Singularity categories and singular loci of certain quotient singularities

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Abstract

Let V be a finite dimensional k-vector space, where k is an algebraic closed field of characteristic zero. Let $G \subseteq \mathrm{SL}(V)$ be a finite abelian group, and denote by S the G-invariant subring of the polynomial ring $k[V]$. It is shown that the singularity category $D_{sg}(S)$ recovers the reduced singular locus of $\mathrm{Spec}(S)$.

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1 Introduction

Let A be an associative algebra over a base field k of characteristic zero. Its singularity category $D_{sg}(A)$ is the Verdier quotient $D^b(A)/\mathrm{Perf}(A)$, where $\mathrm{Perf}(A)$ is the full subcategory consisting of perfect complexes over A. It was first introduced by Buchweitz in [2] in his study of algebraic representations of Gorenstein rings. $D_{sg}(A)$ measures the smoothness of A in the sense that it is homologically smooth if and only if $D_{sg}(A)$ is a trivial category. Moreover, Buchweitz showed that $D_{sg}(A)$ is equivalent to the stable category $\underline{\mathrm{CM}}(A)$ of Cohen-Macaulay A-modules as triangulated categories when A is Gorenstein. Later, in [18, 19, 20], Orlov rediscovered this notion from the perspective of algebraic geometry and mathematical physics, which has a deep relationship with Homological Mirror Symmetry.

In recent years, the singular equivalent invariants of singularity categories have attracted much attention. For example, in [3], Chen and Sun introduced the notion of singular equivalence of Morita type. The famous Knörrer periodicity theorem can be realized by singular equivalence of Morita type. In [32], Zhou and Zimmermann showed that for two Noetherian algebras A and B which are singular equivalent of Morita type, their n-th Hochschild cohomology groups are isomorphic, for n big enough. Later, Wang generalized this result to singular equivalence of Morita type with level (see [28]). He also showed that the Tate-Hochschild cohomology together with the Gerstenhaber bracket is invariant under such equivalence (see [29]). Another example is the result of Hua and Keller obtained in [7]. They showed that the singularity category of a local hypersurface ring with isolated singularity recovers the algebra itself via the isomorphism between the zeroth Tate-Hochschild cohomology and the Tyurina algebra of this hypersurface. For more results on the singular equivalent invariants, one may refer to, just to name a few, [6, 11, 14, 17, 30] and references therein.

In particular, recently there has been an increasing study of the relationships between the singular locus and the singularity category of a given algebra. In [2], Buchweitz showed that the Jacobian ideal of a quotient S of a formal power series ring modulo regular sequence annihilates the singularity category of S. Later, in [10], Iyengar and Takahashi extended this result to more general rings, including equicharacteristic complete Cohen-Macaulay local rings. Recently, in [13], Liu proved that when S is either an equidimensional finitely generated algebra over a perfect field, or an equidimensional equicharacteristic complete local ring with a perfect residue field, some power of the Jacobian ideal of S annihilates the singularity category of S.

Among all these results, of special interest is the "reconstruction theorem" for singularity categories. In his research of the topological reconstruction of singular locus, Yu showed that the category of matrix factorizations $\mathrm{MF}(R, f) \cong D_{sg}(R/f)$, equipped with some tensor product structure on it, recovers the spectrum of the singular locus of the hypersurface R/f as a topological space (see [31]). In [5], Hirano extended Yu's result to the case of relative singular locus. Later, for the Gorenstein ring S which is locally a hypersurface on the punctured spectra, Matsui showed that the topological structure of its singular locus can

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be reconstructed by its singularity category $D_{sg}(S)$ (see [16]).

Our objective in this paper is certain Gorenstein normal rings, which are the coordinate rings of the quotients of affine spaces by finite abelian subgroups of the special linear group. In general, the singularities in this case are not isolated. The purpose of this paper is to use the singularity category of S to reconstruct the *reduced* singular locus, denoted by $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$, of $\mathrm{Spec}(S)$. Our main result is the following.

Theorem 1.1. *Let S_i, $i = 1, 2$, be two coordinate rings of the quotients of affine spaces V_i by finite abelian subgroups of $\mathrm{SL}(V_i)$ respectively. Suppose that there is a triangle equivalence $\Upsilon : D_{sg}(S_1) \to D_{sg}(S_2)$ of their singularity categories, then their reduced singular locus are isomorphic:*

$$\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S_1)\big)} \cong \sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S_2)\big)}.$$

In other words, the reduced singular locus of S only depends on its singularity category but nothing else; one can reconstruct the reduced scheme structure of the singular locus from the singularity category.

In [21] Orlov proved that the completion of a variety along its singular locus determines its singularity category, up to the idempotent completion of a triangulated category. However, the converse to Orlov's result does not hold by, for example, the Knörrer periodicity theorem. Nevertheless, our theorem gives a partial answer to this converse problem up to some extent.

To show the above result, we heavily use techniques from the theory of McKay quivers, non-commutative resolutions introduced by Van den Bergh (see [26, 27]), and their contraction algebras introduced by Donovan and Wemyss (see [4]).

The rest of this paper is devoted to the proof of the above theorem. It is organized as follows. In §2, we introduce some necessary notions and results on singularity categories. In §3, we describe the relations between the singular locus of $\mathrm{Spec}(S)$ and the contraction algebra of the canonical noncommutative resolution of S, and prove that the reduced center of this contraction algebra is isomorphic to the coordinate ring of the reduced singular locus of $\mathrm{Spec}(S)$ (see Theorem 3.27). In §4, we introduce a inverse system $\mathcal{R}^S$ from $D_{sg}(S)$, and then show that the inverse limit of $\mathcal{R}^S$ is exactly the coordinate ring of the reduced singular locus. This isomorphism gives the proof of Theorem 1.1. Finally, in §5, we give two examples of our main theorem. This paper also contains several appendices, where the proofs of several results in the main context, usually lengthy and technical, are given.

Convention 1.2. In this paper, we assume the base field k is algebraically closed of characteristic zero. All modules are right modules and all complexes are cochain complexes unless otherwise specified.

2 Preliminaries

In this section, we collect several necessary concepts and notions which will be used in later sections.

2.1 Gorenstein rings and Cohen-Macaulay modules

Definition 2.1. A commutative Noetherian ring S over k is called *Gorenstein* if for any prime ideal $\mathfrak{p} \subseteq S$, $\mathrm{Ext}^i_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{m}, S_{\mathfrak{p}}) = 0$ for all $i \neq d$ and $\mathrm{Ext}^d_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{m}, S_{\mathfrak{p}}) \cong k$ as vector spaces, where $\mathfrak{m}$ is the unique maximal ideal of the local ring $S_{\mathfrak{p}}$ and $d := \dim(S)$ is the Krull dimension of S.

By using the Koszul resolution, it is straightforward to check that the polynomial ring $k[x_1, \cdots, x_n]$ is Gorenstein. More generally, we have the following well-known result.

Proposition 2.2 ([9, §2.2]). *Let $R = k[V] = k[x_1, \cdots, x_n]$ and G be a finite subgroup in $\mathrm{SL}(V)$, which naturally acts on V and hence on R. Then R^G is a Gorenstein normal domain.*

Let S be a Noetherian local ring, $\mathfrak{m}$ be the unique maximal ideal of S and M be a finitely generated S-module. Recall that the *depth* of M, denoted by $\mathrm{depth}(M)$, is defined as follows:

(1) If $M\mathfrak{m} = M$, then $\mathrm{depth}(M) = \infty$;

(2) If $M\mathfrak{m} \neq M$, then $\mathrm{depth}(M)$ is the supremum of the lengths of M-regular sequences in $\mathfrak{m}$.

Here, an M-regular sequence means a sequence elements $\{f_r\}_{1 \leq r \leq m}$ of S such that for any element f_r in this sequence,

(1) f_r is not a zero divisor on $M/M(f_1, \cdots, f_{r-1})$, and

(2) $M/M(f_1, \cdots, f_m)$ is not trivial as S-module.

Definition 2.3. Let S be a commutative Noetherian ring with Krull dimension d and M be a finitely generated S-module. M is called *maximal Cohen-Macaulay* if $\mathrm{depth}(M_{\mathfrak{m}}) = d$ for any maximal ideal $\mathfrak{m}$ of S, or $M \cong 0$. We simply call such modules Cohen-Macaulay.

2.2 The singularity category

Suppose S is an associative k-algebra. Let $\mathrm{mod}(S)$ be the abelian category of finitely generated S-modules. Let $D^b(S)$ be the bounded derived category of $\mathrm{mod}(S)$ and $\mathrm{Perf}(S)$ be the full triangulated subcategory of $D^b(S)$ such that its objects are consist of complexes which are quasi-isomorphic to bounded complexes of projective S-modules. The *singularity category* of S, denoted by $D_{sg}(S)$, is defined to be the Verdier quotient $D^b(S)/\mathrm{Perf}(S)$ of triangulated categories.

Definition 2.4. Let S be a commutative Noetherian ring and $\mathrm{CM}(S)$ be the full subcategory of $\mathrm{mod}(S)$ consisting of Cohen-Macaulay S-modules. The *stable category of Cohen-Macaulay S-modules*, denoted by $\underline{\mathrm{CM}}(S)$, is the category where

(1) the objects of $\underline{\mathrm{CM}}(S)$ are the same as $\mathrm{CM}(S)$;

(2) for any objects X, Y in $\underline{\mathrm{CM}}(S)$,

$$\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(X, Y) := \mathrm{Hom}_{\mathrm{CM}(S)}(X, Y)/I_{X,Y},$$

where $I_{X,Y} \subseteq \mathrm{Hom}_{\mathrm{CM}(S)}(X, Y)$ is the vector subspace consisting of homomorphisms which factor through some projective S-module.

When S is a Gorenstein ring, $\underline{\mathrm{CM}}(S)$ is a triangulated category. Moreover, we have $D_{sg}(S) \cong \underline{\mathrm{CM}}(S)$ as triangulated categories (see [2]).

2.3 Generator of a triangulated category

We next recall the notions of generator and classical generator of a triangulated category.

Definition 2.5. Let $\mathcal{T}$ be a triangulated category. A set $\mathcal{E}$ of objects in $\mathcal{T}$ is said to *generate* $\mathcal{T}$ if for any given $X \in \mathcal{T}$, $\operatorname{Hom}_{\mathcal{T}}(D, X[i]) = 0$ for all $D \in \mathcal{E}$ and all $i \in \mathbb{Z}$ implies $X \cong 0$ in $\mathcal{T}$. $\mathcal{E}$ is called a *generator* of $\mathcal{T}$.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two full triangulated subcategories of $\mathcal{T}$. Denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of $\mathcal{T}$, whose objects are the objects like M of $\mathcal{T}$ such that there exists a distinguished triangle

$$M_1 \longrightarrow M \longrightarrow M_2 \to M_1[1]$$

in $\mathcal{T}$ with $M_i \in \operatorname{Ob}(\mathcal{I}_i)$. Let $\mathcal{E}$ be a set of objects in $\mathcal{T}$. Denote by $\langle \mathcal{E} \rangle_1$ the smallest full subcategory of $\mathcal{T}$ containing the objects in $\mathcal{E}$ and closed under direct summands, finite direct sums and shifts. Let $\langle \mathcal{E} \rangle_0$ be the trivial subcategory of $\mathcal{T}$, $\langle \mathcal{E} \rangle_i := \langle \mathcal{E} \rangle_{i-1} * \langle \mathcal{E} \rangle_1$, and $\langle \mathcal{E} \rangle_\infty := \bigcup_{i \geq 0} \langle \mathcal{E} \rangle_i$ as full subcategories of $\mathcal{T}$.

Definition 2.6. Let $\mathcal{T}$ be a triangulated category. A set $\mathcal{E}$ of objects in $\mathcal{T}$ is said to *classically generate* $\mathcal{T}$ if

$$\mathcal{T} \cong \langle \mathcal{E} \rangle_\infty.$$

$\mathcal{E}$ is called a *classical generator* of $\mathcal{T}$.

For an object M in $\mathcal{T}$, it is classically generated by $\mathcal{E}$ if $M \in \langle \mathcal{E} \rangle_m$ for some $m \in \mathbb{N}$. It is direct to see that a set $\mathcal{E}$ of objects in $\mathcal{T}$ classically generates $\mathcal{T}$ if and only if for any object M in $\mathcal{T}$, it is classically generated by $\mathcal{E}$. A fact is that that a classical generator in a triangulated category must be a generator in this triangulated category (see [25, Lemma 13.36.5]).

Proposition 2.7. *Let* $\pi : D^b(S) \to D^b(S)/\operatorname{Perf}(S) \cong D_{sg}(S)$ *be the Verdier quotient functor. Suppose that* $X \in \operatorname{Ob}(D^b(S))$ *is a classical generator of* $D^b(S)$. *Then* $\pi(X)$ *is a classical generator of* $D_{sg}(S)$.

Proof. Pick any object M in $D_{sg}(S)$. There is an object $\widetilde{M}$ of $D^b(S)$ such that $\pi(\widetilde{M}) = M$. Since X is a classical generator of $D^b(S)$, $\widetilde{M} \in \langle X \rangle_m$ for some $m \in \mathbb{N}$. Next, via the triangle functor π, we obtain that $M = \pi(\widetilde{M}) \in \langle \pi(X) \rangle_m$ in $D_{sg}(S)$. $\square$

3 Contraction algebra and the singular locus

We first recall the definition of the singular locus of an affine variety (see, for example, [24]).

Definition 3.1 (Singular locus). Let S be a commutative Noetherian ring over k. The *singular locus* of $\operatorname{Spec}(S)$, denoted by $\operatorname{Sing}(\operatorname{Spec}(S))$, is the subscheme consisting of prime ideals $\mathfrak{p}$ of S such that the local ring $S_{\mathfrak{p}}$ is not regular.

The purpose of this section is to give an algebraic characterization of the singular locus of S, especially for the invariant subring of the polynomial ring under the action of a finite abelian subgroup of the special linear group.

3.1 Non-commutative resolution and the contraction algebra

We start with the notion of non-commutative resolutions (NCR), which is introduced by Van den Bergh ([26, 27]), and has been intensively studied in recent years.

Let S be a Gorenstein normal domain over k. Recall that an S-module M is called *reflexive* if the natural homomorphism

$$M \to (M^\vee)^\vee \, , m \mapsto (f \mapsto f(m))$$

is an isomorphism of S-modules, where $(-)^\vee$ means $\mathrm{Hom}_S(-, S)$.

Definition 3.2 (Van den Bergh)**.** A *non-commutative resolution* (NCR) of S is a k-algebra of the form $\mathrm{End}_S(M)$ for some reflexive S-module M, such that the global dimension of $\mathrm{End}_S(M)$ is finite.

Example 3.3. Let V be a k-vector space with dimension n. Let G be a finite abelian subgroup in $\mathrm{SL}(V)$ and let $R = k[V]$ and $S = R^G$. Let $\hat{G}$ to be the set of irreducible representations of G (recall that the dimension of any irreducible representation of G is one). For any $W \in \hat{G}$, the S-module $(W \otimes R)^G$ is a Cohen-Macaulay S-module (see [12, Section J]).

It is well-known that if an S-module M is Cohen-Macaulay, then it is reflexive. Let $\Lambda := \mathrm{End}_S\big(\bigoplus_{W \in \hat{G}} (W \otimes R)^G \big)$. Then by Auslander's theorem, $\Lambda \cong G \sharp R$, and is an NCR of S (see [8, Example 2.4]), where $G \sharp R$ is the skew group algebra of G and R.

Let $\mathrm{ref}(S)$ and $\mathrm{ref}(R, G)$ be the categories of reflexive S-modules and of G-equivariant reflexive R-modules respectively. Then we have the following.

Lemma 3.4 ([22, Lemma 3.3])**.** *Let R, G and S be as in Example 3.3. Then the following two functors*

$$\mathrm{ref}(R, G) \to \mathrm{ref}(S), \ M \mapsto M^G$$

and

$$\mathrm{ref}(S) \to \mathrm{ref}(R, G), \ N \mapsto \big((R \otimes_S N)^\vee \big)^\vee$$

are inverse equivalences of two symmetric monoidal categories, where $(-)^\vee := \mathrm{Hom}_R(-, R)$.

By Lemma 3.4, for any $W \in \hat{G}$, we have

$$\begin{aligned}
\mathrm{End}_S\big((W \otimes R)^G \big) &\cong \mathrm{End}_{\mathrm{ref}(R,G)}(W \otimes R) \cong \mathrm{Hom}_{\mathrm{ref}(k,G)}(W \otimes W^*, R) \\
&\cong \mathrm{Hom}_{\mathrm{ref}(k,G)}(k, R) = S,
\end{aligned} \tag{1}$$

where W^* is the linear dual of W.

We next move to the notion of contraction algebras, which is introduced by Donovan and Wemyss in their research of NCR and will play an important role in this paper.

Definition 3.5 (Contraction algebra; see [4])**.** Let S be a Gorenstein commutative ring over k and $\Lambda^M := \mathrm{End}_S(S \oplus M)$, where M is a Cohen-Macaulay S-module. Let $[S]$ be the two-sided ideal of Λ^M consisting of those S-module homomorphisms $(S \oplus M) \longrightarrow (S \oplus M)$ which factor through some object $P \in \mathrm{add}(S)$:

The *contraction algebra* of Λ^M, denoted by Λ^M_{con}, is the quotient algebra $\Lambda^M / [S]$.

In the above definition, add(S) is the full subcategory of mod(S) consisting of direct summands of some direct sums of S, i.e., the subcategory of finitely generated projective S-modules.

From the above definition, it is direct to see that $\Lambda_{\mathrm{con}}^M \cong \Lambda_{\mathrm{con}}^{(M \oplus Q)}$ as algebras, for any projective S-module Q. By the definition of stable categories (see (2.4)), we immediately get the following.

Proposition 3.6. *Let M be a Cohen-Macaulay S-module. Then*

$$\Lambda_{\mathrm{con}}^M \cong \mathrm{End}_{\underline{\mathrm{CM}}(S)}(S \oplus M) \cong \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M)$$

as algebras.

3.2 Singular locus of a Gorenstein domain

In this subsection, we give a description of the singular locus of a Gorenstein domain in terms of its NCR (Theorem 3.9). Let us start with the following.

Lemma 3.7. *Suppose $M \in \mathrm{Ob}\big(\underline{\mathrm{CM}}(S)\big)$ gives an NCR of S and contains a direct summand S as an S-module. Suppose $\mathfrak{p}$ is a prime ideal of S. Denote by $M_\mathfrak{p}$ the localization of M at $\mathfrak{p}$. Then $M_\mathfrak{p}$ is a generator of $\underline{\mathrm{CM}}(S_\mathfrak{p})$.*

Proof. To prove this lemma, it is sufficient to show that $M_\mathfrak{p}$ is a classical generator of $\underline{\mathrm{CM}}(S_\mathfrak{p})$. Since $\underline{\mathrm{CM}}(S_\mathfrak{p}) \cong D_{sg}(S_\mathfrak{p})$ as triangulated categories, this is equivalent to showing that $M_\mathfrak{p}$ is a classical generator of $D_{sg}(S_\mathfrak{p})$. By Proposition 2.7 it is enough to show that $M_\mathfrak{p}$ is a classical generator of $D^b(S_\mathfrak{p})$.

From the definition of NCR, we know that $\Lambda := \mathrm{End}_S(M)$ is homologically smooth. Let $P^\bullet$ be a bounded projective Λ^e-module resolution of Λ. Note that Λ is an S-algebra and thus Λ^e is an S^e-algebra. Localizing at $\mathfrak{p}$, we obtain a bounded projective $\Lambda_\mathfrak{p}^e$-module resolution $P_\mathfrak{p}^\bullet$ of $\Lambda_\mathfrak{p}$, which then implies that $\Lambda_\mathfrak{p}$ is homologically smooth.

We thus obtain a triangle equivalence $D^b(\Lambda_\mathfrak{p}) \cong \mathrm{Perf}(\Lambda_\mathfrak{p})$, which gives $D^b(\Lambda_\mathfrak{p})$ a classical generator $\Lambda_\mathfrak{p}$. Next, let $e_\mathfrak{p}$ be the indecomposable idempotent of $\Lambda_\mathfrak{p}$ corresponding to the direct summand $S_\mathfrak{p}$ of $M_\mathfrak{p}$. Then since mod($S_\mathfrak{p}$) is equivalent to the Serre quotient mod($\Lambda_\mathfrak{p}$)/mod($\Lambda_\mathfrak{p} e_\mathfrak{p} \Lambda_\mathfrak{p}$) as abelian categories, there is the following localization functor

$$(-) \otimes_{\Lambda_\mathfrak{p}}^{\mathbb{L}} \Lambda_\mathfrak{p} e_\mathfrak{p} : D^b(\Lambda_\mathfrak{p}) \to D^b(S_\mathfrak{p}).$$

Thus similarly to the proof of Proposition 2.7, $\Lambda_\mathfrak{p} e_\mathfrak{p} \cong M_\mathfrak{p}$ is a classical generator of $D^b(S_\mathfrak{p})$. $\square$

Remark 3.8. *With the same argument, one can show that M is a generator of $\underline{\mathrm{CM}}(S)$.*

Theorem 3.9. *Let S be a Gorenstein normal domain, and $\Lambda = \mathrm{End}_S(\bigoplus_{i=0}^m M_i)$ be an NCR of S such that $M_0 = S$ and M_i's are indecomposable Cohen-Macaulay S-modules for all i. Let $\varphi : S \hookrightarrow \Lambda$ be the canonical injection of algebras induced by the S-module structure of $\bigoplus_{i=0}^m M_i$, and let $e \in \Lambda$ be the idempotent corresponding to the direct summand S. Then*

$$\mathrm{Sing}\big(\mathrm{Spec}(S)\big) = \mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S) \cap (\Lambda e \Lambda)\big)\Big).$$

as subschemes of $\mathrm{Spec}(S)$.

In the above theorem, note that since $\bigoplus_{i=0}^m M_i$ is a reflexive S-module and S is a domain, there is no zero divisor in $\bigoplus_{i=0}^m M_i$, and therefore φ is an injection.

