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Selective Inference for Time-Varying Effect Moderation

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Abstract

Causal effect moderation investigates how the effect of interventions (or treatments) on outcome variables changes based on observed characteristics of individuals, known as potential effect moderators. With advances in data collection, datasets containing many observed features as potential moderators have become increasingly common. High-dimensional analyses often lack interpretability, with important moderators masked by noise, while low-dimensional, marginal analyses yield many false positives due to strong correlations with true moderators. In this paper, we propose a two-step method for selective inference on time-varying causal effect moderation that addresses the limitations of both high-dimensional and marginal analyses. Our method first selects a relatively smaller, more interpretable model to estimate a linear causal effect moderation using a Gaussian randomization approach. We then condition on the selection event to construct a pivot, enabling uniformly asymptotic semi-parametric inference in the selected model. Through simulations and real data analyses, we show that our method consistently achieves valid coverage rates, even when existing conditional methods and common sample splitting techniques fail. Moreover, our method yields shorter, bounded intervals, unlike existing methods that may produce infinitely long intervals.

Keywords: effect moderation, selective inference, randomization, semi-parametric inference

1 Introduction

Causal effect moderation investigates how the effect of interventions (or treatments) on outcome variables changes based on observed characteristics of individuals, known as potential effect moderators. The motivating example for this paper comes from mobile health (mHealth) studies in which real-time interventions are provided and scientists wish to evaluate their time-varying effect on health outcomes. The motivating example is based on VALENTINE, an mHealth clinical trial involving individuals enrolled in a cardiac rehabilitation program. Individuals were repeatedly randomized to receive either a digital intervention (e.g., a push notification) designed to increase physical activity during the program or nothing. As an indicator of exercise, the study recorded step count in the hour following potential treatment, the outcome of interest. A scientific goal is to understand the impact of features such as individual traits, external variables, and previously recorded treatment responses for the effect of digital interventions on the target outcome.

Current methods consider low-dimensional moderation analysis in which the set of potential moderators is chosen beforehand and effects are considered marginal over observed and unobserved variables not included in this set. With advancements in data acquisition, datasets with many observed features as potential moderators have become more common. For example, in the VALENTINE study, the set of potential moderators includes individual traits such as demographics, time of enrollment, activity summaries, external readings such as time and day of the week, environmental readings such as weather and type of locations, and past treatment responses. In such cases, researchers want to answer questions regarding high-dimensional effect moderation, requiring identification of the important moderators from the large pool of potential ones. Using penalized regression, like the lasso, for feature selection can help identify relevant effect moderators. While this data-driven approach would result in an interpretable moderation analysis, naive causal inferences based on the

selected effect moderators would lead to overly optimistic p-values and confidence intervals that are narrower than desired.

One way to conduct valid inference after data-driven model selection is conditional selective inference. First introduced for inference after lasso (Lee et al., 2016), this approach utilizes a truncated normal distribution for exact inference. Suppose that the lasso selects a set of variables E and our goal is to infer for a set of post-selection parameters $\{\beta^{E\cdot j}: j \in E\}$, which represent the effects of the selected variables. Then, a selectionadjusted distribution is obtained by conditioning on the selection event $\{\hat{E} = E\}$, where \hat{E} represents the random variable for the selected set of variables. This conditional distribution enables the construction of selective confidence intervals $\{\hat{C}^{\hat{E}\cdot j}: j \in \hat{E}\}$, where $\hat{C}^{\hat{E}\cdot j}$ is the interval constructed for $\beta^{E\cdot j}$. The fundamental principle behind this conditioning, which ensures valid inference, is that

$$\mathbb{P}\left[\beta^{\widehat{E}\cdot j}\in\widehat{C}^{\widehat{E}\cdot j}\mid\widehat{E}=E\right]\geq 1-\alpha\implies\mathbb{P}\left[\beta^{\widehat{E}\cdot j}\in\widehat{C}^{\widehat{E}\cdot j}\right]\geq 1-\alpha,\tag{1}$$

due to the tower property of expectation. Obviously, the conditional guarantee on the lefthand side ensures coverage at the desired level for each individual post-selection parameter. As shown by Lee et al. (2016), it also controls the false coverage rate (FCR), defined in Benjamini and Yekutieli (2005) as

$$\mathbb{E}\left[\frac{\left|\left\{j\in\widehat{E}:\beta^{\widehat{E}\cdot j}\notin\widehat{C}^{\widehat{E}\cdot j}\right\}\right|}{\max(|\widehat{E}|,1)}\right],\$$

representing the expected proportion of miscoverage for the |E| post-selection parameters.

Building on this conditional approach, initially proposed for a normal response, asymptotic selective inference for effect modification problems was developed in Zhao et al. (2021). However, as demonstrated in Table 1, we note that inference based on this approach fails to remain valid for certain parameter values. For this first example, we compute average coverage of interval estimates with nominal FCR level of 0.10 for the proposed and existing methods over 500 Monte-Carlo simulations. Data was generated with a sparse signal on a

	Proposed 'SI'			Polyhedral		Splitting	
	Coverage	CI Length	Coverage	Finite Length	% of Finite CI	Coverage	CI Length
Low Signal	0.895	0.778	0.779	1.53	84 %	0.86	0.75
High Signal	0.901	0.552	0.894	0.497	94~%	0.872	0.734

300-by-30 design with Laplace errors. Details of the setting are discussed later in Section 6 of the paper.

Table 1: Comparison of Coverage and CI Length under Proposed, Polyhedral, and Splitting Methods

In the high signal scenario, all three methods achieve nominal FCR but the polyhedral method produces some infinitely long intervals. In the low signal scenario, the polyhedral method fails to achieve nominal FCR. We note that our new selective inference method, abbreviated as "SI", achieves the desired false coverage rate (FCR) in both scenarios, with intervals that are always bounded and significantly narrower than those produced by the polyhedral method. Another conditional method for selective inference is the widely used data splitting, which conditions on all data used for selection, forming a superset of the selection event $\{\hat{E} = E\}$. As a result, data splitting discards all the data used during selection, leading to intervals that do not adapt to the observed selection event. As shown in Table 1, the intervals derived from data splitting are significantly longer than those produced by our "SI" method and even exhibit slight undercoverage in the low signal regime, possibly due to insufficient samples for inference. This table provides strong evidence that "SI" offers a more reliable as well as a more powerful alternative when compared to the existing conditional methods.

The new methodology relies on a randomization scheme that adds independent Gaussian noise to the penalized estimation objective, as proposed in the work of Panigrahi et al. (2023b,c). Similar to data splitting, the added Gaussian randomization provides analysts with a flexible lever to choose the amount of data used for model selection versus infer-

ence, enabling a trade-off between the predictive accuracy of the model, and inferential reliability and power in the selected model. However, unlike splitting, our method conditions on significantly less information, resulting in narrower intervals for the post-selection parameters than the non-adaptive intervals produced by data splitting. Following the approach in Panigrahi et al. (2023a) for Gaussian linear regression, we marginalize over the additional randomization to obtain a simple pivot from a bivariate truncated normal distribution. In contrast to the aforementioned references, our paper develops an asymptotic theory to demonstrate the weak convergence of this pivot to a uniform random variable, without imposing parametric assumptions on the distribution of the observed data. More precisely, this allows for valid semi-parametric guarantees of inference for the causal moderation analysis that holds uniformly across a large class of distributions with mild moment conditions.

2 Preliminaries

2.1 Notations and framework

Consider data with n i.i.d samples where each sample, for $i \in [n] := \{1, \ldots, n\}$, contains longitudinal observations $\{X_{i,t}, A_{i,t}, Y_{i,t}\}_{t=1}^{T}$ collected on the *i*-th individual for the decision times $t = 1, \cdots, T$. Here $Y_{i,t} \in \mathbb{R}$ is the response, $A_{i,t} \in \{0,1\}$ is a binary treatment and $X_{i,t} \in \mathbb{R}^p$ contains covariates measured prior to the treatment assignment. Let the overbar notation represent a sequence of variables (or their realized values) through time, for example, $\bar{A}_{i,t} = \{A_{i,1}, \ldots, A_{i,t}\}$ denotes the sequence of treatment variables up to time t. Then the complete history of an individual i till time t, is denoted as $H_{i,t} = \{\bar{X}_{i,t}, \bar{A}_{i,t-1}, \bar{Y}_{i,t-1}\}$, i.e., all information prior to the t-th treatment. For a given subject i, at each $t \in [T], X_{i,t}$ denotes the covariates information collected from time t - 1 to t, prior to the treatment assignment $A_{i,t}$. The treatment $A_{i,t} \in \{0, 1\}$ may depend on the complete history $H_{i,t}$ and is designed to impact the target response $Y_{i,t} \in \mathbb{R}$ which is observed just after the treatment

intervention at time t. This is a typical set-up for examining time-varying causal effect moderation, which is particularly relevant in mHealth applications where inference about which factors moderate the response to treatments is desired. In the VALENTINE Study, for example, the scientific team is interested in understanding the impact of different push notifications on a patient's proximal step counts and whether the effect is moderated by observed individual characteristics and potentially time-varying contextual factors.

A standard way to collect longitudinal data to answer these types of questions is by designing a sequentially randomized trial, where treatments are assigned randomly through time. Alternatively such data can come from an observational study, in which case the treatment distribution $p_{i,t} = (A_{i,t}|H_{i,t})_{i=1}^n$ is unknonw and must be estimated from the data. We adopt the framework of Micro-randomized trials (MRTs), which are experimental trials specifically designed to collect data to answer scientific questions concerning causal moderation. In an MRT, each participant *i* is assigned sequentially randomized treatments $A_i = (A_{i,t})_{t=1}^T$ following randomization probabilities $\mathbf{p}_i = \{p_{i,t} (A_{i,t} | H_{i,t})\}_{t=1}^T$. These probabilities are typically pre-specified and so are assumed to be known or correctly specified by a parametric family.

In order to define the causal estimand, we adopt the standard potential outcomes framework Rubin (2005). For the *i*-th individual, define $X_{i,t}(\bar{a}_{i,t-1})$ as the potential information that would have been observed if the individual had been assigned the treatment sequence $\bar{a}_{i,t-1} \in \{0,1\}^{\otimes(t-1)}$. Under the sequence of treatments $\{\bar{a}_{i,t-1}, a_{i,t}\}$, let $Y_{i,t}(\bar{a}_{i,t-1})$ denote the potential outcome at time *t*. The potential value of the history variable under the treatment sequence $\bar{a}_{i,t-1}$ at time *t* is denoted by $H_{i,t}(\bar{a}_{i,t-1})$. Now let $S_{i,t}(\bar{a}_{t-1})$ be a vector of deterministic summaries derived from $H_{i,t}(\bar{a}_{t-1})$ which we consider as the potential time-varying effect moderators.

In the rest of the paper, we assume that the potential outcomes are i.i.d. over individuals. Then, we define the causal moderated effect on $Y_{i,t}$ as:

$$\beta^*(t;s) = \mathbb{E}_{\mathbf{p}} \left[Y_{i,t} \left(\bar{A}_{i,t-1}, A_{i,t} = 1 \right) - Y_{i,t} \left(\bar{A}_{i,t-1}, A_{i,t} = 0 \right) \mid S_{i,t} \left(\bar{A}_{i,t-1} \right) = s \right].$$
(2)

The expectation in this estimand is taken with respect to the joint distribution of the treatment sequence $\bar{A}_{i,t-1} = \{A_{i,1}, A_{i,2}, \ldots, A_{i,t-1}\}$. This is emphasized through the subscript $\mathbf{p} = \{p_t (A_{i,t} \mid H_{i,t})\}_{t=1}^T$, which is the full sequence of randomization probabilities in the MRT. The causal estimand in (2), first introduced in Boruvka et al. (2018), has been termed a causal excursion effect in the literature.

Next, we state the three fundamental assumptions to reformulate the causal excursion effect in terms of the observed data Robins (1997):

Assumption 1. We assume consistency, positivity, and sequential ignorability. For $t = 1, \ldots, T$:

- 1. Consistency: $\{X_{i,t}(\bar{A}_{i,t-1}), A_{i,t}(\bar{A}_{i,t-1}), Y_{i,t}(\bar{A}_{i,t})\} = \{X_{i,t}, A_{i,t}, Y_{i,t}\}, i.e., observed values equal the corresponding potential outcomes;$
- 2. **Positivity:** $\mathbb{P}(A_{i,t} = a \mid H_{i,t} = h) > 0$ given that the joint density for $\{A_{i,t}, H_{i,t}\}$ when evaluated at (a, h), is greater than zero;
- 3. Sequential ignorability: Conditional on the observed history $H_{i,t}$,

 $\{Y_{i,t}(\bar{a}_{i,t}), X_{i,t+1}(\bar{a}_{i,t}), A_{i,t+1}(\bar{a}_{i,t}), \dots, Y_{i,T}(\bar{a}_{i,T-1})\} \perp A_{i,t}, \}$

i.e., the potential outcomes are independent of the current treatment given the past history.

Under Assumption 1, the causal excursion effect can be expressed as

$$\beta^{*}(t;s) = \mathbb{E}\left[\mathbb{E}_{\mathbf{p}}\left[Y_{i,t+1} \mid A_{i,t} = 1, H_{i,t}\right] - \mathbb{E}_{\mathbf{p}}\left[Y_{i,t+1} \mid A_{i,t} = 0, H_{i,t}\right] \mid S_{i,t} = s\right], \quad (3)$$

where the inner expectations are over the outcome conditional on the history H_t and current treatment A_t , and the outer expectation marginalizes over the history except for those variables included in $S_{i,t}$.

Note that if T = 1, i.e., there is no longitudinal dependence, our set-up resembles that of *effect modification*, as discussed in Zhao et al. (2021). With data n i.i.d observations

 $\{X_i, A_i, Y_i\}_{i=1}^n$, using the potential outcomes framework (Rubin, 2005), let $Y_i(a)$ denote the counterfactual outcome if the treatment is set to $a \in \{0, 1\}$. Then the conditional average treatment effect (CATE) is defined as $\Delta(x) = \mathbb{E}[Y_i(1) - Y_i(0)|X_i = x]$. When T = 1, Assumption 1 corresponds with the standard causal inference assumptions made in Zhao et al. (2021), and (3) corresponds to the CATE. When T > 1, the causal excursion effect marginalizes over treatments not contained in $S_{i,t}$ and thus depends on the treatment assignment distribution $\mathbf{p} = (\mathbf{p}_i)_{i=1}^n$. Unlike the standard CATE, marginalization over different probabilistic assignment of treatments may yield different results.

2.2 Excursion effect estimation

We assume a linear model for the causal excursion effect:

$$\beta^*(t;s) = f_t(s)^\top \beta,$$

where $f_t(s) \in \mathbb{R}^p$ is a *p*-dimensional feature vector constructed from the effect moderators $S_t = s$ and time *t*. Typically, $f_t(S_{i,t})$ involves the linear and non-linear functions of each moderator, their potential interactions, and interactions with time. In this model, a consistent estimator for β can be obtained from a sample of *n* individuals by minimizing a weighted and centered least squares objective (Boruvka et al., 2018; Shi et al., 2022), in short called the WCLS. As the name suggests, this estimation method relies primarily on two principles: centering of the treatment indicators and weighting of the estimating function. As the moderated effects are marginalized over the history $H_{i,t}$ which is not part of the summary $S_{i,t}$, the weights are just the ratio of the randomization probabilities given $H_{i,t}$ as denominator and pseudo-randomization probabilities given $S_{i,t} \subset H_{i,t}$ as numerator. Note that as long as the randomization probabilities **p** depend on history $H_{i,t}$ through the summary variables $S_{i,t}$ only, the weights can be expressed as $W_{i,t} = \tilde{p}(A_{i,t}|S_{i,t})/p(A_{i,t}|H_{i,t})$, with an arbitrary choice of $\tilde{p}(A_{i,t}|S_{i,t}) \in (0, 1)$.

A main challenge in effect moderation inference is the necessary estimation of nuisance parameters. In Boruvka et al. (2018), action centering allows simultaneous estimation of the

causal parameter β and a linear working model for the nuisance function, $\mathbb{E}[W_{i,t}Y_{i,t}|H_{i,t}] = g_t(H_t)^{\top}\alpha$ where $g_t(H_t)$ are features extracted from the history. In this paper, we use Neyman orthogonality as in Shi and Dempsey (2023) to eliminate the nuisance parameter. We partition our data independently into two parts based on individuals. Then we use the first independent part of data estimate the above nuisance parameter by estimating $\mathbb{E}[Y_{i,t}|H_{i,t}, A_{i,t}]$. Estimation can leverage machine learning methods with good prediction performance. Let $\tilde{g}_t(h, a)$ denote the estimated nuisance parameter and define $\tilde{g}_t(h) = \sum_{a \in \{0,1\}} \tilde{p}(a|s)\tilde{g}_t(h, a)$ to be the estimate of the nuisance function. Then, we plug in these estimates to the WCLS objective and solve:

$$\widehat{\beta} = \underset{b}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(Y_{i,t} - \tilde{g}_t(H_{i,t}) - (A_{i,t} - \tilde{p}(1|S_{i,t})) f_t(S_{i,t})^\top b \right)^2 W_{i,t}$$
(4)

using the rest data. The above procedure can accommodate sample splitting in which the roles of data parts are reversed and (4) uses all data, as well as a larger, fixed number of splits K. We focus on the two-stage approach for simplicity. In either case, WCLS provides a consistent estimate for the moderated effect β even if the nuisance function is misspecified, assuming known randomization probabilities. However, good estimates of the nuisance function will improve efficiency of the estimator.

In the observational setting, one must estimate the randomization probabilities. In such settings, the error

$$B_n = \sum_{i=1}^n \sum_{t=1}^T \left\{ \|\tilde{p}(1|H_{i,t}) - p(1|H_{i,t})\| \sum_{a \in \{0,1\}} \|\tilde{g}(H_{i,t},a) - g(H_{i,t},a)\| \right\}$$

must scale as $o_p(n^{-1/2})$ to ensure asymptotic normality of the WCLS estimator (Shi and Dempsey, 2023). This is guaranteed under known randomization probabilities (i.e., MRTs) or correctly specified parametric models for the randomization probabilities. Specifically,

if
$$\widehat{B}_n = o_p(n^{-1/2})$$
 then $\sqrt{n}(\widehat{\beta} - \beta) \to \mathcal{N}(0, \Sigma)$ with covariance $\Sigma = Q^{-1}WQ^{-1}$ where

$$O = \mathbb{E}\left[\sum_{j=1}^{T} \widetilde{z}_j^2(S_j) f(S_j) f(S_j)^{\top}\right]$$

$$Q = \mathbb{E} \left[\sum_{t=1}^{T} \sigma_t(S_{i,t}) f_t(S_{i,t}) f_t(S_{i,t}) \right],$$

$$W = \mathbb{E} \left[\sum_{t=1}^{T} \tilde{\sigma}_t^2(S_{i,t}) (\beta(t; S_{i,t}) - f_t(S_{i,t})^\top \beta) f_t(S_{i,t}) \times \sum_{t=1}^{T} \tilde{\sigma}_t^2(S_{i,t}) (\beta(t; S_{i,t}) - f_t(S_{i,t})^\top \beta) f_t(S_{i,t})^\top \right]$$

where $\tilde{\sigma}^2(S_{i,t}) := \tilde{p}_t(1|S_{i,t})(1 - \tilde{p}_t(1|S_{i,t})).$

2.3 Why is excursion effect modelled linearly?

As emphasized in the introduction, when there is a large number of moderators, i.e., $f_t(s) \in \mathbb{R}^p$ with large p, there is a trade-off between prediction accuracy and interpretability of $\beta^*(t; s)$. If the only goal is prediction, one might want to use more flexible machine learning methods to model the causal excursion effects in order to achieve better prediction accuracy. However, these black box models lack interpretability and it remains unclear how to quantify their uncertainty. To step away from black box approaches, one may propose a high-dimensional linear model. Such models, however, still suffer from lack of interpretability, and can lead to important moderators being masked by noise covariates. An alternative is to consider each component of $S_{i,t}$ separately, performing marginal analyses for each component. This approach is incredibly common in current analysis of MRTs. However, such marginal analyses can discover many false positives due to strong correlations with true effect moderators.

In this paper, we adopt a two-step method to address the limitations of both highdimensional and marginal analyses. First, we select a relatively smaller, more interpretable model to estimate the causal excursion effect. Then, we develop selective inference tools to construct confidence intervals for the coefficients in this selected model. In the next section, we provide an outline of this two-step method.

3 Two-step method for modeling and excursion effect estimation

In the first step, we use penalized regression to select relevant moderators

$$\widehat{E}\left(\{\bar{X}_{i,T}, \bar{A}_{i,T}, \bar{Y}_{i,T}\}_{i=1}^n\right) \subseteq \{1, \dots, p\}$$

where $\bar{X}_{i,T} = (X_{i,1}, \ldots, X_{i,T})$ is the observed sequence of covariates through time T and $\bar{A}_{i,T}$ and $\bar{Y}_{i,T}$ are defined similarly. A popular penalized regression approach for conducting variable selection is the Least Absolute Shrinkage and Selection Operator (LASSO). In our problem, this involves adding the lasso penalty to the WCLS criterion and solving the minimization objective

$$\min_{\beta} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(Y_{i,t} - \tilde{g}_t(H_{i,t}) - (A_{i,t} - \tilde{p}(1|S_{i,t})) f_t(S_{i,t})^\top \beta \right)^2 W_{i,t} + \lambda \|\beta\|_1 \right\}.$$

We select the moderators in $f_t(S_{i,t}) \in \mathbb{R}^p$ based on the variables with non-zero lasso coefficients. Let

$$\left\{\widehat{E}\left(\{\bar{X}_{i,T}, \bar{A}_{i,T}, \bar{Y}_{i,T}\}_{i=1}^n\right) = E\right\}$$

denote the identified a set of potentially important moderators, and $f_t^E(S_{i,t}) \in \mathbb{R}^{|E|}$ denote the restriction of the moderators to the selected set. We then consider the working model

$$\beta^{E}(t;s) = \left(f_{t}^{E}(s)\right)^{\top} \beta^{E}$$

Although this approach results in a relatively simpler and easier-to-interpret low-dimensional model, making inferences for β^E requires adjustments for the data-driven model selection procedure in the first step. To estimate the CATE in the effect modification model, Zhao et al. (2021) applied the lasso for feature selection and proposed a polyhedral based approach Lee et al. (2016) for selective inference. As shown in our first example in Section 1, this approach can fail to attain the desired level of coverage and may lead to infinitely long confidence intervals.

