

PERVERSE SHEAVES ON A LOOP GROUP AND LANGLANDS' DUALITY

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Introduction

Grothendieck associated to any complex of sheaves \mathcal{F} on a variety X over a finite field \mathbb{F} the function

$$x \mapsto \chi_{\mathcal{F}}(x) = \sum (-1)^i \text{Tr}(\text{Fr}; \mathcal{H}_x^i \mathcal{F}), \quad x \in X^{\text{Fr}}$$

on the set of \mathbb{F} -rational points of X given by the alternating sum of traces of Fr , the Frobenius action on stalks of the cohomology sheaves $\mathcal{H}^i \mathcal{F}$. He then went on to initiate an ambitious program of giving *geometric* (= sheaf theoretic) meaning to various classical *algebraic* formulas via the above “functions-faisceaux” correspondence $\mathcal{F} \mapsto \chi_{\mathcal{F}}$. This program got a new impetus with the discovery of perverse sheaves [BBD], for it happens for certain mysterious reasons that most of the functions encountered ‘in nature’ arise via the “functions-faisceaux” correspondence from perverse sheaves. In the present paper Grothendieck’s philosophy is applied to what may be called the Geometric Langlands duality.

The relevance of the intersection cohomology technique to our problem was first pointed out by Drinfeld [D] and Lusztig [Lu 1]. Later, in the remarkable paper [Lu], Lusztig established algebraically a surprising connection between finite-dimensional representations of a semisimple Lie algebra and the Kazhdan-Lusztig polynomials for an affine Weyl group. It is one

of the purposes of the present paper to provide a geometric interpretation of [Lu]. It should be mentioned however that the results of [Lu] are used in a very essential way in the proof of our main theorem in section 2.

To begin with, recall the classical notion of a *spherical function*. Fix a semisimple group G . Given a ring R , let $G(R)$ denote the corresponding group of R -rational points. Let p be a prime, \mathbb{Q}_p the p -adic field, and \mathbb{Z}_p its ring of integers. Let $\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]$, be the Hecke algebra with respect to convolution of complex valued $G(\mathbb{Z}_p)$ -biinvariant, compactly supported functions on $G(\mathbb{Q}_p)$. Let further $T(\mathbb{C}) \subset G(\mathbb{C})$ be a maximal torus in the corresponding complex group and $X_*(T) = \text{Hom}_{\text{alg}}(\mathbb{C}^*, T(\mathbb{C}))$ the lattice of one-parameter subgroups of $T(\mathbb{C})$. Let $\mathbb{C}[X_*(T)]$ be the group algebra of this lattice acted upon naturally by W , the Weyl group of G . Then, one has the following classical result due to Satake:

There is an algebra isomorphism (explicitly constructed by Macdonald)

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \simeq \mathbb{C}[X_*(T)]^W \quad (0.0)$$

Our aim is to produce a sheaf-theoretic counterpart of isomorphism (0.1). This is done in several steps by reinterpreting both the left hand and the right hand sides of (0.1). First, we introduce the dual complex torus T^\vee such that $X_*(T) = X^*(T^\vee)$ is the weight lattice of T^\vee . Let G^\vee be the Langlands dual of G , the complex semisimple group having T^\vee as a maximal torus and having the root system dual to that of G . Then, the RHS of (0.1) can be rewritten as

$$\mathbb{C}[X^*(T^\vee)]^W \simeq \mathbb{C}[G^\vee]^{G^\vee} \quad (0.0)$$

where $\mathbb{C}[G^\vee]^{G^\vee}$ stands for the algebra of polynomial class functions on G^\vee . Let Rep_{G^\vee} be the abelian category of finite dimensional rational complex representations of G^\vee , and $K(\text{Rep}_{G^\vee})$ its Grothendieck group. Assign to a representation $V \in \text{Rep}_{G^\vee}$ its character, a class function on G^\vee . This yields a natural algebra isomorphism $\mathbb{C} \otimes_{\mathbb{Z}} K(\text{Rep}_{G^\vee}) \xrightarrow{\sim} \mathbb{C}[G^\vee]^{G^\vee}$. By (0.2), the isomorphism (0.1) can thus be written as

$$\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \simeq \mathbb{C} \otimes_{\mathbb{Z}} K(\text{Rep}_{G^\vee}) \quad (0.0)$$

We next turn to the LHS of (0.1). Let $\mathbb{F} = \mathbb{Z}_p/p \cdot \mathbb{Z}_p$ be the residue class field and $\overline{\mathbb{F}}$ an algebraic closure of \mathbb{F} . First we use the well known analogy between the fields \mathbb{Q}_p and $\mathbb{F}((z))$ (=the Laurent formal power series field). The algebra $\mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]$ is unaffected by the substitution

$$\mathbb{Q}_p \longleftrightarrow \mathbb{F}((z)), \quad \mathbb{Z}_p \longleftrightarrow \mathbb{F}[[z]], \quad p \longleftrightarrow z$$

Thus, the LHS of (0.3) can be rewritten as

$$G(\mathbb{F}[[z]])\text{-invariant functions on } G(\mathbb{F}((z)))/G(\mathbb{F}[z])$$

Now, following Grothendieck, we view the discrete set $G(\mathbb{F}((z)))/G(\mathbb{F}[z])$ as the set of \mathbb{F} -rational points of $\text{Gr} = G(\overline{\mathbb{F}}((z)))/G(\overline{\mathbb{F}}[[z]])$, an infinite dimensional algebraic variety. We introduce a category $P(\text{Gr})$ whose objects are semisimple $G(\overline{\mathbb{F}}[[z]])$ -equivariant perverse sheaves on Gr . There is a natural tensor product structure on the category $P(\text{Gr})$ given by a sort of convolution. Moreover, the ‘‘function-faisceaux’’ correspondence assigns to $\mathcal{F} \in P(\text{Gr})$ a function $\chi_{\mathcal{F}}$ on $G(\mathbb{F}((z)))/G(\mathbb{F}[z])$. That gives an isomorphism of the Grothendieck group, $\mathbb{C} \otimes_{\mathbb{Z}} K(P(\text{Gr}))$, with the algebra of $G(\mathbb{F}[[z]])$ -invariant functions on $G(\mathbb{F}((z)))/G(\mathbb{F}[z])$. Thus, the algebra isomorphism (0.1) takes the following final form

$$\mathbb{C} \otimes_{\mathbb{Z}} K(P(\text{Gr})) \simeq \mathbb{C} \otimes_{\mathbb{Z}} K(\text{Rep}_{G^{\vee}}) \quad (0.0)$$

The main result of this paper (theorem 1.4) can now be formulated as follows:

There is an equivalence of tensor categories $P(\text{Gr}) \simeq \text{Rep}_{G^{\vee}}$ that induces isomorphism (0.4) on the corresponding Grothendieck rings. Moreover, the underlying vector space of a representation $V \in \text{Rep}_{G^{\vee}}$ may be identified canonically with the hyper-cohomology of the corresponding perverse sheaf $\mathcal{P}(V) \in P(\text{Gr})$.

In the main body of the paper we replace the field $\overline{\mathbb{F}}$ by the field \mathbb{C} of complex numbers. This replacement does not affect the formulation of the theorem above, except that Gr is now regarded as an (infinite-dimensional) complex algebraic variety equipped with the usual Hausdorff topology. Thus, finite and p -adic fields serve only as a motivation and will never appear in the rest of the paper.

A few words on the organization of the paper are in order.

In the first chapter we formulate most of the results of the paper without going into any technical details and sometimes without even giving proper definitions. This is made to simplify the reading of the paper, for working with perverse sheaves on an infinite-dimensional variety like Gr requires a lot of extracare in technical details to make our approach rigorous. Thus, the first chapter contains the statement of the main theorem and its basic applications.

Chapters 2 and 3 are devoted to the proof of the main theorem. The key ingredient of the proof is a construction of a *fiber functor* on the tensor category $P(\text{Gr})$ in terms of the equivariant cohomology.

There are two kinds of applications of the main theorem that we consider. The first one is of ‘global’ nature, concerning a smooth compact complex curve X of genus > 1 . We set up a Langlands-type correspondence between ‘*automorphic perverse sheaves*’ on the moduli space of principal holomorphic G -bundles on X and homomorphisms $\pi_1(X) \rightarrow G^\vee$, respectively. The detailed treatment of this subject including definition of the *modular stack* of principal G -bundles on a smooth complex curve is given in chapter 6. We include here new proofs of two (known) basic geometric results underlying the theory. The first one says that any algebraic G -bundle on an affine curve is trivial. The second one says that the *global Nilpotent variety* is a Lagrangian substack in the cotangent bundle to the moduli space. We then show how these results can be combined with the main theorem to obtain the Langlands correspondence (one way).

Applications of the second type related to the topology of the Grassmannian Gr are discussed in chapters 4–5. It is shown there that the cohomology, $H^*(\text{Gr}, \mathbb{C})$, is isomorphic to the Symmetric algebra of the centralizer of the *principal nilpotent* in $\text{Lie}(G^\vee)$. The principal nilpotent itself turns out to be nothing but the first Chern class of the Determinant bundle on Gr . Among other things, that enables us to give natural proofs of the results of Lusztig [Lu] and Brylinski [Br] concerning the q -analogue of the weight multiplicity, and also to establish connections with various interesting questions in representation theory.

This paper is a slightly revised TeX-version of the typewritten manuscript with the same title distributed in 1989 (without chapter 6).

I am greatly indebted to V. Drinfeld without whom this work would have never been carried out. He initiated the whole project by raising the question whether the category $P(\text{Gr})$ studied below is a tensor category. I am grateful to A. Beilinson who explained me a lot about the geometry of modular stacks which is crucial for chapter 6. The present paper may be seen, in fact, as a preparatory step towards the Geometric Langlands correspondence [BeDr].

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1 Main results

1.1 Duality

Given a complex torus T , let T^\vee denote the group of 1-dimensional local systems on T with the tensor product group structure. In ground to earth terms, T^\vee is the group of all homomorphisms: $\pi_1(T) \rightarrow \mathbb{C}^*$. This is again a complex torus which is called the dual of T . More explicitly, let \mathfrak{t} be the Lie algebra of T , \mathfrak{t}^* the dual space, $X_*(T)$ the lattice of holomorphic homomorphisms: $\mathbb{C}^* \rightarrow T$, and $X^*(T)$ the (dual) lattice of holomorphic homomorphisms: $T \rightarrow \mathbb{C}^*$. Taking the differential of such a homomorphism at the identity defines embeddings: $X_*(T) \hookrightarrow \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathfrak{t}) = \mathfrak{t}$ and $X^*(T) \hookrightarrow \text{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C}) = \mathfrak{t}^*$. Then, we have canonically:

$$T \cong \mathfrak{t}/X_*(T), \quad T^\vee \cong \mathfrak{t}^*/X^*(T)$$

so that there are canonical identifications:

$$(\mathfrak{t}^\vee)^* \simeq \mathfrak{t}, \quad X^*(T^\vee) \simeq X_*(T) \tag{1.1}$$

Now, let G be a split connected semisimple group over \mathbb{Z} , and T a maximal torus in G . We let G^\vee denote the connected complex semisimple group having the dual torus T^\vee as a maximal torus and having the root system dual to that of G . The group G^\vee is said to be the Langlands dual to G [Lan]. Our immediate goal is to give an intrinsic new construction of G^\vee which does not appeal to root systems, maximal tori, etc.

1.2 Infinite Grassmannian

Given the semisimple group G and a \mathbb{C} -algebra R write $G(R)$ for the group (over \mathbb{C}) of R -rational points of G . Our basic choices of the ring R are

$\mathbb{K} = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$. Below we will more often use polynomial rings $\mathbb{C}[z, z^{-1}]$ and $\mathbb{C}[z]$ instead of the rings \mathbb{K} and \mathcal{O} , respectively. We will use shorthand notation LG and L^+G instead of $G(\mathbb{C}[z, z^{-1}])$ and $G(\mathbb{C}[z])$ respectively. Choosing an imbedding $G \rightarrow GL_n$ presents $G(\mathbb{K})$ as a subgroup of \mathbb{K} -valued $n \times n$ matrices and the group LG as the group of maps $f : \mathbb{C}^* \rightarrow G(\mathbb{C}) \subset \text{Mat}_n(\mathbb{C})$ that can be written as a finite Laurent polynomial:

$$f(z) = \sum_{i=-m}^m A_i \cdot z^i, \quad A_i \text{ are } n \times n \text{ matrices, and } f(z) \in G(\mathbb{C}), \forall z \in \mathbb{C}^* \quad (1.2.0)$$

Neither topological nor algebraic structure of the groups $G(\mathbb{K})$ and LG themselves will be of any importance for us. What *is* important is that the corresponding ‘integral’ groups $G(\mathcal{O})$ and L^+G each has a decreasing chain of normal subgroups:

$$L^0 \supset L^1 \supset L^2 \supset \dots$$

In either case, the subgroup $L^m, m > 0$, is defined to consist of the following loops f regular at the origin and such that

$$f(0) = 1 \text{ and } f \text{ has vanishing derivatives at } 0 \text{ up to order } m \quad (1.2.1^m)$$

(if $m = 0$ we put $L^0 = \{f \mid f(0) = 1\}$). For each $m \geq 0$, the quotient $G(\mathcal{O})/L^m$, resp. L^+G/L^m , clearly has the natural structure of a finite-dimensional algebraic group. Observe that, for $m > 0$, the quotient L^m/L^{m+1} is an abelian group isomorphic to the additive group $\mathfrak{g} := \text{Lie } G$.

We are interested in this paper in the coset space $\text{Gr} := G(\mathbb{K})/G(\mathcal{O})$. It is called the Infinite Grassmannian for, if $G = SL_n$, the set Gr gets identified naturally with the variety of unimodular \mathcal{O} -lattices of maximal rank in \mathbb{K}^n . The ‘finiteness’ properties of the Infinite Grassmannian listed in the following proposition allow us to apply to Gr standard algebro-geometric constructions as if it was a finite-dimensional algebraic variety.

Proposition 1.2.2 (i). *The set Gr is the union of an infinite sequence of $G(\mathcal{O})$ -stable subsets: $\text{Gr}_1 \subset \text{Gr}_2 \subset \dots$*

(ii). *These subsets have a compatible structure of finite-dimensional projective varieties of increasing dimension, i.e. $\text{Gr}_i \hookrightarrow \text{Gr}_j$ is a projective imbedding for any $i < j$.*

(iii). *For any i the $G(\mathcal{O})$ -action on Gr_i factors through an algebraic action of the group $G(\mathcal{O})/L^m$, where $m = m(i)$ is large enough.*

(iv). Each piece Gr_i consists of finitely many $G(\mathcal{O})$ -orbits.

Comments on Proof: Part (i) of the proposition is due to Lusztig [Lu, n. 11]). Let Grass be the set of all \mathcal{O} -submodules in the vector space $\mathfrak{g}(\mathbb{K})$, viewed as an \mathcal{O} -module. Lusztig considered a map $G(\mathbb{K}) \rightarrow \text{Grass}$ given by the assignment $f \mapsto (\text{Ad } f)\mathfrak{g}(\mathcal{O})$. He showed that this map yields a bijection of the coset space $G(\mathbb{K})/G(\mathcal{O})$ with a set \mathcal{B} of Lie subalgebras in $\mathfrak{g}(\mathbb{K})$ subject to certain algebraic conditions. Further, for each $i \geq 1$, following Lusztig put

$$\text{Gr}_i := \{L \in \mathcal{B} \mid z^i \cdot \mathfrak{g}(\mathcal{O}) \subset L \subset z^{-i} \cdot \mathfrak{g}(\mathcal{O})\} \quad (1.2.0)$$

With this definition, parts (i) and (ii) of the proposition are immediate. Part (iv) follows by comparing the known (cf. [PS, ch. 8] and section 1.4 below) parametrization of $G(\mathcal{O})$ -orbits with the above definition of the sets Gr_i . It remains to prove (iii). This can be done for each $G(\mathcal{O})$ -orbit separately, due to (iv). Furthermore, it suffices to check the claim for a single point of each $G(\mathcal{O})$ -orbit, for the groups L^m are normal in $G(\mathcal{O})$. But every orbit contains a point represented by a group homomorphism $\lambda : \mathbb{C}^* \rightarrow G$, see n. 1.4. For such a point claim (iii) is clear. Indeed, the image of λ is contained in a maximal torus $T \subset G$. Then, λ may be viewed, cf. n°1.1, as an element of the lattice $X_*(T) \subset \mathfrak{t} = \text{Lie } T$. Then, in property (iii), one may take m to be an integer, such that $m > \alpha(\lambda)$, for any root $\alpha \in \mathfrak{t}^*$. \square

One may introduce a similar Grassmannian in the polynomial setup, putting $\text{Gr}' = LG/L^+G$. It turns out that the object one gets in this way is not only *similar* but is in fact *identical* to the previous one. Specifically, the natural imbeddings

$$LG \hookrightarrow G(\mathbb{K}) \quad , \quad L^+G \hookrightarrow G(\mathcal{O})$$

give a map $j : \text{Gr}' \rightarrow \text{Gr}$. This map is injective, for it is clear that $LG \cap G(\mathcal{O}) = L^+G$. Moreover, the following comparison result shows that $j(\text{Gr}') = \text{Gr}$ and that we may (and will) make no distinction between Gr and Gr' .

Proposition 1.2.4 *The map j is an LG -equivariant isomorphism $j : \text{Gr}' \xrightarrow{\sim} \text{Gr}$. Any $G(\mathcal{O})$ -orbit in Gr is the image of a single L^+G -orbit in Gr' .*

Proof: Define an exhaustion $\text{Gr}'_1 \subset \text{Gr}'_2 \subset \dots = \text{Gr}'$ similar to the one introduced in the proposition 1.2.2. Equation (1.2.3) shows that Gr_i may

be viewed as a variety of vector subspaces in the finite dimensional \mathbb{C} -vector space $z^{-i} \cdot \mathfrak{g}(\mathcal{O})/z^i \cdot \mathfrak{g}(\mathcal{O})$, subject to certain algebraic conditions. Observe now that replacing power series by polynomials affects neither the space

$$z^{-i} \cdot \mathfrak{g}(\mathcal{O})/z^i \cdot \mathfrak{g}(\mathcal{O}) = z^{-i} \cdot \mathfrak{g}(\mathbb{C}[z])/z^i \cdot \mathfrak{g}(\mathbb{C}[z])$$

nor the algebraic conditions (the ‘infinite tail’ of all the powers of z greater than i disappears in the quotient). It follows that $j(\mathrm{Gr}'_i) = \mathrm{Gr}_i$ for any i . Moreover, it follows that $G(\mathcal{O})$ -orbits in Gr_i are the same things as L^+G -orbits in Gr'_i . That completes the proof. \square

There is another realization of the Infinite Grassmannian often used in topology. Let K be a maximal compact subgroup of G and S^1 the unit circle. Write Ω for the group of based polynomial loops $f : S^1 \rightarrow K$, i.e. expressions like (1.2.1) with $f(z) \in K$ for all $z \in S^1$, and $f(1) = 1$. Such a loop extends uniquely to a polynomial map $f : \mathbb{C}^* \rightarrow G$ such that $f(\bar{z}) = \overline{f(z)}$ where ‘bar’ stands for the complex conjugation (in \mathbb{C}) on the LHS and for the involution on G corresponding to the real form K on the RHS. This way one gets a group imbedding $\Omega \hookrightarrow LG$. One has an ‘Iwasawa’ decomposition, see [PS]: $LG = \Omega \cdot L^+G$, $\Omega \cap L^+G = \{1\}$. Thus, the Grassmannian $\mathrm{Gr} = LG/L^+G$ can be identified with the group Ω . We endow both Gr and Ω with the topology of direct limit of closed finite-dimensional subvarieties $\mathrm{Gr}_1 \subset \mathrm{Gr}_2 \subset \dots$ and let $m : \mathrm{Gr} \times \mathrm{Gr} \rightarrow \mathrm{Gr}$ denote the map corresponding to the multiplication in Ω . We’ll show later that, for any $i, j \geq 1$, there exists an integer $k = k(i, j) \gg 0$ such that the product of Gr_i and Gr_j is contained in Gr_k and, moreover, the multiplication map $m : \mathrm{Gr}_i \times \mathrm{Gr}_j \rightarrow \mathrm{Gr}_k$ is a morphism of real-analytic sets. This makes $\mathrm{Gr} \simeq \Omega$ a topological group (with an additional sub-analytic structure).

Let $D^b(\mathrm{Gr})$ be the bounded derived category of constructible complexes on Gr whose support is contained in a big enough subset Gr_i , i.e., the direct limit of the categories $D^b(\mathrm{Gr}_i)$. Given $L, M \in D^b(\mathrm{Gr})$, define $L * M \in D^b(\mathrm{Gr})$ by the formula: $L * M = m_*(L \boxtimes M)$. The assignment $L, M \mapsto L * M$ is called *convolution*. The convolution is associative, i.e., for any $M_1, M_2, M_3 \in D^b(\mathrm{Gr})$, there is a functorial isomorphism: $M_1 * (M_2 * M_3) = (M_1 * M_2) * M_3$.

1.3 Tensor category $P(\mathrm{Gr})$

There is an obvious L^+G -action on Gr on the left. Each L^+G -orbit is contained in a certain Gr_i , and it is a smooth locally closed algebraic subvariety of Gr_i (cf. [PS, ch. 8]). Each orbit is known [PS] to be isomorphic to a vector

bundle over a partial flag variety (for the group G). In particular, the orbits are simply-connected. By proposition 1.2.2, all the Gr_i are L^+G -stable subsets of Gr , and the orbits form a finite stratification of each of the Gr_i .

Let $IC(O)$ denote the Intersection cohomology complex of the closure of a L^+G -orbit O , extended by 0 to all Gr . As explained, $IC(O)$ is a well-defined object of $D^b(\text{Gr})$. Let $P(\text{Gr})$ be the abelian full subcategory of $D^b(\text{Gr})$ whose objects are perverse sheaves on Gr isomorphic to finite direct sums of complexes $IC(O)$ (for various orbits O). Any object $L \in P(\text{Gr})$ has finite-dimensional hyper-cohomology $H^*(L)$.

Let Vect denote the abelian tensor category of finite-dimensional vector spaces over \mathbb{C} with the standard tensor product. The following result will be proved in the next chapter.

Theorem 1.3.1 (i). *If $L, M \in P(\text{Gr})$ then $L * M \in P(\text{Gr})$;*

(ii). *The pair $(P(\text{Gr}), *)$ is a semisimple rigid tensor category (cf. [DM, def. 1.7]);*

(iii). *The assignment: $M \rightsquigarrow H^\bullet(M)$ yields an exact fully-faithful tensor functor: $P(\text{Gr}) \rightarrow \text{Vect}$.*

1.4 Equivalence of categories

We remind the reader one of the main results of [DM]. It says [DM, thm. 2.1] that any tensor category having the properties as in theorem 1.3 is equivalent to the category $\text{Rep}(G^*)$ of finite-dimensional rational representations of a reductive group G^* . This applies, in particular, to the category $P(\text{Gr})$. Thus, starting from a semisimple group G we have constructed another reductive group G^* . The group G^* turns out to be isomorphic to G^\vee , the dual group of G . Thus, we get a new construction of the dual group.

We turn to the formulation of the main result of the paper. Let $T \subset G$ be a maximal torus. Any homomorphism $\lambda : \mathbb{C}^* \rightarrow T$, viewed as a map: $S^1 \rightarrow G$, determines the coset $\lambda \cdot L^+G \subset LG$ and, hence, a point in Gr . Let O_λ be the L^+G -orbit of that point. Any L^+G -orbit in Gr is known to be an orbit of the form O_λ for some $\lambda \in X_*(T)$, and $O_\lambda = O_\mu$ iff λ and μ are conjugate by W , the Weyl group of the pair (G, T) . For $\lambda \in X_*(T)$ we set $IC_\lambda := IC(O_\lambda)$, but it should be understood that λ is only determined here up to the action of W .

It is well known that irreducible finite-dimensional rational representations of the group G^\vee are labeled by their extreme weights. Let V_λ denote

the irreducible representation with extreme weight λ . Clearly, it depends only on the W -orbit of λ . Thus, we see from (1.1) that simple objects of the categories $P(Gr)$ and Rep_{G^\vee} are both indexed by the same set: $X_*(T)/W = X^*(T^\vee)/W$ (note that pairs (G, T) and (G^\vee, T^\vee) have identical Weyl groups).

Theorem 1.4.1 *There is an equivalence of the tensor categories $P(Gr)$ and Rep_{G^\vee} which sends IC_λ to V_λ . Furthermore, the hyper-cohomology functor H^\bullet on $P(Gr)$ goes, under the equivalence, to the forgetful functor: $\text{Rep}_{G^\vee} \rightarrow \text{Vect}$.*

1.5 A conjecture on automorphic sheaves

The purpose of this section is to explain at a heuristic level the main idea of the connection between theorem 1.4.1 above and Langlands' parametrization of so-called 'automorphic sheaves' on the moduli space of principal G -bundles. Rigorous approach to the moduli spaces in question requires some special techniques, which will be worked out later in chapter 6. It seems instructive at this point, however, to outline the perspective without waiting till chapter 6.

Let X be a smooth compact complex (algebraic) curve. Fix a point $x \in X$, and let \mathcal{O}_x be the local ring at x , $\mathfrak{m}_x \subset \mathcal{O}_x$ the maximal ideal of functions vanishing at x , and \mathbb{K}_x the fields of fractions of \mathcal{O}_x , that is the field of germs of rational functions on a punctured neighborhood of x .

Let G be a connected semisimple algebraic group, and $G(\mathbb{K}_x)$ and $G(\mathcal{O}_x)$ the corresponding groups of rational points, viewed as infinite dimensional complex groups. Set $Gr_x := G(\mathbb{K}_x)/G(\mathcal{O}_x)$, and write $P(Gr_x)$ for the category of $G(\mathcal{O}_x)$ -equivariant perverse sheaves on Gr_x with compact support. Any such perverse sheaf turns out to be automatically semisimple (for there are no extensions between simple objects of $P(Gr_x)$, due to the fact that all $G(\mathcal{O}_x)$ -orbits on Gr_x are even dimensional). Thus, the setup is very similar to that of section 1.2. Moreover, we claim that it is not only similar, but in fact identical:

Lemma 1.5.1 *There is a natural equivalence of tensor categories*

$$P(Gr_x) \simeq P(Gr).$$

To prove the lemma, one first replaces the rings \mathbb{K}_x and \mathcal{O}_x by the corresponding \mathfrak{m}_x -adic completions. Repeating the argument of proposition

1.2.4, one shows that this doesn't change the grassmannian Gr_x . It remains to note, that the curve X being smooth, choosing a local parameter z at the point $x \in X$ yields an isomorphism of rings $\hat{\mathcal{O}}_x \simeq \mathbb{C}[[z]]$. Whence, an isomorphism of the grassmannians $Gr_x \simeq Gr$, and the induced equivalence $P(Gr_x) \simeq P(Gr)$. Observe finally, that this equivalence is in effect independent of the choice of a local coordinate. Indeed, choosing another local coordinate, z' , at the point $x \in X$ amounts to choosing an isomorphism $\mathbb{C}[[z']] \simeq \mathbb{C}[[z]]$; the latter gives an automorphism of Gr that preserves L^+G -orbits. \square .

Further, let \mathcal{O}_{out} denote the ring of regular functions on the punctured curve $X \setminus \{x\}$. Restricting functions to a neighborhood of the puncture yields an algebra imbedding $\mathcal{O}_{\text{out}} \hookrightarrow \mathbb{K}_x$. Whence, the group imbedding $G(\mathcal{O}_{\text{out}}) \hookrightarrow G(\mathbb{K}_x)$. In chapter 6 we will endow the coset space $\mathcal{M} = G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}})$ with a certain structure of an (infinite dimensional) algebraic variety. This variety has a natural left $G(\mathcal{O}_x)$ -action. Moreover, we will show that the orbit space $G(\mathcal{O}_x) \backslash \mathcal{M}$ has a reasonable algebraic structure (though not of a 'variety' but rather of a 'stack', see e.g. [LS]). The well-known 'double-coset construction' provides the following natural isomorphism of stacks, see [LS] or theorem 6.3.1 and proposition 6.3.8 of chapter 6:

$$G(\mathcal{O}_x) \backslash \mathcal{M} \simeq G(\mathcal{O}_x) \backslash G(\mathbb{K}_x) / G(\mathcal{O}_{\text{out}}) \simeq \text{Bun}_G \quad (1.5.0)$$

where Bun_G is the modular stack of the (isomorphism classes of) algebraic principal G -bundles on X .

To any perverse sheaf $A \in P(Gr_x)$ we associate a functor $D^b(\text{Bun}_G) \rightarrow D^b(\text{Bun}_G)$, called the *local Hecke functor* at the point x . To that end, we identify (or rather *define*, cf. §6.1.4) the derived category, $D^b(G(\mathcal{O}_\S) \backslash \mathcal{M})$, with $D^b_{G(\mathcal{O}_x)}(\mathcal{M})$, the $G(\mathcal{O}_x)$ -equivariant derived category on \mathcal{M} , see [BL] or chapter 8 below for the definition of an equivariant derived category. Mimicing the construction of sheaf-theoretic convolution given in section 3.2, one introduces a convolution pairing

$$* : P(Gr_x) \times D^b_{G(\mathcal{O}_x)}(\mathcal{M}) \rightarrow D^b_{G(\mathcal{O}_x)}(\mathcal{M}).$$

We now use double-coset isomorphism (1.5.2) and lemma 1.5.1, to reinterpret this convolution as a bi-functor

$$* : P(Gr) \times D^b(\text{Bun}_G) \rightarrow D^b(\text{Bun}_G) \quad (1.5.0)$$

Given $M \in P(Gr)$, we define the corresponding local (at x) Hecke functor $D^b(\text{Bun}_G) \rightarrow D^b(\text{Bun}_G)$ by the formula $A \mapsto M * A$.