Proof of Theorem 3.9. (1) We first show

$$\mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\Big)\subseteq\mathrm{Sing}\big(\mathrm{Spec}(S)\big).$$

To this end, let $x\in\mathrm{Spec}(S)$ be a smooth point which corresponds to a prime ideal $\mathfrak{p}$. Consider the regular ring $S_{\mathfrak{p}}$ and the $S_{\mathfrak{p}}$-algebra $\Lambda_{\mathfrak{p}}$. To simplify the notations, set $\overline{M}:=\bigoplus_{i=0}^{m}M_i$. Since $S_{\mathfrak{p}}$ is regular and $\overline{M}_{\mathfrak{p}}$ is a Cohen-Macaulay $S_{\mathfrak{p}}$-module, $\overline{M}_{\mathfrak{p}}$ is free over $S_{\mathfrak{p}}$. Thus we have

$$\Lambda_{\mathfrak{p}}\cong\mathrm{End}_S(\overline{M})_{\mathfrak{p}}\cong\mathrm{End}_{S_{\mathfrak{p}}}(\overline{M}_{\mathfrak{p}})\cong M_{r\times r}(S_{\mathfrak{p}}), \tag{2}$$

where r is rank of $\overline{M}_{\mathfrak{p}}$. Moreover, if we denote by $\Psi:\Lambda e\otimes_S e\Lambda\cong\mathrm{Hom}_S(S,\overline{M})\otimes_S\mathrm{Hom}_S(\overline{M},S)\twoheadrightarrow\Lambda e\Lambda\subseteq\mathrm{End}_S(\overline{M})$ the composition of homomorphisms, we get

$$\begin{aligned}(\Lambda e\Lambda)_{\mathfrak{p}}&\cong\Psi\big((\Lambda e\otimes_S e\Lambda)\big)_{\mathfrak{p}}\cong\Psi_{\mathfrak{p}}\big((\Lambda e)_{\mathfrak{p}}\otimes_{S_{\mathfrak{p}}}(e\Lambda)_{\mathfrak{p}}\big)\\&\cong\Psi_{\mathfrak{p}}\Big(\mathrm{Hom}_{S_{\mathfrak{p}}}\big(S_{\mathfrak{p}},(S_{\mathfrak{p}})^{\oplus r}\big)\otimes_{S_{\mathfrak{p}}}\mathrm{Hom}_{S_{\mathfrak{p}}}\big((S_{\mathfrak{p}})^{\oplus r},S_{\mathfrak{p}}\big)\Big)\\&\cong M_{r\times r}(S_{\mathfrak{p}}),\end{aligned} \tag{3}$$

where $\Psi_{\mathfrak{p}}$ is the localization of Ψ at $\mathfrak{p}$. Combining (2) and (3) we obtain that $\Lambda_{\mathfrak{p}}/(\Lambda e\Lambda)_{\mathfrak{p}}\cong 0$. In the meantime, there is a natural injection

$$\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\hookrightarrow\Lambda/(\Lambda e\Lambda)$$

induced by φ. This injection induces the following injection after localization

$$\varphi(S)_{\mathfrak{p}}/\big(\varphi(S)_{\mathfrak{p}}\cap(\Lambda e\Lambda)_{\mathfrak{p}}\big)\hookrightarrow\Lambda_{\mathfrak{p}}/(\Lambda e\Lambda)_{\mathfrak{p}}.$$

Since $\Lambda_{\mathfrak{p}}/(\Lambda e\Lambda)_{\mathfrak{p}}\cong 0$, we get that $\Big(\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\Big)_{\mathfrak{p}}\cong\varphi(S)_{\mathfrak{p}}/\big(\varphi(S)_{\mathfrak{p}}\cap(\Lambda e\Lambda)_{\mathfrak{p}}\big)\cong 0$.

Thus x is not in the subscheme $\mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\Big)$ of $\mathrm{Spec}(S)$, which implies that

$$\mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\Big)\subseteq\mathrm{Sing}\big(\mathrm{Spec}(S)\big).$$

(2) We next show

$$\mathrm{Sing}\big(\mathrm{Spec}(S)\big)\subseteq\mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)\Big).$$

To this end, let $\mathfrak{q}$ be an arbitrary prime ideal of S corresponding to a point in $\mathrm{Sing}\big(\mathrm{Spec}(S)\big)$. To show the above inclusion, it is sufficient to show that the algebra $\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)_{\mathfrak{q}}$ is nontrivial.

In fact, $\Lambda_{\mathfrak{q}}/(\Lambda e\Lambda)_{\mathfrak{q}}$ is an algebra over $\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)_{\mathfrak{q}}\cong\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)_{\mathfrak{q}}\cap(\Lambda e\Lambda)_{\mathfrak{q}}\big)$. Meanwhile, $\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)_{\mathfrak{q}}\cap(\Lambda e\Lambda)_{\mathfrak{q}}\big)$ as a subalgebra contains the unit of $\Lambda_{\mathfrak{q}}/(\Lambda e\Lambda)_{\mathfrak{q}}$. Then $\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)_{\mathfrak{q}}\cap(\Lambda e\Lambda)_{\mathfrak{q}}\big)$ is trivial if and only if $\Lambda_{\mathfrak{q}}/(\Lambda e\Lambda)_{\mathfrak{q}}$ is trivial. Thus, to prove that $\varphi(S)_{\mathfrak{q}}/\big(\varphi(S)\cap(\Lambda e\Lambda)\big)_{\mathfrak{q}}$ is nontrivial, it suffices to show that

$$\Lambda_{\mathfrak{q}}/(\Lambda e\Lambda)_{\mathfrak{q}}\cong\Lambda_{\mathfrak{q}}/\Lambda_{\mathfrak{q}}e_{\mathfrak{q}}\Lambda_{\mathfrak{q}}\cong\mathrm{End}_{S_{\mathfrak{q}}}(\overline{M}_{\mathfrak{q}})/[S_{\mathfrak{q}}]\cong\mathrm{End}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}})$$

is nontrivial. We prove this by contradiction.

Assume that $\mathrm{End}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}})=0$. By Lemma 3.7 we know that $\overline{M}_{\mathfrak{q}}$ is a generator of $\underline{\mathrm{CM}}(S_{\mathfrak{q}})$. Now for any object $M\in\mathrm{Ob}\big(\underline{\mathrm{CM}}(S_{\mathfrak{q}})\big)$, $\mathrm{Hom}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}},M)$ is an $\mathrm{End}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}})$-module. Furthermore, the following composition

$$\mathrm{Hom}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}},M[i])\otimes\mathrm{End}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}})\to\mathrm{Hom}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}},M[i])$$

is surjective for any $i \in \mathbb{Z}$. It implies that $\mathrm{Hom}_{\underline{\mathrm{CM}}(S_{\mathfrak{q}})}(\overline{M}_{\mathfrak{q}}, M[i]) = 0$ for all $i \in \mathbb{Z}$. Meanwhile, since $\overline{M}_{\mathfrak{q}}$ is a generator of $\underline{\mathrm{CM}}(S_{\mathfrak{q}})$, we get that $M \cong 0$ in $\underline{\mathrm{CM}}(S_{\mathfrak{q}})$. Hence, the objects in $\underline{\mathrm{CM}}(S_{\mathfrak{q}})$ are all trivial. Therefore, the triangulated category $\underline{\mathrm{CM}}(S_{\mathfrak{q}})$ is trivial, and thus $S_{\mathfrak{q}}$ is a homologically smooth ring. However, this contradicts to that $\mathfrak{q}$ is a point in $\mathrm{Sing}(\mathrm{Spec}(S))$. Therefore $\Lambda_{\mathfrak{q}}/(\Lambda e \Lambda)_{\mathfrak{q}} \cong \underline{\mathrm{End}}_{S_{\mathfrak{q}}}(\overline{M}_{\mathfrak{q}})$ is nontrivial, which then implies that

$$\mathrm{Sing}(\mathrm{Spec}(S)) \subseteq \mathrm{Spec}\Big(\varphi(S)/\big(\varphi(S) \cap (\Lambda e \Lambda)\big)\Big). \qquad \square$$

By taking the reduced schemes of the isomorphism in Theorem 3.9, we immediately get the following (in this paper, for a scheme X, we use $\sqrt{X}$ to denote its reduced scheme):

Corollary 3.10. *The isomorphism in Theorem 3.9 induces the following isomorphism*

$$\sqrt{\mathrm{Sing}(\mathrm{Spec}(S))} \cong \mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)}\big).$$

3.3 The singular locus of $\mathrm{Spec}(S)$

From now on, we focus on the case in Example 3.3. For reader's convenience, let us list some notations that we will be repeatedly using:

Notation 3.11. – Let $\widehat{M} := \bigoplus_{W \in \hat{G}} (W \otimes R)^G$, which, by Auslander's theorem, is isomorphic to R as S-modules.

  – Let $\Lambda := \mathrm{End}_S(\widehat{M})$ and let e be the indecomposable idempotent of Λ corresponding to the direct summand S of $\widehat{M}$. Note that when W is the trivial G-representation, then $(W \oplus R)^G = R^G = S$.

  – Let $\varphi : S \hookrightarrow \Lambda$ be the canonical injection of algebras induced by the S-module structure on $\widehat{M}$.

  – Let $\Lambda_{\mathrm{con}} := \Lambda_{\mathrm{con}}^{\widehat{M}} \cong \Lambda/[S] \cong \mathrm{End}_{\underline{\mathrm{CM}}(S)}(\bigoplus_\chi M_\chi)$ (see Definition 3.5 and Proposition 3.6). Here, $[S]$ is the ideal of $\mathrm{End}_S(\widehat{M})$ consisting of homomorphisms which factor through some finitely generated projective S-module. Note that $\Lambda_{\mathrm{con}} \cong\cong \Lambda/\Lambda e \Lambda$.

  – Let $R^{\widehat{M}}$ be the reduced ring of the center $Z(\Lambda_{\mathrm{con}})$ of Λ_{con}.

The purpose of this subsection is to describe the reduced singular locus of $\mathrm{Spec}(S)$ by means of $\widehat{M}$. In Theorem 3.26 we show that $\mathrm{Spec}(R^{\widehat{M}}) \cong \sqrt{\mathrm{Sing}(\mathrm{Spec}(S))}$ as subschemes of $\mathrm{Spec}(S)$. Recall that from Corollary 3.10, $\sqrt{\mathrm{Sing}(\mathrm{Spec}(S))} = \mathrm{Spec}(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)})$ as subschemes of $\mathrm{Spec}(S)$. Thus by combining these two theorems and considering the associated rings, we obtain that $\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)} \cong R^{\widehat{M}}$, which is in fact induced by the natural map $\varphi : S \to \Lambda$ (see Theorem 3.27).

3.3.1 Characters of G and components of the singular locus

Recall that the characters of G are in one-one correspondence with the indecomposable idempotents of the group algebra kG, which is further in one-one correspondence with the indecomposable idempotents of Λ. Let $\chi : G \to k^*$ be a character of G. Denote the corresponding idempotent by e_χ, then $e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)(g \otimes 1) \in \Lambda$. Let χ_0 be the trivial

character of G. It is direct to see that e is the indecomposable idempotent corresponding to χ_0.

Since G is a finite abelian subgroup of $\mathrm{SL}(V)$, G can be viewed as a group consisting of diagonal matrices in $\mathrm{SL}(V)$. Fix such a diagonalization for G, and denote by $\{E_i\}_{1 \le i \le n}$ the basis of V with respect to this diagonalization. Notice that any E_i is associated to a character, denoted by χ_{E_i}, of G, given by $\chi_{E_i}(g) = g_i \in k^*$ for any $g \in G$, where $g(E_i) = g_i E_i$ and $g(-)$ is the action of g on V.

Dually, let $R = k[V] = k[x_1, \cdots, x_n]$, where $\{x_i\}$ is the set of basis dual to $\{E_i\}$. Then each x_i is also associated to a character, denoted by χ_{x_i} of G, such that $\chi_{x_i} = \chi_{E_i}^{-1}$. Moreover, for any monomial $f \in R$, define a character χ_f of G as $\chi_f(g) := g_f \in k$, for any $g \in G$, where $g(f) = g_f f$ and $g(-)$ is the dual action of g on $R = k[V]$.

For any character χ of G, let V_χ be the one dimensional irreducible representation of G given by $\chi : G \to k^*$. Let $M_\chi := (V_\chi \otimes R)^G$ be an indecomposable direct summand of $\widehat{M}$. Note that $(V_\chi \otimes R)^G \subseteq R$ as S-modules. Moreover, since $(V_\chi \otimes R)^G = \mathrm{Hom}_{\mathrm{ref}(k,G)}(V_{\chi^{-1}}, R)$, M_χ is generated by monomials like f such that $\chi_f = \chi^{-1}$ as S-modules.

At the same time, it is obvious that for any monomial $f \in R$,

$$e_\chi(1 \otimes f)e_{\chi\chi_f} = e_\chi e_\chi(1 \otimes f) = e_\chi(1 \otimes f) \quad \text{and} \quad (1 \otimes f)e_\chi = e_{\chi\chi_f^{-1}}(1 \otimes f)e_\chi, \qquad (4)$$

where $\chi\chi_f$ and $\chi\chi_f^{-1}$ are the products of characters.

Furthermore, let $\Lambda e_\chi \subseteq \Lambda$ be product of Λ with e_χ. Then we have $\Lambda e_\chi = (G \sharp R)e_\chi \cong R \cong \widehat{M}$ as S-modules. In the meantime, since $e \in \Lambda$ is the idempotent corresponding to the summand S of $\widehat{M}$,

$$\Lambda e \cong \mathrm{Hom}_S(S, \widehat{M}) \cong \widehat{M} \qquad (5)$$

as Λ^{op}-modules. Hence, we may identify Λe with $\widehat{M}$ as $S \otimes \Lambda^{op}$-modules.

Consider the direct summand $e_\chi \widehat{M} := e_\chi \Lambda \otimes_\Lambda \widehat{M}$ of $\widehat{M}$ for any e_χ. We have

$$e_\chi \widehat{M} \cong e_\chi \Lambda e \cong e_\chi(G \sharp R)e \cong (e_\chi \otimes R)e \cong (e_\chi \otimes \widehat{M})e$$
$$\cong \left(e_\chi \otimes \left(\bigoplus_\lambda M_\lambda\right)\right)e \cong \bigoplus_\lambda \left(e_\chi \otimes M_\lambda\right)e$$
$$\cong \bigoplus_\lambda \left(e_\chi e_\lambda \otimes M_\lambda\right) \cong e_\chi \otimes M_\chi \cong M_\chi$$

as S-modules. In above identities, we identify the idempotents of Λ with the ones of kG. Thus, for any two characters χ and χ' of G,

$$e_\chi \Lambda e_{\chi'} \cong e_\chi \Lambda \otimes_\Lambda \mathrm{End}_S(\widehat{M}) \otimes_\Lambda \Lambda e_{\chi'}$$
$$\cong \mathrm{Hom}_S\left(e_{\chi'}\Lambda \otimes_\Lambda \widehat{M}, e_\chi \Lambda \otimes_\Lambda \widehat{M}\right)$$
$$\cong \mathrm{End}_S(M_{\chi'}, M_\chi) \qquad (6)$$

as S-modules. In particular, $e_\chi \Lambda e_\chi \cong S$ for any χ (see (1)), and thus in the following we may identify $e_\chi \Lambda e_\chi$ with S. Note that the inverse isomorphism $S \cong e_\chi \Lambda e_\chi$ maps $f \in S$ to $e_\chi(1 \otimes f)e_\chi \in e_\chi \Lambda e_\chi$.

Replacing e_χ by e, by the above argument we get that $e\widehat{M} \cong M_{\chi_0} \cong S$ as S-modules. It is direct to see that

$$e_\chi \Lambda e \cong \mathrm{Hom}_S(S, M_\chi) \qquad (7)$$

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as S-modules. By directly summing up all $\{M_\chi\}_\chi$, it follows that $e\Lambda \cong \mathrm{Hom}_S(\widehat{M}, S)$ as $\Lambda \otimes S^{op}$-modules. Now recall that from the proof of Theorem 3.9, $\Psi : \Lambda e \otimes_S e\Lambda \to \Lambda e\Lambda$ is given by the composition map of $\mathrm{End}_S(\widehat{M})$. (Note Ψ is a surjection.) From the above two isomorphisms (5) and (7) we then obtain that

$$\Lambda e\Lambda \cong \Psi(\Lambda e \otimes_S e\Lambda) \cong \Psi\big(\mathrm{Hom}_S(S, \widehat{M})\big) \otimes_S \mathrm{Hom}_S(\widehat{M}, S) \cong [S] \subseteq \Lambda$$

as Λ^e-modules, and then $\Lambda_{\mathrm{con}} \cong \Lambda / \Lambda e\Lambda$ as Λ^e-algebras. Thus, by $\Lambda e\Lambda \cong [S]$ and by Proposition 3.6, we get that

$$e_\chi \Lambda_{\mathrm{con}} e_\chi \cong e_\chi \Lambda e_\chi / e_\chi \Lambda e\Lambda e_\chi \cong \mathrm{End}_S(M_\chi) / \big(\mathrm{End}_S(M_\chi) \cap \Lambda e\Lambda\big)$$
$$\cong \mathrm{End}_S(M_\chi) / \big(\mathrm{End}_S(M_\chi) \cap [S]\big) \cong \Lambda_{\mathrm{con}}^{M_\chi} \cong \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(M_\chi, M_\chi) \qquad (8)$$

as S-algebras.

Lemma 3.12. *Let $\bar{I}_\chi := e_\chi \Lambda e\Lambda e_\chi \subseteq e_\chi \Lambda e_\chi \cong S$ be the ideal of S and $I_\chi := \sqrt{\bar{I}_\chi} \subseteq S$ be the radical of $\bar{I}_\chi$. Then*

$$\mathrm{Spec}\big(S/I_\chi\big) \subseteq \sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$$

as subschemes of $\mathrm{Spec}(S)$.

Proof. First, since $\mathrm{Spec}(S) = V/G$ is an irreducible scheme, it is easy to see that S is an equidimensional finitely generated ring over k. Set ideals

$$\mathrm{ann}_S(X) := \{ f \in S \mid \mathrm{End}_{D_{sg}(S)}(X)\, f = 0 \}$$

for any object X of $D_{sg}(S)$, and

$$\mathrm{ann}_S\big(D_{sg}(S)\big) := \bigcap_{X \in D_{sg}(S)} \mathrm{ann}_S(X)$$

(see [13, §3]). Here, we view $\mathrm{End}_{D_{sg}(S)}(X, X)$ as an S-modules. From [13, Corollary 4.9], we know that

$$\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)} = \mathrm{Spec}\Big(S/\sqrt{\mathrm{ann}_S\big(D_{sg}(S)\big)}\Big).$$

Therefore, to prove this lemma, it suffices to show that

$$\sqrt{\mathrm{ann}_S(D_{sg}(S))} \subseteq I_\chi,$$

which is further enough to show that

$$\mathrm{ann}_S(D_{sg}(S)) \subseteq e_\chi \Lambda e\Lambda e_\chi.$$

Since $D_{sg}(S) \cong \underline{\mathrm{CM}}(S)$ and $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_\chi) \cong S/e_\chi \Lambda e\Lambda e_\chi$ (by (8) above), we have $e_\chi \Lambda e\Lambda e_\chi = \mathrm{ann}_S(M_\chi)$. This implies that $\mathrm{ann}_S(D_{sg}(S)) \subseteq e_\chi \Lambda e\Lambda e_\chi$, and the lemma follows. $\qquad\square$

3.3.2 The quiver description

In this paper, we shall also use the quiver presentations of Λ and Λ_{con}, which we denote by Q_Λ and Q_{con} respectively. Indeed, Q_Λ is the McKay quiver associated to the faithful representation V of G, and Q_{con} is obtained from Q_Λ by removing the vertex that corresponds to idempotent e and the arrows that start from or end at this vertex from Q_Λ. For any indecomposable idempotent e', we also use e' to denote its corresponding vertex.

Proposition 3.13. Q_{con} *is connected.*

Proof. We prove by contradiction. Assume that Q_{con} is not connected. Decompose Q_{con} into two disjoint quivers Q_{con}^1 and Q_{con}^2.

Note that Q_{con} is obtained from Q_Λ by removing the vertex e and the arrows that start from or end at e. Since Q_Λ is the McKay quiver associated to the faithful representation V of G, it is well-known that Q_Λ is connected (see, for example, [15, Theorem 3.1]). This means that, as subquivers of Q_Λ, Q_{con}^1 and Q_{com}^2 are connected only by paths that go through the vertex e in Q_Λ.

Now since $\Lambda = G \sharp R$ and $\hat{G}$ consists of one-dimensional representations, each arrow in Q_Λ is represented by some formal variable x_i, and hence each arrow in Q_{con} is also given by x_i.

By (4), for any x_i in $\{x_r\}_r$, e is connected to $e_{\chi_{x_i}}$ by an arrow, denoted by $\iota_{x_i}^e$, corresponding to $e(1 \otimes x_i)e_{\chi_{x_i}} \in \Lambda$. Moreover, e is connected with $e_{\chi_{x_i}^{-1}}$ by an arrow, denoted by $\tau_{x_i}^e$, corresponding to $e_{\chi_{x_i}^{-1}}(1 \otimes x_i)e \in \Lambda$. Now, by the connectedness of Q_Λ, it is easy to see that there are only two types of possible relations between $\{\iota_{x_r}^e\}_r$ and $\{\tau_{x_r}^e\}_r$:

(1) $\chi_{x_i} = \chi_{x_j}$ for any $x_i \neq x_j$. We then have that for any x_i, either arrow $\tau_{x_i}^e$ starts from some vertex in Q_{con}^2 and $\iota_{x_i}^e$ ends at some vertex in Q_{con}^1, or arrow $\tau_{x_i}^e$ starts from some vertex in Q_{con}^1 and $\iota_{x_i}^e$ ends at some vertex in Q_{con}^2.

(2) There are two x_i and x_j such that $\chi_{x_i} \neq \chi_{x_j}$ and one of the following holds:

   (i) $\tau_{x_i}^e$ starts from some vertex in Q_{con}^2 and $\iota_{x_j}^e$ ends at some vertex in Q_{con}^1;

   (ii) $\tau_{x_i}^e$ starts from some vertex in Q_{con}^1 and $\iota_{x_j}^e$ ends at some vertex in Q_{con}^2.

For case (1), since $\chi_{x_i} = \chi_{x_j}$ for any $x_i \neq x_j$, the group G must be the acyclic group $\mathrm{diag}(\epsilon, \epsilon, \cdots, \epsilon)$ with the natural action on V, where ϵ is an n-th root of unit. Then, in this case, the underlying graph of Q_Λ is a cycle with length n, which implies that Q_{con} is connected. It contradicts to our assumption.

