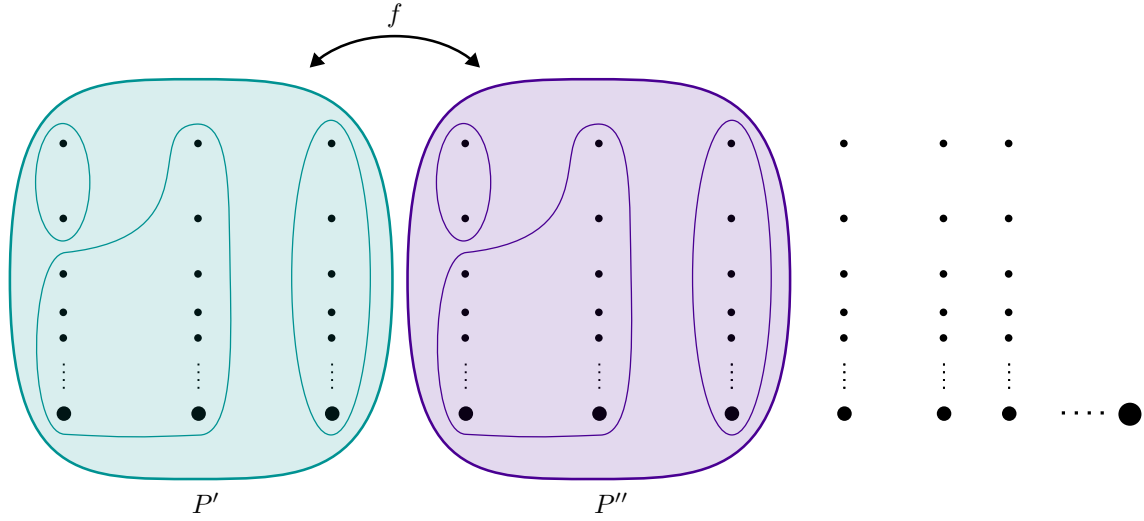


Figure 2: An example of the sets P' and P'' and the involution f for $\omega^2 + 1$. The subsets of P' are meant to represent the partition sets P_1, P_2 , and P_3 . Note that P_0 is simply the complement of P' and hence not drawn explicitly.

We note that the above corollary also follows from [MR23, Proposition 3.1] by considering $\text{Homeo}(\omega^\alpha + 1)$ as a continuous quotient of the mapping class group of the genus zero surface with end space homeomorphic to $\omega^\alpha + 1$.

This has as an immediate corollary that all of the homeomorphism groups we are considering are locally bounded.

Corollary 15. *The group $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is locally bounded.*

Proof. Consider a partition of $\omega^\alpha \cdot n + 1$ given by $\mathcal{P} = P_1 \sqcup \dots \sqcup P_n$ where each P_i contains exactly one maximal point. That is, each P_i is homeomorphic to $\omega^\alpha + 1$. Then $\mathcal{U}_{\mathcal{P}}$ is a neighborhood of the identity in $\text{Homeo}(\omega^\alpha \cdot n + 1)$. Furthermore, $\mathcal{U}_{\mathcal{P}}$ has an index $n!$ open subgroup, the subgroup that fixes each partition set, that is topologically isomorphic to $\text{Homeo}(\omega^\alpha + 1)^n$. It is clear from the definition of coarse boundedness in terms of action on metric spaces that a product of coarsely bounded groups is coarsely bounded, and that if $H \leq G$ is an open subgroup of finite index, then G will be coarsely bounded if and only if H is. \square

4.3 Cayley-Abels-Rosendal Graphs for Successor Ordinals

Next we turn to the case when $n \geq 2$ and α is a successor ordinal. We will build an unbounded Cayley-Abels-Rosendal graph for $\text{Homeo}(\omega^\alpha \cdot n + 1)$, proving the following.

