Independence and indifferent points imply continuity

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Abstract

The expected utility theorem of Von Neumann and Morgenstern (1947) has been a milestone in economics, describing rational behavior by two axioms on a weak preference on lotteries on a finite set of outcomes: the Independence Axiom and the Continuity Axiom. For a weak preference fulfilling the Independence Axiom, I prove that continuity is equivalent to the existence of a set indifferent lotteries spanning a hyperplane.

Let X be a finite set of cardinality n and let $Ł := \Delta(X)$ be the set of lotteries over X, which is a mixture space and can be seen as a simplex and a subset of $\mathbb{R}^{n-1}$.[1] I consider a weak preference relation $\succeq$ on Ł, i.e., $\succeq$ is complete and transitive. As usual, $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succeq$, indicating indifference and a strict preference. For a fixed $P \in Ł$ the strictly better, indifferent and strictly worse sets are defined as $Ł_{\succ P} := \{R \in Ł \mid R \succ P\}$, $Ł_{\sim P} := \{R \in Ł \mid R \sim P\}$, $Ł_{\prec P} := \{R \in Ł \mid P \succ R\}$.

Classical expected utility theory is based on two more axioms - independence and continuity. The Independence Axiom asserts that the preference ranking of two lotteries P and Q remains the same if equally mixed with any third lottery R.

Axiom 1 (Independence Axiom)**.** *For all $P, Q, R \in Ł$ and $\alpha \in (0,1]$ we have*

$$P \succeq Q \iff \alpha P + (1-\alpha)R \succeq \alpha Q + (1-\alpha)R. \tag{IA}$$

Continuity ensures that there are "no jumps" in the preference and thus rule out lexicographic orderings.

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[1] An element $P \in Ł$ is identified with the vector $(P(x_1), \dots, P(x_{n-1})) \in \mathbb{R}^{n-1}$ and can be recovered from it via $P(x_n) = 1 - \sum_{i=1}^{n-1} P(x_i)$.

Axiom 2 (Continuity Axiom)**.** *For all $P \in L$, $L_{\succ P}$ and $L_{\prec P}$ are open.*

For a weak preference relation on Ł that fulfills the Independence Axiom, I show that continuity is equivalent to the existence of a set of indifference points that span a hyperplane in $\mathbb{R}^{n-1}$.

Axiom 3 (indifferent points)**.** *There exist $P_1, \dots, P_{n-1} \in Ł_{\sim P_1}$ that span a hyperplane in $\mathbb{R}^{n-1}$.*

In other words, the set of directional vectors $\{P_k - P_1\}_{k=2}^{n-1}$ is linearly independent in $\mathbb{R}^{n-1}$. In the special case of $n = 3$, Axiom 3 simply requires the existence of two indifferent lotteries,i.e., $P_1 \neq P_2, P_1 \sim P_2$.

Theorem 1. *Let $\succeq$ be a weak preference on $Ł = \Delta(X)$ that fulfills the Independence Axiom 1. Then the following statements are equivalent:*

(i) $\succeq$ is continuous, i.e., Axiom 2 holds.

(ii) There exist $P_1, \dots, P_{n-1} \in Ł_{\sim P_1}$ that span a hyperplane in $\mathbb{R}^{n-1}$, i.e., Axiom 3 holds.

As an immediate consequence, we obtain the following variant of the expected utility theorem.

Corollary 1. *For a binary preference relation $\succeq$ on $Ł = \Delta(X)$, the following statements are equivalent:*

(i) $\succeq$ is a weak preference and fulfills Axioms 1 and 3.

(ii) $\succeq$ is a weak preference and fulfills Axioms 1 and 2.

(iii) There exists a utility function $u\colon X \to \mathbb{R}$ such that $\succeq$ is represented by the functional

$$U(P) := \sum_{x \in X} P(x) \cdot u(x).$$

Remark 1. On Ł there are weak preferences that fulfill Axiom 3, but not the Continuity Axiom 2. For instance, let $X = \{x_1, \dots, x_n\}$ and let $\succeq_{\text{lex}}$ denote the lexicographic ordering on Ł that prioritizes the weight on x_1. Define the preference $P \succeq Q :\iff P(x_1) = Q(x_1) = \frac{1}{2}$ or $P \succeq_{\text{lex}} Q$. Then, $\succeq$ is complete, transitive and fulfills Axiom 3, but is not continuous.
Consequently, Axiom 3 does not generally imply continuity on the simplex. Continuity is thus either stronger than Axiom 3 or the two axioms cannot be generally compared.