For the subcase (i) in case (2), the arrow $\tau_{x_i}^e$ starts from the vertex $e_{\chi_{x_i}^{-1}} \in Q_{\mathrm{con}}^1$ and $\iota_{x_j}^e$ ends at the vertex $e_{\chi_{x_j}} \in Q_{\mathrm{con}}^2$. In the meantime, x_i gives an arrow which starts from the vertex $e_{\chi_{x_j}\chi_{x_i}^{-1}}$ and ends at the vertex $e_{\chi_{x_j}}$ by (4). Moreover, x_j gives an arrow which starts from the vertex $e_{\chi_{x_i}^{-1}}$ and ends at the vertex $e_{\chi_{x_j}\chi_{x_i}^{-1}}$. Thus, due to $e_{\chi_{x_i}^{-1}} \in Q_{\mathrm{con}}^1$ and $e_{\chi_{x_j}} \in Q_{\mathrm{con}}^2$, Q_{con}^1 is connected with Q_{con}^2 through the vertex $e_{\chi_{x_j}\chi_{x_i}^{-1}}$ in Q_Λ. But $e_{\chi_{x_j}\chi_{x_i}^{-1}} \neq e$ due to $\chi_{x_i} \neq \chi_{x_j}$. This means there is a path which connects Q_{con}^1 and Q_{con}^2 and does not pass through the vertex e. It follows that Q_{con}^1 and Q_{con}^2 are not disjoint in Q_{con}. It contradicts to our assumption again.

For the subcases (ii) in (2), applying the same method, it contradicts to our assumption, too. The proposition now follows. $\square$

This proposition implies that for any two characters χ' and χ'', there is a path in Q_{con}, connecting the vertices $e_{\chi'}$ and $e_{\chi''}$.

3.3.3 Decomposition of Λ_{con}

In this subsection, we study the relations between the coordinate ring of the irreducible components of the singular locus and the algebra $R^{\widehat{M}}$.

Recall that by Proposition 3.13, Q_{con} is connected. It suggests that Λ_{con} cannot be viewed as a direct sum of algebras. Nevertheless, in this subsection we construct an algebra homomorphism ζ_0 (see (20)) from Λ_{con} to a direct sum of some skew group algebras (see Lemma 3.19). This algebra homomorphism induces an injection of algebras (see Theorem 3.23), which, interpreted geometrically, says that the irreducible components of $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$ gives a cover of $\mathrm{Spec}(R^{\widehat{M}})$ (see Lemma B.1).

To this end, let us first introduce a finite set $\{(H,\chi)\}$, where H is a nontrivial subgroup of G such that it is maximal among all subgroups which have the same invariant subspace as that of H, and χ is a character of H. Let $\tilde{G} := \{(H,\chi)\}/\sim$, where $(H,\chi) \sim (H',\chi')$ if and only if $H = H'$ in G and $\chi'|_H = \chi|_H$. Let $\tilde{G}_0 \subseteq \tilde{G}$ be the subset consisting of classes (H,χ) such that $\chi|_H$, the restriction of χ on H, is nontrivial.

Suppose (H,χ) represents an element in $\tilde{G}_0$. Let W_H be the H-invariant subspace of V. Let $R_H := k[W_H]$ and $S_H := R_H^{G/H}$. We are going to show in Theorem 3.23 that there is an injection of algebras

$$\bar{\zeta}_0 : R^{\widehat{M}} \to \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H.$$

We proceed to construct $\bar{\zeta}_0$ in several steps, which is the composition of several homomorphisms (from the first to the fifth homomorphisms below). Let us start with some preparations.

Fix a pair (H,W_H) as above. There is a set $\{\lambda_i^H\}_i$ of characters of G consisting of all characters satisfied that $(H,\lambda_i^H) \sim (H,\chi_0)$. It means that $\lambda_i^H(H) = \{1\}$. For any λ_i^H, it induces a character of G/H. In the meantime, by the group homomorphism $G \to G/H$, any character of G/H gives a character of G. Furthermore, it is obvious that there is a bijection between the characters in $\{\lambda_i^H\}_i$ and the characters of G/H. Hence, we also use λ_i^H to represents the induced character of G/H for simplicity.

Since for any pair (H,W_H), H is also viewed as a group consisting of diagonal matrixes in $\mathrm{SL}(V)$, the vector space W_H has set of a basis, denoted by $\mathfrak{S}_H$, which is a subset of $\{E_i\}_i$. Then there is a G-subrepresentation $W_H' \subseteq V$, which is spanned by vectors in $\{E_i\}_i\backslash\mathfrak{S}_H$. Hence, we have $W_H' \cong V/W_H$ and $V \cong W_H \oplus W_H'$ as G-representations.

Now, for any two characters λ_i^H, λ_j^H, by Lemma 3.4, we have

$$\mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_j^H}) \cong \Big(V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes k[W_H] \otimes k[W_H']^H\Big)^{G/H} \tag{9}$$

as S-modules. Recall that $R_H = k[W_H]$ and $S_H = R_H^{G/H}$. Let $M_{\lambda_i^H}^H := (V_{\lambda_i^H} \otimes R_H)^{G/H}$ and $M_{\lambda_j^H}^H := (V_{\lambda_j^H} \otimes R_H)^{G/H}$, then by Lemma 3.4 again, we have

$$\mathrm{Hom}_{S_H}(M_{\lambda_i^H}^H, M_{\lambda_j^H}^H) \cong (V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H)^{G/H} \tag{10}$$

as S_H-modules.

Consider the following natural G/H-equivariant surjection

$$V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k[W_H']^H \twoheadrightarrow V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k \cong V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H,$$

which is induced by the canonical surjection $k[W_H']^H \twoheadrightarrow k$. Combining with (9) and (10), we obtain the following surjection of algebras

$$\tau_{\lambda_i^H,\lambda_j^H}^H : \mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_j^H}) \cong \Big(V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k[W_H']^H\Big)^{G/H}$$

$$\twoheadrightarrow (V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H)^{G/H} \cong \mathrm{Hom}_{S_H}(M_{\lambda_i^H}^H, M_{\lambda_j^H}^H).$$

Now let $\widehat{M}^H := \bigoplus_i M_{\lambda_i^H}$, we have the following algebra surjection:

$$\tau^H := \bigoplus_{i,j} \tau_{\lambda_i^H,\lambda_j^H}^H : \bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^H}, M_{\lambda_j^H}\big)$$
$$\twoheadrightarrow \bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^H}^H, M_{\lambda_j^H}^H\big) \cong \mathrm{Hom}_{S_H}(\widehat{M}^H, \widehat{M}^H). \tag{11}$$

Now let M' be a Cohen-Macaulay S-module such that M is a direct summand of M', i.e., $M' \cong M \oplus N$ as S-modules for some S-module N. Then there is a natural S-module injection $M \hookrightarrow M'$ and a natural S-module surjection $M' \twoheadrightarrow M$. It follows that there is an S-module surjection

$$\mathrm{Hom}_S(M', M') \twoheadrightarrow \mathrm{Hom}_S(M, M).$$

Now, let M' be $\widehat{M}$ and M be $\bigoplus_i M_{\lambda_i^H}$. Then there is an S-module surjection

$$Pr_H : \mathrm{Hom}_S(\widehat{M}, \widehat{M}) \twoheadrightarrow \mathrm{Hom}_S\big(\bigoplus_i M_{\lambda_i^H}, \bigoplus_i M_{\lambda_i^H}\big) \cong \bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^H}, M_{\lambda_j^H}\big). \tag{12}$$

With these preparations, in the following, we introduce five algebra homomorphisms, one is based on the other, and finally reach the homomorphism given in Theorem 3.23.

Definition 3.14 (The first homomorphism). Let

$$\zeta_H : \Lambda \cong \mathrm{Hom}_S(\widehat{M}, \widehat{M}) \twoheadrightarrow \mathrm{Hom}_{S_H}(\widehat{M}^H, \widehat{M}^H) \tag{13}$$

be the composition of $\tau^H \circ Pr_H$.

Lemma 3.15. ζ_H *is a surjection of algebras.*

The proof is quite long, and is postponed to Appendix A.

Remark 3.16. *Similarly to the definition of τ^H given by (11), we give an algebra injection, which we denote by ρ^H, as follows. Start with the natural G/H-equivariant injection*

$$V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \cong V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k \hookrightarrow V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k[W_H']^H,$$

which is induced by the canonical injection $k \hookrightarrow k[W_H']^H$. By (9) and (10), we obtain the following injection of algebras

$$\rho_{\lambda_i^H,\lambda_j^H}^H : \mathrm{Hom}_{S_H}(M_{\lambda_i^H}^H, M_{\lambda_j^H}^H) \cong (V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H)^{G/H}$$
$$\hookrightarrow \Big(V_{(\lambda_i^H)^{-1}\lambda_j^H} \otimes R_H \otimes k[W_H']^H\Big)^{G/H} \cong \mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_j^H}). \tag{14}$$

Let $\widehat{M}^H = \bigoplus_i M_{\lambda_i^H}$ as before, we have an algebra injection:

$$\rho^H := \bigoplus_{i,j} \rho_{\lambda_i^H,\lambda_j^H}^H : \mathrm{Hom}_{S_H}(\widehat{M}^H, \widehat{M}^H) \cong \bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^H}^H, M_{\lambda_j^H}^H\big)$$
$$\hookrightarrow \bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^H}, M_{\lambda_j^H}\big). \tag{15}$$

In the following, we fix a character χ of G such that $\chi(H) \neq \{1\}$. Let $\{\lambda_i^{(H,\chi)}\}_i$ be the set of characters of G consisting of all characters such that $(H, \chi) \sim (H, \lambda_i^{(H,\chi)})$. On the one hand, for any $\lambda_i^{(H,\chi)}$, $\lambda_i^{(H,\chi)}\chi^{-1}$ is in $\{\lambda_i^H\}_i$. On the other hand, for any λ_i^H, $\lambda_i^H\chi$ is in $\{\lambda_i^{(H,\chi)}\}_i$. Thus it is obvious that there is a bijection between characters in $\{\lambda_i^{(H,\chi)}\}_i$ and characters in $\{\lambda_i^H\}_i$. In the meantime, by Lemma 3.4 we have

$$
\begin{aligned}
\mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_j^H}) &\cong \mathrm{Hom}_{kG}\big((\lambda_j^H)^{-1}\lambda_i^H, R\big) \\
&\cong \mathrm{Hom}_{kG}\big((\lambda_j^H\chi)^{-1}\lambda_i^H\chi, R\big) \\
&\cong \mathrm{Hom}_S(M_{\lambda_i^{(H,\chi)}}, M_{\lambda_j^{(H,\chi)}})
\end{aligned}
\tag{16}
$$

as S-modules, where $\lambda_i^{(H,\chi)} = \lambda_i^H\chi$ and $\lambda_j^{(H,\chi)} = \lambda_j^H\chi$. Furthermore, there is a commutative diagram of S^e-module homomorphisms:

$$
\begin{array}{ccc}
\mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_j^H}) \times \mathrm{Hom}_S(M_{\lambda_j^H}, M_{\lambda_r^H}) & \longrightarrow & \mathrm{Hom}_S(M_{\lambda_i^H}, M_{\lambda_r^H}) \\
\Big\downarrow \wr & & \Big\downarrow \wr \\
\mathrm{Hom}_S(M_{\lambda_i^{(H,\chi)}}, M_{\lambda_j^{(H,\chi)}}) \times \mathrm{Hom}_S(M_{\lambda_j^{(H,\chi)}}, M_{\lambda_r^{(H,\chi)}}) & \longrightarrow & \mathrm{Hom}_S(M_{\lambda_j^{(H,\chi)}}, M_{\lambda_r^{(H,\chi)}}),
\end{array}
$$

where the vertical morphisms are given by the isomorphisms in (16) and the horizontal morphisms are the compositions of homomorphisms. Hence, we get that

$$
\mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^{(H,\chi)}}, \bigoplus_i M_{\lambda_i^{(H,\chi)}}\Big) \cong \mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^H}, \bigoplus_i M_{\lambda_i^H}\Big),
\tag{17}
$$

as algebra. Denote this homomorphism by $\Delta_{(H,\chi)}$.

Composing the isomorphism in (17) with the canonical projection:

$$
\mathrm{Hom}_S(\widehat{M}, \widehat{M}) \twoheadrightarrow \mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^{(H,\chi)}}, \bigoplus_i M_{\lambda_i^{(H,\chi)}}\Big),
$$

which is denoted by $Pr_{(H,\chi)}$, we obtain the S-module surjection

$$
\mathrm{Hom}_S(\widehat{M}, \widehat{M}) \twoheadrightarrow \mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^H}, \bigoplus_i M_{\lambda_i^H}\Big),
$$

which is denoted by $\Pi_{(H,\chi)}$.

Definition 3.17 (The second homomorphism)**.** Let

$$
\zeta_H^\chi : \Lambda \cong \mathrm{Hom}_S(\widehat{M}, \widehat{M}) \rightarrow \mathrm{Hom}_{S_H}(\widehat{M}^H, \widehat{M}^H)
\tag{18}
$$

be $\tau^H \circ \Pi_{(H,\chi)}$.

Note that $\zeta_H^\chi = \zeta_H$ if χ is the trivial character of G. By Auslander's theorem for the representation W_H of G/H, we have

$$
\mathrm{Hom}_{S_H}(\widehat{M}^H, \widehat{M}^H) \cong \mathrm{Hom}_{S_H}\Big(\bigoplus_i M_{\lambda_i^H}^H, \bigoplus_i M_{\lambda_i^H}^H\Big) \cong (G/H)\sharp R_H
$$

as algebras. We thus thus re-write the above homomorphism ζ_H as $\zeta_H : \Lambda \rightarrow (G/H)\sharp R_H$.

Lemma 3.18. ζ_H^χ *is a surjection of algebras.*

The proof is the same as Lemma 3.15, and is left to the interested readers. Now recall that $\Lambda_{\mathrm{con}} = \Lambda/\Lambda e\Lambda$. We have the following.

Lemma 3.19. *For any pair (H, χ) such that $\chi(H) \neq \{1\}$, ζ_H^χ induces a well-defined surjection of algebras (**the third homomorphism**)*

$$\tilde{\zeta}_H^\chi : \Lambda_{\mathrm{con}} \twoheadrightarrow (G/H) \sharp R_H. \tag{19}$$

Proof. By Lemma 3.18, to prove this lemma, we only need to show that $\zeta_H^\chi(\Lambda e\Lambda) = 0$. Since the direct summand M_{χ_0} corresponds to the idempotent e, to prove $\zeta_H^\chi(\Lambda e\Lambda) = 0$ it suffices to show that for any characters χ_1, χ_2 of G and $\alpha \in \mathrm{Hom}_S(M_{\chi_0}, M_{\chi_2})$ and $\beta \in \mathrm{Hom}_S(M_{\chi_1}, M_{\chi_0})$, $\zeta_H^\chi(\alpha\beta) = 0$.

Since $\chi(H) \neq \{1\}$, $\chi|_H \neq \chi_0|_H$. Thus, χ_0 is not in $\{\lambda_i^{(H,\chi)}\}_i$. By $\zeta_H^\chi = \tau^H \circ \Pi_{(H,\chi)} = \tau^H \circ \Delta_{(H,\chi)} \circ Pr_{(H,\chi)}$, we know that $\zeta_H^\chi(\alpha) = \zeta_H^\chi(\beta) = 0$ since $Pr_{(H,\chi)}(\alpha) = Pr_{(H,\chi)}(\beta) = 0$. Thus, by Lemma 3.15, we get that

$$\zeta_H^\chi(\alpha\beta) = \zeta_H^\chi(\alpha)\zeta_H^\chi(\beta) = 0.$$

This completes the proof. $\qquad\square$

Note that ζ_H^χ is determined by the set $\{\lambda_i^{(H,\chi)}\}_i$ of characters and the map τ^H. It follows that ζ_H^χ is determined by the pair (H, χ). Then, there is bijection between the algebra homomorphisms in $\{\zeta_H^\chi\}$ and the classes in $\tilde{G}$. We get that $\tilde{\zeta}_H^\chi = \tilde{\zeta}_H^{\chi'}$ if and only if $(H, \chi) \sim (H, \chi')$ in $\tilde{G}_0$.

Definition 3.20 (The fourth homomophism)**.** Let

$$Z(\tilde{\zeta}_H^\chi) : Z(\Lambda_{\mathrm{con}}) \to Z\big((G/H)\sharp R_H\big)$$

be the homomorphism induced from (19) by taking the centers of both sides' algebras.

Lemma 3.21. $Z(\tilde{\zeta}_H^\chi)$ *is an algebra homomorphism.*

Proof. To prove this lemma, it is enough to prove that

$$\tilde{\zeta}_H^\chi\big(Z(\Lambda_{\mathrm{con}})\big) \subseteq Z\big((G/H)\sharp R_H\big).$$

Let $h \in (G/H)\sharp R_H$ and $f \in Z(\Lambda_{\mathrm{con}})$. Since ζ_H^χ is a surjection, $\tilde{\zeta}_H^\chi$ is also a surjection. Then there exists $\tilde{h} \in \Lambda_{\mathrm{con}}$ such that $\tilde{\zeta}_H^\chi(\tilde{h}) = h$. Thus, we have

$$\tilde{\zeta}_H^\chi(f)h = \tilde{\zeta}_H^\chi(f)\tilde{\zeta}_H^\chi(\tilde{h}) = \tilde{\zeta}_H^\chi(f\tilde{h}) = \tilde{\zeta}_H^\chi(\tilde{h}f) = \tilde{\zeta}_H^\chi(\tilde{h})\tilde{\zeta}_H^\chi(f) = h\tilde{\zeta}_H^\chi(f).$$

It implies that $\tilde{\zeta}_H^\chi(f) \in Z\big((G/H)\sharp R_H\big)$, and it follows that

$$\tilde{\zeta}_H^\chi\big(Z(\Lambda_{\mathrm{con}})\big) \subseteq Z\big((G/H)\sharp R_H\big)$$

and hence the lemma. $\qquad\square$

In the above lemma, since $Z\big((G/H)\sharp R_H\big) = S_H$, we can write $Z(\tilde{\zeta}_H^\chi)$ as a map $Z(\tilde{\zeta}_H^\chi) : Z(\Lambda_{\mathrm{con}}) \to S_H$.

In the following, let

$$\zeta_0 := \bigoplus_{(H,\chi)\in\tilde{G}_0} \tilde{\zeta}_H^\chi : \Lambda_{\mathrm{con}} \to \bigoplus_{(H,\chi)\in\tilde{G}_0} (G/H)\sharp R_H \tag{20}$$

and

$$Z(\zeta_0) := \bigoplus_{(H,\chi)\in\tilde{G}_0} Z(\tilde{\zeta}_H^\chi) : Z(\Lambda_{\mathrm{con}}) \to \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H. \tag{21}$$

be the induced map on the centers. Since $S_H \subseteq R_H$ as algebras, S_H is a reduced ring for any pair (H, W_H). Hence, $\bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$ is also a reduced ring.

Definition 3.22 (The fifth homomorphism). Let $\sqrt{Z(\Lambda_{\mathrm{con}})}$ be the reduced ring of $Z(\Lambda_{\mathrm{con}})$. Let

$$\tilde{\zeta}_0 : \sqrt{Z(\Lambda_{\mathrm{con}})} \to \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H, \tag{22}$$

be the map induced from (21) by taking the reduced rings on both sides.

Recall that $R^{\widehat{M}} = \sqrt{Z(\Lambda_{\mathrm{con}})}$ (see the beginning of §3.3.3). In the following we write $\tilde{\zeta}_0$ as $\tilde{\zeta}_0 : R^{\widehat{M}} \to \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$. The following is the main result of this subsection.

Theorem 3.23. $\tilde{\zeta}_0 : R^{\widehat{M}} \to \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$ *is an injection of algebras.*

The proof of this theorem is quite technical, and we postpone it to Appendix B.

3.3.4 The singular locus of $\mathrm{Spec}(S)$ and $\mathrm{Spec}(R^{\widehat{M}})$

In this subsection, we show that the affine scheme of $R^{\widehat{M}}$ is isomorphic to the reduced singular locus $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$ (see Theorem 3.26). The proof roughly goes as follows:

(1) We first construct a canonical cover of $\mathrm{Spec}\big(\sqrt{\varphi(S)/\varphi(S)\cap\Lambda e\Lambda}\big)$ (see surjection (29));

(2) We then construct the canonical cover of $\mathrm{Spec}(R^{\widehat{M}})$ (see surjection (30));

(3) After that, we give the morphism $\mu^\# : \mathrm{Spec}(R^{\widehat{M}}) \to \mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S)\cap\Lambda e\Lambda}\big)$ of schemes, and show that this morphism takes the cover (30) to the cover (29) and is surjective;

(4) Finally we show $\mu^\#$ is injective.

We do these step by step. First, let $\tilde{\zeta}_H^\chi$ and $Z(\tilde{\zeta}_H^\chi)$ be as in Lemmas 3.19 and 3.21. We first have the following.

Lemma 3.24. *For any* $(H,\chi), (H,\chi') \in \tilde{G}_0$, $Z(\tilde{\zeta}_H^\chi) = Z(\tilde{\zeta}_H^{\chi'})$. *In other words, the algebra homomorphism* $Z(\tilde{\zeta}_H^\chi)$ *only depends on* H .

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Again, since the proof of this lemma is quite long, we postpone it to Appendix C. Now, for the pair $(H, \chi) \in \tilde{G}_0$, taking the reduced rings, $Z(\tilde{\zeta}_H^\chi)$ induces the following surjection of algebras:

$$R^{\widehat{M}} \cong \sqrt{Z(\Lambda_{\text{con}})} \to Z(G/H\sharp R_H) \cong S_H. \tag{23}$$

By Lemma 3.24, we know that the above algebra homomorphism (23) is independent of the character χ. Hence, it only depends on pair (H, W_H). Thus, we can write the above surjection (23) just as $\tilde{\xi}_H$. Since $Z(\zeta_0) := \bigoplus_{(H,\chi) \in \tilde{G}_0} Z(\tilde{\zeta}_H^\chi)$, considering the associated reduced rings, it is obvious that

$$\tilde{\zeta}_0 = \bigoplus_{(H,\chi) \in \tilde{G}_0} \tilde{\xi}_H \tag{24}$$

by the definition of $\tilde{\zeta}_0$ (see (22)) and $\tilde{\xi}_H$.

