Kac–Moody Symmetries of Critical Ground States

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Abstract

The symmetries of critical ground states of two-dimensional lattice models are investigated. We show how mapping a critical ground state to a model of a rough interface can be used to identify the chiral symmetry algebra of the conformal field theory that describes its scaling limit. This is demonstrated in the case of the six-vertex model, the three-coloring model on the honeycomb lattice, and the four-coloring model on the square lattice. These models are critical and they are described in the continuum by conformal field theories whose symmetry algebras are the $su(2)_{k=1}$, $su(3)_{k=1}$, and the $su(4)_{k=1}$ Kac-Moody algebra, respectively. Our approach is based on the Frenkel–Kac–Segal vertex operator construction of level one Kac–Moody algebras.

I. INTRODUCTION

Over the years the ground state ensembles of certain two dimensional classical spin models have been found to exhibit critical behavior. The first such ground state ensemble was encountered in the antiferromagnetic Ising model on the triangular lattice [\[1,2](#page-36-0)]. The correlation functions of operators constructed from the Ising spins, in the ground state ensemble, were found to decay with distance as power laws with various exponents. Typically, critical ground states are found in classical two-dimensional antiferromagnetic spin models, and in other models that exhibit frustration. The existence of critical ground states was first argued on rather general grounds, by Berker and Kadanoff [\[3](#page-36-0)].

Other than the Ising antiferromagnet on the triangular lattice, known models with critical ground states are the six-vertex model[[4](#page-36-0)], the closely related three-state antiferromagnetic Potts model on the square lattice[[5](#page-36-0)], the three-state antiferromagnetic Potts model on the Kagomé lattice $[6-8]$, the four-state antiferromagnetic Potts vertex model on the square lattice [\[9,10](#page-36-0)],and the $O(n)$ model on the honeycomb lattice [[11–13](#page-36-0)]. New models that are of this type are the antiferromagnetic Ising model of general spin on the triangular lattice [\[14,15](#page-36-0)], the non-crossing dimer model and the dimer-loop model [\[16\]](#page-36-0), both defined on the square lattice.

It has recently been shown that the ground state properties of all the known classical spin models with critical ground states can be analyzed by mapping the spin model onto an interface model[[17](#page-36-0)]. This analysis is equivalent, by a duality, to the Coulomb gas method, which has been used successfully for calculating exact values of critical exponents for many two dimensional lattice models [\[18\]](#page-36-0). The free energy of the interface is entropic in origin and the strength of the height fluctuations are governed by the stiffness of the interface. If the interface is in the rough phase then correlation functions in the spin model decay as power laws, with exponents whose numerical values are completely determined by the stiffness. This scenario is also found in the Coulomb gas approach to critical phenomena, where the scaling dimensions of all the electric and magnetic type operators are fixed by the value of the coupling constant.

Critical ground states are completely defined by constraints on the allowed configurations. An interesting question that has been raised recently[[11\]](#page-36-0) is how, and whether, the introduction of frustration, or of constraints, can lead to large values of the conformal charge $(c > 1)$ in two dimensional lattice models. It should be mentioned that it is a widely held belief that conformal field theories with large values of the conformal charge have rather limited application in describing critical phenomena in two dimensions. Nonetheless, large values of c have been observed in the fully frustrated XY model on the square lattice $[19]$, and more recently in the fully packed loop model on the honeycomb lattice [\[11\]](#page-36-0) and the four-coloring model on the square lattice [\[10](#page-36-0)]. In the loop model, and the coloring model, the large values of the conformal charge can be understood by mapping these models to interface models. The basic idea is that the constraints on the allowed states in these models require the height, which defines the interface, to have more than one component. Therefore the interface can be thought of as an embedding of the two-dimensional lattice in a higher dimensional target space. If the interface is rough, then the effective field theory of the interface, which describes the long-wavelength fluctuations of the height, is conformal, and its conformal charge is equal to the dimensionality of the target space; this is well known from the theory of the bosonic string[[20](#page-37-0)]. Furthermore, if the target space is compactified to a torus, then this can give rise to a nonabelian symmetry of the effective field theory.

In this paper we analyze the symmetry algebra of the conformal field theory that describes the scaling limit of the six-vertex model, the three-coloring model on the honeycomb lattice, and the four-coloring model on the square lattice, from a unifying perspective. This is provided by a mapping of these models to models of rough interfaces. The heights that define the interface are shown to be compactified on the root lattice of the $su(2)$, $su(3)$, and $su(4)$ Lie algebra, respectively. The *stiffness* of the interface plays the role of the compactification radius, and for special values of the stiffness a Kac-Moody algebra appears in the scaling limit of the interface model. We calculate the stiffness exactly using the loop correlation function[[21](#page-37-0)], which measures the probability that two points lie on the same contour loop of the interface; when the height has more than one component contours of a particular component need to be considered. Once the value of the stiffness is known then the scaling dimensions of all the operators in the six-vertex, the three-coloring, and the four-coloring model, can be calculated exactly. Furthermore, for each of these models, the currents of the Kac-Moody symmetry algebra can be written down in terms of the heigh field using the Frenkel–Kac–Segal [\[22\]](#page-37-0) vertex operator construction.

This paper is organized as follows: In Section II we describe the mapping of critical ground states to interface models. This allows us to introduce the concepts of the ideal state, the ideal state graph, and the repeat lattice, which play an important role in the later sections where we study specific models. In this section we also introduce an effective field theory for the long-wavelength fluctuations of the interface, which is conformally invariant. The conformal properties of this field theory, and in particular its chiral symmetry algebra, are discussed in Section [III](#page-9-0). These first two sections are devoted to critical ground states in general, and in the remaining sections we turn to specific examples. In Section [IV](#page-13-0) we review the mapping of the six-vertex model to an interface model, we calculate the stiffness, and we show that this model has a chiral $su(2)_{k=1}$ symmetry. We follow the same procedure in Sections [V](#page-19-0) and [VI,](#page-26-0) where we show that the the scaling limits of the three-coloring, and the four coloring model are the $su(3)_{k=1}$, and the $su(4)_{k=1}$ Wess–Zumino–Witten model [\[23\]](#page-37-0), respectively.

II. INTERFACE REPRESENTATIONS OF CRITICAL GROUND STATES

Here we give a brief summary of the construction of an interface model equivalent to the critical ground state of a spin model; the details of this construction, for the specific models, will be given in Sections [IV B,](#page-14-0) [V B,](#page-20-0) and [VI B.](#page-27-0) First we discuss the mapping itself which, after coarse graining, leads to an effective field theory for the long-wavelength fluctuations of the interface. In the second part of this section we show how operators in the ground state ensemble can be expressed in terms of the height field, and how their scaling dimensions can be calculated once the stiffness of the interface is known. In the third part we outline the calculation of the stiffness, which relies on the so-called loop correlation function. This correlation function measures the probability that two points on a rough interface lie on the same contour loop[[21\]](#page-37-0).

A. Mapping to an interface model and the effective field theory

Critical ground states are defined by the constraints on the allowed spin configurations. For example, in every ground state of the antiferromagnetic Ising model on the triangular lattice, all triangular plaquettes have exactly one pair of spins pointing in the same direction, or in other words, one pair of "frustrated" spins. Every ground state is given the same statistical weight.

In order to map a critical ground state of a spin model onto an interface model we start by defining a *height rule*. The height rule allows us to map a given ground state spin configuration $\{\sigma(\mathbf{x}_i)\}\)$ to a *microscopic height* configuration $\{\mathbf{z}(\mathbf{x}_i)\}\$. In general, the microscopic heights take their values in \mathbb{Z}^D , and they define a two-dimensional interface in $D + 2$ dimensions. Each allowed height configuration of the interface is given the same statistical weight.

Typically the height rule is such that the difference in **z** between neighboring sites \mathbf{x}_i and x_i on the lattice, is determined by the spins $\sigma(x_i)$ and $\sigma(x_j)$. In order for the height rule to be well defined, the change in the microscopic height must be zero when a closed loop on the lattice is traversed. Furthermore, we consider a mapping given by the height rule to be a *height mapping* if one can always find a spin configuration that will have any given average slope of the microscopic height; this ensures that **z** can take any value in \mathbb{Z}^D , i.e., the height is not restricted.

In order to define an effective field theory for the interface model, we introduce a coarsegraining procedure for the microscopic heights; see Fig. [1](#page-39-0) for a summary. First, among the ground states of the spin model we identify ideal states which are flat, that is they have zero average slope and they minimize deviations of the microscopic height away from its average. These ground states also have a maximum entropy density, that is they allow for a maximum number of local rearrangements of the spins which are in accord with the ground state constraints. The implicit assumption being made is that states with the maximum entropy density are flat. We will find this to be the case for all the models studied here.

Second, we coarse-grain the original spin model using the ideal states. We think of the lattice as being broken up into *ideal state domains*; in each domain fluctuations, which simply correspond to rearrangements of the spins allowed by the ground state constraints, occur about a particular ideal state. On the level of the interface, the microscopic heights z are replaced by the *coarse grained heights* **h**, which are defined for every ideal state domain, and equal to the average microscopic height in a particular domain (see Fig. [1](#page-39-0)(b)): $h = \langle z \rangle$.

Third, we define the *ideal state graph* $\mathcal{I} \subset \mathbb{R}^D$. Every node of \mathcal{I} represents an ideal state, and its position in \mathbb{R}^D is given by the coarse grained height $h \in \mathbb{R}^D$ of the ideal state it represents. Two nodes of the graph $\mathcal I$ are connected if the two ideal states they represent differ by the minimum number of local spin rearrangements needed to transform one ideal state to the other.

