ON ASYMPTOTIC PROPERTIES OF LARGE RANDOM MATRICES WITH INDEPENDENT ENTRIES

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Abstract

We study the normalized trace $g_n(z) = n^{-1}\mathrm{tr}\,(H - zI)^{-1}$ of the resolvent of $n \times n$ real symmetric matrices $H = [(1+\delta_{jk})W_{jk}/\sqrt{n}]_{j,k=1}^{n}$ assuming that their entries are independent but not necessarily identically distributed random variables. We develop a rigorous method of asymptotic analysis of moments of $g_n(z)$ for $|\Im z| \geq \eta_0$ where η_0 is determined by the second moment of W_{jk}. By using this method we find the asymptotic form of the expectation $\mathbf{E}\{g_n(z)\}$ and of the connected correlator $\mathbf{E}\{g_n(z_1)g_n(z_2)\} - \mathbf{E}\{g_n(z_1)\}\mathbf{E}\{g_n(z_2)\}$. We also prove that the centralized trace $ng_n(z) - \mathbf{E}\{ng_n(z)\}$ has the Gaussian distribution in the limit $n = \infty$. Basing on these results we present heuristic arguments supporting the universality property of the local eigenvalue statistics for this class of random matrix ensembles.

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I. INTRODUCTION

Since the pioneer works of Wigner and Dyson random matrix theory (RMT) has been successfully used to describe the energy levels of complex quantum systems: heavy nuclei, quantum chaotic systems, mesoscopic samples, etc (see e. g. Refs. 1–5). Another rather broad field of the RMT applications is related to quantum field theory: large colour limit of QCD, 2D quantum gravity and bosonic strings (see e.g. Refs. 6–8).

The phenomenological nature of the RMT approach that may be regarded as its certain drawback on the one hand, provides, on the other hand, the model independent frameworks, that make the approach applicable to a wide variety of systems having different microscopic natures and origins. These frameworks assume certain amount of "robustness" of the RMT models and results. In other words, it is believed that "sufficiently large" number of them should have no dependence or a rather weak one on a random matrix ensemble used. This belief explains partly the fact that majority of the RMT ideas and applications are based on results obtained for the archetype Gaussian Ensembles (GE's) and the Circular Ensembles (CE's). On the other hand, this belief requires certain justification, in particular extending the results known for the GE's and for the CE's to other classes of ensembles.

The most referred to are the Gaussian Orthogonal Ensemble (GOE) of random $n \times n$ symmetric matrices and the Gaussian Unitary Ensemble (GUE) of random $n \times n$ Hermitian matrices. The density of the probability distribution in these ensembles has the form

$$P_n(H) = Z_n^{-1} \exp[-n\mathrm{tr}\ F(H)], \tag{I.1}$$

where $F(x) = x^2/4w^2$ and Z_n is the normalization constant.

The probability distribution (I.1) possesses two important properties: i) it is invariant with respect to either orthogonal or unitary transformations of $\mathbf{R}^n$ or $\mathbf{C}^n$, respectively; and ii) the matrix elements are independent random variables (modulo the obvious symmetry conditions).

These properties of the GE's determine them uniquely and motivate two classes of generalisations of the GE's.

The first class consists of ensembles having an orthogonal or unitary invariant but not necessarily matrix-element-independent probability distribution. The typical representatives are the ensembles with the probability distribution of the form (I.1) in which $F(x)$ is an arbitrary bounded below and growing fast enough on infinity function. These invariant ensembles can be used to describe physical systems having no preferential basis. They arose also in studying the large-n limit in quantum field theory[6–8] and later found other applications[3,9,10].

Random matrices with invariant distributions show remarkable "robustness" (known as the universality) of spectral properties in the microscopic regime. In this regime one scales the energy so that the mean distance between nearest eigenvalues remains of order unity as the dimension of matrices increases[1,11]. Thus one is able to study properties of a finite number of eigenvalues. The universality of the level spacing distribution and other microscopic (local) spectral characteristics has been extensively discussed in recent theoretical physics and mathematical literature. We refer the reader to a number of publications: Refs. 11–16.

The second class consists of ensembles whose matrix elements in a certain basis are independent random variables, i. e. the ensemble probability distribution factorizes into a product of distributions of the matrix elements in this basis. The corresponding random matrices can be associated with physical systems having a preferential basis and appear, in particular, in condensed matter physics and theory of disordered systems. This second class goes back to Wigner[17] and we shall refer to the corresponding ensembles as Wigner ensembles (or Wigner matrices).

The subject of the present paper is the Wigner ensemble of $n \times n$ real symmetric matrices of the form

$$H = [H_{jk}]_{j,k=1}^{n}, \quad H_{jk} = (1+\delta_{jk})W_{jk}/\sqrt{n}, \tag{I.2}$$

where $W_{jk}, j \le k$ are independent random variables such that

$$\mathbf{E}\{W_{jk}\} = 0, \quad \mathbf{E}\{W_{jk}^2\} = w^2. \tag{I.3}$$

Here and thereafter $\mathbf{E}\{\cdot\}$ denotes averaging over all $W_{jk}, j \le k$.

The distributions of W_{jk}'s may depend on (j,k), but we assume that they are independent of n. We make the latter assumption mainly for the sake of technical simplicity. On the other hand, this assumption allows one to consider all W_{jk} on the same probability space and to find an optimal form of a number of important facts related to the Wigner ensembles (for example, the convergence with probability 1 in formulae (I.7) and (I.11) below). If W_{jk}'s are independent Gaussian random variables, then the ensemble (I.2)-(I.3) coincides with the GOE. This justifies the presence of the term with δ_{jk} in (I.2).

Macroscopic properties of Wigner ensembles are more or less well understood. We call macroscopic the asymptotic regime in which the number of eigenvalues in unit energy interval is proportional to n. Discussing macroscopic properties of random matrices we have to mention first of all the density of states (DOS) which is the simplest macroscopic characteristic of the ensemble eigenvalue statistics. It turns out that under rather natural and mild conditions on the distributions of W_{jk} the DOS in the Wigner ensemble (I.2)-(I.3) does not depend on the form of the distributions of W_{jk}. This DOS is known as the Wigner semi-circle law (see Eq. (I.5) below). Other macroscopic spectral quantities such as the conductivity and the interband light absorption coefficient show the same "robustness"[19–21]. Definition of these quantities requires some care for Wigner matrices. However, the conductivity and the interband light absorption coefficient can be defined and computed for the so-called band random matrices and random operators with independent matrix elements that are quite close to the Wigner ensemble (I.2)-(I.3) in their macroscopic properties both technically and by results (see e.g. Ref. 22). As for the microscopic scale, supersymmetry calculations[18] suggest "robustness" (universality) of spectral properties of the Wigner ensemble (I.2)-(I.3) as well but evidence of this has not been rigorously established so far.

Introduce the normalized eigenvalue counting function

$$N_n(E) = n^{-1}\#\{E_j : E_j \text{ is an eigenvalue of } H \text{ and } E_j \le E\}. \tag{I.4}$$

Wigner in the end of fifties proved[17] that in the case of identically distributed W_{jk} having all moments $N_n(E)$ converges in probability as $n \to \infty$ to a non-decreasing function $N_{sc}(E)$ (the semicircle law) whose derivative (DOS) is

$$\rho(E) = \begin{cases} \dfrac{1}{2\pi w^2}\sqrt{4w^2 - E^2} & |E| \le 2w \\ 0 & |E| > 2w. \end{cases} \tag{I.5}$$

The modern formulation of Wigner's result is as follows. Let us consider random matrices (I.2)-(I.3) with mutually independent arbitrary distributed entries defined on a common probability space. Then the condition (the matrix analogue of the Lindeberg condition of probability theory)

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j \le k} \int_{|x| > \nu n^{1/2}} x^2 d\text{Prob}[W_{jk} \le x] = 0, \text{ for any } \nu > 0, \tag{I.6}$$

is sufficient[23] and necessary[24] for the following limiting relation

$$\lim_{n\to\infty} N_n(E) = N_{sc}(E). \tag{I.7}$$

to hold for every E with probability 1[25]. If we will not assume that Wjk are defined on the same probability space or if their probability distributions depend on n, then the same condition (I.6) will imply the convergence in probability in (I.7)

As usual in spectral theory, this result admits a natural reformulation in terms of the resolvent (Green's function). Indeed, the normalized trace of the resolvent

$$g_n(z) = n^{-1}\text{tr } (H - zI)^{-1} \tag{I.8}$$

is simply the Stieltjes transform of $N_n(E)$:

$$g_n(z) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{E_j - z} = \int \frac{dN_n(E)}{E - z}. \tag{I.9}$$

Denote the Stieltjes transform of the Wigner law (I.5) by $r(z)$,

$$r(z) = \int \frac{N_{sc}(dE)}{E - z} = \frac{-z + \sqrt{z^2 - 4w^2}}{2w^2}. \tag{I.10}$$

The obvious condition $\Im r(z)\Im z \geq 0$ determines the branch of the square root in (I.10). Due to one-to-one correspondence between non-decreasing functions and their Stieltjes transforms[26] (I.7) is equivalent to the following limiting relation

$$\lim_{n\to\infty} g_n(z) = r(z), \tag{I.11}$$

which holds with probability 1 for any non-real z.

The relation (I.11) and the obvious bound $|g_n(z)| \leq |\Im z|^{-1}$ imply that the variance of $g_n(z)$ vanishes as $n \to \infty$ and hence the moments

$$m_n^{(p)}(z_1,\ldots,z_p) = \mathbf{E}\Big\{\prod_{l=1}^{p} g_n(z_l)\Big\} \tag{I.12}$$

factorize:

$$m_n^{(p)}(z_1,\ldots,z_p) = \prod_{l=1}^{p} m_n^{(1)}(z_l) + o(1),\ n \to \infty \tag{I.13}$$

This factorization, that follows already from the convergence in probability in (I.7) or in (I.11) , is typical for the macroscopic regime and can be found hidden behind many calculations in this regime. It is known in fact since papers Refs. 17, 27, 23 and 28.

Since according to (11) $m_n^{(1)}(z) = \mathbf{E}\{g_n(z)\} = r(z)+o(1)$, the leading term of $m_n^{(p)}(z_1,\ldots,z_p)$ is $\prod_{l=1}^{p} r(z_l)$. It seems interesting from a number of points of view to find also sub-leading terms and their dependence on the probability distributions of matrix elements. For instance these sub-leading terms are important when we would like to go beyond the macroscopic regime, when we are computing connected correlators of $g_n(z)$, etc.

For the Gaussian entries respective corrections were studied in Ref. 29 where the formal perturbation theory with respect to H_{jk} and the respective diagrammatic technique were applied. This approach is an adaptation of the technique developed in Ref. 30 in order to construct the $1/n$ expansion for the random operator describing disordered systems on $\mathbf{Z}^d$ with n orbitals per site.

In Ref. 31 we suggested an approach that allows for the rigorous treatment of this problem in the general case of independent and arbitrary not necessarily identically distributed matrix elements. Our approach allows us to estimate remainders in respective asymptotic formulae and we show that these estimates are in a sense optimal. The approach is also free to the large extent from the cumbersome combinatorial problem of rearranging diagrams which is neccessary in order to carry out various "dressing" procedures. In particular, the dressing procedure that replaces the "bare" Green function $-1/z$ by $\lim_{n\to\infty} \mathbf{E}\{g_n(z)\}$ is automatic in our approach. Following Ref. 31 one is able to find as many terms in the asymptotic expansion of $m_n^{(p)}(z_1,\ldots,z_p)$ as needed, though the technical difficulties increase with the order.