Similarly to Lemma 3.21, by taking the centers of the surjection $\zeta_H : \Lambda \twoheadrightarrow G/H\sharp R_H$ for the pair (H, W_H), we get the following algebra homomorphism

$$Z(\zeta_H) : S = Z(\Lambda) \to Z(G/H\sharp R_H) = S_H. \tag{25}$$

Now, for any two pairs (H, W_H) and $(H', W_{H'})$, there is a unique pair $(G_{H,H'}, W_H \cap W_{H'})$, where $(G_{H,H'}, W_H \cap W_{H'})$ is the maximal subgroup whose invariant subspace is $W_H \cap W_{H'}$. It is direct to see that $H, H' \subseteq G_{H,H'}$. Next, replace G by G/H, V by W_H, H by $G_{H,H'}/H$, W_H by $W_H \cap W_{H'}$. Analogously to τ_{χ_i, χ_j}^H for any two characters χ_i and χ_j of G, we obtain

$$\tau_{\lambda_i, \lambda_j}^{(G_{H,H'}/H)} : \text{Hom}_{S_H}\big((V_{\lambda_i} \otimes R_H)^{(G/H)}, (V_{\lambda_j} \otimes R_H)^{(G/H)}\big)$$
$$\twoheadrightarrow \text{Hom}_{S_{G_{H,H'}}}\big((V_{\lambda_i} \otimes R_{G_{H,H'}})^{(G_{G,H'}/H)}, (V_{\lambda_j} \otimes R_{G_{H,H'}})^{(G_{H,H'}/H)}\big)$$

for any two characters λ_i and λ_j in $\{\lambda_i^{(G_{H,H'}/H)}\}_i$. Then analogously to ζ_H and $Z(\zeta_H)$ respectively (see (13) and (25)), we obtain the corresponding algebra homomorphisms

$$\zeta_{(G_{H,H'}/H)} : G/H\sharp R_H \twoheadrightarrow G/G_{H,H'}\sharp R_{G_{H,H'}} \quad \text{and} \quad Z(\zeta_{(G_{H,H'}/H)}) : S_H \to S_{G_{H,H'}}.$$

Lemma 3.25. *For any two pairs (H, W_H) and $(H', W_{H'})$, we have the following commutative diagram of algebra homomorphisms*

$$
\begin{array}{ccc}
R^{\widehat{M}} & \xrightarrow{\quad \tilde{\xi}_H \quad} & S_H \\
 & \searrow{\scriptstyle \tilde{\xi}_{G_{H,H'}}} & \downarrow{\scriptstyle Z(\zeta_{(G_{H,H'}/H)})} \\
 & & S_{G_{H,H'}}
\end{array}
\tag{26}
$$

We postpone the proof of this lemma to D. Before introducing the main result of this subsection, we should note that $R^{\widehat{M}}$ is a commutative Noetherian ring. In fact, $\widehat{M}$ is a finitely generated S-module. Hence $\Lambda = \text{End}_S(\widehat{M})$ is finitely generated S-module, and then Λ_{con} and $Z(\Lambda_{\text{con}})$ are both finitely generated S-modules. Thus $R^{\widehat{M}}$ is a finitely generated S-module. Since S is a commutative Noetherian ring, $R^{\widehat{M}}$ is also a commutative Noetherian ring.

Theorem 3.26. $\text{Spec}(R^{\widehat{M}}) \cong \sqrt{\text{Sing}(\text{Spec}(S))}$ *as subschemes of* $\text{Spec}(S)$.

Proof. The proof consists of four steps.

Step 1: *We construct a canonical cover of* $\mathrm{Spec}\big(\sqrt{\varphi(S)/(\varphi(S) \cap \Lambda e\Lambda)}\big)$.

To this end, we consider the following composition of two algebra injections:

$$\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \hookrightarrow R^{\widehat{M}} \xrightarrow{\tilde{\zeta}_0} \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$$

(for the injection of the second homomorphism see Theorem 3.23). We show the first homomorphism is also an injection. In fact, φ gives the injection:

$$\varphi(S)/\big(\varphi(S) \cap (\Lambda e\Lambda)\big) \hookrightarrow \Lambda/\Lambda e\Lambda \cong \Lambda_{\mathrm{con}}$$

and the image of this injection is contained in $Z(\Lambda_{\mathrm{con}})$. Then by considering the associated reduced rings, we get an injection

$$\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \hookrightarrow R^{\widehat{M}}. \tag{27}$$

Fixing a pair $(H,\chi) \in \tilde{G}_0$ and composing the above injection $\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \hookrightarrow \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$ with the canonical projection $\bigoplus_{(H,\chi)\in\tilde{G}_0} S_H \twoheadrightarrow S_H$, we get an algebra homomorphism

$$\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \hookrightarrow R^{\widehat{M}} \xrightarrow{\tilde{\zeta}_0} \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H \twoheadrightarrow S_H, \tag{28}$$

which we denote by φ_H. Next, when composing φ_H with the quotient map

$$S \cong \varphi(S) \to \varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)},$$

we get another algebra homomorphism $S \to S_H$. It is easy to check that this algebra homomorphism is exactly the quotient

$$S = k[V]^G \twoheadrightarrow k[W_H]^{G/H} = S_H$$

associated to the canonical injection $W_H/(G/H) \hookrightarrow (W_H \oplus W'_H)/G \cong V/G$ of schemes. Since $S \to S_H$ is a surjection of algebras, so is φ_H. Thus, φ_H gives a closed injection of schemes, say

$$\varphi_H^{\#} : \mathrm{Spec}(S_H) \hookrightarrow \mathrm{Spec}\sqrt{\varphi(S) \cap (\Lambda e\Lambda)}.$$

In the meantime, $\bigoplus_{(H,\chi)} \varphi_H$ is exactly the injection $\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \hookrightarrow R^{\widehat{M}} \xrightarrow{\tilde{\zeta}_0} \bigoplus_{(H,\chi)\in\tilde{G}_0} S_H$. Then the induced homomorphism of schemes

$$\bigsqcup_{(H,\chi)} \varphi_H^{\#} : \bigsqcup_{(H,\chi)} \mathrm{Spec}(S_H) \twoheadrightarrow \mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)}\big)$$

is a cover of $\mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)}\big)$. Moreover, by replacing (H,χ) by H for any $(H,\chi) \in \tilde{G}_0$, the above cover reduces to a new cover of $\mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)}\big)$:

$$\bigsqcup_H \varphi_H^{\#} : \bigsqcup_H \mathrm{Spec}(S_H) \twoheadrightarrow \mathrm{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)}\big). \tag{29}$$

Step 2: *We construct a canonical cover of* $\operatorname{Spec}(R^{\widehat{M}})$.

Note that the composition

$$R^{\widehat{M}} \xrightarrow{\tilde{\zeta}_0} \oplus_{(H,\chi)\in\tilde{G}_0} S_H \twoheadrightarrow S_H$$

is exactly $\tilde{\xi}_H$ by definition (see (23)). By (28), $\tilde{\xi}_H$ is surjective since φ_H is surjective. Then for any pair (H, W_H),

$$(\tilde{\xi}_H)^\# : \operatorname{Spec}(S_H) \hookrightarrow \operatorname{Spec}(R^{\widehat{M}})$$

is an injection of schemes, where $(\tilde{\xi}_H)^\#$ is the homomorphism of schemes induced by $\tilde{\xi}_H^\chi$. Since $\tilde{\zeta}_0 = \bigoplus_{(H,\chi)} \tilde{\xi}_H$ is an injection of algebras (see Theorem 3.23 and (24)), we also get a cover of $\operatorname{Spec}(R^{\widehat{M}})$

$$\bigsqcup_{(H,\chi)} (\tilde{\xi}_H)^\# : \bigsqcup_{(H,\chi)} \operatorname{Spec}(S_H) \twoheadrightarrow \operatorname{Spec}(R^{\widehat{M}}).$$

By definition of $\tilde{\xi}_H$ (see (23)), it is independent of the character χ. Hence, if we only consider the subcover $\bigsqcup_H (\tilde{\xi}_H)^\#$ instead of $\bigsqcup_{(H,\chi)} (\tilde{\xi}_H)^\#$, since $\operatorname{Im}(\bigsqcup_H(\tilde{\xi}_H)^\#) = \operatorname{Im}(\bigsqcup_{(H,\chi)} (\tilde{\xi}_H)^\#)$, then we get a new cover of $\operatorname{Spec}(R^{\widehat{M}})$

$$\bigsqcup_H (\tilde{\xi}_H)^\# : \bigsqcup_H \operatorname{Spec}(S_H) \twoheadrightarrow \operatorname{Spec}(R^{\widehat{M}}). \tag{30}$$

Step 3: *We give a morphism* $\mu^\# : \operatorname{Spec}(R^{\widehat{M}}) \to \operatorname{Spec}(\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda})$ *of schemes, and show that this morphism takes the cover (30) to the cover(29) and is surjective.*

In the following, denote by μ the following composition of natural algebra homomorphisms

$$\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)} \hookrightarrow Z(\Lambda_{\mathrm{con}}) \twoheadrightarrow R^{\widehat{M}},$$

the first induced by φ and the second by quotient by the nilradical in $Z(\Lambda_{\mathrm{con}})$. Let

$$\mu^\# : \operatorname{Spec}(R^{\widehat{M}}) \to \operatorname{Spec}(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)})$$

be the homomorphism of schemes given by μ. Now, by the definition of $\tilde{\xi}_H$ (see (23)) and (28), we have $\tilde{\xi}_H \circ \mu = \varphi_H$ for any pair (H, W_H). Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\bigsqcup_H \operatorname{Spec}(S_H) & \xrightarrow{\bigsqcup_H \varphi_H^\#} & \operatorname{Spec}(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)}) \\
& \bigsqcup_H (\tilde{\xi}_H)^\# \searrow \qquad \nearrow \mu^\# & \\
& \operatorname{Spec}(R^{\widehat{M}}). &
\end{array}
$$

Since $\bigsqcup_H \varphi_H^\#$ is a cover of $\operatorname{Spec}(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)})$, it implies that $\mu^\#$ is a surjection of schemes.

Step 4: *We show that* $\mu^\#$ *is injective.*

Since for any pair (H, W_H), $\mu^\# \circ (\tilde{\xi}_H)^\# = \varphi_H^\#$ is an injection, $\mu^\#|_{\operatorname{Im}((\tilde{\xi}_H)^\#)}$ is an injection when restricting $\mu^\#$ on $\operatorname{Im}((\tilde{\xi}_H)^\#)$. Note that $\bigsqcup_H (\tilde{\xi}_H)^\#$ is a cover of $\operatorname{Spec}(R^{\widehat{M}})$. Hence, to

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prove that $\mu^\#$ is an injection of schemes, it suffices to show that for any two different pairs (H, W_H) and $(H', W_{H'})$, and any $p \in \text{Im}((\tilde{\xi}_H)^\#)$ and $p' \in \text{Im}((\tilde{\xi}_{H'})^\#)$, $\mu^\#(p) = \mu^\#(p')$ if and only if $p = p'$ in $\text{Spec}(R^{\widehat{M}})$.

In fact, by Theorem 3.9 and Lemma B.1, we have

$$\bigcup_{(H, W_H)} \text{Spec}(S_H) \cong \sqrt{\text{Sing}(\text{Spec}(S))} \cong \text{Spec}\left(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)}\right).$$

The composition of the above two isomorphisms coincides with

$$\bigcup_H \text{Im}\left((\varphi_H)^\#\right) \cong \text{Spec}\left(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)}\right)$$

induced by $\bigoplus_H \varphi_H^\#$. Here, $\text{Spec}(S_H) \cong \text{Im}((\varphi_H)^\#)$ in scheme $\sqrt{\text{Sing}(\text{Spec}(S))}$ for any pair (H, W_H). Then for any two different pairs (H, W_H) and $(H', W_{H'})$, the intersection of $\text{Im}(\varphi_H^\#)$ and $\text{Im}(\varphi_{H'}^\#)$ in $\sqrt{\text{Sing}(\text{Spec}(S))}$ is isomorphic to the intersection of $\text{Spec}(S_H)$ and $\text{Spec}(S_{H'})$ in $\sqrt{\text{Sing}(\text{Spec}(S))}$.

Recall that $S_H = S/I_H$ and S_H is associated to the subscheme $W_H/(G/H)$ of V/G. In $\sqrt{\text{Sing}(\text{Spec}(S))}$, the the intersection of $\text{Spec}(S_H)$ and $\text{Spec}(S_{H'})$ is given by the following canonical pullback commutative diagram:

$$
\begin{array}{ccc}
\text{Spec}(S/(I_H, I_{H'})) & \longrightarrow & \text{Spec}(S/I_H) \\
\downarrow & & \downarrow \\
\text{Spec}(S/I_{H'}) & \longrightarrow & \text{Spec}(S).
\end{array}
\tag{31}
$$

Thus, the intersection of $\text{Im}(\varphi_H^\#)$ and $\text{Im}(\varphi_{H'}^\#)$ is isomorphic to

$$
\begin{aligned}
\text{Spec}(S/(I_H, I'_H)) &\cong W_H/(G/H) \bigcap W_{H'}/(G/H') \\
&\cong (W_H \cap W_{H'})/(G/G_{H,H'}) \\
&\cong \text{Spec}(S_{G_{H,H'}})
\end{aligned}
$$

in $\sqrt{\text{Sing}(\text{Spec}(S))}$, which is exactly $\text{Im}(\varphi_{G_{H,H'}}^\#)$. Let $\tilde{p} \in \text{Spec}(S_H)$ and $\tilde{p}' \in \text{Spec}(S_{H'})$ such that $(\tilde{\xi}_H)^\#(\tilde{p}) = p$ and $(\tilde{\xi}_{H'})^\#(\tilde{p}') = p'$. If $\mu^\#(p) = \mu^\#(p')$, then $\varphi_H^\#(\tilde{p}) = \varphi_{H'}^\#(\tilde{p}')$. It implies that

$$\mu^\#(p) = \mu^\#(p') \in \text{Im}(\varphi_H^\#) \cap \text{Im}(\varphi_{H'}^\#) = \text{Im}(\varphi_{G_{H,H'}}^\#) \cong \text{Spec}(S_{G_{H,H'}}). \tag{32}$$

In the meantime, by the commutative diagram (26) in Lemma 3.25, we know that

$$Z(\zeta_{(G_{H,H'}/H)}) \circ \tilde{\xi}_H = \tilde{\xi}_{G_{H,H'}}.$$

It implies that $Z(\zeta_{(G_{H,H'}/H)})$ is a surjection of algebras, since $\tilde{\xi}_{G_{H,H'}}$ is surjective. This induces an injection

$$\left(Z(\zeta_{(G_{H,H'}/H)})\right)^\# : \text{Im}\left((\tilde{\xi}_{G_{H,H'}})^\#\right) \hookrightarrow \text{Im}\left((\tilde{\xi}_H)^\#\right)$$

of schemes. Meanwhile, there is a natural injection

$$\text{Im}\left((\varphi_{G_{H,H'}})^\#\right) \cong W_{G_{H,H'}}/(G/G_{H,H'}) \hookrightarrow W_H/(G/H) \cong \text{Im}\left((\varphi_{H'})^\#\right)$$

of schemes. Furthermore, since $\varphi_H^\#$ is an injection, when $\mu^\#$ restricts on $\mathrm{Im}\big((\tilde{\xi}_H)^\#\big)$, $\mu^\#$ is also an injection by $\mu^\# \circ (\tilde{\xi}_H)^\# = \varphi_H^\#$. Then $\mu^\#$ gives an isomorphism between $\mathrm{Im}\big((\tilde{\xi}_H)^\#\big)$ and $\mathrm{Im}\big((\varphi_H)^\#\big)$. In the same way, $\mu^\#$ also gives an isomorphism between $\mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big)$ and $\mathrm{Im}\big((\varphi_{G_{H,H'}})^\#\big)$. By combining the above morphisms, we then get the following pullback commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big) & \xrightarrow{\ \sim\ }_{\mu^\#} & \mathrm{Im}\big(\varphi_{G_{H,H'}}^\#\big) \\
{\scriptstyle\big(Z(\zeta_{(G_{H,H'}/H)})\big)^\#}\Big\uparrow & & \Big\uparrow \\
\mathrm{Im}\big((\tilde{\xi}_H)^\#\big) & \xrightarrow[\ \sim\ ]{\mu^\#} & \mathrm{Im}\big(\varphi_H^\#\big).
\end{array}
\tag{33}
$$

Since $p \in \mathrm{Im}\big((\tilde{\xi}_H)^\#\big)$ and $\mu^\#(p) \in \mathrm{Im}\big(\varphi_{G_{H,H'}}^\#\big)$ by (32), we have $p \in \mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big)$ by (33). In the same way, we also have $p' \in \mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big)$. Meanwhile, we know that $\mu^\#$ is also an injection when it restricts on $\mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big)$. Thus, considering this injection $\mu^\# : \mathrm{Im}\big((\tilde{\xi}_{G_{H,H'}})^\#\big) \hookrightarrow \mathrm{Spec}(R^{\widehat{M}})$, we obtain that $p = p'$ since $\mu^\#(p) = \mu^\#(p')$. $\qquad\square$

Summarizing the above Steps 1-4, we get the theorem.

Combining Theorem 3.26 and Corollary 3.10 we get the following theorem.

Theorem 3.27. *There is an algebra isomorphism $\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e\Lambda)} \cong R^{\widehat{M}}$ induced from the map $\varphi : S \to \Lambda$.*

4 Proof of the main result

In this section, we prove Theorem 1.1.

For any Cohen-Macaulay S-module M, recall that $\Lambda_{\mathrm{con}}^M \cong \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M)$ (see Proposition 3.6). Denote by $Z(\Lambda_{\mathrm{con}}^M)$ the center of Λ_{con}^M, and denote by R^M the reduced ring of $Z(\Lambda_{\mathrm{con}}^M)$. Note that since M is a finitely generated S-module, $\mathrm{End}_S(M)$ is laso a finitely generated S-module. It implies that Λ_{con}^M and R^M are both finitely generated S-modules. Therefore, R^M is a commutative Noetherian ring since S is a commutative Noetherian ring.

Our strategy to the proof of Theorem 1.1 is as follows.

(1) First, we show that $\mathcal{R}^S := \big\{ R^M, \varphi_M^{M'} \big\}_{M,M' \in \underline{\mathrm{CM}}(S)}$, depending only on the triangulated category $\underline{\mathrm{CM}}(S)$, forms an inverse system;

(2) Then we prove that the inverse limit is exactly $\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e\Lambda}$.

(3) Finally, since by Theorem 3.9 the reduced ring of the singular locus of S is $\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e\Lambda}$, it only depends on $\underline{\mathrm{CM}}(S)$, and the theorem follows.

Let us accomplish them step by step.

4.1 An inverse system

We first claim that there exists the following inverse system:

$$
\mathcal{R}^S := \big\{ R^M, \varphi_M^{M'} \big\}_{M,M' \in \underline{\mathrm{CM}}(S)},
$$

where $\varphi_M^{M'}$ is given as follows.

Let M' be a Cohen-Macaulay S-module such that M is a direct summand of M', i.e., $M' \cong M \oplus N$ for some S-module N. Then there is a natural S-module injection $M \hookrightarrow M'$ and a natural S-module surjection $M' \twoheadrightarrow M$. It follows that there is an S-module surjection

$$\mathrm{Hom}_S(M', M') \twoheadrightarrow \mathrm{Hom}_S(M, M)$$

given by composing with these two S-module homomorphisms. This S-module homomorphism naturally induces the S-module surjection $\Lambda_{\mathrm{con}}^{M'} \twoheadrightarrow \Lambda_{\mathrm{con}}^{M}$, which we denote by $\phi_M^{M'}$.

In the meantime, by taking the centers of the algebras, we have

$$Z(\Lambda_{\mathrm{con}}^{M'}) \cong Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M')\big) \cong Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M \oplus N)\big)$$

$$\cong \Big\{ (f_1, f_2) \in Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M)\big) \oplus Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(N)\big) \Big| \forall\, g \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(M, N) \,\&$$

$$h \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M), \ g \circ f_1 = f_2 \circ g, \ \& \ f_1 \circ h = h \circ f_2 \Big\}.$$

Thus we obtain a natural S-algebra homomorphism

$$\widetilde{\phi}_M^{M'} : Z(\Lambda_{\mathrm{con}}^{M'}) \to Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M)\big) \cong Z(\Lambda_{\mathrm{con}}^{M}).$$

Moreover, it is easy to check that this S-algebra homomorphism gives the following commutative diagram of S-module homomorphisms:

$$
\begin{array}{ccc}
Z(\Lambda_{\mathrm{con}}^{M'}) & \lhook\joinrel\longrightarrow & \Lambda_{\mathrm{con}}^{M'} \\
\widetilde{\phi}_M^{M'} \downarrow & & \downarrow \phi_M^{M'} \\
Z(\Lambda_{\mathrm{con}}^{M}) & \lhook\joinrel\longrightarrow & \Lambda_{\mathrm{con}}^{M}.
\end{array}
$$

From the S-algebra homomorphism $\widetilde{\phi}_M^{M'}$, by taking the corresponding reduced rings, we get the S-algebra homomorphism $R^{M'} \twoheadrightarrow R^M$, which we denote by $\varphi_M^{M'}$.

Lemma 4.1. $\mathcal{R}^S := \big\{ R^M, \varphi_M^{M'} \big\}_{M, M' \in \underline{\mathrm{CM}}(S)}$ *is an inverse system.*

Proof. We only need to show that $\{\varphi_M^{M'}\}$ defined above satisfies the cocycle condition. In fact, to show $\varphi_M^{M''} = \varphi_M^{M'} \circ \varphi_{M'}^{M''}$, it is enough to show

$$\widetilde{\phi}_M^{M''} = \widetilde{\phi}_M^{M'} \circ \widetilde{\phi}_{M'}^{M''},$$

for any M, M', M'' such that M is a direct summand of M' and M' is a direct summand of M''.