Two different points on the ideal-state graph **h** and **h'** can correspond to the same spin configuration (ideal state). The set of vectors $\mathbf{b} = \mathbf{h} - \mathbf{h}'$ form a lattice in \mathbb{R}^D called the repeat lattice $\mathcal R$. This lattice plays an important role in calculating scaling dimensions of lattice operators, which are defined as local functions of the spins, and it holds the key to identifying the symmetry algebra of the scaling limit of the critical ground states.

Finally, we consider the continuum limit of the interface model, where the heights defined over particular ideal state domains are replaced with a continuously varying height field $h(\mathbf{x}) \equiv (h_1(\mathbf{x}), h_2(\mathbf{x}), \dots, h_D(\mathbf{x}))$, as seen in Fig. [1](#page-39-0)(c). The dimensionless free energy (action) of the interface, which is entropic in origin, is assumed to be of the form

$$
f = \int d^2 \mathbf{x} \left[\left(\frac{K}{2} \sum_{i=1}^{D} |\nabla h_i(\mathbf{x})|^2 + V(\mathbf{h}) \right) ; \right]
$$
 (2.1)

K is the *stiffness* of the interface,¹ and $V(h)$ is a periodic potential with the periodicity given by the ideal state graph, i.e.,

$$
V(\mathbf{h} + \mathcal{I}) = V(\mathbf{h}) \tag{2.2}
$$

¹Here we assume that the stiffness is an isotropic tensor, which is not always the case. Critical ground states with more complicated stiffness tensors have been defined and analyzed in[[16](#page-36-0)].

The free energy f defines an effective field theory of the critical ground state; the assumption being made is that it correctly describes the long-wavelength fluctuations of the microscopic height z. The periodic potential $V(\bf{h})$, which is usually referred to as the *locking* potential [[24](#page-37-0)], favors the heights to take their values on I , while the first term represents fluctuations around the flat ideal states. Therefore, the assumption that the effective field theory of the critical ground state is given by Eq. [\(2.1](#page-5-0)), is directly related to the intuitive idea put forward earlier, that the free energy (entropy) of the ground state is entirely due to fluctuations around the ideal states.

The locking potential is periodic with the periodicity of $\mathcal I$. Thus, the ground state, in its interface representation, undergoes a roughening transition for some value of the stiffness $K = K_r$ $K = K_r$ $K = K_r$ [\[24\]](#page-37-0). If the stiffness K satisfies $K < K_r$, then $V(\mathbf{h})$ in Eq. ([2.1\)](#page-5-0) is irrelevant, in the renormalization group sense, the ground state is critical, and its scaling limit is described by a Gaussian model, with the free energy

$$
f = \frac{K}{2} \int d^2 \mathbf{x} \left(\sum_{i=1}^D |\nabla h_i(\mathbf{x})|^2 \right).
$$
 (2.3)

In the case that the locking term is relevant $(K > K_r)$, the ground state will lock into long range order in one of the ideal states.²

B. Operators and critical exponents

Here we discuss the calculation, in the ground state ensemble, of two point correlation functions of lattice operators. Our analysis is equivalent to the Coulomb gas approach [\[18\]](#page-36-0), only now, as will be shown below, the magnetic and electric charges are vectors in the repeat lattice and its reciprocal. Coulomb gas methods with vector charges have been used

²Note that our analysis implies that a ground state ensemble, for which a height mapping can be defined, is either critical (i.e. the interface is rough) or long range ordered (i.e. the interface is smooth); in particular, it can *not* be disordered.

previously by Fateev and Zamolodchikov [\[25\]](#page-37-0) to calculate correlation functions in the \mathbb{Z}_3 models. Their work was extended by Pasquier[[26](#page-37-0)] who considered the continuum limit of lattice models with quantum group symmetries.

A local operator $O(x)$ constructed from the spins and spatially uniform in the ideal states, can be written in terms of the coarse grained height as $\overline{O}({\bf h}({\bf x}))$. Since **h** and ${\bf h} + R$ represent the same ideal state on the ideal state graph, we can identify them

$$
\mathbf{h} \equiv \mathbf{h} + \mathcal{R} \,, \tag{2.4}
$$

and the operator $\overline{O}(\mathbf{h})$ is necessarily periodic with the periods forming the repeat lattice \mathcal{R} . Therefore, we can write $O(x)$ as a Fourier series

$$
O(\mathbf{x}) = \sum_{\mathbf{G} \in \mathcal{R}^*} O_{\mathbf{G}} e^{i\mathbf{G} \cdot \mathbf{h}(\mathbf{x})}
$$
(2.5)

where \mathcal{R}^* is the lattice *reciprocal* to the repeat lattice.³ The scaling dimension of $O(\mathbf{x})$ is equal to the scaling dimension of the most relevant vertex operator $\exp(i\mathbf{G} \cdot \mathbf{h}(\mathbf{x}))$ in the above expansion. In this sense, lattice operators constructed from the spins can be associated with vertex operators in the Gaussian model.

From the assumed form of the free energy, Eq.([2.1\)](#page-5-0), we can calculate the height-height correlation function:

$$
\langle (h_i(0) - h_j(\mathbf{x}))^2 \rangle = \frac{\delta_{ij}}{\pi K} \ln |\mathbf{x}| + \text{const} , \qquad (2.6)
$$

for $|x| \gg a$, the lattice spacing. Using Eq. (2.6) we can calculate the two-point correlation function

$$
< e^{i\mathbf{G}\cdot\mathbf{h}(0)} e^{-i\mathbf{G}\cdot\mathbf{h}(\mathbf{x})} > = e^{-\frac{1}{2} \langle [\mathbf{G}\cdot(\mathbf{h}(0) - \mathbf{h}(\mathbf{x}))]^2 \rangle} \sim \frac{1}{|\mathbf{x}|^{2x(\mathbf{G})}},
$$
 (2.7)

where the exponent $x(G)$ is the scaling dimension of the vertex operator $\exp(i\mathbf{G}\cdot\mathbf{h}(\mathbf{x}))$, and it is related to the stiffness by

³By definition, the dot product of any vector in the repeat lattice with any vector in the reciprocal lattice is an integer multiple of 2π .

$$
x(\mathbf{G}) = \frac{1}{2\pi K} \frac{\mathbf{G}^2}{2}.
$$
\n(2.8)

The other type of operators we consider correspond to violations of the ground state condition that is placed on the spins. This typically corresponds to a topological defect (vortex) in the height language, where the topological charges (Burgers vectors) b take their values in the repeat lattice.⁴ Using the equation for the free energy, Eq. (2.1) , we find that the vortex-antivortex correlation function decays algebraically[[18\]](#page-36-0), with the exponent:

$$
x_v(\mathbf{b}) = \frac{K}{2\pi} \frac{\mathbf{b}^2}{2} \,. \tag{2.9}
$$

This result follows from the logarithmic form of the dimensionless interaction energy between two vortices ($\pm \mathbf{b}$), separated by $r: E_{\text{int}} = \frac{K}{2\pi}$ $\frac{K}{2\pi}$ **b**² ln *r* + const; the vortex-antivortex correlation function is given by the Boltzmann factor $\exp(-E_{\text{int}})$.

We see that all the exponents describing critical correlations in the ground state of the spin model can be related to the stiffness K of the interface. Here the stiffness plays the role of the coupling constant in the Coulomb gas approach[[18\]](#page-36-0), or the radius of compactification of the appropriate conformal field theory [\[27](#page-37-0)].

C. Contour loops

We have argued recently [\[21](#page-37-0)] that geometrical exponents associated with contour loops on a Gaussian interface have universal values which are independent of the stiffness. In particular, the loop correlation function $\mathcal{G}(x)$, which measures the probability that two points on the interface, separated by x , belong to the same contour loop, falls off with distance as

$$
\mathcal{G}(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{2x_{\text{loop}}}}\,,\tag{2.10}
$$

⁴In the language of conformal field theory these would be the winding modes, while the vertex operators are related to the momentum modes[[27\]](#page-37-0).

where the loop correlation-function exponent

$$
x_{\rm loop} = \frac{1}{2},\tag{2.11}
$$

is independent of the stiffness. Therefore, if we could somehow calculate the loop correlation function in a particular spin model in terms of the stiffness, then the above result would clearly allow us to find the exact value of the stiffness. For the critical ground states studied in this paper this is indeed the case. It will be shown for the six-vertex model (for a particular choice of vertex weights), the three-coloring model, and the four-coloring model, that the loop correlation function is equal to a vortex-antivortex correlation function for a particular value of the Burgers charge b_{loop} . As was discussed in the previous section the vortexantivortex correlation function falls of with distance as a power law, with the exponent $2x_v(\mathbf{b}_{\text{loop}})$ which depends on the stiffness K; see Eq. [\(2.9](#page-8-0)). From Eq. (2.11) $x_v(\mathbf{b}_{\text{loop}}) = 1/2$ follows, and the stiffness can be calculated.

III. CONFORMAL INVARIANCE OF CRITICAL GROUND STATES

Here we discuss the conformal properties of the field theory which describes the continuum limit of a critical ground state. In the previous section we found, under certain assumptions, this to be a simple Gaussian field theory, and we therefore first give a lightning review of conformal invariance in the Gaussian model, mostly following Ginsparg [\[27\]](#page-37-0). We also found that the Gaussian fields, namely the heights, are compactified on the repeat lattice, which naturally leads into the discussion of the vertex operator (VO) construction of affine Lie algebras. We briefly review this construction for $su(N)_{k=1}$ Kac-Moody (KM) algebras in the second part of this section; details of the VO construction can be found in the original papers by Frenkel, Kac, and Segal [\[22\]](#page-37-0), and in the review by Goddard and Olive [\[28\]](#page-37-0).

A. Gaussian model

The Gaussian field theory described by Eq.([2.3\)](#page-6-0) is conformally invariant; the free energy fcan be thought of as the Euclidean action for D free massless bosons [[27\]](#page-37-0). Consequently, the conformal charge is $c = D$.