Based on the selected model, denote the post-selection WCLS estimator as $\widehat{\beta}^{E}(t;s)$. Note that the post-selection WCLS estimator and its target causal effect $\beta^{E}(t;s)$ both depend on the data-driven selection of $\{\widehat{E} = E\}$. Currently, however, there are no tools for uncertainty quantification of the post-selection estimators. Data splitting could be employed but we demonstrate in the first example that this results in less efficient inference relative to our proposed method as it discards all the data used for selection. In the next section, the randomized LASSO is used for selection of moderators to include in the causal model and a selective inference method with asymptotic conditional guarantees is presented.

3.1 Moving from excursion effect estimation to a general framework

Here, we translate the linear causal excursion effect procedure to a generic notional framework for our selective inference method, so that it can be applied in more general problems of effect moderation. For now, we focus on the WCLS, deferring to Section 5 the extensions of our methodology to these other problems.

With a slight abuse of notation, we let $\{Y_i, X_i\}_{i \in [n]}$ represent n independent and identically distributed observations drawn from an unknown data-generating distribution \mathbb{F}_n . Here, $Y_i \in \mathbb{R}^T$ is the response and $X_i \in \mathbb{R}^{T \times p}$ are stacked temporal observations for each individual, obtained through a suitable transformation of the original data. We define these variables explicitly when minimizing the WCLS objective. We assume that target causal moderated effect is consistently estimated by minimizing a generic loss function linear in the coefficients, i.e., the loss can be expressed as $\psi(Y_i, X_i b)$ where $b \in \mathbb{R}^p$.

Example 3.1. Recall that a consistent estimate of the linear causal excursion effect was obtained by minimizing the WCLS objective 4, which can be rewritten as

$$\operatorname*{argmin}_{b \in \mathbb{R}^p} \sum_{i,t} \frac{1}{2} \left(\tilde{Y}_{i,t} - \tilde{X}'_{i,t} b \right)^2$$

where $\tilde{Y}_{i,t} = \sqrt{W_t} (Y_{i,t} - \tilde{g}_t(H_{i,t})) \in \mathbb{R}, \tilde{X}_{i,t} = \sqrt{W_t} (A_t - \tilde{\rho} (1 | S_{i,t})) f_t (S_{i,t}) \in \mathbb{R}^p$. Stacking the temporal observations of *i*-th individual as

$$X_{i}_{(T \times p)} = \begin{bmatrix} \tilde{X}'_{i,1} \\ \vdots \\ \tilde{X}'_{i,T} \end{bmatrix} and Y_{i}_{(T \times p)} = \begin{bmatrix} \tilde{Y}_{i,1} \\ \vdots \\ \tilde{Y}_{i,T} \end{bmatrix},$$

and set $\psi(X_i b, Y_i) = \frac{1}{2} ||Y_i - X_i b||_2^2$. This shows that WCLS falls under the generic framework as $(Y_i, X_i)_{i \in [n]}$ are i.i.d observations and the loss ψ is linear in β .

4 Selective inference for excursion effect

4.1 Selection via a randomized lasso

Consider a *p*-dimensional Gaussian randomization variable $\sqrt{n}\omega_n$ from $\mathcal{N}(0_p, \Omega)$, drawn independently of the observed data, where Ω is a $p \times p$ predefined covariance matrix. When the randomization variable $\sqrt{n}\omega_n$ is added to the lasso optimization objective, this gives rise to the randomized lasso, given by

$$\underset{b \in \mathbb{R}^p}{\text{minimize}} \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i b; Y_i) + \lambda \|b\|_1 - \sqrt{n} \omega_n^\top b \right\}.$$
(5)

We use $\widehat{\beta}_n^{(\lambda)} \in \mathbb{R}^{p \times 1}$ to represent the solution to the randomized lasso in (5) and denote the selected set of non-zero lasso entries by

$$\widehat{E} = \left\{ j \in \{1, 2, \dots, p\} : |\operatorname{sign}(\widehat{\beta}_{n,j}^{(\lambda)})| = 1 \right\}.$$

For our given data, assume that we observe $\{\widehat{E} = E\}$, where E represents the observed value of the selected set \widehat{E} . Hereafter, let E' denote the complement set of E, and let |E| = q and |E'| = q' = p - q.

A simple randomization scheme, also used later in our simulations, involves adding pi.i.d. Gaussian noise variables to the lasso objective, i.e., $\Omega = \tau^2 \cdot I_p$. The value of τ^2

in this randomization scheme is similar to the split ratio in data splitting, determining how much information in the data is used for selecting moderators versus how much is exclusively reserved for inference. However, note that our randomization scheme differs from randomly splitting the data into two parts as it involves using all samples in the model selection process. Furthermore, as we show in this section, the additive form of Gaussian randomization enables us to obtain a pivot for $\beta_n^{E,j}$, for $j \in E$, from the full data without discarding any samples during inference.

Before proceeding further, we can rearrange the columns of X and write

$$X_{i} = \begin{bmatrix} X_{i,E} & X_{i,E'} \end{bmatrix}, \text{ and } \widehat{\beta}_{n}^{(\lambda)} = \begin{bmatrix} \widehat{\beta}_{n,E}^{(\lambda)} \\ 0_{q'} \end{bmatrix} \text{ such that } X_{i} \widehat{\beta}_{n}^{(\lambda)} = X_{i,E} \widehat{\beta}_{n,E}^{(\lambda)}$$

without loss of generality. We note that the K.K.T. (Karush–Kuhn–Tucker) stationarity conditions of the randomized lasso are given as:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i}^{\top}\nabla\psi(X_{i}\widehat{\beta}_{n,E}^{(\lambda)};Y_{i}) + \lambda S = \sqrt{n}\omega_{n}, \qquad (6)$$

where S is the observed value of the sub-gradient of the ℓ_1 -penalty at the solution

$$\widehat{S}_n = \{\partial \|b\|_1\}_{b=\widehat{\beta}_n^{(\lambda)}}.$$

Let the sub-vectors S_E and $S_{E'}$ be the observed values of $\widehat{S}_{n,E}$ and $\widehat{S}_{n,E'}$ respectively, which collect the entries of S in the sets E and E'.

It is easy to see a description of the selection event $\{\widehat{E} = E\}$ in terms of the subgradient variables of the ℓ_1 -penalty at the solution of randomized lasso, as stated in Proposition 4.1. This description holds true even when there is no added randomization.

Proposition 4.1. It holds that

$$\left\{\widehat{E} = E\right\} = \bigcup_{s \in \{-1,1\}^q} \left\{\widehat{S}_{n,E} = S_E, \|\widehat{S}_{n,E'}\|_{\infty} \le 1\right\}.$$

Even with a description of the selection event, it is often not feasible to compute the distribution obtained by conditioning on the selection event. Instead, conditioning on additional information beyond the variables in the selected set can lead to truncation sets that yield simpler conditional distributions. As shown in (1), conditioning on extra information still ensures valid inference, once again using the tower property of expectations. Later in this section, we show that we can instead condition on a subset of the selection event, which leads to a much simpler conditioning set.

4.2 Master statistics

In the rest of this section, we focus on selective inference for $\beta_n^{E,j}$ and introduce our master statistics for this task. Similar to the master statistics in Tibshirani et al. (2018), the selection event $\{\hat{E} = E\}$ depends on our observed data only through these statistics. Furthermore, for any fixed E, these statistics have an asymptotic normal distribution in the fixed p and growing n regime.

In order to define these statistics, we fix some notations. For a $p \times p$ matrix A and a set $\mathcal{E} \subset [p]$, we use $\mathbb{S}_{\mathcal{E}}A$ to denote the $|\mathcal{E}| \times p$ submatrix of A that collects the rows corresponding to \mathcal{E} . In the special case where we have singleton sets $\mathcal{E} = \{j\}$, we abuse notations and simply use \mathcal{S}_jA to denote the *j*th row of A. Furthermore, we use

$$A = \begin{bmatrix} A_{\mathcal{E}} & A_{\mathcal{E}'} \end{bmatrix} = \begin{bmatrix} A_{\mathcal{E},\mathcal{E}} & A_{\mathcal{E},\mathcal{E}'} \\ A_{\mathcal{E}',\mathcal{E}} & A_{\mathcal{E}',\mathcal{E}'} \end{bmatrix}$$

to denote submatrices that partitions the columns and rows of A based on the set \mathcal{E} and its complement $\mathcal{E}' = \mathcal{E}^c$.

We start by considering the following $p \times p$ matrices

$$H = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}X_{i}^{\top}\nabla^{2}\psi\left(X_{i,E}\beta_{n}^{E};Y_{i}\right)X_{i}\right], \ K = \operatorname{Cov}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i}^{\top}\nabla\psi\left(X_{i,E}\beta_{n}^{E};Y_{i}\right)\right).$$

Both these matrices are derived from the gradient and hessian of the twice differentiable loss function $\psi(\cdot; y)$. Assuming that $H_{E,E}$, the (E, E) block submatrix of H is invertible,

let

$$\Sigma_{E,E} = H_{E,E}^{-1} K_{E,E} H_{E,E}^{-1}$$

Additionally, let $\Sigma_{E,j} = \Sigma_{E,E} \mathbb{S}_j^T$ and $\sigma^{E,j} = (\mathbb{S}_j \Sigma_{E,E} \mathbb{S}_j^T)^{1/2}$, where $\Sigma_{E,j}$ is the *j*-th column of $\Sigma_{E,E}$ and $(\sigma^{E,j})^2$ is the *j*-th diagonal entry of $\Sigma_{E,E}$.

Now, we let

$$\widehat{b}_{n}^{E} = \underset{b \in \mathbb{R}^{q}}{\operatorname{argmin}} \sum_{i=1}^{n} \psi(X_{i,E}b; Y_{i})$$
(7)

and let $\hat{b}_n^{E,j} \in \mathbb{R}$ represent the j^{th} component of \hat{b}_n^E . In the case where our loss matches with the quadratic loss, which results in the WCLS criterion in Section 2, \hat{b}_n^E denotes the WCLS estimator obtained with moderators in the set E.

Our master statistics, however, do not simply involve \hat{b}_n^E . This is because the selection of E depends not just on the moderators included in our model, but also on those that were excluded from it. Therefore, we consider the statistic

$$\widehat{g}_{n}^{E,j} = \begin{bmatrix} \mathbb{S}_{[E]\setminus j} \left(\widehat{b}_{n}^{E} - \frac{1}{(\sigma^{E,j})^{2}} \Sigma_{E,j} \widehat{b}_{n}^{E,j} \right) \\ \frac{1}{n} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi \left(X_{i,E} \widehat{b}_{n}^{E}; Y_{i} \right) - (H_{E',E} - K_{E',E} K_{E,E}^{-1} H_{E,E}) \widehat{b}_{n}^{E} \end{bmatrix} \in \mathbb{R}^{p-1},$$

where the first q-1 components are obtained from \hat{b}_n^E and the last p-q components involve the moderators in E' as well as the residuals from the fit in (7). When inferring for $\beta_n^{E,j}$, the expected value of $\gamma_n^{E,j} = \mathbb{E}[\hat{g}_n^{E,j}]$ plays the role of nuisance parameters in our inferential task.

In Proposition 4.2, we first derive the asymptotic normal distribution of our master statistics

$$\sqrt{n} \left((\widehat{b}_n^{E,j})^\top \quad (\widehat{g}_n^{E,j})^\top \right)^\top \in \mathbb{R}^p$$

for a fixed set E and a fixed $j \in E$. Define the matrices

$$M_{1} = \begin{bmatrix} -\mathbb{S}_{j}H_{E,E}^{-1} & 0_{q'}^{\top} \end{bmatrix} K^{1/2} \in \mathbb{R}^{1 \times p}$$
$$M_{2} = \begin{bmatrix} M_{2,1} \\ M_{2,2} \end{bmatrix} = \begin{bmatrix} \mathbb{S}_{[E]\setminus j} \left(\frac{1}{(\sigma^{E,j})^{2}} \Sigma_{E,j} \mathbb{S}_{j} H_{E,E}^{-1} - H_{E,E}^{-1} \right) & 0_{q-1,q'} \\ -K_{E',E} K_{E,E}^{-1} & I_{q',q'} \end{bmatrix} K^{1/2} \in \mathbb{R}^{p-1 \times p},$$

and the variable

$$\zeta_n = \frac{1}{\sqrt{n}} K^{-1/2} \begin{bmatrix} \sum_{i=1}^n X_{i,E}^\top \nabla \psi \left(X_{i,E} \widehat{b}_n^E; Y_i \right) \\ \sum_{i=1}^n X_{i,E'}^\top \nabla \psi \left(X_{i,E} \widehat{b}_n^E; Y_i \right) \end{bmatrix} \in \mathbb{R}^{p \times 1}.$$

Proposition 4.2. We have that:

1.
$$\sqrt{n} \begin{pmatrix} \widehat{b}_{n}^{E,j} - \beta_{n}^{E,j} \\ \widehat{g}_{n}^{E,j} - \gamma_{n}^{E,j} \end{pmatrix} = \begin{pmatrix} M_{1} \\ M_{2} \end{pmatrix} \zeta_{n} + R_{n,1}, \text{ where } R_{n,1} = o_{p}(1),$$

2. $\sqrt{n} \begin{pmatrix} \widehat{b}_{n}^{E,j} - \beta_{n}^{E,j} \\ \widehat{g}_{n}^{E,j} - \gamma_{n}^{E,j} \end{pmatrix} \xrightarrow{d} \mathcal{N}_{p} \begin{pmatrix} 0 \\ 0_{p-1} \end{pmatrix}, \begin{bmatrix} (\sigma^{E,j})^{2} & 0 \\ 0 & \Sigma^{E,j} \end{bmatrix} \end{pmatrix}, \text{ where } \Sigma^{E,j} = M_{2}M_{2}^{\top}$

In the next Proposition, we derive the relationship of our master statistics with the randomized lasso solution.

Proposition 4.3. The K.K.T. stationarity conditions at the randomized lasso can be rewritten as

$$P_1^{E \cdot j} \sqrt{n} \widehat{b}_n^{E \cdot j} + P_2^{E \cdot j} \sqrt{n} \widehat{g}_n^{E \cdot j} + H_E \sqrt{n} \widehat{\beta}_{n,E}^{(\lambda)} + \lambda \begin{bmatrix} S_E \\ S_{E'} \end{bmatrix} = \sqrt{n} \omega_n + R_{n,2}$$

where

$$P_{1}^{E \cdot j} = -\frac{1}{(\sigma^{E \cdot j})^{2}} K_{E} H_{E,E}^{-1} \mathbb{S}_{j}^{\top} \in \mathbb{R}^{p}, \ P_{2}^{E \cdot j} = \begin{bmatrix} -H_{E,E} \mathbb{S}_{[E] \setminus j}^{\top} & 0_{q-1,p-q} \\ -K_{E',E} K_{E,E}^{-1} H_{E,E} \mathbb{S}_{[E] \setminus j}^{\top} & I_{p-q,p-q} \end{bmatrix} \in \mathbb{R}^{p \times (p-1)},$$

and $R_{n,2} = o_p(1)$.

Hereafter, we define

$$\sqrt{n}\tilde{\omega}_n = \sqrt{n}\omega_n + R_{n,2},\tag{8}$$

which is asymptotically distributed as a $\mathcal{N}(0_p, \Omega)$ random variable.

As a direct consequence of Proposition 4.3, we have the following corollary.

Corollary 1. It holds that

$$\{\widehat{S}_{n,E} = S_E\} = \left\{ \operatorname{sign} \left(H_{E,E}^{-1} \left\{ \sqrt{n}\widetilde{\omega}_n - \mathbb{S}_{[E]} \left(P_1^{E \cdot j} \sqrt{n}\widehat{b}_n^{E \cdot j} + P_2^{E \cdot j} \sqrt{n}\widehat{g}_n^{E \cdot j} \right) - \lambda S_E \right\} \right) = S_E \right\}$$

$$\left\{ \|\widehat{S}_{n,E'}\|_{\infty} = S_{E'} \right\} = \left\{ \frac{1}{\lambda} \left[\sqrt{n}\widetilde{\omega}_n - \mathbb{S}_{[-E]} \left(P_1^{E \cdot j} \sqrt{n}\widehat{b}_n^{E \cdot j} + P_2^{E \cdot j} \sqrt{n}\widehat{g}_n^{E \cdot j} \right) - H_{-E,E}H_{E,E}^{-1} \left\{ \sqrt{n}\widetilde{\omega}_n - \mathbb{S}_{[E]} \left(P_1^{E \cdot j} \sqrt{n}\widehat{b}_n^{E \cdot j} + P_2^{E \cdot j} \sqrt{n}\widehat{g}_n^{E \cdot j} \right) - \lambda \operatorname{sign}(\widehat{\beta}_{n,E}^{(\lambda)}) \right\} \right] = S_{E'} \right\}.$$

Clearly, the selection event $\{\widehat{E} = E\}$ depends on both $\widetilde{\omega}_n$ and the observed data. Together with Proposition 4.1, the corollary above shows that the selection event depends on the observed data through the defined master statistics.

Variable	Definition	Dimension	
$\widehat{eta}_n^{(\lambda)}$	Randomized lasso solution	р	
$\widehat{E} \subseteq [p]$	$\left\{ j \in \{1, 2, \dots, p\} : \operatorname{sign}(\widehat{\beta}_{n,j}^{(\lambda)}) = 1 \right\}$	q	
$\widehat{eta}_{n,E}^{(\lambda)}$	$(\widehat{\beta}_{n,j}^{(\lambda)}: j \in E)$	q	
$\widehat{S}_n = [\widehat{S}_{n,E} \ \widehat{S}_{n,E'}]^\top$	$\left\{\partial\ b\ _1 ight\}_{b=\widehat{eta}_n^{(\lambda)}}$	p	
$\widehat{U}_n^{E\cdot j}$	$\sqrt{n}(\eta^{E\cdot j})^ op \widehateta_{n,E}^{(\lambda)} $	1	
$\widehat{V}_n^{E\cdot j}$	$\sqrt{n} \left(I_q - Q^{E \cdot j} (\eta^{E \cdot j})^\top \right) \widehat{\beta}_{n,E}^{(\lambda)} $	q	

Table 2: A list of notations for estimators based on the randomized lasso solution.

4.3 Conditioning event

The conditioning event in our paper, as mentioned earlier, is a subset of the selection event, allowing for valid selective inference and a much simpler conditioning set. We describe this event below. Fixing some more notations, let $\Lambda = (H_E^{\top} \Omega^{-1} H_E)^{-1}$, and let

$$\eta^{E \cdot j} = \Lambda^{-1} H_E^T \Omega^{-1} P_1^{E \cdot j} \in \mathbb{R}^q, Q^{E \cdot j} = \frac{\Lambda \eta^{E \cdot j}}{(\eta^{E \cdot j})^\top \Lambda \eta^{E \cdot j}} \in \mathbb{R}^q.$$

Define

$$\widehat{U}_{n}^{E\cdot j} = \sqrt{n} (\eta^{E\cdot j})^{\top} |\widehat{\beta}_{n,E}^{(\lambda)}| \in \mathbb{R}, \ \widehat{V}_{n}^{E\cdot j} = \sqrt{n} \left(I_{q} - Q^{E\cdot j} (\eta^{E\cdot j})^{\top} \right) |\widehat{\beta}_{n,E}^{(\lambda)}| \in \mathbb{R}^{q}, \tag{9}$$

which are estimators obtained from the nonzero entries of randomized lasso solution. To make it easier for readers, we have collected a list of the variables from our randomized lasso solution in Table 2.

In Proposition 4.4, we present our conditioning event and show that the conditioning set is equivalent to truncating $\widehat{U}_n^{E,j}$ to a simple interval.

Proposition 4.4. Let the estimators $\widehat{U}_n^{E \cdot j}$ and $\widehat{V}_n^{E \cdot j}$ be as defined in (9). For $j \in E$, we have that

$$\left\{\widehat{S}_{n}=S, \ \widehat{V}_{n}^{E\cdot j}=V^{E\cdot j}\right\}=\left\{\widehat{U}_{n}^{E\cdot j}\in[I_{-}^{E\cdot j},I_{+}^{E\cdot j}], \ \widehat{S}_{n,E'}=S_{E'}, \ \widehat{V}_{n}^{E\cdot j}=V^{E\cdot j}\right\},$$

where

$$I_{-}^{E \cdot j} = \max_{k: \mathbb{S}_k Q^{E \cdot j} > 0} \left\{ \frac{-\mathbb{S}_k V^{E \cdot j}}{\mathbb{S}_k Q^{E \cdot j}} \right\}, I_{+}^{E \cdot j} = \min_{k: \mathbb{S}_k Q^{E \cdot j} < 0} \left\{ \frac{-\mathbb{S}_k V^{E \cdot j}}{\mathbb{S}_k Q^{E \cdot j}} \right\}.$$

Note that the event on the left-hand side of Proposition 4.4 is a subset of $\{\widehat{E} = E\}$ and is based on additional information using the randomized lasso solution. Conditioning on this event yields a simpler truncation region, resulting in a closed-form pivot, as shown in the next section.

4.4 Pivot via conditioning

By conditioning on the event in Proposition 4.4, we obtain a pivot for $\beta_n^{E.j}$ from the conditional distribution of our master statistics, given as

$$\sqrt{n} \left((\widehat{b}_n^{E,j})^\top \quad (\widehat{g}_n^{E,j})^\top \right)^\top \left| \left\{ \widehat{U}_n^{E,j} \in [I_-^{E,j}, I_+^{E,j}], \widehat{V}_n^{E,j} = V^{E,j}, \widehat{S}_{n,E'} = S_{E'} \right\}.$$

In Theorem 4.1, we first present the asymptotic marginal density of the variables involved in this conditional distribution before conditioning. In order to do this, given a fixed

set $E \subseteq [p]$ and a fixed vector of signs $S_E \in \mathbb{R}^q$, we define the mapping

$$\Pi_{\sqrt{n}\widehat{b}_{n}^{E,j},\sqrt{n}\widehat{g}_{n}^{E,j}}(u,v,z) = L^{E,j}u + H_{E}v + Gz + \lambda \begin{bmatrix} S_{E} \\ 0 \end{bmatrix} + P_{1}^{E,j}\sqrt{n}\widehat{b}_{n}^{E,j} + P_{2}^{E,j}\sqrt{n}\widehat{g}_{n}^{E,j}$$
(10)

where $L^{E \cdot j} = H_E Q^{E \cdot j}, G = \begin{bmatrix} 0_{(p-q) \times q} \\ I_{q \times q} \end{bmatrix}$.