In fact, notice that we have the following two commutative diagrams of S-module homomorphisms:

$$
\begin{array}{ccc}
Z(\Lambda_{\mathrm{con}}^{M''}) & \lhook\joinrel\longrightarrow & \Lambda_{\mathrm{con}}^{M''} \\
\widetilde{\phi}_{M'}^{M''} \downarrow & & \downarrow \phi_{M'}^{M''} \\
Z(\Lambda_{\mathrm{con}}^{M'}) & \lhook\joinrel\longrightarrow & \Lambda_{\mathrm{con}}^{M'} \\
\widetilde{\phi}_{M}^{M'} \downarrow & & \downarrow \phi_{M}^{M'} \\
Z(\Lambda_{\mathrm{con}}^{M}) & \lhook\joinrel\longrightarrow & \Lambda_{\mathrm{con}}^{M}
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\Lambda_{\mathrm{con}}^{M''} \\
\phi_{M'}^{M''} \downarrow \quad \phi_{M}^{M''} \\
\Lambda_{\mathrm{con}}^{M'} \\
\phi_{M}^{M'} \downarrow \\
\Lambda_{\mathrm{con}}^{M}
\end{array}
$$

Since in the left diagram the horizontal morphisms are all injective, combining it with the right commutative diagram we then get $\widetilde{\phi}_M^{M''} = \widetilde{\phi}_M^{M'} \circ \widetilde{\phi}_{M'}^{M''}$. $\qquad\square$

Now set $\widehat{M} = \bigoplus_{W \in \hat{G}} (W \otimes R)^G$ as before. In the next two subsections, we are going to show that $R^{\widehat{M}} \cong \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}$ is the inverse limit of $\mathcal{R}^S$.

4.2 The universal algebra homomorphism

In this subsection, we show that for any Cohen-Macaulay S-module M, there is a (universal) algebra homomorphism

$$\varphi_M : \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \to R^M.$$

We shall also prove a technical result (see Proposition 4.3), which says that for any Cohen-Macaulay S-module N, $\varphi_{\widehat{M}}$ can be extended to a new algebra isomorphism $\varphi_{\widehat{M}(l,l') \oplus N}$ for some integers $l' > 0 > l$.

To this end, notice first that we have a natural algebra homomorphism

$$\phi_M : S \to \text{End}_{\underline{\text{CM}}(S)}(M), \tag{34}$$

given by the S-module structure of M. It is obvious that the image of (34) is in the center of $\text{End}_{\underline{\text{CM}}(S)}(M)$. By slightly abusing the notations, we still denote this morphism by $\phi_M : S \to Z\big(\text{End}_{\underline{\text{CM}}(S)}(M)\big)$. Next, from the proof of Theorem 3.9, the support of the S-module $\text{End}_{\underline{\text{CM}}(S)}(M)$ is contained in the singular locus of $\text{Spec}(S)$. When identifying S with $\varphi(S)$, we have

$$\text{End}_{\underline{\text{CM}}(S)}(M)\big(\varphi(S) \cap \Lambda e \Lambda\big) = 0,$$

since by Theorem 3.9, $\text{Spec}\big(\varphi(S)/(\varphi(S) \cap \Lambda e \Lambda)\big)$ is the singular locus of $\text{Spec}(S)$. Hence,

$$Z\big(\text{End}_{\underline{\text{CM}}(S)}(M)\big)\big(\varphi(S) \cap \Lambda e \Lambda\big) = 0.$$

Thus, ϕ_M induces the following well-defined algebra homomorphism

$$\widetilde{\phi}_M : \varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big) \to Z\big(\text{End}_{\underline{\text{CM}}(S)}(M)\big).$$

Taking the reduced rings of the above algebras, we get the following algebra homomorphism

$$\varphi_M : \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \to R^M.$$

Note that in the special case $M = \widehat{M}$, $\varphi_{\widehat{M}}$ is the algebra isomorphism given by Theorem 3.27. φ_M thus defined satisfies the following.

Lemma 4.2. *For any two Cohen-Macaulay S-modules M and M' such that M is a direct summand of M',*

$$\varphi_M = \varphi_M^{M'} \circ \varphi_{M'}.$$

Proof. Recalling that

$$\phi_M^{M'} : \text{End}_{\underline{\text{CM}}(S)}(M') \twoheadrightarrow \text{End}_{\underline{\text{CM}}(S)}(M)$$

is the canonical projection, we have $\phi_M = \phi_M^{M'} \circ \phi_{M'}$ by definition (see (34)). By taking the centers of algebras, and since the images of ϕ_M and $\phi_{M'}$ are both in the centers respectively, we have

$$\phi_M = \widetilde{\phi}_M^{M'} \circ \phi_{M'}.$$

Since both centers are supported on the singular locus $\text{Spec}\big(\varphi(S)/\sqrt{\varphi(S) \cap (\Lambda e \Lambda)}\big)$ of $\text{Spec}(S)$ (see the proof of Theorem 3.9), ϕ_M and $\phi_{M'}$ induce the following algebra homomorphisms

$$\widetilde{\phi}_M : \varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big) \to Z\big(\text{End}_{\underline{\text{CM}}(S)}(M)\big)$$

and

$$\widetilde{\phi}_{M'} : \varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big) \to Z\big(\text{End}_{\underline{\text{CM}}(S)}(M')\big).$$

Moreover, the natural algebra homomorphism

$$S \to \varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big),$$

induced by φ, is a surjection. In summary, we have the following commutative diagram:

$$
\begin{array}{c}
S \\
\phi_{M'} \swarrow \quad \downarrow \quad \searrow \phi_M \\
\varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big) \\
\tilde{\phi}_{M'} \swarrow \qquad \searrow \tilde{\phi}_M \\
Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M')\big) \xrightarrow{\quad \widetilde{\phi}_M^{M'} \quad} Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M)
\end{array}
$$

Then, by $\phi_M = \widetilde{\phi}_M^{M'} \circ \phi_{M'}$, we have $\widetilde{\phi}_M = \widetilde{\phi}_M^{M'} \circ \widetilde{\phi}_{M'}$, since $S \to \varphi(S)/\big(\varphi(S) \cap \Lambda e \Lambda\big)$ is surjective. Taking the reduced rings for these rings, we get $\varphi_M = \varphi_M^{M'} \circ \varphi_{M'}$. $\qquad \square$

Next, we introduce a result (Proposition 4.3), which will be to be used in next section. The background is that, for a Cohen-Macauley module N, we expect to have an isomorphism $R^{\widehat{M \oplus N}} \cong R^{\widehat{M}}$; however, at present we are not able to show this isomorphism. The idea to solve this issue is that, we may find two integers l, l' and consider $\widehat{M}(l, l') := \bigoplus_{i=l}^{l'} \widehat{M}[i]$, and then we have: (a) $R^{\widehat{M}} \cong R^{\widehat{M}(l,l')}$ and (b) $R^{\widehat{M}(l,l') \oplus N} \cong R^{\widehat{M}}$. On the other hand, considering $\widehat{M}(l, l')$ instead of $\widehat{M}$ is natural in the sense that $\widehat{M}$ is a generator of $\underline{\mathrm{CM}}(S)$, where the degree shifts appear naturally.

Proposition 4.3. *For any Cohen-Macaulay S-module N, there exist $l, l' \in \mathbb{Z}$ with $l' > 0 > l$, and a commutative diagram*

$$
\begin{array}{ccc}
\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} & \xrightarrow{\quad \varphi_{\widehat{M}} \quad} & R^{\widehat{M}} \\
\varphi_{\widehat{M}(l,l') \oplus N} \searrow & & \nearrow \varphi_{\widehat{M}}^{\widehat{M}(l,l') \oplus N} \\
& R^{\widehat{M}(l,l') \oplus N} &
\end{array}
$$

of algebra isomorphisms.

The proof is postponed to Appendix E.

4.3 The limit of the inverse system

Now we are ready to show the main result of this section.

Theorem 4.4. *The inverse limit of $\mathcal{R}^S$ is*

$$\big(\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}, \{\varphi_M\}_{M \in \underline{\mathrm{CM}}(S)}\big).$$

Proof. To prove this theorem, we only need to show the universal property of

$$\big(\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}, \{\varphi_M\}_{M \in \underline{\mathrm{CM}}(S)}\big).$$

That is, for any $\big(T, \{\psi_M\}_{(M \in \underline{\mathrm{CM}}(S))}\big)$, where $\psi_M : T \to R^M$, satisfying that if M is a direct summand of M' as S-modules and $\psi_M = \varphi_M^{M'} \circ \psi_{M'}$, there is an unique algebra homomorphism $\eta_T : T \to \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}$ such that

$$\psi_M = \varphi_M \circ \eta_T. \tag{35}$$

To this end, define η_T to be $(\varphi_{\widehat{M}})^{-1} \circ \psi_{\widehat{M}}$. Let N be an arbitrary Cohen-Macaulay S-module. Note that by Proposition 4.3, $\varphi_{\widehat{M}}$, $\varphi_{\widehat{M(l,l')}\oplus N}$ and $\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N}$ are all invertible. We then have

$$
\begin{aligned}
\varphi_N \circ \eta_T &= \varphi_N \circ \left((\varphi_{\widehat{M}})^{-1} \circ \psi_{\widehat{M}}\right) \\
&= \left(\varphi_N^{\widehat{M(l,l')}\oplus N} \circ \varphi_{\widehat{M(l,l')}\oplus N}\right) \circ \left((\varphi_{\widehat{M}})^{-1} \circ \psi_{\widehat{M}}\right) \\
&= \left(\varphi_N^{\widehat{M(l,l')}\oplus N} \circ \varphi_{\widehat{M(l,l')}\oplus N}\right) \circ \left((\varphi_{\widehat{M(l,l')}\oplus N})^{-1} \circ (\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N})^{-1} \circ \psi_{\widehat{M}}\right) \\
&= \varphi_N^{\widehat{M(l,l')}\oplus N} \circ (\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N})^{-1} \circ \psi_{\widehat{M}}.
\end{aligned}
$$

Moreover, by the assumption of $\left(T, \{\psi_M\}_{M \in \underline{\mathrm{CM}}(S)}\right)$,

$$
\psi_N = \varphi_N^{\widehat{M(l,l')}\oplus N} \circ \psi_{\widehat{M(l,l')}\oplus N}.
$$

Hence, to prove $\psi_N = \varphi_N \circ \eta_T$, it suffices to show that

$$
\psi_{\widehat{M(l,l')}\oplus N} = (\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N})^{-1} \circ \psi_{\widehat{M}}.
$$

In fact, this is true by composing $(\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N})^{-1}$ on both sides of the following equality

$$
\varphi_{\widehat{M}}^{\widehat{M(l,l')}\oplus N} \circ \psi_{\widehat{M(l,l')}\oplus N} = \psi_{\widehat{M}}.
$$

Now letting $N = M$, we get $\psi_M = \varphi_M \circ \eta_T$.

Finally, we show the uniqueness of η_T. Assume that there is another algebra homomorphism η_T' satisfying that $\psi_M = \varphi_M \circ \eta_T'$ for any Cohen-Macaulay S-module M. Choosing the Cohen-Macaulay S-module $\widehat{M}$, we have

$$
\varphi_{\widehat{M}} \circ \eta_T' = \psi_{\widehat{M}} = \varphi_{\widehat{M}} \circ \eta_T.
$$

Next, composing the inverse $\varphi_{\widehat{M}}^{-1}$ on the both sides of the above equality, we get that $\eta_T' = \eta_T$. This shows the uniqueness of η_T. QED. $\qquad\square$

Combining Theorem 4.4 with Corollary 3.10 we get the following.

Corollary 4.5. *The coordinate ring of $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$, which is $\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}$, is isomorphic to the ring of the inverse limit of $\mathcal{R}^S$.*

4.4 Proof of Theorem 1.1

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. Let S be the Gorenstein commutative Noetherian ring as in Example 3.3. By Corollary 4.5, we get the coordinate ring of $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$ from the inverse limit of $\mathcal{R}^S$. Note that the inverse system $\mathcal{R}^S$ only depends on the triangulated category $D_{sg}(S) \cong \underline{\mathrm{CM}}(S)$, so does the coordinate ring of $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$.

If $\Upsilon : D_{sg}(S_1) \to D_{sg}(S_2)$ is a triangle equivalence, then the inverse systems $\mathcal{R}^{S_1}$ and $\mathcal{R}^{S_2}$ are isomorphic, and so are their limits. Thus by Corollary 4.5 we obtain that

$$
\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S_1)\big)} \cong \sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S_2)\big)}
$$

as schemes. $\qquad\square$

5 Examples

In this section, we give two simple examples of our main theorem.

Example 5.1. Let $R = k[x_1, x_2, x_3]$ and $G \subseteq \mathrm{SL}(3, k)$ be generated by elements

$$f_1 := \mathrm{Diag}(1, -1, -1) , f_2 := \mathrm{Diag}(-1, 1, -1),$$

where f_1 sents x_1 to x_1, x_2 to $-x_2$ and x_3 to $-x_3$. f_2 sents x_1 to $-x_1$, x_2 to x_2 and x_3 to $-x_3$. Then $S = R^G = k[x_1^2, x_2^2, x_3^2, x_1 x_2 x_3] \cong k[A, B, C, D]/(ABC - D^2)$, where $A = x_1^2$, $B = x_2^2$, $C = x_3^2$ and $D = x_1 x_2 x_3$.

Consider the Jacobi ring $k[A, B, C, D]/(AB, BC, CA, D)$ of the hypersurface $\mathrm{Spec}(S)$. Since $ABC - D^2 \subseteq (AB, BC, CA, D)$, we have that

$$\mathrm{Sing}\big(\mathrm{Spec}(S)\big) \cong \mathrm{Spec}\big(k[A, B, C, D]/(AB, BC, CA, D)\big).$$

We now apply the method in the previous sections to compute $\mathrm{Sing}\big(\mathrm{Spec}(S)\big)$. To this end, consider the commutative ring

$$Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\bigoplus_i M_i)\big) \cong Z\big(\Lambda/\Lambda e \Lambda\big).$$

It is given as follows: observe that $\widehat{M} \cong S \oplus (V_1 \otimes R)^G \oplus (V_2 \otimes R)^G \oplus (V_3 \otimes R)^G$ and $\Lambda = \mathrm{End}_S(\widehat{M})$, where

- V_1 is the irreducible representation of G given by $f_1 v_1 = v_1$ and $f_2 v_1 = -v_1$ for any $v_1 \in V_1$,

- V_2 is the irreducible representation of G given by $f_1 v_2 = -v_2$ and $f_2 v_2 = v_2$ for any $v_2 \in V_2$, and

- V_3 is the irreducible representation of G given by $f_1 v_3 = -v_3$ and $f_2 v_3 = -v_3$ for any $v_3 \in V_3$.

Let $M_i = (V_i \otimes R)^G$ and $M_0 = S$, which are Cohen-Macaulay S-modules. It is easy to check that

- $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_1) = S/I_1$, where $I_1 := (x_1^2, x_2^2 x_3^2, x_1 x_2 x_3) \cong (A, BC, D)$;

- $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_2) = S/I_2$, where $I_2 := (x_2^2, x_1^2 x_3^2, x_1 x_2 x_3) \cong (B, AC, D)$;

- $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_1) = S/I_3$, where $I_3 := (x_3^2, x_1^2 x_2^2, x_1 x_2 x_3) \cong (C, AB, D)$.

For $i = 1, 2, 3$, $\mathrm{Spec}(S/I_i)$ is a subscheme of the singular locus $\mathrm{Spec}\big(S/(AB, BC, CD, D)\big)$. The intersection of $\mathrm{Spec}(S/I_i)$ and $\mathrm{Spec}(S/I_j)$ is $\mathrm{Spec}\big(S/(I_i, I_j)\big)$ for any $i, j \in \{1, 2, 3\}$, and the intersection of the three schemes is the origin in $\mathrm{Spec}(S)$.

By Auslander's theorem, $\Lambda \cong G \sharp R$. The associated quiver Q_Λ is
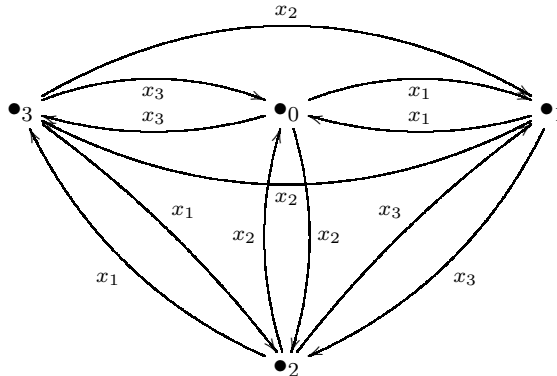
In the above diagram, vertex i corresponds to the S-module M_i. The quiver of $\Lambda_{\mathrm{con}} \cong \Lambda/\Lambda e\Lambda$ is given as follows:
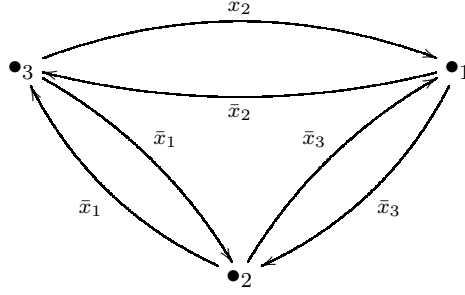
where $\bar{x}_2\bar{x}_3 = \bar{x}_3\bar{x}_2 = 0$, $\bar{x}_2\bar{x}_1 = \bar{x}_1\bar{x}_2 = 0$ and $\bar{x}_1\bar{x}_3 = \bar{x}_3\bar{x}_1 = 0$. Let $J_{1,2}$ be the annihilator of $e_1\bar{x}_3 e_2 \in \Lambda/\Lambda e\Lambda$. It is easy to check that $J_{1,2} = (\bar{x}_1^2, \bar{x}_2^2, \bar{x}_1\bar{x}_2\bar{x}_3) \cong (A, B, D) \cong (I_1, I_2)$. Analogously,

$$J_{2,1} \cong J_{1,2} \cong (A, B, D) \cong (I_1, I_2),$$
$$J_{2,3} \cong J_{3,2} \cong (B, C, D) \cong (I_2, I_3),$$
$$J_{3,1} \cong J_{3,1} \cong (A, C, D) \cong (I_3, I_1).$$

Thus, we see that

$$Z(\Lambda/\Lambda e\Lambda) = \left\{ (\bar{h}_1, \bar{h}_2, \bar{h}_3) \in S/I_1 \oplus S/I_2 \oplus S/I_3 \,\middle|\, h_j - h_{j'} \in J_{j,j'}, \forall j, j' \in \{1, 2, 3\} \right\}$$
$$\cong S/(AB, BC, CD, D),$$

where $h_j \in S$. The above second equality is given by the following. Since $I_1 \cap I_2 \cap I_3 = (AB, BC, CD, D)$, there is an injection

$$S/(AB, BC, CD, D) \hookrightarrow S/I_1 \oplus S/I_2 \oplus S/I_3,$$

given by the three natural projections

$$S/(AB, BC, CD, D) \twoheadrightarrow S/(A, BC, D) = S/I_1,$$
$$S/(AB, BC, CD, D) \twoheadrightarrow S/(B, AC, D) = S/I_2,$$
$$S/(AB, BC, CD, D) \twoheadrightarrow S/(C, AB, D) = S/I_3.$$

It is clear that the image of this injection is exactly the subring:

$$\left\{ (\bar{h}_1, \bar{h}_2, \bar{h}_3) \in S/I_1 \oplus S/I_2 \oplus S/I_3 \,\middle|\, h_j - h_{j'} \in J_{j,j'}, \forall j, j' \in \{1, 2, 3\} \right\},$$

since the intersection of $\mathrm{Spec}(S/I_j)$ and $\mathrm{Spec}(S/I_{j'})$ is exactly $\mathrm{Spec}(S/J_{j,j'})$.

Thus we have

$$\mathrm{Sing}\big(\mathrm{Spec}(S)\big) \cong \mathrm{Spec}\big(S/(AB, BC, CD, D)\big) \cong \mathrm{Spec}\big(Z(\Lambda/\Lambda e\Lambda)\big).$$

Hence, the reduced scheme of $\mathrm{Spec}(Z(\Lambda/\Lambda e\Lambda))$ is exactly $\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)}$.

Example 5.2. Let V be a three-dimensional vector space, $R = k[V] = k[x_1, x_2, x_3]$ and $G \subseteq \mathrm{SL}(V)$ be generated by elements

$$g := \mathrm{Diag}(\sigma, \sigma, \sigma^2),$$

where σ is a 4-th primitive root of unit. Here, g sents x_1 to σx_1, x_2 to σx_2 and x_3 to $\sigma^2 x_3$. Then

$$S = R^G = k[x_1^4, x_2^4, x_3^2, x_1 x_2^3, x_1^2 x_2^2, x_1^3 x_2, x_1^2 x_3, x_2^2 x_3, x_1 x_2 x_3].$$

Since G does not contain any pseudo-reflection, by the Chevalley-Shephard-Todd theorem (see Appendix B), the reduced scheme of the singular locus in $\mathrm{Spec}(S)$ consists of the images under the quotient map of the points whose isotropy subgroup is nontrivial. Let $W_3 \subseteq V$ be the one-dimensional subspace with the associated surjection of algebras $\pi_3 : k[V] \twoheadrightarrow k[W_3] \cong k[x_3]$ given by $\pi_3(x_3) = x_3$ and $\pi_3(x_1) = \pi_3(x_2) = 0$. It is direct to see that the set of points whose isotropy subgroups are nontrivial is exactly W_3. Now, the isotropy subgroup of W_3 is

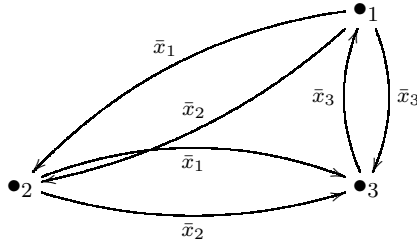$$G_3 := \langle \mathrm{Diag}(-1,-1,1) \rangle \subseteq G.$$

Hence,

$$\sqrt{\mathrm{Sing}\big(\mathrm{Spec}(S)\big)} \cong W_3/(G/G_3) = \mathrm{Spec}(k[x_3^2]).$$

Consider the skew group algebra Λ, the quiver Q_Λ is given as follows:
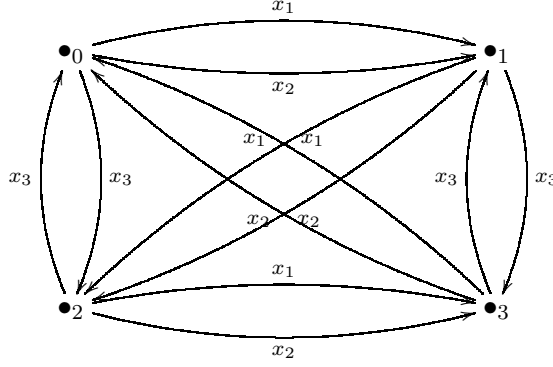
The quiver Q_{con} associated to $\Lambda_{\mathrm{con}} \cong \Lambda/\Lambda e\Lambda$ is the following:

Let e_r be the indecomposable idempotent corresponding to vertex $\bullet_r$ in the quiver Q_Λ. Then from the above quiver, we obtain

$$e_r(1 \otimes \bar{x}_i \bar{x}_j \bar{x}_3) e_{r'} = 0 \quad \text{and} \quad e_r(1 \otimes \bar{x}_i \bar{x}_3^2) e_{r'} = 0 \tag{36}$$

in $\Lambda/\Lambda e\Lambda$, for any $r, r' \in \{1, 2, 3\}$ and any $i, j \in 1, 2$. In fact, one of the following paths

$$x_i x_j x_3, \ x_i x_3 x_j \ x_3 x_i x_j, \ x_i x_3 x_3 \ x_3 x_i x_3 \ x_3 x_3 x_i$$

starting from vertex $\bullet_r$ must pass through vertex $\bullet_0$ in Q_Λ. By (36), it is then straightforward to check that

$$\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_1) = k[\bar{x}_3^2], \ \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_2) = k, \ \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_1) = k[\bar{x}_3^2],$$

29

where M_i is the direct summand of R corresponding to the idempotent e_i. Thus, by (36) again, it is easy to verify that

$$Z(\Lambda/\Lambda e\Lambda) = Z\big(\mathrm{End}_{\underline{\mathrm{CM}(S)}}(\bigoplus_{i=1}^{3} M_i)\big) = k[\bar{x}_3^2]$$

as algebras. Hence, the reduced scheme of $\mathrm{Spec}(Z(\Lambda/\Lambda e\Lambda))$ is $\sqrt{\mathrm{Sing}(\mathrm{Spec}(S))}$.