A complex $A \in D^b(\text{Bun}_G)$ is called a *local Hecke eigen-sheaf* if, for each $M \in P(\text{Gr})$, there exists a finite dimensional vector space L_M and an isomorphism

$$M * A = L_M \otimes A. \quad (1.5.0)$$

Remark. If $G = \text{SL}_n$, then the orbit $O_\lambda \subset \text{Gr}$ associated to a fundamental weight λ is closed in Gr . Hence the corresponding simple object $IC_\lambda \in P(\text{Gr})$ reduces (up to shift) to the constant sheaf (this is false for G simple, $G \neq \text{SL}_n$). Furthermore, we will see later that if equation (1.5.4) holds for $M = IC_\lambda$, for each fundamental weight λ , then it holds for all $M \in P(\text{Gr})$. Thus, if $G = \text{SL}_n$, we have only to check $(n-1)$ equations out of the infinite family of equations (1.5.4) and these involve no intersection cohomology whatsoever (in agreement with the classical theory). \square

The constructions that we have made so far depended on the choice of an arbitrary point $x \in X$. One can modify the construction slightly by letting the point x to vary within the curve X . In the special case of $G = \text{SL}_n$ and of the complexes IC_λ where λ is a fundamental weight, this was done by Laumon [La1] (cf. remark above). In the general case, one considers the stack of quadruples:

$$\text{Heck}_X = \{(P_1, P_2, x, \phi) \mid P_1, P_2 \in \text{Bun}_G, x \in X, \phi : P_1|_{X \setminus \{x\}} \xrightarrow{\sim} P_2|_{X \setminus \{x\}}\}$$

where P_1 and P_2 are G -bundles on X and ϕ is an isomorphism of their restrictions to $X \setminus \{x\}$. The stack Heck_X is not algebraic. It has, however, a natural stratification by the algebraic substacks $\text{Heck}_{X,\lambda}$, $\lambda \in X_*(T)/W$, labeled by L^+G -orbits in the grassmannian Gr . For each stratum, the projection $pr_2 : \text{Heck}_{X,\lambda} \rightarrow \text{Bun}_G \times X$, $(P_1, P_2, x, \phi) \mapsto (P_2, x)$ is a smooth morphism with fiber isomorphic to O_λ , the corresponding L^+G -orbit in Gr . Thus, there is a natural functor $M \mapsto M_X$ assigning to an object $M \in P(\text{Gr})$ the corresponding perverse sheaf M_X on Heck_X .

Write $pr_1 : \text{Heck}_X \rightarrow \text{Bun}_G$ for the projection $(P_1, P_2, x, \phi) \mapsto P_1$. We define a convolution type functor

$$\star : D^b(\text{Heck}_X) \times D^b(\text{Bun}_G) \rightarrow D^b(X \times \text{Bun}_G), \quad N, A \mapsto N \star A = (pr_2)_*(N \otimes_{pr_1^!} A) \quad (1.5.0)$$

Setting in particular, $N = M_X$, we thus associate with any $M \in P(\text{Gr})$, the corresponding *global Hecke functor* given by the formula

$$T_M : D^b(\text{Bun}_G) \rightarrow D^b(X \times \text{Bun}_G), \quad A \mapsto T_M(A) = M_X \star A \quad (1.5.0)$$

We will now show, using Theorem 1.4.1, that to some Hecke eigen-sheaves on Bun_G one can associate naturally a homomorphism from, $\pi_1(X)$, the fundamental group of the curve, into the Langlands dual group G^\vee . This way one gets the geometric Langlands correspondence.

Given a principal G^\vee -bundle on X with flat connection, and a representation $V \in \text{Rep}_{G^\vee}$, write V_P for the associated vector bundle, i.e. a local system on X . On the other hand, let $\mathcal{P} : \text{Rep}_{G^\vee} \xrightarrow{\sim} P(\text{Gr})$ be the equivalence inverse to the equivalence of theorem 1.4.1. We write $\mathcal{P}(V)$ for the perverse sheaf on Gr arising, via the equivalence, from $V \in \text{Rep}_{G^\vee}$. Associated to $\mathcal{P}(V)$, is the corresponding *global* Hecke functor $T_{\mathcal{P}(V)} : D^b(\text{Bun}_G) \rightarrow D^b(X \times \text{Bun}_G)$.

Here is our main result on the geometric Langlands correspondence.

Theorem 1.5.7 *Let $A \in D^b(\text{Bun}_G)$ be a local Hecke eigen-sheaf at a certain point $x \in X$. Assume the following two properties hold:*

- (i). *A is an irreducible perverse sheaf of geometric origin (see [BBD]);*
- (ii). *The characteristic variety of A (see e.g. [KS] or [Gi1]) is contained in the global nilpotent cone $\text{Nilp} \subset T^*\text{Bun}_G$, defined by Laumon [La2].*

Then, A is a global Hecke eigen-sheaf. Moreover, there exists a unique (up to isomorphism) flat G^\vee -bundle $P = P(A)$ on X such that, for any $V \in \text{Rep}_{G^\vee}$, we have

$$T_{\mathcal{P}(V)}(A) = V_P \boxtimes A. \quad (1.5.0)$$

Proof of the theorem will be outlined in $n^\circ 6.6$.

The correspondence $A \mapsto P(A)$ established in the theorem may be thought of as the (one way) geometric Langlands correspondence. To explain this, fix a point $x \in X$. Recall that taking the monodromy representation in the fiber at x of a flat G^\vee -bundle P on X sets-up a bijection between the isomorphism classes of flat G^\vee -bundles on X , and the conjugacy classes of group homomorphisms $\phi : \pi_1(X, x) \rightarrow G^\vee$. Writing $\phi(A)$ for the homomorphism associated to the flat G^\vee -bundle $P(A)$, via the theorem, we get the geometric Langlands correspondence: *Hecke eigen-sheaves satisfying (i)-(ii) of 1.5.7 \leftrightarrow Homomorphisms $\pi_1(X, x) \rightarrow G^\vee$ up to conjugacy.*

Constructing a correspondence in the opposite direction turns out to be a much more complicated task. Let Λ be the set of isomorphism classes of pairs (P, χ) , where P is a flat G^\vee -bundle on X such that, $\text{Aut}(P)$, the group of automorphisms of P , is finite, and χ is an irreducible representation

of $\text{Aut}(P)$. The conjecture below is an extension to general semisimple groups of the geometric Langlands reciprocity conjecture due to Laumon [La1, conjecture 2.1.1] for $G = \text{GL}_n$. A proof of a \mathcal{D} -module analogue of this conjecture (at least of an essential part of (i) below) is going to appear in the series of papers of Beilinson and Drinfeld [BeDr].

Conjecture (i) *To any $(P, \chi) \in \Lambda$ one can associate a finite set, called an ‘L-packet’, consisting of $\dim \chi$ perverse sheaves A on Bun_G whose characteristic variety is contained in Nilp , and such that equation 1.5.8 holds.*

(ii) *If the flat bundle P carries a variation of mixed Hodge structure in the sense of Deligne, then the corresponding perverse sheaves on Bun_G have an additional structure of mixed Hodge modules in the sense of Saito [Sa].*

1.6 “Topological” gradation

By the last sentence of theorem 1.4.1, there is a canonical identification of the underlying vector space of an irreducible representation $V_\lambda \in \text{Rep}_{G^\vee}$ with the hyper-cohomology $H^\bullet(IC_\lambda)$. This raises some new questions:

Question 1. What is the representation-theoretic meaning (in terms of V_λ) of the natural gradation on $H^\bullet(IC_\lambda)$ by cohomology degree?

Question 2. What is the representation-theoretic meaning of the natural $H^\bullet(\text{Gr}, \mathbb{C})$ -action on $H^\bullet(IC_\lambda)$?

To answer these questions we fix a principal nilpotent n in \mathfrak{g}^\vee , the Lie algebra of the group G^\vee . By the Jacobson-Morozov theorem, we can (and will) choose a Lie algebra homomorphism:

$$j : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{g}^\vee \quad \text{such that} \quad j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = n \quad (1.6.0)$$

We set:

$$h = j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.6.0)$$

Clearly, h is a semisimple regular element of \mathfrak{g}^\vee that has integral eigenvalues in any finite-dimensional \mathfrak{g}^\vee -module. (Moreover, it can be assumed without loss of generality, that h is the sum of positive coroots in a Cartan subalgebra of \mathfrak{g}^\vee and the nilpotent n is the sum of root vectors corresponding to all simple roots.)

The answer to Question 1 above is given by the following

Theorem 1.6.3 *The natural gradation on $H^\bullet(IC_\lambda)$ corresponds, via theorem 1.4.1, to the gradation on V_λ by the eigenvalues of h , i.e. to the gradation:*

$$V = \bigoplus_{i \in \mathbb{Z}} V^h(i) \quad \text{where} \quad V^h(i) = \{v \in V \mid h \cdot v = i \cdot v\} \quad (1.6.0)$$

Remark 1.6.4 *Theorem 1.6.3 looks ambiguous as stated, for gradation (1.6.4) clearly depends on the choice of h , that cannot be made canonical. To resolve this “contradiction”, observe that the correspondence between the two gradations in Theorem 1.6.3 depends on the equivalence of Theorem 1.4.1. Such an equivalence is not unique and can only be fixed up to automorphism of the category Rep_{G^\vee} . Any group automorphism of G^\vee gives rise to an automorphism of Rep_{G^\vee} . Now, the element h in (1.6.2) is uniquely determined up to conjugacy. The effect of conjugating h by $y \in G^\vee$ amounts to applying the automorphism of Rep_{G^\vee} induced by the conjugation by y . That resolves the ‘contradiction’. Similar meaning is assigned to the term ‘correspond’ in many other results stated in this paper, the objects on the ‘topological’ side are usually defined in a canonical way while their representation theoretic counterparts are only determined up to simultaneous conjugation. It should be emphasized however that if more than one object are involved, then the ‘relative position’ of the objects may still make an ‘absolute’ sense.*

1.7 The $H^\bullet(\text{Gr}, \mathbb{C})$ -action

To answer Question 2, we have to describe the cohomology of the Grassmannian fist. Let $\text{Gr}_1 \hookrightarrow \text{Gr}_2 \hookrightarrow \dots$ be the exhaustion of Gr mentioned in n. 1.2, and $H^k(\text{Gr}_1) \leftarrow H^k(\text{Gr}_2) \leftarrow \dots$ the corresponding projective system of cohomology. For each $k \geq 0$ this system stabilizes, for all Gr_i are known to have compatible Bruhat cell decompositions so that Gr_{i+1} is obtained from Gr_i by attaching cells of dimensions $> k$ provided $i = i(k)$ is big enough. We set by definition $H^k(\text{Gr}) = \varprojlim_i H^k(\text{Gr}_i)$.

To compute $H^\bullet(\text{Gr})$ we identify the Grassmannian with the Loop group Ω viewed as a topological group (n. 1.2). Hence, $H^*(\Omega)$ is a graded-commutative and cocommutative Hopf algebra. By a well-known theorem $H^*(\Omega)$ is freely generated by the subspace *prim* of its primitive elements. These elements can be obtained [B] by transgressing primitive cohomology classes of the compact group K , i.e. by pulling them back to $S^1 \times \Omega$ via the evaluation

map

$$\text{ev} : S^1 \times \Omega \rightarrow K, \quad (1.7.0)$$

and then integrating over S^1 . We will give another description of the primitive subspace $\text{prim} \subset H^\bullet(\Omega)$ after the following remark.

Remark. Let O be the *regular* coadjoint orbit in $(\mathfrak{g}^\vee)^*$, i.e., an orbit of maximal dimension in the dual of the Lie algebra \mathfrak{g}^\vee . Let $G^\vee(x)$ denote the isotropy group of $x \in O$ and $\mathfrak{g}^\vee(x) := \text{Lie } G^\vee(x)$. Then $\mathfrak{g}^\vee(x)$ is an abelian Lie subalgebra of \mathfrak{g}^\vee whose dimension equals $\text{rk } G$. (The easiest way to prove these facts is to identify \mathfrak{g}^\vee with $(\mathfrak{g}^\vee)^*$ via an invariant bilinear form on \mathfrak{g}^\vee , and to view x as a regular element in \mathfrak{g}^\vee so that the algebra $\mathfrak{g}^\vee(x)$ becomes the centralizer of x in \mathfrak{g}^\vee . This centralizer can be clearly viewed as the limit of a sequence of Cartan subalgebras of \mathfrak{g}^\vee .) Let x' be another point of the orbit O and let $u \in G^\vee$ be any element such that $\text{Ad } u(x) = x'$. Then the operator $\text{Ad } u$ gives an isomorphism $\mathfrak{g}^\vee(x) \xrightarrow{\text{sim}} \mathfrak{g}^\vee(x')$. Moreover, this isomorphism does not depend on the choice of u , for u is determined up to an element of the group $G^\vee(x)$ and the latter, being commutative, acts trivially on $\mathfrak{g}^\vee(x)$. Thus, all the algebras $\mathfrak{g}^\vee(x)$, $x \in O$, can be canonically identified with each other.

Further, it is well-known, see [Ko2], that there is a canonical bijection between regular and semisimple coadjoint orbits in $(\mathfrak{g}^\vee)^*$, respectively. The latter are parametrized naturally by the orbit space \mathfrak{t}/W . Thus, associated with each point $t \in \mathfrak{t}/W$, is a regular orbit O as above, hence, a *canonically defined* abelian Lie algebra $\mathfrak{g}^\vee(x)$, $x \in O$. We let \mathfrak{a}_t denote this ‘‘universal’’ abelian Lie algebra, $\mathfrak{g}^\vee(x)$, associated to $t \in \mathfrak{t}/W$. The family $\{\mathfrak{a}_t, t \in \mathfrak{t}/W\}$ may be thought of as the family of fibers of a vector bundle on \mathfrak{t}/W . \square

For $t = 0$, i.e. in the case of a regular nilpotent x , we write \mathfrak{a} instead of \mathfrak{a}_t , for short. In particular, take $x = n$, cf. (1.6.4), and endow the vector space \mathfrak{a} with a grading induced by the grading on $\mathfrak{g}^\vee(x)$ by the eigenvalues of the adjoint h -action on $\mathfrak{g}^\vee(n)$.

Proposition 1.7.2 *There is a canonical graded space isomorphism: $\text{prim} = \mathfrak{a}$.*

We sketch a construction of the isomorphism: $\text{prim} \cong \mathfrak{a}$. Let $\mathbb{C}[\mathfrak{g}]^G$ and $\mathbb{C}[(\mathfrak{g}^\vee)^*]^{G^\vee}$ denote the graded algebras of invariant polynomials on \mathfrak{g} and $(\mathfrak{g}^\vee)^*$ respectively. These algebras are canonically isomorphic via the following chain of isomorphisms:

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[(\mathfrak{h}^\vee)^*]^W \xleftarrow{\sim} \mathbb{C}[(\mathfrak{g}^\vee)^*]^{G^\vee} \quad (1.7.0)$$

where the first and the third isomorphisms are due to Chevalley's restriction theorem. We shall now construct the following diagram

$$\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}]_+^G & \xlongequal{(1.7.3)} & \mathbb{C}[(\mathfrak{g}^\vee)^*]_+^{G^\vee} \\
\downarrow c & & \downarrow d \\
\text{prim} & \overset{\sim}{\dashrightarrow} & \mathfrak{a}
\end{array} \tag{1.7.0}$$

where $\mathbb{C}[\mathfrak{g}]_+^G$ stands for the augmentation ideal in $\mathbb{C}[\mathfrak{g}]^G$, etc.

To define the map c decompose the 2-dimensional sphere as the union of two disks D_+ and D_- with the common boundary S^1 . Take trivial principal K -bundles on $D_+ \times \Omega$ and $D_- \times \Omega$ and identify them over the boundary $S^1 \times \Omega$ by means of the map (1.7.1). This way we obtain a principal K -bundle on $S^2 \times \Omega$. Given an invariant polynomial $P \in \mathbb{C}[\mathfrak{g}]^G$, let $ch(P) \in H^\bullet(S^2 \times \Omega)$ be the corresponding characteristic class of our K -bundle, and let $c(P)$ be the integral of $ch(P)$ along the factor S^2 . One can show that $c(P)$ is a primitive class in $H^*(\Omega)$. The assignment $P \mapsto c(P)$ defines the map c in (1.7.4). It is clear from the construction that: $\deg c(P) = \deg ch(P) - 2 = 2 \cdot \deg P - 2$.

To define the map d in (1.7.4), notice first that the differential of any polynomial P^\vee on $(\mathfrak{g}^\vee)^*$ may be viewed as a \mathfrak{g}^\vee -valued function on $(\mathfrak{g}^\vee)^*$. If P^\vee is an invariant polynomial then, for any $x \in (\mathfrak{g}^\vee)^*$, we have: $dP^\vee(x) \in \mathfrak{g}^\vee(x)$. Now, take x to be a regular nilpotent and identify $\mathfrak{g}^\vee(x)$ with \mathfrak{a} . We define the map d in (1.7.4) by the formula $\mathbb{C}[(\mathfrak{g}^\vee)^*]^{G^\vee} \ni P^\vee \mapsto dP^\vee(x) \in \mathfrak{a}$.

Let $I \subset \mathbb{C}[\mathfrak{g}]^G$ and $I^\vee \subset \mathbb{C}[(\mathfrak{g}^\vee)^*]^{G^\vee}$ denote the augmentation ideals. Proposition 1.7.2 clearly follows from diagram (1.7.4) and the following result which will be proved in chapter 4.

Lemma 1.7.5 *The maps c and d in (1.7.4) both vanish on the squares of the augmentation ideals. The resulting maps:*

$$I/I \cdot I \xrightarrow{c} \text{prim} \quad \text{and} \quad I^\vee/I^\vee \cdot I^\vee \xrightarrow{d} \mathfrak{a}$$

are isomorphisms.

In the simply laced case a result similar to Proposition 1.7.2 has been also obtained by D. Peterson (unpublished).

Let $U(\mathfrak{a})$ denote the enveloping algebra of the (commutative) Lie algebra \mathfrak{a} , and let \tilde{u} denote the element of $U(\mathfrak{a})$ corresponding to a cohomology class $u \in H^\bullet(\Omega)$ via the isomorphism $H^\bullet(\Omega) \cong U(\mathfrak{a})$ induced by that of Proposition 1.7.2.

We are now ready to give an answer to Question 2 in n. 1.6.

Theorem 1.7.6 *For any $u \in H^\bullet(\text{Gr})$, the natural action of u on the hypercohomology of a perverse sheaf from the category $P(\text{Gr})$ corresponds (cf. Remark 1.6.4) to the natural action of $\tilde{u} \in \mathfrak{a}$ in the G^\vee -module that corresponds to the perverse sheaf via Theorem 1.4.1.*

1.8 Kostant theorem and the generalized exponents

The imbedding (1.6.1) makes \mathfrak{g}^\vee an $\mathfrak{sl}_2(\mathbb{C})$ -module with respect to the adjoint action. One can decompose \mathfrak{g}^\vee into irreducible $\mathfrak{sl}_2(\mathbb{C})$ -submodules \mathfrak{g}_i^\vee so that the corresponding highest weight vectors $\check{a}_i \in \check{\mathfrak{g}}_i^\vee$ form a base of the subalgebra \mathfrak{a} (in particular, the decomposition is known to have $r = \text{rk}(G^\vee)$ direct summands).

Furthermore, all the eigenvalues of the operator $\text{ad } h$ in \mathfrak{g}_i^\vee are *even* integers for otherwise we would have: $\dim \mathfrak{a} > \text{rk } G$. The structure theory of \mathfrak{sl}_2 -modules then yields an equality: $\dim \mathfrak{g}_i^\vee = \deg \check{a}_i + 1$. On the other hand, proposition 1.7.2 combined with theorem 1.7.6 implies: $\deg \check{a}_i = \deg \mathfrak{a}_i = 2 \cdot \deg P_i - 2$. Thus, we have proved the following numerical identity first discovered by Kostant [Ko1]:

$$\deg \mathfrak{g}_i^\vee = 2 \cdot \deg P_i - 2$$

Let $\mathfrak{t} = Z_{\mathfrak{g}^\vee}(h)$ be a Cartan subalgebra of \mathfrak{g}^\vee and T^\vee the corresponding Cartan subgroup. Let $C \subset X^*(T^\vee)$ be a coset with respect to the root lattice of G^\vee and $C^{++} = \{\mu \in C \mid \mu(h) \geq 0\}$, a dominant Weyl chamber. There is a unique weight $\mu_C \in C^{++}$ which is the minimal element in C^{++} , i.e. such that for any other $\lambda \in C^{++}$, $\lambda - \mu_C$ is a sum of positive roots. The weight μ_C is *minuscule*, that is all the weights of the irreducible representation V_{μ_C} form a single W -orbit. Any minuscule weight is known to be the minimal element in some coset C .

Let v_μ denote a lowest weight vector in the representation V_μ . The following result was suggested to me by B. Kostant.

Proposition 1.8.1 *The module V_{μ_C} is cyclically generated by the action of the algebra $\mathfrak{a} = Z_{\mathfrak{g}^\vee}(n)$, that is $V_{\mu_C} = U(\mathfrak{a}) \cdot v_{\mu_C}$.*

Proof: The orbit O_{μ_C} is the unique closed orbit in the connected component of the Grassmannian Gr that corresponds to the coset C (connected components are parameterized by elements of the finite group $X_*(T)/\text{root lattice}$). Hence, $\overline{O}_\mu = O_\mu$ is a smooth variety so that IC_{μ_C} is a constant sheaf on O_{μ_C} (up to shift). By Theorem 1.7.2, proving the Proposition amounts to showing that the restriction map $H^*(\text{Gr}) \rightarrow H^*(O_{\mu_C})$ is surjective. This is equivalent, by duality, to the injectivity of the map $H_*(O_{\mu_C}) \rightarrow H_*(\text{Gr})$. But the orbit O_{μ_C} has an even-dimensional cell decomposition which is part of a similar decomposition of Gr (Bruhat decomposition). The injectivity follows. \square

For any $V \in \text{Rep}_{G^\vee}$, let $V(\mu)$ denote the weight subspace corresponding to a weight $\mu \in X^*(T^\vee)$, and $V^\mathfrak{a}$ the subspace annihilated by the subalgebra \mathfrak{a} . In chapter 5 we will prove the following result.

Proposition 1.8.2 *Let $\lambda \in C^{++}$. Then, $\dim V_\lambda^\mathfrak{a} = \dim V_\lambda(\mu_C)$; moreover, there exists a basis v_1, \dots, v_m of the weight space $V_\lambda(\mu_C)$ and non-negative integers (called the generalized exponents): k_1, \dots, k_m such that the elements: $n^{k_1} \cdot v_1, \dots, n^{k_m} \cdot v_m$ form a base of $V_\lambda^\mathfrak{a}$.*

If C is the root lattice then $\mu_C = 0$. In that case the dimension equality of the first part of the Proposition is proved in [Ko2] and the second part is proved in [Br]. Proposition 1.8.2 is closely related to some ideas of R. K. Brylinski who stated it as a conjecture.

1.9 Integrals over Schubert cycles

We now describe (partially) the pairing between homology and cohomology of the Grassmannian. Namely, for any $u \in H^{2k}(\text{Gr})$ and any L^+G -orbit O_λ (see n. 1.4) such that $\dim_{\mathbb{C}} O_\lambda = k$, we shall give a representation-theoretic formula for $\langle u, [O_\lambda] \rangle$, the value of u on the fundamental cycle of the closure of O_λ . To that end, let v_λ denote a lowest weight vector in the irreducible representation V_λ and v^λ a lowest weight vector of the contragredient representation. Further, let \tilde{u} be the element of $U(\mathfrak{a})$ corresponding to the cohomology class u via the isomorphism of Corollary 1.7.3. The element \tilde{u} acts naturally in the representation V_λ , and we have

Proposition 1.9 $\langle u, [O_\lambda] \rangle = \langle v^\lambda, \tilde{u} \cdot v_\lambda \rangle$.

Proof: Let $S : V_\lambda \xrightarrow{\sim} H^\bullet(IC_\lambda)$ denote the canonical isomorphism provided by theorem 1.4.1, and V_λ^\pm the highest (resp. the lowest) weight subspaces

in V_λ . We'll see later that the isomorphism S sends V_λ^\pm into $H^{\pm k}(IC_\lambda)$, the top (resp. the lowest) intersection cohomology. We use the map S and the natural map from intersection cohomology to ordinary homology to obtain the following commutative digram:

$$\begin{array}{ccccc}
V_\lambda^+ & \xrightarrow{\tilde{S}} & H^k(IC_\lambda) & \xrightarrow{\sim} & H_0(\overline{O}_\lambda) \\
\uparrow \tilde{u} & & \uparrow u \cdot & & \uparrow u \cap \\
V_\lambda^- & \xrightarrow{\tilde{S}} & H^{-k}(IC_\lambda) & \xrightarrow{\sim} & H_{2k}(\overline{O}_\lambda)
\end{array}$$

The choice of linear function $v^\lambda : V_\lambda^+ \rightarrow \mathbb{C}$ corresponds via the above isomorphism, to a map $\varphi : H_0(\overline{O}_\lambda) \rightarrow \mathbb{C}$. Hence, the commutative diagram yields:

$$\langle u, [\overline{O}_\lambda] \rangle = \varphi(u \cap [\overline{O}_\lambda]) = \langle v^\lambda, \tilde{u} \cdot v_\lambda \rangle$$

□

1.10 Computation of Ext-groups

We are interested in the groups $\text{Ext}^\bullet(IC_\lambda, IC_\mu)$, computed in $D^b(\text{Gr})$ (but not in the category $P(\text{Gr})$ where Ext's are trivial). To find these groups observe first that, for any $M, N \in D^b(\text{Gr})$, one has, by functoriality, a natural morphism:

$$\text{Ext}^i(M, N) \rightarrow \text{Hom}_{H^\bullet(\text{Gr})}^i(H^\bullet(M), H^\bullet(N)) \quad (1.10.0)$$

where Hom's on the right are taken in the category of graded $H^\bullet(\text{Gr})$ -modules and Hom^i denotes the space of homomorphisms shifting gradation by i .

Now, let $M = IC_\lambda$ and $N = IC_\mu$, $\lambda, \mu \in X_*(T)$. In that case the right-hand side of (1.10.1) turns, by theorem 1.7.2, into $\text{Hom}_{\mathfrak{a}}^i(V_\lambda, V_\mu)$, so that we obtain a map:

$$\text{Ext}^\bullet(IC_\lambda, IC_\mu) \rightarrow \text{Hom}_{\mathfrak{a}}^\bullet(V_\lambda, V_\mu) \quad (1.10.0)$$

Theorem 1.10.3 *If the strata O_λ and O_μ are contained in the same connected component of the Grassmannian, then the morphism (1.10.2) is an isomorphism.*

This theorem follows from the main result of [Gi3] (see Remark at the end of [Gi3]).

We shall now give a reformulation of Theorem 1.10.3 in slightly more invariant terms, not involving a choice of principal nilpotent.

Let \mathcal{N} be the cone of nilpotent elements of the Lie algebra \mathfrak{g}^\vee , and $\mathbb{C}[\mathcal{N}]$ the algebra of regular functions on \mathcal{N} . There is a natural action on \mathcal{N} of the group $\mathbb{C}^* \times G^\vee$ (the factor \mathbb{C}^* acts by multiplication and G^\vee by conjugation). Let $\mathcal{Coh}(\mathcal{N})$ denote the abelian category of $\mathbb{C}^* \times G^\vee$ -equivariant coherent $\mathcal{O}_{\mathcal{N}}$ -sheaves. Any such sheaf is characterized by the $\mathbb{C}[\mathcal{N}]$ -module of its global sections, for \mathcal{N} is an affine variety.

To each representation $V \in \text{Rep}_{G^\vee}$ we attach the free $\mathcal{O}_{\mathcal{N}}$ -sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{N}}$ whose global sections form the free $\mathbb{C}[\mathcal{N}]$ -module of V -valued regular functions on \mathcal{N} . The \mathbb{C}^* -action on $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{N}}$ is induced from that on \mathcal{N} , and the G^\vee -action on $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{N}}$ arises from the simultaneous action of G^\vee both on V and on \mathcal{N} . In particular, for $\lambda \in X_*(T)$ we set $\mathcal{V}_\lambda = V_\lambda \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{N}} \in \mathcal{Coh}(\mathcal{N})$.

Theorem 1.10.3 turns out to be equivalent to the following

Proposition 1.10.4 *For any strata O_λ and O_μ contained in the same connected component of Gr one has a canonical isomorphism:*

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(IC_\lambda, IC_\mu) \cong \text{Hom}_{\mathcal{Coh}(\mathcal{N})}^\bullet(\mathcal{V}_\lambda, \mathcal{V}_\mu).$$

Recently, Kashiwara-Tanisaki [KT] proved a version of the Kazhdan-Lusztig conjecture for affine Lie algebras. The complexes IC_λ correspond, by that conjecture, to irreducible highest weight representations L_λ (of negative level) of the Lie algebra $\hat{\mathfrak{g}}$ (= central extension of the algebra $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$). Correspondingly, the left-hand side of (1.10.2) turns out to be isomorphic to $\text{Ext}^\bullet(L_\lambda, L_\mu)$, the Ext-group in an appropriate category of $\hat{\mathfrak{g}}$ -modules. Thus, Theorem 1.10.3 gives an expression for $\text{Ext}^\bullet(L_\lambda, L_\mu)$ in terms of finite-dimensional representations of the dual Lie algebra. This is quite surprising, for there is no apparent connection between representations of $\hat{\mathfrak{g}}$ and those of \mathfrak{g}^\vee . To make a hint on what an explanation of such a connection might be, notice that the sheaves $\mathcal{V}_\lambda, \mathcal{V}_\mu$ appearing in Proposition 1.10.4 are *projective* objects of the category $\mathcal{Coh}(\mathcal{N})$. So, the Proposition manifests some instances of the Koszul duality (see [BGS]). Moreover, it is likely that Proposition 1.10.4 is a special case of an extension to affine Lie algebras [BGSo] of the Koszul duality conjecture for semisimple Lie algebras. The latter one has been proved by W. Soergel [So]. Our Theorem 1.10.3 is similar in spirit to Soergel's result (cf. also [BGSo]).

A result somewhat related to Proposition 1.10.4 was in effect obtained much earlier in [FP] in the context of modular representations of a semisimple group G^\vee in positive characteristic. The corresponding analogue for quantum groups at roots of unity was later proved in [GK]. In the last section of [GK] we proposed a conjectural interpretation of the intersection cohomology $H^*(IC_\lambda)$ in terms of quantum groups. The latter gives way to translating theorem 1.10.3 into a purely algebraic claim concerning simple modules over the finite-dimensional quantum algebra introduced by Lusztig. Finding an algebraic proof of that claim, independent of the intersection cohomology methods, presents a very interesting problem.

We would like to close this discussion with a conjecture that would generalize theorems 1.3.1 and 1.4.1. Fix an Iwahori (= affine Borel) subgroup $I \subset L^+G$ and write $P_I(Gr)$ for the category of I -equivariant perverse sheaves on Gr with compact support. This is a non-semisimple abelian category containing $P(Gr)$ as a subcategory. Furthermore, category $P_I(Gr)$ is known, through the combination of the works of Kazhdan-Lusztig [KL2] and Kashiwara-Tanisaki [KT], to be equivalent to a (regular block of the) category of finite dimensional representations of a quantum group at a root of unity. The following conjecture may be approached, we believe, using combinatorial results of [Lu 5] (I am grateful to Lusztig for pointing out this reference).