Theorem 4.1. Let E and S_E be a given set of indices and vector of signs, respectively. Let $\Pi_{\sqrt{n}\widehat{b}_n^{E,j},\sqrt{n}\widehat{g}_n^{E,j}}(\cdot)$ be defined according to (10). Then, the asymptotic marginal density of

$$\left(\sqrt{n}\widehat{b}_{n}^{E.j},\sqrt{n}\widehat{g}_{n}^{E.j},\widehat{U}_{n}^{E.j},\widehat{V}_{n}^{E.j},\widehat{S}_{n,E'}\right)$$

is equal to

$$\begin{split} \phi(\sqrt{n}\widehat{b}_{n}^{E.j};\sqrt{n}\beta_{n}^{E.j},(\sigma^{E.j})^{2})\times\phi\left(\sqrt{n}\widehat{g}_{n}^{E.j};\sqrt{n}\gamma_{n}^{E.j},\Sigma^{E.j}\right)\\ \times\phi\left(\Pi_{\sqrt{n}\widehat{b}_{n}^{E.j},\sqrt{n}\widehat{g}_{n}^{E.j}}(\widehat{U}_{n}^{E.j},\widehat{V}_{n}^{E.j},\widehat{S}_{n,E'});0_{p},\Omega\right)\times\det(J), \end{split}$$
where $J = \begin{bmatrix} H_{E,E} & 0_{q,q'} \\ H_{E',E} & \lambda I_{q',q'} \end{bmatrix}$ and $\det(J)$ is the determinant of J .

The derivation of this result takes advantage of the use of independent randomization variables to facilitate a change of variables based on the K.K.T. conditions of stationarity. We defer a detailed proof of this result to the Appendix.

When $\{p(Y_i|X_i)\}_{i\in[n]}$ is a Gaussian density and we are in a fixed-X regime, the marginal density of our master statistics in Proposition 4.2 and their joint density with the randomized lasso variables in Theorem 4.1 are exact (i.e., non-asymptotic). In other words, the $o_p(1)$ terms in the asymptotic representations in both these results vanish, i.e., $R_{n,1} = R_{n,2} = 0$.

In the above-stated special case, conditioning the marginal density on the event in Proposition 4.4 immediately gives us an exact pivot for $\beta_n^{E.j}$. This problem was addressed in

Panigrahi et al. (2023a) for Gaussian linear regression with a fixed design matrix. Observe that the conditional density

$$\left(\sqrt{n}(\widehat{b}_n^{E.j})^\top \quad \sqrt{n}(\widehat{g}_n^{E.j})^\top \quad \widehat{U}_n^{E.j}\right)^\top \left| \left\{ \widehat{U}_n^{E.j} \in [I_-^{E.j}, I_+^{E.j}], \widehat{V}_n^{E.j} = V^{E.j}, \widehat{S}_{n,E'} = S_{E'} \right\}$$

is proportional to

$$\phi(\sqrt{n}\widehat{b}_{n}^{E.j};\sqrt{n}\beta_{n}^{E.j},(\sigma^{E.j})^{2}) \times \phi\left(\sqrt{n}\widehat{g}_{n}^{E.j};\sqrt{n}\gamma_{n}^{E.j},\Sigma^{E.j}\right) \\ \times \phi\left(\Pi_{\sqrt{n}\widehat{b}_{n}^{E.j},\sqrt{n}\widehat{g}_{n}^{E.j}}(\widehat{U}_{n}^{E.j},V^{E\cdot j},S_{E'});0_{p},\Omega\right) \times 1_{[I_{-}^{E\cdot j},I_{+}^{E\cdot j}]}(\widehat{U}_{n}^{E.j}).$$

We apply two simple steps from here. In the first step, we further condition on the observed value of $\sqrt{n}\hat{g}_n^{E.j}$ to get rid of the nuisance parameters in $\gamma_n^{E.j}$ and then marginalize over $\hat{U}_n^{E.j}$. This gives us a conditional density for $\sqrt{n}\hat{b}_n^{E.j}$ only involving the parameter of interest, $\beta_n^{E.j}$, which equals

$$C^{-1} \cdot \phi(\sqrt{n}\widehat{b}_n^{E.j}; \sqrt{n}\beta_n^{E.j}, (\sigma^{E.j})^2) \cdot F(\sqrt{n}\widehat{b}_n^{E.j}, \sqrt{n}\widehat{g}_n^{E.j})$$

where

$$F(\sqrt{n}\widehat{b}_{n}^{E.j}, \sqrt{n}\widehat{g}_{n}^{E.j}) = \int_{I_{-}^{E.j}}^{I_{+}^{E.j}} \phi\left(L^{E.j}t + P_{1}^{E.j}\sqrt{n}\widehat{b}_{n}^{E.j} + P_{2}^{E.j}\sqrt{n}\widehat{g}_{n}^{E.j} + T^{E.j}; 0_{p}, \Omega\right) dt, \quad (11)$$

with $T^{E \cdot j} = H_{,E} V^{E \cdot j} + \lambda G S_{E'}$ and

$$C = \int_{-\infty}^{\infty} \phi(x; \sqrt{n}\beta_n^{E.j}, (\sigma^{E.j})^2) F(x, \sqrt{n}\widehat{g}_n^{E.j}) dx.$$

Second, applying a probability integral transform to this conditional density yields an exact pivot

$$P^{E,j}(\widehat{b}_{n}^{E,j},\widehat{g}_{n}^{E,j};\beta_{n}^{E,j}) = \frac{\int_{-\infty}^{\sqrt{n}\widehat{b}_{n}^{E,j}}\phi(x;\sqrt{n}\beta_{n}^{E,j},(\sigma^{E,j})^{2})F(x,\sqrt{n}\widehat{g}_{n}^{E,j})dx}{\int_{-\infty}^{\infty}\phi(x;\sqrt{n}\beta_{n}^{E,j},(\sigma^{E,j})^{2})F(x,\sqrt{n}\widehat{g}_{n}^{E,j})dx},$$
(12)

which is distributed as a Unif(0, 1) random variable.

The superscript $E \cdot j$ in the pivot and the matrices involved in its expression indicate the dependence of these quantities on $j \in E$. To simplify our notation for the asymptotic theory, we will, with a slight abuse of notation, drop the superscript $E \cdot j$ from here on.

In what follows, our theory shows how we can utilize the same pivot in (12), as though the data were normally distributed, to conduct asymptotic selective inference.

4.5 Uniformly asymptotic selective inference

To obtain confidence intervals for $\beta_n^{E,j}$, we invert our proposed pivot. For example, twosided confidence intervals at level α are constructed as

$$\left(L_{\alpha.n}^{j}, U_{\alpha.n}^{j}\right) = \left\{\beta_{n}^{E.j} : P^{E.j}(\widehat{b}_{n}^{E.j}, \widehat{g}_{n}^{E.j}; \beta_{n}^{E.j}) \in \left[\frac{\alpha}{2}, 1 - \frac{\alpha}{2}\right]\right\}.$$

In this section, we demonstrate that these intervals achieve the desired coverage probability as $n \to \infty$ when $(X_i, Y_i)_{i \in [n]}$ are i.i.d. variables from \mathbb{F}_n . This leads us to our main result in Theorem 4.2, which offers asymptotic conditional guarantees for these selective confidence intervals, uniformly across all distributions in a large collection $\mathcal{F}_n = \{\mathbb{F}_n\}$, for $n \in \mathbb{N}$ that meet the following assumptions.

For our theory, we consider a sequence of parameters that depend on n as follows:

$$\sqrt{n} \begin{bmatrix} (\beta_n^{E.j})^\top & (\gamma_n^{E.j})^\top \end{bmatrix}^\top = O(r_n),$$

where r_n is a non-negative sequence of real numbers and $r_n = o(n^{1/6})$. The sequence r_n determines how the probability of the selection event changes with increasing sample size. This probability is bounded away from zero when r_n does not grow with n. Note that the asymptotic selective inference theory in Zhao et al. (2021) for effect modification problems considers sequences of parameters where $r_n = O(1)$. In contrast, our theory in this section accommodates a larger set of parameters, including sequences where r_n is allowed to grow to ∞ with increasing sample size.

Assumption 2. There exist a constant $B_0 > 0$, such that it holds that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}\left[\exp\left(t \left| \left| X_{i}^{\top} \nabla \psi\left(X_{i,E} \widehat{b}_{n}^{E}; Y_{i}\right) \right| \right| \right)\right] \leq \exp(B_{0} t^{2}),$$

for any $t \in \mathbb{R}$ and $i \in [n]$.

Assumption 3. Let $\tilde{\omega}_n$ be a random variable defined in (8), with a Lebesgue density q_n for $n \geq n_0$. Define,

$$G_n(x,y) = \int_{I_-^{E\cdot j}}^{I_+^{E\cdot j}} q_n \left(Lt + P_1 x + P_2 y + T; 0_p, \Omega \right) dt.$$

Then, assume that

$$\lim_{n \to \infty} \sup_{(x,y)} \left| \frac{G_n(x,y)}{F(x,y)} - 1 \right| = 0.$$

Assumption 4. For any $\epsilon > 0$,

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}} \frac{1}{r_{n}^{2}} \log \mathbb{P}\left[\frac{||R_{n,1}||}{r_{n}} > \epsilon\right] = -\infty,$$

if $r_n \to \infty$ as $n \to \infty$.

Remark 1. All the expectations and probabilities in this section are taken over the generating distribution, $\mathbb{F}_n \in \mathcal{F}$.

The condition in Assumption 2 states that the variable $X_i^{\top} \nabla \psi \left(X_{i,E} \hat{b}_n^E; Y_i \right)$ must be a sub-Gaussian variable. This condition is satisfied if the variables $X_i \in \mathbb{R}^p$ are uniformly bounded, i.e., $\|X_i\|_{\infty} \leq C$ for some C > 0, and if the score variable derived from the loss function is sub-Gaussian. The condition in Assumption 3 ensures that $\sqrt{n}\tilde{\omega}_n$ in (8) behaves similarly to its limiting normal counterpart $\sqrt{n}\omega_n$. Since $\sqrt{n}(\tilde{\omega}_n - \omega_n) = o_p(1)$, this condition is automatically satisfied when the sequence r_n does not grow with n. Finally, Assumption 4 places conditions on the tail behavior of the error variable $R_{n,1}$, and is only required when r_n grows to ∞ with increasing n.

Theorem 4.2. For each $n \in \mathbb{N}$, let $\mathcal{F}_n = \{\mathbb{F}_n\}$ denotes a collection of data-generating distributions. Suppose that this collection of distributions satisfies Assumptions 2, 3, 4. Then, we have that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}} \left| \mathbb{P} \left[\beta_{n}^{E.j} \in \left(L_{\alpha.n}^{j}, U_{\alpha.n}^{j} \right) \left| \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} = V^{E.j} \right\} \right] - (1 - \alpha) \right| = 0$$

Put in words, our main result establishes that for any fixed $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that the asymptotic interval $(L_{\alpha.n}^j, U_{\alpha.n}^j)$ will have a coverage probability of at least $1 - \alpha - \epsilon$ for any $n \ge N(\epsilon)$ and for any data-generating distribution in \mathcal{F}_n . Additionally, this coverage guarantee is based on an event that is a strict subset of $\{\widehat{E} = E\}$. Therefore, the total probability law ensures that the same asymptotic assurance for the coverage probability applies to the selection event $\{\widehat{E} = E\}$.

5 Extensions

Here we present some extensions and alternate settings under which our general methodology of selective inference for causal effects can be applied. We mainly relax some key assumptions made early in our paper and discuss their implications.

5.1 Effect moderation for binary outcomes

Recall while defining causal excursion effect (2), we only considered continuous outcomes. When the proximal response is binary or a zero-inflated count outcome, it can be more interpretable to consider causal excursion effects on the relative risk scale [Qian et al. (2020)]:

$$\beta_{\mathbf{p}}^{(RR)}(t;s) = \log \left(\frac{\mathbb{E}_{\mathbf{p}} \left[Y_{i,t+1} \left(\bar{A}_{i,t-1}, W_{i,t} = 1 \right) \mid S_t \left(\bar{A}_{i,t-1} \right) = s \right]}{\mathbb{E}_{\mathbf{p}} \left[Y_{i,t+1} \left(\bar{A}_{i,t-1}, W_{i,t} = 0 \right) \mid S_t \left(\bar{A}_{i,t-1} \right) = s \right]} \right) \\ = \log \left(\frac{\mathbb{E} \left[\mathbb{E} \left[Y_{i,t+1} \mid H_{i,t}, A_{i,t} = 1 \right] \mid S_t = s \right]}{\mathbb{E} \left[\mathbb{E} \left[Y_{i,t+1} \mid H_{i,t}, A_{i,t} = 0 \right] \mid S_t = s \right]} \right).$$

The RR causal excursion effect is also modeled via a linear form, i.e., $\beta_{\mathbf{p}}^{(RR)}(t;s) = f_t(s)^{\top}\beta$, and a consistent estimator of the marginal excursion effect (EMEE) by solving a set of weighted estimating equations:

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} W_{i,t} e^{-A_{i,t} f_t(S_{i,t})^\top \beta} \left(Y_{i,t+1} - e^{\tilde{g}_t(H_{i,t}) + A_{i,t} f_t(S_{i,t})^\top \beta} \right) \begin{bmatrix} g_t(H_{i,t}) \\ (A_{i,t} - \hat{p}(1|S_{i,t})) f_t(S_{i,t}) \end{bmatrix} \right] = 0,$$
(13)

where $\tilde{g}_t(H_{i,t})$ is the initial nuisance plugin. While the WCLS criterion can be written in a loss formulation that can naturally incorporate penalization and randomization, the EMEE is derived from a set of weighted estimating equations with no equivalent loss formulation. In Appendix 13, we present a loss minimization framework to causal contrast estimation for selection that includes both settings above. This reformulation comes handy for selection of moderators, as we directly add penalty to the minimization objective and get sparse solutions.

Let $\mu_t(h, a) := \mathbb{E}_{\mathbf{p}}[Y_{i,t+1}|H_{i,t} = h, A_{i,t} = a]$ denotes conditional mean proximal outcome given history h and action a, and $\tilde{\sigma}_t^2(S_{i,t}) = \tilde{p}_t(1|S_{i,t})(1 - \tilde{p}_t(1|S_{i,t}))$. Motivated by recent work on empirical risk minimization van der Laan et al. (2024), we consider a one-step debiased estimation procedure to ensure minimizing the empirical risk yield oracle efficiency similar to orthogonal learning strategies Shi and Dempsey (2023). First, we consider a doubly-robust initial estimator

$$\hat{\mu}_t^{(DR)}(H_{i,t},a) = \hat{\mu}_t(H_{i,t},a) + \frac{1[A_{i,t}=a]}{p(A_{i,t}|H_{i,t})} \left(Y_{i,t+1} - \hat{\mu}_t(H_{i,t},a)\right),$$

Given the initial estimate, we propose to construct a refined estimator $\mu_t^{\star}(h, a) = \hat{\mu}_t(h, a) + (a - \hat{p}_t(1|S_{i,t}))f_t(S_{i,t})^{\top}\hat{\theta}$ where $\hat{\theta}$ solves the following estimating equations:

$$\sum_{i=1}^{n} \left[\sum_{t=1}^{T} \frac{1}{p(A_{i,t}|H_{i,t})} \tilde{\sigma}_{t}^{2}(S_{i,t})(Y_{i,t+1} - \mu^{\star}(H_{i,t+1}, A_{i,t})) \left\{ \begin{array}{c} \psi(S_{i,t}) \\ -A_{i,t}\psi(S_{i,t}) \end{array} \right\} \right] = 0.$$

where $\psi(S_{i,t}) \in \mathbb{R}^d \supset f_t(S_{i,t})$ is a basis such that $\log\left(1 + e^{f_t(S_{i,t})^\top\beta}\right)$ can be well approximated by $\psi(S_{i,t})^\top \alpha$ for some $\alpha \in \mathbb{R}^d$. Then we can use the plug-in estimator and minimize

the empirical loss:

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} \tilde{\sigma}_{i,t}^{2}(S_{i,t}) \left(\mu_{t}^{\star}(H_{i,t}, 1) + \mu_{t}^{\star}(H_{i,t}, 0) \right) \times \left\{ \log \left(1 + e^{f_{t}(S_{i,t})^{\top}\beta} \right) - \frac{\mu_{t}^{\star}(H_{i,t}, 1)}{(\mu_{t}^{\star}(H_{i,t}, 1) + \mu_{t}^{\star}(H_{i,t}, 0))} f_{t}(S_{i,t})^{\top}\beta \right\} \right],$$
(14)

which is equivalent to logistic regression with weight $\tilde{\sigma}_t^2(S_{i,t})$, $(\mu_t^*(H_{i,t}, 1) + \mu_t^*(H_{i,t}, 0))$ and outcome $\frac{\mu_t^*(H_{i,t}, 1)}{\mu_t^*(H_{i,t}, 1) + \mu_t^*(H_{i,t}, 0)}$. Having used this generic formulation to develop our selective inference method in Section 4, we apply our method directly to our examples of interest and also any other causal estimand having similar form.

5.2 Nuisance Parameter Estimation

So far, we have focused on one-way sample splitting to estimate nuisance parameters; however, our method can incorporate other approaches to nuisance estimation. Recall, both WCLS ((4)) and EMEE ((13)) have an unknown nuisance parameter α in their estimating equation, similar to the nuisance functions $\hat{\mu}_t(H_{i,t}, 1)$, $\hat{\mu}_t(H_{i,t}, 0)$ for risk based estimation. Before applying the penalized regression for selection we need to plug in initial estimates for the the nuisance parameters or functions.

These nuisance parameters (or functions) can be directly estimated from data, using simple regression models, or some machine learning methods with good prediction performance as done by Chernozhukov et al. (2018). The main ideas for efficient estimation for semi-parametric model in presence of nuisance parameters goes back to the works of Robinson (1988), Schick (1986). The most commonly used strategy here is cross-fitting or sample-splitting (Newey and Robins (2018), Chernozhukov et al. (2018)), where for the i-th data point we plug-in predicted value (of the nuisance functions) using a subsample of the data that does not contain the i-th data point. This can be seen as a two-stage estimation procedure, on two independent folds of data. For our data experiments, we use a random one-third sample to fit a least squares model to estimate the nuisance parameters and plug

in the values to our minimization objectives.

It's important to recognize that while estimating nuisance parameters is crucial for obtaining the desired estimate, the estimation process itself has minimal bearing on the subsequent selective inference for the target parameter. Provided that the nuisance parameters, or functions, were estimated independently, our inferential guarantees for the casual target β holds irrespective of our nuisance estimation method.

6 Simulation Study

In this section, we assess the performance of our method compared to existing approaches using synthetic MRT data. Our data generating model is motivated from the experimental setup described in Boruvka et al. (2018). We simulate n = 120 i.i.d. synthetic MRT participants over T = 30 time points, mimicking a real-world scenario where individual data is collected at 30 decision times. For each $i \in [n]$, synthetic MRT data $(X_{i,t}, A_{i,t}, Y_{i,t})_{t=1}^{T}$ is generated based on a predefined logistic model for action selection, $p_t(1|H_t)$. The potential moderators $S_{i,t} \in \mathbb{R}^p$ follow a vector auto-regressive process $S_{i,t} = \Gamma_0 S_{i,t-1} + A_{i,t} \Gamma_1 + e_t$, where $\Gamma_0 \in \mathbb{R}^{p \times p}$ and $\Gamma_1 \in \mathbb{R}^p$ and $e_t \sim \mathcal{N}(0, \sigma^2 I_p)$ is the white noise term.

Assume that a sparse subset $E \subset [p]$ represents the true moderators of treatment. Define $\tilde{S}_{i,t} = (S_{i,t} - E[S_{i,t}|H_{t-1}, A_{i,t-1}])$ and let $\tilde{S}_{t,E} \in \mathbb{R}^{|E|}$ extract the true sparse support from the larger vector of potential moderators. Then our outcomes are generated from the following model:

$$Y_{i,t} = \theta_1^{\top} \tilde{S}_{i,t} + \theta_2 (A_{i,t-1} - p_t(1|H_{t-1})) + (A_{i,t} - p_t(1|H_{i,t})) \left(\beta_{11}^{\top} \tilde{S}_{t,E} - 0.2\right) + \epsilon_{i,t},$$

where $\{\epsilon_{i,t}\}_{i=1}^{n}$ are *n* i.i.d longitudinally correlated noise terms with $\operatorname{Corr}(\epsilon_{i,t}, \epsilon_{i,s}) = \rho^{|t-s|}$.

We set p = 50 and set E to be the first 5 moderators in $S_{i,t}$. Additionally, we fix $\rho = 0.5, \theta_1 = 0.8\mathbf{1}_p$ and $\sigma = 1.5$. Recall that the goal of our methods is to infer the sparse p-dimensional vector $\beta^* = (\beta_{11}^{\top}, \mathbf{0}_{p-|E|})^{\top}$. To evaluate our method's performance

across different settings, we vary our sample size through n and signal strength through $\beta_{11} = \frac{c}{5} \mathbf{1}_5$, where β_{11} represents the true effects associated with the moderators in E.

Finally, we consider three experimental settings, each defined by a different error distribution:

- 1. Gaussian noise: Model errors $(\epsilon_{i,t})_{t \in [T]}$ are generated from a Gaussian auto-regressive process
- 2. Added Laplace noise: Model errors $\epsilon_{i,t} = \epsilon_{i,t}^g + \tilde{\epsilon}_{i,t}$, where $(\tilde{\epsilon}_{i,t})_{t \in [T]} \stackrel{i.i.d.}{\sim}$ Laplace(0, 1.5) and $(\epsilon_{i,t}^g)_{t \in [T]}$ are generated from a Gaussian auto-regressive process.
- 3. Added Exponential noise: Model errors $\epsilon_{i,t} = \epsilon_{i,t}^g + \tilde{\epsilon}_{i,t}$ where $(\tilde{\epsilon}_{i,t})_{t \in [T]} \stackrel{i.i.d.}{\sim}$ Exponential(1.5) and $(\epsilon_{i,t}^g)_{t \in [T]}$ are generated from a Gaussian auto-regressive process.