A Proof of Lemma 3.15

Proof of Lemma 3.15. First, from the constructions of τ^H and Pr_H, we know that they are both surjective maps, and thus so is ζ_H. Thus to prove this lemma, it is enough to prove that ζ_H is an algebra homomorphism, which is further sufficient to show that for any $\alpha, \beta \in \Lambda$,

$$\zeta_H(\alpha\beta) = \zeta_H(\alpha)\zeta_H(\beta).$$

Without loss of generality, let $\alpha \in \mathrm{Hom}_S(M_{\chi_2}, M_{\chi_3})$ and $\beta \in \mathrm{Hom}_S(M_{\chi_1}, M_{\chi_2})$ for characters χ_1, χ_2, χ_3 of G. Moreover, suppose that α and β are monomials in $(V_{\chi_3\chi_2^{-1}}\otimes R)^G \subseteq R$ and $(V_{\chi_2\chi_1^{-1}} \otimes R)^G \subseteq R$ respectively. We have the following two possibilities about χ_1, χ_2 and χ_3:

(1) χ_1, χ_2 and χ_3 are all in $\{\lambda_i^H\}_i$;

(2) at least one of $\{\chi_1, \chi_2, \chi_3\}$ is not in $\{\lambda_i^H\}_i$.

For case (1), by the definition of Pr_H (see (12)), the images of α, β and $\alpha\beta$ under the projection

$$Pr_H : \mathrm{Hom}_S(\widehat{M}, \widehat{M}) \twoheadrightarrow \mathrm{Hom}_S\big(\bigoplus_i M_{\lambda_i^H}, \bigoplus_i M_{\lambda_i^H}\big)$$

are also $\alpha, \beta, \alpha\beta \in \mathrm{Hom}_S\big(\bigoplus_i M_{\lambda_i^H}, \bigoplus_i M_{\lambda_i^H}\big)$ as monomials. Then we have $Pr_H(\alpha) = \alpha$, $Pr_H(\beta) = \beta$ and $Pr_H(\alpha\beta) = \alpha\beta$.

In the meantime, we know that χ_1, χ_2 and χ_3 are all in $\{\lambda_i^H\}_i$. Then by (9), we have $\alpha \in \big(V_{\chi_3\chi_2^{-1}}\otimes k[W_H]\otimes k[W_H']^H\big)^{G/H}$ and $\beta \in \big(V_{\chi_2\chi_1^{-1}} \otimes k[W_H]\otimes k[W_H']^H\big)^{G/H}$ as monomials. Thus, we have $\alpha = \alpha^H \alpha'$ and $\beta = \beta^H \beta'$ as monomials, where $\alpha^H \in V_{\chi_3\chi_2^{-1}}\otimes k[W_H] \cong k[W_H]$, $\beta^H \in V_{\chi_2\chi_1^{-1}} \otimes k[W_H] \cong k[W_H]$, and $\alpha', \beta' \in k[W_H']^H$. Moreover we have $\alpha\beta = \alpha^H \beta^H \alpha' \beta'$ as monomials. By the definition of τ_{χ_2,χ_3}^H, we have

$$\tau_{\chi_2,\chi_3}^H(\alpha) = \begin{cases} \alpha, & \alpha' \in k \subseteq k[W_H']^H, \\ 0, & \alpha' \in (k[W_H']^H)^+, \end{cases}$$

where $(k[W_H']^H)^+$ is the augmentation ideal of $k[W_H']^H$. Similarly, we have

$$\tau_{\chi_1,\chi_2}^H(\beta) = \begin{cases} \beta, & \beta' \in k \subseteq k[W_H']^H, \\ 0, & \beta' \in (k[W_H']^H)^+, \end{cases}$$

and

$$\tau_{\chi_1,\chi_3}^H(\alpha\beta) = \begin{cases} \alpha\beta, & \alpha'\beta' \in k \subseteq k[W_H']^H, \\ 0, & \alpha'\beta' \in (k[W_H']^H)^+. \end{cases}$$

If $\alpha', \beta' \in k$, then $\tau^H_{\chi_2,\chi_3}(\alpha)\tau^H_{\chi_1,\chi_3}(\beta) = \alpha\beta = \tau^H_{\chi_1,\chi_3}(\alpha\beta)$. If one of α', β' is not in k, $\alpha'\beta'$ is not in k, either. Then $\tau^H_{\chi_2,\chi_3}(\alpha)\tau^H_{\chi_1,\chi_3}(\beta) = 0 = \tau^H_{\chi_1,\chi_3}(\alpha\beta)$. It follows that

$$\tau^H_{\chi_2,\chi_3}(\alpha)\tau^H_{\chi_1,\chi_3}(\beta) = \tau^H_{\chi_1,\chi_3}(\alpha\beta).$$

It suggests that $\tau^H(\alpha)\tau^H(\beta) = \tau^H(\alpha\beta)$ since $\tau^H = \bigoplus_{i,j} \tau^H_{\lambda^H_i,\lambda^H_j}$. By composing with the projection Pr_H, we have

$$\begin{aligned}
\zeta_H(\alpha\beta) &= \tau^H \circ Pr_H(\alpha\beta) = \tau^H(\alpha\beta) \\
&= \tau^H(\alpha)\tau^H(\beta) = \big(\tau^H \circ Pr_H(\alpha)\big)\big(\tau^H \circ Pr_H(\beta)\big) \\
&= \zeta_H(\alpha)\zeta_H(\beta)
\end{aligned}$$

in this case.

For case (2), if one of χ_1, χ_3 is not in $\{\lambda^H_i\}_i$, then one of the images of α and β is trivial under the projection Pr_H. In this case, the image of $\alpha\beta$ is also trivial under Pr_H. Hence, when one of χ_1, χ_3 is not in $\{\lambda^H_i\}_i$, we have $Pr_H(\alpha\beta) = 0 = Pr_H(\alpha)Pr_H(\beta)$. By composing with τ^H, we obtain that

$$\zeta_H(\alpha\beta) = 0 = \zeta_H(\alpha)\zeta_H(\beta).$$

If χ_2 is not in $\{\lambda^H_i\}_i$ and χ_1, χ_3 are both in $\{\lambda^H_i\}_i$, we first have $\zeta_H(\alpha) = \zeta_H(\beta) = 0$ since the images of α and β is trivial under projection Pr_H. In the meantime, by Lemma 3.4 we have

$$\alpha \in \mathrm{Hom}_S(M_{\chi_2}, M_{\chi_3}) \cong \big(V_{\chi_2^{-1}\chi_3} \otimes k[W_H] \otimes k[W'_H]\big)^G.$$

Then we can write $\alpha = \alpha^*\alpha^\vee$ as monomials, where $\alpha^* \in V_{\chi_2^{-1}\chi_3} \otimes k[W_H] \cong k[W_H]$ and $\alpha^\vee \in k[W'_H]$. At this time, neither of $\chi_1^{-1}\chi_2, \chi_2^{-1}\chi_3$ is in $\{\lambda^H_i\}_i$. Then $V_{\chi_2^{-1}\chi_3}$ is not invariant under action of H. Meanwhile, $k[W_H]$ is invariant under action of H. Hence, α^* is not invariant under action of H. It implies that α^* is not invariant under action of G. But $\alpha^*\alpha^\vee$ is invariant under action of G. Thus, $\alpha^\vee \in (k[W'_H]^H)^+$ since k is invariant under action of G.

On the other hand, let $\beta = \beta^*\beta^\vee$ as monomials, where $\beta^* \in V_{\chi_1^{-1}\chi_2} \otimes k[W_H]$ and $\beta^\vee \in k[W'_H]$. In the same way, we get that $\beta^\vee \in (k[W'_H]^H)^+$ as a monomial.

Thus, we have $\alpha\beta = \alpha^*\beta^*\alpha^\vee\beta^\vee$ as monomials such that $\alpha^\vee\beta^\vee \in (k[W'_H]^H)^+$. Then $\tau^H_{\chi_1,\chi_3}(\alpha\beta) = 0$ by the definition of $\tau^H_{\chi_1,\chi_3}$. It implies that $\tau^H(\alpha\beta) = 0$. Finally, from $\zeta_H = \tau^H \circ Pr_H$, we have $\zeta_H(\alpha\beta) = 0$. It follows that if χ_2 is not in $\{\lambda^H_i\}_i$ and χ_1, χ_3 are both in $\{\lambda^H_i\}_i$,

$$\zeta_H(\alpha\beta) = 0 = \zeta_H(\alpha)\zeta_H(\beta).$$

Thus we have $\zeta_H(\alpha\beta) = 0 = \zeta_H(\alpha)\zeta_H(\beta)$ in this case.

In summary, in both cases $\zeta_H(\alpha\beta) = \zeta_H(\alpha)\zeta_H(\beta)$. This completes the proof. $\square$

B   Proof of Theorem 3.23

Before proving this theorem, let us first recall the Chevalley-Shephard-Todd theorem (see, for example, [1]): *Let G be a finite group acting faithfully on an affine space V. For $x \in V$, let G_x be the stabilizer of x. Let $\bar{x}$ be the image of x in the quotient scheme V/G. Then $\bar{x}$ is a non-singular point if and only if G_x is generated by pseudo-reflections.*

Here, a pseudo-reflection is a linear isomorphism $s : V \xrightarrow{\sim} V$ that leaves a hyperplane $W \subseteq V$ pointwise invariant. When G is a finite subgroup of $\mathrm{SL}(V)$, the unique pseudo-reflection is the identity. Thus by the Chevalley-Shephard-Todd theorem, we know that $\bar{x}$ is not a singular point if and only if G_x is trivial in our case, and obtain the following.

Lemma B.1. *With the notations from §3.3,*

$$\sqrt{\operatorname{Sing}\big(\operatorname{Spec}(S)\big)} \cong \bigcup_{(H,W_H)} W_H/(G/H) \cong \bigcup_{(H,W_H)} \operatorname{Spec}(S_H)$$

as subschemes of $\operatorname{Spec}(S)$.

Now, let $I_H \subseteq S$ be the ideal associated to the component $W_H/(G/H)$ of singular locus in $\operatorname{Spec}(S)$. By Lemma B.1,

$$\sqrt{\operatorname{Sing}\big(\operatorname{Spec}(S)\big)} = \bigcup_{(H,W_H)} \operatorname{Spec}(S_H) = \operatorname{Spec}\big(S/\bigcap_{(H,W_H)} I_H\big)$$

as subschemes of $\operatorname{Spec}(S)$. We have the following lemma.

Lemma B.2. *Let $\mathfrak{S}$ be a set of formal variable in R such that $\mathfrak{S} \not\subset R_H$ for any (H, W_H). Then the image of $\bigodot_{x_i \in \mathfrak{S}} x_i^{\#(G)} \in S$ is trivial in the coordinate ring of $\sqrt{\operatorname{Sing}\big(\operatorname{Spec}(S)\big)}$, where $\bigodot$ is the product of R.*

Proof. To prove this lemma, it is sufficient to show that $\bigodot_{x_i \in \mathfrak{S}} x_i^{\#(G)} \in I_H$ for any pairs (H, W_H) as above. Note that $\bigodot_{x_i \in \mathfrak{S}} x_i^{\#(G)} \in S$ since $x_i^{\#(G)}$ is invariant under the action of G.

By the definition of $S_H \cong k[W_H]^G$, I_H is the kernel of the natural quotient map

$$S \cong k[W_H \oplus W_H']^G \twoheadrightarrow k[W_H]^G \cong k[W_H]^{G/H} = S_H.$$

It implies that

$$I_H = S \cap k[W_H'].$$

Thus to prove $\bigodot_{x_i \in \mathfrak{S}} x_i^{\#(G)} \in I_H$, it is enough to show that $\mathfrak{S} \cap k[W_H'] \neq \emptyset$. Here, $\mathfrak{S}$ is a set of formal variable in R_H. Note that $R_H = k[W_H]$ and $k[W_H']$ are both generated by formal variables as algebras. Then a formal variable in R is either in R_H, or in $k[W_H']$. Since $\mathfrak{S} \not\subset R_H = k[W_H]$, we have $\mathfrak{S} \cap k[W_H'] \neq \emptyset$. $\square$

Proof of Theorem 3.23. Since

$$\operatorname{Ker}\big(Z(\zeta_0)\big) \cong \operatorname{Ker}(\zeta_0) \cap Z(\Lambda_{\mathrm{con}}) \quad \text{and} \quad \operatorname{Ker}(\tilde{\zeta_0}) \cong \operatorname{Ker}\big(Z(\zeta_0)\big)/\big(\operatorname{Ker}\big(Z(\zeta_0)\big) \cap N_\Lambda\big),$$

where N_Λ is the radical of $Z(\Lambda_{\mathrm{con}})$, we have

$$\operatorname{Ker}(\tilde{\zeta_0}) \cong \operatorname{Ker}(\zeta_0) \cap Z(\Lambda_{\mathrm{con}})/\big(\operatorname{Ker}(\zeta_0) \cap Z(\Lambda_{\mathrm{con}}) \cap N_\Lambda\big).$$

Thus, to prove this theorem, it suffices to show that

$$\operatorname{Ker}(\zeta_0) \cap Z(\Lambda_{\mathrm{con}}) \subseteq N_\Lambda.$$

In fact, since any element $g \in Z(\Lambda_{\mathrm{con}}) \cong Z\big(\operatorname{End}_{\underline{\mathrm{CM}}(S)}(\bigoplus_\chi M_\chi)\big)$, we have

$$g = \sum_\chi g \circ \operatorname{Id}_{M_\chi} = \sum_\chi g \circ \operatorname{Id}_{M_\chi} \circ \operatorname{Id}_{M_\chi} = \sum_\chi (\operatorname{Id}_{M_\chi} \circ g \circ \operatorname{Id}_{M_\chi}) \in \bigoplus_\chi \operatorname{End}_{\underline{\mathrm{CM}}(S)}(M_\chi).$$

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Here, Id_{M_χ} is the identity map in $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_\chi)$. And then we have

$$Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big) \cong Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\oplus_\chi M_\chi)\big) \subseteq \bigoplus_\chi \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_\chi) \qquad (37)$$

as algebras. Now by the argument in §3.3.1 and Lemma 3.12, we have

$$\bigoplus_\chi \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_\chi) \cong \bigoplus_\chi e_\chi \Lambda_{\mathrm{con}} e_\chi \cong \bigoplus_\chi S/\bar{I}_\chi,$$

and thus for any $\bar{h} \in Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\bigoplus_\chi M_\chi)\big)$, $\bar{h}$ can be written in the form $\sum_\chi \bar{h}_\chi$, where $\bar{h}_\chi \in \mathrm{End}_{\underline{\mathrm{CM}}(S)}(M_\chi) \cong S/\bar{I}_\chi$.

Let $\bar{h}$ be an element in $\mathrm{Ker}(\zeta_0) \cap Z(\Lambda_{\mathrm{con}})$ and χ be a character of G. Without loss of generality, suppose that there is a nontrivial monomial $f \in S$ such that its image in $e_\chi \Lambda_{\mathrm{con}} e_\chi \cong S/\bar{I}_\chi$ is exactly $\bar{h}_\chi$. Then to prove this theorem, it is enough to prove that

$$f \in I_\chi.$$

In fact, if $f \in I_\chi$, then $f^{m_\chi} \in \bar{I}_\chi$ for some $m_\chi \in \mathbb{N}$. Then $\bar{h}_\chi^{m_\chi} = 0$. Let m be the maximal number in $\{m_\chi\}_\chi$. Then $\bar{h}^m = \big(\sum_\chi \bar{h}_\chi\big)^m = \sum_\chi \bar{h}_\chi^m = 0$. It implies that $\bar{h} \in N_\Lambda$.

We have the following two possibilities of χ:

(1) There is no pair (H', χ) in $\tilde{G}_0$ such that $W_{H'}$ is nontrivial;

(2) There is a pair (H', χ) in $\tilde{G}_0$ such that $W_{H'}$ is nontrivial.

We shall show that in each case, $f \in I_\chi$.

Case *(1): There is no pair (H', χ) in $\tilde{G}_0$ such that $W_{H'}$ is nontrivial.*

Note that $W_{H'}$ is nontrivial means that the dimension of $W_{H'}$ is not zero.

Let $\{f_r\}_r$ be the set of degree-one factors of f, i.e., $\{f_r\}_r \subseteq \{x_i\}_i$. For this case, without loss of generality, suppose that $x_1 \in \{f_r\}_r$. Let $G^\vee$ be the dual group of G. It consists of all character of G. Also let $\langle \chi_{x_1} \rangle \subseteq G^\vee$ be the subgroup generated by χ_{x_1}. Since

$$\#\big(\langle \chi_{x_1} \rangle\big) \#\big(\mathrm{Ker}(\chi_{x_1})\big) = \#(G),$$

where $\#(-)$ represents the number of elements in group, to prove the theorem for case (1), we only need to consider the following two situations: either (i) $\mathrm{Ker}(\chi_{x_1})$ is trivial, or (ii) $\mathrm{Ker}(\chi_{x_1})$ is non-trivial. Let us study them case by case.

Subcase *(i): $\mathrm{Ker}(\chi_{x_1})$ is trivial.*

When $\mathrm{Ker}(\chi_{x_1})$ is trivial, we have $\#\big(\langle \chi_{x_1} \rangle\big) = \#(G)$. It follows that $\langle \chi_{x_1} \rangle$ generates $G^\vee$ since $\#(G) = \#(G^\vee)$. Then $\chi_{x_1}^{m_1} = \chi_0 \chi^{-1}$ for some $m_1 \in \mathbb{N}$. It implies that $\chi \chi_{x_1}^{m_1} = \chi_0$. It means that $e_\chi(1 \otimes x_1^{m_1}) = e_\chi(1 \otimes x_1^{m_1})e$ by (4). Thus, we get

$$e_\chi(1 \otimes x_1^{m_1}) \in e_\chi \Lambda e.$$

Then

$$e_\chi(1 \otimes x_1^{n_1 \#(G)}) \in e_\chi \Lambda e \Lambda e_\chi$$

for some $n_1 \in \mathbb{N}$ such that $n_1 \#(G) \geq m_1$. Hence, when we view f as a monomial, $e_\chi(1 \otimes f^{n_1 \#(G)}) \in e_\chi \Lambda e \Lambda e_\chi$, since x_1 is a factor of f. By the isomorphism $S \cong e_\chi \Lambda e_\chi$, it suggests that $f^{n_1 \#(G)} \in \bar{I}_\chi$. Then we have $f \in I_\chi$.

Subcase *(ii):* $\mathrm{Ker}(\chi_{x_1})$ *is non-trivial.*

If $\mathrm{Ker}(\chi_{x_1})$ is non-trivial, letting $K := \mathrm{Ker}(\chi_{x_1})$, then we have $E_1 \in W_K$. Hence, since $K = \mathrm{Ker}(\chi_{x_1})$, the maximal group whose invariant subspace is W_K is exactly K. Then we get a pair (K, W_K). Now, set

$$\varpi^K : (G/K)\sharp R_K \cong \mathrm{End}_{S_K}(\widehat{M^K}) \to \mathrm{End}_S(\widehat{M}) \cong \Lambda \qquad (38)$$

be the composition ρ^K (see (15)) with the canonical injection $\bigoplus_{i,j} \mathrm{Hom}_S\big(M_{\lambda_i^K}, M_{\lambda_j^K}\big) \hookrightarrow \mathrm{End}_S(\widehat{M})$. Since ρ^K is an injection of algebras, ϖ^K is also an injection of algebras.

Since $\chi_0(K) = \{1\}$, the trivial character χ_0 of G is in $\{\lambda_i^K\}$. Hence, by the definition of $\rho^K = \bigoplus_{i,j} \rho^K_{\lambda_i^K, \lambda_j^K}$, it is obvious that e is in the image of ϖ^K. In fact, we have $\varpi^K(e^K) = e$, where e^K is the idempotent in $(G/K)\sharp R_K$ corresponding to summand S_K of $\widehat{M^K}$. Let

$$\big((G/K)\sharp R_K\big)_{\mathrm{con}} := (G/K)\sharp R_K / \big((G/K)\sharp R_K\big)e^K\big((G/K)\sharp R_K\big).$$

Then ϖ^K induces another injection of algebras:

$$\widetilde{\varpi}^K : \big((G/K)\sharp R_K\big)_{\mathrm{con}} \hookrightarrow \Lambda_{\mathrm{con}}$$

such that the following diagram

$$\begin{array}{ccc} (G/K)\sharp R_K & \longrightarrow & \big((G/K)\sharp R_K\big)_{\mathrm{con}} \\ {\scriptstyle \varpi^K} \Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow {\scriptstyle \widetilde{\varpi}^K} \\ \Lambda & \longrightarrow\!\!\!\longrightarrow & \Lambda_{\mathrm{con}} \end{array} \qquad (39)$$

commutes, where the morphisms of horizontal direction are given by quotients by ideal (e^K) and (e) respectively.