In the present paper we use the general scheme of Ref. 31 in order to compute first terms in the asymptotic expansion of $\mathbf{E}\{g_n(z)\}$. We also prove that if the distributions of W_{jk} satisfy the Lindeberg condition (I.6) with x^2 in the integral replaced by x^4, and if in addition to (I.3) the fourth moments of W_{jk} do not depend on (j,k), then

$$F_n(z_1,z_2) = m_n^{(2)}(z_1,z_2) - m_n^{(1)}(z_1)m_n^{(1)}(z_2) = n^{-2}f(z_1,z_2) + o(n^{-2}), \tag{I.14}$$

where

$$f(z_1,z_2) = \tag{I.15}$$
$$\frac{2w^2}{[1-w^2r^2(z_1)][1-w^2r^2(z_2)]}\left[\frac{r(z_1)-r(z_2)}{z_1-z_2}\right]^2 + \frac{2\sigma r^3(z_1)r^3(z_2)}{[1-w^2r^2(z_1)][1-w^2r^2(z_2)]}.$$

and $\sigma = \mathbf{E}\{W_{jk}^4\} - 3\mathbf{E}^2\{W_{jk}^2\}$ is the excess of W_{jk}. We also establish a limit theorem for the centralized trace $ng_n(z) - \mathbf{E}\{ng_n(z)\}$ of the resolvent.

Unfortunately, our approach gives a bound for the remainder term in (I.14) containing a power of $|\Im z_1 \Im z_2|^{-1}$ as a factor. Thus we cannot treat rigorously the microscopic regime which requires $\Im z \propto 1/n$. On the other hand, the first term (15) of the asymptotic formula (14) is well defined in this regime and coincides with respective exact expression known for the GOE, provided that the latter is considered for large level spacings and is smoothed over an interval Δ such that $1/n \ll |\Delta| \ll 1$ in proper units (see Section VI). We feel therefore, that by using our procedure of the computing of corrections, i.e. keeping the imaginary part of energy fixed when n goes to infinity and then letting $\Im z$ go to zero , one may treat energy intervals that are very large on the microscopic scale. On this intermediate scale the second term in the r.h.s. of (I.15) which contains the probability distribution excess σ vanishes and the above mentioned universality restores.

Our article is organized as follows. In section II we present our basic tools. In sections III we calculate first terms of the asymptotic expansion for $\mathbf{E}\{g_n(z)\}$. In section IV we give a simple proof of (I.14)-(I.15) with $o(n^{-2})$ replaced by $O(n^{-5/2})$ provided that 5th absolute moment of W_{jk} is uniformly bounded. This result was cited without proof in Ref. 31. In section V we treat the general case of W_{jk} satisfying the higher order Lindeberg condition mentioned above. We prove that the fluctuations of $ng_n(z)$ around its mean value become Gaussian in the limit $n\to\infty$ and that the covariance of the limiting Gaussian function is $f(z_1,z_2)$, thus proving (I.14)-(I.15) in the general case. Section VI contains a discussion of some implications of our results.

II. PRELIMINARIES

In this section we present our basic technical tools.

(i) If ξ is a real-valued random variable such that $\mathbf{E}\{|\xi|^{p+2}\} < \infty$ and if $f(t)$ is a complex-valued function of a real variable such that its first $p+1$ derivatives are continuous and bounded, then

$$\mathbf{E}\{\xi f(\xi)\} = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!}\mathbf{E}\{f^{(a)}(\xi)\} + \varepsilon, \tag{II.16}$$

where κ_a are the semi-invariants (cumulants) of ξ, $|\varepsilon| \leq C \sup_t |f^{(p+1)}(t)| \, \mathbf{E}\{|\xi|^{p+2}\}$ and the quantity C depends on p only.

The semi-invariants can be expressed in terms of the moments. If $\mathbf{E}\{\xi\} = 0$ (the case we shall deal with) and $\mu_a = \mathbf{E}\{\xi^a\}$, then few first such relations are: $\kappa_1 = \mu_1 = 0$, $\kappa_2 = \mu_2$, $\kappa_3 = \mu_3$, $\kappa_4 = \mu_4 - 3\mu_2^2$, $\kappa_5 = \mu_5 - 10\mu_3\mu_2$, $\kappa_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 - 30\mu_2^3$, etc. For a Gaussian random variable with zero mean, all semi-invariants but κ_2 vanish and (II.16) reduces to the exact relation

$$\mathbf{E}\{\xi f(\xi)\} = \mathbf{E}\{\xi^2\}\mathbf{E}\{f'(\xi)\}, \tag{II.17}$$

which can directly be checked integrating the l.h.s. of (II.17) by parts. This is only the case , when formula (II.16) contains finite number of terms for non-polynomial f's. Indeed, according to the Marcynkiewicz theorem[32] if all but finite number of cumulants are zero, then only first and second can be nonzero.

(ii) For any matrix $A = [A_{\alpha\beta}]_{\alpha,\beta=1}^n$

$$\frac{\partial}{\partial A_{jk}} \left(A^{-1}\right)_{\alpha\beta} = -\left(A^{-1}\right)_{\alpha j} \left(A^{-1}\right)_{k\beta}$$

provided A^{-1} exists. For the resolvent G of a real symmetric matrix H this becomes

$$\frac{\partial G_{\alpha\beta}}{\partial H_{jk}} = \begin{cases} -G_{\alpha j}G_{k\beta}, & j = k \\ -G_{\alpha j}G_{k\beta} - G_{\alpha k}G_{j\beta}, & j \neq k \end{cases} \tag{II.18}$$

(iii) For any two real symmetric matrices and any non-real z the resolvent identity

$$(H_2 - zI)^{-1} = (H_1 - zI)^{-1} - (H_1 - zI)^{-1}(H_2 - H_1)(H_2 - zI)^{-1} \tag{II.19}$$

is valid. In particular, if $H_2 = H$, $H_1 = 0$ and $G = (H - zI)^{-1}$, then

$$G_{jm} = z^{-1}\delta_{jm} + z^{-1}\sum_{k=1}^{n} G_{jk}H_{km} \tag{II.20}$$

Let H belong to the Wigner ensemble (I.2)-(I.3). For a fixed complex z consider complex-valued random variable $g_n(z) = n^{-1}\text{tr }(H - zI)^{-1}$. Define its variance as

$$\mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\} = F_n(z, z^\dagger). \tag{II.21}$$

and define also the domain in the complex plane as follows

$$U_0 = \{z \in \mathbf{C}_\pm : |\Im z| \geq 2w\}. \tag{II.22}$$

We use † to denote complex conjugate and the sub-index C to indicate centering to zero mean. For instance, $g_n^C(z) = g_n(z) - \mathbf{E}\{g_n(z)$ and thus we can rewrite (II.21) as

$$F_n(z, z^\dagger) = \mathbf{E}\{|g_n^C(z)|^2\} = \mathbf{E}\{g_n(z)g_n^C(z^\dagger)\}. \tag{II.23}$$

We will write $O(n^{-p})$ in asymptotic formulae for the remainders having an uniform (with respect $z \in U_0$) upper bound of the form Cn^{-p} where C does not depend on n. In fact the bounds we are able to derive contain $1/(1 - |\Im z|^2/2w^2)$ (see e.g. formula (29) below for the simplest case). Thus C is finite for any fixed z satisfying $|\Im z| > \sqrt{2}w$. But we prefer to use $|\Im z| \geq 2w$ in favour of uniformity of the bounds with respect to $z \in U_0$.

(iv) Let H belong to the Wigner ensemble (I.2)-(I.3). Assume that the 5th absolute moment of the random variables W_{jk} is uniformly bounded, i. e. $\sup_{j\le k}\mathbf{E}\{|W_{jk}|^5\} < +\infty$, and that $z \in U_0$. Then

$$\mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\} = \mathbf{E}\{|g_n^C(z)|^2\} = O(n^{-2}), \text{ as } n \to \infty \tag{II.24}$$

Let us comment on (i)-(iv). Facts (ii) and (iii) are well known. The "decoupling" formula (II.16) is simple to understand in the case when ξ has all moments and $f(x)$ belongs to the Schwartz space. Indeed, by using the Parseval relation for the Fourier transforms we can rewrite the l.h.s. of (II.16) as

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}F^{\dagger}(t)\Pi(t)dt = -\frac{i}{2\pi}\int_{-\infty}^{\infty}F^{\dagger}(t)\frac{d}{dt}\Pi(t)dt \tag{II.25}$$

where

$$F(t) = \int_{-\infty}^{\infty} e^{i\xi t}f(\xi)d\xi \text{ and } \Pi(t) = \int_{-\infty}^{\infty} e^{i\xi t}dP(\xi)$$

are the Fourier transforms of $f(\xi)$ and of the probability distribution $P(\cdot)$ of ξ, respectively. Now, if we take into account that

$$\Pi(t) = \sum_{a=0}^{\infty}\frac{(it)^a\mu_a}{a!} \text{ and } u(t) \equiv \log\Pi(t) = \sum_{a=1}^{\infty}\frac{(it)^a\kappa_a}{a!}$$

we can rewrite the r.h.s. of (II.25) as

$$\begin{aligned} -\frac{i}{2\pi}\int_{-\infty}^{\infty}F^{\dagger}(t)u'(t)e^{u(t)}dt &= -\frac{i}{2\pi}\sum_{a=0}^{\infty}\frac{\kappa_{a+1}}{a!}\int_{-\infty}^{\infty}(it)^{a+1}F^{\dagger}(t)\Pi(t)dt \\ &= \sum_{a=0}^{\infty}\frac{\kappa_{a+1}}{a!}\mathbf{E}\{f^{(a)}(\xi)\} \end{aligned}$$

where we again used the Parseval relation. The latter formula is obviously (II.16) for $p = \infty$. The case when ξ has a finite number of moments and $f(\xi)$ has respective number of derivatives requires certain technicalities which we will not discuss here.

The bound (iv) plays an important role in many questions of random matrix theory and its applications. In fact, it is the simplest of bounds for connected correlators (cumulants) of $g_n(z)$ or, more generally, for cumulants of linear statistics of the eigenvalues (i.e. sums $n^{-1}\sum_{j=0}^{n}\phi(E_j)$ where $\phi(E)$ is smooth enough). We are going to present a detailed derivation of these bounds and asymptotics (both, for the Wigner ensembles and unitary invariant ensembles) in a subsequent publication.

Here we only outline the scheme of the derivation of the bound (iv) considering the simplest case of the GOE and treating it as a representative of the Wigner ensembles, i.e. ensembles with independent entries. Set $r_n(z) = \mathbf{E}\{g_n(z)\}$. Then, according to (II.17), (II.18) and (II.20) we have the relation

$$r_n(z) = -\frac{1}{z} - \frac{w^2}{z}\mathbf{E}\{g_n^2(z)\} + \frac{w^2}{n^2 z}\mathbf{E}\{\mathrm{tr}\ G^2\} \tag{II.26}$$

Applying similar arguments to $\mathbf{E}\{|g_n^2(z)|\}$ and using (II.26) we obtain the analogous relation for the variance (II.23)

$$F_n = -\frac{w^2}{z} E\{g_n^2(z) g_n^C(z^\dagger)\} - \frac{w^2}{n^2 z} \mathbf{E}\{[g_n^C(z)]^\dagger \text{tr } G^2\} - \frac{2w^2}{n^3 z} \mathbf{E}\{\text{tr } G(G^*)^2\} \tag{II.27}$$

where $G^* = (H - z^\dagger I)^{-1}$. By using the identity $E\{g_n^2(z) g_n^C(z^\dagger)\} = E\{(g_n(z) + E\{g_n(z)\}) |g_n^C(z)|^2\}$, Cauchy-Schwarz inequality and the inequality (III.33) below we can show that the first term in the r.h.s of (II.27) is bounded above by $2w^2\eta^{-2}F_n$, where $\eta = |\Im z|$, the second is bounded by $w^2(n\eta^2)^{-1}F_n^{1/2}$ and the third is bounded by $2w^2(n^2\eta^4)^{-1}$. As the result we obtain the following inequality for $\eta^2 > 2w^2$

$$(1 - \frac{2w^2}{\eta^2}) F_n - \frac{w^2}{n\eta^2} F_n^{1/2} - \frac{2w^2}{n^2\eta^4} \leq 0. \tag{II.28}$$

which implies that

$$F_n = \mathbf{E}\{|g_n^C(z)|^2\} \equiv \mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|^2\} \leq \frac{C_1}{n^2} \tag{II.29}$$

where $C_1 = \eta^{-2}(\epsilon - 1)^{-2} C_1(\epsilon)$, $\epsilon = \eta^2/2w^2$, $1 < \epsilon < \infty$ and $C_1(\epsilon)$ is finite for $1 \leq \epsilon < \infty$ Thus we have obtained (II.24) for the GOE. This is a simplest but typical bound that can be obtained by our method. In the general case of non-Gaussian W_{jk}'s one has to iterate the resolvent identity (20) and use (16) instead of (17), truncating this procedure on the proper step and estimating remainders by variants of arguments presented above.