Proposition 16. *Let α be a successor ordinal. Then $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is boundedly generated and not coarsely bounded.*

Suppose that $X \cong \omega^\alpha \cdot n + 1$ is a countable Stone space, where $n \geq 2$ and $\alpha = \beta + 1$ is a successor ordinal. The space X therefore has n maximal points. Say that a partition \mathcal{P} is *good* when it comprises exactly n clopen sets, each containing a single maximal point. Each clopen set in the partition is therefore homeomorphic to $\omega^\alpha + 1$.

We consider the operation of *shifting* a good partition \mathcal{P} : choose a pair of maximal points x_i and x_j , write P_i and P_j for the clopen sets in the partition containing x_i and x_j respectively. Remove from P_i a clopen subset homeomorphic to $\omega^\beta + 1$ and add it to P_j . The prototypical

example is when $X \cong \omega \cdot 2 + 1$ is the end compactification of \mathbb{Z} . Here $\beta = 0$, so sets homeomorphic to $\omega^\beta + 1$ are single points. One pair of good partitions \mathcal{P} and \mathcal{Q} is given by $\mathcal{P} = [-\infty, 0] \sqcup [1, \infty]$ and $\mathcal{Q} = [-\infty, -1,] \sqcup [0, \infty]$. The operation “add one” on \mathbb{Z} extends to a homeomorphism of the end compactification which takes \mathcal{Q} to \mathcal{P} .

Consider the graph $\Gamma = \Gamma(n, \alpha)$ whose vertices are the good partitions of X , where two vertices are connected by an edge when the corresponding partitions differ by a shift. See Figure 3 for an example.