If instead of the Cartesian coordinates $\mathbf{x} = (x_1, x_2)$, we introduce complex coordinates in the plane, $w = x_1 + ix_2$ and $\bar{w} = x_1 - ix_2$, the free energy in Eq. [\(2.3](#page-6-0)) can be written as:

$$
f = K \int d^2w \, \partial \mathbf{h}(w, \bar{w}) \cdot \bar{\partial} \mathbf{h}(w, \bar{w}) \,, \tag{3.1}
$$

where $\partial = 1/2(\partial_1 + \partial_2)$, $\bar{\partial} = 1/2i(\partial_1 - \partial_2)$, and $d^2w = 2d^2\mathbf{x}$ due to the Jacobian factor. Written in this form the invariance of the free energy under a conformal mapping $w' =$ $f(w), \bar{w}' = \bar{f}(\bar{w})$ is apparent, if we postulate that the height field transforms as a scalar quantity, i.e.,

$$
\mathbf{h}'(w', \bar{w}') = \mathbf{h}(w, \bar{w}) \tag{3.2}
$$

The Euler–Lagrange equation for the height field, with the Euclidean action given by Eq. (3.1), is $\bar{\partial}\partial h(w,\bar{w})=0$. We can therefore write the height field as a sum of a holomorphic and an antiholomorphic field,

$$
\mathbf{h}(\mathbf{x}) \equiv \mathbf{h}(w, \bar{w}) = \mathbf{h}(w) + \bar{\mathbf{h}}(\bar{w}). \tag{3.3}
$$

Now the holomorphic and antiholomorphic components of the stress-energy tensor can be expressed in terms of $h(w)$ and $h(\bar w)$ as:

$$
T(w) = -2\pi K \; : \partial \mathbf{h}(w) \cdot \partial \mathbf{h}(w) : \quad \bar{T}(\bar{w}) = -2\pi K \; : \bar{\partial} \bar{\mathbf{h}}(\bar{w}) \cdot \bar{\partial} \bar{\mathbf{h}}(\bar{w}) : , \tag{3.4}
$$

where the symbol : \dots : denotes the normal ordering prescription defined by the point splittingoperation [[27\]](#page-37-0). The modes of $T(w)$,

$$
L_m = \oint_C \frac{dw}{2\pi i} w^{m+1} T(w) , \qquad (3.5)
$$

satisfy the commutation relations of the Virasoro algebra with central charge $c = D$

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} ,
$$
\n(3.6)

whichis the symmetry algebra of the CFT defined by Eq. (3.1) (3.1) ; the contour integral in Eq.([3.5\)](#page-10-0) is taken around the origin.

The symmetry algebra of the Gaussian field theory, Eq. [\(3.1](#page-10-0)), which describes a critical ground state, can be larger than Virasoro, which will necessarily be present as a subalgebra. This will be the case if the repeat lattice is equivalent, up to an overall scale factor, to the root lattice of some simple Lie algebra \overline{q} , and the stiffness (compactification radius) has the correct value. Namely, in this case the symmetry algebra associated with the scaling limit of a critical ground state is the chiral Kac–Moody algebra g at level one, whose horizontal subalgebra is \overline{g} [\[29](#page-37-0)]. The currents of g can be constructed from the height field using the Frankel–Kac–Segal vertex operator construction [\[22\]](#page-37-0), which we review next.

B. Vertex operator construction

For the purposes of this section we will assume that $\mathcal{R} \subset \mathbb{R}^D$ is equivalent, up to a scale factor, to the root lattice of an $su(D+1)$ Lie algebra; the $D(D+1)/2$ positive roots form a D simplex: equilateral triangle for $D = 2$, tetrahedron for $D = 3$, and so on. We show, for a particular value of the stiffness K , that the symmetry algebra of the Gaussian field theory, Eq. [\(3.1](#page-10-0)), where the height field is compactified on \mathcal{R} , is the $su(D+1)_{k=1}$ Kac-Moody algebra.

The $su(D+1)_{k=1}$ Kac–Moody (KM) algebra is an infinite dimensional Lie algebra, whose generators $\{T_m^a: a=1,\ldots,(D+1)^2-1; m \in \mathbb{Z}\}\$ satisfy the commutation relations

$$
[T_m^a, T_n^b] = f_c^{ab} T_{m+n}^c + m \delta_{m+n,0} , \qquad (3.7)
$$

where f_c^{ab} are the structure functions of the $su(D+1)$ Lie algebra. The defining commutation relations of an $su(D+1)_{k=1}$ KM algebra can be conveniently rewritten in the so-called Cartan–Weyl basis as

$$
[H_m^i, H_n^j] = m\delta^{i,j}\delta_{m+n,0}
$$
\n
$$
[H_m^i, E_n^{\alpha}] = \alpha^i E_{m+n}^{\alpha}
$$
\n
$$
[E_m^{\alpha}, E_n^{\beta}] = \varepsilon(\alpha, \beta) E_{m+n}^{\alpha+\beta}, \quad \text{for } \alpha + \beta \text{ a root}
$$
\n
$$
[E_n^{\alpha}, E_m^{-\alpha}] = \frac{2}{\alpha^2}(\alpha \cdot \mathbf{H}_{m+n} + n\delta_{m+n,0})
$$
\n(3.8)

where H_m^i are the Cartan generators, while E_m^{α} are the step operators associated with the roots (α) of the su(D+1) Lie algebra; the constants $\varepsilon(\alpha, \beta)$ are antisymmetric in α and β , and can be normalized to ± 1 . The number of Cartan generators for an $su(D+1)$ Lie algebra is D while the number of roots is $D(D+1)$; this gives a total of $(D+1)^2 - 1$ generators which is the dimension of $su(D+1)$.

It can be shown that the commutation relations, in the Cartan-Weyl basis, of an $su(D +$ $1)_{k=1}$ KM algebra are also the commutation relations of the modes of the *currents*

$$
H^{i}(w) = i\partial h^{i}(w)
$$
\n
$$
E^{\alpha}(w) = \cdot e^{i\alpha \cdot \mathbf{h}(w)} \cdot c_{\alpha},
$$
\n(3.9)

where c_{α} are the cocycle factors which are needed to give the proper signs in the commutation relations of the current modes (for details see Sec. 6.5 in[[28\]](#page-37-0)).

The derivatives of the height field are currents of the Gaussian field theory given by Eq. $(3.1)^5$ regardless of the stiffness K, while for this to be true of the vertex operators $E^{\alpha}(w)$ it is required that their conformal dimensions (h, \bar{h}) satisfy

$$
(\frac{1}{2\pi K}\frac{\alpha^2}{4},0) = (1,0).
$$
 (3.10)

We see that the value of the stiffness has to be fine tuned if the above equation is to hold, since the vectors α are completely determined by the height construction. Namely, the roots α generate the root lattice, and they are also vectors in \mathcal{R}^* , which is to be identified with

⁵They are generators of D copies of the $u(1)_{k=1}$ Kac-Moody algebra, which is defined by the first relation in Eq. (3.8) .

the weight lattice of an $su(D+1)$ Lie algebra. The quotient of the weight lattice with the root lattice is \mathbb{Z}_{D+1} , which completely specifies the $D(D+1)$ reciprocal vectors α , once \mathcal{R}^* is known.

To conclude, two ingredients are necessary, and sufficient, for the Gaussian field theory, which describes the continuum limit of a critical ground state, to have a chiral symmetry given by the $su(D+1)_{k=1}$ Kac-Moody algebra: 1) the repeat lattice has to be equivalent, up to a scale factor, to the root lattice of the $su(D+1)$ Lie algebra, and 2) the stiffness has to be such that the vertex operators associated with the root vectors are dimension-one currents. In the remaining sections of this paper we show that both ingredients are to be found in the six-vertex model, the three-coloring, and the four-coloring model. Finally, we note that the stress-energy tensor in Eq.([3.4\)](#page-10-0) is in the Sugawara form, which means that the Gaussian field theory described here is nothing but the $SU(D+1)_{k=1}$ Wess-Zumino-Witten model.⁶

IV. SIX-VERTEX MODEL

In this section it is shown that the symmetry algebra of the scaling limit of the six-vertex model, for a particular choice of vertex weights, is the $su(2)_{k=1}$ Kac–Moody algebra. Namely, the effective field theory of the six vertex model turns out to be a Gaussian field theory with a one-component height compactified on the root lattice of the $su(2)$ Lie algebra.

A. Definitions

The six-vertex model on the square lattice is defined by placing arrows along the bonds of the lattice, with the constraint that at each vertex the number of arrows pointing in, is

⁶Here, and throughout this paper, we adopt the definition of a Wess-Zumino-Witten model as a conformal field theory with a chiral Kac–Moody symmetry algebra and a stress-energy tensor of the Sugawara form[[29\]](#page-37-0).

equal to the number of arrows pointing out. This constraint is the so called ice rule which has its origins in the charge neutrality condition for square ice, for which this model was originally proposed[[4\]](#page-36-0). Namely, in square ice, we think of the oxygen atoms as sitting at the vertices and the hydrogen atoms on the bonds of a square lattice. Each arrow points from the hydrogen to the oxygen atom to which it is bonded, and local charge neutrality requires that the divergence of arrows at each vertex be zero.

There are six possible vertex configurations and to each one we assign a Boltzmann weight, a, b , or c , as shown in Fig. [2](#page-39-0). This model was solved exactly by Lieb [[30](#page-37-0)]. Recently Affleck[[31](#page-37-0)] has shown that the six-vertex model with isotropic vertex weights, $a = 1, b = \lambda$, and $c = 1 + \lambda$, has a chiral $su(2)_{k=1}$ Kac–Moody symmetry. This symmetry is hidden, and it can be uncovered by mapping the six-vertex model to a quantum spin chain [\[31](#page-37-0)]. Here we demonstrate the same result for the symmetry algebra of the six-vertex model with vertex weights $a = b = c/2 = 1$, by mapping it to an interface model. We mainly do this as an example of the general construction outlined in the previous section, and to set the stage for the sections to come.