The bound (II.29) and (II.26)allows us to prove (I.7) and (I.11) for the GOE. Indeed, combining (II.26) and (II.29) we obtain

$$|r_n(z) + \frac{1}{z} + \frac{w^2}{z} r_n^2(z)| \leq \frac{C_2}{n}$$

where C_2 has same properties as C_1 in (II.29). This bound and standard compactness arguments show that any limit point $r(z)$ of the sequence $\{r_n(z)\}$ satisfies the equation

$$w^2 r^2(z) + z r(z) + 1 = 0. \tag{II.30}$$

for $|\Im z| \geq \eta_0 > 0$. Since this equation has the unique solution (I.10) satisfying $\Im r(z) \Im z \geq 0$, we conclude that uniformly in $|\Im z| \geq \eta_0 > 0$ $\lim_{n\to\infty} r_n(z) = r(z)$ where $r(z)$ is given by (I.10). Besides, since the Gaussian W_{jk} satisfying (I.3) can be defined on the same probability space we conclude from (II.29) and the Borel-Cantelli lemma that (I.11) and (I.7) and are valid.

III. ASYMPTOTIC EXPANSION FOR $\mathbf{E}\{g_n(z)\}$

We recall our notation $m_n^{(1)}(z)$ for the mean value of $g_n(z) = n^{-1}\text{tr } (H - zI)^{-1}$. In this section we prove the following

Theorem 1 *Consider the Wigner ensemble of random real symmetric matrices with independent entries defined by (I.2)-(I.3). Assume additionally that the third and fourth moments of W_{jk} do not depend on j and k and that $\widehat{\mu_5} = \sup_{j\leq k} \mathbf{E}\{|W_{jk}|^5\} < +\infty$.*

Then the following asymptotic formula

$$m_n^{(1)}(z) = r(z) \left\{ 1 + \frac{1}{n} \left[\frac{w^2 r^2(z)}{[1 - w^2 r^2(z)]^2} + \frac{\sigma r^4(z)}{1 - w^2 r^2(z)} \right] \right\} + O(n^{-3/2}). \tag{III.31}$$

holds for any $z \in U_0$ (U_0 is defined in (II.22))

Proof. By the resolvent identity (II.19)-(II.20),

$$m_n^{(1)}(z) = -z^{-1} + (zn)^{-1} \sum_{j,m=1}^{n} \mathbf{E}\{G_{jm} H_{mj}\}. \tag{III.32}$$

If we were following the conventional perturbational-diagrammatic approach trying to develop the asymptotic expansion for $\mathbf{E}\{g_n(z)\}$, we would repeatedly iterate the resolvent identity selecting on each step the terms that contribute to the leading and sub-leading terms. The obvious drawback of such approach is that infinitely many iterations are needed and in the non-Gaussian case, when there is no analogue of the Wick theorem, the diagrammatic approach is rather complicated.

We propose making use of (II.16) instead of iterating the resolvent identity. For each pair (j, m), G_{jm} is a smooth function of H_{mj} and its derivatives are bounded because of (II.18) and the inequality

$$|G_{jm}| \le \|G\| \le |\Im z|^{-1} \tag{III.33}$$

which holds for the resolvent of any real symmetric matrix. In particular, $|D_{mj}^4 G_{jm}| \le C|\Im z|^{-5}$ where C is an absolute constant. Here and thereafter we use notation D_{mj} for $\partial/\partial H_{mj}$.

According to (I.2)-(I.3) and our assumptions, the fifth absolute moment of H_{mj} is of order $n^{-5/2}$. Thus applying (II.16) (with $p = 3$) to each of the summands in the r.h.s. of (III.32) one finds that

$$zm_n^{(1)}(z) = -1 + \sum_{a=1}^{3} \frac{1}{n^{(a+3)/2}} \sum_{j,m=1}^{n} \frac{\kappa_{a+1}(1+\delta_{jm})^{(a+1)/2}}{a!} \mathbf{E}\{D_{mj}^a G_{jm}\} + \varepsilon_n, \tag{III.34}$$

where κ_a are the semi-invariants of W_{mj} and

$$|\varepsilon_n| \le \frac{C}{n^{3/2}} \frac{\widehat{\mu_5}}{|\Im z|^5}.$$

Obviously, G is a complex symmetric matrix, i. e. $G_{jm} = G_{mj}$. By (II.18), $D_{mm}^a G_{mm} = a! G_{mm}^{a+1}$ and

$$-D_{mj}^1 G_{jm} = G_{jm}^2 + G_{jj} G_{mm} \tag{III.35}$$

$$D_{mj}^2 G_{jm} = 2G_{jm}^3 + 6G_{jm} G_{jj} G_{mm} \tag{III.36}$$

$$-D_{mj}^3 G_{jm} = 6G_{jm}^4 + 36 G_{jm}^2 G_{jj} G_{mm} + 6 G_{jj}^2 G_{mm}^2 \tag{III.37}$$

for distinct j and m. Let us set $\kappa_2 = w^2$ and $\kappa_4 = \sigma$ in (III.34). Then, as a consequence of (III.35)-(III.37),

$$zm_n^{(1)}(z) = -1 - w^2 m_n^{(2)}(z, z) - n^{-1}[w^2 \mathbf{E}\{c_n(z)\} + \sigma \mathbf{E}\{d_n^2(z)\}] + \varepsilon_n, \tag{III.38}$$

where

$$c_n(z) = \frac{1}{n} \sum_{j,m=1}^{n} G_{jm}^2, \quad d_n(z) = \frac{1}{n} \sum_{m=1}^{n} G_{mm}^2 \tag{III.39}$$

and

$$|\varepsilon_n| \le \frac{C}{n^{3/2}} \left(\frac{|\kappa_3|}{|\Im z|^3} + \frac{|\kappa_4|}{|\Im z|^4} + \frac{\widehat{\mu_5}}{|\Im z|^5} \right). \tag{III.40}$$

provided $|\Im z| 2w$ and n is large enough.

To infer (III.38)-(III.40) from (III.34), notice first that for the sum over $j = m$ in the r.h.s. of (III.34) we have the bound

$$\frac{1}{n^{(a+3)/2}}\left|\sum_{m=1}^{n} D^a_{mm} G_{mm}\right| \le \frac{C}{n^{(a+1)/2}|\Im z|^{a+1}} \propto \frac{1}{n^{(a+1)/2}} \le \frac{1}{n^{3/2}}, \quad a = 2, 3.$$

for all realizations of W_{jk}. Therefore, being interested in the leading-order and $1/n$-order terms of $m_n^{(1)}(z)$ we can omit δ_{jm} from the factor in front of the second and third derivatives. As for the first derivatives, it follows from (III.35),that for all j and m $(1+\delta_{jm})D^1_{jm}G_{jm} = G^2_{jm} + G_{jj}G_{mm}$ and the term arising from δ_{jm} contributes to $1/n$-order term in the asymptotic expansion of $m_n^{(1)}(z)$.

Now, G^2_{jm} in the r.h.s. of (III.35) makes $\mathbf{E}\{c_n(z)\}$ in (III.38) and $G_{jj}G_{mm}$ does $m_n^{(2)}(z,z)$.The term containing $G^2_{jj}G^2_{mm}$ in the r.h.s. of (III.37) leads to $\mathbf{E}\{d^2_n(z)\}$ in (III.38) and the rest in the r.h.s. of (III.36) and (III.37) contributes to ε_n in (III.38). Corresponding bounds for the terms coming from G^3_{jm} in (III.36) and from G^4_{jm} and $G^2_{jm}G_{jj}G_{mm}$ in (III.37) result from the simple inequality

$$n^{-1}\sum_{j,m=1}^{n} |G_{jm}|^p \le |\Im z|^{-p}, \quad p \ge 2 \tag{III.41}$$

which holds for the resolvent of any real symmetric matrix. Estimating the term coming from $G_{jm}G_{jj}G_{mm}$ in the r.h.s. of (III.36) requires a longer calculation. Set

$$h_n(z) = \frac{1}{n}\sum_{j,m=1}^{n} G_{jm}G_{jj}G_{mm}. \tag{III.42}$$

Substitute the r.h.s. of (II.20) for G_{jm} in $h_n(z)$. Then

$$z\mathbf{E}\{h_n(z)\} = -\frac{1}{n}\sum_{m=1}^{n}\mathbf{E}\{G^2_{mm}\} - w^2\mathbf{E}\{g_n(z)h_n(z)\} + O(n^{-1/2}) \tag{III.43}$$

as it follows from (II.16), (III.35)-(III.37) and simple resolvent bounds like (III.33) or (III.41). Here and below we use notation $O(n^{-p})$ for remainders admitting the upper bound Cn^{-p}, where C does not depend on n for $|\Im z| \ge 2w$.

According to (24) the variance of $g_n(z)$ is of order n^{-2} under our assumptions. In other words

$$m_n^{(2)}(z,z) = [m_n^{(1)}(z)]^2 + O(n^{-2}) \tag{III.44}$$

if $z \in U_0$. Obviously,

$$\mathbf{E}\{h_n(z)g_n(z)\} - \mathbf{E}\{h_n(z)\}\mathbf{E}\{g_n(z)\} = \mathbf{E}\{h_n(z)[g_n(z) - \mathbf{E}\{g_n(z)\}]\}$$

and by the Cauchy-Schwarz inequality

$$\mathbf{E}\{h_n(z)g_n(z)\} = \mathbf{E}\{h_n(z)\}m_n^{(1)}(z) + O(n^{-1}).$$

Therefore by (III.43),

$$[z - w^2 m_n^{(1)}(z)]\mathbf{E}\{h_n(z)\} = -\frac{1}{n}\sum_{m=1}^{n}\mathbf{E}\left\{G^2_{mm}\right\} + O(n^{-1/2}) \tag{III.45}$$

and $\mathbf{E}\{h_n(z)\}$ is of order unity. The term we wish to estimate is

$$\frac{2\kappa_3}{n^{5/2}}\sum_{j,m=1}^{n}\mathbf{E}\{G_{jm}G_{jj}G_{mm}\} = \frac{2\kappa_3}{n^{3/2}}\mathbf{E}\{h_n(z)\}.$$

and from (III.45) we see it is of order $n^{-3/2}$. This proves (III.38)-(III.40).

The calculation above is typical of our approach and uses (II.16) and (II.20) combined with simple resolvent bounds on different stages. In what follows we shall (I.14) and (I.15)often use similar calculations omitting details.

Equations (III.38)-(III.40) and (III.44) imply that

$$m_n^{(1)}(z) = r(z) + O(n^{-1}), \tag{III.46}$$

where $r(z)$ solves the equation (II.30). Because of (I.9) $m_n^1(z)$ as a function of z must satisfy the inequality $\Im r(z)\Im z \geq 0$. This restriction fix the branch of the square root in the expression for the solutions of (II.30). Thus $r(z)$ coincides with the Stieltjes transform (I.10) of the semi-circle law (I.5), as expected.