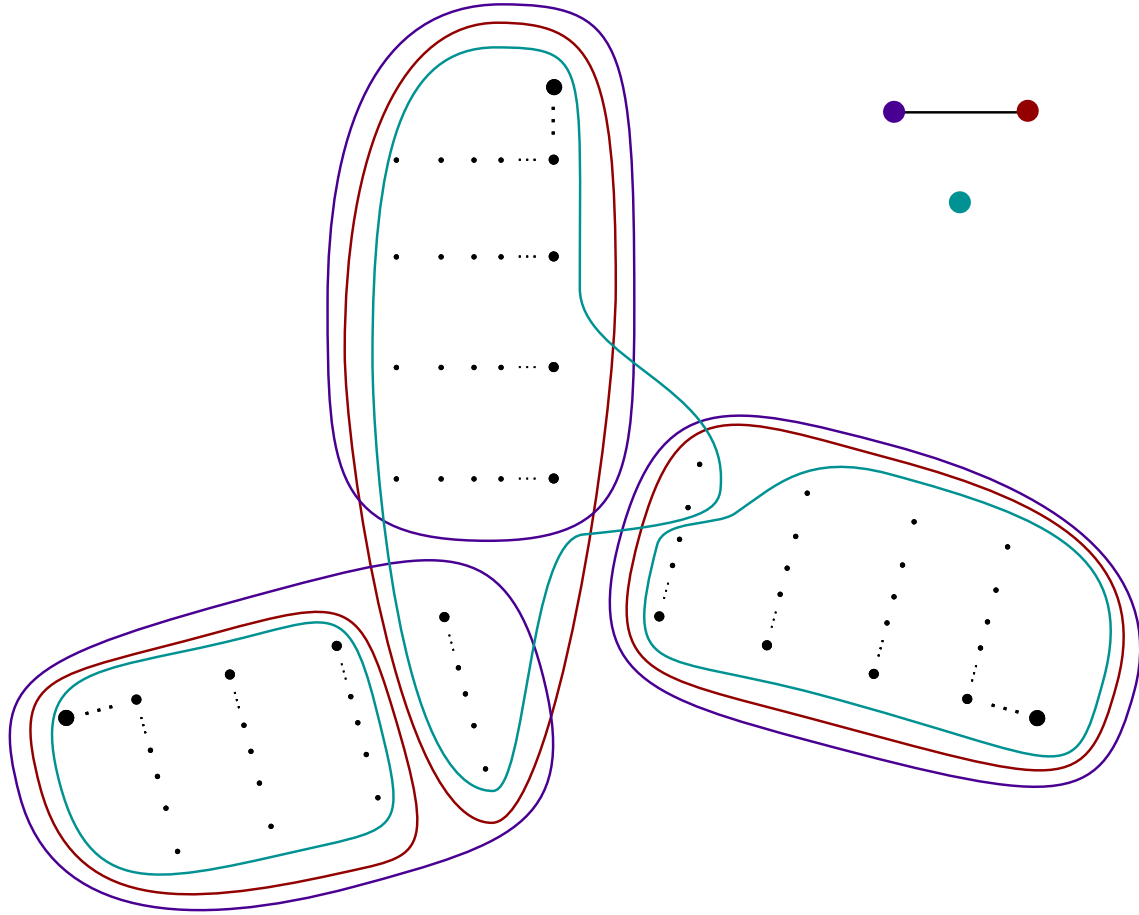


Figure 3: An example of three partitions in $\Gamma = \Gamma(3, 2)$. The red and purple vertices are connected by an edge and the teal vertex is not connected by an edge to either of the red or purple vertices.

Proposition 17. *When $n \geq 2$ and $\alpha = \beta + 1$ is a successor ordinal, the graph $\Gamma(n, \alpha)$ is connected and has infinite diameter.*

Proof. Consider two good partitions \mathcal{P} and \mathcal{Q} . For each maximal point x_i , consider the clopen set $R'_i = P_i \cap Q_i$, where P_i and Q_i are the partition sets in \mathcal{P} and \mathcal{Q} respectively containing x_i . The complement $R'_0 = X - \coprod_{i=1}^n R'_i$ is a clopen set containing none of the maximal points, so it can contain at most finitely many, say d , points of rank β .

We will connect \mathcal{P} to \mathcal{Q} by a path of length at most $d + 2n(n - 1)$ by beginning with \mathcal{P} and progressively altering it until we have produced \mathcal{Q} . This process involves considering each ordered pair of maximal points x_i and x_j . By only altering P_i and P_j by shifts, we will make it so that the

set of points in P_i which are in Q_j is empty. If P_i contains some points of Q_j , the number of shifts we will use is equal to either d_{ij} , the number of rank β points in $P_i \cap Q_j$, if $d_{ij} > 0$, or to two.

Assuming $d_{ij} > 0$, the set of points in $P_i \cap Q_j$ is homeomorphic to $\omega^\beta \cdot d_{ij} + 1$; choosing a good partition of this set, we can shift one element of this partition at a time out of P_i and into P_j , producing a path of length d_{ij} from \mathcal{P} to a new partition \mathcal{P}' which satisfies that $P'_i \cap Q_j = \emptyset$. See Figure 4 for an example of this.