For each of these three error distributions, we evaluate our methods for three different values of $c \in \{1.2, 2.4, 4.4\}$, which we term "Low", "Medium" and "High" signal strengths.

6.1 Methods and findings

We compare the performance of our proposed method with three other approaches: (i) the polyhedral method for selective inference developed by Zhao et al. (2021), (ii) the data splitting approach, which involves dividing the data into two independent parts—using the first part for variable selection and reserving the second part for inference; in our simulations, we use 70% of the data for variable selection and the remaining 30% for making inferences, and (iii) the naive approach, which does not account for the selection of moderators and uses the standard confidence intervals without accounting for the selection process.

We conduct 500 independent Monte Carlo simulations for each experimental setting we consider. In each setting, we produce confidence intervals in the selected model using four different methods, all aiming to achieve a False Coverage Rate (FCR) of 0.1% or a coverage rate of 90%. We evaluate the inferential validity of different methods by computing the empirical coverage rate, which is the ratio of the number of selective intervals that cover the post-selection parameter to the total number of intervals constructed. To compare the power of these different inferential methods, we report the average lengths of their respective confidence intervals. In most of the settings we consider, the polyhedral method produces intervals that are infinitely long, resulting in an infinite average length for the confidence intervals across all the Monte Carlo simulations. Therefore, when comparing with other methods, we only plot its bounded intervals. Additionally, we provide a histogram indicating the percentage of infinitely long intervals produced by the polyhedral method, noting that all other methods we implement yield bounded confidence intervals.

The coverage rates and averaged lengths of the confidence intervals produced by the four methods across the three signal regimes are shown in Figure 1, Figure 2, and Figure 3 for the three types of error distributions.



Figure 1: Average Coverage (left), Average Length of Finite CIs (mid), % infinitely long polyhedral intervals (right) for varying signal strength with Gaussian errors

Across all settings, we observe that the naive approach consistently fails to achieve the



Figure 2: Average Coverage (left), Average Length of Finite CIs (mid), % infinitely long polyhedral intervals (right) for varying signal strength with Laplace errors

desired coverage rate, underscoring the unreliability of inference when the selection process is not accounted for. As demonstrated in the first example in Section 1, the polyhedral method exhibits a substantial lack of coverage at low signal strengths but successfully attains the desired coverage as the signal strength increases.

In contrast, both data splitting and our proposed method consistently achieve close to 90% coverage across all signal regimes. However, under non-Gaussian errors and low signal strengths, data splitting occasionally falls slightly short of the desired coverage, likely due to insufficient sample sizes. In these scenarios, our selective inference method consistently achieves the desired coverage rate by making use of the full dataset for inference, unlike data splitting that uses only a portion of the full dataset for this task.

When comparing statistical power in terms of average bounded interval length, our proposed method consistently outperforms the polyhedral approach, particularly for Laplace and exponential errors in low signal regimes. Notably, for lower signal values, the polyhedral method produces more than 25% infinitely long intervals. Furthermore, our intervals



Figure 3: Average Coverage (left), Average Length of Finite CIs (mid), % infinitely long polyhedral intervals (right) for varying signal strength with Exponential errors

are significantly shorter than those from data splitting as signal strength increases. Notably, the intervals produced by our method adapt to the signal strength, becoming shorter as the signal increases. In contrast, the intervals generated by data splitting fail to adapt to signal strength, remaining unchanged across different signal regimes.

7 Data Application

In this section, we demonstrate an application of our selective inference method on data from the Virtual Application-supported Environment To Increase Exercise (VALENTINE) Study (Golbus et al., 2023).

7.1 VALENTINE Study Description

The VALENTINE Study is a remotely managed, randomized controlled clinical trial that aims to assess the potential of mobile health (mHealth) interventions to increase exercise adherence among low-to-medium risk patients enrolled in a cardiac rehabilitation program. Patients are randomly assigned to either a treatment or control group after screening. Both groups receive a compatible smartwatch (either a Fitbit Versa 2 for Android users or an Apple Watch 4 for iPhone users) alongside the standard care. The study employs Just-in-Time Adaptive Interventions (JITAIs) to leverage contextual information from wearable devices, examining the short-term effects of notifications on physical activity through microrandomization.

Subjects in the intervention arm undergo daily activity tracking, goal setting via a mobile application, receive contextually tailored notifications, and weekly activity summaries. Notifications are inspired by behavioral health theories, such as goal setting and implementation intentions, and tailored to users based on contextual factors to encourage physical activity. The notifications come in two functional types—activity notifications and exercise planning notifications—and are delivered at one of four time points throughout the day (morning, lunchtime, mid-afternoon, evening), each with a 25% probability. Personalization factors include weather, day of the week, time of day, and the patient's phase within the cardiac rehabilitation program.

Data collected through the smartwatch app include biometric data like heart rate, activity levels, and potentially respiratory metrics. The study aims to identify important contextual moderators that could enhance the effectiveness of these notifications, selecting from a large set of potential moderators derived from wearable data features and their interactions with time and phases of rehabilitation. Effect modification, rather than marginal effect estimation, is of primary interest to understand the conditions under which notifications are most beneficial.

7.2 Data Processing & Selection of Moderators

To demonstrate our selective inference method, we focus on data from Apple Watch users. After excluding rows with missing values, the dataset comprises 23,462 observa-

tions from 62 unique participants, with each participant contributing between 36 and 592 observations. The dataset includes 61 variables, out of which we use 23 relevant variables for our analysis. The primary response variable is Log transformed step-counts after notification (value_tran) and 23 potential moderator variables like centered treatment indicator(notification_c), Phases of the study period (phases), (Baseline_steps), (WalkDistanceAgg_m), and (StepsAgg_priorweek0).

We are particularly interested in moderated treatment effect analysis rather than estimating marginal effects alone. This requires selecting a set of key moderators from the observed features and their interactions. To achieve this, we use a lasso-based approach to select from a large pool of potential moderators, allowing us to narrow down to a smaller, impactful set. We then apply selective inference methods to compute confidence intervals for these selected moderators, offering insights into how each moderator influences treatment effectiveness.

For comprehensive analysis, we consider three different sets of potential moderators:

- Relevant Variables Only: This set includes only the most relevant observed variables, without interaction terms, yielding total 25 moderators.
- Phase-Time Interactions: This set includes interactions between phases of rehabilitation and time-related variables, resulting in 58 moderators.
- Extended Interactions: This set builds on the Phase-Time Interactions, adding other important interaction terms for a total of 96 moderators.

In the following section, we detail the application of our selective inference method, showcasing how it enables post-selection confidence intervals for the chosen moderators, allowing us to rigorously assess which contextual factors most effectively modulate treatment outcomes in the VALENTINE Study. We provide results bellow for the first setting where all three methods selected the same 4 variables from the set of 25 potential moderators through their respective LASSO-based selection. Selective inference of an MRT in cardiac rehab population reveals several important effect moderators. The processed dataset with detailed description and results for the remaining two settings are available in the supplement 12.



Figure 4: All the three method selected the same 4 variables. The 90% confidence intervals for the selected parameters are plotted above, our method (in blue) consistently provides shorter intervals than data splitting. Polyhedral method for this setting fails to provide finite intervals, hence not included above.

7.3 Methods and Results

To carry out lasso-based moderator selection, recall that we need to incorporate a nuisance estimate within the regularization process. We employ a simple sample-splitting technique by randomly dividing participants based on id into two groups: 30% and 70%. First, we use the 30% group to fit an Ordinary Least Squares (OLS) model, which provides the nuisance estimator required for the lasso procedure. This estimator is then used as a plug-in value in the penalized estimation to identify a more focused subset of important moderators. to the remaining 70% of the data, where we This approach typically results in a selection of fewer than 10 key moderators with appropriately chosen penalty/tuning parameter. With this final subset, we proceed to conduct inference for the selected sub-model, allowing for a detailed analysis of the effects of each selected moderator. Along with our proposed

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method which uses randomized LASSO for selection, we also demonstrate *polyhedral*, *data-splitting* methods using standard LASSO selection (splitting using even part of 70%) to provide a fair comparison. Finally, we construct 90% confidence intervals for each of the selected moderators, which we plot in Figure 4. The polyhedral method generates infinitely long interval for all 4 selected parameters. Intervals produced using the entire data by our method is consistently shorter than the respective splitting intervals which uses 30% data for inference.

8 Discussion

Our paper develops a selective inference approach for time-varying causal effect moderation, enabling researchers to first select a simpler, interpretable model through penalized regression and then construct uniformly valid semiparametric inference for effects of the selected moderators. Our two-step method differs from traditional selective inference approaches like data splitting by employing a Gaussian randomization scheme, which allows analysts to utilize the full set of observed samples for both steps of the analysis, selection of moderators and inference on their effect sizes. This randomization introduces an analyst-controlled tradeoff between predictive modeling accuracy and inferential quality. As a result, even with limited sample sizes—such as in expensive experiments where data splitting may fall short of the desired coverage probability—our approach consistently achieves the desired coverage probability and produces significantly shorter intervals than splitting-based methods. Compared to previous conditional methods for selective inference in effect moderation, such as the approach by Zhao et al. (2021), our method not only continues to ensure valid inference in challenging scenarios with low signal-to-noise ratios but also delivers bounded, substantially shorter intervals, overcoming the lack of coverage and infinitely long intervals associated with the earlier technique.

The asymptotic framework for inference presented in this paper can be extended in several ways. For instance, our method can easily accommodate misspecified models by

using the classical M-estimation framework, as described in Huang et al. (2023). Penalties such as the group lasso, which can aggregate variables measured across multiple time points within a specified window, can be applied to identify important moderators. While previous work by Panigrahi et al. (2023b) and Huang et al. (2023) have introduced an approximate likelihood-based framework for applying such penalties, adopting a pivot-based approach will eliminate the need of such approximations in computing selective inference and is another natural extension of this approach. All the python code used for empirical analysis and results in this paper can be found in the following github repository.

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10 Appendix- Proofs for results in Sections 4.1-4.4

Proof of Proposition 4.2. Observe that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E}^{\top} \nabla \psi \left(X_{i,E} \widehat{b}_{n}^{E}; Y_{i} \right) = 0$. From Taylor expanding the expression on the left-hand side display around β_{n}^{E} , we have that:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i,E}^{\top}\nabla\psi\left(X_{i,E}\beta_{n}^{E};Y_{i}\right) - \frac{1}{n}\sum_{i=1}^{n}X_{i,E}^{\top}\nabla^{2}\psi\left(X_{i,E}\beta_{n}^{E};Y_{i}\right)X_{i,E}\sqrt{n}(\hat{b}_{n}^{E} - \beta_{n}^{E}) = 0 \quad (15)$$

$$\implies \sqrt{n}(\widehat{b}_n^E - \beta_n^E) = -H_{E,E}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,E}^\top \nabla \psi \left(X_{i,E} \beta_n^E; Y_i \right) + o_p(1). \tag{16}$$

Multiplying both sides of (16) with the matrix \mathbb{S}_j , we obtain

$$\sqrt{n}(\hat{b}_{n}^{E,j} - \beta_{n}^{E,j}) = -\mathbb{S}_{j}H_{E,E}^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i,E}^{\top}\nabla\psi\left(X_{i,E}\beta_{n}^{E};Y_{i}\right) + o_{p}(1)$$
$$= M_{1}\zeta_{n} + o_{p}(1).$$

Next, observe that $\widehat{g}_n^{E,j} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathbb{R}^{p-1}$, where

$$T_1 = \mathbb{S}_{[E]\setminus j} \left(\widehat{b}_n^E - \frac{1}{(\sigma^{E.j})^2} \Sigma_{E,j} \widehat{b}_n^{E.j} \right) \in \mathbb{R}^{q-1} \text{ and}$$
(17)

$$T_{2} = \frac{1}{n} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi \left(X_{i,E} \widehat{b}_{n}^{E}; Y_{i} \right) + \left(K_{E',E} K_{E,E}^{-1} H_{E,E} - H_{E',E} \right) \widehat{b}_{n}^{E} \in \mathbb{R}^{q'}.$$
(18)

Multiplying both sides of (16) with $\mathbb{S}_{[E]\setminus j}$, we obtain that

$$\sqrt{n} (T_1 - \mathbb{E}[T_1]) = \mathbb{S}_{[E] \setminus j} \sqrt{n} (\hat{b}_n^E - \beta_n^E) - \frac{1}{(\sigma^{E.j})^2} \mathbb{S}_{[E] \setminus j} \Sigma_{E,j} \sqrt{n} (\hat{b}_n^{E.j} - \beta_n^{E.j}) \\
= \mathbb{S}_{[E] \setminus j} \left(-H_{E,E}^{-1} + \frac{1}{(\sigma^{E.j})^2} \Sigma_{E,j} \mathbb{S}_j H_{E,E}^{-1} \right) \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,E}^\top \nabla \psi \left(Y_i, X_{i,E} \beta_n^E \right) + o_p(1).$$

Next, observe that Taylor expanding $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(X_{i,E} \hat{b}_n^E; Y_i) \in \mathbb{R}^q$ around β_n^E yields:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(X_{i,E} \widehat{b}_{n}^{E}; Y_{i})$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(X_{i,E} \beta_{n}^{E}; Y_{i}) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla^{2} \psi(X_{i,E} \beta_{n}^{E}; Y_{i}) X_{i,E} \left(\widehat{b}_{n}^{E} - \beta_{n}^{E}\right)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(X_{i,E} \beta_{n}^{E}; Y_{i}) - H_{E',E} \sqrt{n} \left(\widehat{b}_{n}^{E} - \beta_{n}^{E}\right) + o_{p}(1).$$

Adding $\sqrt{n}(K_{E',E}K_{E,E}^{-1}H_{E,E} - H_{E',E})\hat{b}_n^E$ to both sides of the last display, we obtain that:

$$\sqrt{n}T_2 = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,E'}^\top \nabla \psi(X_{i,E}\beta_n^E;Y_i) + K_{E',E}K_{E,E}^{-1}H_{E,E}\sqrt{n}\hat{b}_n^E - H_{E',E}\sqrt{n}\beta_n^E.$$

This leads us to note that

$$\begin{split} \sqrt{n} \left(T_2 - \mathbb{E}[T_2] \right) &= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^n X_{i,E'}^\top \nabla \psi(X_{i,E}\beta_n^E;Y_i) - \mathbb{E}\left[\sum_{i=1}^n X_{i,E'}^\top \nabla \psi(X_{i,E}\beta_n^E;Y_i) \right] \right) \\ &+ K_{E',E} K_{E,E}^{-1} H_{E,E} \sqrt{n} (\widehat{b}_n^E - \beta_n^E) \\ &= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^n X_{i,E'}^\top \nabla \psi(X_{i,E}\beta_n^E;Y_i) - \mathbb{E}\left[\sum_{i=1}^n X_{i,E'}^\top \nabla \psi(X_{i,E}\beta_n^E;Y_i) \right] \right) \\ &- K_{E',E} K_{E,E}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n X_{i,E}^\top \nabla \psi\left(X_{i,E}\beta_n^E;Y_i\right) + o_p(1). \end{split}$$

To complete our proof, we have that

$$\mathbb{E}\left[\sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(X_{i,E}\beta_{n}^{E};Y_{i})\right] = 0.$$

The second part of the claim is a direct consequence of the asymptotic representation in the first part. $\hfill \Box$

Proof of Proposition 4.3. A Taylor expansion of K.K.T. stationarity conditions around \hat{b}_n^E yields:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i}^{\top}\nabla\psi(X_{i,E}\widehat{b}_{n}^{E};Y_{i}) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_{i}^{\top}\nabla^{2}\psi(X_{i}\widehat{b}_{n};Y_{i})X_{i,E}\left(\widehat{\beta}_{n,E}^{(\lambda)} - \widehat{b}_{n}^{E}\right) + \lambda S = \sqrt{n}\omega_{n} + R_{n,2},$$

where $R_{n,2}$ is an $o_p(1)$ term with the Taylor series remainder. Using the fact that $\sum_{i=1}^n X_{i,E}^\top \nabla \psi(X_{i,E} \hat{b}_n^E; Y_i) = 0$ and replacing the sample average $\frac{1}{n} \sum_{i=1}^n X_i^\top \nabla^2 \psi(X_i \hat{b}_n; Y_i) X_{i,E}$ with its expected value H_E , the above equation can be expressed as

$$\begin{bmatrix} 0_E \\ \frac{1}{\sqrt{n}\sum_{i=1}^n X_{i,E'}^\top \nabla \psi(Y_i, X_{i,E}\hat{b}_n^E) \end{bmatrix} + \begin{bmatrix} H_{E,E} \\ H_{E',E} \end{bmatrix} \sqrt{n}\hat{\beta}_{n,E}^{(\lambda)} - \begin{bmatrix} H_{E,E} \\ H_{E',E} \end{bmatrix} \sqrt{n}\hat{b}_n^E + \lambda \begin{bmatrix} S_E \\ S_{E'} \end{bmatrix} = \sqrt{n}\omega_n + R_{n,2}$$

where $R_{n,2}$ is an $o_p(1)$ term.

Recall the definition of $\widehat{g}_n^{E,j} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathbb{R}^{p-1}$ in (18). Since,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_{i,E'}^{\top} \nabla \psi(Y_i, X_{i,E} \widehat{b}_n^E) = \sqrt{n}T_2 + \left(H_{E',E} - K_{E',E} K_{E,E}^{-1} H_{E,E}\right) \widehat{b}_n^E$$

we further obtain that

$$\begin{bmatrix} 0_q \\ \sqrt{n}T_2 \end{bmatrix} - \begin{bmatrix} H_{E,E} \\ K_{E',E}K_{E,E}^{-1}H_{E,E} \end{bmatrix} \sqrt{n}\widehat{b}_n^E + H_E\sqrt{n}\widehat{\beta}_{n,E}^{(\lambda)} + \lambda \begin{bmatrix} S_E \\ S_{E'} \end{bmatrix} = \sqrt{n}\omega_n + R_{n,2}.$$

For the second term on the left-hand side display, note that

$$\begin{bmatrix} H_{E,E} \\ K_{E',E}K_{E,E}^{-1}H_{E,E} \end{bmatrix} \sqrt{n}\widehat{b}_{n}^{E} = \begin{bmatrix} H_{E,E} \\ K_{E',E}K_{E,E}^{-1}H_{E,E} \end{bmatrix} \left(\sqrt{n}\widehat{b}_{n}^{E} - \frac{1}{(\sigma^{E,j})^{2}}\Sigma_{E,j}\sqrt{n}\widehat{b}_{n}^{E,j} + \frac{1}{(\sigma^{E,j})^{2}}\Sigma_{E,j}\sqrt{n}\widehat{b}_{n}^{E,j} \right)$$
$$= \begin{bmatrix} H_{E,E}\mathbb{S}_{[E]\setminus j}^{\mathsf{T}} \\ K_{E',E}K_{E,E}^{-1}H_{E,E}\mathbb{S}_{[E]\setminus j}^{\mathsf{T}} \end{bmatrix} \mathbb{S}_{[E]\setminus j}\sqrt{n} \left(\widehat{b}_{n}^{E} - \frac{1}{(\sigma^{E,j})^{2}}\Sigma_{E,j}\widehat{b}_{n}^{E,j} \right)$$
$$+ \begin{bmatrix} H_{E,E} \\ K_{E',E}K_{E,E}^{-1}H_{E,E} \end{bmatrix} \frac{1}{(\sigma^{E,j})^{2}}\Sigma_{E,j}\sqrt{n}\widehat{b}_{n}^{E,j}$$
$$= \begin{bmatrix} H_{E,E}\mathbb{S}_{[E]\setminus j}^{\mathsf{T}} \\ K_{E',E}K_{E,E}^{-1}H_{E,E}\mathbb{S}_{[E]\setminus j}^{\mathsf{T}} \end{bmatrix} \sqrt{n}T_{1} + \frac{1}{(\sigma^{E,j})^{2}}K_{E}H_{E,E}^{-1}\mathbb{S}_{j}^{\mathsf{T}}\sqrt{n}\widehat{b}_{n}^{E,j}$$

Therefore, we have

$$\begin{bmatrix} 0_q \\ \sqrt{n}T_2 \end{bmatrix} - \begin{bmatrix} H_{E,E} \\ K_{E',E}K_{E,E}^{-1}H_{E,E} \end{bmatrix} \sqrt{n}\widehat{b}_n^E = P_1^{E\cdot j}\sqrt{n}\widehat{b}_n^{E\cdot j} + P_2^{E\cdot j}\sqrt{n}\widehat{g}_n^{E\cdot j},$$
to our assertion.

which leads to our assertion.