Once the leading term of $m_n^{(1)}(z)$ is found, we can proceed with finding the sub-leading term. From (III.38) it is clear that performing this task requires calculating the leading-order terms of $\mathbf{E}\{c_n(z)\}$ and $\mathbf{E}\{d_n^2(z)\}$. Substitute the r.h.s. of (II.20) for one of G_{jm} in $c_n(z)$ and apply (II.16). As a result,

$$z\mathbf{E}\{c_n(z)\} = -m_n^{(1)}(z) - 2w^2\mathbf{E}\{c_n(z)g_n(z)\} + O(n^{-1}).$$

By (II.24),

$$z\mathbf{E}\{c_n(z)\} = -m_n^{(1)}(z) - 2w^2 m_n^{(1)}(z)\mathbf{E}\{c_n(z)\} + O(n^{-1}) \tag{III.47}$$

and in the leading order

$$\mathbf{E}\{c_n(z)\} = -m_n^{(1)}(z)[z + 2w^2 m_n^{(1)}(z)]^{-1}.$$

Taking into account (III.46)-(II.30) we conclude that

$$\mathbf{E}\{c_n(z)\} = r^2(z)[1 - w^2 r^2(z)]^{-1} + O(n^{-1}). \tag{III.48}$$

Now, calculate the leading-order term of $\mathbf{E}\{d_n^2(z)\}$. Recall that according to (II.24) the variance of $g_n(z)$ is of order n^{-2}. By (I.11), $g_n(z) = n^{-1}\sum_{m=1}^{n} G_{mm}$ converges almost surely to $r(z)$ as $n \to \infty$. Or, put it into another way, the Cesaro limit of G_{mm} is $r(z)$. This suggests that the Cesaro limit of G_{mm}^2 should be equal to $r^2(z)$, or in other words $d_n(z)$ should converge almost surely to $r^2(z)$. Therefore $\mathbf{E}\{d_n^2(z)\}$ should converge to $r^2(z)$.

To prove the convergence rigorously and to estimate its rate, we first note that the variance of $d_n(z)$ is of order n^{-1} if $z \in U_0$ (this can be proved following calculations of Appendix B). Therefore

$$\mathbf{E}\{d_n^2(z)\} = \mathbf{E}\{d_n(z)\}^2 + O(n^{-2}). \tag{III.49}$$

Thus, it suffices to find the leading-order term of $\mathbf{E}\{d_n(z)\}$.

Again, as in the case of $c_n(z)$, substitute the r.h.s. of (II.20) ($j = m$) for one of G_{mm} in $d_n(z)$ and apply (II.16). As a result,

$$z\mathbf{E}\{d_n(z)\} = -m_n^{(1)}(z) - w^2\mathbf{E}\{d_n(z)g_n(z)\} + O(n^{-1/2}).$$

By (II.24),

$$z\mathbf{E}\{d_n(z)\} = -m_n^{(1)}(z) - w^2 m_n^{(1)}(z)\mathbf{E}\{d_n(z)\} + O(n^{-1/2})$$

and

$$\mathbf{E}\{d_n(z)\} = -m_n^{(1)}(z)[z + w^2 m_n^{(1)}(z)]^{-1} + O(n^{-1/2}).$$

Finally by (III.46) and (II.30),

$$\mathbf{E}\{d_n(z)\} = r^2(z) + O(n^{-1/2}) \tag{III.50}$$

and by (III.49),

$$\mathbf{E}\{d_n^2(z)\} = r^4(z) + O(n^{-1/2}). \tag{III.51}$$

Now we are in a position to find the sub-leading term of $m_n^{(1)}(z)$. Collect (III.38), (III.44), (III.48) and (III.51) and write

$$zm_n^{(1)}(z) = -1 - w^2[m_n^{(1)}(z)]^2 - \frac{1}{n}\left[\frac{w^2r^2(z)}{1 - w^2r^2(z)} + \sigma r^4(z)\right] + O(n^{-3/2}).$$

In view of (III.46) and (II.30) this relation is obviously equivalent to the statement of the theorem, i.e. to the asymptotic formula (III.31). The theorem is proved.

Remarks

1) Our bound for the remainder in (III.31) is an optimal one. For assuming the 6th absolute moment of W_{jk} to be uniformly bounded and keeping one more term when applying (II.16), we can find the term of order $n^{-3/2}$ in the asymptotic expansion of $m_n^{(1)}(z)$. This term is proportional to $\kappa_3 = \mathbf{E}\{W_{jk}^3\}$

2) If the distributions of W_{jk} are such that $\mathbf{E}\{W_{jk}^3\} = 0$, then the bound $O(n^{-3/2})$ for the remainder in (III.31) can be strengthened to $O(n^{-2})$. For terms of order $n^{-3/2}$ appear in (III.31) due to contribution of $\kappa_3 n^{-3/2}\mathbf{E}\{h_n(z)\}$ to ε_n in (III.38) and also because of (III.51). If $\mathbf{E}\{W_{jk}^3\} = 0$, then $\kappa_3 = 0$ and we can prove that the remainder in (III.51) is of order n^{-1} (terms of order $n^{-1/2}$ in the r.h.s. of (III.51) are proportional to κ_3).

3) For Gaussian W_{jk} the excess σ is zero and (III.31) reduces to the asymptotic formula

$$\mathbf{E}\{g_n(z)\} = r(z)\left[1 + \frac{1}{n}\frac{w^2r^2(z)}{[1 - w^2r^2(z)]^2}\right] + O(n^{-2}).$$

that has been derived earlier by the formal diagrammatic approach[29].

IV. LEADING ORDER OF $F_n(z_1, z_2)$

Let us recall our notation $F_n(z_1, z_2)$ (see (I.14)) for the covariance function of $g_n(z) = n^{-1}\text{tr}\,(H - zI)^{-1}$,

$$F_n(z_1, z_2) = \mathbf{E}\{g_n^C(z_1)g_n^C(z_2)\} = \mathbf{E}\{g_n(z_1)g_n^C(z_2)\}$$

In this section we prove the following

Theorem 2 *Consider the Wigner ensemble of random real symmetric matrices with independent entries defined by (I.2)-(I.3). Assume additionally that the third and fourth moments of W_{jk} do not depend on j and k and that $\widehat{\mu_5} = \sup_{j\le k}\mathbf{E}\{|W_{jk}|^5\} < +\infty$*

Let $f(z_1, z_2)$ be the function given by (I.15). If z_1 and z_2 belong to U_0 (II.22), then the following asymptotic relation

$$F_n(z_1,z_2) = n^{-2} f(z_1,z_2) + O(n^{-5/2}) \tag{IV.52}$$

is valid.

Proof. Let us first prove (IV.52) under assumption

$$\widehat{\mu_7} = \sup_{j\le k} \mathbf{E}\{|W_{jk}|^7\} < +\infty. \tag{IV.53}$$

Let $G_{jm}(z)$ denote a matrix element of $(H - zI)^{-1}$. By (II.20),

$$z_1 F_n(z_1,z_2) = \frac{1}{n}\sum_{j,m=1}^{n} \mathbf{E}\{H_{mj} G_{jm}(z_1) g_n^C(z_2)\}.$$

For each pair (j,m) $G_{jm}(z_1)g_n^C(z_2)$ is a smooth function of H_{mj} and its derivatives are bounded because of (III.33). In particular, $|D^6_{mj}[G_{jm}(z_1)g_n^C(z_2)]| \le C(|\Im z_1|^{-1} + (|\Im z_2|^{-1})^8$. Therefore by (II.16),

$$zF_n(z_1,z_2) = \tag{IV.54}$$
$$\sum_{a=1}^{5} \frac{1}{n^{(a+3)/2}} \sum_{j,m=1}^{n} \frac{\kappa_{a+1}(1+\delta_{jm})^{(a+1)/2}}{a!} \mathbf{E}\{D^a_{mj}[G_{jm}(z_1)g_n^C(z_2)]\} + \varepsilon_n,$$

where κ_a are semi-invariants of W_{mj}, as in (III.34), and

$$|\varepsilon_n| \le n^{-5/2} C\hat{\mu}_7(|\Im z_1|^{-1} + (|\Im z_2|^{-1})^8$$

Performing differentiating in the r.h.s of (IV.54) one finds that the sums of fifth, fourth and second derivatives in the r.h.s. of (IV.54) contribute to $z_1F_n(z_1,z_2)$ terms of order $n^{-9/2}$, $n^{-7/2}$ and $n^{-5/2}$, respectively (corresponding bounds can be obtained using (II.24), (III.33) and (III.41)). So, these derivatives give no contribution to the leading-order term of $F_n(z_1,z_2)$. It remains to estimate the contributions coming from first and third derivatives.

The contribution of third derivatives to $z_1F_n(z_1,z_2)$ consists of several terms which we shall label by integer a and b satisfying $0 \le a, b \le 3$ and $a + b = 3$. These terms are

$$s_n^{(a,b)}(z_1,z_2) = \frac{\kappa_4}{6n^3} \sum_{j,m=1}^{n} \mathbf{E}\{D^a_{mj} G_{jm}(z_1) D^b_{mj} g_n^C(z_2)\}.$$

First estimate $s_n^{(3,0)}(z_1,z_2)$. After differentiating it takes the form

$$\begin{aligned} s_n^{(3,0)}(z_1,z_2) = -n^{-2}\kappa_4 \mathbf{E}\Big\{n^{-1}\sum_{j,m=1}^{n} G^4_{jm}(z_1) g_n^C(z_2) + 6h_n(z_1)g_n^C(z_2)\Big\} \\ -n^{-1}\kappa_4 \mathbf{E}\{d_n^2(z_1) g_n^C(z_2)\} \end{aligned} \tag{IV.55}$$

($h_n(z)$ and $d_n(z)$ are defined in (III.42) and (III.39), respectively). As it follows from (II.24) and (III.41), the mean value in the r.h.s. of the equation above is $O(n^{-1})$, so

$$s_n^{(3,0)}(z_1,z_2) = -n^{-1}\kappa_4 \mathbf{E}\{d_n^2(z_1) g_n^C(z_2)\} + O(n^{-3}).$$

Now we employ the obvious algebraic relation (in the below sub-index C indicates subtracted mean value)

$$\mathbf{E}\{\eta^2 \xi^C\} = 2\mathbf{E}\{\eta^C \xi^C\}\mathbf{E}\{\eta\} + \mathbf{E}\{(\eta^C)^2 \xi^C\} \tag{IV.56}$$

and write $\mathbf{E}\{d_n^2(z_1)g_n^C(z_2)\}$ as

$$2\mathbf{E}\{d_n^C(z_1)g_n^C(z_2)\}\mathbf{E}\{d_n(z_1)\} + \mathbf{E}\{[d_n^C(z_1)]^2 g_n^C(z_2)\}.$$

If $z \in U_0$, variances of $g_n(z)$ and $d_n(z)$ are of order n^{-2}. In addition to this, $d_n(z)$ is bounded in absolute value by $C|\Im z|^{-2}$ for all realizations of W_{jk}. Therefore by the Cauchy-Schwarz inequality, $\mathbf{E}\{d_n^2(z_1)g_n^C(z_2)\} = O(n^{-2})$, provided $z_1, z_2 \in U_0$. Thus we have proved that $s_n^{(3,0)}(z_1, z_2) = O(n^{-3})$. A similar argument shows that $s_n^{(0,3)}(z_1, z_2)$ and $s_n^{(2,1)}(z_1, z_2)$ are $O(n^{-3})$, too. The last term we need to estimate is $s_n^{(1,2)}(z_1, z_2)$. It is easy to see that

$$s_n^{(1,2)}(z_1, z_2) =$$
$$n^{-1}2\kappa_4\mathbf{E}\{n^{-1}\sum_{m=1}^{n} G_{mm}(z_1)G_{mm}(z_2)\}\mathbf{E}\{n^{-1}\sum_{j,m=1}^{n} G_{mm}(z_1)G_{jm}^2(z_2)\} + O(n^{-3}).$$

Mean values in the above are calculated in exactly the same way as $\mathbf{E}\{c_n(z)\}$ and $\mathbf{E}\{d_n(z)\}$ have been done. For large n:

$$n^{-1}2\kappa_4\mathbf{E}\{n^{-1}\sum_{m=1}^{n} G_{mm}(z_1)G_{mm}(z_2)\} = r(z_1)r(z_2) + O(n^{-1/2})$$

and

$$\mathbf{E}\{n^{-1}\sum_{j,m=1}^{n} G_{mm}(z_1)G_{jm}^2(z_2)\} = r(z_1)r^2(z_2) + O(n^{-1})$$

(compare with (III.48) and (III.50)). Thus we conclude that the contribution of third derivatives is

$$-\frac{1}{n^2}\frac{\sigma r^2(z_1)r^3(z_2)}{1 - w^2 r^2(z_2)} \tag{IV.57}$$

(II.17) (we recall using σ for κ_4).