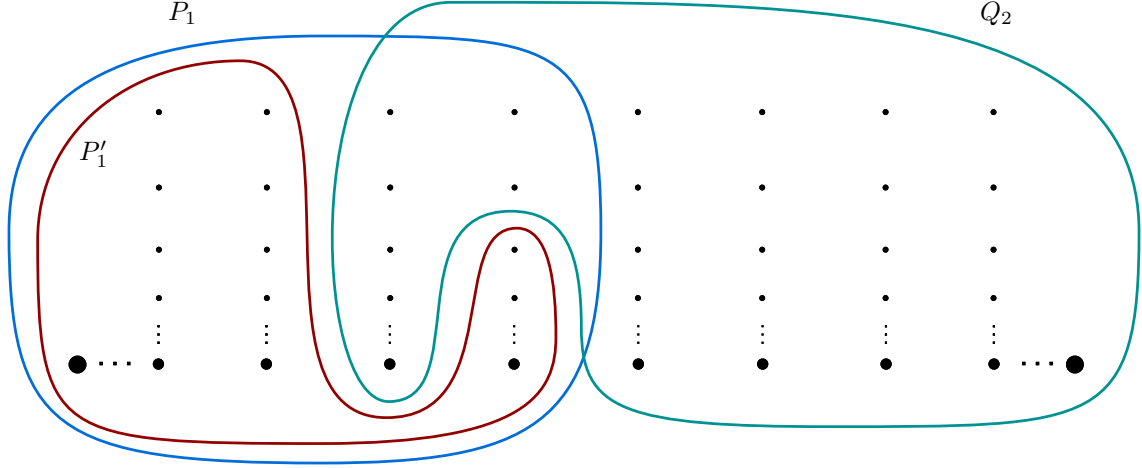


Figure 4: An example of two partitions \mathcal{P} , in blue, and \mathcal{Q} , in teal, with $d_{12} = 1$ in $\Gamma(2, 2)$. Pictured in red is \mathcal{P}'_1 obtained by shifting $P_1 \cap Q_2$ out of P_1 and into P_2 . Note that the left maximal point is x_1 and the right is x_2 .

If instead $d_{ij} = 0$, we may freely choose a single rank β point y of P_i . This point y has two clopen neighborhoods U and V , both homeomorphic to $\omega^\beta + 1$, such that $U = V \sqcup (P_i \cap Q_j)$. First shifting U out of P_i and into P_j , and then shifting V back in produces a path of length two in Γ between \mathcal{P} and a new partition \mathcal{P}' which satisfies that $P'_i \cap Q_j = \emptyset$. See Figure 5 for an example.

Repeating this process for each ordered pair (i, j) yields the desired path, proving that Γ is connected.

To see that the graph Γ has infinite diameter, notice that because a shift moves one point of rank β at a time, if the set R'_0 constructed above contains d points of rank β , any path from \mathcal{P} to \mathcal{Q} must have length at least d . Any natural number d is realized as the number of rank β points in R'_0 , so Γ has infinite diameter. (Put another way, this observation and the existence of the path above proves that counting the number of rank β points different between \mathcal{P} and \mathcal{Q} is a coarse measure of their distance in Γ .) \square

Now, the stabilizer $\text{Stab}(\mathcal{P})$, for any good partition \mathcal{P} , is open and sits in a (continuous) short exact sequence

$$1 \longrightarrow \prod_{i=1}^n \text{Homeo}(\omega^\alpha + 1) \longrightarrow \text{Stab}(\mathcal{P}) \longrightarrow S_n \longrightarrow 1.$$

(Here the topology on the finite symmetric group S_n is discrete.) One can directly show that the topological group $\text{Stab}(\mathcal{P})$ is coarsely bounded, since the kernel and quotient are coarsely bounded, so $\text{Stab}(\mathcal{P})$ is open and coarsely bounded in $\text{Homeo}(X)$. Also, given two good partitions \mathcal{P} and \mathcal{Q} , by Theorem 12, there exists an element $g \in \text{Homeo}(X)$ which takes \mathcal{P} to \mathcal{Q} . In particular, this holds if \mathcal{P} and \mathcal{Q} differ by a shift. Fix such a pair \mathcal{P} and \mathcal{Q} and $g \in \text{Homeo}(X)$. In the example of the end compactification of \mathbb{Z} , this homeomorphism is the extension of $x \mapsto x + 1$ to the end compactification.