References

Von Neumann, J. and O. Morgenstern (1947). *Theory of games and economic behavior, 2nd rev.* Princeton university press.

A Proofs

For completeness sake, I provide a self-contained proof that Axiom 3 implies the Continuity Axiom 2 for a weak preference that fulfills the Independence Axiom 1. To this end, some well-known auxiliary lemmata are provided.

Our first observation identifies indifference classes as convex subsets of Ł.

Lemma A1. *The set $Ł_{\sim P}$ is convex for every $P \in Ł$.*

Proof of Lemma A1. Let $Q, Q' \in Ł_{\sim P}$ and $\alpha \in (0,1)$. By the (IA) we have

$$\alpha Q + (1-\alpha)Q' \sim \alpha P + (1-\alpha)Q' \sim \alpha P + (1-\alpha)P = P.$$

Hence, $Ł_{\sim P}$ is closed under taking convex combinations. □

Note that Lemma A1 reveals that the convex hull $\text{conv}(\mathcal{P})$ of any set indifferent lotteries $\mathcal{P} \subseteq Ł_{\sim P}$ is contained in $Ł_{\sim P}$.
Let $p_1, \dots, p_m$ be points in $\mathbb{R}^{n-1}$. Recall that the *affine hull* of $p_1, \dots, p_m$ in $\mathbb{R}^{n-1}$ is

$$\text{aff}(p_1, \dots, p_m) := \left\{ \sum_{k=1}^{m} \lambda_k \cdot p_k \,\middle|\, \sum_{k=1}^{m} \lambda_k = 1, \lambda_k \in \mathbb{R} \right\} \tag{1}$$

$$= \left\{ p_1 + \sum_{k=2}^{m} \lambda_k \cdot (p_k - p_1) \,\middle|\, \lambda_k \in \mathbb{R} \right\}. \tag{2}$$

From expression (1) it is easy to see that the convex hull $\text{conv}(p_1, \dots, p_m)$ is contained in the affine hull by simply restricting the coefficients λ_k to be non-negative. Expression (2) represents the affine hull as the translation of the vector space, spanned by the set of directional vectors $\{p_k - p_1\}_{k=2}^m$, by (and thus through) p_1.

The following observation shows that all lotteries in the affine hull defined by a set of indifferent lotteries is contained in the corresponding indifference class.

Lemma A2. *Let $P_1, \dots, P_m \in Ł_{\sim P_1}$. Then $\text{aff}(P_1, \dots, P_m) \cap Ł \subseteq Ł_{\sim P_1}$.*

Proof of Lemma A2. Let $A := \text{aff}(P_1, \dots, P_m)$. If P_k is contained in the affine hull of the remaining points, we can discard it from the set of considered points. Assume thus that A is an affine space of dimension $m-1$, i.e., the vectors $P_k - P_1$ for $k = 2, \dots, m$ are linearly independent in $\mathbb{R}^{n-1}$. Especially, the representation of any element $P \in A$ in the form of expression (1) is unique. If $m = 1$, $\text{conv}(P_1) = A \cap Ł = \{P_1\} \subseteq Ł_{\sim P_1}$. Assume thus $m \geq 2$ in the following.

Consider now any $P = \sum_{k=1}^m \lambda_k P_k \in Ł$, $\sum_{k=1}^m \lambda_k = 1$. If $\lambda_k \geq 0$ for all $k = 1, \dots, m$, we have $P \in \text{conv}(P_1, \dots, P_M)$ and thus $P \in Ł_{\sim P_1}$ by Lemma A1. Assume thus that $\lambda_k < 0$ for at least one $k \in \{1, \dots, m\}$ and let $k^* \in \arg\min_k \lambda_k$ and $\lambda^* := -\lambda_{k^*}$, which is positive. Set $\overline{P} := \sum_{k=1}^m \frac{1}{m} P_k \in$