First derivatives in the r.h.s. of (IV.54) contribute to $z_1 F_n(z_1, z_2)$ the term

$$\begin{aligned} t_n(z_1, z_2) = &-w^2\mathbf{E}\{g_n^2(z_1)g_n^C(z_2)\} - n^{-1}w^2\mathbf{E}\{c_n(z_1)g_n^C(z_2)\} \\ &-n^{-2}2w^2\mathbf{E}\{n^{-1}\mathrm{tr}\ (H - z_1 I)^{-1}(H - z_2 I)^{-2}\}. \end{aligned} \tag{IV.58}$$

By the resolvent identity (II.19)

$$(H - z_1 I)^{-1}(H - z_2 I)^{-1} = (z_1 - z_2)^{-1}[(H - z_1 I)^{-1} - (H - z_2 I)^{-1}]$$

Thus one reduces $\mathbf{E}\{n^{-1}\mathrm{tr}\ (H - z_1 I)^{-1}(H - z_2 I)^{-2}\}$ to

$$(z_1 - z_2)^{-1}[(z_1 - z_2)^{-1}\mathbf{E}\{g_n(z_1) - g_n(z_2)\} - \mathbf{E}\{c_n(z_2)\}].$$

Now recalling (III.46), (III.48) and (II.30),

$$\mathbf{E}\{n^{-1}\mathrm{tr}\ (H - z_1 I)^{-1}(H - z_2 I)^{-2}\} =$$
$$\frac{1}{r(z_1)[1 - w^2 r^2(z_2)]}\left[\frac{r(z_1) - r(z_2)}{z_1 - z_2}\right]^2 + O(n^{-1}). \tag{IV.59}$$

Clearly, $\mathbf{E}\{c_n(z_1)g_n^C(z_2)\} = \mathbf{E}\{c_n^C(z_1)g_n^C(z_2)\}$ and the corresponding summand in the r.h.s. of (IV.58) is $O(n^{-3})$. So it remains to find $\mathbf{E}\{g_n^2(z_1)g_n^C(z_2)\}$.

Use (IV.56) to write

$$\mathbf{E}\{g_n^2(z_1)g_n^C(z_2)\} = 2F_n(z_1,z_2)m_n^{(1)}(z_1) + \mathbf{E}\{[g_n^C(z_1)]^2 g_n^C(z_2)\}$$
$$= 2F_n(z_1,z_2)m_n^{(1)}(z_1) + O(n^{-5/2}) \quad \text{(IV.60)}$$

The latter equality uses $\mathbf{E}\{[g_n^C(z_1)]^2 g_n^C(z_2)\} = O(n^{-5/2})$, the bound which can be obtained following calculations of Appendix B.

Now we are in a position to find the leading order of $F_n(z_1,z_2)$. Collecting (IV.57)-(IV.60), we find that

$$z_1 F_n(z_1,z_2) = -2w^2 F_n(z_1,z_2)m_n^{(1)}(z_1) - \frac{1}{n^2}\frac{\sigma r^2(z_1)r^3(z_2)}{1-w^2r^2(z_2)}$$
$$-\frac{2w^2}{r(z_1)[1-w^2r^2(z_2)]}\left[\frac{r(z_1)-r(z_2)}{z_1-z_2}\right]^2 + O(n^{-5/2}).$$

As it is clear from (III.46) and (II.30),

$$[z_1 + 2w^2 m_n^{(1)}(z_1)]^{-1} = -r(z_1)[1-w^2r^2(z_1)]^{-1} + O(n^{-1})$$

and we end up with (IV.52).

The standard truncation technique of probability theory allows to prove (IV.52) in the case when only 5th absolute moment of the random variables W_{jk} is uniformly bounded. Calculations using the truncation technique are similar to those used in next section in proof of Theorem 3 and we omit them. Theorem 2 proved.

One can consider the covariance function $F_n(z_1,z_2)$ for the Wigner ensemble of random Hermitian matrices (see remark 3 after the statement theorem 3 in next section). Repeating almost literally calculations used in proof of theorem 2, one can prove that for the Wigner ensemble of Hermitian matrices (IV.52) is still valid. The only difference is that now $f(z_1,z_2)$ is given by the r.h.s. of Eq. (I.15) multiplied by factor 1/2.

V. GAUSSIAN FLUCTUATIONS OF THE CENTRALISED TRACE OF THE RESOLVENT

In this section we prove the statement which is analogous to the central limit theorem in the same sense in which the result (I.11) is analogous to the law of large numbers. Indeed, we can rewrite (I.11) as following limiting relation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} G_{mm} = r(z) \quad \text{(V.61)}$$

valid with probability 1. Since the l.h.s. here has the form of the arithmetic (Cesaro) mean, this relation is obviously similar to the strong law of large numbers (or more generally to the ergodic theorem). Common wisdom of probability and ergodic theory suggests that (V.61) should imply that the probability distribution of the random variable

$$n^{1/2}\Big[n^{-1}\sum_{m=1}^{n}(G_{mm} - \mathbf{E}\{G_{mm})\}\Big] = n^{1/2}[g_n(z) - \mathbf{E}\{g_n(z)\}] \quad \text{(V.62)}$$

has the Gaussian form in the limit $n=\infty$. We prove that under rather natural conditions on W_{jk} this is indeed the case provided that we use non-standard normalisation, replacing $n^{1/2}$ in (V.62) by n, i. e. we consider just the centralised trace of the resolvent

$$\gamma^{(n)}(z) = \sum_{m=1}^{n}(G_{mm} - \mathbf{E}\{G_{mm}\}) = ng_n(z) - \mathbf{E}\{ng_n(z)\} \quad \text{(V.63)}$$

instead of $n^{1/2}\gamma^{(n)}(z)$. This normalisation can of course be anticipated from the formula (I.14) giving the order of magnitude (in fact, the asymptotics) of the variance of $g_n(z)$. This decay of the variance, which is "twice" more strong than in the standard central limit theorem setting, is rather typical for a number of problems of the theory of disordered systems with non-local interaction and is known as the strong self-averaging property (see e. g. Refs. 28 and 33).

Theorem 3 *Consider the Wigner ensemble of random real symmetric matrices with independent entries defined by (I.2)-(I.3) assuming additionally that the fourth moments of W_{jk} exist and are independent of j and k and that the probability distribution functions $P_{jk}(w)$ of W_{jk} satisfy the condition: for any fixed fixed $\nu > 0$*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j\le k} \int_{|x|>\nu n^{1/2}} x^2 dProb[W_{jk} \le x] = 0, \quad \textit{for any} \ \ \nu > 0. \tag{V.64}$$

Then for any z from $U_0 = \{z \in \mathbf{C}_\pm : |\Im z| \ge 2w\}$ the random function $\gamma^{(n)}(z)$ (V.63) converges in distribution as $n \to \infty$ to the Gaussian random function $\gamma(z)$ with zero mean and the covariance function $f(z_1, z_2)$ given by (I.15). In other words, for any integer q and arbitrary collection $z_1, \dots, z_q$ of complex numbers from U_0 the joint probability distribution of random variables $\gamma^{(n)}(z_1), \dots, \gamma^{(n)}(z_q)$ converges as $n \to \infty$ to the q-dimensional Gaussian distribution with zero mean and the covariance matrix $[f(z_s, z_t)]_{s,t=1}^q$

Remarks

1. Limit theorems concerning $\gamma^{(n)}(z)$ for the Wigner ensemble were established for the first time by Girko (see Ref. 24 and references therein) under assumption that there exist a positive δ such that

$$\sup_{j\le k} \mathbf{E}\,|W_{jk}|^{4+\delta} < \infty, \tag{V.65}$$

which is slightly more restrictive than (V.64). For example in the case of identically distributed W_{jk}, (V.64) is obviously satisfied if $w_4 \equiv \mathbf{E}\{W_{jk}^4\}$ is finite. However, the more important in our opinion improvement of the result of Ref. 24 is that we calculate the covariance matrix of the limiting Gaussian process in the explicit form while in Ref. 24 this matrix was given in the implicit form as a solution of a system of cumbersome partial differential equations.

2. For the random variables W_{jk} satisfying (V.65) we can estimate the rate of convergence:

$$\sup_{z_1,z_2\in U_0} |\,\mathbf{E}\,\{\gamma^{(n)}(z_1)\gamma^{(n)}(z_2)\} - f(z_1, z_2)| = O(n^{-\delta/2}). \tag{V.66}$$

3. Consider the Wigner ensemble of the $n \times n$ random Hermitian matrices defined as in (I.2) with $W_{jk} = A_{jk} + iB_{jk}, \ \ j \le k$, $W_{jk} = W_{kj}^\dagger$, where A_{jk} and B_{jk} are mutually independent random variables with zero mean, variance $w^2/2$ and excess $\sigma/2$. It can be proved by analogous technique that in this ensemble the fluctuations of the trace of the resolvent around its mean become Gaussian in the limit $n \to \infty$. The corresponding covariance function is given by (I.15) in which the factor 2 is replaced by 1 in the denominator of both terms.

Proof. We shall work with real-valued variables $\alpha^{(n)}(z) = \Re\gamma^{(n)}(z)$ and $\beta^{(n)}(z) = \Im\gamma^{(n)}(z)$. Then we have to prove that the limiting random functions $\alpha(z)$ and $\beta(z)$ are jointly Gaussian, i.e. if

$$X(z,c) = \begin{cases} \alpha(z) & \text{if } c = \alpha; \\ \beta(z) & \text{if } c = \beta, \end{cases}$$

and

$$(a(c), b(c)) = \begin{cases} (1/2, 1/2) & \text{if } c = \alpha; \\ (1/2i, 1/2i) & \text{if } c = \beta, \end{cases}$$

then $\mathbf{E}\{X(z,c)\} = 0$ and for any integer q and arbitrary collections $z_s, s = 1, ..., q,\ z_s \in U_0$ and $c_s, , s = 1, ..., q,\ c_s \in \{\alpha, \beta\}$ the joint probability distribution of random variables $X(z_1, c_1), \dots, X(z_q, c_q)$ is the q-dimensional Caussian distribution with zero mean and covariance matrix

$$\begin{aligned} \mathbf{E}\{X(z_s, c_s)X(z_t, c_t)\} = a(c_s)a(c_t)f(z_s, z_t) + a(c_s)b(c_t)f(z_s, z_t^\dagger) + \\ a(c_t)b(c_s)f(z_s^\dagger, z_t) + b(c_s)b(c_t)f(z_s^\dagger, z_t^\dagger), \end{aligned} \tag{V.67}$$

Let us consider the characteristic function of random variables $X(z_1, c_1), .., X(z_q, c_q)$ which we shall write in the form

$$e_q^{(n)}(T_q, C_q, Z_q) = \mathbf{E}\{\prod_{s=1}^{q} \exp\{i\tau_s[a(c_s)\gamma^{(n)}(z_s) + b(c_s)\gamma^{(n)}(z_s^\dagger)]\},$$

where $T_q = (\tau_1, \dots, \tau_q),\ C_q = (c_1, \dots, c_q),\ Z_q = (z_1, \dots, z_q)$

Recall that we designate the complex conjugate by the symbol $\dagger$. Also writing the characteristic function we shall often omit indices indicating its dependence on n and some other variables provided there will arise no confusion.

Obviously

$$\frac{\partial}{\partial \tau_s}\mathbf{E}\{e_q(T_q)\} = i\ \mathbf{E}\{e_q[a(c_s)\gamma^{(n)}(z_s) + b(c_s)\gamma^{(n)}(z_s^\dagger)]\}.$$

Our aim is to show that there exists a set of the "covariance" coefficients $A_{st}^{(n)},\ s, t = 1, ..., q$ such that for each fixed T_q

$$\lim_{n\to\infty} \left| \mathbf{E}\{e_q^{(n)}[a_s\gamma^{(n)}(z_s) + b_s\gamma^{(n)}(z_s^\dagger)]\} - i\sum_{t=1}^{q} \tau_s A_{st}^{(n)}\ \mathbf{E}\{e_q^{(n)}\} \right| = 0, \quad z \in U_0,$$

to show that limits of all these coefficients exist

$$A_{st} = \lim_{n\to\infty} A_{st}^{(n)}, \tag{V.68}$$

and correspond to the r.h.s. of (V.67). Then standard arguments will allow us to prove that the limit characteristic function has the Gaussian form $\exp(-1/2\sum_{s,t=1}^{q} A_{st}\tau_s\tau_t)$.