B. The interface model

The six-vertex model can be mapped to an interface model, which is the so called bodycentered solid-on-solid model (BCSOS) introduced by van Beijeren [\[32](#page-37-0)]. It describes the surface of a body centered cubic structure when viewed in the [100] direction.

The microscopic heights of the BCSOS model $z \in \mathbb{Z}$, are defined at the centers of the square plaquettes in such a way that when crossing a bond of the square lattice, z is increased (decreased) by 1, depending on whether we cross from left to right (right to left), as seen when looking in the direction of the arrow on the bond; see Fig. [3](#page-39-0). The ice rule ensures that the microscopic heights are well defined, i.e., the total change in height when going around a closed loop is zero. In other words, the zero divergence property of the arrows in the six-vertex model becomes the zero curl property of the height increments in the BCSOS model.

The smallest change on the lattice, that does not violate the ice rule, is accomplished by reversing the direction of the arrows on all the bonds along a loop of constant arrow direction. This observation prompts us to define ideal states of the six-vertex model as ones that maximize the number of such loops. There are two ideal states related to each other by translations of the lattice, and both are defined by a c-type vertex state which is periodically repeated throughout the lattice; see Fig. [3](#page-39-0). The ideal states also have the important property that they are flat, in the sense that they minimize the variance of the microscopic height, and we use them to coarse grain the six-vertex model.

The ideal state graph \mathcal{I} , is a one-dimensional lattice with the lattice spacing equal to 1. Each vertex corresponds to one of the two ideal states, and its position in the graph is determined by the average microscopic height in the ideal state it represents. The same ideal state repeats every two vertices on \mathcal{I} , and these vertices form the repeat lattice \mathcal{R} , which has a lattice spacing of 2; see Fig. [4](#page-39-0). The ideal state graph and the repeat lattice play a crucial role in constructing an effective field theory of the BCSOS model, which is what we turn to next.

In order to describe the long-wavelength fluctuations of the microscopic height in the BCSOS model we introduce a coarse-graining procedure. Namely, we think of dividing up the square lattice into ideal state domains, and to each of these domains we assign a coarse grained height h , which is equal to the microscopic height averaged over the domain. The effective field theory for the long-wavelength fluctuations of the BCSOS model is given by the dimensionless free energy (action), which is assumed to be of the form:

$$
f = \int d^2 \mathbf{x} \left[\frac{K}{2} (\nabla h)^2 + V(h) \right]. \tag{4.1}
$$

The height field $h(\mathbf{x})$ is the continuum limit of the discrete coarse grained heights defined over ideal state domains, and K is the dimensionless stiffness of the interface. Since h and $h + \mathcal{R}$ correspond to the same ideal state, we consider the height field compactified on the circle, i.e.,

$$
h \in \mathbb{R}/\mathcal{R} \tag{4.2}
$$

The potential term $V(h)$ favors heights on the ideal state graph and is therefore a periodic function of the height field,

$$
V(h+\mathcal{I}) = V(h) \tag{4.3}
$$

The scaling dimension of the periodic potential is given by Eq. [\(2.8](#page-8-0))

$$
x(\mathbf{G}_V) = \frac{\pi}{K} \,,\tag{4.4}
$$

where $G_V = 2\pi$ is the periodicity of the ideal state graph; see Fig. [4](#page-39-0). Therefore, adding the potential term to the Gaussian free energy is an irrelevant perturbation, i.e., $x(G_V) > 2$, for

$$
K < \frac{\pi}{2} \tag{4.5}
$$

If this condition is satisfied then the effective field theory of the BCSOS model is purely Gaussian, defined by a dimensionless free energy

$$
f = \frac{K}{2} \int d^2 \mathbf{x} (\nabla h)^2 . \tag{4.6}
$$

If on the other hand $K > \frac{\pi}{2}$, the periodic potential is relevant, and the BCSOS model will be in the smooth phase, i.e. the six-vertex model will lock into long-range order in one of the ideal states.

C. Calculation of the stiffness

In order to calculate the stiffness for the six-vertex model we map this model to a loop model, in which the loops are contour loops of the BCSOS model.

The six-vertex model with vertex weights $a = b = c/2 = 1$ can be mapped to a loop model using the break up procedure of Evertz *et al.* [[33](#page-37-0)]. Every vertex of type a and b is broken up into two corners of a loop in a unique way, while the c type vertices are broken up into two possible corners with equal probability, Fig. [5](#page-39-0). Loops generated in this way are closed and non-intersecting, and they cover all the bonds of the square lattice. The direction of the arrows around each loop can be clockwise or counterclockwise, independent of the other loops, giving a loop fugacity equal to two.

The loops in this model are nothing but *contour* loops of the BCSOS model. This implies that the correlation function which measures the probability that two points are on the same loop in the loop model, is equal to the loop correlation function for a Gaussian interface, givenby Eq. (2.10) (2.10) . This observation can be used to calculate the value of the stiffness K.

The loop correlation function is equal to the vortex-antivortex correlation function for vortices of Burgers charge $b_{\text{loop}} = 2$, as seen in Fig. [6](#page-40-0). As shown in the figure, the vortexantivortex configuration in the BCSOS model is generated by simply reversing the arrows along the bonds on one half of the loop. This "loop trick" was previously used by Saleur and Duplantier to calculate the fractal dimension of a percolation hull [\[34\]](#page-37-0), and the fractal dimension of an Ising cluster [\[35\]](#page-38-0) in two dimensions. Now if we use the formula for the scaling dimension associated with a vortex-antivortex correlation function, Eq.([2.9\)](#page-8-0), and we identify it with the universal value of the loop correlation function exponent $x_{\text{loop}} = 1/2$, we find

$$
K = \frac{\pi}{2} \tag{4.7}
$$

Thisvalue of the stiffness implies $x(G_V) = 2$ (see Eq. ([4.4\)](#page-16-0)) which means that the locking potential $V(h)$ in Eq. ([4.1\)](#page-15-0) is *marginal*, and the BCSOS model is exactly at the roughening transition. This is in agreement with the exact solution of the six-vertex model[[4\]](#page-36-0), which predicts a Kosterlitz-Thouless type transition at $a = b = c/2 = 1$.

It's interesting to note that the exponent x_{loop} has been measured indirectly by Evertz et al. [\[33](#page-37-0)]. Namely, in their numerical simulations of the six-vertex model, among other measurements, they measure the average length $\lt s >$ of a loop of constant arrow direction as a function of the system size L. For $a = b = c/2 = 1$ they find

$$
\langle s \rangle \sim L^{1.06} \tag{4.8}
$$

We can calculate the average length of a loop inside a circle of radius L , using the loop correlation function $\mathcal{G}(\mathbf{x})$;

$$
\langle s \rangle = \int_0^L d^2 \mathbf{x} \, \mathcal{G}(\mathbf{x}) \,. \tag{4.9}
$$

If we substitute into this equation the expression for the loop correlation function, Eq.([2.10\)](#page-8-0), and take $x_{\text{loop}} = 1/2$, we find

$$
\langle s \rangle \sim L^{2-2x_{\text{loop}}}=L^1 \,, \tag{4.10}
$$

which is in good agreement with the numerical result, Eq. (4.8) , once systematic errors are taken into account.⁷

D. Symmetry algebra of the six-vertex model

The effective field theory of the six-vertex model is a Gaussian field theory defined by the Euclidean action in Eq. [\(4.6](#page-16-0)), and compactified on the circle \mathbb{R}/\mathbb{R} . This is a conformal field theory (CFT) with a stress energy tensor whose holomorphic and antiholomorphic components are given by Eq.([3.4\)](#page-10-0) where the height field has one component. The conformal charge is $c = 1$, independent of the stiffness K. The modes of the stress-energy tensor are generators of conformal transformations in the plane, and they form the Virasoro algebra.

For the special value of the stiffness $K = \pi/2$, that we find for the six-vertex model with vertex weights $a = b = c/2 = 1$, the symmetry algebra of the CFT is the chiral $su(2)_{k=1}$ Kac–Moody algebra. The currents of the holomorphic half of this algebra are defined in terms of the holomorphic component of the height $h(w)$, Eq. [\(3.3](#page-10-0)), as:

$$
J^3 = i\partial h(w) , \quad J^\pm = :e^{\pm i\alpha h(w)} : c_{\pm\alpha} . \tag{4.11}
$$

The currents that generate the antiholomorphic part are defined in the exact same fashion, the only difference being that $h(w)$ is replaced by $h(\bar{w})$. The reciprocal lattice vector $\alpha =$

⁷Ref.[[33](#page-37-0)] gave an exponent of 1.060(2), but this error includes only statistical errors [\[36](#page-38-0)].

 2π is chosen so that the currents $J^{\pm}(z)$ have the required conformal dimension $(1,0)$; see Eq.([3.10\)](#page-12-0). The integral multiples of α generate the *root lattice* of an $su(2)$ Lie algebra, while the reciprocal lattice \mathcal{R}^* can be identified with the *weight lattice* of this algebra; note that the quotient of the latter with the former is \mathbb{Z}_2 , as required for $su(2)$.

The stress energy tensor is given by Eq.([3.4\)](#page-10-0), where the height field has one component. The stress energy tensor is in the Sugawara form [\[29\]](#page-37-0), and therefore the effective field theory of the six-vertex model with vertex weights $a = b = c/2 = 1$, is the $SU(2)_{k=1}$ Wess-Zumino-Witten (WZW) model. In other words, the Gaussian field theory that we have proposed as the effective field theory of the BCSOS model, is the free field representation of this WZW model.