$\mathrm{conv}(P_1,\dots,P_m) \subseteq \text{Ł}_{\sim P_1}$. Define $Q(\alpha) := \alpha\overline{P} + (1-\alpha)P$, which can be written as $Q(\alpha) = \sum_{k=1}^{m} \left(\frac{\alpha}{m} + (1-\alpha)\lambda_k\right) \cdot P_k$. For $\alpha^* := \frac{m\lambda^*}{1+m\lambda^*}$, the coefficients of $Q := Q(\alpha^*)$ are all non-negative and the one for k^* is equal to zero. Thus, $Q \in \mathrm{conv}(P_1,\dots,P_m) \subseteq \text{Ł}_{\sim P_1}$ and $Q \neq \overline{P}$. By $Q \sim \overline{P} \sim P_1$ and applying the (IA), we find $Q = \alpha^*\overline{P} + (1-\alpha^*)P \sim \alpha^* Q + (1-\alpha^*)P$ and thus $P \sim Q \sim P_1$. □

The following observation shows that translations of indifference sets remain indifference sets.

Lemma A3. *Let $P, Q \in \text{Ł}$. Then $\{R + (Q-P) \mid R \in \text{Ł}_{\sim P}\} \cap \text{Ł} \subseteq \text{Ł}_{\sim Q}$.*

Proof of Lemma A3. Note that $Q \in \{R + (Q-P) \mid R \in \text{Ł}_{\sim P}\} \cap \text{Ł}$. If the set is a singleton, there is nothing to prove. Let now $Q \neq Q' = P' + (Q-P) \in \text{Ł}$ for some $P' \in \text{Ł}_{\sim P}$. Define $Z := \frac{1}{2}P + \frac{1}{2}Q' = \frac{1}{2}P' + \frac{1}{2}Q$. Since $P \sim P'$, we have $Z \sim \frac{1}{2}P + \frac{1}{2}Q$ and thus $Q' \sim Q$ by the (IA). □

The line between two points $p, q \in R^{n-1}$ is defined by

$$\mathrm{line}(p,q) := \{q + t \cdot (p-q) \mid t \in \mathbb{R}\}, \tag{3}$$

and partitions into points to the left of q ($t < 0$), between p, q ($t \in [0,1]$) and to the right of p ($t > 1$). The following observation classifies how a strict preference translates to all points on the line connecting two lotteries.

Lemma A4. *Let $P, Q \in \text{Ł}$, $P \succ Q$ and $P_t := Q + t \cdot (P-Q) \in \mathrm{line}(P,Q)$. Then, $P_0 = Q, P_1 = P$ and*

$$\begin{cases} Q \succ P_t & , t < 0, \\ P \succ P_t \succ Q & , 0 < t < 1, \\ P_t \succ P & , 1 < t. \end{cases} \tag{4}$$

Proof of Lemma A4. The cases for $t = 0, 1$ are clear. *Case* $t \in (0,1)$: Since $P \succ Q$ we have by the (IA) that $P \succ P_t = tP + (1-t)Q \succ Q$. *Case* $1 < t$: P is equal to the convex combination $\alpha P_t + (1-\alpha)Q$ for $\alpha := \frac{1}{t}$. Assume by means of contradiction that $P \succeq P_t$. Then, by applying the (IA) twice, we find the contradiction $P \succeq \alpha P_t + (1-\alpha)P \succ \alpha P_t + (1-\alpha)Q = P$. *Case* $t < 0$: Now, Q is the convex combination $\alpha P_t + (1-\alpha)P$ for $\alpha := \frac{1}{1-t}$. Assume by means of contradiction that $P_t \succeq Q$. Then, by applying the IA, we find the contradiction $Q = \alpha P_t + (1-\alpha)P \succ \alpha P_t + (1-\alpha)Q \succeq Q$. □

I now show that Axiom 3 implies the Continuity Axiom 2.

Theorem A1. *Let $P_1, \dots, P_{n-1} \in \text{Ł}_{\sim P_1}$ such that $\{P_k - P_1\}_{k=2}^{n-1}$ are linearly independent in $\mathbb{R}^{n-1}$. Then, $\succeq$ is continuous.*

Proof of Theorem A1. Let $H_{P_1} := \mathrm{aff}(P_1,\dots,P_{n-1})$ which is a hyperplane of $\mathbb{R}^{n-1}$ by the assumption on the P_k. Hence, there exists a normal vector $\vec{n} \in \mathbb{R}^{n-1}$ such that $H_{P_1} = \left\{q \in \mathbb{R}^{n-1} \,\middle|\, \langle q - P_1, \vec{n}\rangle = 0\right\}$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product of $\mathbb{R}^{n-1}$.