Thus, we have to compute

$$\mathbf{E}\{e_q\gamma^{(n)}(z)\} = \sum_{j=1}^{n}\ \mathbf{E}\{e_q^C(Z_q)G_{jj}\},$$

for large n (we recall that sub-index C indicate centering to zero mean). Then putting one of $z_1, \dots, z_q$ or one of their conjugates in place of z we calculate the limits in (V.68)

We have complex energies $z, z_1, \dots, z_q$ and we introduce notation $G(z_s)$ for the resolvent corresponding to z_s keeping notation G for the resolvent corresponding to z.

By the resolvent identity (II.20),

$$\sum_{j=1}^{n} \mathbf{E}\{e_q^C G_{jj}\} = z^{-1} \sum_{j,m=1}^{n} \mathbf{E}\{e_q^C G_{jm} H_{mj}\}. \tag{V.69}$$

We compute the average in the r.h.s of (V.69) following the scheme described in Section 2. However its direct application requires too strong conditions on the distribution of W_{jk}. Thus, we modify slightly the general scheme and carry out more accurate estimates.

Denote by $\mathbf{E}_{mj}$ the conditional expectation $\mathbf{E}\{\cdot|W_{mj} = w\}$ and rewrite the right-hand side of (V.69) in the form

$$z^{-1} n^{-1/2} \sum_{j,m=1}^{n} \int \mathbf{E}_{mj}\{e_q^C G_{jm}\} w dP_{mj}(w).$$

We split the integral into the two ones over sets $\Gamma_1(n) = \{\omega : |W_{mj}| \le \delta n^{1/2}\}$ and $\Gamma_2(n) = \{\omega : |W_{mj}| > \delta n^{1/2}\}$. Now the inequalities

$$\left| n^{-1/2} \sum_{j,m=1}^{n} \int_{\Gamma_2(n)} \mathbf{E}_{mj}\{e_q^C G_{jm}\} w P_{mj}(w) \right| \le \eta^{-1} n^{-1/2} \sum_{j,m=1}^{n} \int_{\Gamma_2(n)} |w| dP_{mj}(w)$$
$$\le \eta^{-1} \nu^{-3} n^{-2} \sum_{j,m=1}^{n} \int_{\Gamma_2(n)} w^4 dP_{mj}(w)$$

and the assumption (V.64) imply that only the integrals over $\Gamma_1(n)$ gives the a non-vanishing contribution contribution to (V.69).

Following our general scheme, we expand the function $e_q^C G_{jm}$ in powers of random variable $H_{mj} = n^{-1/2} W_{mj}$ restricted to $\Gamma_1(n)$. Since it is bounded in absolute value by ν we can write the relation

$$n^{-1/2} \sum_{j,m=1}^{n} \int_{\Gamma_1} \mathbf{E}_{mj}\{e_q^C G_{jm}\} w dP_{mj}(w) = \sum_{k=1}^{5} S_k(n), \tag{V.70}$$

where

$$S_k(n) = n^{k/2} \sum_{j,m=1}^{n} \mathbf{E}\left[e_q^C G_{jm}\right]_{mj}^{(k-1)} \int_{\Gamma_1} w^k dP_{mj}(w),$$

and $[...]_{mj}^{(k)}$ denotes that the k-th derivatives with respect to H_{mj} is taken and then H_{mj} is replaced by zero. Let us note also that in S_5 the expression in square brackets is taken at some point $\tilde{H}_{mj} \in (0, \nu)$.

The term $S_1(n)$ vanishes as $n \to \infty$ due our assumption (V.64):

$$|S_1(n)| \le \left| n^{-1/2} \sum_{j,m=1}^{n} \mathbf{E}\left[e_q^C G_{jm}\right]_{mj} \int_{\Gamma_1} w dP_{mj}(w) \right|$$
$$\le \eta^{-1} \nu^{-3} n^{-2} \sum_{j,m=1}^{n} \int_{\Gamma_1} w^4 dP_{mj}(w)$$

where $\eta \equiv |\Im z|$.

The term $S_5(n)$ vanishes as $n \to \infty$ because it can be estimated by

$$\frac{B_4(T_p)}{\eta^6 n^{5/2}} \sum_{j,m=1}^{n} \int_{\Gamma_1} |w|^5 dP_{mj}(w) \le \frac{B_4(T_p)\nu}{\eta^6 n^2} \sum_{j,m=1} \int_{\Gamma_1} w^4 dP_{mj}(w),$$

where $B_4(T_p)$ is the upper bound of absolute value of the fourth derivative in (V.70). For any fixed T_p, $B_4(T_p)$ is finite and recalling (V.64) we see that $S_5(n)$ goes to zero as $n \to \infty$

The term $S_3(n)$ also vanishes as $n \to \infty$. We establish this fact at the end of the proof.

Terms $S_2(n)$ and $S_4(n)$ give main contribution to (V.69). Let us first consider $S_2(n)$. The resolvents $G(z_s)$ and G are complex symmetric matrices and we have:

$$- \sum_{j,m=1}^{n} \left[e_q^C G_{jm}\right]_{mj}^{(1)} = \left[e_q^C G_{jj} G_{mm}\right]_{mj} + \sum_{j,m=1}^{n} \left[e_q^C G_{jm}^2\right]_{mj} + \\ i \sum_{j,m=1}^{n} \left[\sum_{s=1}^{q} 2\tau_s \left(a_s [G^2(z_s)]_{jm} + b_s [G^2(z_s^\dagger)]_{jm}\right) G_{jm} e_q\right]_{mj}. \qquad \text{(V.71)}$$

Each term of the right-hand side of this relation is a function of H in which H_{mj} is replaced by zero and we have to come "back" to expressions dependent on the whole matrix H. To this end, we use again the resolvent identity but now in the "opposite " direction. We obtain for the first term of (V.71)

$$w^2 n^{-1} \sum_{j,m=1}^{n} \mathbf{E}\left\{\left[e_q^C G_{jj} G_{mm}\right]_{mj}\right\} = v^2 n^{-1} \sum_{j,m=1}^{n} \mathbf{E}\left\{e_q^C G_{jj} G_{mm}\right\} - w^2 \Psi^{(n)},$$

where

$$\Psi^{(n)} = n^{-3/2} \sum_{j,m=1}^{n} \mathbf{E}\left\{\left[e_q^C G_{jj}\, G_{mm}\right]_{mj}^{(1)}\right\} \int_{\Gamma_1} w dP_{mj}(w) - \\ 2^{-1} n^{-2} \sum_{j,m=1}^{n} \mathbf{E}\left\{\left[e_q^C G_{jj}\, G_{mm}\right]_{mj}^{(2)}\right\} \int_{\Gamma_1} w^2 dP_{mj}(w) - \\ 6^{-1} n^{-5/2} \sum_{j,m=1}^{n} \mathbf{E}\left\{\left[e_q^C G_{jj}\, G_{mm}\right]^{(3)} W_{mj}^3\right\}.$$

It is easy to see that the first and the last terms of $\Psi^{(n)}$ vanish as $n \to \infty$ due to our assumption (V.64).

Using (V.64) and (III.37), we can rewrite the second term of $\Psi^{(n)}$ in the form

$$\frac{-2iw^2}{n^2} \sum_{j,m=1}^{n} \mathbf{E} \left[e_q \sum_{s=1}^{q} \tau_s \left(a_s [G^2(z_s)]_{mm} [G(z_s)]_{jj} + b_s [G^2(z_s^\dagger)]_{mm} [G(z_s^\dagger)]_{jj}\right) G_{jj} G_{mm}\right]_{mj} + \Phi^{(n)},$$

where the remainder $\Phi^{(n)}$ includes terms which have one or more factors G_{jm} or terms of the form

$$n^{-2} \sum_{j,m=1}^{n} \mathbf{E}\left\{\left[e_q^C [G(z_s)_{jj}]^2 [G(z_j)_{jj}]^2\right]_{mj}\right\}.$$

It is clear that in all these expressions we can remove square brackets $[...]_{mj}$ because this procedure will add terms of order $O(n^{-1/2})$ to the sums under consideration. Using the estimate (III.41), taking into account the self-averaging property

$$\mathbf{E}\, |g_n^C(z)|^2 = o(n^{-1}), \text{ as } n \to \infty \qquad \text{(V.72)}$$

(see Lemma 1 of Appendix B for the proof), and relation

$$\lim_{n\to\infty} \mathbf{E}\, |(n^{-1} \sum_j G_{jj}^{\alpha} G_{jj}^{\beta} n^{-1} \sum_j G_{mm}^{\mu} G_{mm}^{\nu})^C| = 0 \qquad \text{(V.73)}$$

with some $\alpha, \beta, \mu, \nu = 0, 1, 2$ (Lemma 2 of Appendix B), it is easy to prove that $\Phi^{(n)}$ also vanishes as $n \to \infty$.

We obtain finally that among terms coming from first summand in the right-hand side of (V.71) only the following

$$w^2 z^{-1} \mathbf{E}\{e_q^C \sum_{j=1}^n G_{jj}\} g_n(z) - \tag{V.74}$$

$$\frac{-2iw_4}{n^2} \mathbf{E}\{e_q\} \sum_{s=1}^q \tau_s \mathbf{E}\{\sum_{j,k=1}^n \left(a_s [G^2(z_s)]_{mm} [G(z_s)]_{jj} + b_s [G^2(z_s^\dagger)]_{mm} [G(z_s^\dagger)]_{jj}\right) G_{jj} G_{mm}\}.$$

does not vanish as $n \to \infty$.

The second summand in the r.h.s. of (V.71) vanishes as $n \to \infty$. This becomes clear after applying the same procedure of removing square brackets $[\ldots]_{mj}$ to the expression $n^{-2} \sum_{j,m=1}^n [e_q^C (G_{jm})^2]_{mj}^{(k)}$ and using the estimates (III.41) and (V.73).