Proof of Proposition 4.4. Firstly observe

$$\left\{\widehat{S}_n = S, \widehat{V}_n^{E \cdot j} = V^{E \cdot j}\right\} = \left\{\operatorname{Diag}(S_E)\widehat{\beta}_{n,E}^{(\lambda)} > 0, \widehat{S}_{n,E'} = S_{E'}, \widehat{V}_n^{E \cdot j} = V^{E \cdot j}\right\}.$$

Now, basic algebra leads to

$$\sqrt{n} \operatorname{Diag}(S_E) \widehat{\beta}_{n,E}^{(\lambda)} = \sqrt{n} |\widehat{\beta}_{n,E}^{(\lambda)}| = \left(I_q - Q^{E \cdot j} (\eta^{E \cdot j})^\top \right) |\widehat{\beta}_{n,E}^{(\lambda)}| + Q^{E \cdot j} (\eta^{E \cdot j})^\top |\widehat{\beta}_{n,E}^{(\lambda)}|$$
$$= \widehat{V}_n^{E \cdot j} + Q^{E \cdot j} \widehat{U}_n^{E \cdot j}.$$

Thus when $\widehat{V}_n^{E\cdot j} = V^{E\cdot j}$ the constraint $\text{Diag}(S_E)\widehat{\beta}_{n,E}^{(\lambda)} > 0$ becomes

$$Q^{E \cdot j} \widehat{U}_n^{E \cdot j} > -V^{E \cdot j} \iff \max_{k: \mathbb{S}_k Q^{E \cdot j} > 0} \left\{ \frac{-\mathbb{S}_k V^{E \cdot j}}{\mathbb{S}_k Q^{E \cdot j}} \right\} \le \widehat{U}_n^{E \cdot j} \le \min_{k: \mathbb{S}_k Q^{E \cdot j} < 0} \left\{ \frac{-\mathbb{S}_k V^{E \cdot j}}{\mathbb{S}_k Q^{E \cdot j}} \right\}$$

Proof of Theorem 4.1. First note for a fixed set E and S_E , $\sqrt{n}\hat{b}_n^{E,j}$, $\sqrt{n}\hat{g}_n^{E,j}$, and $\sqrt{n}\omega_n$ are independent variables and have joint normal density

$$\phi(\sqrt{n}\widehat{b}_n^{E.j};\sqrt{n}\beta_n^{E.j},\sigma_j^2)\phi\left(\sqrt{n}\gamma_n^{E.j};\sqrt{n}\widehat{g}_n^{E.j},\Sigma^{E.j}\right)\phi(\sqrt{n}w_n;0,\Omega).$$

Using the mapping in (10) and Proposition 4.4, we know that

$$\Pi_{\sqrt{n}\widehat{b}_{n}^{E\cdot j},\sqrt{n}\widehat{g}_{n}^{E\cdot j}}(\widehat{U}_{n}^{E\cdot j},\widehat{V}_{n}^{E\cdot j},\widehat{S}_{n,E'}) = \sqrt{n}\widetilde{\omega}_{n} = \sqrt{n}\omega_{n} + R_{n,2}.$$

Also as $\sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}| = H_E Q^{E \cdot j} \widehat{U}_n^{E.j} + H_E \widehat{V}_n^{E.j}$, the function can be seen as

$$\begin{split} \widetilde{\Pi}_{\sqrt{n}\widehat{b}_{n}^{E,j},\sqrt{n}\widehat{g}_{n}^{E,j}}(\sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}|,\widehat{S}_{n,E'}) &= \sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}| + \lambda \begin{bmatrix} S_{E} \\ 0 \end{bmatrix} + P_{1}^{E,j}\sqrt{n}\widehat{b}_{n}^{E,j} + P_{2}^{E,j}\sqrt{n}\widehat{g}_{n}^{E,j} \\ &= \Pi_{\sqrt{n}\widehat{b}_{n}^{E,j},\sqrt{n}\widehat{g}_{n}^{E,j}}(\widehat{U}_{n}^{E,j},\widehat{V}_{n}^{E,j},\widehat{S}_{n,E'}). \end{split}$$

To derive the density for the optimization variables, consider the following change of variables $f: \mathbb{R}^{2p} \to \mathbb{R}^{2p}$, for $(\sqrt{n}\widehat{b}_n^{E.j}, \sqrt{n}\widehat{g}_n^{E.j}, \sqrt{n}|\widehat{\beta}_{n,E'}^{(\lambda)}|, \widehat{S}_{n,E'}) \in \mathbb{R} \times \mathbb{R}^{p-1} \times \mathbb{R}^q \times \mathbb{R}^{q'}$

$$f(\sqrt{n}\widehat{b}_{n}^{E.j}, \sqrt{n}\widehat{g}_{n}^{E.j}, \sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}|, \widehat{S}_{n,E'}) = \begin{bmatrix} \sqrt{n}\widehat{b}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \widetilde{\Pi}_{\sqrt{n}\widehat{b}_{n}^{E.j}, \sqrt{n}\widehat{g}_{n}^{E.j}}(\sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}|, \widehat{S}_{n,E'}) \end{bmatrix}$$
so
$$f: \begin{bmatrix} \sqrt{n}\widehat{b}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n,E'}^{E.j} \end{bmatrix} \mapsto \begin{bmatrix} \sqrt{n}\widehat{b}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \sqrt{n}\widehat{g}_{n}^{E.j} \\ \sqrt{n}\omega_{n} \end{bmatrix}$$

Hence, asymptotically the joint density of the variables $\sqrt{n}\widehat{b}_{n}^{E.j}, \sqrt{n}\widehat{g}_{n}^{E.j}, \sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}|, \widehat{S}_{n,E'}$ is equal to

$$= |J|f\left(\pi(\sqrt{n}\widehat{b}_{n}^{E.j}, \sqrt{n}\widehat{g}_{n}^{E.j}, \sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}|, \widehat{S}_{n,E'})\right)$$

$$= |J|\phi(\sqrt{n}\widehat{b}_{n}^{E.j}; \sqrt{n}\beta_{n}^{E.j}, (\sigma^{E\cdot j})^{2})\phi\left(\sqrt{n}\widehat{g}_{n}^{E.j}; \sqrt{n}\gamma_{n}^{E.j}, \Sigma^{E.j}\right)$$

$$\times \phi\left(P_{1}^{E\cdot j}\sqrt{n}\widehat{b}_{n}^{E.j} + P_{2}^{E\cdot j}\sqrt{n}\widehat{g}_{n}^{E.j} + H_{E}\sqrt{n}|\widehat{\beta}_{n,E}^{(\lambda)}| + \lambda \begin{bmatrix} S_{E}\\ \widehat{S}_{n,E'} \end{bmatrix}; 0_{p}, \Omega \right),$$

where

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \mathbb{S}_E P_1^{E \cdot j} & \mathbb{S}_E P_2^{E \cdot j} & H_{E,E} & 0 \\ \mathbb{S}_{-E} P_1^{E \cdot j} & \mathbb{S}_{-E} P_2^{E \cdot j} & H_{-E,E} & \lambda I \end{bmatrix}, \text{ so the determinant is}$$
$$\implies |J| = \begin{vmatrix} H_{E,E} & 0 \\ H_{-E,E} & \lambda I \end{vmatrix}$$

11 Appendix- Asymptotic Theory in Section 4.5

11.1 Main Results

Before developing the proofs for the asymptotic theory in this paper, we establish a few more notations. Recall from Proposition 4.2 that our master statistics satisfy the following representation:

$$\sqrt{n} \begin{pmatrix} \widehat{b}_n^{E,j} - \beta_n^{E,j} \\ \widehat{g}_n^{E,j} - \gamma_n^{E,j} \end{pmatrix} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \zeta_n + R_{n,1},$$

where $R_{n,1} = o_p(1)$.

For the proofs in this section, we will express the pivot in (12) in terms of the standardized variable, Υ_n , which is defined as:

$$\Upsilon_n = \sqrt{n} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^{-1} \begin{bmatrix} \widehat{b}_n^{E,j} - \beta_n^{E,j} \\ \widehat{g}_n^{E,j} - \gamma_n^{E,j} \end{bmatrix} = \zeta_n + \Delta_{n,1}, \text{ with } \Delta_{n,1} := \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^{-1} R_{n,1}.$$
(19)

Then, since $\sqrt{n}\hat{b}_n^{E.j} = M_1\Upsilon_n + \sqrt{n}\beta_n^{E.j}$, $\sqrt{n}\hat{g}_n^{E.j} = M_2\Upsilon_n + \sqrt{n}\gamma_n^{E.j}$, our pivot in terms of the standardized variables Υ is equal to:

$$P^{E,j}(\Upsilon_n) = \frac{\int_{-\infty}^{M_1\Upsilon_n + \sqrt{n}\beta_n^{E,j}} \phi(x; \sqrt{n}\beta_n^{E,j}, (\sigma^{E,j})^2) F(x, M_2\Upsilon_n + \sqrt{n}\gamma_n^{E,j}) dx}{\int_{-\infty}^{\infty} \phi(x; \sqrt{n}\beta_n^{E,j}, (\sigma^{E,j})^2) F(x, M_2\Upsilon_n + \sqrt{n}\gamma_n^{E,j}) dx}.$$

Furthermore, we express the function $F(\cdot)$, which adjusts for the selection process, in terms of Υ as:

$$F(\Upsilon_n) = \int_{I_-^{E,j}}^{I_+^{E,j}} \phi\left(Lt + \widetilde{P}\Upsilon_n + T_n; 0_p, \Omega\right) dt,$$
(20)

where

$$\widetilde{P} = (P_1 M_1 + P_2 M_2), \ T_n = P_1 \sqrt{n} \beta_n^{E.j} + P_2 \sqrt{n} \gamma_n^{E.j} + T.$$
(21)

To prove our main result in Theorem 4.2, we start with a supporting result that provides sufficient conditions for the main result to hold.

Theorem 11.1. For $\mathcal{H} \in \mathbb{C}^3(\mathbb{R}, \mathbb{R})$ an arbitrary function with bounded derivatives up to the third order and for an increasing sequence of sets $\{D_n : n \in \mathbb{N}\}$ in \mathbb{R}^p such that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{P}_{\mathbb{F}_{n}} \left[\zeta_{n} \in D_{n}^{c} \right] = 0,$$

define the relative difference terms

$$\operatorname{RD}_{n}^{(1)} = \frac{\left|\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right] - \mathbb{E}_{\mathcal{N}}\left[F(Z)\mathbb{1}_{D_{n}}(Z)\right]\right|}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]}$$
$$\operatorname{RD}_{n}^{(2)} = \frac{\left|\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E.j}\left(\zeta_{n}\right)\times F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right] - \mathbb{E}_{\mathcal{N}}\left[\mathcal{H}\circ P^{E.j}(Z)\times F(Z)\mathbb{1}_{D_{n}}(Z)\right]\right|}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]}.$$

Suppose that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(1)} = 0, \quad \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(2)} = 0.$$

Then under Assumption 3, Theorem 4.2 holds.

Proof. To prove the claim in Theorem 4.2, it is sufficient to show that for any arbitrary function $\mathcal{H} \in \mathbb{C}^3(\mathbb{R}, \mathbb{R})$ with bounded derivatives up to the third order, the following statement holds:

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} | \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\zeta_{n} \right) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} = V^{E.j} \right\} \right] \\ - \mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E.j}(Z) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} = V^{E.j} \right\} \right] | = 0.$$

By applying Lemma 11.6, we observe that a sufficient condition for the convergence mentioned above is to establish a bound on

$$\left|\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E.j}\left(\zeta_{n}\right)\times F\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]}-\frac{\mathbb{E}_{\mathcal{N}}\left[\mathcal{H}\circ P^{E.j}(Z)\times F(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]}\right|.$$
(22)

In the rest of the proof, we obtain a bound for (22) in terms of the relative difference terms in our assertion.

Note that the difference in (22) is bounded as

$$Bd_0 + Bd_1 + Bd_2 + Bd_3$$

where

$$Bd_{0} = \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} (\zeta_{n}) \times F (\zeta_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F (\zeta_{n}) \right]} \right|$$

$$Bd_{1} = \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \mathbb{1}_{D_{n}} \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \mathbb{1}_{D_{n}} \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} \right| ,$$

$$Bd_{2} = \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \mathbb{1}_{D_{n}} \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} - \frac{\mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E.j} \left(Z \right) \times F(Z) \mathbb{1}_{D_{n}} \left(Z \right) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} \right| ,$$

$$Bd_{3} = \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \mathbb{1}_{D_{n}^{c}} \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \right]} - \frac{\mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E.j} \left(Z \right) \times F(Z) \mathbb{1}_{D_{n}^{c}} \left(Z \right) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} \right| .$$

To begin with, according to Lemma 11.5, we can conclude that:

$$\lim_{n} \sup_{\mathbb{F}_n \in \mathcal{F}_n} \mathrm{Bd}_0 = 0.$$

Moving to the next term, we observe that

$$Bd_{1} \leq \left| \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j}\left(\zeta_{n}\right) \times F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}}\left(\zeta_{n}\right) \right] \right| \times \left| \frac{1}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} - \frac{1}{\mathbb{E}_{\mathcal{N}}\left[F\left(\zeta_{n}\right)\right]} \right|$$
$$\leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \left| \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathcal{N}}\left[F\left(\zeta_{n}\right)\right]} \right|.$$

By applying the triangle inequality once more, we obtain the following bound for Bd₁:

$$\begin{aligned} \operatorname{Bd}_{1} &\leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} |\mathcal{H}| \times \left\{ \left| \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} - \frac{\mathbb{E}_{\mathcal{N}}\left[F(Z)\mathbb{1}_{D_{n}}(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]} \right. \\ &+ \left| \frac{\mathbb{E}_{\mathcal{N}}\left[F(Z)\mathbb{1}_{D_{n}}(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathcal{N}}\left[F\left(\zeta_{n}\right)\right]} \right| \right\} \\ &= \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} |\mathcal{H}| \times \left\{ \frac{\left| \mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} - \frac{\mathbb{E}_{\mathcal{N}}\left[F(Z)\mathbb{1}_{D_{n}^{c}}(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]} \right| \\ &+ \left| \frac{\mathbb{E}_{\mathcal{N}}\left[F(Z)\mathbb{1}_{D_{n}}(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathcal{N}}\left[F\left(\zeta_{n}\right)\right]} \right| \right\} \\ &\leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} |\mathcal{H}| \times \left\{ 2\sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}}\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} + \operatorname{RD}_{n}^{(1)} \right\}. \end{aligned}$$

It is easy to see that Bd_2 is equal to $RD_n^{(2)}$, and that

$$\operatorname{Bd}_{3} \leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} |\mathcal{H}| \times 2 \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \right]}$$

Thus, we conclude that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \right]} - \frac{\mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E \cdot j} (Z) \times F(Z) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} \right] \\
\leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(1)} + \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(2)} \\
+ \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times 4 \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \mathbb{1}_{D_{n}^{c}} \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \right]}.$$

Using Lemma 11.4, which proves that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \frac{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]} = 0,$$

we conclude that the difference in (22) uniformly converges to 0. This completes our proof. $\hfill \Box$

To prove our main result on the asymptotic validity of the pivot, we establish the convergence of the two relative difference terms identified in Theorem 11.1. We divide the proof of our result into two cases based on how the sequence of parameters $\sqrt{n} \left[(\beta_n^{E.j})^\top \quad (\gamma_n^{E.j})^\top \right]^\top$ grows with increasing *n*. First, Theorem 11.2 demonstrates the convergence of both relative differences when r_n does not grow with *n*. Then, Theorem 11.3 extends this proof to the case where we allow r_n to grow to ∞ .

Theorem 11.2. Suppose that $r_n \leq C$ for some constant $C \in \mathbb{R}$. Under Assumptions 2 and 3, we have that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(1)} = 0, \quad \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(2)} = 0,$$

we let $D_n = \mathbb{R}^p$ for $n \in \mathbb{N}$.

Proof. We start by analyzing the denominator of the two relative difference terms and note that

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right] \propto \mathbb{E}_{\mathcal{N}}\left[\int_{I_{-}^{E\cdot j}}^{\infty} \exp\left\{-\frac{1}{2}\left(Lt + \widetilde{P}Z + T_{n}\right)^{\top}\Omega^{-1}\left(Lt + \widetilde{P}Z + T_{n}\right)\right\}dt\right].$$

Let $\mathcal{D}_{C_1} = [-C_1 \cdot \mathbf{1}_p, C_1 \cdot \mathbf{1}_p] \subseteq \mathbb{R}^p$ for a large enough $C_1 > 0$ such that

$$\mathbb{P}_{\mathcal{N}}\left[Z \in \mathcal{D}_{C_1}\right] \geq \frac{1}{2}.$$

Then for $Z \in \mathcal{D}_{C_1}$ and $r_n \leq C$, we have that

$$\exp\left\{-\frac{1}{2}\left(Lt+\widetilde{P}Z-r_nb-LI_{-}^{E\cdot j}\right)^{\top}\Omega^{-1}\left(Lt+\widetilde{P}Z-r_nb-LI_{-}^{E\cdot j}\right)\right\}\geq d_2,$$

where d_2 is some positive constant. This leads us to observe that

$$\mathbb{E}_{\mathcal{N}}[F(Z)] \ge C_2 \times \mathbb{P}_{\mathcal{N}}[Z \in \mathcal{D}_{C_1}] \ge C_2 \times \frac{1}{2}.$$

Our assertion holds if we show that the numerators in the two relative difference terms converge to 0. In the remaining proof, we show that:

1. $\sup_{\mathbb{F}_n \in \mathcal{F}_n} |\mathbb{E}_{\mathbb{F}_n} [F(\zeta_n)] - \mathbb{E}_{\mathcal{N}} [F(Z)]| \leq \frac{1}{\sqrt{n}}$ 2. $\sup_{\mathbb{F}_n \in \mathcal{F}_n} |\mathbb{E}_{\mathbb{F}_n} [\mathcal{H} \circ P^{E.j}(\zeta_n) \times F(\zeta_n)] - \mathbb{E}_{\mathcal{N}} [\mathcal{H} \circ P^{E.j}(Z) \times F(Z)]| \leq \frac{1}{\sqrt{n}}.$

We then apply the Stein bound from Lemma 11.7 to the real-valued functions $G^{(1)}(Z) = F(Z)$, $G^{(2)}(Z) = H \circ P^{E,j}(Z) \times F(Z)$, using $\mathcal{W}_{\epsilon_1,\epsilon_2}, \zeta_n[-1], a_{1,n}$, and $a_{1,n}^*$ as defined in this result. This yields:

$$\begin{aligned} \left\| \mathbb{E}_{\mathbb{F}_{n}} \left[G(\zeta_{n}) \right] - \mathbb{E}_{\mathcal{N}}[G(Z)] \right\| \\ \lesssim \frac{1}{\sqrt{n}} \sum_{\substack{\lambda, \gamma \in \mathbb{N}: \\ \lambda + \gamma \leq 3}} \sum_{i_{1}, i_{2}, i_{3} \in [p]} \mathbb{E}_{\mathbb{F}_{n}} \left[\left\| a_{1,n} \right\|^{\lambda} \left\| a_{1,n}^{*} \right\|^{\gamma} \right. \\ \left. \times \sup_{\epsilon_{1}, \epsilon_{2} \in [0,1]} \int_{0}^{1} \frac{\sqrt{t}}{2} \mathbb{E}_{\mathcal{N}} \left[\left| \partial_{i_{1}, i_{2}, i_{3}}^{3} G\left(\sqrt{t} \mathcal{W}_{\epsilon_{1}, \epsilon_{2}} + \sqrt{1 - t} Z\right) \right| \right] dt \right]. \end{aligned}$$

$$(23)$$

Using the behavior of the pivot and weight function in Propositions 11.2 and 11.3, we have that

$$\begin{aligned} \left| \partial_{i_1, i_2, i_3}^3 G^{(l)} \left(\sqrt{t} \mathcal{W}_{\alpha, \kappa} + \sqrt{1 - t} Z \right) \right| &\lesssim \sum_{l=0}^3 \left\| L \sqrt{t} \mathcal{W}_{\alpha, \kappa} + L \sqrt{1 - t} Z + T_n \right\|^l \\ &\lesssim \sum_{l=0}^3 \sqrt{t} \left\| \mathcal{W}_{\alpha, \kappa} \right\|^l + \sqrt{1 - t} \|Z\|^l, \end{aligned}$$

since $||T_n|| = O(1)$ in this case.

Revisiting the Stein's bound in (23), we conclude that:

$$\frac{1}{\sqrt{n}} \sum_{\substack{\lambda,\gamma \in \mathbb{N}: \\ \lambda+\gamma \leq 3}} \sum_{i_{1},i_{2},i_{3}} \mathbb{E}_{\mathbb{F}_{n}} \left[\|a_{1,n}\|^{\lambda} \|a_{1,n}^{*}\|^{\gamma} \sup_{\alpha,\kappa \in [0,1]} \int_{0}^{1} \frac{\sqrt{t}}{2} \mathbb{E}_{\mathcal{N}} \left[\sum_{l=0}^{3} \sqrt{t} \|\mathcal{W}_{\alpha,\kappa}\|^{l} + \sqrt{1-t} \|Z\|^{l} \right] dt \right] \\
\lesssim \frac{1}{\sqrt{n}} \sum_{\lambda,\gamma \in \mathbb{N}: \lambda+\gamma \leq 3} \mathbb{E}_{\mathbb{F}_{n}} \left[\|a_{1,n}\|^{\lambda} \|a_{1,n}^{*}\|^{\gamma} \sup_{\alpha,\kappa \in [0,1]} \|\mathcal{W}_{\alpha,\kappa}\|^{3} \right] \\
\lesssim \frac{1}{\sqrt{n}} \sup_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[\|a_{1,n}\|^{6} \right] \lesssim \frac{1}{\sqrt{n}},$$

which proves our claim.

Now we consider the set-up, where the parameters grow as $O(r_n)$, where $r_n \to \infty$, i.e.,

$$\sqrt{n} \left[(\beta_n^{E,j})^\top \quad (\gamma_n^{E,j})^\top \right]^\top = r_n \beta,$$

for some given constant vector $\beta \in \mathbb{R}^p$. Observe that $T_n = O(r_n)$. To simplify the notations in our proof, we will hereafter simply write

$$T_n + LI_-^{E \cdot j} = P_1 \sqrt{n} \beta_n^{E \cdot j} + P_2 \sqrt{n} \gamma_n^{E \cdot j} + T + LI_-^{E \cdot j} = -r_n b \in \mathbb{R}^p,$$

where $b \in \mathbb{R}^p$ is a fixed vector.

Theorem 11.3. Let $r_n \to \infty$ as $n \to \infty$, with a rate of $o(n^{\frac{1}{6}})$. Let the increasing sequence of sets D_n be as defined according to Proposition 11.4. Then under Assumptions 2, 3, 4, it holds that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(1)} = 0, \quad \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathrm{RD}_{n}^{(2)} = 0.$$

Proof. For this proof, we consider two cases based on the sign of $L^{\top}\Omega^{-1}b$. As shown in Proposition 11.4, the denominator in the relative differences bound, has different rates of decay for the two cases. We provide a proof when $L^{\top}\Omega^{-1}b > 0$. The same proof strategy applies similarly to the other case when $L^{\top}\Omega^{-1}b < 0$.