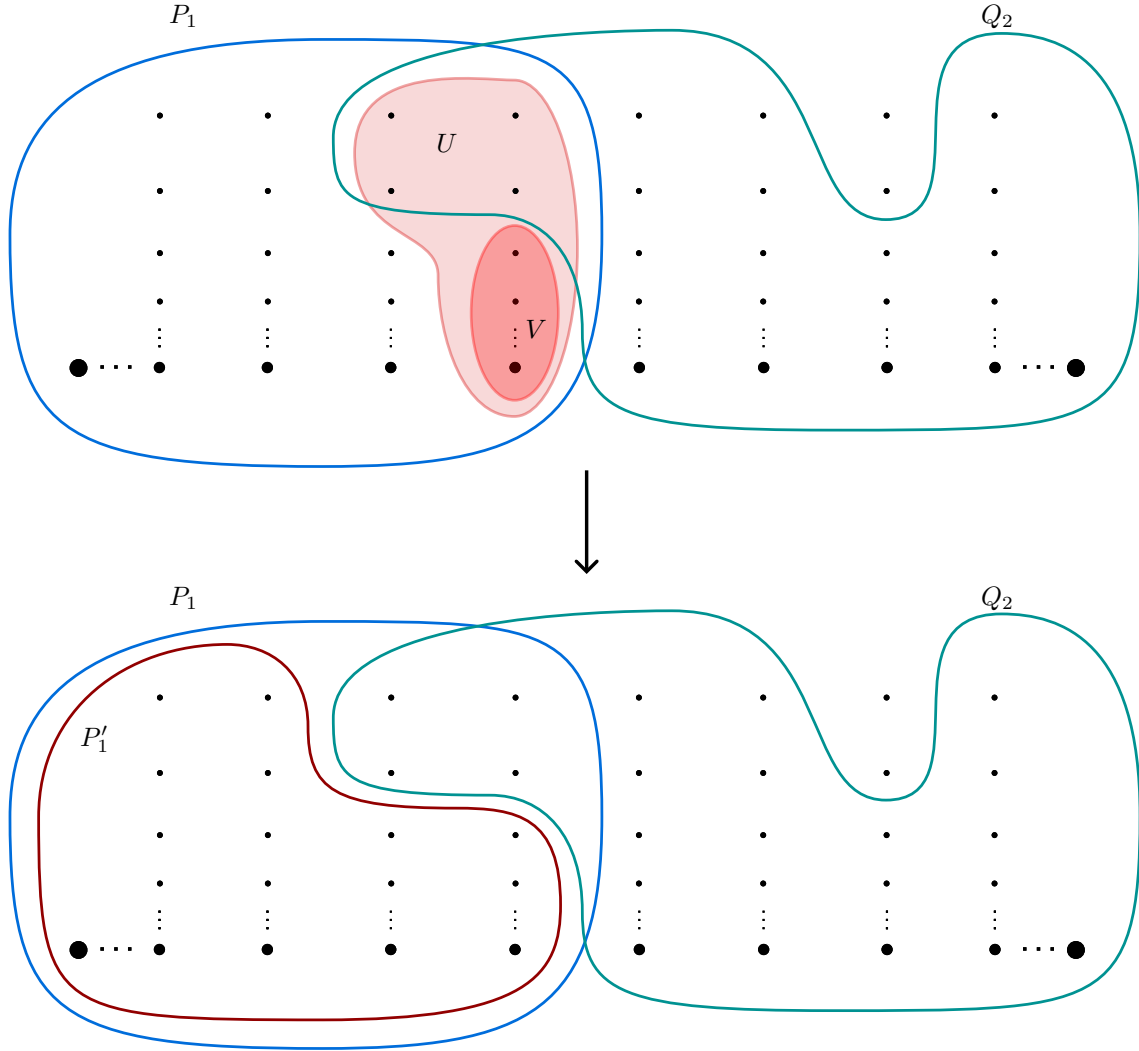


Figure 5: An example of two partitions \mathcal{P} , in blue, and \mathcal{Q} , in teal, with $d_{12} = 0$ in $\Gamma(2, 2)$. The sets U and V are represented in the top figure. In the bottom figure we have, in red, P'_1 obtained by shifting U out of P_1 and into P_2 and then shifting V back. Again, the left maximal point is x_1 and the right is x_2 .

Proposition 18. *The graph Γ is of the form $\Gamma(\text{Stab}(\mathcal{P}), A)$, where $A = \{g^{\pm 1}\}$ and is thus a Cayley–Abels–Rosendal graph for $\text{Homeo}(X)$.*