Let us consider the contribution of the last summand in r.h.s of (V.71) for a fixed parameter z_s, $s = 1, ..., q$. Taking into account (V.64) and repeating the "returning" procedure, we obtain for this term

$$n^{-1} \sum_{j,m=1}^n \left[[G^2(z_s)]_{jm} G_{mj} e_q \right]_{mj} =$$

$$n^{-1} \sum_{j,m=1}^n [G^2(z_s)]_{jm} G_{mj} e_q - n^{-3/2} \sum_{j,m=1}^n \left[[G^2(z_s)]_{jm} G_{mj} e_q \right]_{mj}^{(1)} \int_{\Gamma_1} w dP_{mj}(w) -$$

$$n^{-2} \sum_{j,m=1}^n \left[[G^2(z_s)]_{jm} G_{mj} e_q \right]_{mj}^{(} 2) \int_{\Gamma_1} w^2 dP_{mj}(w) - n^{-5/2} \sum_{j,m=1}^n \mathbf{E}\{\left[[G^2(z_s)]_{jm} G_{mj} e_q \right]^{(3]} W_{mj}^3\}.$$

It is easy to see that in this equality terms with the first and the third derivatives vanish as $n \to \infty$. The second derivative gives

$$2\left[\{ [G^2(z_s)]_{jj}\, G_{jj}\, G_{mm}\, G_{mm} + [G^2(z_s)]_{mm}\, G_{mm}\, [G(z_s)]_{jj}\, G_{jj} \} e_q \right]_{mj}$$

and 24 terms having a factor of the form $(G^\alpha)_{jm}$, $\alpha = 1, 2$. Omitting brackets $[...]_{mj}$ and using (V.72)-(V.73), we see that the last term of (V.71) gives the leading contribution

$$\mathbf{E}\{e_q^C\} \sum_{s=1}^q \tau_s \mathbf{E}\{a_s n^{-1} \mathrm{tr}\ G^2(z_s) G + b_s n^{-1} \mathrm{tr}\ G^2(z_s^\dagger) G\} \tag{V.75}$$

$$-iw_4 \mathbf{E}\{e_q^C\} \sum_{s=1}^q \tau_s \mathbf{E}\{n^{-2} \sum_{j,m=1}^n (a_s [G^2(z_s)]_{jj} [G(z_s)]_{mm} + b_s [G(z_s)]_{jj} [G^2(z_s)]_{mm}) G_{jj} G_{mm}\}.$$

Consider now the term $S_4(n)$ of (V.70). The third derivative $[e_q^C G_{jm}]_{mj}'''$ consists of 140 terms. One part of them vanishes as $n \to \infty$ due to the property (V.72)-(V.73), another part - due to the presence of the factor of the form $(G^\alpha)_{jm}$, $\alpha = 1, 2, 3$. Only six terms of the form

$$iw_4 \sum_{s=1}^q \mathbf{E}\{e_q^C\} \mathbf{E}\{n^{-2} \sum_{j,m=1}^n (a_s [G^2(z_s)]_{jj} [G(z_s)]_{mm} + b_s [G(z_s)]_{jj} [G(z_s)^2]_{mm}) G_{jj} G_{mm}\}$$

are non-vanishing in the limit $n \to \infty$. These terms arise when we differentiate G_{jm} once and e_q twice with respect to H_{mj}. Combining these terms with (V.74) and (V.75), we finally obtain that

$$\mathbf{E}\{e_q^C \mathrm{tr}\ G\} = i\frac{1}{z - 2w^2 g_n(z)}\ \mathbf{E}\{e_q^C\} \sum_{s=1}^{q} \tau_s (2w^2 n^{-1}\ \mathbf{E}\{\mathrm{tr}\ (a_s G(z_s)^2 + b_s G(z_s^\dagger)^2)\, G\} + \quad \text{(V.76)}$$
$$2\sigma\ \mathbf{E}\{n^{-2} \sum_{j,m=1} (a_s G^2(z_s)_{jj} G(z_s)_{mm} + b_s G^2(z_s^\dagger)_{jj} G(z_s^\dagger)_{mm}) G_{jj} G_{mm}\}).$$

Notice, that the denominator in the first term of the r.h.s. of this expression is bounded away from zero because z belongs to the domain (II.22). Now, combining (I.11) and (I.10) with relations

$$\lim_{n\to\infty}\ \mathbf{E}\{n^{-1} \sum_{j=1}^{m} [G^2(z_s)]_{jj} G_{jj}\} = r^2(z_s)[1 - w^2 r^2(z_s)]^{-1} r(z), \quad \text{(V.77)}$$

we derive from (V.76) the final form of the covariance.

Relation (V.77) can be easily deduced from our proof of (V.72)-(V.73).

Let us briefly discuss now the proof of the fact that $S_3(n)$ of (V.70) vanishes as $n \to \infty$. The second derivative $[e_q^C G_{jm}]_{mj}^{(2)}$ gives terms each having the factor of the form $[(G^\alpha)_{jj}(G^k)_{jm}(G^\beta)_{mm}]_{mj}$. The brackets can be simply omitted because the "returning" procedure adds terms of order $O(n^{-1/2})$. Now, regarding $n^{-1/2}(G^\alpha)_{jj}$ as vectors and $(G^m)_{mj}$ as the kernel, we can write inequality

$$|n^{-3/2} \sum_{j,m=1}^{n} (G^\alpha)_{jj} (G^m)_{mj} (G^\beta)_{mm}| \le \eta^{-\alpha-\beta-m} n^{-1/2}$$

which completes the proof of Theorem 3.

VI. SCALING LIMIT AND UNIVERSALITY CONJECTURE

We have presented above the rigorous derivation of asymptotic corrections (in fact expansions) for moments and more complex quantities constructed from the traces of the Green functions of the Wigner random matrix ensembles. Now we use our result to draw certain non-rigorous conclusions on the form of the leading term of the correlation function $S_n(E_1, E_2)$ of the formal level density $\rho_n(E) = n^{-1}\mathrm{tr}\ \delta(H - EI)$. Since $\rho_n(E) = N_n'(E)$ where $N_n(E)$ is defined in (I.4), then basing on the relation (I.7) one can conclude that the number of eigenvalues lying inside the interval (E_1, E_2) with the center at E will be $N(E_2) - N(E_1) \sim n\rho_n(E)(E_2 - E_1)$, i.e. that the mean distance between levels is $[n\rho_n(E)]^{-1}$. Thus the scaling $E_2 - E_1 = O(n^{-1})$ defines the microscopic or local regime in which one deals with a finite numbers of eigenvalues[1,11].

Consider the density-density correlation function

$$S_n(E_1, E_2) = \ \mathbf{E}\{\rho_n(E_1)\rho_n(E_2)\} - \ \mathbf{E}\{\rho_n(E_1)\}\ \mathbf{E}\{\rho_n(E_2)\}. \quad \text{(VI.78)}$$

By using (8) and (9) we obtain from (14) and (V.63) that the Stieltjes transform of $S_n(E_1, E_2)$

$$F_n(z_1, z_2) = \int\!\!\int \frac{S_n(E_1, E_2)}{(E_1 - z_1)(E_2 - z_2)} dE_1 dE_2, \quad \Im z_i \neq 0$$

is

$$F_n(z_1, z_2) = n^{-2}\ \mathbf{E}\{\gamma^{(n)}(z_1)\gamma^{(n)}(z_2)\}.$$

It follows from the inversion formula for the Stieltjes transform $f(z) = \int (E - z)^{-1}\rho(E)dE$

$$\rho(E) = \pi^{-1} \lim_{\epsilon\downarrow 0} \Im f(E + i\epsilon) \equiv I_{E_1}\{f(z)\} \quad \text{(VI.79)}$$

that to find $S_n(E_1, E_2)$, one has to know $F_n(z_1, z_2)$ up to the real axis in both variables because

$$S_n(E_1, E_2) = I_{E_1} \circ I_{E_2}\{F_n(z_1, z_2)\} \tag{VI.80}$$

On the other hand, we have found the form (I.14) and (I.15) of $F_n(z_1, z_2)$ only in the domain $|\Im z| \geq 2w$. However, since the function $f(z_1, z_2)$ given by (I.15) can obviously be continued up to the real axis with respect to the both variables z_1 and z_2 we can apply to the first term of (I.14) the operation $I_{E_1} I_{E_2}$, $E_1 \neq E_2$ to compute formally the "leading" term of the density-density correlation function. This means that we perform first the limit $n \to \infty$ and then the limits $\epsilon_1, \epsilon_2 \downarrow 0$. This order of limiting transitions is inverse with respect to that prescribed by the definition of this correlation function.

To make these computations , we use the identity

$$\frac{r_1 - r_2}{z_1 - z_2} = \frac{r_1 r_2}{1 - w^2 r_1 r_2}$$

which follows from (I.10) or (II.30). The identity yields the relations $\varepsilon |r(E + i\varepsilon)|^2 = \Im r(E + i\varepsilon)(1 - w^2 |r(E + i\varepsilon)|^2)$ and $|r(E + i\ 0)|^2 = w^{-2}$ for E such that $\Im r(E + i0) > 0$. Combining these relations with (I.5), we obtain that

$$w^2[\Re\, r(E + i0)]^2 = \frac{E^2}{4w^2} \text{ and } w^2[\Im r(E + i0)]^2 = 1 - \frac{E^2}{4w^2}.$$

Using these equalities, we derive from our result (I.14) and (I.15) and from (VI.79) and (VI.80)that

$$\begin{aligned} S_n(E_1, E_2) = & -\frac{1}{\beta\pi^2[n(E_1 - E_2)]^2} \frac{4w^2 - E_1 E_2}{(4w^2 - E_1^2)^{1/2}(4w^2 - E_2^2)^{1/2}} \\ & + \frac{\sigma}{2n^2\pi^2 w^8} \frac{(2w^2 - E_1^2)(2w^2 - E_2^2)}{(4w^2 - E_1^2)^{1/2}(4w^2 - E_2^2)^{1/2}}. \end{aligned} \tag{VI.81}$$

with $\beta = 1$. It can be shown that for the Hermitian matrices with independent entries (see Remark 3 to Theorem 3) the density-density correlator has the same form with $\beta = 2$. For the Gaussian orthogonal and unitary ensembles (GOE and GUE) $\sigma = 0$, and we recover the result

$$S_n(E_1, E_2) = -\frac{1}{\beta\pi^2[n(E_1 - E_2)]^2} \frac{4w^2 - E_1 E_2}{(4w^2 - E_1^2)^{1/2}(4w^2 - E_2^2)^{1/2}}$$

obtained in Ref. 34 and Ref. 35.

We see that in a general non-Gaussian case the respective expression depends not only on the second moment of entries , but also on their fourth moment via the excess σ.

The remarkable fact is that this dependence vanishes in the microscopic (called also scaling) limit

$$E_1, E_2 \to E,\ n(E_2 - E_1) \to s \tag{VI.82}$$

Indeed, it easy to see that in this limit we obtain from (VI.81) very simple expression:

$$\lim_{n(E_2 - E_1) \to s} S_n(E_1, E_2) = -\frac{1}{\beta\pi^2 s^2}. \tag{VI.83}$$

According to Wigner and Dyson (see e.g. Ref. 1), the exact large-s asymptotics for the limiting correlation function of the Gaussian ensembles are: $-1/(\pi^2 s^2)$ (GOE) and $-\sin^2 \pi\rho(E)s/(\pi^2 s^2)$ (GUE). Comparing these expressions with our results, we see that the procedure of computing of the correlation function yields for the general case the expression coinciding with the large-s asymptotics of the Gaussian ensembles correlation function smoothed over energy intervals whose length is much smaller than the macroscopic scale $w = \mathbf{E}\{W^2\}^{1/2}$ but much bigger than the microscopic scale given by the mean level spacing $[n\rho(E)]^{-1}$. It is natural to think that in our computations the smoothing has been implemented "automatically" due to the nonzero imaginary part of the spectral parameter $\Im z_j$. We notice that the same procedure is widely used in the mesocsopic calculations based on the Kubo formula, weak disorder perturbation theory, etc.

The independence of the scaling limit expressions (VI.83) on the excess σ can be regarded as a support of the universality conjecture for the Wigner ensembles. Let us mention supports of this conjecture for other ensembles.

The first one[18] concerns the so-called sparse (or diluted) random matrices whose entries are independently distributed random variables such that $\Pr\{H_{k,l} = 0\} = p/n$. The authors of Ref. 18 used the Grassman integral technique and found the Wigner-Dyson universal form of the density-density correlator if p is large enough.

The second[36] concerns the ensemble $H = \sum_{\mu=1}^{p} \tau_\mu(\cdot, \xi^\mu)\xi^\mu$, where τ_μ and $\xi^\mu = \{\xi_1^\mu, \dots, \xi_n^\mu\}$ are independent identically distributed random variables (the ensemble was introduced in Ref. 27). For this ensemble, whose entries are dependent random variables, the analogue of (VI.81) is obtained and it is shown that its scaling limit is the same as above.

The third follows from Appendix A below. We consider there case of the deformed GOE (see definitions below). For this ensemble the analog of Theorem 3 was proved in Ref. 37. In the appendix we present a short derivation of the density-density correlator and show that in the scaling limit it has the form (VI.83).

We mention also that for the unitary invariant ensembles of the form (I.1) the universality conjecture is rigorously proved in Ref. 16 for a rather broad class of functions $F(x)$.