Conjecture (i) *For any $M \in P_I(Gr)$ and $A \in P(Gr)$ we have $M * A \in P_I(Gr)$, so that we get a bi-functor $* : P_I(Gr) \times P(Gr) \rightarrow P_I(Gr)$. Moreover,*

(ii) *This bi-functor corresponds, via the Kazhdan-Lusztig-Kashiwara-Tanisaki equivalence, to the standard tensor product of representations of the quantum group.*

(iii) *In particular, for any fixed $A \in P(Gr)$, the functor $M \rightsquigarrow M * A$ on the (non-semisimple) category $P_I(Gr)$ is exact.*

Observe that it is essential in the conjecture to convolve with an object $A \in P(Gr)$ "on the right", i.e to take $M * A$ and not $A * M$.

1.11

The action of the principal nilpotent n in a representation $V \in \text{Rep}_{G^\vee}$ yields the filtration on V by the kernels of powers of n . We let $V_i(\mu) := V(\mu) \cap \ker(n^{i+1})$, $i = 0, 1, 2, \dots$, denote the induced filtration on a weight

subspace $V(\mu)$ (notation of 1.8). Write the Poincaré polynomial:

$$P_\mu(V, q) = \sum_{i \geq 0} q^{2i} \cdot \dim(V_i(\mu)/V_{i-1}(\mu)) \quad (1.11.0)$$

On the other hand, for any pair of dominant weights λ and μ , Lusztig considered the affine Kazhdan-Lusztig polynomial $P_{\mu, \lambda}(q)$ and proved (see [Lu]) the weight multiplicity formula: $\dim V_\lambda(\mu) = P_{\mu, \lambda}(1)$. We'll show in section 5 that the Localization theorem for equivariant cohomology yields the following q -analogue of Lusztig's formula, involving polynomials (1.11.1).

Theorem 1.11.2 *For any dominant λ and μ we have:*

$$P_\mu(V_\lambda, q) = q^{\lambda(h) - \mu(h)} \cdot P_{\mu, \lambda}(q^2).$$

A similar result was proved (under certain restrictions) in [Br] by totally different means.

2 Tensor category of Perverse Sheaves

This section is entirely devoted to the proof of Theorem 1.3.

2.1

There is a natural (1-1)-correspondence between orbits of the L^+G -action on $\text{Gr} = LG/L^+G$ and orbits of the diagonal LG -action on $\text{Gr} \times \text{Gr}$. If these latter orbits were finite-dimensional we would have been able to define the Intersection cohomology complexes of their closures. The abelian category of perverse sheaves on $\text{Gr} \times \text{Gr}$ isomorphic to direct sums of such complexes would have been clearly equivalent to the category $P(\text{Gr})$ and would have had a natural convolution structure defined by the formula (cf. [Spr]):

$$M * N = (p_{13})_*(p_{12}^*M \otimes p_{23}^*N)$$

where $p_{ij} : \text{Gr} \times \text{Gr} \times \text{Gr} \rightarrow \text{Gr} \times \text{Gr}$ denotes the projection along the factor not named in the subscript.

Unfortunately, all LG -orbits in $\text{Gr} \times \text{Gr}$ have infinite dimension. That is why we were forced in n. 1.2 to use the loop group Ω in order to define the convolution $*$. We will now give a purely algebraic construction of the convolution. Another, more canonical, algebraic construction based on equivariant technique will be given in the next chapter.

Let O_μ^2 denote the LG -orbit in $\text{Gr} \times \text{Gr}$ corresponding to an L^+G -orbit $O_\mu \subset \text{Gr}$, $\mu \in X_*(T)$, and let $p_{1,2} : \text{Gr} \times \text{Gr} \rightarrow \text{Gr}$ denote the first and the second projections. Given $\lambda, \mu \in X_*(T)$, set $O_{\lambda,\mu} = p_1^{-1}(O_\lambda) \cap O_\mu^2$. Clearly, $O_{\lambda,\mu}$ is a finite-dimensional subvariety of $\text{Gr} \times \text{Gr}$. Furthermore, the first projection makes $O_{\lambda,\mu}$ a locally-trivial fibration over O_λ with fibre O_μ . Let $IC_{\lambda,\mu}$ denote the Intersection cohomology complex on $\overline{O}_{\lambda,\mu}$, the closure of $O_{\lambda,\mu}$.

Proposition 2.1.1 *There is natural isomorphism: $IC_\lambda * IC_\mu \cong (p_2)_* IC_{\lambda,\mu}$.*

Proof: Identify Gr with the Loop group Ω and define an automorphism f of variety $\Omega \times \Omega$ by the formula $f : (x_1, x_2) \mapsto (x_1, x_1^{-1} \cdot x_2)$, $x_1, x_2 \in \Omega$. The map f fits into the commutative triangle:

$$\begin{array}{ccc} \Omega \times \Omega & \xrightarrow[\sim]{f} & \Omega \times \Omega \\ & \searrow p_2 & \swarrow m \\ & \Omega & \end{array}$$

The triangle yields: $IC_\lambda * IC_\mu = m_*(IC_\lambda \times IC_\mu) = (p_2)_* \circ f^*(IC_\lambda \times IC_\mu)$. Hence, proving the Proposition amounts to showing that $f^*(IC_\lambda \times IC_\mu) \cong IC_{\lambda,\mu}$, i.e. that $\overline{O}_{\lambda,\mu} = f^{-1}(\overline{O}_\lambda \times \overline{O}_\mu)$. It suffices to check that $O_{\lambda,\mu} = f^{-1}(O_\lambda \times O_\mu)$.

To prove the last equality we introduce the map $q : \Omega \times \Omega \rightarrow \Omega$ defined by $q(x_1, x_2) := x_1^{-1} \cdot x_2$. We claim that $O_\mu^2 = q^{-1}(O_\mu)$. Indeed, the sets O_μ^2 and $q^{-1}(O_\mu)$ are, clearly, stable under the diagonal Ω -action on $\Omega \times \Omega$. Hence, each of these sets is completely determined by its intersection with $\Omega \times \{1\}$. But we have:

$$O_\mu^2 \cap (\Omega \times \{1\}) = O_\mu \times \{1\} = q^{-1}(O_\mu) \cap (\Omega \times \{1\}),$$

and the claim follows. Thus, one finds:

$$O_{\lambda,\mu} = p_1^{-1}(O_\lambda) \cap O_\mu^2 = p_1^{-1}(O_\lambda) \cap q^{-1}(O_\mu) = f^{-1}(O_\lambda \times O_\mu).$$

This completes the proof of the Proposition. □

Remark. We have shown in the course of the proof of Proposition 2.1.1 that the product $m_*(O_\lambda \times O_\mu)$ of any two orbits O_λ and O_μ is a finite union

of orbits. It is clear, on the other hand, that each of the sets Gr_i forming an exhaustion of Gr (see n. 1.2) is a finite union of orbits. Hence, for any i, j , the set $m_*(\text{Gr}_i \times \text{Gr}_j)$ is contained in a big enough Gr_k , as claimed in n. 1.2.

2.2

The following result implies the first part of Theorem 1.3. Its proof is based in a very essential way on a result of Lusztig [Lu]. The geometric construction of the Hecke algebra used in the proof of Proposition 2.2.1 below is nowadays well known, cf. [Spr]. It was independently discovered in the course of the proof of the Kazhdan-Lusztig conjecture by a number of people (including Beilinson-Bernstein, MacPherson, Lusztig and others).

Proposition 2.2.1 *For any $\lambda, \mu \in \mathbf{X}_*(T)$, the complex $IC_\lambda * IC_\mu$ is isomorphic to a finite direct sum of the complexes IC_ν , $\nu \in \mathbf{X}_*(T)$.*

Proof: By Proposition 2.1.1 we have: $IC_\lambda * IC_\mu = (p_2)_* IC_{\lambda, \mu}$. The latter complex is isomorphic to a finite direct sum of shifted Intersection cohomology complexes, by the Decomposition theorem [BBD]. Note further, that the variety $O_{\lambda, \mu}$ is stable under the diagonal action of the group L^+G . So, the complex $(p_2)_* IC_{\lambda, \mu}$, and hence all of its direct summands, are locally-constant along L^+G -orbits in Gr . Hence, these direct summands are of the form $IC_\nu[n_\nu]$ for some $\nu \in \mathbf{X}_*(T)$, where $[n_\nu]$ denotes the "shift" in the derived category. Thus, we obtain:

$$IC_\lambda * IC_\mu = \sum_{\nu} IC_\nu[n_\nu], \quad \nu \in \mathbf{X}_*(T) \quad (2.2.0)$$

Recall next that associated to the semisimple group G is the *affine Hecke algebra* H , cf. [Spr]. This is a $\mathbb{Z}[t, t^{-1}]$ -algebra which has a base formed by certain distinguished elements c_w , $w \in W_a$ (= affine Weyl group), the so-called Kazhdan-Lusztig basis. The algebra H is known to have a geometric realization as the Grothendieck group of a semisimple category whose objects are direct sums of certain (shifted) Intersection cohomology complexes $I_w[n]$, $w \in W_a$, $n \in \mathbb{Z}$, on the Flag variety associated to an affine Lie algebra. The complex $I_w[n]$ corresponds to the element $t^n \cdot c_w \in H$. The ring structure on H corresponds to a convolution of complexes.

There is a natural inclusion $\lambda \mapsto w(\lambda)$ of the set $\mathbf{X}_*(T)/W$ into W_a . Furthermore, the assignment $IC_\lambda \mapsto I_{w(\lambda)}$ is compatible with convolutions so that one has the formula:

$$I_{w(\lambda)} * I_{w(\mu)} = \sum_{\nu} I_{w(\nu)}[n_\nu], \quad (2.2.0)$$

corresponding to (2.2.2) term by term.

Now, Lusztig has proved [Lu, Corollary 8.7] that for any $\lambda, \mu \in \mathbf{X}_*(T)$ one has an equality in H :

$$c_{w(\lambda)} \cdot c_{w(\mu)} = \sum_{\nu} m_{\lambda, \mu}^{\nu} \cdot c_{w(\nu)}, \quad \nu \in \mathbf{X}_*(T) \quad (2.2.0)$$

where $m_{\lambda, \mu}^{\nu}$ are some non-negative integers (they are equal to the multiplicity of the irreducible representation V_ν in $V_\lambda \otimes V_\mu$). What is important for us in (2.2.4) is that the coefficients $m_{\lambda, \mu}^{\nu}$ are independent of t . This implies that there are no "shifts" in the right-hand side of (2.2.3), i.e. $n_\nu = 0$ for all ν . Hence, the same is true for the right-hand side of (2.2.2) and the Proposition follows. \square

2.3

We will show now that the convolution on $P(Gr)$ is commutative. More precisely, one has

Proposition 2.3.1 *There is a natural functor isomorphism:*

$$A * B \cong B * A, \quad A, B \in \text{Ob } P(Gr).$$

Proof of the Proposition is essentially standard. Let θ be a Cartan anti-involution on G such that $\theta(K) = K$ where K is a maximal compact subgroup of G (choosing θ amounts to choosing a maximal torus $T \subset G$ such that $T \cap K$ is a maximal torus in K). The induced (pointwise) anti-involution on LG will be also denoted by θ . By definition, the anti-involution θ preserves the subgroup Ω and acts identically on the set $\mathbf{X}_*(T)$, viewed as a subset of LG . It is clear also that $\theta(L^+G) = L^+G$. Hence, for any $\lambda \in \mathbf{X}_*(T)$, the map θ preserves the set

$$(L^+G \cdot \lambda \cdot L^+G) \cap \Omega.$$

But this set may be identified with the stratum O_λ in Gr . So, we have an isomorphism $\theta^*IC_\lambda \cong IC_\lambda$ for any $\lambda \in \mathbf{X}_*(T)$ and, hence, an isomorphism $\theta^*A \cong A$ for any $A \in P(\text{Gr})$.

The isomorphism $\theta^*A \cong A$ above is not canonical, in general. To choose a canonical one we argue as follows. For each $\lambda \in \mathbf{X}_*(T)$, viewed as a point in Gr , let $i_\lambda : \{\lambda\} \hookrightarrow \text{Gr}$ denote the inclusion. Since λ is fixed by θ , for any complex $A \in P(\text{Gr})$, there is a *canonical* isomorphism $I_\lambda : i_\lambda^*A \xrightarrow{\sim} i_\lambda^*(\theta^*A)$. We now define a *canonical isomorphism* $I : \theta^*A \xrightarrow{\sim} A$ to be the unique isomorphism such that for any $\lambda \in \mathbf{X}_*(T)$ the composite:

$$i_\lambda^*A \xrightarrow{I_\lambda} i_\lambda^*(\theta^*A) \xrightarrow{I} i_\lambda^*A$$

is the identity morphism on i_λ^*A .

Finally, we define a functor isomorphism $A * B \cong B * A$ to be the composite of isomorphisms:

$$A * B \xrightarrow{I^{-1}} (\theta^*A) * (\theta^*B) \xrightarrow{\sim} \theta^*(B * A) \xrightarrow{I} B * A,$$

where the isomorphism in the middle arises from the fact that θ is an anti-involution on the group Ω . \square

It is clear that the complex IC_e supported on e , the unit of the group Ω , is the unit in the category $P(\text{Gr})$, i.e., for any $M \in P(\text{Gr})$, there are functorial isomorphisms: $M * IC_e \cong M \cong IC_e * M$. Furthermore, we will show in the next chapter that the associativity constraint and the above defined commutativity constraint satisfy the hexagon axiom [DM, (1.0.2)]. That will prove that $(P(\text{Gr}), *)$ is a semisimple abelian tensor category.

2.4

Let σ be an involutive automorphism of the Grassmannian arising from the involution: $x \mapsto x^{-1}$ on the group Ω . Let D denote the Verdier duality functor on $D^b(\text{Gr})$. Define the *transposition* functor on $D^b(\text{Gr})$ by $A^t := \sigma^*(DA)$, $A \in D^b(\text{Gr})$.

For any $M, N, L \in D^b(\text{Gr})$ one has natural isomorphisms:

$$\begin{aligned} \text{(i)} \quad & (M^t)^t \cong M; \\ \text{(ii)} \quad & (M * N)^t \cong N^t * M^t; \\ \text{(iii)} \quad & \text{RHom}(M * N, L) \cong \text{RHom}(M, L * N^t). \end{aligned} \tag{2.4.0}$$

Extending σ to the anti-involution $x \mapsto x^{-1}$ on the group LG , we observe that:

$$\sigma(\Omega \cap (L^+G \cdot \lambda \cdot L^+G)) = \Omega \cap (L^+G \cdot \lambda^{-1} \cdot L^+G)$$

for any $\lambda \in \mathbf{X}_*(T)$. It follows that one has an isomorphism: $(IC_\lambda)^t \cong IC_{\lambda^{-1}}$. Hence, the category $P(Gr)$ is stable under the transposition $M \mapsto M^t$.

Further, given $M, N \in P(Gr)$ define an object $\mathcal{H}om(M, N) \in P(Gr)$ by the formula: $\mathcal{H}om(M, N) := M^t * N$.

Proposition 2.4.2 *For any $M, N, L, T \in P(Gr)$ we have:*

- (i) $M^t \cong \mathcal{H}om(M, IC_e)$;
- (ii) *The functor: $T \mapsto \text{Hom}(T * N, L)$ is represented by the object $\mathcal{H}om(N, L)$, that is:*

$$\text{Hom}_{P(Gr)}(T * N, L) \cong \text{Hom}_{P(Gr)}(T, \mathcal{H}om(N, L)).$$

Proof: (i) follows from definitions. To prove (ii) note that for any objects $X, Y \in P(Gr)$ we have: $\text{Hom}_{P(Gr)}(X, Y) \cong \text{Hom}_{\mathcal{D}^b(Gr)}(X, Y)$. Hence, the isomorphism (2.4.1 (iii)) yields: $\text{Hom}_{P(Gr)}(T * N, L) \cong \text{Hom}_{P(Gr)}(T, L * N^t)$. But $L * N^t \cong N^t * L$, by the commutativity of the convolution. Thus, $L * N^t = \mathcal{H}om(N, L)$ and statement (ii) follows. \square

Lemma 2.4.3 *There is natural functor isomorphism:*

$$\mathcal{H}om(M_1, N_1) * \mathcal{H}om(M_2, N_2) \cong \mathcal{H}om(M_1 * M_2, N_1 * N_2)$$

for $M_i, N_i \in P(Gr)$. \square

Proposition 2.4.2, Lemma 2.4.3 and the results of n. 3.7 below show that $P(Gr)$ is an abelian rigid tensor category.

2.5

Let us make a general remark. Let $f : X \rightarrow Y$ be a morphism of topological spaces and J a constructible complex on X . Then, one has a natural isomorphism of hyper-cohomology:

$$H^\bullet(Y, f_*J) \cong H^\bullet(X, J). \quad (2.5.0)$$

This isomorphism is obtained as the composite of the following isomorphisms:

$$H^\bullet(Y, f_*J) \cong R^\bullet p_*(f_*J) \cong R^\bullet(p \circ f)_*J \cong H^\bullet(X, J)$$

where p denotes the constant map: $Y \rightarrow \text{pt.}$

We shall now prove the third claim of theorem 1.3.1.

The exactness of the functor H is obvious since $P(Gr)$ is a semisimple category. Next, let $M, N \in P_\bullet(G)$. We have:

$$\begin{aligned} H^*(M * N) &\cong \text{(definition of the convolution)} \\ H^*(\Omega, m_*(M \boxtimes N)) &\cong \text{((2.5.1) applied to } J = M \boxtimes N) \\ H^*(\Omega \times \Omega, M \boxtimes N) &\cong \text{(Kunneth formula)} \\ H^*(\Omega, M) \otimes H^*(\Omega, N). &\quad \square \end{aligned}$$

3 Application of the equivariant cohomology.

Throughout this chapter the reader is assumed to be familiar with definitions and results on equivariant hyper-cohomology, proved in [BL] (see also [Lu 3], [Lu 4]) and summarized in chapter 8 below.

3.1 Convolution construction

Let M be a Lie group, $L \subset M$ a closed subgroup and X an M -variety. Let $D_L^b(X)$ be the L -equivariant derived category on X , cf. [BL], and write $D_L^b(M/L)_c$ for the L -equivariant derived category of complexes on M/L with compactly supported cohomology sheaves. In this setup, we are going to define a convolution pairing

$$* : D_L^b(M/L)_c \times D_L^b(X) \rightarrow D_L^b(X) \quad (3.1.0)$$

The construction of this pairing is essentially due to Lusztig, who considered the special case $X = M/L$.

First, define a ‘right-left’ L -action on $M \times X$ by the formula: $l : (m, x) \mapsto (m \cdot l^{-1}, l \cdot x)$. The right-left action is clearly a free left action, and we write $M \times_L X$ for the corresponding orbit space. Observe that M -action-map $M \times X \rightarrow X$ descends to a well-defined map $a : M \times_L X \rightarrow X$.

Second, let $p : M \rightarrow M/L$ be the projection. Then the pull-back functor p^* gives, by equivariant descent property, cf. §8.2, an equivalence between $D_L^b(M/L)$ and the equivariant derived category on M with respect

to the L -action by right translation. In particular, we get an equivalence of $D_L^b(M/L)_c$ with the equivariant derived category of complexes on M with an additional support condition. Observe also that, for a subset $S \subset M/L$, the restriction map

$$a : p^{-1}(S) \times_L X \rightarrow X \quad \text{is proper whenever } S \text{ is compact.} \quad (3.1.0)$$

Now, let $\mathcal{M} \in D_L^b(M/L)_c$ and $\mathcal{A} \in D_L^b(X)$. By the second remark above, we view $p^*\mathcal{M}$ as an L -equivariant complex on M with respect to right translation. Then, we may regard $(p^*\mathcal{M}) \boxtimes \mathcal{A}$ as an element of $D_{L_{rl}}^b(M \times X)$, the equivariant derived category on $M \times X$ with respect to the right-left L -action. The latter is equivalent, by the equivariant descent, to $D^b(M \times_L X)$. Thus, there is a uniquely determined complex $\overline{\mathcal{A} \boxtimes \mathcal{M}}$ on $M \times_L X$ such that its pull-back to $M \times X$ is isomorphic to $(p^*\mathcal{M}) \boxtimes \mathcal{A}$. The support condition on $\mathcal{M} \in D_L^b(M/L)_c$ combined with 3.1.2 ensure that the restriction of the map a to the support of any cohomology sheaf $\mathcal{H}^\bullet(\overline{\mathcal{A} \boxtimes \mathcal{M}})$ is proper. We put

$$\mathcal{M} * \mathcal{A} = a_* (\overline{\mathcal{A} \boxtimes \mathcal{M}}) = a! (\overline{\mathcal{A} \boxtimes \mathcal{M}}) \quad (3.1.0)$$

It is easy to verify that $\mathcal{M} * \mathcal{A} \in D_L^b(X)$, and that one has functorial isomorphisms

$$(\mathcal{M}_1 * \mathcal{M}_2) * \mathcal{A} = \mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{A}) \quad , \quad \forall \mathcal{M}_1, \mathcal{M}_2 \in D_L^b(M/L)_c, \mathcal{A} \in D_L^b(X)$$

In the next subsection we will adapt the above construction to the infinite dimensional set-up: $M = LG$, $L = L^+G$ and $X = LG/L^+G$ to get an algebraic definition of the convolution on the category $P(Gr)$. With similar modifications, the above construction may be applied, using the formalism of §8.2, in the case: $M = LG$, $L = L^+G$ and $X = G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}})$ to give a rigorous definition of the Hecke operators exploited in $n^\circ 1.5$. The condition allowing reduction to finite dimensions in either of those cases is essentially the following: there exists a normal subgroup $L' \subset L$ such that : (i) *The quotient L/L' is a finite-dimensional Lie group;* (ii) *The group L' acts trivially on X .*

3.2

Recall the subgroups $L^i \subset L^+G$, $i = 0, 1, \dots$ introduced in (1.2.1^m). It follows from proposition 1.2.2(iii) that, for each Gr_i , the L^+G -action on Gr_i factors through the action of a finite-dimensional algebraic group $L_k := L^+G/L^k$, $k = k(i)$. Thus, we may speak of L^+G -equivariant sheaves on

Gr_i , meaning the sheaves that are equivariant with respect to the group L_k where k is chosen so that the L^k -action on Gr_i is trivial. That does not depend on a particular choice of k .

We can now construct the convolution $*$ on $P(\text{Gr})$ using the approach of the previous n° as follows Let $\mathcal{M}, \mathcal{A} \in P(\text{Gr})$. Choose positive integers i and j so that the complexes \mathcal{M} and \mathcal{A} are supported on Gr_i and Gr_j respectively. Let LG_i denote the inverse image of Gr_i under the projection $LG \rightarrow LG/L^+G = \text{Gr}$. Clearly $L^+G \subset LG_i$. Moreover, by n. 1.2, there exists an integer $m = m(i, j) \gg 0$ such that $LG_i \cdot \text{Gr}_j \subset \text{Gr}_m$; furthermore, there exists another integer, $k = k(i, j) \gg 0$, such that the group L^k acts trivially on the subsets Gr_j and Gr_m . We can therefore replace in the setup of $n^\circ 3.1$ the action map $M \times X \rightarrow X$ by the following well defined morphism of *finite-dimensional* varieties induced by the action

$$LG_i/L^k \times \text{Gr}_j \rightarrow \text{Gr}_m \quad (3.2.0)$$

Recall the group L^k is normal in L^+G , and set $L_k := L^+G/L^k$, a finite dimensional Lie group. There is a natural free L_k -action on LG_i/L^k on the right making the projection $LG_i/L^k \rightarrow LG_i/L^+G = \text{Gr}_i$ a principal L_k -bundle over Gr_i . Thus, the ‘right-left’ L_k -action on the product $LG_i/L^k \times \text{Gr}_j$ is free, and the orbit-space

$$LG_i/L^k \times_{L_k} \text{Gr}_j = LG_i \times_{L^+G} \text{Gr}_j$$

is a well-defined finite dimensional variety. Moreover, the map 3.2.1 induces the following morphism

$$a : LG_i/L^k \times_{L_k} \text{Gr}_j \rightarrow \text{Gr}_m$$

which is an analogue of the map $a : M \times_L X \rightarrow X$ from $n^\circ 3.1$. With that understood, the convolution-construction of $n^\circ 3.1$ goes through verbatim. One can show that this way we obtain the same convolution on the category $P(\text{Gr})$ as the one defined in n. 1.2 (for $\mathcal{M} = IC_\lambda$ and $\mathcal{A} = IC_\mu$ our present construction is identical to that of n. 2.1 (Proposition 2.1.1)).

3.3

Now let K be any *connected* subgroup of L^+G , that is a subgroup that projects to a connected Lie subgroup $K_i \subset L_i$, for all large enough $i \gg 0$. Then, by (8.2.4), the constant sheaf on each L^+G -orbit O_λ has a unique structure of K -equivariant complex. Hence, the Intersection cohomology

extension of such a complex has a unique K -equivariant structure. Observe next that any object of the category $P(\text{Gr})$ is a direct sum of such complexes and, moreover, the direct sum can be chosen so that there are no non-trivial morphisms between different summands. Thus, we have proved the following

Proposition 3.3.1 *For any connected subgroup $K \subset L^+G$ as above, every object of the category $P(\text{Gr})$ has a unique structure of K -equivariant perverse sheaf.* \square

We apply the proposition to a maximal torus $T \subset G$, viewed as a subgroup of constant loops. We get

Corollary 3.3.2 *Any object of the category $P(\text{Gr})$ has a canonical (unique) structure of T -equivariant complex on Gr .* \square

Recall that any algebraic homomorphism $\lambda : \mathbb{C}^* \rightarrow T$ gives a point in Gr denoted by the same symbol λ . These points form a countable discrete subset $\mathbf{X}_*(T) \subset \text{Gr}$ that turns out to be exactly the set of T -fixed points.

Lemma 3.3.3 $\text{Gr}^T = \mathbf{X}_*(T)$.

To prove the Lemma, choose a maximal compact subgroup $K \subset G$ such that $T_c := K \cap T$ is a maximal (compact) torus in K . Identify the Grassmannian with the Loop group Ω as in n. 1.2. The action of the compact torus T_c on Gr corresponds to the natural T_c -action on Ω by conjugation.

Proof of the Lemma: It is clear that $\text{Gr}^T = \text{Gr}^{T_c} \cong \Omega^{T_c}$. Further, a loop $f \in \Omega$ is fixed by the T_c -action iff the image of f is contained in $T_c \subset K$. But any *polynomial* loop: $S^1 \rightarrow T_c$ must be a group homomorphism. Hence, $f \in \mathbf{X}_*(T)$. The inverse inclusion: $\mathbf{X}_*(T) \subset \text{Gr}^T$ is obvious. \square

3.4

We now apply the machinery of equivariant cohomology. To any perverse sheaf $M \in P(\text{Gr})$ we associate the space $H_T^*(M)$, the T -equivariant cohomology group with respect to the T -equivariant structure on M provided by Corollary 3.3.1. The assignment: $M \mapsto H_T^*(M)$ gives rise to a functor:

$$H_T^* : P(\text{Gr}) \rightarrow \text{mod}^\bullet\text{-}\mathbb{C}[\mathfrak{t}],$$

where $\text{mod}^\bullet\text{-}\mathbb{C}[\mathfrak{t}]$ denotes the tensor category of finitely-generated graded $\mathbb{C}[\mathfrak{t}]$ -modules.

Proposition 3.4.1 *The functor H_T^* is a fibre functor.*

(Proposition says that H_T^* is an exact fully-faithful tensor functor such that $H_T^*(M)$ is a projective $\mathbb{C}[\mathfrak{t}]$ -module for any $M \in P(Gr)$).

Proof: The functor H_T^* is obviously exact. Theorem 8.4.1 implies that it is fully-faithful and that the module $H_T^*(M)$ is free for any $M \in P(Gr)$. It remains to show that H_T^* is a tensor functor. This can be done by repeating the argument of n. 2.4 as follows.

Identify Gr with the Loop group Ω and replace T -equivariant cohomology by T_c -equivariant cohomology (see 8.3.4). By abuse of notation, we shall drop the subscript "c" and will always write " T " instead of " T_c ". Given $M, N \in P(Gr)$, set $J = M \boxtimes N$. This is a T -equivariant complex on $\Omega \times \Omega$. To such a complex J we have associated in n. 3.1 a complex J_T on the variety $(\Omega \times \Omega)_T = ET \times_T (\Omega \times \Omega)$.

Further, note that the compact torus acts on the Loop group Ω by group automorphisms, so that the multiplication map $m : \Omega \times \Omega \rightarrow \Omega$ commutes with the T -action. Hence, the functor $J \mapsto J_T$ commutes with the direct image functor m_* . Hence, we obtain:

$$\begin{aligned}
H_T^*(M * N) &= H_T^*(m_*(M \boxtimes N)) = \\
H_T^*(m_*J) &= H^*(\Omega_T, (m_*J)_T) = \\
&H^*(\Omega_T, (m_T)_*(J_T)) = (2.4.1) & (3.4.0) \\
H^*((\Omega \times \Omega)_T, J_T) &= H_T^*(M \boxtimes N) = (\text{Corollary 8.4.3}) \\
&H_T^*(M) \otimes_{\mathbb{C}[\mathfrak{t}]} H_T^*(N).
\end{aligned}$$

□

We can specialize the equivariant cohomology functor H_T^* at various points $t \in \mathfrak{t}$. By Proposition 3.4.1, we obtain a family of exact fully-faithful tensor functors $H_t : P(Gr) \rightarrow \text{Vect}$. All these functors are non-canonically isomorphic to each other (corollary 7.2.2) and, in particular, isomorphic to the ordinary cohomology functor $H^*(\cdot) \cong H_0^*(\cdot)$ (cf. n. 8.5).

3.5

Recall that Gr^T , the fixed point subvariety, is the discrete set consisting of isolated points $\lambda \in \mathbf{X}_*(T)$. Let $i_\lambda : \{\lambda\} \hookrightarrow Gr$ denote the inclusion. Fix a regular $t \in \mathfrak{t}$. For any complex $M \in P(Gr)$, the Localization theorem 8.6 yields an isomorphism:

$$H_t(M) \cong \bigoplus_{\lambda \in \mathbf{X}_*(T)} H_t(i_\lambda^! M) \quad (3.6.0)$$

This isomorphism will be referred to as the *fixed point decomposition* and will be viewed as a gradation on $H_t(M)$ by the lattice $\mathbf{X}_*(T)$.