By the above argument, there is a pair (K, W_K) which means that K is maximal among all subgroups which have the same invariant subspace as that of K. Then $(K, \chi) \in \tilde{G}$. But $(K, \chi) \notin \tilde{G}_0$ by our assumption. Thus $\chi(K) = \{1\}$. Hence, e_χ is in the image of ϖ^K. Let $\varpi^K(e_\chi^K) = e_\chi$. Thus, we get that $x_1 \in k[W_K] = R_K$ by $E_1 \in W_K$, and $e_\chi^K \in (G/K)\sharp R_K$. Hence, by the definition of ϖ^K, this algebra homomorphism maps $e_\chi^K(1 \otimes x_1^m)$ to $e_\chi(1 \otimes x_1^m)$ for any $m \in \mathbb{N}$. Since $\widetilde{\varpi}^K$ is injective,

$$e_\chi^K(1 \otimes x_1^m) \in e_\chi^K(G\sharp R)e^K$$

for some $m \in \mathbb{N}$ if and only if $e_\chi(1 \otimes x_1^m) \in e_\chi \Lambda e$. Moreover, considering the character χ_{x_1} of G/K, we know that the kernel of χ_{x_1} is trivial in G/K. From the above argument in subcase (i) and the fact that $\mathrm{Ker}(\chi_{x_1})$ is trivial in G/K, by replacing Λ by $(G/K)\sharp R_K$, we have

$$e_\chi^K(1 \otimes x_1^m) \in e_\chi^K\big((G/K)\sharp R_K\big)e_i^K$$

for some $m_1 \in \mathbb{N}$. It implies that $e_\chi(1 \otimes x_1^m) \in e_\chi \Lambda e$. Thus, we get that $f \in I_\chi$ in the same way as in subcase (i).

Combining the above two subcases (i) and (ii), we have $f \in I_\chi$ for case (1).

Case *(2): There is a pair (H', χ) in $\tilde{G}_0$ such that $W_{H'}$ is nontrivial.*

For this case, observing that there is no pair (H, χ) in $\tilde{G}_0$ such that $\{f_r\}_r \subseteq R_H$, we know that $\{f_r\}_r \not\subset R_H$ for any $(H, \chi) \in \tilde{G}$ unless $\{f_r\}_r \subseteq R_{\bar{H}}$ for some pair $(\bar{H}, \chi) \in \tilde{G}\backslash\tilde{G}_0$. Thus to prove the theorem in this case, we only need to consider the following two situations:

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either (i) $\{f_r\}_r \not\subset R_H$ for any $(H, \chi) \in \tilde{G}$, or (ii) $\{f_r\}_r \subseteq R_{\bar{H}}$ for some $(\bar{H}, \chi) \in \tilde{G} \backslash \tilde{G}_0$. Again, we study them case by case.

***Subcase** (i): $\{f_r\}_r \not\subset R_H$ for any $(H, \chi) \in \tilde{G}$.*

When $\{f_r\}_r \not\subset R_H$ for any $(H, \chi) \in \tilde{G}$, by Lemma B.2, the image of $\bigodot_r f_r^{\#(G)} \in S$ is trivial in the coordinate ring of $\sqrt{\mathrm{Sing}(\mathrm{Spec}(S))}$. It implies that

$$\bigodot_r f_r^{\#(G)} \in \sqrt{\varphi(S) \cap \Lambda e \Lambda} \cong \bigcap_\chi \sqrt{\bar{I}_\chi} \cong \bigcap_\chi I_\chi$$

by Theorem 3.9. Then we have $\bigodot_r f_r^{\#(G)} \in \sqrt{e_\chi \Lambda e \Lambda e_\chi} = I_\chi$. Since $\bigodot_r f_r^{\#(G)}$ is a factor of f, $f^{\#(G)} \in I_\chi$. It follows that $f \in I_\chi$.

***Subcase** (ii): $\{f_r\}_r \subseteq R_{\bar{H}}$ for some $(\bar{H}, \chi) \in \tilde{G} \backslash \tilde{G}_0$.*

If $\{f_r\}_r \subseteq R_{\bar{H}}$ for some $(\bar{H}, \chi) \in \tilde{G} \backslash \tilde{G}_0$, then we have $\chi(\bar{H}) = \{1\}$ and $f \in R_H$ since $\{f_r\}_r$ is the set of degree one factors of f. Let $e_\chi^{\bar{H}}$ and $e^{\bar{H}}$ be the idempotents of $G/\bar{H} \sharp R_{\bar{H}}$ corresponding to characters $\chi|_{G/\bar{H}}$ and $\chi_0|_{G/\bar{H}}$ respectively, and $\varpi^{\bar{H}}$ be the algebra homomorphism in (38) by replacing K by $\bar{H}$. Then we have $\varpi^{\bar{H}}(e_\chi^{\bar{H}}) = e_\chi$ and $\varpi^{\bar{H}}(e^{\bar{H}}) = e$. Thus, since $f \in R_H$,

$$e_\chi(1 \otimes f) = \varpi^{\bar{H}}\big(e_\chi^{\bar{H}}(1 \otimes f)\big) \in \varpi^{\bar{H}}\big(G/\bar{H} \sharp R_{\bar{H}}\big)$$

by the definition of $\varpi^{\bar{H}}$(see (38)). By the same argument and the commutative diagram (39) as in case (1), $e_\chi^{\bar{H}}(1 \otimes f) \in e_\chi^{\bar{H}}(G/\bar{H} \sharp R_{\bar{H}})e^{\bar{H}}$ if and only if $e_\chi(1 \otimes f) \in e_\chi \Lambda e$.

Now, we consider $G/\bar{H} \sharp R_{\bar{H}}$ instead of Λ. Applying Lemma B.2 to $G/\bar{H} \sharp R_{\bar{H}}$, we then get $f \in I_\chi$ with the same argument as in Case (1) or in Subcase (i) of Case (2), or reduce the discussion from $G/\bar{H} \sharp R_{\bar{H}}$ to a even smaller skew group algebra. Apply this procedure recursively, and finally we get that, for any pair (H, χ) in $\tilde{G}$, $\chi(H) = \{1\}$. This case is exactly the case (1). Thus in this case we again have $f \in I_\chi$.

Combining the above subcases (i) and (ii), we have $f \in I_\chi$ for case (2).

In summary, we get that $f \in I_\chi$ for both cases, which finishes the proof. $\qquad\square$

C Proof of Lemma 3.24

Proof of Lemma 3.24. Fix a pair (H, W_H). The set of morphisms $\{Z(\tilde{\zeta}_H^\chi)\}_\chi$ is in one to one correspondence with the classes of characters in $\tilde{G}_0$. Moreover, by definition of $\tilde{G}_0$, for any $(H, \chi), (H, \chi') \in \tilde{G}_0$, $(H, \chi) \sim (H, \chi')$ in $\tilde{G}_0$ if and only if $\chi|_H = \chi'|_H$. It means that the classes of characters in $\tilde{G}_0$ is in one to one correspondence with the set of characters of H. Thus the set of morphisms $\{Z(\tilde{\zeta}_H^\chi)\}_\chi$ is in one to one correspondence with the set of characters of H, where this correspondence takes a character λ of H to morphism $Z(\tilde{\zeta}_H^{\tilde{\lambda}})$ and $\tilde{\lambda}$ is a preimage of λ under the group homomorphism $G^\vee \twoheadrightarrow H^\vee$ given by restriction of characters on H. Note that for any preimages $\tilde{\lambda}_1, \tilde{\lambda}_2$ of λ, $(\tilde{\lambda}_1, H) \sim (\tilde{\lambda}_2, H)$ in $\tilde{G}_0$.

In the meantime, the set of characters of H is also in one to one correspondence with the set of indecomposable idempotents of $\Lambda_H := H \sharp k[W/W_H]$. Then the set of morphisms $\{Z(\tilde{\zeta}_H^\chi)\}_\chi$ is again one to one correspondence with the set of indecomposable idempotents of Λ_H.

Since the set of indecomposable idempotents of Λ_H is in one to one correspondence with the vertices of the quiver Q_{Λ_H} whose path algebra is Λ_H, the set of morphisms $\{Z(\tilde{\zeta}_H^\chi)\}_\chi$ is in one to one correspondence with the vertices of the quiver Q_{Λ_H}.

Let $(H, \chi), (H, \chi')$ be two different pairs in $\tilde{G}_0$. They correspond to two vertices of Q_{Λ_H}, say, e_χ' and $e_{\chi'}'$ respectively. Note that $H \subseteq \mathrm{SL}(W/W_H)$. Let Q_{con}^H be the quiver which is associated to the algebra $\Lambda_H / \Lambda_H e^H \Lambda_H$, where e^H is the idempotent corresponding to the trivial character of H. Consider Λ_H instead of Λ, by Proposition 3.13, we know that the two vertices corresponding to e_χ' and $e_{\chi'}'$ in Q_{con}^H are connected by a path, say l, in Q_{con}^H.

The path l may be the concatenation of several arrows. To prove $Z(\tilde{\zeta}_H^\chi) = Z(\tilde{\zeta}_H^{\chi'})$, it is enough to show that for any arrow $\mathfrak{r}$ in the path l,

$$Z(\tilde{\zeta}_H^{\tilde{\lambda}_s^{\mathfrak{r}}}) = Z(\tilde{\zeta}_H^{\tilde{\lambda}_t^{\mathfrak{r}}}),$$

where the characters $\tilde{\lambda}_s^{\mathfrak{r}}$ and $\tilde{\lambda}_t^{\mathfrak{r}}$ of G are defined as follows. Let $\lambda_s^{\mathfrak{r}}$ and $\lambda_t^{\mathfrak{r}}$ be characters of H such that they correspond to idempotents $e_{\lambda_s^{\mathfrak{r}}}'$ and $e_{\lambda_t^{\mathfrak{r}}}'$ respectively, where $e_{\lambda_s^{\mathfrak{r}}}'$ is the source of $\mathfrak{r}$ and $e_{\lambda_t^{\mathfrak{r}}}'$ is the target of $\mathfrak{r}$ in quiver Q_{con}^H. Then $\tilde{\lambda}_s^{\mathfrak{r}}$ and $\tilde{\lambda}_t^{\mathfrak{r}}$ are two preimages of $\lambda_s^{\mathfrak{r}}$ and $\lambda_t^{\mathfrak{r}}$ under the surjection of groups $G^\vee \twoheadrightarrow H^\vee$.

Next, note that $W/W_H \cong W_H'$ as G-representations. Then $\Lambda_H \cong H \sharp k[W_H']$. Since $\{E_i\}_i \backslash \mathfrak{S}_H$ gives a basis of W_H', each arrow in the McKay quiver Q_{Λ_H} corresponds to a formal variable, say $x_r \in \{x_i\}_i \backslash \mathfrak{S}_H^*$, where $\mathfrak{S}_H^*$ is given by the canonical dual of $\mathfrak{S}_H$ (which sends x_i to E_i). Hence, each arrow in Q_{con}^H corresponds to a formal variable, say $x_r \in \{x_i\}_i \backslash \mathfrak{S}_H^*$,

Fix an arrow $\mathfrak{r}$ in l. Now, we suppose that $\mathfrak{r}$ corresponds to formal variable x_r. For simplicity, we also suppose that $\lambda_s^{\mathfrak{r}} = \chi|_H$ and $\tilde{\lambda}_s^{\mathfrak{r}} = \chi$. Then, it is easy to check that the image of $\chi\chi_{x_r}$ under $G^\vee \twoheadrightarrow H^\vee$ is exactly $\lambda_t^{\mathfrak{r}} = \lambda_s^{\mathfrak{r}}\chi_{x_r}' = \chi|_H\chi_{x_r}'$, since the group homomorphism $G^\vee \twoheadrightarrow H^\vee$ takes χ_{x_r} to χ_{x_r}', where χ_{x_r}' is the character of H given by the formal variable x_r. Hence, we can choose $\chi\chi_{x_r}$ to be $\tilde{\lambda}_t^{\mathfrak{r}}$. Thus, it is left to show that

$$Z(\tilde{\zeta}_H^\chi) = Z(\tilde{\zeta}_H^{\chi\chi_{x_r}}).$$

Let $\bar{h} = \sum_\lambda \bar{h}_\lambda$ be an element in $Z(\Lambda_{\mathrm{con}})$. In the following, for any character λ of G, let h_λ be an element in S such that its image is exactly $\bar{h}_\lambda$ in $S/\bar{I}_\lambda = e_\lambda \Lambda_{\mathrm{con}} e_\lambda$. From the construction of $Z(\tilde{\zeta}_H^\chi)$, we have

$$Z(\tilde{\zeta}_H^\chi)(\bar{h}) = \zeta_H^\chi(\sum_\lambda h_\lambda) = \tau^H \circ \Pi_{(H,\chi)}(\sum_\lambda h_\lambda)$$

$$= \tau^H\Big(\sum_{\chi^{-1}\lambda \in \{\lambda_i^H\}_i} h_\lambda\Big) = \sum_{\chi^{-1}\lambda \in \{\lambda_i^H\}_i} \tau_{\chi^{-1}\lambda, \chi^{-1}\lambda}^H(h_\lambda) \in Z(G/H \sharp R_H).$$

Note that the center of $G/H \sharp R_H$ is the images of canonical injection $S_H \hookrightarrow G/H \sharp R_H \cong \mathrm{End}_{S_H}(\widehat{M}^H)$, which is given by its S_H-module structure. This injection induces the isomorphism of algebras

$$S_H \cong Z(G/H \sharp R_H).$$

Under the inverse of this isomorphism we identify $\sum\limits_{\chi^{-1}\lambda \in \{\lambda_i^H\}_i} \tau_{\chi^{-1}\lambda, \chi^{-1}\lambda}^H(h_\lambda)$ with $\tau_{\chi_0, \chi_0}^H(h_\chi) \in$ S_H by letting $\lambda = \chi$. Here, note that $\tau_{\chi^{-1}\lambda, \chi^{-1}\lambda}^H(h_\lambda) = \tau_{\chi^{-1}\lambda', \chi^{-1}\lambda'}^H(h_{\lambda'})$ as elements in S_H for any $\chi^{-1}\lambda, \chi^{-1}\lambda' \in \{\lambda_i^H\}$. Then we get that

$$Z(\tilde{\zeta}_H^\chi)(\bar{h}) = \tau_{\chi_0, \chi_0}^H(h_\chi) \in S_H.$$

In the same way, we get that

$$Z(\tilde{\zeta}_H^{\chi\chi_{x_r}})(\bar{h}) = \tau_{\chi_0,\chi_0}^H(h_{\chi\chi_{x_r}}) \in S_H.$$

Hence, to prove that $Z(\tilde{\zeta}_H^{\chi}(\bar{h})) = Z(\tilde{\zeta}_H^{\chi\chi_{x_r}}(\bar{h}))$, it is sufficient to show that $\tau_{\chi_0,\chi_0}^H(h_\chi) = \tau_{\chi_0,\chi_0}^H(h_{\chi\chi_{x_r}})$. Since the algebra homomorphism τ_{χ_0,χ_0}^H is given by the canonical quotient map

$$S \cong (k[W_H] \otimes k[W_H'])^G \twoheadrightarrow k[W_H]^{(G/H)} = S_H$$

with kernel I_H, it is left to prove that $h_\chi - h_{\chi\chi_{x_r}} \in I_H$.

Recall that $\bar{I}_\chi = e_\chi \Lambda e \Lambda e_\chi$. Note that $\bar{I}_\chi \cong e_\chi \Lambda e \Lambda e_\chi \cong e \Lambda e_{\chi^{-1}} \Lambda e$ as ideals of S, since $e_\chi \Lambda e \cong e \Lambda e_{\chi^{-1}}$ and $e \Lambda e_\chi \cong e_{\chi^{-1}} \Lambda e$ by Lemma 3.4. Since

$$\tau_{\chi_0,\chi_0}^H(\bar{I}_\chi) \cong \tau_{\chi_0,\chi_0}^H(e \Lambda e_{\chi^{-1}} \Lambda e) \cong \tau^H(e \Lambda e_{\chi^{-1}} \Lambda e)$$

$$\cong \tau^H \circ \Pi_{(H,\chi)}(e_\chi \Lambda e \Lambda e_\chi) \cong \zeta_H^\chi(e_\chi \Lambda e \Lambda e_\chi) \overset{(3.19)}{=} \{0\},$$

we have $\bar{I}_\chi \subseteq \mathrm{Ker}(\tau_{\chi_0,\chi_0}^H) \cong I_H$. In the same way, we also have $\bar{I}_{\chi\chi_{x_r}} \subseteq I_H$. It is straightforward to see that either $h_\chi - h_{\chi\chi_{x_r}} \in \bar{I}_\chi$, or $h_\chi - h_{\chi\chi_{x_r}} \in \bar{I}_{\chi\chi_{x_r}}$. Then we get that $h_\chi - h_{\chi\chi_{x_r}} \in I_H$ since both $\bar{I}_\chi \subseteq I_H$ and $\bar{I}_{\chi\chi_{x_r}} \subseteq I_H$. It follows that $Z(\tilde{\zeta}_H^\chi) = Z(\tilde{\zeta}_H^{\chi\chi_{x_r}})$. We thus completed the proof. $\qquad\square$

D   Proof of Lemma 3.25

Proof of Lemma 3.25. First, let $(H,\chi) \in \tilde{G}_0$. Then since $H \subseteq G_{H,H'}$, $(G_{H,H'},\chi) \in \tilde{G}_0$. Thus to prove this lemma, it suffices to show that

$$Z(\zeta_{(G_{H,H'}/H)}) \circ Z(\zeta_H^\chi) = Z(\zeta_{G_{H,H'}}^\chi). \tag{40}$$

And to prove (40), it suffices to show that

$$\zeta_{(G_{H,H'}/H)} \circ \zeta_H^\chi = \zeta_{G_{H,H'}}^\chi. \tag{41}$$

To this end, denote by $Pr_{(G_{H,H'},\chi)}^H$ the canonical projection

$$\mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^{(H,\chi)}}, \bigoplus_i M_{\lambda_i^{(H,\chi)}}\Big) \twoheadrightarrow \mathrm{Hom}_S\Big(\bigoplus_i M_{\lambda_i^{(G_{H,H'},\chi)}}, \bigoplus_i M_{\lambda_i^{(G_{H,H'},\chi)}}\Big),$$

and let $Pr_{G_{H,H'}}^H := Pr_{(G_{H,H'},\chi_0)}^H$ due to $\{\lambda_i^{(G_{H,H'},\chi)}\} \subseteq \{\lambda_i^{(H,\chi)}\}$. Then we have

$$\begin{aligned}
Pr_{G_{H,H'}}^H \circ \Pi_{(H,\chi)} &= Pr_{G_{H,H'}}^H \circ \Delta_{(H,\chi)} \circ Pr_{(H,\chi)} \\
&= \Delta_{(G_{H,H'},\chi)} \circ Pr_{(G_{H,H'},\chi)}^H \circ Pr_{(H,\chi)} \\
&= \Delta_{(G_{H,H'},\chi)} \circ Pr_{(G_{H,H'},\chi)} \\
&= \Pi_{(G_{H,H'},\chi)}.
\end{aligned} \tag{42}$$

In the meantime, it is easy to check that

$$Pr_{G_{H,H'}/H} \circ \tau^H = \bigoplus_{\chi_j,\chi_r \in \left\{\lambda_i^{G_{H,H'}}\right\}_i} \tau_{\chi_j\chi_r}^H \circ Pr_{G_{H,H'}}^H. \tag{43}$$

37

Moreover, for any two $\chi_j, \chi_r \in \{\lambda_i^{G_{H,H'}}\}_i$, both of them are in $\{\lambda_i^H\}_i$ since $\{\lambda_i^{G_{H,H'}}\}_i \subseteq \{\lambda_i^H\}_i$. From the following commutative diagram of canonical projections of polynomial rings

$$
\begin{array}{ccc}
k[V] & \longrightarrow & k[W_{G_{H,H'}}] \\
& \searrow \quad \nearrow & \\
& k[W_H] &
\end{array}
$$

we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\left(V_{\chi_j^{-1}\lambda_r} \otimes k[V]\right)^G & \xrightarrow{\;\;\tau_{\chi_j,\chi_r}^{G_{H,H'}}\;\;} & \left(V_{\chi_j^{-1}\lambda_r} \otimes k[W_{G_{H,H'}}]\right)^{G/G_{H,H'}} \\
\quad \searrow \tau_{\chi_j,\chi_r}^H & & \nearrow \tau_{\chi_j,\chi_r}^{(G_{H,H'}/H)} \\
& \left(V_{\chi_j^{-1}\chi_r} \otimes k[W_H]\right)^{G/H} &
\end{array}
$$

That is,

$$
\tau_{\chi_j,\chi_r}^{G_{H,H'}} = \tau_{\chi_j,\chi_r}^{(G_{H,H'}/H)} \circ \tau_{\chi_j,\chi_r}^H. \tag{44}
$$

It suggests that

$$
\tau^{(G_{H,H'}/H)} = \tau^H \circ \bigoplus_{\chi_j,\chi_r \in \left\{\lambda_i^{G_{H,H'}}\right\}_i} \tau_{\chi_j\chi_r}^H. \tag{45}
$$

By the definition of $\zeta_{G_{H,H'}}^{\chi}$ (see (18)), we know that

$$
\zeta_{G_{H,H'}}^{\chi} = \tau^{G_{H,H'}} \circ \Pi_{(G_{H,H'},\chi)} \tag{46}
$$

and

$$
\zeta_{(G_{H,H'}/H)} \circ \zeta_H^{\chi} = \left(\tau^{(G_{H,H'}/H)} \circ Pr_{G_{H,H'}/H}\right) \circ \left(\tau^H \circ \Pi_{(H,\chi)}\right). \tag{47}
$$

Combining (42), (43) and (45), we have

$$
\begin{aligned}
&\left(\tau^{(G_{H,H'}/H)} \circ Pr_{G_{H,H'}/H}\right) \circ \left(\tau^H \circ \Pi_{(H,\chi)}\right) \\
&= \tau^{(G_{H,H'}/H)} \circ \bigoplus_{\chi_j,\chi_r \in \left\{\lambda_i^{G_{H,H'}}\right\}_i} \tau_{\chi_j\chi_r}^H \circ Pr_{G_{H,H'}}^H \circ \Pi_{(H,\chi)} \\
&= \tau^{G_{H,H'}} \circ \Pi_{(G_{H,H'},\chi)}.
\end{aligned}
$$

It follows that $\zeta_{(G_{H,H'}/H)} \circ \zeta_H^{\chi} = \zeta_{G_{H,H'}}^{\chi}$ from (46) and (47). We thus get (41), and the lemma follows. $\qquad\square$

E   Proof of Proposition 4.3

Let us first recall the following fact on $\underline{\mathrm{CM}}(S)$. For any Cohen-Macaulay S-module M, let $\Omega(M)$ be the object in $\underline{\mathrm{CM}}(S)$, which is the kernel of a(ny) surjection $\pi_M : S^{\oplus n} \twoheadrightarrow M$. Note that $\Omega(M)$ is unique in $\underline{\mathrm{CM}}(S)$, but not in $\mathrm{mod}(S)$. It is well-known that $[-1] = \Omega(-)$ in $\underline{\mathrm{CM}}(S)$.