Consider any $P \in Ł$. To prove continuity of $\succeq$, I show that $Ł_{\succ P}$ is open. That $Ł_{\prec P}$ is open follows analogously.

To this end, I study the hyperplane parallel to H_{P_1} that passes through P and partitions $\mathbb{R}^{n-1}$ (and thus Ł through intersection) in three convex parts: $H_P^+ := \{q \in \mathbb{R}^{n-1} \,|\, \langle q - P, \vec{n}\rangle > 0\}$, $H_P := \{q \in \mathbb{R}^{n-1} \,|\, \langle q - P, \vec{n}\rangle = 0\}$ and $H_P^- := \{q \in \mathbb{R}^{n-1} \,|\, \langle q - P, \vec{n}\rangle < 0\}$. If $Ł_{\succ P} = \emptyset$, there is nothing to prove. Assume thus that there is $P^* \in Ł_{\succ P}$. Since $H_P \subseteq Ł_{\sim P}$ by Lemma A3, I can assume $P^* \in H_P^+$ (otherwise consider $-\vec{n}$ instead). I claim $Ł_{\succ P} = H_P^+$, which is an open set and is thus going to conclude the proof.

"$\subseteq$": Let $Q \in Ł_{\succ P}$ and assume by means of contradiction $Q \in H_P^-$. Then, by continuity of the scalar product, there exists $\alpha \in (0,1)$ such that $\alpha P^* + (1-\alpha)Q \in H_P \subseteq Ł_{\sim P}$, which violates $\alpha P^* + (1-\alpha)Q \succ P$ by the (IA).

"$\supseteq$": Let $Q \in H_P^+$. Consider the hyperplanes H_Q, H_{P^*} parallel to H_P and passing through Q, P^*. If $\langle P^* - P, \vec{n}\rangle \geq \langle Q - P, \vec{n}\rangle$, Q lies between H_P and H_{P^*}. Connecting P and P^*, there exists and $\alpha \in [0,1)$ with $\alpha P + (1-\alpha)P^* \in H_Q \cap Ł$, which is thus indifferent to Q by Lemma A3. By Lemma A4, we thus have $P^* \succeq Q \succ P$. Finally, if $\langle Q - P, \vec{n}\rangle > \langle P^* - P, \vec{n}\rangle$, P^* lies between H_P and H_Q. Connecting P and Q, we find an $\alpha \in (0,1)$ with $\alpha P + (1-\alpha)Q \in H_{P^*} \cap Ł \subseteq Ł_{\sim P^*}$. Applying Lemma A4, we find $Q \succ P^* \succ P$. □

We now conclude with the proof of Theorem 1 and Corollary 1

Proof of Theorem 1 and Corollary 1. By Theorem A1, a weak preference that fulfills the Independence Axiom 1 and Axiom 3 is continuous. By the classical result of von Neumann-Morgenstern, conditions (ii) and (iii) are equivalent. It thus suffices to show that (iii) implies the existence of indifferent points that span a hyperplane, i.e., Axiom 3. To this end, let $X = \{x_1, \ldots, x_n\}$, define $\overline{u} := \sum_{k=1}^{n} \frac{1}{n} \cdot u(x_k)$ and consider the following system of linear equations for $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$.

$$\underbrace{\begin{pmatrix} u(x_1) & u(x_2) & \ldots & u(x_n) \\ 1 & 1 & \ldots & 1 \end{pmatrix}}_{=:M} \cdot \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} \overline{u} \\ 1 \end{pmatrix},$$

Note that $\overline{P} := (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is a particular solution. Since $\operatorname{rank}(M) \leq 2$, the kernel $\ker(M)$ of the linear map induced by M is a vector space of at least dimension $n-2$ and the solution set of the system is given by $\overline{P} + \ker(M)$. Let $b_2, \ldots, b_{n-1}$ be a basis of $\ker(M)$. Since all entries of $\overline{P}$ are positive, there is an $\varepsilon > 0$ such that $P_1 := \overline{P}, P_k := \overline{P} + \varepsilon \cdot b_k$, $k = 2, \ldots, n-1$ spans a hyperplane of points that are indifferent with utility level $\overline{u}$. □