V. THREE-COLORING MODEL

In this section we show that the symmetry algebra associated with the three coloring model is the $su(3)_{k=1}$ Kac–Moody algebra. This is accomplished by mapping the coloring model to an interface model. We find that the effective field theory of the interface model is Gaussian with a height field that is compactified on the root lattice of the $su(3)$ Lie algebra.

A. Definitions

The three-coloring model is defined by coloring the bonds of the honeycomb lattice with three different colors, say A , B , and C , in such a way that no two bonds of equal color meet at a vertex.⁸ Each coloring is given equal statistical weight, and the partition function can be written as $Z_0 = \sum_{\mathcal{C}} 1$, where the sum goes over all allowed coloring configurations $\mathcal C$ of the honeycomb lattice. If we think of the colors as representing spins in the three-

⁸ This type of coloring is referred to in the mathematical literature as an *edge coloring*.

state antiferromagnetic Potts model on the Kagomé lattice ⁹, then the coloring model is the ground state of the Potts model [\[7](#page-36-0)].

The three coloring model was considered by Baxter [\[38\]](#page-38-0), who calculated Z_0 exactly. For a honeycomb lattice of N sites he found a non-zero entropy per site $s = \lim_{N \to \infty} \frac{\ln Z_0}{N}$ $0.1895...$ Below we argue that the three coloring model is critical, with power law correlations which are described in the continuum by the $SU(3)_{k=1}$ Wess-Zumino-Witten model.

B. The interface model

The three coloring model can be mapped to a solid-on-solid model which describes a two dimensional interface in four spatial dimensions [\[7,13](#page-36-0)]. This is accomplished by placing a two component microscopic height $z = (z_1, z_2)$ at the center of each plaquette of the honeycomb lattice. The change in z when going from one plaquette to the neighboring one is given by A, B, or C, depending on the color of the bond that is crossed; see Fig. [7](#page-40-0). The vectors A, B, and C point to the vertices of an equilateral triangle:

$$
\mathbf{A} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \quad \mathbf{B} = (1, 0), \quad \mathbf{C} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right). \tag{5.1}
$$

This ensures that the coloring constraint becomes the zero curl condition for the height increments, i.e. the change in height when going around a plaquette of the triangular lattice, on which the heights are defined, is $\Delta z = \mathbf{A} + \mathbf{B} + \mathbf{C} = 0$. Each allowed height configuration is given equal statistical weight. We will be interested in the long-wavelength fluctuations of the interface for which we introduce an effective field theory. We emphasize that the height fluctuations are entropy driven, in the sense that they are solely due to the different ways of edge-coloring the honeycomb lattice with three different colors.

We motivate the long-wavelength theory of the interface model by a coarse-graining procedure for the microscopic heights z, which was described for a general critical ground

⁹The bond midpoints of the honeycomb lattice form the Kagomé lattice

state in Sec. [II A.](#page-4-0) Here the general procedure is implemented as follows: First, we define the ideal states of the three-coloring model as states in which every elementary plaquette of the honeycomb lattice is colored with two colors only, Fig. [7](#page-40-0). These states are flat, in the sense that they have the smallest possible variance of the microscopic height. We argue that the free energy of the coloring model (which is purely entropic in origin) is dominated by fluctuations around the ideal states. Namely, the smallest change on the lattice, that is allowed by the constraints of the three-coloring model, is an exchange of colors along a loop of alternating color (eg. $A-B-A...$ to $B-A-B...$). The ideal states maximize the number of loops that allow for these loop exchanges, and it is this property that selects them out. This entropic selection effect is close in spirit to the "order by disorder" effect, introduced by Villain [\[39](#page-38-0)].

Second, we divide the honeycomb lattice into domains so that each domain represents a fluctuation away from a different ideal (flat) state. To each domain we assign a coarse grained height h, which is equal to the microscopic height averaged over the domain: $h = \langle z \rangle$. The coarse grained heights associated with different ideal states form a honeycomb lattice which is the ideal state graph \mathcal{I} , of the three-coloring model; see Fig. [8.](#page-40-0) The side of the elementary hexagon of $\mathcal I$ is $\sqrt{3}/3$, in the units chosen for the vectors representing the colors; see Eq. (5.1) . Nodes of $\mathcal I$ that correspond to the *same* ideal state form a triangular lattice with an elementary triangle of side $\sqrt{3}$. This is the repeat lattice R of the three-coloring model; points on the ideal state graph separated by vectors in R are identified,

$$
\mathbf{h} \equiv \mathbf{h} + \mathcal{R} \tag{5.2}
$$

Finally, we consider the continuum limit of the interface model in which the discrete heights, defined over different ideal state domains, are replaced with a continuously varying height field $h(\mathbf{x}) \equiv (h_1(\mathbf{x}), h_2(\mathbf{x}))$. Since nodes of the ideal state graph separated by repeat lattice vectors are identified Eq. (5.2), we take the height field to be compactified on the torus, $h(x) \in \mathbb{R}^2/\mathcal{R}$. This property of the height field will be important in analyzing the symmetry algebra of the three coloring model. The dimensionless free energy (action) of the interface, which is entropic in origin, is assumed to be of the form

$$
f = \int d^2 \mathbf{x} \left[\frac{K}{2} (|\nabla h_1|^2 + |\nabla h_2|^2) + V(\mathbf{h}) \right],
$$
 (5.3)

where K is the stiffness, and $V(\bf{h})$ is a periodic potential with the periodicity given by the ideal state graph,

$$
V(\mathbf{h} + \mathcal{I}) = V(\mathbf{h}) \tag{5.4}
$$

The free energy f defines the effective field theory of the three-coloring model; the assumption being made is that it correctly describes the long-wavelength fluctuations of the microscopic height z. The periodic (locking) potential $V(\bf{h})$ favors the heights to take their values on \mathcal{I} , while the first term represents fluctuations around the flat ideal states.

If the locking potential is irrelevant, in the renormalization group sense, than the effective field theory of the three-coloring model is a Gaussian field theory with a dimensionless free energy (action) given by:

$$
f = \frac{K}{2} \int d^2 \mathbf{x} \left(|\nabla h_1|^2 + |\nabla h_2|^2 \right).
$$
 (5.5)

In the case that the locking potential is relevant, the three-coloring model will lock into long range order in one of the ideal states. Which of the two possibilities is actually realized in this model is determined by the value of the stiffness K in Eq. (5.3).

The scaling dimension of the locking potential can be calculated using the procedure outlined in Sec. [II B](#page-6-0). Namely, this operator has the periodicity of the ideal state graph which, as we saw earlier in this section, is a honeycomb lattice with a lattice constant $\sqrt{3}/3$. The Bravais lattice[[40\]](#page-38-0) of the ideal state graph is a triangular lattice with lattice constant 1, and its reciprocal lattice is also triangular, with lattice constant $\frac{4\pi}{3}$ $\sqrt{3}$. This is also the length of the shortest vector $\mathbf{G}_V \in \mathcal{R}^*$ that appears in the Fourier expansion of $V(\mathbf{h})$. Therefore, from Eq.([2.8\)](#page-8-0), the scaling dimension of the locking potential is

$$
x(\mathbf{G}_V) = \frac{4\pi}{3K} \,. \tag{5.6}
$$

The locking potential will be irrelevant, in the renormalization group sense, if $x(\mathbf{G}_V) > 2$, or if the stiffness

$$
K < \frac{2\pi}{3} \tag{5.7}
$$

For $K=\frac{2\pi}{3}$ $\frac{2\pi}{3}$ the three-coloring model, in the interface representation, undergoes a roughening transition; for $K > \frac{2\pi}{3}$ the model will lock into long range order in one of the six ideal states.

C. Calculation of the stiffness

Here we examine the correlation function that measures the probability that two points on the honeycomb lattice are on the same loop of alternating color. The crucial observation that these loops are contour loops for a particular component of the microscopic height, will allow us to calculate the stiffness exactly.

If we choose two colors, for instance A and C , then due to the edge coloring constraint there will be a A-C-A-. . . loop passing through every vertex of the honeycomb lattice. The correlation function that measures the probability that two points, say one at 0 and the other at \bf{R} , belong to the *same AC* loop is

$$
\mathcal{G}(\mathbf{R}) = \frac{Z(\mathbf{R})}{Z_0} \,. \tag{5.8}
$$

The restricted partition function $Z(\mathbf{R})$ is simply the number of colorings with an AC loop passing through 0 and \mathbf{R} , while Z_0 is the total number of colorings. Now, if we exchange the two colors on the loop along one half of the loop going from 0 to \bf{R} , then this will generate a vortex and an antivortex in the interface model at these two points; see Fig. [9](#page-40-0). If we do this for all the configurations entering $Z(\mathbf{R})$, then $\mathcal{G}(\mathbf{R})$ becomes the vortex-antivortex correlation function. The Burgers charges associated with these vortices are:

$$
b_{\text{loop}} = \pm (A - C) = \pm (0, -\sqrt{3})
$$
\n(5.9)

and the loop correlation function is given by

$$
\mathcal{G}(\mathbf{R}) \sim R^{-2x_v(\mathbf{b}_{\text{loop}})}\,. \tag{5.10}
$$

The exponent $x_v(\mathbf{b}_{\text{loop}})$ is the scaling dimension of a magnetic type operator, and from Eqs. (2.9) and (5.9) we find

$$
x_v(\mathbf{b}_{\text{loop}}) = \frac{3K}{4\pi} \,. \tag{5.11}
$$

In the interface representation of the three-coloring model loops of alternating color become loops of constant height. This is most readily understood from an example: take an AC loop and consider the points at the centers of the hexagonal plaquettes along the inside of the loop (Fig. [9\)](#page-40-0). These points are separated by B colored bonds, and therefore the component of the microscopic height $\mathbf{z} \cdot \mathbf{e}_2$ is unchanged as we go around the loop, since the projection $\mathbf{B} \cdot \mathbf{e}_2 = 0$; $\{\mathbf{e}_1, \mathbf{e}_2\}$ are the orthonormal basis vectors in the height space. In general, every pair of colors defines a loop of alternating colors on the lattice, which is also a contour line of the component of z in the direction which is perpendicular to the vector representing the third color.