Proposition 3.6.2 *The fixed point decomposition is compatible (in the sense of (7.1.2)) with the convolution $*$ on $P(Gr)$.*

Proof: We use the notation $\Omega^2 := \Omega \times \Omega$. Given $\lambda \in \mathbf{X}_*(T)$, set:

$$(\Omega_\lambda^2)^T := \{(\mu, \nu) \in \Omega^2 \mid \mu, \nu \in \mathbf{X}_*(T), \mu + \nu = \lambda\} \quad (3.6.0)$$

and let $i_\lambda^2 : (\Omega_\lambda^2)^T \hookrightarrow \Omega^2$ denote the natural inclusion. Proving the Proposition amounts to showing that, for each $\lambda \in \mathbf{X}_*(T)$, the image of the push-forward morphism

$$(i_\lambda)_! : H_t(i_\lambda^!(M * N)) \rightarrow H_t(\Omega, M * N) \quad (3.6.0)$$

is identified – via the isomorphisms (3.6.5) – with the image of the similar morphism:

$$(i_\lambda^2)_! : H_t(i_\lambda^2!(M \boxtimes N)) \rightarrow H_t(\Omega^2, M \boxtimes N) \quad (3.6.0)$$

In order to compare (3.6.4) with (3.6.5) we introduce the subvariety $\Omega_\lambda^2 := m^{-1}(\lambda) \subset \Omega^2$ where m is the multiplication map. Clearly, Ω_λ^2 is a T_c -stable subvariety of Ω^2 and we have an equivariant Cartesian square:

$$\begin{array}{ccc} \Omega_\lambda^2 & \xrightarrow{m} & \{\lambda\} \\ \downarrow j & & \downarrow i_\lambda \\ \Omega^2 & \xrightarrow{m} & \Omega \end{array}$$

The Base change theorem for this square yields an isomorphism:

$$\text{Image}(i_\lambda)_! \simeq \text{Image}[H_t(\Omega_\lambda^2, j^!(M \boxtimes N)) \xrightarrow{j_!} H_t(\Omega^2, M \boxtimes N)] \quad (3.6.0)$$

Next, note that the set of T_c -fixed points in Ω_λ^2 coincides with the set (3.6.3) (this agrees with the notation $(\Omega_\lambda^2)^T$). We have the following commutative triangles of maps

$$\begin{array}{ccc} (\Omega_\lambda^2)^T & \xrightarrow{\varepsilon} & \Omega_\lambda^2 \\ & \searrow i_\lambda^2 & \swarrow j \\ & \Omega^2 & \end{array} \quad (3.6.0)$$

That induces the corresponding morphisms on cohomology:

$$\begin{array}{ccc}
H_t(i_\lambda^2)!(M \times N) & \xrightarrow[\sim]{\varepsilon_!} & H_t(\Omega_\lambda^2, j^!(M \boxtimes N)) \\
& \searrow^{(i_\lambda^2)!} & \swarrow_{j_!} \\
& & H_t(\Omega^2, M \boxtimes N)
\end{array} \tag{3.6.0}$$

The horizontal arrow $\varepsilon_!$ in the latter triangle is an isomorphism by localization theorem 3.3. So, we obtain from (3.6.6) and (3.6.7) that

$$\mathrm{Im}(i_\lambda)_! \cong \mathrm{Im}(j_!) \cong \mathrm{Im}(j_! \circ \varepsilon_!) \cong \mathrm{Im}(i_\lambda^2)_! \tag{3.6.0}$$

The composition of the isomorphisms above coincides, in the case under consideration, with isomorphism (7.1.2). That completes the proof of Proposition 3.6.2. \square

3.7

We will now use Proposition 3.6.2 to show that the associativity and commutativity constraints (see n. 2.3) satisfy the hexagon axiom, i.e., for any $X, Y, Z \in P(Gr)$ the following natural diagram commutes (cf. [DM]):

$$\begin{array}{ccccc}
X * (Y * Z) & \xrightarrow{\sim} & (X * Y) * Z & \xrightarrow{\sim} & Z * (X * Y) \\
\sim \downarrow & & & & \sim \downarrow \\
X * (Z * Y) & \xrightarrow{\sim} & (X * Z) * Y & \xrightarrow{\sim} & (Z * X) * Y
\end{array} \tag{3.7.0}$$

Assume the notation and assumptions of n. 3.6, in particular, $t \in \mathfrak{t}$ is a regular element.

Lemma 3.7.2 *Let f be an endomorphism of an object $M \in P(Gr)$, such that the induced morphism on cohomology $H_t(f) : H_t(M) \rightarrow H_t(M)$ is the identity morphism. Then f is the identity morphism.*

Proof: Decompose M into the direct sum of disjoint isotypical components $M = M_1 + \dots + M_r$. Clearly, f maps each M_i into itself and the induced morphism on cohomology is the identity. Hence, we may assume without loss of generality that M is isotypical. Then, proving that f is the identity, suffices it to show that for some point λ in the nonsingular locus of $\mathrm{supp}(M)$

the induced morphism $H_t i_\lambda^!(f) : H_t(i_\lambda^! M) \rightarrow H_t(i_\lambda^! M)$ is the identity. But the latter assertion follows from the assumptions of the Lemma and the Fixed-point decomposition (3.6.1). \square

Corollary 3.7.3 *Let $f, g : M \rightarrow N$ be two isomorphisms in $P(\text{Gr})$ such that $H_t(f) = H_t(g)$. Then $f = g$.* \square

Let $u : A * B \xrightarrow{\sim} B * A$ be the commutativity (functor) isomorphism defined in Proposition 2.3.1 and let $H_t(u) : H_t(A * B) \xrightarrow{\sim} H_t(B * A)$ be the corresponding isomorphism on cohomology. Further, let $v : H(A * B) \xrightarrow{\sim} H(A) \otimes H(B)$ denote the functor isomorphism arising from Proposition 3.5.4. Finally, for any vector spaces V and W , let $s : V \otimes W \xrightarrow{\sim} W \otimes V$ denote the standard isomorphism.

Lemma 3.7.4 *For any $A, B \in P(\text{Gr})$ the following natural diagram commutes*

$$\begin{array}{ccc} H_t(A * B) & \xrightarrow{v} & H_t(A) \otimes H_t(B) \\ \downarrow H_t(u) & & \downarrow s \\ H_t(B * A) & \xrightarrow{v} & H_t(B) \otimes H_t(A) \end{array}$$

Proof: By the fixed-point decomposition and Proposition 3.6.2, proving the Lemma amounts to showing that, for each $\lambda \in \mathbf{X}_*(T)$, the following diagram commutes:

$$\begin{array}{ccc} H_t(i_\lambda^!(A * B)) & \xrightarrow{v_\lambda} & \bigoplus_{\mu+\nu=\lambda} H_t(i_\mu^! A) \otimes H_t(i_\nu^! B) \\ \downarrow H_t(i_\lambda^! u) & & \downarrow s \\ H_t(i_\lambda^!(B * A)) & \xrightarrow{v_\lambda} & \bigoplus_{\nu+\mu=\lambda} H_t(i_\nu^! B) \otimes H_t(i_\mu^! A) \end{array}$$

where the horizontal isomorphisms v_λ are the compositions of the isomorphisms arising from (3.6.8). The commutativity of this latter diagram follows immediately from the very definition of the morphism u and the construction of the isomorphisms (3.6.8). \square

Proof of the hexagon axiom: We have to show that diagram (3.7.1) commutes. By Corollary 3.7.3, this amounts to showing that the corresponding diagram of the cohomology groups $H_t(\cdot)$ commutes. Since $H_t(\cdot)$ is a tensor functor, this latter diagram can be rewritten as follows (we write H instead of H_t for short):

$$\begin{array}{ccccc}
H(X) \otimes (H(Y) \otimes H(Z)) & \xrightarrow{b} & (H(X) \otimes H(Y)) \otimes H(Z) & \xrightarrow{c} & H(Z) \otimes (H(X) \otimes H(Y)) \\
\downarrow a & & & & \downarrow d \\
H(X) \otimes (H(Z) \otimes H(Y)) & \xrightarrow{e} & (H(X) \otimes H(Z)) \otimes H(Y) & \xrightarrow{f} & (H(Z) \otimes H(X)) \otimes H(Y)
\end{array}$$

The maps b , d and e in the diagram are the standard associativity isomorphisms for the tensor product of vector spaces. The maps a , c and f arising from the commutativity constraint in the category $P(Gr)$ coincide, by Lemma 3.7.4, with standard isomorphisms arising from the commutativity constraint in the category of vector spaces. Thus, the diagram above is nothing but the hexagon diagram in the tensor category of vector spaces. Hence, this diagram commutes and the commutativity of (3.7.1) follows. \square

This completes the proof of Theorem 1.3.1.

3.8

The rest of this chapter is devoted to the proof of Theorem 1.4.1.

From now on we fix a regular element $t \in \mathfrak{t}$. The argument of n. 1.4 shows that there is a reductive group G^* such that the tensor category $P(Gr)$ is equivalent to the tensor category $\text{Rep}(G^*)$ in such a way that the functor H goes into the forgetful functor: $\text{Rep}(G^*) \rightarrow \text{Vect}$.

The main result of this section is the following

Theorem 3.8.1 *The group G^* is isomorphic to G^\vee , the dual group.*

Theorem 1.4.1 is an immediate consequence of Theorem 3.8.1 since the ordinary cohomology functor H^* is isomorphic to H_t .

Lemma 3.8.2 *G^* is a connected semisimple Lie group.*

Proof: For any $\lambda \in \mathbf{X}_*(T)$ the sequence of perverse sheaves: $IC_\lambda, IC_\lambda * IC_\lambda, IC_\lambda * IC_\lambda * IC_\lambda, \dots$ obviously has the infinite set $\{IC_{\lambda^n}, n = 1, 2, \dots\}$ among their irreducible constituents. The general criterium [DM, Corollary 2.22]

shows now that the group G^* is connected. This group must be semisimple, for it has the only one 1-dimensional representation. \square

Let T^\vee be the torus dual to T , so that we have a canonical identification: $\mathbf{X}_*(T) = \mathbf{X}^*(T^\vee)$. The fixed point gradation 3.6.1 yields, by corollary 8.1.3 and proposition 3.6.2, an algebraic homomorphism:

$$f : T^\vee \longrightarrow G^*. \quad (3.8.0)$$

Lemma 3.8.4 *The homomorphism f is injective and its image is a maximal torus in G^* .*

Proof: To prove injectivity, it suffices to show that for any $\lambda \in \mathbf{X}_*(T)$ there is a perverse sheaf $M \in P(Gr)$ such that $i_\lambda^! M \neq 0$. But one obviously has: $i_\lambda^!(IC_\lambda) \neq 0$.

The first assertion of the Lemma being proved, to prove the second suffices it to show that $\text{rk}(G) = \text{rk}(G^*)$. To that end, observe that, for any abelian tensor category Cat , its Grothendieck group $K(\text{Cat})$ has a natural ring structure induced by the tensor product. Let $Q(\text{Cat})$ denote the field of fractions of the ring $K(\text{Cat})$. Set:

$$\text{rk}(\text{Cat}) := \text{Deg. transc. } (Q(\text{Cat})/\mathbb{Q})$$

It is clear that $\text{rk}(P(Gr)) = \text{rk}(G)$. On the other hand, for any semisimple group, and hence for G^* , we have: $\text{rk}(\text{Rep}(G^*)) = \text{rk}(G^*)$. Since the category $\text{Rep}(G^*)$ is equivalent to $P(Gr)$ we find that: $\text{rk}(G^*) = \text{rk}(\text{Rep}(G^*)) = \text{rk}(P(Gr)) = \text{rk}(G)$. \square

3.9

We introduce now the following subsets in $\mathbf{X}^*(T)$:

${}^G R$ = the root system of the pair (G, T) ;

${}^G R_+$ = the set of positive roots corresponding to a choice of simple roots;

To this set of data one associates the dual data:

${}^G R^\vee$ = the root system dual to ${}^G R$, which is viewed as the set in $\mathbf{X}_*(T)$ of coroots of G ;

${}^G R_+^\vee =$ the set of positive roots in ${}^G R^\vee$ corresponding to the subset ${}^G R_+$ in ${}^G R$.

For $\lambda, \mu \in \mathbf{X}_*(T)$, we write: $\mu \leq \lambda$ iff $\lambda - \mu =$ a sum of positive coroots of ${}^G R^\vee$. Also, set:

${}^G D^\vee = \{\lambda \in \mathbf{X}_*(T) \mid \langle \lambda, x \rangle \geq 0 \quad \forall x \in {}^G R_+\}$, the dominant Weyl chamber.

Let \mathfrak{t} denote the Lie algebra of the torus T and \mathfrak{t}^* the dual space. We shall often view $\mathbf{X}^*(T)$, ${}^G R$, ${}^G R_+$ as subsets of \mathfrak{t}^* and $\mathbf{X}_*(T)$, ${}^G R^\vee$, ${}^G R_+^\vee$ as subsets of \mathfrak{t} . Let:

$h =$ the sum of positive roots in ${}^G R$.

Thus, h is an element of \mathfrak{t}^* having the following properties:

- (i) h is a regular element in \mathfrak{t}^* ;
- (ii) $\langle \lambda, h \rangle$ is an integer for any $\lambda \in \mathbf{X}_*(T)$;
- (iii) $\langle \mu, h \rangle < \langle \lambda, h \rangle$ if $\mu \leq \lambda, \mu \neq \lambda, \mu, \lambda \in \mathbf{X}_*(T)$.

We have a canonical identification of \mathfrak{t}^\vee , the Lie algebra of the dual torus T^\vee , with \mathfrak{t}^* . Further, we identify the torus T^\vee with its image in G^* . Thus, T^\vee is a maximal torus in G^* , \mathfrak{t}^\vee is a Cartan subalgebra in $\text{Lie}(G^*)$, and h is an element of \mathfrak{t}^\vee .

Lemma 3.9.2 *h is a regular element in $\text{Lie}(G^*)$.*

Remark 3.9.3 *The element h is regular, of course, as an element of \mathfrak{t}^* (see 3.9.1 (i)), but this does not necessarily imply that it is regular as an element of \mathfrak{t}^\vee since we know nothing about the root system of the pair (G^*, T^\vee) , which we are going to introduce now. \square*

Let ${}^{G^*} R \subset \mathbf{X}^*(T^\vee)$ be the root system of the pair (G^*, T^\vee) . We view the set ${}^{G^*} R$ as a subset of $\mathbf{X}_*(T)$ via the identification: $\mathbf{X}^*(T^\vee) \cong \mathbf{X}_*(T)$. Assuming that Lemma 3.9.2 is true and taking (3.9.1 (ii)) into account, we define the system of positive roots in ${}^{G^*} R$ by

$${}^{G^*} R_+ = \{\alpha \in {}^{G^*} R \mid \langle \alpha, h \rangle > 0\}. \quad (3.9.0)$$

The choice of positive roots being made, we can define the set ${}^{G^*} R_+^\vee$ of positive roots in ${}^{G^*} R^\vee$ ($=$ the root system dual to ${}^{G^*} R$) and, hence, the dominant Weyl chamber ${}^{G^*} D \subset \mathbf{X}^*(T^\vee)$. Note that the sets ${}^G D^\vee$ and ${}^{G^*} D$ are contained in the same lattice $\mathbf{X}_*(T)$.

Proposition 3.9.5 *Let $\lambda \in {}^G D^\vee$ and V_λ the irreducible representation of G^* corresponding to the complex IC_λ via the equivalence of the categories $\text{Rep}(G^*)$ and $P(Gr)$. Then $\lambda \in {}^G D$ and V_λ is representation with highest weight λ .*

Remark. The Proposition ensures consistency of the notation V_λ which a priori means either the representation of G^* with highest weight λ , or the simple object of $\text{Rep}(G^*)$ corresponding to IC_λ .

Notation. Given a representation $V \in \text{Rep}(G^*)$, we let $V(\nu)$ denote the weight subspace of V corresponding to a weight $\nu \in \mathbf{X}^*(T^\vee)$ and set:

$$\text{Spec}(V) = \{\nu \in \mathbf{X}^*(T^\vee) \mid V(\nu) \neq 0\}.$$

Similarly, for a perverse sheaf $M \in P(Gr)$ we set:

$$\text{Spec}(M) = \{\nu \in \mathbf{X}_*(T) \mid i_\nu^! M \neq 0\}.$$

Proof of Lemma 3.9.2 and Proposition 3.9.5: We know that $\langle \lambda, h \rangle \in \mathbb{Z}$ for any $\lambda \in \mathbf{X}^*(T^\vee)$ (by 3.9.1 (ii)). Hence, forgetting (3.9.4) for a moment, we can (and will) make a choice of the set ${}^{G^*}R_+$ of positive roots in such a way that $\langle \alpha, h \rangle \geq 0$ for any $\alpha \in {}^{G^*}R_+$. Let ${}^{G^*}D$ stand for the corresponding dominant Weyl chamber (we'll show later that these ${}^{G^*}R_+$ and ${}^{G^*}D$ coincide with those defined by (3.9.4)).

We have the following two key observations:

For an irreducible representation $V \in \text{Rep}(G^*)$, the function $\text{Spec}(V) \ni \nu \longmapsto \langle \nu, h \rangle$ attains its maximum at the highest weight of V . (3.9.0)

And similarly:

(a) For $\lambda \in {}^G D^\vee$, the function $\text{Spec}(IC_\lambda) \ni \nu \mapsto \langle \nu, h \rangle$ attains its maximum at the point $\nu = \lambda$; and

(b) $\langle \nu, h \rangle < \langle \lambda, h \rangle$ for any $\nu \neq \lambda$; $\nu \in \text{Spec}(IC_\lambda)$. (3.9.0)

Claim (3.9.7) follows immediately from (3.9.2 (iii)) and the implication: $\nu \in \overline{O}_\lambda \Rightarrow \nu \leq \lambda$.

Now, let $\lambda \in {}^G D^\vee$, let V_λ be the irreducible representation of G^* corresponding to the complex IC_λ via the equivalence of the categories $P(Gr)$

and $\text{Rep}(G^*)$, and let $\mu \in {}^{G^*}D$ be the highest weight of V_λ . We know that the fixed-point gradation on $H_t(IC_\lambda)$ corresponds to the weight gradation on V_λ . Hence, comparison of (3.9.6) with (3.9.7(a)) shows that $\langle \mu, h \rangle = \langle \lambda, h \rangle$. Then, the statement (3.9.7(b)) yields an equality: $\mu = \lambda$. This completes proof of Proposition 3.9.5.

Finally, suppose that h is not a regular element of \mathfrak{t}^\vee . Then, one can find an irreducible representation V_λ (with highest weight $\lambda \in {}^{G^*}D$) and a simple root $\alpha \in {}^{G^*}R_+$ such that

$$\langle \alpha, h \rangle = 0 \quad \text{and} \quad \lambda - \alpha \in \text{Spec}(V_\lambda).$$

Translating this into topological language, and setting $\nu = \lambda - \alpha$, we get

$$\langle \nu, h \rangle = \langle \lambda, h \rangle \quad \text{and} \quad \nu \in \text{Spec}(IC_\lambda).$$

Again, this contradicts to (3.9.7(b)), completing the proof of Lemma 3.9.2. \square

Corollary 3.9.8 *We have: ${}^G D^\vee = {}^{G^*} D$.*

Proof: It follows from Proposition 3.9.5 that ${}^G D^\vee \subset {}^{G^*} D$. Let $\lambda \in {}^{G^*} D$ and let IC_μ , $\mu \in {}^G D^\vee$ be the object of $P(Gr)$ corresponding to the irreducible representation V_λ . Then, $\lambda = \mu$ by the Proposition, hence $\lambda \in {}^G D^\vee$. \square

3.10

We remind the reader that, given a semisimple group G with a maximal torus T , one associates to the pair (G, T) three natural lattices in $\mathfrak{t} = \text{Lie}(T)$ and three natural lattices in \mathfrak{t}^* :

$${}^G Q \subset \mathbf{X}^*(T) \subset {}^G P \subset \mathfrak{t}^* \quad \text{and} \quad {}^G Q^\vee \subset \mathbf{X}_*(T) \subset {}^G P^\vee \subset \mathfrak{t}.$$

Here ${}^G Q$ is the lattice generated by the roots of (G, T) (the root lattice), ${}^G Q^\vee$ is the lattice generated by coroots (the coroot lattice), ${}^G P$ is the lattice dual to ${}^G Q^\vee$ (the weight lattice), and ${}^G P^\vee$ is the lattice dual to ${}^G Q$ (the coweight lattice). Note that replacing G by G^\vee (= the dual group) exchanges the roles of \mathfrak{t} and \mathfrak{t}^* and reverses the order of lattices. Further, the center of a semisimple group can be expressed in terms of the lattices as follows:

$$Z(G) \cong {}^G P^\vee / \mathbf{X}_*(T) \quad \text{and} \quad Z(G^\vee) \cong {}^G P / \mathbf{X}^*(T). \quad (3.10.0)$$

Proof of Theorem 3.8.1: We shall view the groups G^* and G^\vee as having the *same* vector space $\mathfrak{t}^\vee (= \mathfrak{t}^*)$ as a Cartan subalgebra. This space contains various lattices associated to G , G^* and G^\vee .

In n. 3.8 we have fixed the set ${}^G R_+$ of positive roots of the pair (G, T) and showed that it determines the choice of positive roots for the pair (G^*, T) . On the other hand, the choice of positive roots in ${}^G R$ leads to the choice of positive roots in the dual root system. Hence, we may speak freely about positive roots, simple roots, dominant Weyl chambers, etc. for all three groups: G , G^* , G^\vee . We claim that the following sets in $(\mathfrak{t}^\vee)^* = \mathfrak{t}$ coincide:

$$\text{simple roots of } G^* = \text{simple roots of } G^\vee. \quad (3.10.0)$$

To prove this claim we let ${}^G Q_+$ denote the semigroup in the root lattice ${}^G Q$ generated by positive roots and let ${}^G Q_+^\vee$, ${}^{G^*} Q_+$, \dots denote the objects defined in a similar way. We can characterize these objects in terms of the categories $P(Gr)$ and $\text{Rep}(G^*)$ and the corresponding dominant Weyl chambers. as follows:

$${}^G Q_+^\vee = \{\alpha \in \mathbf{X}_*(T) \mid \exists \lambda \in {}^G D^\vee : \lambda - \alpha \in \text{Spec}(IC_\lambda)\} \quad (3.10.0)$$

and

$${}^{G^*} Q_+ = \{\alpha \in \mathbf{X}^*(T^\vee) \mid \exists \lambda \in {}^{G^*} D : \lambda - \alpha \in \text{Spec}(V_\lambda)\}. \quad (3.10.3^*)$$

We know that the sets $\text{Spec}(IC_\lambda)$ and $\text{Spec}(V_\lambda)$ correspond to each other via the equivalence of the categories $P(Gr)$ and $\text{Rep}(G^*)$, and we have shown that ${}^G D^\vee = {}^{G^*} D$ (Corollary 3.9.8). Hence, the right-hand sides of (3.10.3) and (3.10.3*) coincide, and we obtain that

$${}^{G^*} Q_+ = {}^G Q_+^\vee. \quad (3.10.0)$$

We now derive some consequences of (3.10.4). First, recall that each side in (3.10.4) is a free semigroup generated by the corresponding set of simple roots. Therefore, simple roots can be characterized as indecomposable elements of the semigroup in question (an element λ is said to be indecomposable if $\lambda = \mu + \nu \implies \mu = 0$ or $\nu = 0$). Hence, (3.10.4) yields: *simple roots of G^* = simple roots of G* , and (3.10.2) follows.

Next, note that each of the lattices Q is generated by the corresponding semigroup $Q_+ \subset Q$, so that the lattice Q is completely determined by its positive part Q_+ . Passing to the dual lattices and using (3.10.4) again, we obtain:

$${}^{G^*} P^\vee = {}^G P.$$

Hence, (3.10.1) implies that

$$Z(G^*) \cong Z(G^\vee). \quad (3.10.0)$$

Now, we will (temporarily) make an additional assumption, that the group G is of *adjoint* type, i.e. has no center. Then, the group G^\vee is simply-connected and we claim that (compare with (3.10.2)):

$$\text{simple coroots of } G^* = \text{simple coroots of } G^\vee. \quad (3.10.0)$$

To prove the claim note first that $\mathbf{X}^*(T^\vee) = {}^{G^\vee}P$ (since G^\vee is simply-connected) and, hence, any Weyl chamber in $\mathbf{X}^*(T^\vee)$ (of the group G^\vee) is the free semigroup generated by the corresponding fundamental weights. Therefore, for any Weyl chambers $G_{D_1}^\vee$ and $G_{D_2}^\vee$ separated by a wall we have:

$$\begin{aligned} & \text{The wall between } G_{D_1}^\vee \text{ and } G_{D_2}^\vee \text{ is the hyperplane in } (\mathfrak{t}^\vee)^* = \mathfrak{t} \\ & \text{spanned by } G_{D_1}^\vee \cap G_{D_2}^\vee. \end{aligned} \quad (3.10.0)$$

Furthermore, the line $L = \mathbb{R} \cdot \lambda$ generated by a fundamental weight λ has the following characterization:

$$L = \text{intersection of the walls containing } L. \quad (3.10.0)$$

Let us now turn to the group G^* . Notice that Corollary 3.9.8 holds, of course, for any choice of positive roots in ${}^G R$. Hence, the Corollary implies that the groups G^* and G^\vee give rise to identical partitions of $\mathbf{X}^*(T^\vee)$ into Weyl chambers. (One should be careful here since we do not yet know that the group G^* is simply-connected, so that $\mathbf{X}^*(T^\vee) \neq {}^{G^*}P$ in general. Speaking about Weyl chambers of G^* in $\mathbf{X}^*(T^\vee)$ we have in mind intersections of Weyl chambers in ${}^{G^*}P$ with $\mathbf{X}^*(T^\vee)$.) Let \tilde{D}_1, \tilde{D}_2 be Weyl chambers in ${}^{G^*}P$ and let $D_i := \tilde{D}_i \cap \mathbf{X}^*(T^\vee)$. We have: $D_1 \cap D_2 \subset \tilde{D}_1 \cap \tilde{D}_2$. Suppose now that D_1 and D_2 are separated by a wall, as Weyl chambers of G^\vee . Then, (3.10.7) shows that the set $D_1 \cap D_2$ spans a hyperplane in $(\mathfrak{t}^\vee)^*$. Hence, this hyperplane must be the wall between \tilde{D}_1 and \tilde{D}_2 . It follows, that the groups G^* and G^\vee give rise to the same collection of walls in $(\mathfrak{t}^\vee)^*$. Thus, fundamental weights of G^* are proportional to fundamental weights of G^\vee , by (3.10.8).

We fix a choice of positive roots, let $\alpha_1, \dots, \alpha_r$ be the corresponding simple roots of G^\vee and G^* (they are the same by (3.10.2)), and let $\alpha_1^\vee, \dots, \alpha_r^\vee$ (resp. $\alpha_1^*, \dots, \alpha_r^*$) be the corresponding simple coroots of G^\vee (resp. G^*). It

follows from the preceding paragraph (for, simple roots are dual to the fundamental weights) that the collection of lines generated by the $\{\alpha_i^*\}$ is the same as that generated by the $\{\alpha_i^\vee\}$. Whence, $\alpha_i^* = \alpha_i^\vee$, for the coroots α_i^* and α_i^\vee are uniquely determined by the following conditions:

$$\langle \alpha_i^*, \alpha_i \rangle = 2 = \langle \alpha_i^\vee, \alpha_i \rangle \quad \text{and} \quad \langle \alpha_i^*, \alpha_j \rangle < 0 \text{ for } i \neq j.$$

Now, it follows from (3.10.2) and (3.10.6), that for an adjoint group G , the groups G^\vee and G^* have the same Cartan matrices and therefore have isomorphic Lie algebras. Hence, there exists a surjective homomorphism $G^\vee \rightarrow G^*$ with finite kernel. This kernel must be trivial, by (3.10.5).

Finally, let us drop the assumption that the group G has no center and let G' denote the derived group of G . The projection $G \rightarrow G'$ gives rise to an inclusion of Grassmannians: $\text{Gr}_G \hookrightarrow \text{Gr}_{G'}$, and hence to a fully faithful imbedding of tensor categories: $P(\text{Gr}) \hookrightarrow P(\text{Gr}')$ which may be viewed as an imbedding $\text{Rep}(G^*) \hookrightarrow \text{Rep}((G')^*)$. The last imbedding is induced (Proposition 7.1.1) by a homomorphism $f : (G')^* \rightarrow G^*$ which, being restricted to the maximal tori, gives a surjective homomorphism $(T')^\vee \rightarrow T^\vee$ with finite kernel. Hence, f is also a surjective homomorphism with finite kernel. It remains to note that the Theorem is already proved for the group G' , and to apply (3.10.5). \square

Remark 3.11. The proof of Theorem 3.8.1 shows that the fixed-point decomposition 3.6.1 corresponds to the weight decomposition for representations of the group G with respect to the maximal torus T embedded into G via homomorphism (3.8.3).

4 Loop group cohomology and the Principal nilpotent

In this chapter we will answer Question 1 of n. 1.6. (Theorem 1.6.3) and will approach Question 2 of n. 1.6.

4.1

For representation $V \in \text{Rep}(G^\vee)$ let $V = \bigoplus_i V^{\text{top}}(i)$, $i \in \mathbb{Z}$ be the gradation on the underlying vector space of the representation, arising — via Theorem 1.4.1 — from the natural gradation on hyper-cohomology. It will be referred to as the “topological” gradation.