Proof. Observe that because any two good partitions of X recognize it as being homeomorphic to $\omega^\alpha \cdot n + 1$, the group $\text{Homeo}(X)$ acts transitively on good partitions; the stabilizer of the partition \mathcal{P} is the group $\text{Stab}(\mathcal{P})$, which is open in $\text{Homeo}(X)$ and coarsely bounded. Now, supposing that \mathcal{Q} and \mathcal{Q}' are good partitions of X which differ from \mathcal{P} by a shift, notice that there is a homeomorphism preserving \mathcal{P} taking one of the shifted sets, which is homeomorphic to $\omega^\beta + 1$, to the other. See Figure 6 for an example of such a shift. Thus the group $\text{Homeo}(X)$ acts edge-transitively on the graph Γ . □

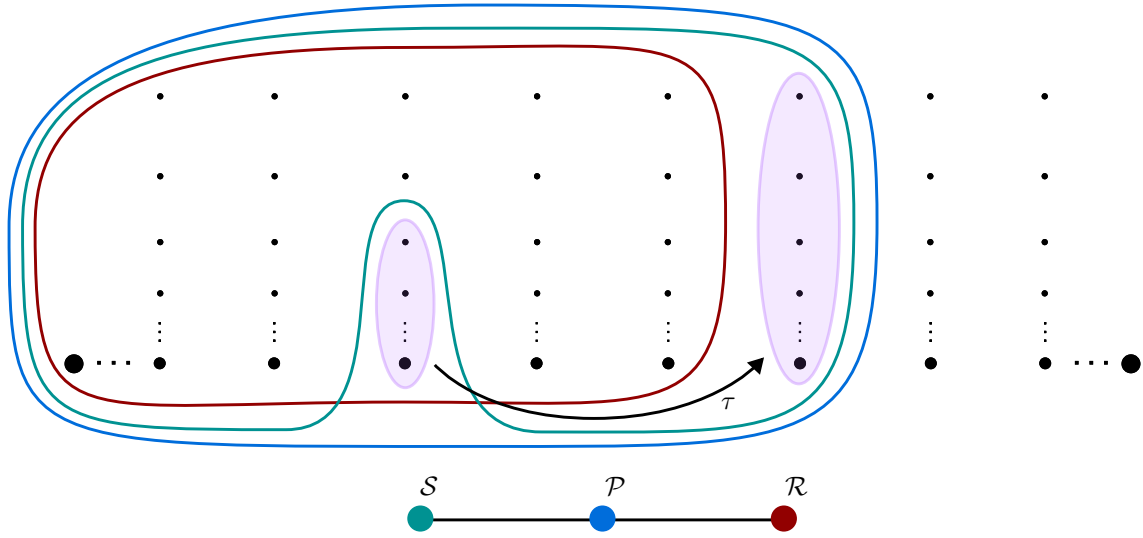


Figure 6: An example of two edges in $\Gamma(2, 2)$ sharing the vertex \mathcal{P} and the map $\tau \in \text{Stab}(\mathcal{P})$ that maps one edge to the other.

4.4 Limit Ordinals

Finally we turn to limit ordinals and prove the following.

Proposition 19. *Let α be a limit ordinal. If $n \geq 2$, then $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is not boundedly generated.*

We will prove this by using Lemma 6. That is, we will show that $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is a union of a countably infinite chain of proper open subgroups. We again label the maximal points of $\omega^\alpha \cdot n + 1$ as x_1, \dots, x_n and let \mathfrak{P} be the set of all good partitions. We will define this chain of subgroups by first defining a height function of \mathfrak{P} . Then the subgroups will be defined as stabilizers of sublevel sets of this height function.

Remark 20. Note that the set \mathfrak{P} is defined in the same way as our vertex set in the previous section. One interpretation of the arguments in this section is that we are showing that there is no way to make a Cayley-Abels-Rosendal graph out of this vertex set. Of course, the actual result we prove at the end of the day is stronger. It implies that there cannot exist a Cayley-Abels-Rosendal graph for *any* choice of vertex set.

Given any clopen subset $A \subset \omega^\alpha \cdot n + 1$ and ordinal $\beta < \alpha$ we let $[A]_\beta$ denote the set of points of type β in A . We now define a relative height function on partitions as follows. For $\mathcal{P}, \mathcal{Q} \in \mathfrak{P}$,

$$h(\mathcal{P}, \mathcal{Q}) := \sup \{ \beta \mid \beta < \alpha \text{ and } [P_i \Delta Q_i]_\beta \neq \emptyset \text{ for all } i = 1, \dots, n \}.$$

We first check some basic properties of this function.