For the numerators of the relative differences, we will show in the rest of the proof that: **(B1)**. $\sup_{\mathbb{F}_n \in \mathcal{F}_n} |\mathbb{E}_{\mathbb{F}_n} [F(\zeta_n) \mathbb{1}_{\mathcal{D}_n}(\zeta_n)] - \mathbb{E}_{\mathcal{N}} [F(Z) \mathbb{1}_{\mathcal{D}_n}(Z)]| \leq \frac{r_n^2}{\sqrt{n}} \exp\left(-r_n b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right);$ **(B2)**. $\sup_{\mathbb{F}_n \in \mathcal{F}_n} |\mathbb{E}_{\mathbb{F}_n} [\mathcal{H} \circ P^{E \cdot j}(\zeta_n) \times F(\zeta_n) \mathbb{1}_{\mathcal{D}_n}(\zeta_n)] - \mathbb{E}_{\mathcal{N}} [\mathcal{H} \circ P^{E \cdot j}(Z) \times F(Z) \mathbb{1}_{\mathcal{D}_n}(Z)]$

$$\leq \frac{r_n^2}{\sqrt{n}} \operatorname{Exp}\left(-r_n b, \left[\Omega + \widetilde{P}\widetilde{P}^{\mathsf{T}}\right]^{-1}\right).$$

Define the real-valued functions

$$\bar{G}^{(1)}(Z) = \operatorname{Exp}\left(\tilde{P}Z + T_n + LI_{-}^{E\cdot j}, \Omega^{-1}\right) \times \mathcal{K}_n^1(Z) \mathbb{1}_{\mathcal{D}_n}(Z)$$
$$\bar{G}^{(2)}(Z) = \mathcal{H} \circ P^{E\cdot j}(Z) \times \operatorname{Exp}\left(\tilde{P}Z + T_n + LI_{-}^{E\cdot j}, \Omega^{-1}\right) \times \mathcal{K}_n^1(Z) \mathbb{1}_{\mathcal{D}_n}(Z)$$

where \mathcal{K}_n^1 and the increasing sequence of sets \mathcal{D}_n are as defined in Proposition 11.4. Note that, $\bar{G}^{(1)}(Z)$ and $\bar{G}^{(2)}(Z)$ are equal to F(Z) and $\mathcal{H} \circ P^{E \cdot j}(Z) \times F(Z)$ respectively on the set \mathcal{D}_n .

Using the Stein bound in Lemma 11.7, for $G = \overline{G}^{(l)}$ for $l \in \{1, 2\}$, and the properties of our pivot, F and \mathcal{K}_n^1 in Propositions 11.2 and 11.3, we note that

$$\left| \partial_{i_1,i_2,i_3}^3 \bar{G}^{(l)} \left(\sqrt{t} \mathcal{W}_{\alpha,\kappa} + \sqrt{1-t} Z \right) \right| \lesssim \sum_{l=0}^3 r_n^{-1} \left\| L \sqrt{t} \mathcal{W}_{\alpha,\kappa} + L \sqrt{1-t} Z + T_n \right\|^l \\ \times \operatorname{Exp} \left(L \sqrt{t} \mathcal{W}_{\alpha,\kappa} + L \sqrt{1-t} Z + T_n, \Omega^{-1} \right).$$

Therefore, the expected value of the above-stated partial derivative satisfies:

$$\begin{split} & \mathbb{E}_{\mathcal{N}}\left[\left|\partial_{i_{1},i_{2},i_{3}}^{3}\bar{G}^{(l)}\left(\sqrt{t}\mathcal{W}_{\alpha,\kappa}+\sqrt{1-t}Z\right)\right|\right] \\ &\lesssim \sum_{\substack{\lambda,\gamma\in\mathbb{N}:\\\lambda+\gamma\leq3}} r_{n}^{-1} \left\|\mathcal{W}_{\alpha,\kappa}\right\|^{\lambda} \left\|T_{n}\right\|^{\gamma} \exp\left(L\sqrt{t}\mathcal{W}_{\alpha,\kappa}+T_{n},\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right) \\ &\lesssim \sum_{\substack{\bar{\lambda},\bar{\kappa}\bar{\lambda},\bar{\kappa}\in\mathbb{N}:\\\bar{\lambda}+\bar{\kappa}+\lambda+\kappa\leq3}} r_{n}^{\bar{\lambda}-1} \left\|\zeta_{n}[-1]\right\|^{\check{}} \left\|\frac{a_{1,n}}{\sqrt{n}}\right\|^{\bar{\lambda}} \left\|\frac{a_{1,n}^{*}}{\sqrt{n}}\right\|^{\bar{\kappa}} \exp\left(L\sqrt{t}\mathcal{W}_{\alpha,\kappa}-r_{n}b,\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right). \end{split}$$

Plugging this into the Stein bound obtained in Proposition 11.7, we have that

$$\begin{split} & \left| \mathbb{E}_{\mathbb{F}_{n}} \left[\bar{G}^{(l)} \left(\zeta_{n} \right) \right] - \mathbb{E}_{\mathcal{N}} \left[\bar{G}^{(l)}(Z) \right] \right| \\ & \lesssim \frac{1}{\sqrt{n}} \sum_{\substack{\lambda, \gamma \in \mathbb{N}: i_{1}, i_{2}, i_{3} \in [p] \\ \lambda + \gamma \leq 3}} \mathbb{E}_{\mathbb{F}_{n}} \left[\left\| a_{1,n} \right\|^{\lambda} \left\| a_{1,n}^{*} \right\|^{\gamma} \sup_{\alpha, \kappa \in [0,1]} \int_{0}^{1} \frac{\sqrt{t}}{2} \right] \\ & \times \sum_{\substack{\bar{\lambda}, \bar{\kappa}, \bar{\lambda}, \bar{\kappa} \in \mathbb{N}: \\ \bar{\lambda} + \bar{\kappa} + \lambda + \bar{\kappa} \leq 3}} r_{n}^{\check{\lambda} - 1} \left\| \zeta_{n} [-1] \right\|^{\check{\kappa}} \left\| \frac{a_{1,n}}{\sqrt{n}} \right\|^{\bar{\lambda}} \left\| \frac{a_{1,n}^{*}}{\sqrt{n}} \right\|^{\bar{\kappa}} \operatorname{Exp} \left(L \sqrt{t} \mathcal{W}_{\alpha, \kappa} - r_{n} b, \left[\Omega + \widetilde{P} \widetilde{P}^{\top} (1 - t) \right]^{-1} \right) dt \end{split} \right]. \end{split}$$

Next, we simplify the bound on the right-hand side display. In particular, we show the simplification for $\bar{\lambda} = \bar{\kappa} = \bar{\kappa} = 0$, $\bar{\lambda} = 3$, noting that a similar approach applies to other values of $\bar{\lambda}, \bar{\kappa}, \bar{\lambda}, \bar{\kappa}$. In this case, applying the result from Proposition 11.7 gives us:

$$\sup_{\mathbb{F}_n \in \mathcal{F}_n} \left| \mathbb{E}_{\mathbb{F}_n} \left[\bar{G}^{(l)} \left(\zeta_n \right) \right] - \mathbb{E}_{\mathcal{N}} \left[\bar{G}^{(l)}(Z) \right] \right| \lesssim \frac{r_n^2}{\sqrt{n}} \exp\left(-r_n b, \left[\Omega + \widetilde{P} \widetilde{P}^\top \right]^{-1} \right)$$

Finally, it is immediate from Proposition 11.6 that

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]^{-1} \ge C \operatorname{Exp}\left(-r_n b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right),$$

for some constant C. Combining the above lower bound with **(B1)** and **(B2)** completes the proof of our claims.

11.2 Supporting Results

Lemma 11.4. Let D_n be a increasing sequence of sets in \mathbb{R}^p such that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{P}_{\mathbb{F}_{n}} \left(\zeta_{n} \in D_{n}^{c} \right) = 0,$$

then we have

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} = 0.$$

Proof. Fix $\varepsilon > 0$. By construct, we have n_0 such that for all $n \ge n_0$,

$$\mathbb{E}_{\mathbb{F}_n}\left[\mathbbm{1}_{D_n^c}\left(\zeta_n\right)\right] < \varepsilon$$

for all $\mathbb{F}_n \in \mathcal{F}_n$. As a result, we note that

$$\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\left(\mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right)-\varepsilon\right)\right] \leq \sup_{\zeta_{n}}F(\zeta_{n})\times\mathbb{E}_{\mathbb{F}_{n}}\left[\mathbb{1}_{D_{n}^{c}}\left(\zeta_{n}\right)-\varepsilon\right] < 0$$

for all $n \ge n_0$ and $\mathbb{F}_n \in \mathcal{F}_n$.

Equivalently, for any $\varepsilon > 0$, there exists a n_0 such that for all $n \ge n_0$ and $\mathbb{F}_n \in \mathcal{F}_n$,

$$\mathbb{E}_{\mathbb{F}_n}\left[F\left(\zeta_n\right)\mathbb{1}_{D_n^c}\left(\zeta_n\right)\right] < \varepsilon \mathbb{E}_{\mathbb{F}_n}\left[F(\zeta_n)\right].$$

This proves our claim.

Lemma 11.5. Under Assumption 4, it holds that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} (\zeta_{n}) \times F (\zeta_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F (\zeta_{n}) \right]} \right| = 0.$$

Proof. To prove this, we apply the triangle inequality:

$$\left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} (\zeta_{n}) \times F (\zeta_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} \right| \\
\leq \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} \right| \\
+ \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} (\zeta_{n}) \times F(\zeta_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} \right|.$$

For the first term on the right-hand side of the display, note that

$$\begin{aligned} &\left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} \right| \\ &\leq \left| \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right] \right| \times \left| \frac{1}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} - \frac{1}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\zeta_{n}) \right]} \right| \\ &\leq \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n} \right) \right]} \right| \\ &= \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n} \right) \right] - \mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n} \right) \right]} \right|. \end{aligned}$$

We will now show that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n} + \Delta_{n,1}\right) - F\left(\zeta_{n}\right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \right]} \right| = 0,$$
(24)

using the fact that $\Upsilon_n = \zeta_n + \Delta_{n,1}$.

Note that, for any $\varepsilon > 0$, there exists n_0 such that for all $n \ge n_0$, it holds that

$$\sup_{\mathbb{F}_{n}\in\mathcal{F}_{n}}\mathbb{E}_{\mathbb{F}_{n}}\left[\left|\frac{F\left(\zeta_{n}+\Delta_{n,1}\right)}{F\left(\zeta_{n}\right)}-1\right|\right]<\varepsilon.$$

Then by a simple rearrangement, we obtain that

$$\sup_{\mathbb{F}_{n}\in\mathcal{F}_{n}} \mathbb{E}\left[F\left(\zeta_{n}\right)\times\left\{\left|\frac{F\left(\zeta_{n}+\Delta_{n,1}\right)}{F\left(\zeta_{n}\right)}-1\right|-\varepsilon\right\}\right]$$
$$\leq \sup_{z}F(z)\times\sup_{\mathbb{F}_{n}\in\mathcal{F}_{n}}\mathbb{E}_{\mathbb{F}_{n}}\left[\left|\frac{F\left(\zeta_{n}+\Delta_{n,1}\right)}{F\left(\zeta_{n}\right)}-1\right|-\varepsilon\right]<0$$

for all $n \ge n_0$. Hence, for any given $\varepsilon > 0$, there exists a n_0 such that for all $n \ge n_0$ and $\mathbb{F}_n \in \mathcal{F}_n$,

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \times \left| \frac{F\left(\zeta_{n} + \Delta_{n,1}\right)}{F\left(\zeta_{n}\right)} - 1 \right| \right] < \lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \varepsilon \mathbb{E}_{\mathbb{F}_{n}} \left[F\left(\zeta_{n}\right) \right],$$

i.e., the claim in (24) holds.

For the second term in this display, note that it is equal to:

$$\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\Upsilon_{n}\right)\times F\left(\Upsilon_{n}\right)\right]-\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\zeta_{n}\right)\times F\left(\zeta_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\zeta_{n}\right)\right]}\right|.$$

A similar proof strategy can be applied to conclude that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j} \left(\zeta_{n} + \Delta_{n,1} \right) \times F \left(\zeta_{n} + \Delta_{n,1} \right) \right] - \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j} \left(\zeta_{n} \right) \times F \left(\zeta_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\zeta_{n} \right) \right]} \right| = 0.$$

Lemma 11.6. Suppose that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E.j}(Z) \times F(Z) \right]}{\mathbb{E}_{\mathcal{N}} \left[F(Z) \right]} \right| = 0,$$

Then the following holds:

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} | \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} = V^{E.j} \right\} \right] \\ - \mathbb{E}_{\mathcal{N}} \left[\mathcal{H} \circ P^{E.j}(Z) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} = V^{E.j} \right\} \right] | = 0.$$

Proof. Suppose that Z follows $\mathcal{N}_p(0_p, I_{p,p})$, then conditional density of Z at the point z is equal to

$$\frac{\phi(z;0_p,I_{p,p})F(z)}{\int \phi(z';0_p,I_{p,p})F(z')\,dz'} = (\mathbb{E}_{\mathcal{N}}[F(Z)])^{-1}\,\phi(z;0_p,I_{p,p})F(z).$$

This means that the ratio between the conditional and unconditional densities of Υ_n , at the point Z, is to

$$R(Z) = \left(\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]\right)^{-1}F(Z),$$

thus

$$\mathbb{E}_{\mathcal{N}}\left[\mathcal{H} \circ P^{E.j}(Z) \middle| \left\{ \widehat{S}_n = S, \widehat{V}_n^{E.j} = V^{E.j} \right\} \right] = \mathbb{E}_{\mathcal{N}}\left[\mathcal{H} \circ P^{E.j}(Z)R(Z)\right] \\ = \frac{\mathbb{E}_{\mathcal{N}}\left[\mathcal{H} \circ P^{E.j}(Z) \times F(Z)\right]}{\mathbb{E}_{\mathcal{N}}\left[F(Z)\right]}$$

Now for general distributions, i.e., Υ_n follows \mathbb{F}_n , the weight F(Z) might not be a Gaussian integral, as the change of variables is not exact. However, from assumption 1, for sufficiently large n, we can use the Lebesgue density q_n and it's corresponding weight $G_n(Z)$ to get:

$$\mathbb{E}_{\mathbb{F}_n}\left[H \circ P^{E.j}(\Upsilon_n) | \left\{\widehat{S}_n = S, \widehat{V}_n^{E.j} = V^{E.j}\right\}\right] = \frac{\mathbb{E}_{\mathbb{F}_n}\left[H \circ P^{E.j}(\Upsilon_n)G_n(\Upsilon_n)\right]}{\mathbb{E}_{\mathbb{F}_n}[G_n(\Upsilon_n)]}.$$

So in order to get

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \mathbb{E}_{\mathbb{F}_{n}} \left[H \circ P^{E.j}(\Upsilon_{n}) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} \right\} \right] - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j}(\Upsilon_{n}) \times F(\Upsilon_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\Upsilon_{n}) \right]} \right| = 0$$

it is sufficient to prove

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times G_{n} \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[G_{n} \left(\Upsilon_{n} \right) \right]} - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j} \left(\Upsilon_{n} \right) \times F(\Upsilon_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\Upsilon_{n}) \right]} \right| = 0.$$

Note that the limit on the left-hand side is further bounded by

$$\underbrace{\lim_{n}\sup_{\mathbb{F}_{n}}\left|\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\Upsilon_{n}\right)\times G_{n}\left(\Upsilon_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[G_{n}\left(\Upsilon_{n}\right)\right]}-\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\Upsilon_{n}\right)\times G_{n}\left(\Upsilon_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\Upsilon_{n}\right)\right]}\right|}_{(F1)}+\underbrace{\lim_{n}\sup_{\mathbb{F}_{n}}\left|\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\Upsilon_{n}\right)\times G_{n}\left(\Upsilon_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\Upsilon_{n}\right)\right]}-\frac{\mathbb{E}_{\mathbb{F}_{n}}\left[\mathcal{H}\circ P^{E\cdot j}\left(\Upsilon_{n}\right)\times F\left(\Upsilon_{n}\right)\right]}{\mathbb{E}_{\mathbb{F}_{n}}\left[F\left(\Upsilon_{n}\right)\right]}\right|}_{(F2)}.$$

Thus, our proof is complete by showing the limits of the two terms in the sum is 0. Observe that

$$(F1) \leq \lim_{n} \sup_{\mathbb{F}_{n}} \int \left| \mathcal{H} \circ P^{E \cdot j} \left(\Upsilon_{n} \right) \right| \left| \frac{G_{n} \left(\Upsilon_{n} \right)}{\mathbb{E}_{\mathbb{F}_{n}} \left[G_{n} \left(\Upsilon_{n} \right) \right]} - \frac{G_{n} \left(\Upsilon_{n} \right)}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} \right| d\mathbb{F}_{n} \left(\Upsilon_{n} \right)$$
$$\leq \lim_{n} \sup_{\mathbb{F}_{n}} \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[G_{n} \left(\Upsilon_{n} \right) - F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} \right|$$

and that

$$(F2) = \lim_{n} \sup_{\mathbb{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j} \left(\Upsilon_{n} \right) \times G_{n} \left(\Upsilon_{n} \right) \right] - \mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E \cdot j} \left(\Upsilon_{n} \right) \times F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} \right|$$
$$\leq \lim_{n} \sup_{\mathbb{F}_{n}} \sup_{\mathcal{H} \in \mathbb{C}^{3}(\mathbb{R},\mathbb{R})} \left| \mathcal{H} \right| \times \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[G_{n} \left(\Upsilon_{n} \right) - F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} \right|$$

Finally, since the condition in Assumption 4 implies that

$$\lim_{n} \sup_{\mathbb{F}_{n}} \left| \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[G_{n} \left(\Upsilon_{n} \right) - F \left(\Upsilon_{n} \right) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F \left(\Upsilon_{n} \right) \right]} \right| = 0,$$

we have that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \left| \mathbb{E}_{\mathbb{F}_{n}} \left[H \circ P^{E.j}(\Upsilon_{n}) | \left\{ \widehat{S}_{n} = S, \widehat{V}_{n}^{E.j} \right\} \right] - \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\mathcal{H} \circ P^{E.j}(\Upsilon_{n}) \times F(\Upsilon_{n}) \right]}{\mathbb{E}_{\mathbb{F}_{n}} \left[F(\Upsilon_{n}) \right]} \right| = 0.$$

This leads us to our claim.

11.2.1 Asymptotic Bounds

Throughout this section, define $a_{i,n} = K^{-1/2} \left(X_i^{\top} \nabla \psi \left(X_{i,E} \widehat{b}_n^E; Y_i \right) \right)$, and let

$$e_{i,n} = a_{i,n} + \frac{\Delta_{n,1}}{\sqrt{n}}.$$
 (25)

Note that

$$\zeta_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n}$$
, and $\Upsilon_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_{i,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,n} + \Delta_{n,1}$.

Proposition 11.1. Under Assumption 3, it holds that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[\|a_{1,n}\|^{6} \right] < \infty.$$

Proof. For $i \in \{1, 2, ..., n\}$, let $b_{i,n} = X_i^\top \nabla \psi \left(X_{i,E} \widehat{b}_n^E; Y_i \right)$. Under Assumption 2, for some constant $B_0 > 0$, it holds that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[\exp\left(t ||b_{i,n}||\right) \right] \le \exp(B_{0}t^{2}),$$

for all $t \in \mathbb{R}$. Note that

$$a_{i,n} = K^{-1/2} \left(b_{i,n} - \mathbb{E}_{\mathbb{F}_n} b_{i,n} \right) = K^{-1/2} b_{i,n}$$

i.e., $a_{i,n}$ is obtained after centering $b_{i,n}$ and then scaling it by the constant matrix $K^{-1/2}$. Since $b_{i,n}$ is sub-Gaussian with the constant B_0 , and centering and scaling by a constant matrix preserve sub-Gaussianity, it follows that there exists $B_1 = ||K^{-1/2}|| B_0 > 0$ such that

$$\lim_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[\exp\left(t ||a_{i,n}||\right) \right] \le \exp(B_{1}t^{2}),$$

for all $t \in \mathbb{R}$. Hence, $a_{i,n} \in \mathbb{R}^p$ is a sub-Gaussian vector, as a consequence, for any $i \in \{1, 2, ..., n\}$, it holds that

$$\sup_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \mathbb{E}_{\mathbb{F}_{n}} \left[\|a_{i,n}\|^{6} \right] < \infty.$$

Lemma 11.7. Let $\zeta_n[-i] = \zeta_n - \frac{a_{i,n}}{\sqrt{n}}$ be a leave-one out variable, after dropping the *i*th variable $a_{i,n}$. Let

$$\mathcal{W}_{\epsilon_1,\epsilon_2} = \zeta_n[-1] + \frac{\epsilon_1}{\sqrt{n}}a_{1,n} + \frac{\epsilon_2}{\sqrt{n}}a_{1,n}^*$$

where $a_{i,n}^*$ is an independent copy of $a_{i,n}$. Suppose that $G : \mathbb{R}^p \to \mathbb{R}$ is a Lebesgue-almost surely three times differentiable function such that $\mathbb{E}_{\mathcal{N}}[|G(Z)|] < \infty$. It holds that

$$\begin{aligned} \left| \mathbb{E}_{\mathbb{F}_{n}} \left[G(\zeta_{n}) \right] - \mathbb{E}_{\mathcal{N}}[G(Z)] \right| \lesssim & \frac{1}{\sqrt{n}} \sum_{\lambda, \gamma \in \mathbb{N}: \lambda + \gamma \leq 3} \sum_{i_{1}, i_{2}, i_{3} \in [p]} \mathbb{E}_{\mathbb{F}_{n}} \left[\left\| a_{1,n} \right\|^{\lambda} \left\| a_{1,n}^{*} \right\|^{\gamma} \times \\ & \sup_{\epsilon_{1}, \epsilon_{2} \in [0,1]} \int_{0}^{1} \frac{\sqrt{t}}{2} \mathbb{E}_{\mathcal{N}} \left[\left| \partial_{i_{1}, i_{2}, i_{3}}^{3} G\left(\sqrt{t} \mathcal{W}_{\epsilon_{1}, \epsilon_{2}} + \sqrt{1 - t} Z\right) \right| \right] dt \right]. \end{aligned}$$

Proof. The Stein bound in Lemma 2 from Panigrahi (2023) yields:

$$\left|\mathbb{E}_{\mathbb{F}_{n}}\left[G\left(\zeta_{n}\right)\right]-\mathbb{E}_{\mathcal{N}}\left[G(Z)\right]\right| \lesssim \frac{1}{\sqrt{n}} \sum_{\substack{\lambda,\gamma \in \mathbb{N}: i_{1}, i_{2}, i_{3} \\ \lambda+\gamma \leq 3}} \mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma} \sup_{\alpha,\kappa \in [0,1]}\left|\partial_{i_{1}, i_{2}, i_{3}}^{3}\mathcal{D}_{G}\left(\mathcal{W}_{\alpha,\kappa}\right)\right|\right].$$

where

$$\mathcal{D}_G(Z^*) = \int_0^1 \frac{1}{2t} \left(\mathbb{E}_{\mathcal{N}}[G(\sqrt{t}Z^* + \sqrt{1-t}Z)] - \mathbb{E}_{\mathcal{N}}[G(Z)] \right) dt$$

Then by taking derivatives we directly get:

$$\left|\partial_{i_1,i_2,i_3}^3 \mathcal{D}_G\left(\mathcal{W}_{\alpha,\kappa}\right)\left[i_1,i_2,i_3\right]\right| = \int_0^1 \frac{\sqrt{t}}{2} \mathbb{E}_{\mathcal{N}}\left[\left|\partial_{i_1,i_2,i_3}^3 G\left(\sqrt{t}\mathcal{W}_{\alpha,\kappa} + \sqrt{1-t}Z\right)\right|\right] dt$$

which gives us the desired Stein's bound on the difference in the two expectations.