Proposition 4.1.1 *There is a unique semisimple element $h \in \mathfrak{g}^\vee$, such that, for any $V \in \text{Rep}(G^\vee)$, the “topological” gradation on V coincides with gradation $V^h(\cdot)$ by the eigenvalues of h (cf. (1.6.4)).*

Proof: The Kunnetth formula shows (cf. n. 2.4) that the “topological” gradation is compatible with tensor product. The result now follows from Corollary 7.1.3 applied to $T = \mathbb{C}^*$. \square

Lemma 4.1.2 *The element h is a regular element of \mathfrak{g}^\vee .*

Proof: of the lemma is similar to that of Lemma 3.9.2. Let \mathfrak{t}^\vee be a Cartan subalgebra of \mathfrak{g}^\vee containing h . First, we choose the set S of simple roots of the pair $(\mathfrak{g}^\vee, \mathfrak{t}^\vee)$ in such a way that $\langle \alpha, h \rangle \geq 0$ for $\alpha \in S$. Then, we have assertion (3.9.6). It holds for the element h we are considering now (although it is different from the one considered in Lemma 3.9.2). Furthermore, an argument in the proof of Lemma 3.9.2 shows that if h is not regular, then one can find an irreducible representation V_λ such that $\dim V_\lambda^h(m) > 1$ where m is the maximum of the function $\text{Spec}(V_\lambda) \ni \nu \mapsto \langle \nu, h \rangle$. That means, in topological terms, that the dimension of the top Intersection cohomology group $IH^m(\overline{O}_\lambda)$ is greater than 1. This is impossible, of course, so that h must be regular. \square

Lemma 4.1.3 *If λ belongs to the coroot lattice, then all the eigenvalues of the element h in the representation V_λ are even.*

Proving the Lemma amounts to showing that $IH^{\text{odd}}(\overline{O}_\lambda) = 0$, provided λ satisfies the assumptions of the Lemma. To prove this cohomology vanishing we recall the following results:

- (a). For any λ , the cohomology sheaves $\mathcal{H}^i(IC_\lambda)$ vanish if $i \not\equiv \dim(O_\lambda) \pmod{2}$ (see [KL]);
- (b). For any λ , we have: $H_c^{\text{odd}}(O_\lambda) = 0$ for O_λ has a stratification by complex affine spaces, the so-called Bruhat decomposition (see [AP]);
- (c). If λ belongs to the coroot lattice, then the stratum O_λ has even dimension (see e.g. [Lu]).

The Lemma now follows from the standard spectral sequence:

$$\bigoplus_{\mu \leq \lambda} \mathcal{H}_\mu^i(IC_\lambda) \otimes H_c^j(O_\mu) \Rightarrow H^{i+j}(IC_\lambda),$$

associated to the stratification $\overline{O}_\lambda = \bigcup_{\mu \leq \lambda} O_\mu$ (we write: $\mu \leq \lambda$ if $O_\mu \subset \overline{O}_\lambda$).
 \square

4.2

Let us interrupt the proof of Theorem 1.5.3 and turn to Question 2 from n° 1.6. We first analyze the relationship between the convolution on the category $P(Gr)$ and the action of the algebra $H^*(\Omega)$ on hyper-cohomology of the complexes $M \in P(Gr)$.

Let $prim$ denote the space of primitive elements of the Hopf algebra $H^*(\Omega)$. We view $prim$ as an abelian Lie algebra acting on $H^*(M)$, $M \in P(Gr)$, by nilpotent transformations. Let $A (\cong A')$ be the unipotent algebraic group corresponding to the Lie algebra $prim$. The $prim$ -action gives rise to an algebraic A -action on $H^*(M)$ by unipotent transformations. This way we obtain a functor $R : P(Gr) \rightarrow \text{Rep}(A)$.

Lemma 4.2.1 *The functor R is an exact tensor functor.*

Proof: First, recall the general formula (2.5.1). One knows that, for any $a \in H^*(Y, \mathbb{C})$, the natural action of a on $H^*(Y, f_*J)$ corresponds, via the isomorphism (2.5.1), to the action of $f^*a \in H^*(X, \mathbb{C})$ on $H^*(X, J)$.

Now let $X = \Omega \times \Omega$, $Y = \Omega$ and let $m : \Omega \times \Omega \rightarrow \Omega$ be the product map. We have an isomorphism (see n° 2.5):

$$H^*(\Omega, m_*(M \boxtimes N)) \cong H^*(\Omega, M) \otimes H^*(\Omega, N) \quad (4.2.0)$$

The action of $a \in prim$ on the left-hand side of (4.2.2) corresponds, by the preceding paragraph, to the action of $m^*(a)$ on the right-hand side. But $m^*(a) = a \otimes 1 + 1 \otimes a$, for a is a primitive element in $H^*(\Omega)$. That means that R is a tensor functor and the Lemma follows. \square

4.3

We now compose the equivalence $\text{Rep}(G^\vee) \xrightarrow{\sim} P(Gr)$ provided by theorem 1.4.1 with the functor R of n° 4.2, to obtain a tensor functor: $\text{Rep}(G^\vee) \rightarrow$

$\text{Rep}(A)$. By Proposition 7.1.1, this functor yields a Lie algebra homomorphism:

$$\text{prim} \rightarrow \mathfrak{g}^\vee \quad (4.3.0)$$

such that the image of prim consists of nilpotent elements in \mathfrak{g}^\vee .

Lemma 4.3.2 *The homomorphism (4.3.1) is injective.*

To prove the Lemma, it suffices to show that for any $a \in \text{prim}$ one can find an irreducible representation $V_\lambda \in \text{Rep}(G^\vee)$ such that a acts on V_λ as a non-zero operator. In topological terms, this amounts to proving the following more general result:

Proposition 4.3.3 *For any $a \in H^k(\text{Gr})$, there is an Intersection cohomology complex $IC_\lambda \in P(\text{Gr})$ such that the element a acts on $H^*(IC_\lambda)$ as a non-zero operator.*

Proof: Choose a Borel subgroup $B \subset G$. Let I denote the ‘‘Iwahori’’ subgroup of LG consisting of those maps $f : S^1 \rightarrow G$ which extend holomorphically to the exterior of the unit disc in the Riemann sphere, and satisfy the condition: $f(\infty) \in B$. Any $\lambda \in X_*(T)$ determines a point in the Grassmannian and we let C^λ denote the I -orbit of that point in Gr . All the orbits C^λ , $\lambda \in X_*(T)$ are disjoint and form a decomposition of Gr by affine cells of finite codimension. The Poincaré duals of the fundamental cycles $[\overline{C}^\lambda]$, $\lambda \in X_*(T)$ form a basis in $H^*(\text{Gr})$.

Now let $a \in H^k(\text{Gr})$. Using the basis $[\overline{C}^\lambda]$ we can write:

$$a = \sum_{\lambda} a_{\lambda} \cdot [\overline{C}^\lambda], \quad \lambda \in X_*(T), \quad a_{\lambda} \in \mathbb{C} \quad (4.3.0)$$

where the sum ranges over k -codimensional cells C^λ . Let $a_{\mu} \neq 0$ for some μ . We’ll show that a acts non-trivially on $H^*(IC_{\mu})$.

It will be convenient for us, while working with cycles, to speak in the language of the Intersection *homology* rather than cohomology. So, the action of $H^*(\text{Gr})$ may be written as a cap-product map:

$$H^k(\text{Gr}) \times IH_i(\overline{O}_{\mu}) \xrightarrow{\cap} IH_{i-k}(\overline{O}_{\mu}) \quad (4.3.0)$$

Let $m = \dim_{\mathbb{R}}(\overline{O}_{\mu})$. By the main property of the Intersection homology there is a non-degenerate pairing:

$$\langle \cdot, \cdot \rangle : IH_{m-k}(\overline{O}_{\mu}) \times IH_k(\overline{O}_{\mu}) \rightarrow H_0(\overline{O}_{\mu}) = \mathbb{C} \quad (4.3.0)$$

The maps (4.3.5) and (4.3.6) are related by the following identity in $H_0(\overline{O}_\mu)$:

$$a \cap c = \langle a \cap [\overline{O}_\mu], c \rangle \quad (4.3.0)$$

where $a \in H^k(\text{Gr})$, $c \in IH_k(\overline{O}_\mu)$.

Formula (4.3.7) shows that if $a \cap [\overline{O}_\mu]$ is a non-zero element in $IH_{m-k}(\overline{O}_\mu)$, then one can always find (because of the non-degeneracy of $\langle \cdot, \cdot \rangle$) a class $c \in IH_k(\overline{O}_\mu)$ such that $a \cap c \neq 0$. Thus, proving the Proposition amounts to showing that for the element a from (4.3.4) we have: $a \cap [\overline{O}_\mu] \neq 0$.

To find $a \cap [\overline{O}_\mu]$ we notice that all the cycles \overline{C}^λ , occurring in (4.3.4), have the same codimension. Hence, none of them, except \overline{C}^μ , has non-empty intersection with \overline{O}_μ (see [PS]). Whence, $\overline{C}^\lambda \cap [\overline{O}_\mu] = 0$ for all $\lambda \neq \mu$ and we may therefore assume that $a = \overline{C}^\mu$.

Now, the stratum O_μ is known to be isomorphic to a vector bundle over G/P , a partial flag manifold for the group G . Let $Z = G/P \hookrightarrow O_\mu$ denote the zero-section of that bundle. One can show [AP] that the cycle \overline{C}^λ has no intersection with the singular locus of the closure \overline{O}_μ , and meets O_μ transversely in a subvariety $S \subset C^\lambda$ (no closure!). Moreover, S is contained in Z and, viewed as a subvariety of G/P , is a Shubert variety, i.e., the closure of a B -orbit. We denote by the same symbol S the corresponding homology class in $H_\bullet(Z)$.

Let $i : Z \hookrightarrow O_\mu$ denote the inclusion. Any cycle in Z may be regarded as an intersection homology cycle in \overline{O}_μ , for Z is contained in the non-singular locus of \overline{O}_μ . Hence, we get a natural morphism

$$i : H_\bullet(Z) \rightarrow IH_\bullet(\overline{O}_\mu) \quad (4.3.0)$$

It is clear from our construction that $\overline{C}^\mu \cap [\overline{O}_\mu] = i_!(S)$. Thus, to prove that $\overline{C}^\mu \cap [\overline{O}_\mu] \neq 0$ it suffices to show that morphism (4.3.8) is injective (for, obviously, S is non-zero class in $H_\bullet(Z)$).

To prove injectivity of (4.3.8) set $M = IC_\mu$, and note that, in the cohomology language, morphism (4.3.8) is nothing but the natural map:

$$H^*(Z, i^!M) \rightarrow H^*(\overline{O}_\mu, M) \quad (4.3.0)$$

Now, the complex M is \mathbb{C}^* -equivariant with respect to the natural \mathbb{C}^* -action on the fibres of the vector bundle: $O_\mu \rightarrow Z$. For such a complex one knows (see [Gi1, Proposition 10.4]) that:

$$H^*(Z, i^!M) \cong H_c^*(\overline{O}_\mu, M) \quad (4.3.0)$$

Furthermore, the morphism (4.3.9) can be identified with the morphism $q : H_c^*(O_\mu, M) \rightarrow H^*(\overline{O}_\mu, M)$ induced by the inclusion: $O_\mu \hookrightarrow \overline{O}_\mu$. To prove injectivity of the latter morphism, write the long exact sequence of cohomology:

$$\dots \rightarrow H^{k-1}(j^*M) \rightarrow H_c^k(O_\mu, M) \xrightarrow{q} H^k(\overline{O}_\mu, M) \rightarrow \dots \quad (4.3.0)$$

where $j : \overline{O}_\mu \setminus O_\mu \hookrightarrow \overline{O}_\mu$ denotes the inclusion. It suffices to show that the connecting homomorphisms in (4.3.11) vanish.

To that end, let us view M as a pure Hodge module (of weight 0, say) in the sense of Saito. Then, the weights of $H^{k-1}(j^*M)$ are $\leq k-1$, since the functor j^* decreases the weights, cf. [BBD]. Further, the functor $i^!$ increases the weights while the functor H_c^* decreases the weights [BBD]. Hence, $H_c^k(O_\mu, M)$ is pure of weight k because of the isomorphism (4.3.10). Hence, each connecting homomorphism is a map between spaces of different weights and therefore must be zero. \square

Because of the injectivity of the morphism (4.3.1) we can (and will) identify the Lie algebra *prim* with its image in \mathfrak{g}^\vee , which will be denoted by \mathfrak{a} .

4.4

Until the end of chapter 5 we assume \mathfrak{g} , the Lie algebra of the group G , is simple. Then, there is a unique (up to constant factor) $\text{Ad } G$ -invariant polynomial P on \mathfrak{g} of degree 2. This is the Killing form $P(x) = (x, x)$. Let $\alpha_1 = \alpha(P)$ be the corresponding primitive class in $H^2(\text{Gr})$ (see *n*° 1.7). The class α_1 is proportional to the first Chern class of the Determinant (line) bundle on the Grassmannian (see e.g [PS]).

Let a_1 denote the image of α_1 under the homomorphism (4.3.1). Clearly, the action of a_1 in any representation $V \in \text{Rep}(G^\vee)$ shifts the topological gradation $V^{\text{top}}(\cdot)$ by 2. Iterating this action, one gets a linear operator:

$$(a_1)^k : V^{\text{top}}(-k) \rightarrow V^{\text{top}}(k) \quad (4.4.0)$$

Lemma 4.4.2 *The map (4.4.1) is an isomorphism for any $k \in \mathbb{Z}$.*

Proof: We may assume that $V = V_\lambda$ is an irreducible representation. The result then follows from the hard Lefschetz theorem [BBD] for the Intersection cohomology (applied to the Chern class of the restriction to \overline{O}_λ of the Determinant bundle on Gr). \square

4.5 Proof of Theorem 1.6.3

Let λ be the maximal root of \mathfrak{g}^\vee so that $V_\lambda = \mathfrak{g}^\vee$ is the adjoint representation (recall that we assumed \mathfrak{g}^\vee to be simple). Let h be the element introduced in Proposition 4.1.1. Obviously, we have: $[\text{ad } h, \text{ad } a_1] = 2\text{ad } a_1$. Hence, $[h, a_1] = a_1$.

The space \mathfrak{g}^\vee is graded by *even* integers via the “topological” grading (Lemma 4.1.3), and lemma 4.4.2 holds for the operator $(\text{ad } a_1)^k$ acting on $V = \mathfrak{g}^\vee$. It follows easily (an exercise in linear algebra) that:

$$\dim \ker(\text{ad } a_1) = \dim V_\lambda^{\text{top}}(0)$$

By definition, $V_\lambda(0) = Z_{\mathfrak{g}^\vee}(h) := \text{centralizer of } h \text{ in } \mathfrak{g}^\vee$. Furthermore, $\dim Z_{\mathfrak{g}^\vee}(h) = \text{rk } \mathfrak{g}^\vee$ by Lemma 4.1.2. Hence, we find that: $\dim Z_{\mathfrak{g}^\vee}(a_1) = \text{rk } \mathfrak{g}^\vee$. Thus a_1 is a regular nilpotent.

To complete the proof of Theorem 1.6.3 we choose a Lie algebra homomorphism $j : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}^\vee$ such that $j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = a_1$. It is known that the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ in \mathfrak{g}^\vee , arising from j , does not contain the trivial 1-dimensional representation as a component. It follows that for $h_1 := j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we have:

$$[Z_{\mathfrak{g}^\vee}(a_1), h_1] = Z_{\mathfrak{g}^\vee}(a_1) \tag{4.5.0}$$

Let A be the unipotent subgroup of G^\vee corresponding to the Lie algebra $Z_{\mathfrak{g}^\vee}(a_1)$. Equation (4.5.1) shows that the affine space $h_1 + Z_{\mathfrak{g}^\vee}(a_1)$ is stable under the adjoint A -action, and that A acts transitively on $h_1 + Z_{\mathfrak{g}^\vee}(a_1)$.

Now, let $h \in \mathfrak{g}^\vee$ be the element arising from Proposition 4.1.1, which was considered at the beginning of the proof. We have: $[h, a_1] = 2 \cdot a_1 = [h_1, a_1]$. Hence, $h \in h_1 + Z_{\mathfrak{g}^\vee}(a_1)$. By the preceding paragraph, one can find an element $u \in A$ such that $h = u \cdot h_1 \cdot u^{-1}$. Conjugating the homomorphism j by the element u , we obtain a new homomorphism $j' : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{g}^\vee$ such that $j' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$. \square

4.6 Proof of Proposition 1.10.4

Let A be the isotropy group of the principal nilpotent $n \in \mathfrak{g}^\vee$ so that $\text{Lie } A = Z_{\mathfrak{g}^\vee}(n)$. Given a locally-free sheaf \mathcal{V} on \mathcal{N} , let \mathcal{V}_n denote its geometric fibre at n . The assignment $\mathcal{V} \rightarrow \mathcal{V}_n$ gives rise to a functor from the category of

G^\vee -equivariant sheaves on \mathcal{N} to the category of A -modules. This way we obtain a natural morphism ($\mathfrak{a} := \text{Lie } A$):

$$\text{Hom}_{\mathcal{C}oh(\mathcal{N})}(\mathcal{V}_\lambda, \mathcal{V}_\mu) \rightarrow \text{Hom}_{\mathfrak{a}}(V_\lambda, V_\mu) \quad (4.6.0)$$

We'll now show that (4.6.1) is an isomorphism.

Identify sheaves on \mathcal{N} with the corresponding $\mathbb{C}[\mathcal{N}]$ -modules of global sections and note that:

$$\text{Hom}_{\mathbb{C}[\mathcal{N}]}(V_\lambda \otimes \mathbb{C}[\mathcal{N}], V_\mu \otimes \mathbb{C}[\mathcal{N}]) \cong V_\lambda^* \otimes V_\mu \otimes \mathbb{C}[\mathcal{N}]$$

Hence, morphism (4.6.1) turns into the map (= restriction to n):

$$(V_\lambda^* \otimes V_\mu \otimes \mathbb{C}[\mathcal{N}])^{G^\vee} \rightarrow (V_\lambda^* \otimes V_\mu)^\mathfrak{a}$$

Note that the highest weight of any irreducible constituent $V \subset V_\lambda^* \otimes V_\mu$ belongs to the root lattice of G^\vee , for orbits O_λ and O_μ are assumed to be in the same connected component of Gr . But for any such module V , Kostant proved in [Ko2] that the restriction map: $(V \otimes \mathbb{C}[\mathcal{N}])^{G^\vee} \rightarrow V^\mathfrak{a}$ is an isomorphism. \square

5 Filtration and the q -analogue of the weight multiplicity

5.1

In this section we extend results of section 4 to the equivariant setting and study their relationship with the fixed point decomposition introduced in section 3.

5.2

Fix an arbitrary element $t \in \mathfrak{t}$. We can identify the categories $P(Gr)$ and $\text{Rep}(G^\vee)$ in such a way that the specialized equivariant cohomology functor H_t on $P(Gr)$ corresponds to the forgetful functor on $\text{Rep}(G^\vee)$. The canonical filtration on H_t (see (8.2.1)) gives rise to a certain increasing filtration $W_\bullet(V)$ on the underlying vector space of any representation $V \in \text{Rep}(G^\vee)$.

Recall the element h defined in (1.6.2). The following result is an equivariant analogue of Theorem 1.6.3.

Theorem 5.2.1 *The filtration W_\bullet coincides with the filtration by the eigenvalues of h , i.e. the space $W_i(V)$ is spanned by all those eigenspaces of h whose eigenvalues are $\leq i$.*

First, we prove the following weakening of the theorem:

Lemma 5.2.2 *There exists a semisimple element h_1 such that the filtration W_\bullet coincides with the filtration by the eigenvalues of h_1 .*

Proof of Lemma: Using the equivalence of categories we can (and will) view the functor H_T^* from n° 3.4 as fiber functor on $\text{Rep}(G^\vee)$. By Proposition 7.2.1, there exists a \mathbb{C}^* -equivariant principal G^\vee -bundle P on \mathfrak{t} such that the functor H_T^* is isomorphic to the functor Γ_P . Hence, the functor H_t is isomorphic to the functor: $V \mapsto$ the fibre of $P \times_{G^\vee} V$ at t .

Restrict the bundle P to the 1-dimensional subspace $\mathbb{C} \cdot t \subset \mathfrak{t}$. We obtain a \mathbb{C}^* -equivariant principal G^\vee -bundle on the line. Now, Proposition 7.3.3 completes the proof. \square

Let \mathfrak{n} denote the nilpotent subalgebra of \mathfrak{g}^\vee spanned by the eigenspaces of the operator $\text{ad } h_1$ corresponding to all negative eigenvalues. Observe that the element h_1 in Lemma 5.2.2 is only determined up to a summand in \mathfrak{n} .

Proof of Theorem 5.2.1: The choice of h_1 gives an isomorphism: $V \cong \text{gr}^W V$, for each $V \in \text{Rep}(G^\vee)$. On the other hand, the associated graded of the functor H_t with respect to the canonical filtration is isomorphic to H^* , the ordinary cohomology functor (see (8.5.2)). Thus, the natural gradation on H^* corresponds, by Lemma 5.2.2, to the gradation on V by the eigenvalues of h_1 . Now, Theorem 1.6.3 yields $h_1 \equiv h \pmod{\mathfrak{n}}$. Since h is regular, we see that \mathfrak{n} is the nilradical of a Borel subalgebra of \mathfrak{g}^\vee . Furthermore, there is an element $u \in \exp \mathfrak{n}$ such that $h_1 = u \cdot h \cdot u^{-1}$. But, conjugating by u does not affect the filtration by the eigenvalues. Thus, we may assume that $h_1 = h$. \square

5.3

We now turn to an equivariant analogue of Theorem 1.7.6, still assuming that t is an arbitrary element of \mathfrak{t} .

First, note that $H_t(\Omega)$, the equivariant cohomology of the Loop group Ω , has a natural structure of a commutative and co-commutative Hopf algebra.

Moreover, one can show by a standard argument that this algebra is isomorphic to the symmetric algebra over the space prim_t of its primitive elements. Further, the construction of primitive elements, given in n° 1.7, carries over to the equivariant setup so that to any $\text{Ad } K$ -invariant polynomial P on $\text{Lie } K$ one can associate a cohomology class $\alpha_T(P) \in H_T^*(\Omega)$. Furthermore, this class specializes to a primitive class $\alpha_t(P) \in H_t(\Omega)$ and the elements $\alpha_t(P)$ form a basis of prim_t when P runs over the set of primitive generators of the algebra of invariant polynomials.

Next, the element t may be viewed as a point of $(\mathfrak{t}^\vee)^*$. Hence, it determines a closed coadjoint orbit $O(t) \subset (\mathfrak{g}^\vee)^*$. Let O be the unique *regular* coadjoint orbit in $(\mathfrak{g}^\vee)^*$ which contains the orbit $O(t)$ in its closure. We fix a point $x \in O$ and let $\mathfrak{a}_t \subset \mathfrak{g}^\vee$ be the Lie algebra of isotropy group of x (under the coadjoint action). To any $\text{Ad } G^\vee$ -invariant polynomial P^\vee on $(\mathfrak{g}^\vee)^*$ one can associate the element of \mathfrak{a}_t defined by $a_t(P^\vee) := dP^\vee(x)$ (cf. n° 1.7).

It is convenient to fix a Killing form on \mathfrak{g} . This form gives rise to an invariant quadratic polynomial on \mathfrak{g} , hence, to a cohomology class $\alpha_T \in H_T^2(\Omega)$, hence to a primitive class $\alpha_t \in \text{prim}_t$. On the other hand, the Killing form gives rise to the linear function (t, \cdot) on \mathfrak{t} . We let t^\vee denote the element of \mathfrak{t}^\vee corresponding to this function via the identification: $\mathfrak{t}^* \cong \mathfrak{t}^\vee$. Further, we pick up a principal nilpotent n_t in $Z_{\mathfrak{g}^\vee}(t^\vee)$ and set $a_t := t^\vee + n_t$. The algebra \mathfrak{a}_t introduced above is (up to conjugation) just the centralizer of a_t in \mathfrak{g}^\vee . So, we assume that $a_t \in \mathfrak{a}_t$.

Recall the bijective correspondence $P \leftrightarrow P^\vee$ between invariant polynomials on \mathfrak{g} and \mathfrak{g}^\vee , see (1.7.3).

Theorem 5.3.1 *The natural action of $\alpha_t(P) \in \text{prim}_t$ on $H_t(IC_\lambda)$, $\lambda \in X_*(T)$, corresponds to the action of $a_t(P^\vee) \in \mathfrak{a}_t$ in the representation V_λ . In particular the class α_t corresponds to the element a_t .*

Corollary 5.3.2 *There is a natural Hopf algebra isomorphism: $H_t(\text{Gr}) \cong U(\mathfrak{a}_t)$.*

Remark. We defined the algebra \mathfrak{a}_t as the Lie algebra of $G^\vee(x)$, the isotropy group of an element $x \in O$. Corollary 5.3.2 says that there is a natural isomorphism between the family $\{\mathfrak{a}_t\}$ and the family $\{\text{prim}_t\}$.

To prove Theorem 5.3.1 we study first the restriction of T -equivariant cohomology classes of the Grassmannian to the lattice $X_*(T) \subset \text{Gr}$. For any point $\lambda \in X_*(T)$ we have: $H_T^*(\lambda) = \mathbb{C}[\mathfrak{t}]$, for λ is a T -fixed point. Hence,

the space $H_T^*(X_*(T))$ may be identified with the space of \mathbb{C} -valued function on $\mathfrak{t} \times X_*(T)$, polynomial with respect to the \mathfrak{t} -factor.

Proposition 5.3.3 *Let P be an invariant polynomial on \mathfrak{g} and $\alpha_T(P) \in H_T^*(\text{Gr})$, the corresponding cohomology class. Then, the restriction of that class to $X_*(T)$ is given by the following function on $\mathfrak{t} \times X_*(T)$:*

$$(t, \lambda) \longmapsto \langle dP(t), \lambda \rangle$$

Here $dP(t)$ stands for the differential at the point t of the restriction of the polynomial P to \mathfrak{t} , and λ is viewed as a vector in \mathfrak{t} , so that the function: $\lambda \mapsto \langle dP(t), \lambda \rangle$ is a linear function.

Proof: Cohomology classes $\alpha_T(P)$ were constructed in n. 1.7 by means of a certain principal G -bundle on $S^2 \times \Omega$. Let \tilde{L} be its restriction to $S^2 \times X_*(T)$. It follows from the construction that the structure group of the bundle \tilde{L} can be reduced from G to T . Hence, the restrictions of the classes $\alpha_T(P)$ to $X_*(T)$ come from a T -bundle L on $S^2 \times X_*(T)$ such that $\tilde{L} = G \times_T L$. Thus, we can forget about the group G and concentrate our attention on the T -bundle L which corresponds, algebraically, to restricting polynomials P to the Cartan subalgebra \mathfrak{t} .

Now, pick up some $\lambda \in X_*(T)$ and let L_λ be the restriction of the bundle L to $S^2 \times \{\lambda\}$. Using a universal T -bundle: $ET \rightarrow BT$, we form the diagram:

$$BT \xleftarrow{\pi} ET \times_T L_\lambda \xrightarrow{p} T \setminus L_\lambda \cong S^2 \quad (5.3.0)$$

where the projection π is a fibration with fiber L_λ and the projection p is a fibration with fiber ET . There is a natural T -action on the space $ET \times_T L_\lambda$ arising from the T -action on L_λ (we used here that T is an abelian group). Coupling the latter T -action with the second copy of ET we obtain from (5.3.4) a diagram:

$$BT \xleftarrow{\tilde{\pi}} ET \times_T (ET \times_T L_\lambda) \xrightarrow{\tilde{p}} BT \times (T \setminus L_\lambda) \cong BT \times S^2 \quad (5.3.0)$$

The map $\tilde{\pi}$ here forgets about the new factor ET , while the map \tilde{p} projects this new factor ET onto BT . The fiber of the projection \tilde{p} remains the same as that of the projection p in (5.3.4), i.e. is isomorphic to ET . The space ET being contractible, we see that \tilde{p} is a homotopy equivalence. Hence, diagram (5.3.5) gives rise to a morphism of cohomology:

$$H^*(BT) \xrightarrow{\tilde{\pi}^*} H^*(ET \times_T (ET \times_T L_\lambda)) \simeq H^*(BT \times S^2) \quad (5.3.0)$$

Upon substituting $H_*(BT) = \mathbb{C}[\mathfrak{t}]$ we finally obtain a map $c_\lambda : \mathbb{C}[\mathfrak{t}] \rightarrow \mathbb{C}[\mathfrak{t}] \otimes H^*(S^2)$. One sees from definitions that the restriction of the class $\alpha_T(P)$ to $\mathfrak{t} \times \{\lambda\}$ equals the integral of the element $c_\lambda(P) \in \mathbb{C}[\mathfrak{t}] \otimes H^*(S^2)$ over the 2-sphere.

To complete the proof of the Proposition, it suffices to prove the following formula for the map c_λ :

$$c_\lambda(P) = P \otimes 1 + \langle dP, \lambda \rangle \otimes u, \quad P \in \mathbb{C}[\mathfrak{t}] \quad (5.3.0)$$

where u denotes a generator of $H^2(S^2, \mathbb{Z})$. To prove this formula write: $c_\lambda(P) = c'_\lambda(P) \otimes 1 + c''_\lambda(P) \otimes u$. We first compute c'_λ . To that end, pick up a point pt in S^2 and compose all the morphisms (5.3.6) with the restriction morphism: $H^*(BT \times S^2) \rightarrow H^*(BT \times pt) = H^*(BT)$. The resulting morphism is induced by the following diagram of maps:

$$BT \xleftarrow{p_1} (ET \times ET)/T \xrightarrow{p_2} BT$$

where p_1, p_2 denote the first and the second projections. One shows easily that this diagram induces the identity morphism on cohomology. Hence, $c'_\lambda(P) = P$.

To proceed further, we may assume without loss of generality that $T = S^1$, so that $H^*(BT) = \mathbb{C}[v]$ where v is a generator of $H^2(BT, \mathbb{Z})$. Note next that both maps (5.3.6) and (5.3.7) are *algebra* homomorphisms. Hence, we have only to check that these maps coincide on the generator v . But $c''_\lambda(v)$ is a scalar. To find it, we pick up a point pt in BT and compose the morphism (5.3.6) with the restriction morphism: $H^*(BT \times S^2) \rightarrow H^*(pt \times S^2) = H^*(S^2)$. The resulting morphism is induced by the diagram (5.3.5). Now, one verifies easily that this diagram sends the generator $v \in H^2(BT)$ to $m \cdot u$ where the integer m is the degree of the homomorphism $\lambda : S^1 \rightarrow T = S^1$. Thus $c''_\lambda(v) = m$, in agreement with (5.3.7). \square

Proof of theorem 5.3.1: The argument of n. 4.2 shows that the action of the abelian Lie algebra $prim_t$ on the equivariant cohomology of complexes $M \in P(Gr)$ gives rise to a tensor functor from the category $P(Gr)$ to the category of $prim_t$ -modules. That gives, via Theorem 1.4.1, a tensor functor on the category $\text{Rep}(G^\vee)$. Proposition 7.1.1 now shows that, for each $a \in prim_t$, the endomorphism $\exp a$ on the underlying vector space of any representation is induced by the action of a uniquely determined element of the group G^\vee . By differentiation, one gets a Lie algebra homomorphism:

$$\varphi_t : prim_t \longrightarrow \mathfrak{g}^\vee \quad (5.3.0)$$

The family $\{\varphi_t\}$ algebraically depends on the parameter $t \in \mathfrak{t}$.