Lemma 21. *The function h takes values strictly less than α , is $\text{Homeo}(\omega^\alpha \cdot n + 1)$ -equivariant, and satisfies a strong triangle inequality. That is, for $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathfrak{P}$,*

$$h(\mathcal{P}, \mathcal{Q}) \leq \max\{h(\mathcal{P}, \mathcal{R}), h(\mathcal{R}, \mathcal{Q})\}.$$

Proof. We first check that $h(\mathcal{P}, \mathcal{Q}) < \alpha$ for all $\mathcal{P}, \mathcal{Q} \in \mathfrak{P}$. Suppose, to the contrary, that it did not. Then, after passing to a subsequence, there would exist an increasing sequence $\beta_1 < \beta_2 < \dots$ such

that $\beta_k \rightarrow \alpha$ and an i so that $[P_i \setminus Q_i]_{\beta_k} \neq \emptyset$ for all k . Let $z_k \in P_i \setminus Q_i$ be a point of type β_k for each k . After passing to a further subsequence we may assume that these points are all contained in some Q_j for $j \neq i$. However, the sequence $\{z_k\}$ must then accumulate onto x_j . This would then imply that $x_j \in P_i$ since P_i is closed. This contradicts the choice of P_i . Therefore we conclude that the maximum is realized.

The function is $\text{Homeo}(\omega^\alpha \cdot n + 1)$ -equivariant since the group acts on $\omega^\alpha \cdot n + 1$ by homeomorphisms.

Finally, to check the strong triangle inequality we use the triangle inequality for symmetric differences. That is, given three sets A, B, C , we have

$$A \Delta B \subset (A \Delta C) \cup (C \Delta B).$$

Therefore, given three partitions $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathfrak{P}$ we have

$$[P_i \Delta Q_i]_\beta \subset [P_i \Delta R_i]_\beta \cup [R_i \Delta Q_i]_\beta$$

for all i and β . Therefore, if both $[P_i \Delta R_i]_\beta = \emptyset$ and $[R_i \Delta Q_i]_\beta = \emptyset$, then $[P_i \Delta Q_i]_\beta = \emptyset$. \square

Next we fix a basepoint partition $\mathcal{P} \in \mathfrak{P}$. This allows us to define a height function on \mathfrak{P} by setting $h(\mathcal{Q}) = h(\mathcal{P}, \mathcal{Q})$. For any $\beta < \alpha$ we define the sublevel sets

$$\mathfrak{P}_\beta := \{\mathcal{Q} \mid h(\mathcal{Q}) \leq \beta\}.$$

We claim that the stabilizers of these sublevel sets form a countable chain of proper open subgroups exhausting $\text{Homeo}(\omega^\alpha \cdot n + 1)$. We break this down into several lemmas.

Lemma 22. *For each $\beta < \alpha$, $\text{Stab}(\mathfrak{P}_\beta)$ is a proper open subgroup of $\text{Homeo}(\omega^\alpha \cdot n + 1)$.*

Proof. We first note that $\text{Stab}(\mathfrak{P}_\beta)$ is a proper subgroup. Indeed, if $g \in \text{Homeo}(\omega^\alpha \cdot n + 1)$ is any homeomorphism mapping a point of type $\beta' > \beta$ from P_1 to P_2 , then $h(g\mathcal{P}) \geq \beta'$ and thus $g \notin \text{Stab}(\mathfrak{P}_\beta)$.