For any two integers $l, l' \in \mathbb{Z}$ with $l' \geq l$, let $\widehat{M}(l, l') = \bigoplus_{i=l}^{l'} \widehat{M}[i]$ as above. Note that if $l' > 0 > l$, then $\widehat{M}[0] = \widehat{M}$ is a direct summand of $\widehat{M}(l, l')$. We have the following lemma.

Lemma E.1. $R^{\widehat{M}} = R^{\widehat{M}(l,l')}$ *for any* $l, l' \in \mathbb{Z}$ *with* $l' > 0 > l$.

Proof. First, similarly to (37) it is easy to check that

$$Z\left(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\left(\widehat{M} \oplus \Omega(\widehat{M})\right)\right) \subseteq Z\left(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\left(\widehat{M}\right)\right) \oplus Z\left(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\left(\Omega(\widehat{M})\right)\right) \qquad (48)$$

as algebras. When considering the associated reduced rings, since $\mathrm{End}_{\underline{\mathrm{CM}}(S)}\left(\Omega(\widehat{M})\right) \cong$ $\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})$ we have

$$R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq R^{\widehat{M}} \oplus R^{\Omega(\widehat{M})} \cong R^{\widehat{M}} \oplus R^{\widehat{M}}. \qquad (49)$$

Here, the compositions

$$R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq R^{\widehat{M}} \oplus R^{\Omega(\widehat{M})} \twoheadrightarrow R^{\widehat{M}} \qquad (50)$$

and

$$R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq R^{\widehat{M}} \oplus R^{\Omega(\widehat{M})} \twoheadrightarrow R^{\Omega(\widehat{M})} \qquad (51)$$

are exactly $\varphi^{\widehat{M} \oplus \Omega(\widehat{M})}_{\widehat{M}}$ and $\varphi^{\widehat{M} \oplus \Omega(\widehat{M})}_{\Omega(\widehat{M})}$ respectively, where the surjections are the natural projections. By Theorem 3.27, from (49) we get that

$$R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \oplus \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \qquad (52)$$

as algebras. In the following, we identify $R^{\widehat{M} \oplus \Omega(\widehat{M})}$ with the image of the above inclusion (52).

Observe that since $\varphi_{\Omega(\widehat{M})} = \varphi^{\widehat{M} \oplus \Omega(\widehat{M})}_{\Omega(\widehat{M})} \circ \varphi_{\widehat{M} \oplus \Omega(\widehat{M})}$ (see (51)), the composition with (51)

$$\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \xrightarrow{\varphi_{\widehat{M} \oplus \Omega(\widehat{M})}} R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq R^{\widehat{M}} \oplus R^{\Omega(\widehat{M})} \twoheadrightarrow R^{\Omega(\widehat{M})} \qquad (53)$$

is exactly $\varphi_{\Omega(\widehat{M})}$. Since the composition

$$\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \xrightarrow{\varphi_{\Omega(\widehat{M})}} R^{\Omega(\widehat{M})} \cong R^{\widehat{M}}$$

is exactly $\varphi_{\widehat{M}}$, by (53) and (50) the composition

$$\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \xrightarrow{\varphi_{\widehat{M} \oplus \Omega(\widehat{M})}} R^{\widehat{M} \oplus \Omega(\widehat{M})} \subseteq R^{\widehat{M}} \oplus R^{\Omega(\widehat{M})}$$
$$\cong R^{\widehat{M}} \oplus R^{\widehat{M}} \cong \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \oplus \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \qquad (54)$$

is the diagonal algebra homomorphism, where the isomorphism $R^{\widehat{M}} \cong \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda}$ is the inverse of $\varphi_{\widehat{M}}$.

In summary, we have the following commutative diagram of homomorphisms:

$$
\begin{array}{ccc}
& \varphi(S)\big/\sqrt{\varphi(S)\cap \Lambda e\Lambda} & \\
\varphi_{\widehat{M}} & \Big\downarrow \varphi_{\widehat{M}\oplus\Omega(\widehat{M})} & \varphi_{\Omega(\widehat{M})} \\
& R^{\widehat{M}\oplus\Omega(\widehat{M})} & \\
& \Big\downarrow \varphi_{\widehat{M}}^{\widehat{M}\oplus\Omega(\widehat{M})}\oplus\varphi_{\Omega(\widehat{M})}^{\widehat{M}\oplus\Omega(\widehat{M})} & \\
& R^{\widehat{M}}\oplus R^{\Omega(\widehat{M})} & \\
R^{\widehat{M}} & \xleftarrow{\quad\sim\quad} & R^{\Omega(\widehat{M})} \\
\Big\downarrow \varphi_{\widehat{M}}^{-1} & & \Big\downarrow \varphi_{\Omega(\widehat{M})}^{-1} \\
\varphi(S)\big/\sqrt{\varphi(S)\cap \Lambda e\Lambda} & \xrightarrow{\quad\mathrm{Id}\quad} & \varphi(S)\big/\sqrt{\varphi(S)\cap \Lambda e\Lambda}.
\end{array}
$$

Now, let $(\bar{f}_1,\bar{f}_2)\in R^{\widehat{M}\oplus\Omega(\widehat{M})}$ such that $\bar{f}_1,\bar{f}_2\in \varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}$ (see (52)). Since the composition (54) is the diagonal algebra homomorphism, the algebra homomorphism $\varphi_{\widehat{M}\oplus\Omega(\widehat{M})}$ takes $\bar{f}_1$ to $(\bar{f}_1,\bar{f}_1)\in R^{\widehat{M}\oplus\Omega(\widehat{M})}$. Then $(\bar{f}_1,\bar{f}_2)-(\bar{f}_1,\bar{f}_1)=(0,\bar{f}_2-\bar{f}_1)$ is also in $R^{\widehat{M}\oplus\Omega(\widehat{M})}$. Let $(\delta,g_{1,2})\in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}\oplus\Omega(\widehat{M})\big)\Big)$ whose image is $(0,\bar{f}_2-\bar{f}_1)$ under the quotient of nilradical. Here, $\delta\in Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big)$ and $g_{1,2}\in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\Omega(\widehat{M})\big)\Big)$.

Note that the nilradical of $Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}\oplus\Omega(\widehat{M})\big)\Big)$ is contained in the nilradical of $Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big)\oplus Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\Omega(\widehat{M})\big)\Big)$ by the inclusion (48). Moreover, the nilradical of $Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big)\oplus Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\Omega(\widehat{M})\big)\Big)$ is the direct sum of the nilradicals of $Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big)$ and $Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\Omega(\widehat{M})\big)\Big)$. It suggests that δ is in the nilradicals of $Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})$. Then, $\delta^{n'}=0$ in $Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(\widehat{M})\big)$ for some $n'\in\mathbb{N}$. It implies that

$$(\delta,g_{1,2})^{n'}=(0,g_{1,2}^{n'})\in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}\oplus\Omega(\widehat{M})\big)\Big).$$

Thus, for any two characters λ,λ' of G and $f\in\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}\big(M_\lambda,\Omega(M_{\lambda'})\big)\subseteq\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}\oplus\Omega(\widehat{M})\big)$, $g_{1,2}^{n'}\circ f=f\circ 0=0$. It is straightforward to see that $g_{1,2}$ is nilpotent, i.e., $\bar{f}_1-\bar{f}_2=0$ in $\varphi(S)/\sqrt{\varphi(S)\cap \Lambda e}$. It follows that $(\bar{f}_1,\bar{f}_2)=(\bar{f}_1,\bar{f}_1)$ in $R^{\widehat{M}\oplus\Omega(\widehat{M})}$. Thus, the image of the inclusion

$$R^{\widehat{M}\oplus\Omega(\widehat{M})}\subseteq \varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}\oplus\varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}$$

is exactly the image of the diagonal algebra homomorphism

$$\Delta_S:\varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}\hookrightarrow \varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}\oplus\varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}.$$

It follows that the image of $R^{\widehat{M}\oplus\Omega(\widehat{M})}$ is isomorphic to $\mathrm{Im}(\Delta_S)\cong \varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}$ as algebras. We then have the isomorphism:

$$R^{\widehat{M}\oplus\Omega(\widehat{M})}\cong \mathrm{Im}(\Delta_S)\cong \varphi(S)/\sqrt{\varphi(S)\cap \Lambda e\Lambda}\cong R^{\widehat{M}},$$

which, by (50), is given by $\varphi_{\widehat{M}}^{\widehat{M}\oplus\Omega(\widehat{M})}$. With the same argument, by induction on i we obtain

$$R^{\left(\bigoplus\limits_{i=l-l'}^{0}\widehat{M}[i]\right)} \cong R^{\widehat{M}}$$

given by $\varphi_{\widehat{M}}^{\left(\bigoplus\limits_{i=l-l'}^{0}\widehat{M}[i]\right)}$. Since $R^{\left(\bigoplus\limits_{i=l-l'}^{0}\widehat{M}[i]\right)} \cong R^{\left(\bigoplus\limits_{i=l}^{l'}\widehat{M}[i]\right)[l']} = R^{\widehat{M}(l,l')}$, we get $R^{\widehat{M}(l,l')} \cong R^{\widehat{M}}$ given by $\varphi_{\widehat{M}}^{\widehat{M}(l,l')}$. $\qquad\square$

Now, let N be a Cohen-Macaulay S-module. From §3.2, we know that $\widehat{M}$ is a classical generator of $\underline{\mathrm{CM}}(S)$. Then we have $N \in \langle \widehat{M} \rangle_m$ for some $m \in \mathbb{N}$. It suggests that there is a sequence of distinguished triangles:

$$\left\{ M_N^{r-1} \to M_N^r \to W_N^{r-1} \to M_N^{r-1}[1] \right\}_{m \geq r \geq 2}$$

in $\underline{\mathrm{CM}}(S)$ where $M_N^m := N$, $M_N^{r-1} \in \langle \widehat{M} \rangle_{r-1}$ and $W_N^{r-1} \in \langle \widehat{M} \rangle_1$. Here, for the triangulated category $\mathcal{T}$ and a class of objects $\mathcal{E}$ in $\mathcal{T}$, recall that $\langle \mathcal{E} \rangle_1$ is the smallest full subcategory of $\mathcal{T}$ containing the objects in $\mathcal{E}$ and closed under direct summands, finite direct sums and shifts.

Suppose that $\bigoplus_{r=2}^m W_N^{r-1} \oplus M_N^1$ is a direct summand of some direct sums of $\bigoplus_{i=v}^{v'} \widehat{M}[i]$ for some $v', v \in \mathbb{Z}$. Let $l', l \in \mathbb{Z}$ such that $l' \geq v' \geq v \geq l$ and $l' > 0 > l$. We have the following.

Lemma E.2. *For any nontrivial $f \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, N)$ which is not nilpotent, there is an element $g \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, \widehat{M}[i])$ for some $i \in [l, l']$ such that $g \circ f^j$ is nontrivial for any $j \in \mathbb{N}$, where f^j is the i-fold self-composition of f.*

Proof. We prove by contradiction. Assume that there is an element $f \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, N)$, which is not nilpotent, such that for any $i \in [l, l']$ and any $g \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, \widehat{M}[i])$, $g \circ f^j = 0$ for some $j \in \mathbb{N}$. It implies that for any $g' \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}\big(N, \bigoplus_{i=l}^{l'} \widehat{M}[i]\big)$, $g \circ f^j = 0$ for some $j \in \mathbb{N}$.

We first consider the distinguished triangle $M_N^1 \to M_N^2 \to W_N^1 \to M_N^1[1]$ in $\underline{\mathrm{CM}}(S)$. By applying the triangle functor $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, -)$ to this distinguished triangle, we have the following long exact sequence

$$\cdots \to \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^1) \to \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^2) \to$$
$$\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, W_N^1) \to \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^1[1]) \to \cdots.$$

Denote by ϱ and ε the above morphisms $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^1) \to \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^2)$ and $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^2) \to \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, W_N^1)$ respectively. Let $h \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^2)$. By assumption, we have $\varepsilon(h) \circ f^j = 0$ for some $j \in \mathbb{N}$, since W_N^1 is a direct summand of $\bigoplus_{i=l}^{l'} \widehat{M}[i]$. It implies that

$$\varepsilon(h \circ f^j) = 0$$

since ε is a $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, N)$-module homomorphism. Then, from the above long exact sequence, there is an element, say $h' \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^1)$, such that $\varrho(h') = h \circ f^j$. By our assumption again, we also have $h' \circ f^{j'} = 0$ for some $j' \in \mathbb{N}$ since M_N^1 is a direct summand of $\bigoplus_{i=l}^{l'} \widehat{M}[i]$. In the meantime, we know that ϱ is also a $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, N)$-module homomorphism. Then we get that

$$h \circ f^{j+j'} = h \circ f^j \circ f^{j'} = \varrho(h') \circ f^{j'} = \varrho(h' \circ f^{j'}) = 0.$$

Next, consider all distinguished triangles in

$$\left\{ M_N^{r-1} \to M_N^r \to W_N^{r-1} \to M_N^{r-1}[1] \right\}_{m \geq r \geq 2}$$

in $\underline{\mathrm{CM}}(S)$. By induction on r, we get that for any M_N^r, and any $\alpha \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, M_N^r)$, $\alpha \circ f^{j_r} = 0$ for some $j_r \in \mathbb{N}$. If $r = m$, let α be the identity morphism in $\mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, N)$ since $N = M_N^m$. Then we obtain that

$$f^{j_m} = \alpha \circ f^{j_m} = 0.$$

It implies that f is a nilpotent element, which contradicts to the assumption that f is not a nilpotent element. $\qquad\square$

Proof of Proposition 4.3. The diagram is naturally commutative. Thus, to prove this proposition, we only need to show that the algebra homomorphisms in this diagram are all isomorphisms.

Fix the Cohen-Macaulay S-module N and keep the above settings for l, l' (see the paragraph above Lemma E.2). First, similarly to (37), it obvious that

$$Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l, l') \oplus N\big)\Big) \subseteq Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l, l')\big)\Big) \oplus Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(N)\Big)$$

as algebras. Moreover, when considering the corresponding reduced rings, by Lemma E.1 we have

$$R^{\widehat{M}(l,l') \oplus N} \subseteq R^{\widehat{M}(l,l')} \oplus R^N \cong R^{\widehat{M}} \oplus R^N, \tag{55}$$

which is exactly $\varphi_{\widehat{M}(l,l')}^{\widehat{M}(l,l') \oplus N} \bigoplus \varphi_N^{\widehat{M}(l,l') \oplus N}$, where $R^{\widehat{M}(l,l')} \cong R^{\widehat{M}}$ is given by $\varphi_{\widehat{M}}^{\widehat{M}(l,l')}$. We identify $R^{\widehat{M}(l,l') \oplus N}$ with the image of the above inclusion (55).

Let $(\bar{g}_1, \bar{g}_2) \in R^{\widehat{M}} \oplus R^N$ be in $R^{\widehat{M}(l,l') \oplus N}$. By Theorem 3.27, we have an isomphism $\varphi_{\widehat{M}} : \varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \cong R^{\widehat{M}}$. Let $\varphi_{\widehat{M}}^{-1}$ be its inverse morphism. The composition of

$$\varphi(S)/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \xrightarrow{\varphi_{\widehat{M}(l,l') \oplus N}} R^{\widehat{M}(l,l') \oplus N}$$

with (55) is $\varphi_{\widehat{M}(l,l')}^{\widehat{M}(l,l') \oplus N} \bigoplus \varphi_N^{\widehat{M}(l,l') \oplus N}$, and the image of $\varphi_{\widehat{M}}^{-1}(\bar{g}_1)$ under this composition is exactly $\big(\bar{g}_1, \varphi_N(\varphi_{\widehat{M}}^{-1}(\bar{g}_1))\big)$. Here, we use the fact that the composition $R^{\widehat{M}(l,l') \oplus N} \subseteq R^{\widehat{M}(l,l')} \oplus R^N \twoheadrightarrow R^{\widehat{M}(l,l')} \cong R^{\widehat{M}}$ is exactly $\varphi_{\widehat{M}}^{\widehat{M}(l,l') \oplus N}$, and the composition $R^{\widehat{M}(l,l') \oplus N} \subseteq R^{\widehat{M}(l,l')} \oplus R^N \twoheadrightarrow R^N$ is exactly $\varphi_N^{\widehat{M}(l,l') \oplus N}$. Thus we obtain that $\big(\bar{g}_1, \varphi_N(\varphi_{\widehat{M}}^{-1}(\bar{g}_1))\big) \in R^{\widehat{M}} \oplus R^N$ is also in $R^{\widehat{M}(l,l') \oplus N}$. Therefore,

$$(\bar{g}_1, \bar{g}_2) - \big(\bar{g}_1, \varphi_N(\varphi_{\widehat{M}}^{-1}(\bar{g}_1))\big) = \big(0, \bar{g}_2 - \varphi_N(\varphi_{\widehat{M}}^{-1}(\bar{g}_1))\big) \in R^{\widehat{M}} \oplus R^N$$

is also in $R^{\widehat{M}(l,l') \oplus N}$.

Let $(\varepsilon, f_{1,2}) \in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l, l') \oplus N\big)\Big)$ whose image is $\big(0, \bar{g}_2 - \varphi_N(\varphi_{\widehat{M}}^{-1}(\bar{g}_1))\big)$ under the quotient by the nilradical

$$Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l, l') \oplus N\big)\Big) \twoheadrightarrow R^{\widehat{M}(l,l') \oplus N}. \tag{56}$$

Here, $\varepsilon \in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l')\big)\Big)$ and $f_{1,2} \in Z\big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}(N)\big)$. From the following commutative diagram

$$
\begin{array}{ccc}
Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l') \oplus N\big)\Big) & \xrightarrow{\;\widetilde{\phi}^{\,\widehat{M}(l,l')\oplus N}_{\widehat{M}(l,l')}\;} & Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l')\big)\Big) \\
\downarrow & & \downarrow \\
R^{\widehat{M}(l,l')\oplus N} & \xrightarrow[\;\varphi^{\widehat{M}(l,l')\oplus N}_{\widehat{M}(l,l')}\;]{} & R^{\widehat{M}(l,l')},
\end{array}
$$

where the vertical morphisms are projections to the quotients by the nilradicals respectively, the morphism $Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l')\big)\Big) \twoheadrightarrow R^{\widehat{M}(l,l')}$ takes ε to $0 \in R^{\widehat{M}(l,l')}$, since the image of $(\varepsilon, f_{1,2})$ is $\Big(0, \bar{g}_2 - \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big)\Big)$ under morphism (56). Here, note that $\widetilde{\phi}^{\,\widehat{M}(l,l')\oplus N}_{\widehat{M}(l,l')}(\varepsilon, f_{1,2}) = \varepsilon$ and $\varphi^{\widehat{M}(l,l')\oplus N}_{\widehat{M}(l,l')}\Big(0, \bar{g}_2 - \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big)\Big) = 0$. Then ε is a nilpotent element in $Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l')\big)\Big)$; that is, $\varepsilon^n = 0$ for some $n \in \mathbb{N}$. Thus

$$
(\varepsilon, f_{1,2})^n = (0, f_{1,2}^n) \in Z\Big(\mathrm{End}_{\underline{\mathrm{CM}}(S)}\big(\widehat{M}(l,l') \oplus N\big)\Big).
$$

Note that $f_{1,2}^n \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N,N)$. Therefore, for any $g \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, \widehat{M}(l,l'))$, we have that $g \circ f_{1,2}^n = g \circ 0 = 0$. It implies that for any $i \in [l, l']$ and for any $g' \in \mathrm{Hom}_{\underline{\mathrm{CM}}(S)}(N, \widehat{M}[i])$, $g' \circ f_{1,2}^n$ is trivial since $\widehat{M}[i]$ is a direct summand of $\widehat{M}(l,l')$.

By Lemma E.2, we have that $f_{1,2}$ is nilpotent, i.e., $f_{1,2}^{n'} = 0$ for some $n' \in \mathbb{N}$. Then we get that $(\varepsilon, f_{1,2})^{n \cdot n'} = 0$. It suggests that $\Big(0, \bar{g}_2 - \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big)\Big) = 0$ in $R^{\widehat{M}(l,l')\oplus N}$. Since $R^{\widehat{M}(l,l')\oplus N} \subseteq R^{\widehat{M}(l,l')} \oplus R^N$, $\Big(0, \bar{g}_2 - \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big)\Big) = 0$ in $R^{\widehat{M}(l,l')} \oplus R^N$. It follows that

$$
\bar{g}_2 - \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big) = 0
$$

in R^N since $(\varepsilon, f_{1,2})$ is a nilpotent. Then $(\bar{g}_1, \bar{g}_2) = (\bar{g}_1, \varphi_N\big(\varphi_{\widehat{M}}^{-1}(\bar{g}_1)\big))$. Thus $(\bar{g}_1, \bar{g}_2)$ is the image of $\varphi_{\widehat{M}}^{-1}(\bar{g}_1)$ under

$$
\varphi(S)\big/\sqrt{\varphi(S) \cap \Lambda e \Lambda} \xrightarrow{\;\varphi_{\widehat{M}(l,l')\oplus N}\;} R^{\widehat{M}(l,l')\oplus N} \subseteq R^{\widehat{M}(l,l')} \oplus R^N \cong R^{\widehat{M}} \oplus R^N.
$$

By the arbitrariness of $(\bar{g}_1, \bar{g}_2)$, we get that $\varphi_{\widehat{M}(l,l')\oplus N}$ is a surjection.

On the other hand, from

$$
\varphi^{\widehat{M}(l,l')\oplus N}_{\widehat{M}} \circ \varphi_{\widehat{M}(l,l')\oplus N} = \varphi_{\widehat{M}},
$$

we get $\varphi_{\widehat{M}(l,l')\oplus N}$ is an injection, since by Theorem 3.27 $\varphi_{\widehat{M}}$ is an isomorphism of algebras.

We thus get that $\varphi_{\widehat{M}(l,l')\oplus N}$ is an isomorphism of algebras. It implies that $\varphi^{\widehat{M}(l,l')\oplus N}_{\widehat{M}}$ is also an isomorphism of algebras by the commutativity of this diagram. $\qquad\square$

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