Recall the canonical filtration on $H_t(\text{Gr})$ and for any $a \in \text{prim}_t$ let $\bar{a} \in \text{gr}_{\bullet}^W H_t(\text{Gr})$ denote the “principal symbol” of a , viewed as an ordinary cohomology class, cf. (8.5.2). We can find a complex $IC_\lambda \in P(\text{Gr})$ such that the element \bar{a} acts on $H^*(IC_\lambda)$ as a non-zero operator (Proposition 4.3.2). Then, a acts on $H_t(IC_\lambda)$ as a non-zero operator, for $H^*(IC_\lambda) = \text{gr}_{\bullet}^W H_t(IC_\lambda)$. Hence, morphism (5.3.8) is injective.

We will show later (in the course of the proof of Proposition 5.4.1) that $\varphi_t(\alpha_t)$, the image of the cohomology class corresponding to the Killing form on \mathfrak{g} , is a regular element of \mathfrak{g}^\vee . Hence, $Z_{\mathfrak{g}^\vee}(\varphi_t(\alpha_t))$, the centralizer of $\varphi_t(\alpha_t)$ in \mathfrak{g}^\vee , has dimension = $\text{rk } \mathfrak{g}^\vee$. We find that $\dim Z_{\mathfrak{g}^\vee}(\varphi_t(\alpha_t)) = \dim(\text{prim}_t) = \dim \varphi_t(\text{prim}_t)$. But the image of morphism (5.3.8), being an abelian subalgebra of \mathfrak{g}^\vee , is contained in $Z_{\mathfrak{g}^\vee}(\varphi_t(\alpha_t))$. Thus, $\varphi_t(\text{prim}_t) = Z_{\mathfrak{g}^\vee}(\varphi_t(\alpha_t))$.

Assume first that t is a regular element in \mathfrak{t} . Then, for each complex $M \in P(\text{Gr})$, we have the fixed-point decomposition 3.6.1 on $H_t(M)$. Any cohomology class in $H_t(\text{Gr})$ acts on $H_t(M)$ as a diagonal operator with respect to this decomposition. Moreover, the class $\alpha_t(P)$ associated to an invariant polynomial P acts on $H_t(M)$ as the semisimple element $dP^\vee(t) \in \mathfrak{t}^\vee$ (by Proposition 5.3.3). This proves the Theorem for all regular $t \in \mathfrak{t}$. For an arbitrary t , by continuity, we have: $\varphi_t(\alpha_t(P)) = dP^\vee(x)$ where $x \in (\mathfrak{g}^\vee)^*$ is an element whose semisimple part belongs to the orbit $O(t)$. But if P is the Killing form, then $\varphi_t(\alpha_t(P))$ was shown to be a regular element. Hence, the point x must be regular and the Theorem follows. \square

Remark. For $t = 0$ Theorem 5.3.1 yields Theorem 1.7.2.

5.4

We shall now analyze the relative position (cf. remark 1.6.4) of the fixed point decomposition 3.6.1 with respect to the canonical filtration on equivariant cohomology $H_t(M)$, $M \in P(\text{Gr})$. In fact, we shall do this simultaneously for all M by finding the relative position inside \mathfrak{g}^\vee of the subalgebra $\mathfrak{a}_t = \text{the image of the homomorphism (5.3.8)}$ with respect to the element h arising from Theorem 5.2.1 (note that the *absolute* position of either \mathfrak{a}_t or h inside \mathfrak{g}^\vee makes no sense since these objects are only determined up to conjugacy).

Recall that the choice of the Killing form on \mathfrak{g} determines a semisimple conjugacy class $O(t^\vee)$ in \mathfrak{g}^\vee and a cohomology class α_t and, hence, its image $a_t \in \mathfrak{a}_t$. We choose a representative $t^\vee \in O(t^\vee)$ which commutes with h .

Note also that conjugating h by an element of $\exp \mathfrak{n}$ (see n° 5.2) doesn't affect the filtration W_\bullet .

The relative position of the element a_t (and, hence, of the subalgebra \mathfrak{a}_t , i.e., the centralizer of a_t) with respect to h is described by the following

Proposition 5.4.1 *Conjugating h by an element of the group $\exp \mathfrak{n}$, if necessary, one can get: $a_t = n + t^\vee$.*

Proof: On the space \mathfrak{g}^\vee of the adjoint representation we have the canonical filtration W_\bullet such that $h \in W_0(\mathfrak{g}^\vee)$ and $\mathfrak{n} = W_{-2}(\mathfrak{g}^\vee)$ (the filtration is indexed by even integers because of Lemma 4.1.3). We know also that $a_t \in W_2(\mathfrak{g}^\vee)$ and Theorem 1.7.2, combined with Theorem 5.2.1, yields: $a_t \equiv n \pmod{W_0(\mathfrak{g}^\vee)}$. Hence, $a_t \in n + W_0(\mathfrak{g}^\vee)$. We set $\mathfrak{b} = W_0(\mathfrak{g}^\vee)$. This is clearly a Borel subalgebra of \mathfrak{g}^\vee .

Let us compare the elements a_t and $n + t^\vee$. Both of them belong to $n + \mathfrak{b}$ and, hence, are regular (see [Ko3, p.108]). The semisimple part of $n + t^\vee$ is clearly G^\vee -conjugate to t^\vee . The same is true for a_t by Theorem 5.3.1. Hence, a_t and $n + t^\vee$ are G^\vee -conjugate. But, Kostant showed in the course of the proof of [Ko3, Theorem 1.2] that any two G^\vee -conjugate elements of $n + \mathfrak{b}$ are conjugate by an element $u \in \exp \mathfrak{n}$. Hence, we have: $a_t = u(n + t^\vee)u^{-1}$. Replacing h by $u \cdot h \cdot u^{-1}$ completes the proof. \square

5.5

Let us now choose the particular element $t \in \mathfrak{t}$ so that $t^\vee = h$. Hence, $a_t = n + h$ is regular and \mathfrak{a}_t is a Cartan subalgebra of \mathfrak{g}^\vee .

Given a representation $V \in \text{Rep}(G^\vee)$ and $\mu \in X_*(T^\vee)$, let $V(\mu)$ denote the weight subspace of V of weight μ with respect to the Cartan subalgebra \mathfrak{a}_t . Further, let

$$V_i(\mu) := \ker(n^{i+1}) \cap V(\mu), \quad i = 0, 1, 2, \dots$$

be the filtration on $V(\mu)$ by the kernels of powers of the n -action on V . On the other hand, let $W_\bullet(V(\mu))$ denote the filtration on $V(\mu)$ induced by restriction of the W -filtration on V .

We have the following elementary result.

Lemma 5.5.1 *The filtrations $W_\bullet(V(\mu))$ and $V_\bullet(\mu)$ coincide up to shift; more precisely, one has:*

$$V_i(\mu) = W_{2i+\mu(h)}(V(\mu)) = W_{2i+\mu(h)+1}(V(\mu)).$$

Proof: Set $u = \exp n$. Then, $a_t = h + n = u \cdot h \cdot u^{-1}$. For $v \in V(\mu)$ and any $i \geq 0$ we have:

$$h \cdot (u^{-1} \cdot n^i \cdot v) = (u^{-1} \cdot a_t \cdot u) \cdot u^{-1} \cdot n^i \cdot v = (\mu(h) + 2i) \cdot (u^{-1} \cdot n^i \cdot v),$$

so that $u^{-1} \cdot n^i \cdot v \in W_{2i+\mu(h)}(V(\mu))$. Now, if $v \in V_k(\mu)$, then:

$$v = u^{-1} \cdot u \cdot v = u^{-1} \cdot v + u^{-1} \cdot n \cdot v + \dots + (1/k!) \cdot u^{-1} \cdot n^k \cdot v$$

The last sum clearly belongs to $W_{2k+\mu(h)}(V(\mu))$, hence, $V_k(\mu) \subset W_{2k+\mu(h)}(V(\mu))$. The opposite inclusion is proved in a similar way. \square

Now let $M = \mathcal{P}(V) \in P(Gr)$ be the perverse sheaf corresponding to a representation $V \in \text{Rep}(G^\vee)$. Recall that the weight decomposition of V with respect to the Cartan subalgebra \mathfrak{a}_t corresponds to the fixed point decomposition (3.7.1) of the equivariant cohomology $H_t(M)$ (via the identification: $V \cong H_t(\mathcal{P}(V))$). Further, the canonical filtration W_\bullet on $H_t(M)$ induces a filtration (which we call ‘‘canonical’’ and denote by W_\bullet again) on each direct summand of the decomposition (3.7.1). Let $M_\mu := H_t(i_\mu^! M)$ denote the summand corresponding to a point $\mu \in X_*(T)$. Theorem 5.2.1, and Proposition 5.4.1, combined with Lemma 5.5.1, yield the following result.

Proposition 5.5.2 *For any $\mu \in X_*(T)$, the canonical filtration on $\mathcal{P}(V)_\mu$ corresponds, up to the shift $\mu(h)$, to the filtration $V_\bullet(\mu)$ on the weight subspace $V(\mu)$. \square*

5.6 Proof of Theorem 1.11.2

Let $V \in \text{Rep}(G^\vee)$ and $M = \mathcal{P}(V) \in P(G^\vee)$. By Proposition 5.5.2 we have:

$$P_\mu(V, q) = \sum_i q^{2i} \cdot \dim(W_{2i+\mu(h)}(M_\mu)/W_{2i+\mu(h)-2}(M_\mu)) \quad (5.6.0)$$

Now, let $i_\mu : \{\mu\} \hookrightarrow \text{Gr}$ denote the embedding. We need the following result which will be proved in the next n° .

Proposition 5.6.2 *The natural morphism: $H_t(i_\mu^! M) \rightarrow H_t(M)$ is strictly compatible with the canonical filtration, for any $M \in P(Gr)$.*

The Proposition implies that the filtration W_\bullet on M_μ corresponds to the canonical filtration on $H_t(i_\mu^! M)$. Next, note that we have a canonical isomorphism: $H_T^*(i_\mu^! M) \cong H^*(i_\mu^! M) \otimes \mathbb{C}[\mathfrak{t}]$ since μ is a fixed point. Evaluation

at $t \in \mathfrak{t}$ yields an isomorphism: $H_t(i_\mu^! M) \cong H^*(i_\mu^! M)$, and it is clear that the canonical filtration on the left-hand side corresponds to the filtration by degree on the right-hand side. Thus, we obtain from (5.6.1), that:

$$P_\mu(V, q) = \sum_i q^{2i} \cdot \dim H^{2i+\mu(h)}(i_\mu^! M) \quad (5.6.0)$$

Assume now that $V = V_\lambda$ is an irreducible representation. Then, $M = IC_\lambda$. The dimensions of stalks of the cohomology sheaves of the complexes IC_λ were computed in [KL]. It was proved there that:

$$\sum_{j \geq 0} q^j \cdot \dim H^j(i_\mu^! IC_\lambda) = q^{\lambda(h)} \cdot P_{\mu, \lambda}(q^2), \quad (5.6.0)$$

where $P_{\mu, \lambda}$ are the Kazhdan-Lusztig polynomials. Inserting (5.6.4) into (5.6.3) completes the proof of the Theorem. \square

5.7

To start proof of Proposition 5.6.2 we need some preparations. First, note that the torus $T \subset G$ can be embedded into a bigger torus \hat{T} , the maximal torus of the semidirect product $\mathbb{C}^* \ltimes LG$, where \mathbb{C}^* acts on LG via ‘rotation of the loop’ (see e.g. [AP]). We have $\hat{T} = \mathbb{C}^* \times T$. Observe that, the subgroup L^+G being \mathbb{C}^* -stable with respect to the rotation of the loop action, the torus \hat{T} acts on $\text{Gr} = LG/L^+G$ in a natural way. Moreover, for any point $\mu \in X_*(T)$, one can find a one-parameter subgroup $\mathbb{C}^* \hookrightarrow \hat{T}$ that contracts a neighbourhood of μ to μ as the parameter $z \in \mathbb{C}^*$ in the subgroup tends to zero. Given μ , fix such a subgroup \mathbb{C}^* .

Further, let L^-G be the subgroup of LG formed by the loops $f : \mathbb{C}^* \rightarrow G$ that are regular at $z = \infty$ and such that $f(\infty) = 1$. The subgroup L^-G should be thought of as ‘complementary’ to L^+G , cf. §8.4. In particular, L^-G -orbits form an (infinite dimensional) cell decomposition of Gr which is ‘transverse’, in a sense, to the stratification by the L^+G -orbits.

Fix a point $\mu \in X_*(T)$ and write $L^-G \cdot \mu$ for the L^-G -orbit through μ . Let O_λ be some other L^+G -orbit in Gr such that $\mu \in \overline{O_\lambda}$. We write $\{U_i\}$ for the collection of the intersections of $L^-G \cdot \mu$ with all the L^+G -orbits O_i that contain μ in its closure and such that $O \subset \overline{O_\lambda}$. Enumerating the pieces U_i in such a way that the dimensions of the U_i form a non-decreasing sequence, we obtain a finite partition: $\overline{O_\lambda} = \bigcup_{i=1}^n U_i$ with the properties (i)-(v) listed below.

- (i). Each U_i is a \hat{T} -stable locally-closed (singular) subvariety of \overline{O}_λ ;
- (ii). U_i contains a single \hat{T} -fixed point, say μ_i , and $\mu_n = \mu$;
- (iii). The subgroup $\mathbb{C}^* \subset \hat{T}$ contracts U_i to μ_i ;
- (iv). The subvariety $Y_k := \bigcup_{i \leq k} U_i$ is closed in \overline{O}_λ (for each $k = 1, 2, \dots, n$) and U_k is a Zariski-open part of Y_k .

Let us view the complex IC_λ as a pure Hodge module (of weight 0, say) in the sense of Saito [Sa]. In addition to the properties (i)–(iv) above one has, due to transversality of L^-G - and L^+G -orbits:

- (v). The restriction of the complex IC_λ to any U_i is pure.

We have following inclusions:

$$Y_k \xrightarrow{j_k} \overline{O}_\lambda \quad , \quad Y_{k-1} \xrightarrow{v} Y_k \xleftarrow{u} U_k$$

Set $L_k := j_k^* IC_\lambda$, $k = 1, 2, \dots, n$.

Lemma 5.7.1 *The equivariant cohomology: (a) $H_T^*(u_! \cdot u^! L_k)$ and (b) $H_T^*(L_k)$ are pure and free $\mathbb{C}[\mathfrak{t}]$ -modules.*

Proof: (a) Let $i_k : \{\mu_k\} \rightarrow Y_k$ denote the inclusion. We have an isomorphism:

$$H^*(u_! \cdot u^! L_k) \cong H^*(i_k^! L_k) \tag{5.7.0}$$

which is due to the fact that L_k is a \mathbb{C}^* -equivariant complex on U_k and the \mathbb{C}^* -action contracts U_k to the point μ_k . But the functor $H^* u_!$ decreases the weights, the functor $H^* i_k^!$ increases weights, and the complex $u^! L_k$ is pure by the property (v) above. Hence, both sides of (5.7.2) are pure.

Further, the classifying space BT can be represented as direct limit of pure varieties (\cong product of copies of $\mathbb{C}\mathbb{P}^d$). Hence, the spectral sequence for the equivariant cohomology (8.1.1) yields an equivariant analogue of (5.7.2) and shows that all the equivariant cohomology involved there are pure. Finally, $H_T^*(i_k^! L_k)$ is obviously a free $\mathbb{C}[\mathfrak{t}]$ -module since the complex $i_k^! L_k$ is supported on a fixed point. This completes the proof of part (a) of the Lemma.

Part (b) follows from part (a) by induction on k using the following long exact sequence of equivariant cohomology:

$$\dots \rightarrow H_T^*(u_! \cdot u^! L_k) \rightarrow H_T^*(L_k) \rightarrow H_T^*(L_{k-1}) \rightarrow \dots \tag{5.7.0}$$

□

Proof of Proposition 5.6.2: First, note that all terms in the long exact sequence (5.7.3) are pure, by Lemma 5.7.1. Hence, all the connecting homomorphisms in (5.7.3) vanish so that the long exact sequence breaks up into short exact sequences. Moreover, these exact sequences are *split* as sequences of $\mathbb{C}[\mathfrak{t}]$ -modules, since $H_T^*(L_{k-1})$ is a free $\mathbb{C}[\mathfrak{t}]$ -module, by Lemma 5.7.1(b). Thus, the map: $H_T^*(u_! \cdot u^! L_k) \rightarrow H_T^*(L_k)$ is an injection onto a direct summand. Note that this injection can be identified with the natural morphism: $H_T^*(i_k^! L_k) \rightarrow H_T^*(L_k)$, due to an equivariant analogue of isomorphism (5.7.2).

Now, put $k = n$ in the above argument. Then, we have $\mu_n = \mu$ (property (ii)), $i_n = i_\mu$, $L_n = IC_\lambda$. Thus, the $\mathbb{C}[\mathfrak{t}]$ -module $H_T^*(i_\mu^! IC_\lambda)$ maps injectively onto a direct summand of $H_T^*(IC_\lambda)$. The Proposition follows. □

6 Moduli spaces and Hecke operators

The purpose of this chapter is to give some definitions and constructions that will put informal arguments of n. 1.5 on a solid mathematical basis. In particular, we give proof of the ‘well-known’ double-coset construction of the moduli space of G -bundles in terms of loop groups, and also give meaning to the Poincaré duality for the Grassmannian Gr , used in the proof of Proposition 4.3.3.

The constructions below have been clarified during my talks with A. Beilinson. I am glad to express to him my deep gratitude. I am also indebted to R. Bezrukavnikov for streamlining the argument in proof of theorem 6.3.1.

6.1 A formal construction

Assume given the following set of data:

- 6.1(i) A partially ordered inductive set D (“inductive” means that, for any $\alpha, \beta \in D$, there exists $\gamma \in D$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$;
- 6.1(ii) A collection $\{\mathcal{M}_{\mu/\lambda}\}$ of *smooth* algebraic varieties $\mathcal{M}_{\mu/\lambda}$ indexed by all pairs $(\lambda, \mu) \in D \times D$ such that $\lambda \leq \mu$;
- 6.1(iii) Zariski-open imbedding $j_{\alpha,\beta} : \mathcal{M}_{\gamma/\alpha} \hookrightarrow \mathcal{M}_{\gamma/\beta}$, for each triple $\alpha, \beta, \gamma \in D$ such that $\alpha < \beta \leq \gamma$;

6.1(iv) A smooth projection $p_{\nu,\mu} : \mathcal{M}_{\nu/\lambda} \longrightarrow \mathcal{M}_{\mu/\lambda}$ making $\mathcal{M}_{\nu/\lambda}$ an affine bundle over $\mathcal{M}_{\mu/\lambda}$ (i.e. $p_{\nu,\mu}$ is a locally-trivial fibration with affine linear space as fibre), for each triple $\lambda, \mu, \nu \in D$ such that $\lambda \leq \mu < \nu$.

The imbeddings $j_{\alpha,\beta}$ and the projections $p_{\nu,\mu}$ should satisfy the following two properties:

For any $\alpha < \beta < \gamma \leq \mu$, (resp. $\alpha \leq \lambda < \mu < \nu$), the following triangles commute:

$$\begin{array}{ccc}
 \mathcal{M}_{\mu/\alpha} & \xrightarrow{j_{\alpha,\gamma}} & \mathcal{M}_{\mu/\gamma} \\
 & \searrow j_{\alpha,\beta} & \nearrow j_{\beta,\gamma} \\
 & & \mathcal{M}_{\mu/\beta}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{M}_{\lambda/\alpha} & \xleftarrow{p_{\nu,\lambda}} & \mathcal{M}_{\nu/\lambda} \\
 & \nwarrow p_{\mu,\lambda} & \swarrow p_{\nu,\mu} \\
 & & \mathcal{M}_{\mu/\alpha}
 \end{array}
 \tag{6.1.0}$$

For any $\alpha < \beta \leq \lambda < \mu$ the following diagram is a cartesian square:

$$\begin{array}{ccc}
 \mathcal{M}_{\mu/\alpha} & \xrightarrow{j_{\alpha,\beta}} & \mathcal{M}_{\mu/\beta} \\
 p_{\mu,\lambda} \downarrow & & \downarrow p_{\mu,\lambda} \\
 \mathcal{M}_{\lambda/\alpha} & \xrightarrow{j_{\alpha,\beta}} & \mathcal{M}_{\lambda/\beta}
 \end{array}
 \tag{6.1.0}$$

Given such data, define an ‘‘infinite-dimensional variety’’ $\widehat{\mathcal{M}}$ as follows. For each $\alpha \in D$, the projections $p_{\nu,\mu} : \mathcal{M}_{\nu/\alpha} \longrightarrow \mathcal{M}_{\mu/\alpha}$, ($\mu \leq \nu$), form a projective system of affine bundles over $\mathcal{M}_{\alpha/\alpha}$, and we set $\widehat{\mathcal{M}}_\alpha := \varprojlim_{\alpha \leq \mu} \mathcal{M}_{\mu/\alpha}$.

Thus, $\widehat{\mathcal{M}}_\alpha$ is a pro-algebraic variety, a bundle over $\mathcal{M}_{\alpha/\alpha}$ whose fibre is a projective limit of affine linear spaces.

Further, the imbeddings $j_{\alpha,\beta} : \mathcal{M}_{\mu/\alpha} \longrightarrow \mathcal{M}_{\mu/\beta}$, ($\alpha \leq \beta$), give rise to an direct system of open imbeddings $\widehat{\mathcal{M}}_\alpha \hookrightarrow \widehat{\mathcal{M}}_\beta$. We set $\widehat{\mathcal{M}} = \varinjlim_{\alpha} \widehat{\mathcal{M}}_\alpha$ and

endow $\widehat{\mathcal{M}}$ with direct limit topology.

To get a better understanding of the structure of $\widehat{\mathcal{M}}$, fix two elements $\alpha, \beta \in D$ such that $\alpha < \beta$. We have the following commutative diagram:

$$\begin{array}{ccccc}
& & \widehat{\mathcal{M}}_\alpha & \xrightarrow{j} & \widehat{\mathcal{M}}_\beta & \\
& & \swarrow p_\alpha & & \downarrow p_\beta & \\
& & \mathcal{M}_{\alpha/\alpha} & & \mathcal{M}_{\beta/\beta} & \\
& & \searrow p & & \swarrow j_{\alpha,\beta} & \\
& & \mathcal{M}_{\beta/\alpha} & \xrightarrow{j_{\alpha,\beta}} & \mathcal{M}_{\beta/\beta} & \\
& & \longleftarrow p_{\beta,\alpha} & & &
\end{array} \tag{6.1.0}$$

In the diagram, the maps p_α , p_β and p are standard projections of a projective limit onto its components. Further, identify $\mathcal{M}_{\beta/\alpha}$ with a Zariski-open part of $\mathcal{M}_{\beta/\beta}$ via the imbedding $j_{\alpha,\beta}$. Then the projection p in (6.1.3) becomes the restriction to $\mathcal{M}_{\beta/\alpha}$ of the affine bundle $p_\beta : \widehat{\mathcal{M}}_\beta \rightarrow \mathcal{M}_{\beta/\beta}$. Thus, the structure of the imbedding $\widehat{\mathcal{M}}_\alpha \hookrightarrow \widehat{\mathcal{M}}_\beta$ is determined, up to fibre of the projection p_β (a non-essential infinite-dimensional affine space), by the low row of (6.1.3). In other words, to construct $\widehat{\mathcal{M}}_\beta$ one starts with $\mathcal{M}_{\alpha/\alpha}$, takes an affine bundle $\mathcal{M}_{\beta/\alpha}$ over $\mathcal{M}_{\alpha/\alpha}$, then attaches to it a closed subvariety ($= \mathcal{M}_{\beta/\beta} \setminus \mathcal{M}_{\beta/\alpha}$), and finally takes an infinite-dimensional affine bundle over the resulting space. It is clear from the description above that $\widehat{\mathcal{M}}_\alpha$ is an open part of $\widehat{\mathcal{M}}_\beta$ so that the $\widehat{\mathcal{M}}$ is the union of the family of increasing open pieces $\{\widehat{\mathcal{M}}_\alpha\}$.

6.1.4 Constructible complexes on $\widehat{\mathcal{M}}$.

For any $\lambda \in D$, define a triangulated category $D_\lambda^b(\widehat{\mathcal{M}})$ as the category formed by all families $\{F_{\alpha/\alpha} \in D^b(\mathcal{M}_{\alpha/\alpha}), \alpha \geq \lambda\}$ that satisfy the following compatibility condition

$$p_{\beta,\alpha}^*(F_{\alpha/\alpha}) \simeq j_{\alpha,\beta}^*(F_{\beta/\beta}) \quad \text{for any } \beta > \alpha \geq \lambda \tag{6.1.0}$$

(where the maps $p_{\beta,\alpha}$ and $j_{\alpha,\beta}$ are the same as in the low row of (6.1.3)).

For any pair $\lambda \leq \mu$ there is an obvious exact functor $D_\lambda^b(\widehat{\mathcal{M}}) \rightarrow D_\mu^b(\widehat{\mathcal{M}})$ assigning to a family $\{F_{\alpha/\alpha}, \alpha \geq \lambda\}$ its part with indices $\alpha \geq \mu$. So the categories $D_\lambda^b(\widehat{\mathcal{M}})$ form an ‘‘direct system’’ and we set $D^b(\widehat{\mathcal{M}}) = \varinjlim_\lambda D_\lambda^b(\widehat{\mathcal{M}})$.

Similar construction works for perverse sheaves. So, there is a well-defined notion of a perverse sheaf on $\widehat{\mathcal{M}}$.

Speaking formally, giving a constructible complex F on $\widehat{\mathcal{M}}$ requires giving a constructible complex $F_{\beta/\alpha} \in D^b(\mathcal{M}_{\beta/\alpha})$, one for each pair $\beta \geq \alpha (\geq \lambda)$,

so that these complexes should be compatible in a natural way with all diagrams (6.1.1) and (6.1.2). However, one can see easily that such a collection $\{F_{\beta/\alpha}\}$ is completely determined by its part $\{F_{\alpha/\alpha}, \alpha > \lambda\}$. Namely, given a collection $\{F_{\alpha/\alpha}\}$ satisfying (6.1.4), one can put $F_{\beta/\alpha} := p_{\beta,\alpha}^*(F_{\alpha/\alpha})$ and objects so defined will be automatically compatible with diagrams (6.1.1) and (6.1.2).

6.1.5 Homology and Cohomology of $\widehat{\mathcal{M}}$.

For any α , the system of affine fibrations $\mathcal{M}_{\mu/\alpha} \leftarrow \mathcal{M}_{\nu/\alpha}, (\mu \leq \nu)$, gives rise to an direct system of cohomology isomorphisms: $H^i(\mathcal{M}_{\mu/\alpha}) \xrightarrow{\sim} H^i(\mathcal{M}_{\nu/\alpha})$. We set $H^i(\widehat{\mathcal{M}}_\alpha) := \varinjlim H^i(\mathcal{M}_{\mu/\alpha})$. Now, the imbeddings $\widehat{\mathcal{M}}_\alpha \hookrightarrow \widehat{\mathcal{M}}_\beta$ give rise to a projective system of cohomology morphisms and we put $H^i(\widehat{\mathcal{M}}_\alpha) := \varprojlim_\alpha H^i(\widehat{\mathcal{M}}_\alpha)$.

Notice that there is an isomorphism $H^\bullet(\widehat{\mathcal{M}}_\alpha) \simeq H^\bullet(\mathcal{M}_{\alpha/\alpha})$ induced by the projection. So, in down to earth terms, giving a cohomology class of $\widehat{\mathcal{M}}$ amounts to giving a collection $\{c_\alpha \in H^\bullet(\mathcal{M}_{\alpha/\alpha})\}$ such that, for any pair $\beta > \alpha$, we have $p_{\beta,\alpha}^*(c_\alpha) = j_{\alpha,\beta}^*(c_\beta)$ where $p_{\beta,\alpha}$ and $j_{\alpha,\beta}$ are as in (6.1.3).

Similarly, one defines $H_k^{BM}(\widehat{\mathcal{M}})$, k -codimensional Borel-Moore homology group, to be formed by collections

$$\{c_\alpha \in H_{d_\alpha - k}^{BM}(\mathcal{M}_{\alpha/\alpha}), d_\alpha = \dim_{\mathbb{R}} \mathcal{M}_{\alpha/\alpha}, \alpha \in D\}$$

such that $p_{\beta,\alpha}^*(c_\alpha) = j_{\alpha,\beta}^*(c_\beta)$ (the pull-back morphism $p_{\beta,\alpha}^*$ is well defined for smooth maps and shifts degree by the fibre dimension).

6.2 Case of a group action

Let \mathcal{M} be an “infinite dimensional” space acted on by a “pro-algebraic” group L . More precisely, assume the following:

- 6.2(i) There is a (decreasing) family of normal subgroups $\{L^\alpha \subset L \mid \alpha \in D\}$ indexed by an inductive set D such that $\alpha \leq \beta \Rightarrow L^\beta \subset L^\alpha, (\alpha, \beta \in D)$, and such that $\bigcap_{\alpha \in D} L^\alpha = \{1\}$;
- 6.2(ii) There is an exhaustion of \mathcal{M} by open subsets $\{\mathcal{M}_\alpha \subset \mathcal{M} \mid \alpha \in D\}$, indexed by the same set D as in (i), such that $\alpha \leq \beta \Rightarrow \mathcal{M}_\alpha \subset \mathcal{M}_\beta$; and moreover $\bigcup_{\alpha \in D} \mathcal{M}_\alpha = \mathcal{M}$;

- 6.2(iii) For every $\alpha \in D$ the group L^α acts freely on \mathcal{M}_α and the orbit-space $L^\alpha \backslash \mathcal{M}_\alpha$ has the structure of a finite-dimensional smooth algebraic variety;
- 6.2(iv) For any $\alpha < \beta$ the quotient L^α/L^β has the structure of a finite dimensional *unipotent* algebraic group; furthermore the induced action of the group L^α/L^β on $L^\beta \backslash \mathcal{M}_\beta$ is algebraic.