We next check that $\text{Stab}(\mathfrak{P}_\beta)$ is an open subgroup. It suffices to see that $\text{Stab}(\mathfrak{P}_\beta)$ contains an open neighborhood of the identity. We claim that $\text{Stab}(\mathcal{P}) \subset \text{Stab}(\mathfrak{P}_\beta)$. Let $g \in \text{Stab}(\mathcal{P})$ and $\mathcal{Q} \in \mathfrak{P}_\beta$. By Lemma 21 we have

$$h(g\mathcal{Q}) = h(\mathcal{P}, g\mathcal{Q}) \leq \sup\{h(\mathcal{P}, g\mathcal{P}), h(g\mathcal{P}, g\mathcal{Q})\} \leq \beta.$$

The final inequality comes from the fact that $h(\mathcal{P}, g\mathcal{P}) = 0$ since $g \in \text{Stab}(\mathcal{P})$ and the equivariance of h . We conclude that $g \in \text{Stab}(\mathfrak{P}_\beta)$ and hence $\text{Stab}(\mathcal{P}) \subset \text{Stab}(\mathfrak{P}_\beta)$. \square

Lemma 23. *If $\delta < \beta < \alpha$, then $\text{Stab}(\mathfrak{P}_\delta) \subsetneq \text{Stab}(\mathfrak{P}_\beta)$.*

Proof. Let $g \in \text{Stab}(\mathfrak{P}_\delta)$ and $\mathcal{Q} \in \mathfrak{P}_\beta$. Again, by Lemma 21 we have

$$h(g\mathcal{Q}) = h(\mathcal{P}, g\mathcal{Q}) \leq \max\{h(\mathcal{P}, g\mathcal{P}), h(g\mathcal{P}, g\mathcal{Q})\} \leq \beta.$$

Here we are using the fact that $g \in \text{Stab}(\mathfrak{P}_\delta)$ implies that $h(\mathcal{P}, g\mathcal{P}) \leq \delta$ and the equivariance of h . Therefore, $g \in \text{Stab}(\mathfrak{P}_\beta)$ and $\text{Stab}(\mathfrak{P}_\delta) \subset \text{Stab}(\mathfrak{P}_\beta)$.

In order to see that $\text{Stab}(\mathfrak{P}_\delta)$ is a proper subgroup of $\text{Stab}(\mathfrak{P}_\beta)$ we note that if $g \in \text{Stab}(\mathfrak{P}_\beta)$ is a homeomorphism that sends a point of type β from P_1 to P_2 then $g\mathcal{P} \notin \mathfrak{P}_\delta$ and hence $g \notin \text{Stab}(\mathfrak{P}_\delta)$. \square

Lemma 24. *We can write $\text{Homeo}(\omega^\alpha \cdot n + 1) = \bigcup_{\beta < \alpha} \text{Stab}(\mathfrak{P}_\beta)$.*

Proof. Let $g \in \text{Homeo}(\omega^\alpha \cdot n + 1)$. By lemma 21, h takes values strictly less than α , so there exists some $\beta < \alpha$ so that $h(g\mathcal{P}) = \beta$. We claim also that $g \in \text{Stab}(\mathfrak{P}_\beta)$. Indeed, by Lemma 21, for $\mathcal{Q} \in \mathfrak{P}_\beta$ we again have

$$h(g\mathcal{Q}) = h(\mathcal{P}, g\mathcal{Q}) \leq \max\{h(\mathcal{P}, g\mathcal{P}), h(g\mathcal{P}, g\mathcal{Q})\} \leq \beta.$$

We conclude that $g \in \text{Stab}(\mathfrak{P}_\beta)$ and hence $\text{Homeo}(\omega^\alpha \cdot n + 1)$ is equal to the countable union $\bigcup_{\beta < \alpha} \text{Stab}(\mathfrak{P}_\beta)$. \square

These three lemmas taken together provide a proof of Proposition 19.

Proof of Proposition 19. The three lemmas above exactly allow one to write $\text{Homeo}(\omega^\alpha \cdot n + 1)$ as the countable union of the proper open subgroups $\{\text{Stab}(\mathfrak{P}_\beta)\}_{\beta < \alpha}$. Therefore, by Lemma 6, we have that $\text{Homeo}(\omega^\alpha \cdot n + 1)$ does not have a coarsely bounded generating set. \square

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