The probability that two points separated by R belong to the same contour loop of a Gaussianinterface scales as R^{-1} for large R (see Section IIC, Eq. ([2.10\)](#page-8-0)). Therefore,

$$
2x_v(\mathbf{b}_{\text{loop}}) = 1 \tag{5.12}
$$

and from Eq. (5.11)

$$
K = \frac{2\pi}{3} \tag{5.13}
$$

is the exact value of the stiffness.

If we compare Eq. (5.13) with Eq. [\(5.7](#page-23-0)) we can conclude that the three-coloring model in the interface representation is exactly at the roughening transition. This observation justifies the Gaussian form of the free energy, Eq.([5.5\)](#page-22-0), and it also follows from Baxter's exact solution of the three coloring model[[38\]](#page-38-0), as was shown by Huse and Rutenberg[[7\]](#page-36-0).

D. The symmetry algebra of the three-coloring model

In this section we analyze the conformal field theory of the three coloring model and we showthat it is given by an $SU(3)_{k=1}$ Wess–Zumino–Witten model [[23\]](#page-37-0), as conjectured by Read [\[9](#page-36-0)].

The effective field theory of the three-coloring model is a Gaussian field theory, Eq.([5.5\)](#page-22-0), compactified on the torus \mathbb{R}^2/\mathcal{R} , Eq. [\(5.2](#page-21-0)). This is a conformal field theory with conformal charge $c = 2$, independent of the value of the stiffness K.

The symmetry algebra of the effective field theory of the three–coloring model is the chiral $su(3)_{k=1}$ Kac-Moody algebra. In order to show this we explicitly construct the currents of the holomorphic half of the algebra from the holomorphic component of the height field $h(w)$, Eq.([3.3\)](#page-10-0), using the Frenkel–Kac–Segal vertex operator construction of level one Kac–Moody algebras [\[22\]](#page-37-0). The construction of the antiholomorphic half of the chiral algebra follows in the same fashion, the only difference being that $h(w)$ is replaced by the antiholomorphic component of the height field $\mathbf{h}(\bar{w})$.

There are eight conserved currents, equal in number to the dimension of the $su(3)$ Lie algebra. The two currents that correspond to the Cartan subalgebra of $su(3)$ are:

$$
H_1(w) = i\partial h_1(w) , \quad H_2(w) = i\partial h_2(w) . \tag{5.14}
$$

The remaining six currents, which are the raising and lowering operators associated with the positive roots α_j , are the vertex operators

$$
J_{\pm \mathbf{\alpha}_j}(w) = :e^{\pm i\mathbf{\alpha}_j \cdot \mathbf{h}(w)} : c_{\pm \mathbf{\alpha}_j} \quad (j = 1, 2, 3) \tag{5.15}
$$

The positive roots are vectors in the reciprocal lattice \mathcal{R}^* shown in Fig. [10](#page-40-0); they are of length $|\boldsymbol{\alpha}_j| = \frac{4\pi}{3}$ 3 $\sqrt{3}$,in units chosen for **A**, **B**, and **C** in Eq. ([5.1\)](#page-20-0). From Eq. ([3.10\)](#page-12-0) their conformal dimension is $(1,0)$ as expected for a current field. The vectors α_i generate a triangular lattice which is the *root* lattice of an $su(3)$ Lie algebra. The quotient of the reciprocal lattice, which we identify with the *weight* lattice of $su(3)$, and the root lattice, is \mathbb{Z}_3 , as required of an $su(3)$ Lie algebra.

Finally, since the stress energy tensor, Eq.([3.4\)](#page-10-0), is in the Sugawara form[[29\]](#page-37-0), we can conclude that the effective filed theory of the three-coloring model is the $SU(3)_{k=1}$ Wess-Zumino-Witten (WZW) model.

The existence of a hidden $SU(3)$ symmetry in the three-coloring model was previously shown by Read[[9\]](#page-36-0), who mapped the three-coloring model to a lattice model in which the symmetry is explicit. Our approach is very different, it shows the emergence of an $su(3)_{k=1}$ Kac–Moody algebra in the continuum, and it is based on the interface representation of the three-coloring model.

VI. THE FOUR COLORING MODEL

In this section we study the ground state of of the four-state antiferromagnetic Potts *vertex* model on the square lattice. This model was first introduced by Read $|9|$ as a generalization of the three-state antiferromagnetic Potts model on the Kagomé lattice. We have recently completed a detailed study of the ground state of this Potts model [\[10](#page-36-0)], both analytically and using Monte-Carlo simulations, and evidence was found in support of the claim that the ground state is indeed critical. Here we focus on the symmetry of the field theory that describes the continuum limit of this critical ground state, and we find that it is given by the chiral $su(4)_{k=1}$ Kac–Moody algebra.

A. Definitions

In this section we introduce the four-coloring model as the ground state of the antiferromagnetic four-state Potts vertex model on the square lattice.

Our starting point is the antiferromagnetic Potts vertex model given by the Hamiltonian

$$
\mathcal{H} = |J| \sum_{\mathbf{x}} \sum_{\substack{i,j=1 \\ i\n(6.1)
$$

where the Potts spins $\sigma_i(\mathbf{x})$ (i = 1, 2, 3, 4), live on the four bonds of the square lattice which share the same vertex \mathbf{x} ; each spin can be in one of four possible states labeled A, B, C , and

D. This Hamiltonian associates an energy penalty $|J|$ to having equal spins on two vertexsharing bonds of the square lattice. An alternative representation of this model, given by Read [\[9](#page-36-0)], involves the "crossed-square" lattice in which diagonal bonds are drawn on every other square plaquette, so that the crossed plaquettes form a checkerboard pattern. In this representation the Potts spins live on the vertices of the crossed-square lattice and have nearest neighbor antiferromagnetic interactions.

At zero temperature, the only allowed spin configurations are ones for which $\{\sigma_i(\mathbf{x}) : i =$ 1, 2, 3, 4} = { A, B, C, D } for every **x**. With { $\sigma_i(\mathbf{x})$: $i = 1, 2, 3, 4$ } we denote the *set* of Potts spins on the four bonds at the vertex **x**, while by the *ordered set* $(\sigma_1(\mathbf{x}), \sigma_2(\mathbf{x}), \sigma_3(\mathbf{x}), \sigma_4(\mathbf{x}))$ we will denote the particular arrangement of spins at **x**.

The ground state ensemble has an extensive entropy. Namely, the ground state entropy per site, defined as [\[4](#page-36-0)] $s = \lim_{N \to \infty} \frac{1}{N}$ $\frac{1}{N}$ ln(Z₀), where Z₀ is the number of ground states and N the number of sites, is non-zero. This can be easily verified by examining the state given in Fig. [11](#page-41-0): In every AB -plaquette the spins A and B can be exchanged independently of the other plaquettes. This gives rise to $2^{N/4}$ states, which puts a lower bound on the entropy per site at $s > \ln 2/4$.

If we think of the four Potts spins as four colors then the ground states correspond to the four-coloring model of the square lattice. In the four-coloring model each bond of the square lattice is colored with one of four different colors A, B, C or D , with the constraint that at each vertex all four colors meet. All such configurations are given the same statistical weight.

B. The interface model

Here we describe the mapping of the four-coloring model to an interface model, and we propose an effective field theory that describes the long-wavelength fluctuations of this interface. Further details of this mapping, and an analysis of the four-coloring operators in the interface representation, can be found in [\[10](#page-36-0)].

We define a height mapping by placing a *three-component* microscopic height $z \in \mathbb{Z}^3$ at the center of each elementary square (plaquette). The change in z, when going from one plaquette to a neighboring one, is given by $\Delta z = \sigma(x)$, where $\sigma(x)$ is the color of the bond that is crossed; see Fig. [11.](#page-41-0) The four possible color values that $\sigma(\mathbf{x})$ can take are represented by vectors pointing to the vertices of a tetrahedron:

$$
\mathbf{A} = (-1, +1, +1), \quad \mathbf{B} = (+1, +1, -1),
$$

$$
\mathbf{C} = (-1, -1, -1), \quad \mathbf{D} = (+1, -1, +1).
$$
 (6.2)

For a given configuration of colors, the set of microscopic heights defines a two dimensional interface in five dimensions. Each allowed microscopic height configuration is given equal statistical weight.

In order to define an effective field theory for the above described interface model, we introduce a coarse-graining procedure for the microscopic heights, following the general procedure outlined in Section [II A.](#page-4-0) First, we define ideal states of the four-coloring model as states in which every plaquette is colored by two colors only; see Fig. [11](#page-41-0). There are $24 = 4!$ ideal states related to each other by lattice symmetries, and each corresponds to a different permutation of the 4 colors. These states are flat, in the sense that the variance of the microscopic height is minimum. Furthermore, they have the important property that they are entropically selected, in the sense that ideal states allow for the maximum number of color rearrangements consistent with the ground state constraints. This point is crucial. Namely, if we wish to change the color of a bond, then the smallest change we can perform on the lattice, without violating the constraints of the four-coloring model, is an exchange of two colors along a loop of alternating color. Just as in the three-coloring model, the ideal states maximize the number of loops of alternating color.