Given a space \mathcal{M} with L -action satisfying conditions 6.2(i)–(iv) above, define a set of data 6.1(i)–6.1(iv) as follows. For each pair $\alpha \leq \beta$ put $\mathcal{M}_{\beta/\alpha} := L^\alpha \backslash \mathcal{M}_\beta$. Conditions 6.2(iii,iv) show that $\mathcal{M}_{\beta/\alpha}$ has the structure of a smooth algebraic variety. Furthermore, there are natural imbeddings $j_{\alpha,\beta} : L^\gamma \backslash \mathcal{M}_\alpha \hookrightarrow L^\gamma \backslash \mathcal{M}_\beta$, ($\alpha < \beta \leq \gamma$), and natural projections $p_{\nu,\mu} : L^\nu \backslash \mathcal{M}_\lambda \longrightarrow L^\mu \backslash \mathcal{M}_\lambda$, ($\lambda \leq \mu < \nu$). These maps clearly satisfy (6.1.1)–(6.1.2). Thus, we are in the setup of n. 6.1, so that the space $\widehat{\mathcal{M}}$ can be defined. Now, any point $x \in \mathcal{M}_\alpha$ gives a point in the projective limit $\widehat{\mathcal{M}}_\alpha = \varprojlim_{\mu} \mathcal{M}_{\mu/\alpha}$. This way one gets an imbedding $\mathcal{M} \hookrightarrow \widehat{\mathcal{M}}$ with dense

image. We endow \mathcal{M} with the topology induced from the topology on $\widehat{\mathcal{M}}$ via the imbedding.

The spaces \mathcal{M} and $\widehat{\mathcal{M}}$ are very much alike. For each $\alpha \in D$, for instance, there are projections $\mathcal{M}_\alpha \longrightarrow \mathcal{M}_{\alpha/\alpha}$ and $\widehat{\mathcal{M}}_\alpha \longrightarrow \mathcal{M}_{\alpha/\alpha}$. Both of them are affine fibrations with infinite-dimensional fibres. The only difference between them is that the fibre of the first fibration is isomorphic to the group L^α while the fibre of the second is isomorphic to $\varprojlim_{\beta} (L^\alpha/L^\beta)$, the completion of L^α .

Assume, in addition to properties 6.2(i)–(iv), that the following holds:

- 6.2(v) For each $\alpha \in D$, the group L/L^α has a structure of a finite-dimensional algebraic group and that structure is compatible with isomorphisms $L/L^\alpha \simeq (L/L^\beta)/(L^\alpha/L^\beta)$, for all pairs $\alpha < \beta$;
- 6.2(vi) For any $\alpha \in D$, \mathcal{M}_α is an L -stable subset of \mathcal{M} , and the induced L/L^α -action on $L^\alpha \backslash \mathcal{M}_\alpha$ is algebraic.

Given an L -action on \mathcal{M} satisfying properties 6.2(i)–(iv), one can make $L \backslash \mathcal{M}$, the orbit-space, into a stack. We will never use the stack approach, however, replacing it by the construction of n. 6.1 instead. For example, we *define* a perverse sheaf on $L \backslash \mathcal{M}$ to be an L -equivariant perverse sheaf

on \mathcal{M} , that is a collection $\{F_\alpha, \alpha \in D\}$ with isomorphisms of $p_{\beta,\alpha}^*(F_\alpha) \simeq j_{\alpha,\beta}^*(F_\beta)[\dim(L^\alpha/L^\beta)]$, for each pair $\alpha < \beta$, (cf. (6.1.4)).

6.3 Moduli space of G-bundles

Let X be a smooth complex connected algebraic curve. By a G -bundle we mean an algebraic principal G -bundle (with G -acting on the right) which is locally trivial in the étale topology on X .

Theorem 6.3.1 *Let X be a smooth affine complex algebraic curve and G a semisimple complex connected group. Then, any algebraic principal G -bundle on X is trivial.*

Remark 6.3.2 (i) *The theorem is false for general connected reductive groups. For example if $G = GL_1$, a G -bundle is a line bundle, and there are plenty of non-trivial line bundles on an affine curve.*

(ii) *If G is semisimple and simply-connected group then a version of Hilbert's 'theorem 90' conjectured by Serre in "Cohomologie Galoisienne" (ch. III.14) says that, for the field $\mathbb{C}(X)$ of rational functions on a curve X , we have*

$$H^1(\text{Gal}(\mathbb{C}(X)), G(\mathbb{C})) = 0 \quad (*)$$

This cohomology vanishing was proved, e.g., in [Bs]. It ensures that a G -bundle which is locally trivial in the étale topology is actually locally trivial in the Zariski topology. The same actually holds for a non-simply group G as well (as was pointed out to me by Telemann). Indeed, if G is semisimple, then $\pi_1(G)$ is a finite abelian group. In such a case, one has $H^2(X, \pi_1(G)) = 0$. Now let \tilde{G} is the simply connected cover of G . The following short exact sequence, viewed as a sequence of constant sheaves on X

$$0 \rightarrow \pi_1(G) \rightarrow \tilde{G} \rightarrow G \rightarrow 0$$

yields the long exact sequence of cohomology. The latter, combined with the vanishing of $H^2(X, \pi_1(G)) = 0$, shows that equation () holds for G provided it holds for \tilde{G} .*

Thus, we may (and will) assume below all G -bundles to be trivial at the generic point of the curve X .

(iii) *Originally, theorem 6.3.1 was deduced from a so-called 'strong approximation theorems' for adèles, cf. [Ha]. That approach gives a bit stronger result but seems to be less geometric.*

Proof of the theorem proceeds in several steps.

STEP 1. We need some simple general facts. Let X be an arbitrary smooth algebraic variety and G a linear algebraic group. Let H be an algebraic subgroup of G and P a principal G -bundle on X . We say that the structure group of P can be reduced from G to H if there is a principal H -bundle P' such that $P \simeq P' \times_H G$.

The following result can be easily derived from definitions.

Lemma 6.3.3 *The structure group of a G -bundle P on X may be reduced to H if and only if the associated bundle $P/H = P \times_G (G/H)$ has a regular global section. \square*

From lemma 6.3.3 one obtains in particular the following result.

Lemma 6.3.4 *Assume that H is a normal subgroup of G and P a G -bundle. If P/H is trivial as a G/H -bundle then P is trivial as a G -bundle. \square*

Now let $H = \mathbf{G}_a$ be the additive group. Then, H -bundles on X are classified by the first sheaf cohomology $H^1(X, \mathcal{O}_X)$. If X is affine, then this group vanishes, by Serre's theorem. It follows, that any \mathbf{G}_a -bundle on an affine algebraic variety is trivial. More generally, we have

Lemma 6.3.5 *Let H be a unipotent group. Then any H -bundle on an affine algebraic variety is trivial.*

Proof of lemma: We may choose a filtration $1 = H_0 \subset H_1 \subset \dots \subset H_n = H$ where the H_i are normal subgroups of H such that $H_i/H_{i-1} \simeq \mathbf{G}_a$ for all $i = 1, 2, \dots$. We prove by descending induction on i that P/H_i is trivial G/H_i -bundle. Assume, this is already proved for some i . To carry out the induction step observe first that since P/H_i is trivial, the structure group of P reduces to H_i , by lemma 1. Hence, we may assume that P is an H_i -bundle. Then the associated bundle P/H_{i-1} has structure group $H_i/H_{i-1} = \mathbf{G}_a$, hence is trivial, by the remark preceding the lemma. Applying lemma 6.3.4 again, we see that the structure group of P can be reduced from H_i to H_{i-1} . That completes the proof. \square

STEP 2. Now let G be a connected semisimple group and \mathcal{B} the Flag manifold for G , i.e. the variety of all Borel subgroups in G . Choosing a Borel subgroup $B \subset G$ yields a G -equivariant identification $\mathcal{B} \simeq G/B$. Recall that \mathcal{B} is a projective variety.

Lemma 6.3.6 *Assume that X is a smooth compact algebraic curve. Then, for any G -bundle P , the associated bundle $P \times_G \mathcal{B} = P/B$ has a regular section.*

Proof: By definition, the bundle P is trivial on U , a Zariski open subset of X . Hence, $(P \times_G \mathcal{B})|_U \simeq U \times \mathcal{B}$, so that there always exists a regular section $s : U \rightarrow P \times_G \mathcal{B}$ over U (e.g. a constant section). We claim that s can be extended to a regular section all over X . To that end, observe that $X \setminus U$ is a finite set of points, for X is 1-dimensional. Thus, we have to extend s to each point $x \in X \setminus U$ separately. The problem being local over a neighborhood of x , we may assume the bundle P to be trivial. Then s becomes a regular map from a punctured neighbourhood of x to \mathcal{B} . But any such map can be uniquely extended, by continuity, to the point x , for \mathcal{B} is a projective variety. \square

Lemma 6.3.7 *(theorem in rank 1 case). Any SL_2 - or PGL_2 -bundle on an affine curve is trivial.*

Proof: We view an SL_2 -bundle as a rank 2 vector bundle, V , with trivial determinant. By lemma 6.3.6, the structure group of this bundle can be reduced to the subgroup of upper-triangular 2×2 -matrices. Hence, V is an extension of line bundles $L \rightarrow V \rightarrow L'$. This extension is controlled by an element of the first cohomology group $H^1(X, \mathcal{H}om(L', L))$. The latter group vanishes, for X is affine. Hence, $V \simeq L \oplus L'$.

Since X is an affine curve, there exist regular sections s and s' of the line bundles L and L' respectively with disjoint zero sets. Then $s \oplus s'$ is a nowhere vanishing section of V . This section gives an imbedding of \mathcal{O}_X as a trivial subbundle in V . The quotient bundle, V/\mathcal{O}_X , has trivial determinant ($\det V = 1$), hence is itself trivial. Therefore, V is an extension of \mathcal{O}_X by \mathcal{O}_X . Such an extension splits by the same cohomology vanishing as above. This proves the lemma for SL_2 -bundles. The PGL_2 -case is reduced to the previous one by remark (ii) after the statement of the theorem. \square

Proof of the Theorem: Choose a Borel subgroup B in G and write U for the unipotent radical of B and $T = B/U$ for the ‘universal maximal torus’. Applying lemma 6.3.3 and lemma 6.3.6 to the subgroup $H = B$ we see that the structure group of P may be reduced from G to B . Assuming this, we view P as a B -bundle and form the associated T -bundle P/U . It suffices to show that this T -bundle is trivial. Indeed, in that case P/U has a regular

section, hence the structure group of the B -bundle P can be reduced to U . But any U -bundle on X is trivial by lemma 6.3.5, and we are done. Thus, the point is to find a reduction of P from G to B such that the associated T -bundle is trivial.

Let $\lambda : T = B/U \rightarrow \mathbb{C}^*$ be a dominant weight, $\lambda : B \rightarrow \mathbb{C}^*$ its pull-back to B , and $\mathcal{O}(\lambda)$ the induced G -equivariant line bundle on $\mathcal{B} = G/B$. Given a principal G -bundle P , form the associated bundle $P \times_G \mathcal{B}$ with fiber \mathcal{B} . The fiber of this bundle is (non-canonically) isomorphic to \mathcal{B} , and there is a canonical line bundle on the total space of $P \times_G \mathcal{B}$ whose restriction to each fiber is \mathcal{O}_λ . Let L_λ denote that line bundle on the total space.

Recall we are looking for a reduction of the structure group of P from G to B such that the associated T -bundle is trivial. It is easy to see via lemma 6.3.3, that getting such a reduction amounts to finding a regular section $s : X \rightarrow P \times_G \mathcal{B}$ such that, for each dominant weight λ , the pull-back $s^*(L_\lambda)$ is a trivial line bundle on X .

Start with any section $s' : X \rightarrow P \times_G \mathcal{B}$, which exists by lemma 6.3.6. This reduces the structure group to B . Fix some fundamental weight α . Let $P_\alpha \supset B$ be the minimal parabolic in G corresponding to α . We may view the B -bundle P as a P_α -bundle. We want to find a new section s'' such that : (1) $(s'')^*(L_\beta) = (s')^*(L_\beta)$ whenever β is a fundamental weight, $\beta \neq \alpha$; and (2) $(s'')^*(L_\alpha)$ is trivial. We look for s'' among the sections, whose image in the associated G/P_α -bundle coincides with the image of s' . Then condition (1) above holds automatically. Note that if U_α denotes the unipotent radical of the parabolic P_α then $P_\alpha/U_\alpha = SL_2$. Lemma 6.3.7 applied to the P_α/U_α -bundle P/U_α yields condition (2). Repeating this process for all fundamental weight one by one, we find a section s such that $s^*(L_\alpha)$ is trivial for all fundamental weights α , hence is trivial for all dominant weights. The theorem is proved. \square

Write \mathfrak{g} for the Lie algebra of the group G . Given a \mathbb{C} -algebra A , let $G(A)$ denote the group of A -rational points of G . Similar notation, $\mathfrak{g}(A)$, will be used in the Lie algebra case even for a not necessarily unital algebra A . Recall the algebras $\mathbb{K}_x, \mathcal{O}_x, \mathfrak{m}_x$ introduced in $n^\circ 1.5$. Thus, we have the Lie algebra $\mathfrak{g}(\mathbb{K}_x)$ with subalgebra $\mathfrak{g}(\mathcal{O}_x)$. The Lie algebra $\mathfrak{g}(\mathcal{O}_x)$ has the chain of ideals $\mathfrak{g}(\mathcal{O}_x) \supset \mathfrak{g}(\mathfrak{m}_x) \supset \mathfrak{g}(\mathfrak{m}_x^2) \supset \dots$. Similarly there are groups $G(\mathbb{K}_x)$ and $G(\mathcal{O}_x)$. The ring homomorphism $\mathcal{O}_x \rightarrow \mathcal{O}_x/\mathfrak{m}_x^m$ induces a group homomorphism $\pi : G(\mathcal{O}_x) \rightarrow G(\mathcal{O}_x/\mathfrak{m}_x^m)$. We put $L^m := \pi^{-1}(1)$, the inverse image of the identity in $G(\mathcal{O}_x/\mathfrak{m}_x^m)$. The groups $L^1 \supset L^2 \supset \dots$ form a chain of normal (congruence) subgroups in $L^0 = G(\mathcal{O}_x)$. Observe

that, for any $j > i > 0$, the quotient L^i/L^j has an obvious structure of a finite-dimensional unipotent group.

Recall (see §1.5), the imbedding $\mathcal{O}_{\text{out}} \hookrightarrow \mathbb{K}_x$ gives rise to a group imbedding $G(\mathcal{O}_{\text{out}}) := G(\mathcal{O}_{\text{out}}) \hookrightarrow G(\mathbb{K}_x)$. We set $\mathcal{M} = G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}})$. There is a natural left $G(\mathbb{K}_x)$ -action on \mathcal{M} . Further, for each $i > 0$, define a subset $\mathcal{M}_i \subset \mathcal{M}$ as follows:

$$\mathcal{M}_i = \{ f \cdot G(\mathcal{O}_{\text{out}}) \in \mathcal{M} = G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}}) \mid \mathfrak{g}(\mathfrak{m}_x^i) \cap (\text{Ad } f)\mathfrak{g}(\mathcal{O}_{\text{out}}) = 0 \}$$

Proposition 6.3.8 (i) *There is a natural bijective correspondence between the set $G(\mathcal{O}_x)\backslash\mathcal{M}$ of $G(\mathcal{O}_x)$ -orbits on \mathcal{M} and the set of isomorphism classes of algebraic principal G -bundles on X ;*

(ii) *For any $i \geq 1$, the group L^i acts freely on \mathcal{M}_i and the orbit space $L^i\backslash\mathcal{M}_i$ has natural structure of a smooth algebraic variety;*

(iii) *Each piece \mathcal{M}_i is a $G(\mathcal{O}_x)$ -stable subset of \mathcal{M} .*

Sketch of Proof: The punctured curve $X \setminus \{x\}$ being affine, any algebraic G -bundle on $X \setminus \{x\}$ is trivial by theorem 6.3.1. Hence, any algebraic G -bundle P on X has regular section, say s_{out} , over $X \setminus \{x\}$. Let s_x be a local section of P on a neighborhood of x . The bundle P is completely determined by the transition function $f = s_{\text{out}} \cdot s_x^{-1} \in G(\mathbb{K}_x)$. Choosing other sections (s_{out}, s_x) changes the element f within the double coset $G(\mathcal{O}_x)\backslash G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}})$. That proves (i). Now, the orbit space $L^i\backslash\mathcal{M}$ can be identified via the correspondence of part (i) with the set of the isomorphism classes of pairs (P, s) , where P is a G -bundle on X and s is an i -th jet of local section of P on a neighborhood of marked points. Part (ii) now follows from [LS]. Part (iii) is clear. \square

Thus, the couple $(G(\mathcal{O}_x), \mathcal{M})$ satisfies conditions 6.2(i)–(vi) so that all the constructions of section 6.1 are applicable. In particular, there is a topology on \mathcal{M} induced from that on $\widehat{\mathcal{M}}$.

6.4 Genus 0 case

Take $X = \mathbb{CP}^1$ with the single marked point $x = \infty$. View S^1 as the unit circle in $\mathbb{C} \subset \mathbb{CP}^1$. That gives a group imbedding $LG \hookrightarrow G(\mathbb{K}_x) = G(\mathbb{K}_\infty)$. Furthermore, we have $L^+G = G(\mathcal{O}_{\text{out}})$ so that one gets an imbedding $\text{Gr} = LG/L^+G \hookrightarrow G(\mathbb{K}_x)/G(\mathcal{O}_{\text{out}}) = \mathcal{M}$. The imbedding has a dense image.

Choose two opposite Borel subgroups B^+ and B^- in G so that $B^+ \cap B^- = T$ is a maximal torus. Define the Iwahori subgroups $I^\pm \subset G(\mathbb{K}_x)$ by

$$I^+ = \{ f \in G(\mathcal{O}_{\text{out}}) \mid \{(\iota) \in \mathcal{B}^+\} \text{ and } \mathcal{I}^- = \{ \{ \in \mathcal{G}(\mathcal{O}_\infty) \mid \{(\infty) \in \mathcal{B}^-\} \}.$$

For each homomorphism $\lambda \in X_*(T)$, viewed as a point of \mathcal{M} , let $C_\lambda = I^+ \cdot \lambda$ and $C^\lambda = I^- \cdot \lambda$ be the corresponding orbits of the Iwahori subgroups. These orbits are called *Bruhat cells*. The orbits C_λ are contained in $\text{Gr} \subset \mathcal{M}$ and form a cell decomposition of the Grassmannian indexed by the lattice $X_*(T)$. This cell decomposition defines the decomposition of Gr into the L^+G -orbits O_λ ; in fact, for any W -orbit $|\lambda| \subset X_*(T)$ we have $O_{|\lambda|} = \sqcup_{\mu \in |\lambda|} C_\mu$. The closures \overline{C}_λ clearly form a base of $H_\bullet(\text{Gr})$.

The orbits C^λ are not contained in Gr and have infinite dimensions. However, \overline{C}^λ , the closure (in \mathcal{M}) of such an orbit, gives a well-defined Borel-Moore homology class in $H_\bullet^{BM}(\mathcal{M})$ in the sense of n. 6.1.

We now define a bilinear cap product map

$$H_i^{BM}(\mathcal{M}) \times H_k(\text{Gr}) \longrightarrow H_{k-i}(\text{Gr}) \quad (6.4.0)$$

where Gr is taken with the direct limit topology as in n. 1.7. Let $Z \in H_i^{BM}(\mathcal{M})$ and $c \in H_k(\text{Gr})$. The cycle c clearly has compact support. Hence, there is $n \gg 0$ such that $\text{supp } c \subset \mathcal{M}_n$ where $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots$ is the exhaustion of \mathcal{M} (cf. n. 6.1). Let Z_n be the representative of Z in $H_\bullet(\mathcal{M}_{n/n})$ and $p_n : \mathcal{M}_n \longrightarrow \mathcal{M}_{n/n}$ the projection (notation of diagram 6.1.3). We assume, replacing the cycle c by a homotopy equivalent cycle if necessary, that p_n maps c isomorphically onto $p_n(c)$ and that Z_n intersects $p_n(c)$ transversely. Then c meets $p_n^{-1}(Z_n)$ transversely and we let $Z \cap c$ denote the cycle in $H_{k-i}(\text{Gr})$ given by that intersection. As a special case of (6.4.1) we have a natural pairing

$$H_i^{BM}(\mathcal{M}) \times H_i(\text{Gr}) \xrightarrow{\cap} H_0(\text{Gr}) = \mathbb{C}$$

which gives rise to a morphism:

$$H_i^{BM}(\mathcal{M}) \longrightarrow H^i(\text{Gr}).$$

In particular, each cycle \overline{C}^λ gives rise to a cohomology class of Gr . The following is known [PS]:

Lemma 6.4.2 *The cell C^λ meets C_λ transversally in a single point λ ; moreover the closure \overline{C}^λ does not meet any other cell C_μ with $\dim C_\mu \leq \dim C_\lambda$. \square*

It follows from the Lemma that the cycles \overline{C}^λ form a basis in $H^*(\text{Gr})$ dual to the basis $\{\overline{C}_\lambda\}$ in homology.

6.5 Global nilpotent variety

Below we write Bun_G for the moduli stack of principal G -bundles on X , a smooth complex compact curve of genus > 1 . Recall that the cotangent space $T_P^*Bun_G$ at a point $P \in Bun_G$ is given by the Kodaira-Spencer formula

$$T_P^*Bun_G = H^1(X, \mathfrak{g}_P) = H^0(X, \mathfrak{g}_P^* \otimes \Omega_X^1) = H^0(X, \mathfrak{g}_P \otimes \Omega_X^1),$$

where the last equality depends on the choice of an invariant bilinear form on \mathfrak{g} .

We call a (possibly singular) algebraic subvariety Y of a smooth symplectic algebraic variety (X, ω) *isotropic*, if for any *smooth* locally closed subvariety $W \subset Y$, we have $\omega|_W = 0$. It can be shown that this definition is equivalent to the more conventional one: the subvariety Y is said to be isotropic if the tangent space, $T_y Y$, at any regular point $y \in Y$ is an isotropic subspace in $T_y X$. The above definitions also apply to smooth stacks that can be locally represented as a quotient of a smooth algebraic variety modulo an action of an algebraic group. The stacks Bun_G and $T_P^*Bun_G$ are of this type, as explained in $n^\circ 6.3$.

Recall that the cotangent space $T_P^*Bun_G$ has a natural symplectic structure. Following Laumon [La2] define $Nilp \subset T^*Bun_G$, the *global nilpotent variety*, as follows

$$Nilp = \{(P, x) \in T^*Bun_G \mid x \in H^0(X, \mathfrak{g}_P \otimes \Omega_X^1), x \text{ is nilpotent section.}\}$$

Theorem 6.5.1 *Regular points of each irreducible component of the variety $Nilp$ form a Lagrangian subvariety in T^*Bun .*

Remark. This theorem was first proved, in the special case $G = SL_n$, by Laumon [La2]. Laumon's argument cannot be generalized to arbitrary semisimple groups. In the general case, the theorem was proved by Faltings [Fa, theorem II.5]. The proof below seems to be more elementary than that of Faltings; it is based on nothing but a few general results of Symplectic geometry.

Let (X_1, ω_1) and (X_2, ω_2) be complex algebraic symplectic manifolds, and $pr_i : X_1 \times X_2 \rightarrow X_i$ the projections.

Lemma 6.5.2 *Let $\Lambda_1 \subset X_1$ and $\Lambda \subset X_1 \times X_2$ be isotropic algebraic smooth subvarieties (the latter with respect to the symplectic form $pr_1^*\omega_1 - pr_2^*\omega_2$).*

Then the smooth locus of $\Lambda_2 := pr_2(pr_1^{-1}(\Lambda_1) \cap \Lambda)$ is an isotropic subvariety of X_2 .

Proof: Set $Y := pr_1^{-1}(\Lambda_1) \cap \Lambda$. Then simple linear algebra shows that, for any $y \in Y$ the image of the tangent map $(pr_2)_* : T_y Y \rightarrow T_{pr_2(y)} X_2$ is isotropic. Let $W \subset \Lambda_2 := pr_2(Y)$ be an irreducible smooth subvariety. Observe that the map $pr_2 : pr_2^{-1}(W) \cap Y \rightarrow W$ is *surjective*. Hence, there exists a non-empty smooth Zariski-open dense subset $U \subset (pr_2^{-1}(W) \cap Y)_{red}$ such that the restriction $pr_2 : U \rightarrow W$ has surjective differential at any point of U . Therefore the tangent space at the generic point of W is isotropic. Whence the tangent space at every point of W is isotropic. Thus, any smooth subvariety of Λ_2 is isotropic, and lemma follows. \square

Let $f : M \rightarrow N$ be a morphism of smooth algebraic varieties. Identify $T^*(M \times N)$ with $T^*M \times T^*N$ via the standard map multiplied by (-1) on the factor T^*N . The cotangent space, T^*X , to any manifold X has a canonical 1-form, usually denoted ‘ pdq ’. The canonical 1-form on $T^*(M \times N)$ gets identified, under the identification above, with ‘ $p_1dq_1 - p_2dq_2$ ’. We endow $T^*M \times T^*N$ with the symplectic form induced from that on $T^*(M \times N)$ via this identification.

Introduce the closed subvariety

$$Y_f = \{(m, \alpha), (n, \beta) \in T^*M \times T^*N \mid n = f(m), f^*\beta = 0\}.$$

Lemma 6.5.3 Y_f is an isotropic subvariety in $T^*M \times T^*N$.

Proof: The conormal bundle to the graph of f is the subvariety

$$\Lambda = \{(m, \alpha), (n, \beta) \in T^*M \times T^*N \mid n = f(m), \alpha = f^*(\beta)\}.$$

Observe that the canonical 1-form ‘ $p_1dq_1 - p_2dq_2$ ’ on $T^*M \times T^*N$ vanishes identically on Λ . Hence, Λ is a Lagrangian, in particular, an isotropic subvariety, and we may apply lemma 6.5.2 to $X_2 = T^*M$, $X_1 = T^*N$, and $\Lambda_1 = T_N^*N$ = the zero-section and the Λ above. Observe now that we have by definition $Y_f = \Lambda \cap pr_1^{-1}(T_N^*N)$. Hence by lemma 6.5.2 the subvariety $p(Y_f)$ is isotropic. \square

Lemma 6.5.4 If M and N are smooth algebraic stacks, and $f : M \rightarrow N$ is a representable morphism of finite type, then the assertion of lemma 6.5.3 still holds.

Proof: Due to locality of the claim we may (and will) assume N is quasi-compact. Let \tilde{N} be a smooth algebraic variety and $\tilde{N} \rightarrow N$ be a smooth surjective equidimensional morphism. Set $\tilde{M} := M \times_N \tilde{N}$. Then we have $Y_f \subset T^*N \times_N M$, and

$$Y_f \times_N \tilde{N} \subset T^*N \times_N \tilde{N} \times_N M \subset T^*\tilde{N} \times_N M$$

We must show that the image of $Y_f \times_N \tilde{N}$ is an isotropic subvariety in $T^*\tilde{N}$. Let $F : \tilde{M} \rightarrow \tilde{N}$ be the natural morphism, and $Y_F \subset T^*\tilde{N} \times_{\tilde{N}} \tilde{M}$ the corresponding subvariety of lemma 6.5.3. Observe that $T^*\tilde{N} \times_{\tilde{N}} \tilde{M} = T^*\tilde{N} \times_N M$ and $Y_f \times_N \tilde{N} = Y_F$. Hence, lemma 6.5.3 shows that the image of $Y_f \times_N \tilde{N}$ in $T^*\tilde{N}$ is isotropic. The claim follows. \square

Proof of theorem 6.5.1: Choose a Borel subgroup $B \subset G$ with Lie algebra \mathfrak{b} . Write \mathfrak{n} for the nilradical of \mathfrak{b} . In the setup of lemma 6.5.4 put $M = Bun_G$, the moduli stack of principal G -bundles on the curve X , and $N = Bun_B$, the moduli stack of principal B -bundles. Let $f : Bun_B \rightarrow Bun_G$ be the natural morphism. Observe that, for any $P \in Bun_B$, by Serre duality we have:

$$T_P^*Bun_B = H^1(X, \mathfrak{b}_P) = H^0(X, \mathfrak{b}_P^* \otimes \Omega_X^1) = H^0(X, \mathfrak{b}_P \otimes \Omega_X^1)$$

It follows, since any nilpotent element of \mathfrak{g} is conjugate to \mathfrak{n} , that in the notation of lemma 6.5.4 we have $Nilp = p(Y_f)$. Observe further that Bun_B is the union of a countable family of open substacks of finite type over Bun_G each. Thus, lemma 6.5.4 implies that $Nilp$ is the union of a countable family of isotropic substacks. But any union of a countable family of isotropic substacks is itself isotropic, for the field of complex numbers is uncountable. It follows that the set $Nilp$ is isotropic.

Finally, Hitchin [Hi] showed that the global nilpotent variety is the special fiber (over 0) of a certain morphism $\pi : T^*Bun_G \rightarrow H$, where H is a complex vector space of dimension $\dim Bun_G$ (at this point we use that genus X is > 1). Furthermore, since Bun_G is an equi-dimensional smooth stack, each irreducible component of T^*Bun_G has dimension $\geq 2 \dim Bun_G$. It follows that any irreducible component of the special fiber $\pi^{-1}(0)$ has dimension $\geq \dim Bun_G$. But we have proved that each component of $Nilp = \pi^{-1}(0)$ is an isotropic subvariety. Thus, $Nilp$ is Lagrangian. \square

Remarks. (i) The above argument shows that NN is a complete intersection in T^*Bun_G , in particular is Cohen-Macaulay. Further, since dimension

of any fiber is \leq dimension of the special fiber, it follows that every irreducible component of any fiber of π has dimension $\dim Bun_G$, hence, π is flat.

(ii) Hitchin actually worked in the setup of *stable* G -bundles and not in the setup of stacks. But his construction of the map π extends to the stack set-up verbatim. Notice that our argument did not use any additional properties of the map π established in [Hi].

6.6 Sketch of proof of theorem 1.5.7

Let A be a perverse sheaf (not necessarily local Hecke eigen-sheaf) satisfying conditions (i) and (ii) of theorem 1.5.7. Observe first that each simple object $IC_\lambda \in P(Gr)$ is of geometric origin, for it may be viewed as the intersection-cohomology extension of a (shifted) constant sheaf on the orbit O_λ . Thus, decomposition theorem (see [BBD]) for the convolution $IC_\lambda * A$ applies. The theorem yields that, for any $M \in P(Gr)$, we have $T_M(A) = \bigoplus L_j[m_j]$, where each L_j is a simple perverse sheaf on $X \times Bun_G$, and $[m_j]$ stands for the shift in the derived category.

Notation: Given a complex $F \in D^b(Y)$, write $SS(F)$ for the characteristic variety of F , cf. e.g. [KS]. Thus $SS(F)$ is a Lagrangian cone-subvariety in T^*Y .