11.2.2 Pivot and Weight Derivatives

For the notations in this section, we denote the partial derivative of a multivariate function $f : \mathbb{R}^p \to \mathbb{R}$, evaluated at $x = (x_1, x_2, \dots, x_p)^\top \in \mathbb{R}^p$, by

$$\partial_{i_1,\dots,i_m}^m f(x) = \frac{\partial^m f(x)}{\partial x_{i_1}\dots\partial x_{i_m}}$$

Proposition 11.2. For $P^{E \cdot j}(Z)$, \widetilde{P} and T_n defined in (20) and (21), it holds that

$$\left|\partial_{i_1,i_2,i_3}^3 P^{E \cdot j}(Z)\right| \le \sum_{l=0}^3 c_l \left\|\widetilde{P}Z + T_n\right\|^l,$$

where c_0, \ldots, c_3 are constants that do not depend on n.

Proof. Recall,

$$P^{E,j}(Z) = \frac{\int_{-\infty}^{M_1 \Upsilon_n + \sqrt{n}\beta_n^{E,j}} \phi(x; \sqrt{n}\beta_n^{E,j}, (\sigma^{E,j})^2) F(x, M_2 Z + \sqrt{n}\gamma_n^{E,j}) dx}{\int_{-\infty}^{\infty} \phi(x; \sqrt{n}\beta_n^{E,j}, (\sigma^{E,j})^2) F(x, M_2 Z + \sqrt{n}\gamma_n^{E,j}) dx}$$

where

$$F(x_1, x_2) = \int_{I_-^{E \cdot j}}^{I_+^{E \cdot j}} \phi(Lt + P_1 x_1 + P_2 x_2 + T; 0_p, \Omega) dt.$$

Applying the Leibniz integral rule, we have that

$$\partial P^{E,j}(Z) = \frac{\int_{-\infty}^{M_1 Z + \sqrt{n} \beta_n^{E,j}} \phi\left(x; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) \partial F\left(x, M_2 Z + \sqrt{n} \gamma_n^{E,j}\right) dx}{\int_{-\infty}^{\infty} \phi\left(x; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) F\left(x, M_2 Z + \sqrt{n} \gamma_n^{E,j}\right) dx} + \frac{\phi\left(M_1 Z + \sqrt{n} \beta_n^{E,j}; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) F\left(M_1 Z + \sqrt{n} \beta_n^{E,j}, M_2 \Upsilon_n + \sqrt{n} \gamma_n^{E,j}\right) M_1}{\int_{-\infty}^{\infty} \phi\left(x; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) F\left(x, M_2 Z + \sqrt{n} \gamma_n^{E,j}\right) dx} - P^{E,j}(Z) \times \frac{\int_{-\infty}^{\infty} \phi\left(x; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) \partial F\left(x, M_2 Z + \sqrt{n} \gamma_n^{E,j}\right) dx}{\int_{-\infty}^{\infty} \phi\left(x; \sqrt{n} b_n^{E,j}, (\sigma^{E,j})^2\right) F\left(x, M_2 Z + \sqrt{n} \gamma_n^{E,j}\right) dx}.$$
(26)

Using the definition of F, we obtain

$$\frac{\partial F\left(x, M_{2}Z + \sqrt{n}\gamma_{n}^{E.j}\right)}{\partial Z} = \int_{I_{-}^{E.j}}^{\infty} -2\left(Lt + P_{1}x + P_{2}M_{2}Z + P_{2}\sqrt{n}\gamma_{n}^{E.j} + T\right)^{\top}\Omega^{-1}\left[\begin{array}{c}P_{1} & P_{2}\end{array}\right]\left[\begin{array}{c}0_{p}^{\top}\\M_{2}\end{array}\right] \\
\times \phi\left(Lt + P_{1}x + P_{2}M_{2}Z + P_{2}\sqrt{n}\gamma_{n}^{E.j} + T; 0_{p}, \Omega\right)dt \\
\leq \left\|\widetilde{P}Z + T_{n}\right\| \times F\left(x, M_{2}Z + \sqrt{n}\gamma_{n}^{E.j}\right).$$

Based on the previous display, we can claim that the first term in (26) is bounded from above by $\|\widetilde{P}Z + T_n\|$. This leads us to claim that

$$\left|\partial_i P^{E \cdot j}(Z)\right| \le c_2 \left\|\widetilde{P}Z + T_n\right\|.$$

The same strategy applied to higher derivatives concludes the proof.

Proposition 11.3. For a vector $x \in \mathbb{R}^p$ and a positive semidefinite matrix $\Sigma \in \mathbb{R}^{p \times p}$, let $\operatorname{Exp}(x, \Sigma)$ denote the function $\exp\left(-\frac{1}{2}x^{\top}\Sigma x\right)$. Then we have that

$$\left|\partial_{i_1,i_2,i_3}^3 F(Z)\right| \le \sum_{l=0}^3 c_l' \left\|\widetilde{P}Z + T_n\right\|^l \times \operatorname{Exp}\left(\widetilde{P}Z + T_n, \Theta\right)$$

for

$$\Theta = \left(\Omega^{-1} - \frac{\Omega^{-1}LL^{\top}\Omega^{-1}}{L^{\top}\Omega^{-1}L}\right).$$

and constants c'_0, \ldots, c'_3 that do not depend on n.

Proof. Observe that

$$F(Z) \propto \int_{I_{-}^{E\cdot j}}^{\infty} \exp\left\{-\frac{1}{2}\left(Lt + \widetilde{P}Z + T_{n}\right)^{\top} \Omega^{-1}\left(Lt + \widetilde{P}Z + T_{n}\right)\right\} dt$$
$$\propto \int_{I_{-}^{E\cdot j}}^{\infty} \exp\left\{-\frac{1}{2}\left(\widetilde{P}Z + T_{n}\right)^{\top} \Theta\left(\widetilde{P}Z + T_{n}\right)\right\}$$
$$\times \exp\left\{-\frac{L^{\top}\Omega^{-1}L}{2}\left[t + \frac{L^{\top}\Omega^{-1}\left(\widetilde{P}Z + T_{n}\right)}{L^{\top}\Omega^{-1}L}\right]^{2}\right\} dt$$

Applying a change of variables $\tilde{t} = t + \mathcal{L}(Z)$ to the previous integral, where $\mathcal{L}(Z) = (L^{\top}\Omega^{-1}L)^{-1}L^{\top}\Omega^{-1}(\widetilde{P}Z + T_n)$, we have that

$$F(Z) \propto \operatorname{Exp}\left(\widetilde{P}Z + T_n, \Theta\right) \times \int_{I_-^{E \cdot j} + \mathcal{L}(Z)}^{\infty} \operatorname{exp}\left\{-\frac{L^{\top} \Omega^{-1} L}{2} \widetilde{t}^2\right\} d\widetilde{t}.$$

Using the Leibniz rule on the integral representation above yields

$$\left|\partial_{i_1,i_2,i_3}^3 F(Z)\right| \le \sum_{l=0}^3 c_l' \left\|\widetilde{P}Z + T_n\right\|^l \times \operatorname{Exp}\left(\widetilde{P}Z + T_n, \Theta\right).$$

11.2.3 Other Key Results

Proposition 11.4. The following assertions hold.

1. Suppose that $L^{\top}\Omega^{-1}b < 0$. Let $\mathcal{D}_n = [-d_0r_n \cdot 1_p, d_0r_n \cdot 1_p]$ for a positive constant d_0 such that

$$d_0 < \frac{1}{2} \frac{\left| L^\top \Omega^{-1} b \right|}{\sqrt{p} \left\| \tilde{P}^\top \Omega^{-1} L \right\|}.$$

Then, there exists a Lebesgue-almost everywhere differentiable function \mathcal{K}_n^1 such that for $m \in \{0, 1, 2, 3\}$:

$$F(Z) = \operatorname{Exp}\left(\widetilde{P}Z - r_n b, \Omega^{-1}\right) \times \mathcal{K}_n^1(Z), \text{ and } \sup_{\mathbb{F}_n \in \mathcal{F}_n} \sup_{Z \in \mathcal{D}_n} |r_n| \left|\partial_{i_1, \dots, i_m}^m \mathcal{K}_n^1(Z)\right| < \infty.$$

2. Suppose that $L^{\top}\Omega^{-1}b > 0$. Then, there exists a Lebesgue-almost everywhere differentiable function \mathcal{K}_n^2 such that for $m \in \{0, 1, 2, 3\}$:

$$F(Z) = \operatorname{Exp}\left(\widetilde{P}Z - r_n b, \Theta\right) \times \mathcal{K}_n^2(Z) \text{ and } \sup_{\mathbb{F}_n \in \mathcal{F}_n} \sup_{Z \in \mathbb{R}^p} \left| \partial_{i_1, \dots i_m}^m \mathcal{K}_n^2(Z) \right| < \infty.$$

Proof. First, consider the case when $L^{\top}\Omega^{-1}b > 0$. We begin by noting that

$$F(Z) \propto \operatorname{Exp}\left(\widetilde{P}Z + T_{n}, \Theta\right) \times \int_{I_{-}^{E \cdot j}}^{\infty} \operatorname{exp}\left\{-\frac{\widetilde{P}^{\top}\Omega^{-1}L}{2} \left[t + \frac{L^{\top}\Omega^{-1}\left(\widetilde{P}Z + T_{n}\right)}{L^{\top}\Omega^{-1}L}\right]^{2}\right\} dt$$
$$\propto \operatorname{Exp}\left(\widetilde{P}Z + T_{n} + LI_{-}^{E \cdot j}, \Theta\right) \times \int_{I_{-}^{E \cdot j}}^{\infty} \operatorname{exp}\left\{-\frac{L^{\top}\Omega^{-1}L}{2} \left[t + \frac{L^{\top}\Omega^{-1}\left(\widetilde{P}Z + T_{n}\right)}{L^{\top}\Omega^{-1}L}\right]_{(27)}^{2}\right\} dt.$$

We can rewrite this function as

$$F(Z) = \operatorname{Exp}\left(\widetilde{P}Z + T_n + LI_{-}^{E \cdot j}, \Omega^{-1}\right) \times \mathcal{K}_n^1(Z)$$

where

$$\mathcal{K}_{n}^{1}(Z) = d_{2} \exp\left\{\frac{1}{2}\left(\widetilde{P}Z + T_{n} + LI_{-}^{E\cdot j}\right)^{\top} \frac{\Omega^{-1}LL^{\top}\Omega^{-1}}{L^{\top}\Omega^{-1}L}\left(\widetilde{P}Z + T_{n} + LI_{-}^{E\cdot j}\right)\right\}$$
$$\times \int_{I_{-}^{E\cdot j}}^{\infty} \exp\left\{-\frac{L^{\top}\Omega^{-1}L}{2}\left[t + \frac{L^{\top}\Omega^{-1}\left(\widetilde{P}Z + T_{n}\right)}{L^{\top}\Omega^{-1}L}\right]^{2}\right\}dt$$

for a constant d_2 . By the variable substitution $\tilde{t} = t - I_{-}^{E \cdot j}$ in the integral involved in the expression of $\mathcal{K}_n^1(Z)$, we have that

$$\mathcal{K}_{n}^{1}(Z) \propto \exp\left\{\frac{1}{2}\left(\widetilde{P}Z - r_{n}b\right)^{\top} \frac{\Omega^{-1}LL^{\top}\Omega^{-1}}{L^{\top}\Omega^{-1}L}\left(\widetilde{P}Z - r_{n}b\right)\right\}$$
$$\times \int_{0}^{\infty} \exp\left\{-\frac{1}{2}\left(\widetilde{P}Z - r_{n}b + L\tilde{t}\right)^{\top} \frac{\Omega^{-1}LL^{\top}\Omega^{-1}}{L^{\top}\Omega^{-1}L}\left(\widetilde{P}Z - r_{n}b + L\tilde{t}\right)\right\}d\tilde{t}$$

Using simple algebra, we simplify the expression for $\mathcal{K}_n^1(Z)$ which leads to the following bound:

$$\mathcal{K}_{n}^{1}(Z) \propto \int_{0}^{\infty} \exp\left\{-\frac{1}{2}\tilde{t}L^{\top}\Omega^{-1}L\tilde{t}\right\} \times \exp\left\{-\tilde{t}L^{\top}\Omega^{-1}\left(\tilde{P}Z - r_{n}b\right)\right\}d\tilde{t}$$
$$\leq \int_{0}^{\infty} \exp\left\{-\tilde{t}L^{\top}\Omega^{-1}\left(\tilde{P}Z - r_{n}b\right)\right\}d\tilde{t}.$$

Furthermore, for $Z \in \mathcal{D}_n$, we have

$$\mathcal{K}_n^1(Z) \lesssim \frac{1}{L^\top \Omega^{-1} \left(\widetilde{P} Z - r_n b \right)} \lesssim \frac{1}{r_n \left| L^\top \Omega^{-1} b \right|}$$

This proves our claim for m = 0.

Now, for m = 1, note that the first derivative of $\mathcal{K}_n^1(\cdot)$, we have

$$\partial^{1} \mathcal{K}_{n}^{1}(Z) \propto \int_{0}^{\infty} -\tilde{t} L^{\top} \Omega^{-1} L \exp\left\{-\frac{1}{2} \tilde{t} L^{\top} \Omega^{-1} L \tilde{t}\right\} \times \exp\left\{-\tilde{t} L^{\top} \Omega^{-1} \left(\tilde{P} Z - r_{n} b\right)\right\} d\tilde{t},$$

which implies the following inequality

$$\begin{aligned} \left|\partial_{i_{1}}^{1}\mathcal{K}_{n}^{1}(Z)\right| &\lesssim \int_{0}^{\infty} -\widetilde{t}\exp\left\{-\widetilde{t}L^{\top}\Omega^{-1}\left(\widetilde{P}Z-r_{n}b\right)\right\}d\widetilde{t}\\ &= \frac{1}{\left[L^{\top}\Omega^{-1}\left(\widetilde{P}Z-r_{n}b\right)\right]^{2}}.\end{aligned}$$

As a result, by definition of sets \mathcal{D}_n , whenever $Z \in \mathcal{D}_n$ our claim holds. A similar strategy is applied to obtain the conclusions for the second and third order partial derivatives $\left|\partial_{i_1,i_2}^2 \mathcal{K}_n^1(Z)\right|$ and $\left|\partial_{i_1,i_2,i_3}^3 \mathcal{K}_n^1(Z)\right|$.

Now we turn to the case when $L^{\top}\Omega^{-1}b > 0$. We revisit (27). If we define

$$\mathcal{K}_n^1(Z) = C_2 \int_{I_-^{E\cdot j}}^\infty \exp\left\{-\frac{L^\top \Omega^{-1} L}{2} \left[t + \frac{L^\top \Omega^{-1} \left(\widetilde{P}Z + T_n\right)}{L^\top \Omega^{-1} L}\right]^2\right\} dt,$$

then we can write

$$F(Z) = \operatorname{Exp}\left(\widetilde{P}Z + T_n + LI_{-}^{E \cdot j}, \Theta\right) \times \mathcal{K}_n^2(Z)$$

for a constant d_2 . Now the uniform bounds on the derivative of \mathcal{K}_n^2 follow directly after we apply the Leibniz integral rule.

Proposition 11.5. Suppose that $L^{\top}\Omega^{-1}b < 0$. Then, for

$$\mathcal{L}_n \sim \mathcal{N}\left(-\left[L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L\right]^{-1}L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}T_n, \left[L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L\right]^{-1}\right),$$

we have that

$$\mathbb{E}_{\mathcal{N}}[F(Z)] = \operatorname{Exp}\left(LI_{-}^{E \cdot j} + T_{n}, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L^{\top}L\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right) \times \mathbb{P}(I_{-}^{E \cdot j} \leq \mathcal{L}_{n} \leq \infty).$$

Proof. Interchanging the order between expectation and integration, we obtain

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right] = \mathbb{E}_{\mathcal{N}}\left[\int_{I_{-}^{E\cdot j}}^{\infty} \phi\left(Lt + \widetilde{P}Z + T_{n}; 0_{p}, \Omega\right) dt\right]$$
$$= \int_{I_{-}^{E\cdot j}}^{\infty} \mathbb{E}_{\mathcal{N}}\left[\phi\left(Lt + \widetilde{P}Z + T_{n}; 0_{p}, \Omega\right)\right] dt$$

Observe that the expectation in the integrand equals

$$\begin{split} &\mathbb{E}_{\mathcal{N}}\left[\phi\left(Lt+\widetilde{P}Z+T_{n};0_{p},\Omega\right)\right]\\ &=\int_{\mathbb{R}^{p}}\phi\left(Lt+\widetilde{P}Z+T_{n};0_{p},\Omega\right)\phi\left(Z;0_{p},I_{p,p}\right)dZ\\ &\propto\int_{\mathbb{R}^{p}}\exp\left\{-\frac{1}{2}\left(Lt+\widetilde{P}Z+T_{n}\right)^{\top}\Omega^{-1}\left(Lt+\widetilde{P}Z+T_{n}\right)-\frac{1}{2}Z^{\top}Z\right\}dZ\\ &\propto\exp\left\{-\frac{1}{2}\left(Lt+T_{n}\right)^{\top}\left[\Omega^{-1}-\Omega^{-1}L\left(I_{p,p}+\widetilde{P}^{\top}\Omega^{-1}\widetilde{P}\right)^{-1}L^{\top}\Omega^{-1}\right]\left(Lt+T_{n}\right)\right\}\\ &\times\int_{\mathbb{R}^{p}}\exp\left\{-\frac{1}{2}\left(Z-\Phi_{n}\right)^{\top}\left(I_{p,p}+\widetilde{P}^{\top}\Omega^{-1}\widetilde{P}\right)\left(Z-\Phi_{n}\right)\right\}dZ, \end{split}$$

where

$$\Phi_n = -\left(I_{p,p} + \widetilde{P}^\top \Omega^{-1} \widetilde{P}\right)^{-1} \widetilde{P}^\top \Omega^{-1} \left(Lt + T_n\right).$$

Then direct application of the Woodbury matrix identity gives us:

$$\Omega^{-1} - \Omega^{-1} \widetilde{P} \left(I_{p,p} + \widetilde{P}^{\top} \Omega^{-1} \widetilde{P} \right)^{-1} \widetilde{P}^{\top} \Omega^{-1} = \left(\Omega + \widetilde{P} \widetilde{P}^{\top} \right)^{-1}.$$

This implies that

$$\mathbb{E}_{\mathcal{N}}\left[\phi\left(Lt+\widetilde{P}Z+T_{n};0_{p},\Omega\right)\right] \propto \exp\left\{-\frac{1}{2}\left(Lt+T_{n}\right)^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(Lt+T_{n}\right)\right\}.$$

Then plugging this expression into our integral, we have that

$$\begin{split} \mathbb{E}_{\mathcal{N}}\left[F(Z)\right] &\propto \int_{I_{-}^{E,j}}^{\infty} \exp\left\{-\frac{1}{2}t^{2}L^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}L - tL^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}T_{n}\right. \\ &\left.-\frac{1}{2}T_{n}^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}T_{n}\right\}dt \\ &\propto \int_{I_{-}^{E,j}}^{\infty} \exp\left\{-\frac{1}{2}L^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}L\left(t + \frac{L^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}T_{n}\right)^{2}\right\} \\ &\times \exp\left\{-\frac{1}{2}T_{n}^{\top}\left[\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega+\widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right]T_{n}\right\}dt. \end{split}$$

At last, we conclude our proof with the observation that

$$\exp\left(T_n, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)$$
$$= \exp\left(LI_{-}^{E\cdot j} + T_n, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right),$$

which yields the claimed expression for $\mathbb{E}_{\mathcal{N}}[F(Z)]$.