Second, we replace the original model with a coarse grained version where the lattice is split into domains, such that in each domain fluctuations occur about a different ideal state. With each ideal-state domain we associate a coarse grained height **h**, which is given by the average microscopic height in that domain, $\mathbf{h} = <\mathbf{z}>$.

Third, we define the ideal state graph $\mathcal{I} \subset \mathbb{R}^3$. Every node of \mathcal{I} represents an ideal state, and its position in \mathbb{R}^3 is given by the coarse grained height **h** of the ideal state it represents. The 24 *different* ideal states form a truncated octahedron with a side of length $\sqrt{2}/2$ in the units chosen for the vectors \bf{A} , \bf{B} , \bf{C} and \bf{D} ; see Fig. [12](#page-41-0). These truncated octahedra are arranged *periodically* in a face centered cubic (FCC) lattice, which is the repeat lattice \mathcal{R} , to form the full ideal state graph. The side of the conventional cubic cell of $\mathcal R$ is 4. Since h and $h + \mathcal{R}$ represent the same ideal state on the ideal state graph, we can identify them

$$
\mathbf{h} \equiv \mathbf{h} + \mathcal{R} \,. \tag{6.3}
$$

In other words the height is compactified on the three-torus \mathbb{R}^3/\mathcal{R} .

Finally, we consider the long-wavelength limit of the interface model, where the heights defined over particular ideal state domains are replaced with a continuously varying height field $h(\mathbf{x}) \equiv (h_1(\mathbf{x}), h_2(\mathbf{x}), h_3(\mathbf{x}))$. The dimensionless free energy (action) of the interface, which is entropic in origin, is assumed to be of the form:

$$
f = \int d^2 \mathbf{x} \left[\frac{K}{2} (|\nabla h_1|^2 + |\nabla h_2|^2 + |\nabla h_3|^2) + V(\mathbf{h}) \right],
$$
 (6.4)

where $V(\mathbf{h})$ is a periodic potential with the periodicity given by the ideal state graph,

$$
V(\mathbf{h} + \mathcal{I}) = V(\mathbf{h}) \tag{6.5}
$$

The free energy f defines an effective field theory of the four-coloring model. The periodic potential $V(\bf{h})$, which is usually referred to as the *locking potential* [[24\]](#page-37-0), favors the heights to take their values on \mathcal{I} , while the first term represents fluctuations around the flat ideal states.

The locking potential is periodic in height space with the periodicity of \mathcal{I} , and therefore its scaling dimension is given by Eq.([2.8\)](#page-8-0), i.e.,

$$
x(\mathbf{G}_V) = \frac{\pi}{2K} ; \qquad (6.6)
$$

 $|\mathbf{G}_V| = \sqrt{2} \pi$ is the shortest reciprocal lattice vector appearing in the Fourier expansion of $V(\mathbf{h})$. The magnitude of \mathbf{G}_V can be deduced from Fig. [12,](#page-41-0) which shows that the Bravais lattice of the ideal state graph is a body centered cubic (BCC) lattice, with a conventional cubic cell whose side is of length 2. The reciprocal of this lattice is an FCC lattice with a conventional cubic cell of side 2π , and consequently the magnitude of its shortest lattice vector (\mathbf{G}_V) is $2\pi/\sqrt{2}$.

If the stiffness K satisfies $K < \pi/4$, then the locking potential in Eq. [\(6.4](#page-29-0)) is irrelevant in the renormalization group sense, i.e., $x(\mathbf{G}_V) > 2$, and the four-coloring model is described by a Gaussian free energy (action)

$$
f = \frac{K}{2} \int d^2 \mathbf{x} \left(|\nabla h_1|^2 + |\nabla h_2|^2 + |\nabla h_3|^2 \right).
$$
 (6.7)

In the case that the locking term is relevant $(K > \pi/4, \text{ i.e., } x(\mathbf{G}_V) < 2)$, the four-coloring model will lock into long range order in one of the ideal states. We will see later that, just as in the six-vertex model with $a = b = c/2 = 1$, and the three-coloring model, the four coloring model is at the roughening transition.

C. Calculation of the stiffness

In perfect analogy with the six-vertex model and the three-coloring model, we calculate the stiffness of the four-coloring model using the loop trick. Namely, we express the loop correlation function exponent x_{loop} , for contour loops on a Gaussian interface, in terms of the stiffness, and then we use the universal value of this exponent, $x_{\text{loop}} = 1/2$ (Eq. [\(2.10\)](#page-8-0)).

We start by calculating the loop correlation function $\mathcal{G}(\mathbf{R})$, for loops of alternating color in the four-coloring model, in the familiar way[[18](#page-36-0)]. If we denote the number of configurations with a loop of alternating color passing through points 0 and **R** as $Z(\mathbf{R})$, then $\mathcal{G}(\mathbf{R})$ can be written as

$$
\mathcal{G}(\mathbf{R}) = \frac{Z(\mathbf{R})}{Z_0} \,,\tag{6.8}
$$

where Z_0 is the total number of configurations; Z_0 is also the partition function, since all configurations in the four-coloring model have equal statistical weight. Now, if we exchange the two colors on the loop along one half of the loop going from 0 to \bf{R} , then this generates a vortex and an antivortex in the interface model, at these two points. For example, if the loop consists of alternating A and B colored bonds, then the color configuration, after flipping half the loop, is (C, D, A, A) at one end, and (B, B, C, D) at the other (see Fig. [13\)](#page-41-0). Using the height rule we find that the Burgers vectors associated with these "defects" are

$$
\pm \mathbf{b}_{\text{loop}} = \pm (\mathbf{A} - \mathbf{B}) = \pm (-2, 0, 2) . \tag{6.9}
$$

Therefore, the loop correlation function is equal to the probability of having a vortexantivortex pair of Burgers charge $\pm b_{\text{loop}}$ separated by **R**, in the interface model. The vortex-antivortex correlation function scales with distance as[[24\]](#page-37-0)

$$
\mathcal{G}(\mathbf{R}) \sim |\mathbf{R}|^{-2x_v(\mathbf{b}_{\text{loop}})},\tag{6.10}
$$

where the exponent $x_v(\mathbf{b}_{\text{loop}})$ is given by Eq. [\(2.9](#page-8-0)).

In the interface representation, loops of alternating color become loops of constant height. This is most readily understood from an example: take an AB loop, and consider the points at the centers of the plaquettes along the inside of the loop (Fig. [13\)](#page-41-0). These points are separated by C and D bonds only, and therefore the component of the microscopic height **z** in the $e_1 - e_3$ direction is unchanged as we go around the loop. This follows from the fact that the projections of both **C** and **D** are zero in this direction; $\{e_1, e_2, e_3\}$ are the orthonormal basis vectors in the height space. In general, every pair of colors defines a loop of alternating color, which is a contour line of the component of z in the direction which is perpendicular to the vectors representing the other two colors.

The observation that loops of alternating color are contour lines leads to the equation

$$
x_v(\mathbf{b}_{\text{loop}}) = x_{\text{loop}} = \frac{1}{2} \,. \tag{6.11}
$$

By making use of Eq. [\(2.9](#page-8-0)) and Eq. (6.9) in the above equation, we find for the exact value of the stiffness

$$
K = \frac{\pi}{4} \tag{6.12}
$$

As advertised earlier, the exact value of K is such that the locking potential is marginal, and we conclude that the interface model is at the roughening transition. The exact scaling dimensions of operators in the four-coloring model are completely determined by K. It is worth noting that we have measured K in Monte-Carlo simulations of the four-coloring model [\[10\]](#page-36-0) and we find $K^{-1} = 1.28 \pm 0.01$, in excellent agreement with the exact result $K^{-1} = 1.273 \dots$, Eq. [\(6.12](#page-31-0)).

D. The symmetry algebra of the four-coloring model

We have seen that the four-coloring model can be mapped onto a Gaussian interface, Eq. (6.7) (6.7) , with a three-component height that is *compactified* on the repeat lattice R, Eq.([6.3\)](#page-29-0). Thus, the conformal field theory (CFT) that emerges in the scaling limit is a rather simple one. It corresponds to three massless free bosons $\mathbf{h} = (h_1, h_2, h_3)$ compactified on the face-centered cubic (FCC) lattice R . The conformal charge of this CFT is $c = 3$.

Theroot lattice of an $su(4)$ Lie algebra is also an FCC lattice [[29](#page-37-0)]. Therefore, if the radius of compactification of the height field has the correct value, then the CFT has an infinite symmetry larger than the usual conformal symmetry, and it is given by the $su(4)_{k=1}$ Kac-Moody algebra. The radius of compactification is equivalent to the stiffness of the interface, which was calculated exactly in the previous section; see Eq.([6.12\)](#page-31-0).

The Kac-Moody algebra associated with the scaling limit of the four-coloring model is chiral. The currents of the holomorphic half of the chiral algebra, that correspond to the Cartan subalgebra of $su(4)$ are

$$
H_1(w) = i\partial h_1(w) , \quad H_2(w) = i\partial h_2(w) , \quad H_3(w) = i\partial h_3(w) , \tag{6.13}
$$

while the remaining twelve, associated with the raising and lowering operators, are given by

$$
J_{\mathbf{\alpha}_j}(w) = :e^{i\mathbf{\alpha}_j \cdot \mathbf{h}(w)} : c_{\mathbf{\alpha}_j} \quad (j = 1, 2, \dots, 12) \tag{6.14}
$$

The currents of the antiholomorphic half of the chiral algebra are also given by Eqs. (6.13) and (6.14), only now $h(w)$ is replaced with the antiholomorphic component of the height field $h(\bar{w})$. The twelve roots α_j are $(1, 1, 0)$ -type vectors in the reciprocal lattice, which we identify with the *weight* lattice of the $su(4)$ Lie algebra; see Fig. [14](#page-41-0). The lattice generated by the roots is the root lattice and the quotient of the weight lattice by the root lattice is \mathbb{Z}_4 , as required of an $su(4)$ Lie algebra. Here we see the special role played by the stiffness K.Its value, Eq. ([6.12](#page-31-0)), ensures that the conformal dimension of the fields $J_{\mathbf{\alpha}_j}(w)$ is $(1,0)$, Eq.([3.10\)](#page-12-0), which is necessary if $J_{\mathbf{\alpha}_j}(w)$ are to be conserved currents.