Assume now that $SS(A) \subset Nilp$. Then, an easy direct calculation, based on a theorem of Kashiwara [KS] giving an estimate on the characteristic variety of a proper direct image, shows that

$$SS(T_M(A)) \subset T_X^*X \times Nilp \subset T^*(X \times Bun_G) \quad , \quad \forall M \in P(Gr)$$

Using the expression for $T_M(A)$ provided by the decomposition theorem we find

$$SS(L_j) \subset T_X^*X \times Nilp \quad , \quad \forall M \in P(Gr), \quad \forall j \quad (6.6.1)$$

By theorem 6.5.1 we know that $Nilp$ is a Lagrangian subvariety in T^*Bun_G . Hence, there exists a stratification $Bun_G = \coprod_\nu Bun_\nu$ such that any complex on $X \times Bun_G$ whose characteristic variety is contained in $T_X^*X \times Nilp$ is locally constant along the strata of the stratification $X \times Bun_G = \coprod_\nu (X \times Bun_\nu)$. Hence, any perverse sheaf L_j that occurs in (6.6.1) is the intersection complex associated with a simple local system on a certain stratum $X \times Bun_\nu$. Observe that $\pi_1(X \times Bun_\nu) = \pi_1(X) \times \pi_1(Bun_\nu)$, and any simple representation of a direct product of groups is the tensor product of simple representations of the factors. We find that each perverse sheaf L_j is of

the form $\mathcal{L} \boxtimes IC(\mathcal{B}\Gamma \backslash \nu, \mathcal{A}_\nu)$ where \mathcal{L} is an irreducible local system on X and $IC(Bun_\nu, \mathcal{A}_\nu)$ is the intersection complex associated with a simple local system \mathcal{A}_ν on Bun_ν . Write $IC_j(M)$ for the possible *non-isomorphic* sheaves $IC(Bun_\nu, \mathcal{A}_\nu)$, that occur above with non-zero multiplicity, and $\mathcal{L}(\mathcal{M})$ for the corresponding local system on X . Thus, we obtain

$$T_M(A) = \bigoplus_j \mathcal{L}_j(\mathcal{M}) \boxtimes IC_j(\mathcal{M})[\uparrow_j] \quad , \quad \forall \mathcal{M} \in \mathcal{P}(\mathcal{G}\nabla) \quad (6.6.2)$$

Next, fix a point $x \in X$, and let $i_x : \{x\} \times Bun_G \hookrightarrow X \times Bun_G$ denote the imbedding. One checks from definitions that a global Hecke operator, T_M , is related to the corresponding local Hecke operator at x by the following natural isomorphism

$$i_x^* T_M(F) = M * F \quad , \quad \forall M \in P(Gr), \forall F \in D^b(Bun_G) \quad (6.6.3)$$

where convolution on the RHS is defined via the double-coset isomorphism (1.5.2) that involves the choice of x in an essential way.

We now use the assumption that the perverse sheaf A is a local Hecke eigen-sheaf (at x). In view of equations (6.6.2) and (6.6.3) the assumption reads

$$\bigoplus_j i_x^* \mathcal{L}_j(\mathcal{M}) \boxtimes IC_j(\mathcal{M})[\uparrow_j] = M * A \simeq \mathcal{L}_M \otimes A \quad , \quad \forall \mathcal{M} \in \mathcal{P}(\mathcal{G}\nabla) \quad (6.6.4)$$

Since all the sheaves $IC_j(M)$ in equation (6.6.2) were assumed to be *pair-wise non-isomorphic* and the eigen-sheaf A is assumed simple, we conclude that isomorphism (6.6.4) can hold only if the direct sum on the left of (6.6.4) contains a single non-zero summand, say for $j = j_o$. Whence, we have $T_M(A) = \mathcal{L}_{j_o}(\mathcal{M}) \boxtimes IC_{j_o}(\mathcal{M})$ and, moreover, $IC_{j_o}(M) \simeq A$. Thus, A is a global Hecke eigen-sheaf, and the first claim of the theorem is proved.

To prove the rest of the theorem, we first argue locally at the point x . Let A be our local Hecke eigen-sheaf, so that $M * A \simeq L_M \otimes A$ holds for any $M \in P(Gr)$. Since A is *simple*, this equation yields $L_M \simeq Hom_{P(Gr)}(A, M * A)$. That shows that we may assume without loss of generality the assignment $M \mapsto L_M$ to be a functor $P(Gr) \rightarrow \text{Vect}$. Observe next that, for any $M, N \in P(Gr)$ we have

$$(M * N) * A = M * (N * A) = M * (L_N \otimes A) = L_N \otimes (M * A) = (L_M \otimes L_N) \otimes A$$

Applying the functor $Hom_{P(Gr)}(A, \bullet)$ to both sides we see that the functor $M \mapsto L_M$ is a tensor functor.

If A is a *global* Hecke eigen-sheaf one can repeat the argument of the previous paragraph ‘globally’, replacing equation $M * A \simeq L_M \otimes A$ by the equation $T_M(A) \simeq \mathcal{L}_M \boxtimes A$. The argument shows that the assignment $\mathcal{L} : \mathcal{M} \mapsto \mathcal{L}_\mathcal{M}$ gives a tensor functor on $P(Gr)$ with values in $\mathcal{L}\lambda](\mathcal{X})$, the tensor category of locally constant sheaves on X with the standard tensor product.

Applying theorem 1.4.1, we obtain the following composition of functors:

$$\mathcal{R} = \mathcal{P} \circ \mathcal{L} : \mathcal{R}]_{\sqrt{G^\vee}} \rightarrow \mathcal{P}(\mathcal{G}\nabla) \rightarrow \mathcal{L}\lambda](\mathcal{X})$$

This composition gives a tensor functor $\mathcal{R} : \mathcal{R}]_{\sqrt{G^\vee}} \rightarrow \mathcal{L}\lambda](\mathcal{X})$. We claim that any such tensor functor \mathcal{R} is isomorphic to the functor $V \mapsto V_P$, for an appropriate principal G^\vee -bundle P on X with flat connection. To prove the claim, view a flat G^\vee -bundle as a representation of the fundamental group, $\pi_1(X, x)$. This way the functor \mathcal{R} may be regarded as a tensor functor

$$\mathcal{R} : \mathcal{R}]_{\sqrt{G^\vee}} \rightarrow \mathcal{R}]_{\sqrt{\pi_\infty(\mathcal{X}, \S)}}$$

But any such functor is known [DM] to be induced by a group homomorphism $\pi_1(X, x) \rightarrow G^\vee$. This proves the claim, and the theorem follows. \square

7 Appendix A: Tensor functors and G -bundles

Throughout the Appendix we let G denote a linear algebraic group over \mathbb{C} , and $F_G : \text{Rep}(G) \rightarrow \text{Vect}$ the forgetful functor. The reader should be warned that the results below involving group G are applied to the ‘‘dual group’’ G^\vee in the main body of the paper. I hope this will not lead to confusion.

7.1

Let T be another group, and F_T the corresponding forgetful functor on $\text{Rep}(T)$. We have the following result [DM, Corollary 2.9]:

Proposition 7.1.1 *Let $S : \text{Rep}(G) \rightarrow \text{Rep}(T)$ be a tensor functor such that $F_G \circ S = F_T$. Then, there exists a unique algebraic homomorphism $s : T \rightarrow G$ such that the functor S is induced by s , i.e., for any representation $r : G \rightarrow \text{GL}(V)$, we have: $S(r) = r \circ s$. \square*

The following special case of Proposition 7.1.1 is particularly useful. Let T be a torus, and $X^*(T)$ the weight lattice. Assume further, that for each representation $V \in \text{Rep}(G)$ we are given a gradation on its underlying vector space by the lattice $X^*(T)$:

$$V = \bigoplus_{\lambda} V(\lambda), \quad \lambda \in X^*(T),$$

which is compatible with tensor product, i.e. for any $V_1, V_2 \in \text{Rep}(G)$ we have:

$$(V_1 \otimes V_2)(\lambda) = \sum_{\mu+\nu=\lambda} V_1(\mu) \otimes V_2(\nu) \quad (7.1.0)$$

Then one has:

Corollary 7.1.3 *There is a unique homomorphism $s : T \rightarrow G$ such that, for any representation $V \in \text{Rep}(G)$, the gradation $V(\cdot)$ coincides with the weight-gradation, that is the gradation:*

$$V(\lambda) = \{v \in V \mid s(t) \cdot v = \lambda(t) \cdot v, t \in T, \lambda \in X^*(T)\} \quad (7.1.0)$$

Proof: For each $V \in \text{Rep}(G)$ define an action of the torus T on V by letting $t \in T$ act on $V(\lambda)$ as multiplication by $\lambda(t)$. In this way we obtain a functor $S : \text{Rep}(G) \rightarrow \text{Rep}(T)$, which is a tensor functor by (7.1.2). The result now follows from Proposition 7.1.1. \square

7.2

Let $A = \bigoplus_{i \geq 0} A_i$ be a finitely-generated commutative graded \mathbb{C} -algebra. The gradation on A yields an algebraic \mathbb{C}^* -action on the affine scheme $\text{Spec} A$. Now, let P be a \mathbb{C}^* -equivariant principal algebraic G -bundle on $\text{Spec} A$ (that means that the group \mathbb{C}^* acts freely on P on the right, the group \mathbb{C}^* acts on P on the left, these two actions commute and the projection: $P \rightarrow \text{Spec} A$ commutes with the \mathbb{C}^* -action). For any representation $V \in \text{Rep}(G)$, we can form the associated bundle $P \times_G V$, which is a \mathbb{C}^* -equivariant algebraic vector bundle on $\text{Spec} A$. The space $\Gamma_P(V)$ of its global sections has a natural structure of a finitely-generated graded A -module. Let $\text{mod}^{\bullet} A$ denote the abelian category of finitely generated graded A -modules. The assignment $V \mapsto \Gamma_P(V)$ clearly defines an exact fully-faithful tensor functor (i.e. a fibre functor):

$$\Gamma_P : \text{Rep}(G) \rightarrow \text{mod}^{\bullet} A$$

The following result is a \mathbb{C}^* -equivariant analogue of [DM, Theorem 3.2].

Proposition 7.2.1 *Any fibre functor: $\text{Rep}(G) \rightarrow \text{mod}^\bullet\text{-}A$ is canonically isomorphic to the functor Γ_P for a uniquely determined \mathbb{C}^* -equivariant principal G -bundle P . \square*

Corollary 7.2.2 *Any fibre functor $F : \text{Rep}(G) \rightarrow \text{Vect}$ is (non-canonically) isomorphic to the forgetful functor F_G .*

Proof: Such a functor F may be regarded as a fiber-functor: $\text{Rep}(G) \rightarrow \text{mod}\text{-}\mathbb{C}$. Then, Proposition 7.2.1 says that there is a principal homogeneous space P such that the functor F is isomorphic to the functor: $V \mapsto P \times_G V$. A choice of a point in P yields an isomorphism: $P \cong G$ and, hence, an isomorphism: $P \times_G V \cong G \times_G V \cong V$. \square

7.3

Assume now that G is a connected reductive Lie group. In this n° we shall give a classification of \mathbb{C}^* -equivariant principal G -bundles on the line \mathbb{C} , where the group \mathbb{C}^* acts on \mathbb{C} in the standard way (by multiplication).

Given a \mathbb{C}^* -equivariant principal G -bundle P on \mathbb{C} , we let P_t denote the fiber of P over a point $t \in \mathbb{C}$. The fiber P_t is a principal homogeneous G -space and the group $\text{Aut}(P_t)$ of its automorphisms is non-canonically isomorphic to G (the map $P_t \rightarrow P_t$ is called an automorphism if it commutes with the G -action).

The fiber P_0 over zero is clearly a \mathbb{C}^* -stable subvariety of P , and the action of \mathbb{C}^* gives rise to a homomorphism $\tau : \mathbb{C}^* \rightarrow \text{Aut}(P_0)$.

Let P_1 be the fiber of P over the unit. To each $g \in \text{Aut}(P_1)$ we associate a family of automorphisms $\{g_t \in \text{Aut}(P_t), t \neq 0\}$ defined as the composition:

$$g_t : P_t \xrightarrow{t^{-1}} P_1 \xrightarrow{g} P_1 \xrightarrow{t} P_t,$$

where the first map is induced by the action of $t^{-1} \in \mathbb{C}^*$, the second one by the action of g , and the last one by the action of $t \in \mathbb{C}^*$. Let Q be the set of those $g \in \text{Aut}(P_1)$ that the family $\{g_t, t \in \mathbb{C}^*\}$ has a limit: $g_t \rightarrow g_0 \in \text{Aut}(P_0)$ as t approaches 0. It is clear that Q is a subgroup of $\text{Aut}(P_1)$ and that the assignment $g \mapsto g_0$ yields a group homomorphism $\lim : Q \rightarrow \text{Aut}(P_0)$.

Lemma 7.3.1 *7.3(i) The group Q is a parabolic subgroup of $\text{Aut}(P_1)$;*

- 7.3(ii) The kernel of the homomorphism 'lim' is equal to $\text{rad } Q$, the unipotent radical of Q ;
- 7.3(iii) The image of the homomorphism 'lim' is equal to L_τ , the centralizer in $\text{Aut}(P_0)$ of the above defined homomorphism $\tau : \mathbb{C}^* \rightarrow \text{Aut}(P_0)$.

Thus, to any \mathbb{C}^* -equivariant principal G -bundle P on \mathbb{C} we can associate a triplet (G_1, Q, τ) where $G_1 := \text{Aut}(P_1)$ is a reductive group isomorphic to G , Q is a parabolic subgroup of G_1 and $\tau : \mathbb{C}^* \rightarrow Q/\text{rad } Q$ is a regular central homomorphism (here ‘‘central’’ means that the image of τ belongs to the center of $Q/\text{rad } Q$, and ‘‘regular’’ means that, for any lifting $\tilde{\tau} : \mathbb{C}^* \rightarrow Q$, the centralizer in G_1 of the image of $\tilde{\tau}$ belongs to Q).

The triplets (G_1, Q, τ) , as above, form a category Tripl . A morphism: $(G_1, Q, \tau) \rightarrow (G'_1, Q', \tau')$ in that category is, by definition, a group isomorphism $G_1 \simeq G'_1$ which maps Q to Q' and τ to τ' .

Proposition 7.3.2 *The functor: $P \mapsto (G_1, Q, \tau)$ gives an equivalence of the category of \mathbb{C}^* -equivariant principal G -bundles on \mathbb{C} and the category Tripl . \square*

Fix a \mathbb{C}^* -equivariant G -bundle on \mathbb{C} . Given $V \in \text{Rep}(G)$, let V_1 denote the fibre over the point 1 of the associated bundle $P \times_G V$. The gradation on the space of global sections of $P \times_G V$ induced by the \mathbb{C}^* -action gives rise to a filtration W on V_1 . Namely, say that $v \in W_i(V_1)$ iff there exists a section $s \in \Gamma_P(V)$ of degree $\leq i$ such that $v = s(1)$.

Next, note that the vector space V_1 can be identified with $P_1 \times_G V$, where P_1 is the fibre of P over 1. Hence, the group $G_1 = \text{Aut}(P_1)$ acts naturally on V_1 . We shall now characterize the filtration W on V_1 in terms of the G_1 -action. To that end, take the homomorphism $\tau : \mathbb{C}^* \rightarrow Q/\text{rad } Q$ associated with the bundle P and let $h = \tau'(1)$ be the derivative of τ at the identity. Thus, h is a semisimple element of the Lie algebra of the group $Q/\text{rad } Q$. Chose $\tilde{h} \in \text{Lie } Q$, a semisimple representative of h . The element \tilde{h} acts on the space V_1 in a natural way and we have

Proposition 7.3.3 *The filtration $W_\bullet(V)$ coincides with the filtration by the eigenvalues of \tilde{h} .*

Remark. The filtration by the eigenvalues of \tilde{h} will not change if \tilde{h} is replaced by $\tilde{h} + x$, $x \in \text{Lie}(\text{rad } Q)$. Hence, it depends only on h and not on its representative \tilde{h} .

8 Appendix B: Equivariant Hyper-Cohomology

In this appendix we recall a number of equivalent constructions of the derived category of equivariant complexes (equivariant derived category, for short) and define equivariant hyper-cohomology of an equivariant complex. The reader is referred to the foundational work of Bernstein-Lunts [BL] for more details. Some of the constructions we are using have been introduced by Lusztig [Lu 3] before [BL] appeared (cf. also [Lu 4] for some more recent additional results).

8.1

Let T be a Lie group and $ET \rightarrow BT$ a universal principal T -bundle, so that T acts freely on ET on the right. There are (at least) two approaches to this bundle.

The first approach, used in topology, is to view the universal bundle as an direct limit of its finite dimensional approximations, i.e. to use a diagram

$$\begin{array}{ccccccc}
 ET^0 & \xrightarrow{i_0} & ET^1 & \xrightarrow{i_1} & ET^2 & \xrightarrow{i_2} & ET^3 \xrightarrow{i_3} \dots \\
 \downarrow T & & \downarrow T & & \downarrow T & & \downarrow T \\
 pt & \longrightarrow & BT^1 & \xrightarrow{i_1} & BT^2 & \xrightarrow{i_2} & BT^3 \xrightarrow{i_3} \dots
 \end{array} \tag{8.1.0}$$

The vertical arrows in this diagram are finite-dimensional principal T -bundles and the horizontal embeddings: $ET^n \hookrightarrow ET^{n+1}$ are T -morphisms. Such a diagram is called an approximation of the universal bundle if the following holds: *For any $k \geq 0$ there is an integer $n(k) \gg 0$ such that all the spaces ET^n , $n \geq n(k)$, are k -contractible, i.e. have vanishing homotopy: $\pi_j(ET^n) = 0$ for all $j \leq k$.* The condition implies that the cohomology of the spaces BT^n stabilize, i.e. the imbeddings $i_n : BT \hookrightarrow BT^{n+1}$ induce isomorphisms

$$i_n^* : H^k(BT^{n+1}) \xrightarrow{\sim} H^k(BT^n) \quad \text{for any } k \leq n.$$

The second, more algebraic, approach to the universal bundle is to view ET as the standard simplicial scheme

$$pt \longleftarrow T \longleftarrow T \times T \longleftarrow T \times T \times T \dots$$

The diagonal T -action on each of the spaces above is obviously free, giving a simplicial model for the universal T -bundle (i.e. a T bundle in the category of simplicial schemes).

8.2

Let Y be a variety with a smooth T -action. Set $Y_T = ET \times_T Y$, viewed either as a direct limit of the finite-dimensional spaces $Y_T^n = ET^n \times_T Y$ or as a simplicial scheme. Anyway, there are diagrams

$$\begin{array}{ccc}
 & ET \times Y & \\
 \pi \swarrow & & \searrow \rho \\
 Y_T & & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 & ET^n \times Y & \\
 \pi_n \swarrow & & \searrow \rho_n \\
 Y_T^n & & Y
 \end{array}
 \tag{8.2.0}$$

and the projection $ET \rightarrow BT$ gives rise to a fibration $Y_T \rightarrow BT$ with fibre Y .

Following Bernstein-Lunts, define a T -equivariant complex on Y as a collection $(M, M_T^n, \phi_n, \psi_n, n \geq 0)$ where $M \in D^b(Y)$, $M_T^n \in D^b(Y_T^n)$ and ϕ_n, ψ_n are isomorphisms $\phi_n : M_T^n \simeq i_n^* M_T^{n+1}$, $\psi_n : \rho_n^* M \simeq \pi_n^* M_T^n$ with certain compatibility conditions. Such collections form a triangulated category $D_G(Y)$, the *equivariant derived category* of Y . An object of $D_G(Y)$ is called an *equivariant complex*. An equivariant morphism $X \rightarrow Y$ of T -varieties induces a compatible system of morphism $X_T^n \rightarrow Y_T^n$, hence, gives rise to all the standard functors on the equivariant derived categories, e.g. direct images, inverse images, etc.

In the simplicial approach, one defines an equivariant complex to be an object of $D^b(Y_T)$, the bounded derived category of constructible complexes on the simplicial scheme Y_T . If the T -action on Y is free so that the orbit space $\bar{Y} = T \backslash Y$ is well-defined, then the second projection $ET \times Y \rightarrow Y$ induces a simplicial morphism $F : Y_T \rightarrow \bar{Y}$ with fiber ET . Since ET is a contractible scheme one deduces the following.

Equivariant descent: *Let $Y \rightarrow \bar{Y}$ be a locally-trivial principal T -bundle. Then the inverse image functor F^* gives an equivalence of $D^b(\bar{Y})$, the ordinary derived category, with $D^b(Y_T)$.*

A connection between the two approaches to equivariant complexes can be established as follows. Given a diagram (8.1.1) and a T -variety Y , view $ET^n \times Y$, as a T -variety with the diagonal T -action. Then for each $n \geq 0$

we get the following simplicial analogue of diagram (8.2.1):

$$\begin{array}{ccc}
 & (ET^n \times Y)_T & \\
 \pi_n \swarrow & & \searrow p_n \\
 ET^n \times_T Y = Y_T^n & & Y_T
 \end{array}$$

where p_n stands for the simplicial map induced by the second projection $ET^n \times Y \rightarrow Y$, and π_n is the “descent map F” for the variety $ET^n \times Y$ (with free action). Now given an object $M \in D^b(Y_T)$, for each $n \geq 0$, take $p_n^*M \in D^b((ET^n \times Y)_T)$. By the equivariant descent property, there is a unique complex $M_T^n \in D^b(Y_T^n)$ such that $p_n^*M = \pi_n^*M_T^n$. There are natural isomorphisms $\phi_n : M_T^n \simeq i_n^*M_T^{n+1}$ and $\psi_n : p_n^*M \simeq \pi_n^*M_T^n$ so that the collection $\{M, M_T^n, \phi_n, \psi_n; n \geq 0\}$ gives an object of $D_G(Y)$. This way one gets a functor $D^b(Y_T) \rightarrow D_G(Y)$ that turns out to be an equivalence of categories.

Restricting an object of $D^b(Y_T)$ to $Y \leftarrow T \times Y \leftarrow T \times T \times Y$, the beginning of the simplicial scheme Y_T , we get an ordinary complex $M \in D^b(Y)$ equipped with an isomorphism

$$I : p^*M \simeq q^*M \quad (8.2.0)$$

($p, q : T \times Y \rightarrow Y$ stand for the second projection and the action, respectively). The isomorphism I satisfies the cocycle condition:

$$p_1^*I \circ q_2^* \simeq m^*I, \quad \text{where } p_1, q_2, m : T \times T \times Y \rightarrow T \times Y$$

are given by the formulas $p_1 : (t_1, t_2, y) \mapsto (t_2, y)$, $q_2 : (t_1, t_2, y) \mapsto (t_1, t_2, y)$ and $m : (t_1, t_2, y) \mapsto (t_1, t_2, y)$. There is also a normalization property: *the restriction of the isomorphism I to $1 \times Y \subset T \times Y$ is the identity morphism $M \rightarrow M$* (notice that both p^*M and q^*M being restricted to $1 \times Y$ become canonically isomorphic to M).

It should be emphasized that, given an object $M \in D^b(Y)$ together with an isomorphism (8.2.2) satisfying the conditions above, it is not possible in general to extend these data to an object of $D^b(Y_T)$ due to the failure of Grothendieck’s descent for derived categories. Grothendieck’s descent holds, however, for perverse sheaves. Hence, we obtain:

Giving an equivariant *perverse sheaf* on Y is the same as giving an ordinary perverse sheaf together with an isomorphism (8.2.2) satisfying the conditions above. (8.2.0)

Let us also mention the following:

The constant sheaf on Y always has the structure of an equivariant complex. Such a structure is unique, provided the group T is connected (because of the normalization $I|_{1 \times Y} = \text{id}$ in (8.2.2)). (8.2.0)

Remark. For a smooth variety Y there is yet another construction of the equivariant derived category based on \mathcal{D} -modules (see [Gi2]).

8.3 Equivariant cohomology

Let $\{M, M_T^n, \phi_n, \psi_n\}$ be an equivariant complex on Y in the sense of Bernstein-Lunts. For each n , the fibration $Y_T^n \rightarrow BT^n$ gives rise to the Leray spectral sequence:

$$E_2^{p,q} = H^p(BT^n) \otimes H^q(Y, M) \implies H^{p+q}(T_T^n, M_T^n). \quad (8.3.0)$$

These spectral sequences form a projective system corresponding to the inductive sequence of T -fibrations $Y_T^0 \hookrightarrow Y_T^1 \hookrightarrow \dots$. The spectral sequences show that, for each $k \geq 0$, the projective system $H^k(M_T^0) \leftarrow H^k(M_T^1) \leftarrow \dots$ stabilizes, due to the stabilization of the cohomology of the spaces BT^n , $n = 0, 1, \dots$.

Define the T -equivariant cohomology by the formula

$$H_T^k(M) = \varprojlim_n H^k(Y_T^n, M_T^n) \quad (8.3.0)$$

(abusing the notation we often denote an equivariant complex $\{M, M_T^n, \phi_n, \psi_n\}$ by a single symbol M). Taking the projective limit of spectral sequences (8.3.1) we obtain the spectral sequence for equivariant cohomology:

$$E_2^{p,q} = H^p(BT) \otimes H^q(Y, M) \implies H_T^{p+q}(M). \quad (8.3.0)$$

For $M = \mathbb{C}_Y$, the constant sheaf, the R.H.S. of (8.3.2) becomes the standard definition of the equivariant cohomology of Y . Moreover, for any equivariant complex M on Y , there is a natural isomorphism:

$$H_T^\bullet(M) = \text{Ext}_{D_G(Y)}^\bullet(\mathbb{C}_Y, M).$$

This isomorphism is one of the main reasons for introducing the equivariant derived category.

Let T_c be a maximal compact subgroup of T . The inclusion $T_c \hookrightarrow T$ induces a system of maps $Y_{T_c}^n \hookrightarrow Y_T^n$ and homotopy equivalence $BT_c^n \approx BT^n$, $n = 1, 2, \dots$. It follows that, for any T -equivariant complex M , one has

$$H_{T_c}^\bullet(M) \cong H_T^\bullet(M). \quad (8.3.0)$$

There is an equivalent definition of equivariant cohomology based on the simplicial approach. Given an object $M \in D^b(Y_T)$, put

$$H_T^\bullet(M) = H^\bullet(Y_T, M)$$

where the R.H.S. stands for the hyper-cohomology of a double-complex with the second differential coming from the combinatorial differential of the simplicial scheme. This definition turns out to be equivalent to (8.3.2).

8.4

Assume now that T is a connected complex reductive Lie group. Then, one can find a diagram (8.1.1) consisting of algebraic varieties and algebraic maps. Moreover, one can find that diagram in such a way that all the spaces BT^n in the diagram are *smooth projective* varieties (e.g. if T is a torus then BT^n can be chosen to be the direct product of $\dim T$ copies of $\mathbb{C}\mathbb{P}^n$).

Further, let Y be projective variety with an algebraic T -action. Then, the projection: $Y_T^n \rightarrow BT^n$ is a projective morphism. Hence, Deligne's theorem on the degeneration of spectral sequences yields the following (cf. Decomposition theorem [BBD]):

Theorem 8.4.1 *Let M be a T -equivariant semisimple perverse sheaf of geometric origin (see [BBD]) on Y . Then, all the spectral sequences (8.3.1) collapse, so that $H_T^*(M)$ is free graded $H^*(BT)$ -module and: $\mathrm{rk}_{H^*(BT)} H_T^*(M) = \dim_{\mathbb{C}} H^*(M)$. Moreover, the cohomology $H_T^*(M)$ is pure.*

We will need the following two corollaries of Theorem 8.4.1 (cf. [Gi2]). Let I_0 be the augmentation ideal in $H^*(BT)$.

Corollary 8.4.2 *The restriction morphism: $H_T^*(M) \rightarrow H^*(M)$ induced by the inclusion $Y \hookrightarrow Y_T$ (as a fibre) vanishes on the submodule $I_0 \cdot H_T^*(M)$ and yields an isomorphism: $H_T^*(M)/I_0 \cdot H_T^*(M) \xrightarrow{\sim} H^*(M)$.*

If M' is another perverse sheaf on a T -variety Y' , satisfying the conditions of the Theorem, then we have:

Corollary 8.4.3 (Kunneth formula) *There is a natural isomorphism:*

$$H_T^*(M \boxtimes M') \cong H_T^*(M) \otimes_{H^*(BT)} H_T^*(M'). \quad \square$$

8.5

Assume from now on that T is a torus. Then, the cohomology $H^*(BT)$ is known to be isomorphic to $\mathbb{C}[\mathfrak{t}]$, the polynomial algebra on the Lie algebra of the torus T . We let I_t denote the maximal ideal in $\mathbb{C}[\mathfrak{t}]$ consisting of all polynomials vanishing at a point $t \in \mathfrak{t}$.

Define the family $H_t(M)$, $t \in \mathfrak{t}$, of specialized equivariant cohomology groups of a complex M by

$$H_t(M) = H_T^*(M)/I_t \cdot H_T^*(M).$$

For $t \neq 0$ the space $H_t(M)$ has no natural grading (for, the ideal I_t is not a homogeneous ideal). Instead, there is a natural increasing filtration W_\bullet on $H_t(M)$, called the *canonical filtration*, which is inherited from the filtration on $H_T^*(M)$ by degree, i.e.:

$$W_i(H_t(M)) = \text{image}(H_T^0(M) \oplus \cdots \oplus H_T^i(M)). \quad (8.5.0)$$

If $t = 0$, then the space $H_0(M)$ has a natural grading and there is a canonical isomorphism: $H_0^*(M) \cong \text{gr}^W H_{t'}(M)$ for any $t' \neq 0$. If, in addition, the complex M satisfies the assumption of Theorem 8.4.1, then Corollary 8.4.2 yields an isomorphism: $H_0^*(M) \cong H^*(M)$ (= the ordinary cohomology of M). Thus, for any $t \neq 0$, we obtain a natural isomorphism:

$$\text{gr}_\bullet^W H_t(M) \cong H^*(M). \quad (8.5.0)$$

8.6

Let Y^T denote the subvariety of T -fixed points in Y and $i : Y^T \hookrightarrow Y$ the inclusion. An element $t \in \mathfrak{t}$ is called *regular* if the zeroes of the vector field on Y generated by t coincide with the fixed point subvariety Y^T .

A connection between fixed points and the equivariant cohomology is provided by the following

Localization theorem 8.6 *For any T -equivariant complex M on Y , the natural push-forward morphism:*

$$i_! : H_t(Y^T, i^! M) \longrightarrow H_t(Y, M)$$

is an isomorphism.

□

□

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