Proposition 11.6. For sufficiently large n, it holds that:

1.
$$\mathbb{E}_{\mathcal{N}}[F(Z)] \gtrsim \operatorname{Exp}\left(-r_{n}b, \left[\Theta^{-1} + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right) \text{ for } L^{\top}\Omega^{-1}b < 0;$$

2. $\mathbb{E}_{\mathcal{N}}[F(Z)] \gtrsim r_{n}^{-1}\operatorname{Exp}\left(-r_{n}b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right) \text{ for } L^{\top}\Omega^{-1}b > 0$

Proof. Starting with the case of $L^{\top}\Omega^{-1}b < 0$, recall we have the form

$$F(Z) \propto \operatorname{Exp}\left(\widetilde{P}Z - r_n b, \Theta\right) \times \int_0^\infty \operatorname{exp}\left\{-\frac{\widetilde{P}^\top \Omega^{-1} \widetilde{P}}{2} \left[t + \frac{L^\top \Omega^{-1} \left(\widetilde{P}Z - r_n b\right)}{\widetilde{P}^\top \Omega^{-1} \widetilde{P}}\right]^2\right\} dt.$$

Then define the set $\mathcal{D}_n = \left\{ Z : L^{\top} \Omega^{-1} \left(\widetilde{P} Z - r_n b \right) < 0 \right\}$. Since

$$\int_0^\infty \exp\left\{-\frac{\widetilde{P}^\top \Omega^{-1} \widetilde{P}}{2} \left[t + \frac{L^\top \Omega^{-1} \left(\widetilde{P} Z - r_n b\right)}{\widetilde{P}^\top \Omega^{-1} \widetilde{P}}\right]^2\right\} dt \gtrsim \frac{1}{2} \quad \text{for } Z \in \mathcal{D}_n$$

it holds that

$$F(Z) \gtrsim \operatorname{Exp}\left(\widetilde{P}Z - r_n b, \Theta\right) \times \mathbb{1}_{Z \in \mathcal{D}_n}$$

Then, we conclude that

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right] \gtrsim \mathbb{E}_{\mathcal{N}}\left[\exp\left(\widetilde{P}Z - r_{n}b,\Theta\right)\mathbb{1}_{Z\in\mathcal{D}_{n}}\right]$$
$$\gtrsim \mathbb{E}_{\mathcal{N}}\left[\exp\left(\widetilde{P}Z - r_{n}b,\Theta\right)\right] - \mathbb{E}_{\mathcal{N}}\left[\exp\left(\widetilde{P}Z - r_{n}b,\Theta\right)\mathbb{1}_{Z\in\mathcal{D}_{n}^{c}}\right]$$
$$\gtrsim \frac{1}{2}\mathbb{E}_{\mathcal{N}}\left[\exp\left(\widetilde{P}Z - r_{n}b,\Theta\right)\right].$$

Then applying the Woodbury matrix identity after integrating with respect to Z and we get

$$\mathbb{E}_{\mathcal{N}}\left[\exp\left(\widetilde{P}Z - r_n b, \Theta\right)\right] = \exp\left(-r_n b, \left[\Theta^{-1} + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right).$$

Our conclusion thus follows. For the other case, note that

$$\mathbb{P}\left(I_{-}^{E\cdot j} \leq \mathcal{L}_{n} \leq \infty\right)$$

$$\propto \int_{I_{-}^{E\cdot j}}^{\infty} \exp\left\{-\frac{1}{2}L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L\left(t - I_{-}^{E\cdot j} + \frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{2}\right\}dt$$

$$\propto \int_{0}^{\infty} \exp\left\{-\frac{1}{2}L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L\left(\widetilde{t} + \frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{2}\right\}d\widetilde{t}.$$

through a change of variable $\tilde{t} = t - I_{-}^{E \cdot j}$. When $L^{\top} \left(\Omega + \tilde{P}\tilde{P}^{\top}\right)^{-1} \left(LI_{-}^{E \cdot j} + T_n\right) > 0$, we apply the Mill's ratio bound to note that

$$\begin{split} & \mathbb{P}\left(I_{-}^{E\cdot j} \leq \mathcal{L}_{n} \leq \infty\right) \\ \geq \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-1} \times \left[1 - \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-2}\right] \\ & \times \operatorname{Exp}\left(LI_{-}^{E\cdot j} + T_{n}, \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right). \end{split}$$

This yields:

$$\begin{split} \mathbb{E}_{\mathcal{N}}\left[F(Z)\right] &\geq \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-1} \times \left[1 - \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-2}\right] \\ &\times \exp\left(LI_{-}^{E\cdot j} + T_{n}, \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right) \\ &\times \exp\left(LI_{-}^{E\cdot j} + T_{n}, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right) \\ &= \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-1} \times \left[1 - \left(\frac{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\left(LI_{-}^{E\cdot j} + T_{n}\right)}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)^{-2}\right] \\ &\times \exp\left(LI_{-}^{E\cdot j} + T_{n}, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right). \end{split}$$

Using the parameterization $LI_{-}^{E \cdot j} + T_n = -r_n b$, we conclude that

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right] \gtrsim r_n^{-1} \operatorname{Exp}\left(-r_n b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right).$$

When $L^{\top} \left(\Omega + \widetilde{P} \widetilde{P}^{\top} \right)^{-1} \left(L I_{-}^{E \cdot j} + T_n \right) < 0$, we note that

$$\mathbb{P}\left(I_{-}^{E \cdot j} \leq \mathcal{L}_{n} \leq \infty\right) \geq \frac{1}{2}$$

Hence, for sufficiently large n, we have

$$\mathbb{E}_{\mathcal{N}}\left[F(Z)\right] \gtrsim \frac{1}{2} \exp\left(-r_{n}b, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1} - \frac{\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}LL^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}}{L^{\top}\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}L}\right)$$
$$\gtrsim \exp\left(-r_{n}b, \left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)^{-1}\right).$$

Proposition 11.7. Let $W_{\alpha,\kappa}$ be as defined in 11.7. Then under Assumptions 2,3,4, we have

$$\sup_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\left\| a_{1,n} \right\|^{\lambda} \left\| a_{1,n}^{*} \right\|^{\gamma} \sup_{\alpha, \kappa \in [0,1]} \left| \int_{0}^{1} \sqrt{t} \operatorname{Exp} \left(\widetilde{P} \sqrt{t} \mathcal{W}_{\alpha,\kappa} - r_{n} b, \left[\Omega + \widetilde{P} \widetilde{P}^{\top} (1-t) \right]^{-1} \right) dt \right| \right]}{\operatorname{Exp} \left(-r_{n} b, \left[\Omega + \widetilde{P} \widetilde{P}^{\top} \right]^{-1} \right)} = O(1)$$

for $\lambda, \gamma \in \mathbb{N}$ such that $\lambda + \gamma \leq 3$.

Proof. Fixing some notations for this proof, let e_{\max} be the largest eigenvalue of $\left(\Omega + \widetilde{P}\widetilde{P}^{\top}\right)$. Define

$$\Pi(t) = \left\{ tL^{\top} \left[\Omega + \widetilde{P}\widetilde{P}^{\top}(1-t) \right]^{-1} \widetilde{P} + I_{p,p} \right\}^{-1} \widetilde{P}^{\top} \left[\Omega + \widetilde{P}\widetilde{P}^{\top}(1-t) \right]^{-1},$$

and let $\Pi_k(t)$ denote the k-th row of $\Pi(t)$, and let $\|\Pi_k(t)\|_{\max} = \max_{k \in [p]} \|\Pi_k(t)\|_2$. Fix

$$c_t > \max\left(e_{\max}^{1/2} \times (\|b\|+1), \sup_{t \in [0,1]} \|\Pi_k(t)\|_{\max} \times (\|b\|+1)\right).$$

Observe that

$$\mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\sup_{\alpha,\kappa\in[0,1]}\left|\int_{0}^{1}\sqrt{t}\operatorname{Exp}\left(\widetilde{P}\sqrt{t}\mathcal{W}_{\alpha,\kappa}-r_{n}b,\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right)dt\right|\right] \\
\leq \mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\sup_{\alpha,\kappa\in[0,1]}\left|\int_{0}^{1}\sqrt{t}\operatorname{Exp}\left(\widetilde{P}\sqrt{t}\mathcal{W}_{\alpha,\kappa}-r_{n}b,\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right)\right. \\
\left.\times\left.\mathbb{1}_{\mathcal{R}_{t}}\left(r_{n}^{-1}\zeta_{n}[-1]\right)dt\right|\right] + \mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\left.\mathbb{1}_{\mathcal{R}_{0}}\left(r_{n}^{-1}\zeta_{n}[-1]\right)\right],$$

where $\mathcal{R}_t = [-c_t \cdot 1_p, c_t \cdot 1_p] \subseteq \mathbb{R}^p$, for $t \in (0, 1]$, and $\mathcal{R}_0 = \mathcal{R}_1^c$. Note that for some positive constant $\chi > 0$, the bound on the right-hand side further simplifies as:

$$\mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\sup_{\alpha,\kappa\in[0,1]}\left|\int_{0}^{1}\sqrt{t}\operatorname{Exp}\left(\widetilde{P}\sqrt{t}\mathcal{W}_{\alpha,\kappa}-r_{n}b,\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right)dt\right|\right] \\
\leq \mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\exp\left(\chi\left\|a_{1,n}\right\|\right)\right] \\
\times \int_{0}^{1}\sqrt{t}\mathbb{E}_{\mathbb{F}_{n}}\left[\operatorname{Exp}\left(\sqrt{t}\widetilde{P}\zeta_{n}[-1]-r_{n}b,\left[\Omega+\widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right)\mathbb{1}_{\mathcal{R}_{t}}\left(r_{n}^{-1}\zeta_{n}[-1]\right)\right]dt \\
+ \mathbb{E}_{\mathbb{F}_{n}}\left[\left\|a_{1,n}\right\|^{\lambda}\left\|a_{1,n}^{*}\right\|^{\gamma}\right]\mathbb{E}_{\mathbb{F}_{n}}\left[\mathbb{1}_{\mathcal{R}_{0}}\left(r_{n}^{-1}\zeta_{n}[-1]\right)\right].$$

Define the function

$$\Phi_t(Z) = \begin{cases} \frac{1}{2} \left(\sqrt{t} \widetilde{P} Z - b \right)^\top \left[\Omega + \widetilde{P} \widetilde{P}^\top (1 - t) \right]^{-1} \left(\sqrt{t} \widetilde{P} Z - b \right), & \text{if } t \in (0, 1] \\ 0, & \text{if } t = 0 \end{cases}$$

.

Then, we obtain that

$$\frac{\int_{0}^{1} \sqrt{t} \mathbb{E}_{\mathbb{F}_{n}} \left[\exp\left(\sqrt{t}R\zeta_{n}[-1] - r_{n}b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}(1-t)\right]^{-1}\right) \mathbb{1}_{\mathcal{R}_{t}} \left(r_{n}^{-1}\zeta_{n}[-1]\right) \right] dt}{\exp\left(-r_{n}b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right)} \\
\leq \sup_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \sup_{t \in (0,1]} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\exp\left(-r_{n}^{2}\Phi_{t} \left(r_{n}^{-1}\zeta_{n}[-1]\right)\right) \mathbb{1}_{\left[-c_{t} \cdot 1_{p}, c_{t} \cdot 1_{p}\right]} \left(r_{n}^{-1}\zeta_{n}[-1]\right)\right] \int_{0}^{1} \sqrt{t} dt}{\exp\left(-r_{n}b, \left[\Omega + \widetilde{P}\widetilde{P}^{\top}\right]^{-1}\right)}.$$

Recall that

$$\zeta_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n a_{i,m}$$

has a finite moment generating function near the origin, since we know that the $\{a_{i,n} : i \in [n]\}$ are sub-Gaussian variables under Assumption 3. Thus, ζ_n satisfies a large deviation principle with rate function $I(Z) = ||Z||^2/2$. Varadhan's large deviation principle ensures that for any $\mathbb{F}_n \in \mathcal{F}_n$ satisfying Assumptions 3 and 4, and for sufficiently large n, the following holds:

$$r_n^{-2}\log \mathbb{E}_{\mathbb{F}_n}\left[\exp\left(-r_n^2\Phi_t\left(\frac{\zeta_n}{r_n}\right)\right)\mathbb{1}_{\mathcal{R}_t}\left(r_n^{-1}\zeta_n\right)\right] \leq \sup_{Z\in\mathcal{R}_t}\left(-\frac{\|Z\|^2}{2} - \Phi_t(Z)\right)$$
$$= -\inf_{Z\in\mathcal{R}_t}\left(\frac{1}{2}Z^{\top}Z + \Phi_t(Z)\right)$$

for $t \in [0, 1]$, where $\mathcal{R}_t = [-c_t \cdot 1_p, c_t \cdot 1_p]$ for $c_t > 0$ and $t \in (0, 1]$, and \mathcal{R}_0 is the complement of $[-c_t \cdot 1_p, c_t \cdot 1_p]$.

Therefore, we conclude that

$$\sup_{n} \sup_{\mathbb{F}_{n} \in \mathcal{F}_{n}} \sup_{t \in (0,1]} \frac{\mathbb{E}_{\mathbb{F}_{n}} \left[\exp \left\{ -r_{n}^{2} \Phi_{t} \left(r_{n}^{-1} \zeta_{n} [-1] \right) \right\} \mathbb{1}_{\left[-c_{t} \cdot 1_{p}, c_{t} \cdot 1_{p} \right]} \left(r_{n}^{-1} \zeta_{n} [-1] \right) \right]}{\exp \left(-r_{n} b, \left[\Omega + \widetilde{P} \widetilde{P}^{\top} \right]^{-1} \right)} < \infty.$$

12 Appendix- Data application details

Variable Name	Description
ParticipantIdentifier	Unique identifier for each participant in the study.
Value_tran	Log transformed step-counts after notification, representing tar- get outcome value.
Notification_c	Count of notifications received by the participant.
${\tt TimeEnrolled_days}$	Total number of days the participant was enrolled in the study.
Phases	Phases of the intervention or study period.
Value_30min_before	Value recorded 30 minutes before a specific event or action.
NotificationType	Type or category of notifications (e.g., morning, afternoon).
Baseline_steps	Number of steps recorded during the baseline period.
IsWeekend	Indicator variable $(1 = Weekend, 0 = Weekday)$.
IsIndoor	Indicator for indoor activity $(1 = \text{Indoor}, 0 = \text{Outdoor})$.
IsLossFramed	Indicator for whether a message was framed as a loss $(1 = \text{Loss-framed}, 0 = \text{Gain-framed}).$
IsSnow	Indicator for snowy weather conditions $(1 = \text{Snow}, 0 = \text{No Snow})$.

Table 3: Descriptions of Variables in the Dataset

Continued on next page...
Variable Name	Description
IsActivity	Indicator for whether the record corresponds to a specific activ- ity $(1 = \text{Yes}, 0 = \text{No}).$
Value_30min_before_tran	Log transformed step counts value recorded 30 minutes prior notification.
${\tt AgeEnrollment_years}$	Age of the participant at the time of enrollment, in years.
Gender	Gender of the participant.
Race	Race or ethnicity of the participant.
ExerciseTimeAgg_min	Aggregated time spent exercising, in minutes.
WalkDistanceAgg_m	Aggregated walking distance, in meters.
StepsAgg_priorweek	Aggregated step counts taken during the prior week.
Value_tran_sd_week	Standard deviation of the transformed step counts over the current week.
Value_tran_sd_priorweek	Standard deviation of the transformed step counts over the prior week.
Distance_m_0	6min walk distance recorded in meters at a specific time (e.g., baseline).

13 Appendix- Extensions

13.1 Linear Risk

For the linear contrast, a population risk function can be defined as

$$R^{lin}(\beta) := \mathbb{E}\left[\sum_{t=1}^{T} \tilde{\sigma}_t^2(S_{i,t}) \left\{ (f_t(S_{i,t})^\top \beta)^2 - 2f_t(S_{i,t})^\top \beta (\mu_t(H_{i,t}, 1) - \mu_t(H_{i,t}, 0)) \right\} \right],$$



Figure 5: Post-selective inference of an MRT in cardiac rehab population reveals several important effect moderators

where $\mu_t(h, a) := \mathbb{E}_{\mathbf{p}}[Y_{i,t+1}|H_{i,t} = h, W_{i,t} = a]$ denotes conditional mean proximal outcome given history h and action a, and $\tilde{\sigma}_t^2(S_{i,t}) = \tilde{p}_t(1|S_{i,t}) (1 - \tilde{p}_t(1|S_{i,t}))$. Recalling that $\beta_{\mathbf{p}}(t;s) := \mathbb{E}[\mu_t(H_{i,t}, 1) - \mu_t(H_{i,t}, 0)|S_{i,t} = s]$, minimizing the risk $R^{lin}(\beta)$ is equivalent to minimizing

$$\sum_{t=1}^{T} \mathbb{E} \left[\tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ (f_{t}(S_{i,t})^{\top}\beta)^{2} - 2f_{t}(S_{i,t})^{\top}\beta\beta(t;S_{i,t}) + \beta(t;S_{i,t})^{2} \right\} \right]$$

=
$$\sum_{t=1}^{T} \mathbb{E} \left[\tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ f_{t}(S_{i,t})^{\top}\beta - \beta(t;S_{i,t}) \right\}^{2} \right]$$

=
$$\sum_{t=1}^{T} \mathbb{E} \left[\tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ f_{t}(S_{i,t})^{\top}\beta - f_{t}(S_{i,t})^{\top}\beta^{\star} \right\}^{2} \right].$$

implying that under the linear causal model being correctly specified, i.e., $\beta_{\mathbf{p}}(t;s) = f_t(S_{i,t})^{\top}\beta^{\star}$, the risk function is minimized by $\beta = \beta^{\star}$. Under model misspecification, the risk minimizer can be thought of as a weighted L_2 -projection of the true causal model as shown in previous work Dempsey et al. (2020).

Computing the population risk requires taking an expectation over the unknown distribution \mathcal{P} and knowing the conditional mean $\mu_t(h, a)$. First, we replace the expectation by an empirical-version and construct an initial estimate $\hat{\mu}_t(h, a)$ of the conditional mean,

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 $\mu_t(h, a)$. In particular, we consider a doubly-robust initial estimator

$$\hat{\mu}_t^{(DR)}(H_{i,t},a) = \hat{\mu}_t(H_{i,t},a) + \frac{1[A_{i,t}=a]}{p(A_{i,t}|H_{i,t})} \left(Y_{i,t+1} - \hat{\mu}_t(H_{i,t},a)\right),$$

Motivated by recent work on empirical risk minimization van der Laan et al. (2024), we consider a one-step debiased estimation procedure to ensure minimizing the empirical risk yield oracle efficiency similar to orthogonal learning strategies Shi and Dempsey (2023). Given the initial estimate, we propose to construct a refined estimator $\mu_t^*(h, a) = \hat{\mu}_t(h, a) + (a - \hat{p}_t(1|S_{i,t}))f_t(S_{i,t})^{\top}\hat{\theta}$ where $\hat{\theta}$ is the minimizer of

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} W_{i,t} (Y_{i,t+1} - \hat{\mu}_t^{(DR)} (H_{i,t}, A_{i,t}) - (A_{i,t} - \tilde{p}_t (1|S_{i,t})) f_t (S_{i,t})^\top \theta)^2 \right] = 0$$

where $W_{i,t} = \tilde{p}(W_{i,t}|S_{i,t})/\hat{p}(W_{i,t}|H_{i,t})$. This guarantees the refined estimator satisfies similar Neyman-orthogonality constraints for consistent causal estimation. This is desirable because we want $\hat{\mu}^*$ to satisfy

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} W_{i,t}(A_{i,t} - \tilde{p}_t(1|S_{i,t}))(Y_{i,t+1} - \mu_t^{\star}(H_{i,t}, A_{i,t}))f_t(S_{i,t}) \right] = 0.$$

The final estimate $\hat{\beta}$ minimizes the empirical version of the population risk with the refined estimator serving as a plug-in estimator for $\mu_t(h, a)$, i.e.,

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} \tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ (f_{t}(S_{i,t})^{\top} \beta)^{2} - 2f_{t}(S_{i,t})^{\top} \beta \left(\mu_{t}^{\star}(H_{i,t},1) - \mu_{t}^{\star}(H_{i,t},0) \right) \right\} \right]$$

which is equivalent to minimizing the loss

$$\sum_{i=1}^{N} \left[\sum_{t=1}^{T} \tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ f_{t}(S_{i,t})^{\top} \beta - \beta^{*}(t;S_{i,t}) \right\}^{2} \right],$$
(28)

where $\tilde{\mu}^{\star}(t; S_{i,t}) := \mu_t^{\star}(H_{i,t}, 1) - \mu_t^{\star}(H_{i,t}, 0).$

14 Relative Risk

For the relative risk contrast, a population-level risk function can be defined as

$$\mathbb{E}\left[\sum_{t=1}^{T} \tilde{\sigma}_{i,t}^{2}(S_{i,t}) \left\{ (\mu_{t}(H_{i,t},1) + \mu_{t}(H_{i,t},0)) \log(1 + e^{f_{t}(S_{i,t})^{\top}\beta}) - \mu_{t}(H_{i,t},1) f_{t}(S_{i,t})^{\top}\beta \right\} \right],$$

where $\tilde{\sigma}_t^2(S_{i,t}) = \tilde{p}_t(1|S_{i,t}) (1 - \tilde{p}_t(1|S_{i,t}))$. Under the linear relative risk causal excursion model, there exists $\beta \in \mathbb{R}^d$ such that $\mathbb{E}[\mu_t(H_{i,t}, 1)|S_{i,t}] = e^{f_t(S_{i,t})^\top \beta} \mathbb{E}[\mu_t(H_{i,t}, 0)|S_{i,t}]$. Thus the population risk can be re-written as

$$\sum_{t=1}^{T} \mathbb{E} \left[\tilde{\sigma}_{i,t}^{2}(S_{i,t}) \times \mathbb{E}[\mu_{t}(H_{i,t}, 0) | S_{i,t}] \left\{ \left(1 + e^{f_{t}(S_{i,t})^{\top}\beta} \right) \log \left(1 + e^{f_{t}(S_{i,t})^{\top}\beta} \right) - e^{f_{t}(S_{i,t})^{\top}\beta} f_{t}(S_{i,t})^{\top}\beta \right\} \right]$$

Differentiating with respect to β , it can be shown that the population risk is minimized by $\beta = \beta$ as desired. For the relative risk, the debiasing term is given by

$$\sum_{i=1}^{n} \left[\sum_{t=1}^{T} \frac{1}{p(A_{i,t}|H_{i,t})} \tilde{\sigma}_{t}^{2}(S_{i,t}) \left\{ \log \left(1 + e^{f_{t}(S_{i,t})^{\top}\beta} \right) - A_{i,t}f_{t}(S_{i,t})^{\top}\beta \right\} (Y_{i,t+1} - \mu^{\star}(H_{i,t+1}, A_{i,t})) \right]$$

Unlike the linear risk setting, an exact solution cannot be derived. Instead we construct a basis $\psi(S_{i,t}) \in \mathbb{R}^d \supset f_t(S_{i,t})$ such that $\log\left(1 + e^{f_t(S_{i,t})^\top\beta}\right)$ can be well approximated by $\psi(S_{i,t})^\top \alpha$ for some $\alpha \in \mathbb{R}^d$. Then the debiasing term yields the following constraints:

$$\sum_{i=1}^{n} \left[\sum_{t=1}^{T} \frac{1}{p(A_{i,t}|H_{i,t})} \tilde{\sigma}_{t}^{2}(S_{i,t})(Y_{i,t+1} - \mu^{\star}(H_{i,t+1}, A_{i,t})) \left\{ \begin{array}{c} \psi(S_{i,t}) \\ -A_{i,t}\psi(S_{i,t}) \end{array} \right\} \right] = 0$$

The first set of constraints imply the action-centering adjustments from the linear setting continue to hold. We then again apply the additive adjustment but now using $\mu_t^*(H_{i,t}, A_{i,t}) = \hat{\mu}_t(H_{i,t}, A_{i,t}) + (\psi(S_{i,t}), A_{i,t}\psi(S_{i,t}))^{\top}\hat{\theta}$ where $\hat{\theta}$ solves the above estimating equation. This does not reflect constraints on the outcome.

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