The idea that the scaling limit of the four-coloring model is given by the $SU(4)_{k=1}$ Wess-Zumino-Witten (WZW) model was first put forward by N. Read [[9](#page-36-0)]. He showed that the four-coloring model is equivalent to a lattice model with explicit $SU(4)$ symmetry and went on to conjecture that the scaling limit is given by a WZW model. From the interface representation of the four-coloring model, we have shown the emergence of a chiral $su(4)_{k=1}$ Kac–Moody algebra in the scaling limit. Taking into account the fact that the stress-energy tensor is in the Sugawara form (see Eq. [\(3.4](#page-10-0))) proves the conjecture put forward by Read.

VII. SUMMARY AND REMARKS

In this paper we have developed a fairly simple geometric approach to analyzing the symmetry properties of discrete spin models with critical ground states. This approach combines certain ideas developed in the study of interface models and the roughening transition, with the vertex operator construction of level-one Kac–Moody algebras.

The first step is to map a critical ground state to an interface model. The stiffness of the interface plays a central role as it determines the values of the scaling dimensions of all the operators in the critical ground state. In order to calculate the stiffness we turn to the loop correlation function $\mathcal{G}(\mathbf{x})$, which is defined as the probability that two points, separated by x, belong to the same contour loop of a Gaussian (rough) interface. In the critical ground states studied here, we were able to identify contour loops of the associated interface models, and express $\mathcal{G}(\mathbf{x})$ as a vortex-antivortex correlation function. From the universal, i.e., stiffness *independent* value of the loop correlation function exponent $x_{\text{loop}} = 1/2$, the

stiffness of the interface could be calculated exactly. This was accomplished by identifying x_{loop} with the vortex-antivortex correlation function exponent, which is stiffness *dependent*. The value of the stiffness, and the fact that the heights are compactified on a lattice, conspire to give a Kac–Moody algebra as the symmetry algebra of the conformal field theory, which describes the long-wavelength fluctuations of the interface. In particular, for the six-vertex model with $a = b = c/2 = 1$, the three-coloring model, and the four-coloring model, we found the value of the stiffness to be such, that the associated interface models, in all three cases, are exactly at the roughening transition. Furthermore, we were able to show that the effective field theories of these interface models are the $SU(2)_{k=1}$, $SU(3)_{k=1}$, and $SU(4)_{k=1}$ Wess-Zumino-Witten model, respectively.

A couple of final remarks are in order. The three models we have studied here share a common property, namely, each one can be mapped to a fully packed loop (FPL) model. The building blocks of an FPL model are loops that run along the bonds of a two-dimensional lattice, with the constraints that every vertex of the lattice belongs to a loop, and the loops do not intersect. The six-vertex model maps to an FPL model on the square lattice via the break up procedure described in Section [IV C.](#page-16-0) Here the loops are loops of constant arrow direction. The three-coloring model maps to the fully packed loop model on the honeycomb lattice[[11\]](#page-36-0) where the loops are loops of alternating color, say A and B [\[13\]](#page-36-0). Finally, the four-coloring model is equivalent to a fully packed loop model with two loop flavors, say AB and CD [\[10\]](#page-36-0); in this case we allow for two loops of different flavor to intersect.

The fugacity of loops in all of the above mentioned models is two, corresponding to the two possible states each loop can be in; there are two choices for the arrow direction along a loop in the six-vertex model, and two ways of coloring a loop of alternating color, in the coloring models. Lowering the loop fugacity leaves the loop models critical; on the level of the effective field theory of the loop model, this corresponds to perturbing the appropriate Wess-Zumino-Witten model by an exactly marginal operator and introducing a background charge [\[41\]](#page-38-0). Increasing the loop fugacity above two, on the other hand, leads to a finite correlation length which is roughly the size of the largest loop. In the limit of vanishing fugacities these loop models define different variants of the self-avoiding walk problem.

Recently Batchelor *et al.* [[12\]](#page-36-0) found a Bethe ansatz solution of the fully packed loop model on the honeycomb lattice. They calculated the so-called watermelon dimensions, and the conformal charge, as a function of the loop fugacity. We have been able to reproduce these results from an analysis based on the mapping to an interface model[[41\]](#page-38-0), and were also able to calculate the so-called temperature dimension which was found numerically by Blöteand Nienhuis [[1](#page-36-0)], and which does not appear in the Bethe ansatz solution. Analogous calculations can be done for the other loop models, and the conformal charge, as well as the complete spectrum of scaling dimensions along the critical line, can be determined.

It is possible to define N-coloring models as a generalization of the 3- and 4-coloring model. For $N = 6$, the simplest realization is given by the 6-coloring model on the triangular lattice, where the bonds are colored with six different colors so that at each vertex all six are represented. It would be interesting to see if these models have an $su(N)$ symmetry associated with them. We hope to address this problem in the near future, both numerically and analytically.

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FIGURES

FIG. 1. The construction of a fluctuating interface equivalent to the ground state ensemble of a discrete spin model: a) The microscopic spin configuration is broken up into ideal-state domains. b) Each ideal state domain is assigned a coarse grained height equal to the average microscopic height of the domain. c) Finally, the discrete heights are replaced by a continuous height field $h(x)$; the interface is assumed to be rough.

FIG. 2. Vertex configurations of the six-vertex model, and their respective Boltzmann weights, $a, b, \text{ and } c.$

FIG. 3. One of the two symmetry related ideal states of the six-vertex model. The heights are defined at the vertices of the dual lattice via the height rule. Note that the ideal state is macroscopically flat.

FIG. 4. The ideal state graph of the six-vertex model. The ideal states are represented by vertex configurations at the origin, and the coarse grained height h assigned to each one is equal to the average microscopic height. The graph is a one-dimensional lattice with a lattice spacing equal to 1. Every two vertices the ideal states repeat, and therefore the repeat lattice vectors are integer multiples of 2.

FIG. 5. The six-vertex model with vertex weights $a = b = c/2 = 1$ can be turned into a model of non-intersecting loops that cover all the bonds of the square lattice. Every vertex configuration is broken up into a loop corner: for the a- and b-type vertices this is done in a unique way, while the two break up possibilities for the c-type vertex are assumed equally likely. This break up procedure gives every one of the eight possible loop configurations at a vertex equal statistical weight. The loops generated in this way are contour loops for the BCSOS model.

FIG. 6. Vortices in the BCSOS model correspond to violations of the ice rule in the six-vertex model. A vortex-antivortex pair (circled) appears when arrows along one half of a loop of constant arrow direction, are flipped. These loops are contour lines in the BCSOS model. The Burgers charge of the vortices is $b_1 = \pm 3$.

FIG. 7. One of six symmetry related ideal states of the three-coloring model. In an ideal state all the plaquettes are colored with two colors only. The microscopic heights z are defined at the centers of the plaquettes, and the change in z, when going from one plaquette to the neighboring one, is determined by the color of the crossed bond. The ideal state is macroscopically flat, in the sense that the variance of the microscopic height is minimal.

FIG. 8. The ideal state graph of the three coloring model is a honeycomb lattice: each vertex is associated with a particular ideal state, and the six different ideal states form a hexagonal plaquette. The ideal states are labeled by the color configuration $(\sigma_1, \sigma_2, \sigma_3)$ of the bonds around the origin. The vertices in the ideal state graph that correspond to the *same* ideal state (say (C, B, A) form a triangular lattice which is the repeat lattice of the three-coloring model.

FIG. 9. Elementary defects in the three-coloring model are associated with loops of alternating color: exchanging the two colors (A and C) along one half of the loop (shown in bold) will generate defects (circled), i.e., violations of the edge coloring constraint, at the two ends. In the interface representation these defects become vortex-antivortex configurations of the height. The loop correlation function is equal to the vortex-antivortex correlation function.

FIG. 10. The three positive roots are identified with the second shortest reciprocal lattice vectors; the reciprocal lattice is a triangular lattice, with the lattice spacing $\frac{4\pi}{3}$. The vertex operators associated with the roots are currents of the $su(3)_{k=1}$ Kac–Moody algebra.

FIG. 11. One of 24 symmetry related ideal states of the four-coloring model. Using the height rule defined in the text, every plaquette is assigned a microscopic height z. Note that in the ideal state the height describes, on average, a flat interface with a fast modulation of the microscopic height.

FIG. 12. The ideal state graph of the four-coloring model. The vertices that correspond to the 24 different ideal states form a truncated octahedron. The full ideal state graph corresponds to a periodic tiling of space with these octahedra. They are arranged in a face-centered cubic lattice, which is the repeat lattice of the four-coloring model.

FIG. 13. A vortex-antivortex pair (circled) in the four-coloring model; the pair belongs to a loop of alternating color (bold). The loop correlation function is equal to the vortex-antivortex correlation function. This can be seen by exchanging the colors $(A \text{ and } B)$ along one part of the loop between the circled vertices.

FIG. 14. The twelve root vectors are $(1,1,0)$ -type lattice vectors in the BCC lattice \mathcal{R}^* ; they lie in planes perpendicular to $(1,0,0)$ -type vectors. The lattice \mathcal{R}^* is identified with the weight lattice of the $su(4)$ Lie algebra, while the root vectors generate the root lattice. Vertex operators associated with the root vectors are currents of the $su(4)_{k=1}$ Kac–Moody algebra.

(c)