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APPENDIX A: SCALING LIMIT FOR THE DEFORMED GOE

In this Appendix we find the exact form of the leading term of the covariance function $F_n(z_1, z_2)$ (I.14) for the ensemble $H_d = H^{(0)} + H$, where $H^{(0)}$ are $n \times n$ nonrandom matrices such that there exists the "unperturbed" IDS

$$N^{(0)}(E) = \lim_{n\to\infty} n^{-1} \#\{e_j : e_j \text{ is an eigenvalue of } H^{(0)} \text{ and } e_j \le E\}$$

and H belongs to the GOE ((I.1) with $F = x^2/4w^2$). This ensemble is called[2] the deformed GOE. Because of the orthogonal invariance of the GOE distribution, we can restrict our considerations to the case of diagonal $H^{(0)}$. So we assume that $H^{(0)} = [\delta_{jk} e_j]_{j,k=1}^n$ and real numbers e_j are such that the limit

$$g^{(0)}(z) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{e_j - z} \tag{A.1}$$

exists for all non-real z. The function $g^{(0)}(z)$ is the Stieltjes transform of $N^{(0)}(E)$. We shall use notation $d_j(z)$ for $(e_j - z)^{-1}$ and $g_n(z)$ for the normalized trace of the resolvent of H_d.

Subsequent arguments are quite similar to those used in derivation of (II.24) and (II.26) - (II.29) for the GOE. By using (17) and (19) for $H_1 = H^{(0)}$ and $H_2 = H_d$ one can derive the following two relations (analogues of (II.26) and (II.27)) :

$$\mathbf{E}\{G_{jk}(z)\} = d_j(z)\delta_{jk} + \mathbf{E}\{g_n(z)G_{jk}(z)\}d_k(z) + n^{-1}\sum_m \mathbf{E}\{G_{jm}(z)G_{km}(z)\}d_k(z), \tag{A.2}$$

and

$$\begin{aligned}\mathbf{E}\{g_n^C(z_1)G_{jj}(z_2)\} = {} & w^2\mathbf{E}\{g_n^C(z_1)G_{jj}(z_2)\}\mathbf{E}\{g_n(z_2)\}d_j(z_2) + \\ & w^2\mathbf{E}\{g_n^C(z_1)g_n^C(z_2)G_{jj}(z_2)\}d_j(z_2) + \\ & w^2 n^{-1}\mathbf{E}\{g_n^C(z_1)[G^2(z_2)]_{jj}\}d_j(z_2) + \\ & 2w^2n^{-2}\sum_m \mathbf{E}\{[G^2(z_1)]_{jm}G_{mj}(z_2)\}d_j(z_2).\end{aligned} \tag{A.3}$$

It follows from (A.2) that if for $z \in U_0$ where U_0 is defined in (II.22)

$$\lim_{n\to\infty} \mathbf{E}\{|g_n(z) - \mathbf{E}\{g_n(z)\}|\} = 0, \tag{A.4}$$

then $\lim_{n\to\infty} \mathbf{E}\{g_n(z)\} = g(z)$, where $g(z)$ is the unique solution of the functional equation[23]

$$g(z) = g^{(0)}(z + w^2 g(z)) \tag{A.5}$$

satisfying $\Im g(z)\Im z \geq 0$. In the equation above, $g^{(0)}(z)$ is given by (A.1).

It is easy to show that (A.2) and (A.4) imply the relation

$$\sup_{j=1,\dots,n} |\mathbf{E}\{G_{jj}(z)\} - g_j^{(n)}(z)| = O(n^{-1}), \quad z \in U_0, \tag{A.6}$$

where $g_j^{(n)}(z)$ solves the equations

$$g_j^{(n)}(z) = \frac{1}{ej - z - w^2 g^{(n)}(z)}, \; j = 1, \dots, n \quad g^{(n)}(z) = n^{-1}\sum_{m=1}^{n} g_m^{(n)}(z). \tag{A.7}$$

Indeed, if $V_j^{(n)} = \mathbf{E}\{g_n^C(z_1)G_{jj}(z_2)\}$ then by (A.3),

$$\begin{aligned}V_j^{(n)} = {} & w^2 V_j^{(n)}\mathbf{E}\{g_n(z_2)d_j(z_2)\} + \\ & n^{-1}\sum_j V_j^{(n)}\mathbf{E}\{g_n(z_2)d_j(z_2)\} + 2w^2n^{-2}\mathbf{E}\{\sum_j [G^2(z_2)]_{jm}G_{mj}(z_2)d_j(z_2)\} + \\ & w^2n^{-1}\mathbf{E}\{g_n^C(z_1)[G^2(z_2)]_{jj}\}d_j(z_2) + w^2\mathbf{E}\{g_n^C(z_1)g_n^C(z_2)[G_{jj}(z_2)]^C\}d_j(z_2).\end{aligned} \tag{A.8}$$

Now, repeating arguments used at the end of Section II, we can easily obtain the estimate $n^{-1}\sum_j V_j^{(n)} = O(n^{-2})$ which proves (A.6). Using this estimate and considering (A.8) once more, we obtain the estimate

$$\sup_j |V_j^{(n)}| = O(n^{-2}). \tag{A.9}$$

It follows from the resolvent identity (II.19) that

$$\sum_j [G^2(z_1)]_{jm}[G(z_2)]_{mj} = \frac{[G^2(z_1)]_{jj}}{z_1 - z_2} - \frac{[G(z_1)]_{jj} - [G(z_2)]_{jj}}{(z_1 - z_2)^2}.$$

Taking into account this relation, (A.6), and (A.9), we obtain that if

$$f_d(z_1,z_2) = \lim_{n\to\infty} n^2 \mathbf{E}\{g^C(z_1)g^C(z_2)\},$$

then

$$|n^{-1}\sum_j V_m^{(n)} - f_d(z_1,z_2)| = o(n^{-2}), \quad z\in U_0,$$

and as the result

$$f_d(z_1,z_2) = -\frac{2w^2}{(z_1-z_2)^2}\lim_{n\to\infty}\frac{n^{-1}\sum_m [g_m^{(n)}(z_1)-g_m^{(n)}(z_2)]g_m^{(n)}(z_2)}{1-w^2n^{-1}\sum_j g_m^{(n)}(z_2)^2}$$
$$+\frac{1}{z_1-z_2}\lim_{n\to\infty}\frac{n^{-1}\sum_j [G^2(z_1)](z_1)g_m^{(n)}(z_2)}{(1-w^2n^{-1}\sum_j g_m^{(n)}(z_1)^2)(1-w^2n^{-1}\sum_j g_m^{(n)}(z_2)^2)}. \qquad \text{(A.10)}$$

Since $\lim_{n\to\infty} n^{-1}\sum_m g_m^{(n)}(z)^2 = \int (E-z-w^2g_0(z))^{-2}dN_0(E)\equiv\Phi_2$, then the second fraction of the last term of (A.10) is not singular for $z_i = E\pm i0$ with E such that $\Im g(E+i0)>0$. Thus, this term vanishes in the scaling limit (VI.82).

Consider now the first term of the right-hand side of (A.10). Simple computation shows that

$$n^{-1}\sum_m g_m^{(n)}(z_1)g_m^{(n)}(z_2) = \frac{g^{(n)}(z_1)-g^{(n)}(z_2)}{z_1-z_2+w^2[g^{(n)}(z_1)-g^{(n)}(z_2)]}.$$

Since, according to (A.4) - (A.7) $\lim_{n\to\infty} g^{(n)}(z) = g(z)$ we find for this term

$$-\frac{2w^2}{(z_1-z_2)^2}\left(\frac{g_0(z_1)-g_0(z_2)}{z_1-z_2+w^2[g_0(z_1)-g_0(z_2)]}-\Phi_2\right)(1-w^2\Phi_2)^{-1} + O(|z_1-z_2|^{-1}).$$

This relation implies that in the scaling limit (VI.82) we obtain again the simple universal expression (VI.83) .

APPENDIX B: AUXILIARY FACTS

Lemma 1. *Self-averageness property (V.72) holds under assumptions of Theorem 3*

Proof. We denote

$$F_n(z,z') = \mathbf{E}\{g^C(g')^C\} \equiv \mathbf{E}\{g^Cg'\} \qquad \text{(B.1)}$$

where $g\equiv g_n(z)$ and $g'\equiv g_n(z')$, $g_n(z) = n^{-1}\text{tr }(H-zI)^{-1}$ and $g^C = g - \mathbf{E}\{g\}$.

Obviously, $G_{jj}(z^\dagger) = G_{jj}^\dagger(z)$ and $F_n(z,z^\dagger) = \mathbf{E}\{|g^C(z)|^2\}$.

Let us apply the resolvent identity (II.20) to the last factor g' in the right-hand side of (B.1). We obtain the relation

$$F_n(z,z') = \frac{1}{z'n}\sum_{jm}\mathbf{E}\{g^CG'_{jm}H_{mj}\}, \qquad \text{(B.2)}$$

where $G'\equiv G(z')$. Comparing relations (B.2) and (V.69), we see that their right-hand sides are similar. The only difference is that the sum in (B.2) has extra factor n^{-1} and e_q of (V.69) is replaced by g. Hence, one can compute the average in (B.2) in the same way as it was done for the right-hand side of (V.69) and come to the expression $F_n(z,z') = \sum_{k=1}^{5} T_k(n)$ where $T_k(n)$ are similar to $S_k(n),\ k = 1,..,5$ in (V.70).

Thus we find that $T_1(n)$ and $T_3(n)$ are of order $o(n^{-1})$, as $n \to \infty$ just as in the case of $S_1(n)$ and $S_3(n)$.

Consider $T_5(n)$ which is analogous to $S_5(n)$ in (V.70). It contains four derivatives of $G_{kk}G_{jm}$ by H_{mj}. It follows from (III.37) that the result of differentiating includes at least one factor G_{jm}. Combining (III.41) with inequalities used to estimate $S_5(n)$, we easily derive that $T_5(n)$ is a quantity of order $O(n^{-3/2})$.

Let us estimate $T_4(n)$ acting in the same way as in the case of $S_4(n)$. As it was mentioned in V, the non-vanishing contribution to $S_4(n)$ comes from terms arising from one derivative of G'_{jm} and two derivatives of e_q. The rest of the terms are of order $o(1)$. Thus, in the corresponding terms of $T_4(n)$ we have to take into account only terms with factors G or G' having coincident arguments. It is easy to see that due to extra factor n^{-1} in front of the whole sum and factor n^{-1} in $g(z)$, these terms are of order n^{-2}. Thus, $T_4(n)$ is of order $o(n^{-1})$.

Turning to $T_2(n)$ and taking into account previous arguments, we arrive at the relation

$$F_n(z,z') = -\frac{w^2}{z'n^2}\sum_{j,m}\mathbf{E}\{g^C G'_{jj}G'_{mm}\} - \frac{w^2}{z'n^2}\sum_{j,m}\mathbf{E}\{g^C G'_{jm}G'_{jm}\} + \Phi'(z,z'), \qquad \text{(B.3)}$$

where $\Phi'(z,z') = o(n^{-1})$. Using (III.41), we easily obtain that

$$\frac{w^2}{n^2}\left|\sum_{j,m}\mathbf{E}\{g^C G'_{jm}G'_{jm}\}\right| \le \frac{w^2}{n^2}\sum_{j}\mathbf{E}\Big\{|g^C|\sum_{m}|G'_{jm}|^2\Big\}$$
$$\le \frac{w^2}{n|\Im z'|^2}\mathbf{E}^{1/2}\{|g^C|^2\}$$

Observing that

$$\mathbf{E}\{g^C g' g'\} = 2\mathbf{E}\{g^C g'\}\mathbf{E}\{g'\} + \mathbf{E}\{g^C[g']^C g'\},$$

we derive from (B.3) that for $z' = z^\dagger$ and $z \in U_0$ (II.22):

$$C_1 F_n(z,z^\dagger) - C_2 n^{-1}|F_n(z,z^\dagger)|^{1/2} - |\Phi'_n| \le O,$$

where C_1 and C_2 are absolute constants (cf.(II.28). This inequality implies (V.72). Lemma proved

Lemma 2. *Under assumptions of Theorem 3 the relation (V.73 is true.*

Proof. It suffices to show that

$$R_n \equiv \mathbf{E}\{|G_2^C|^2\} = o(1),\ n \to \infty, \qquad \text{(B.4)}$$

where $G_2 \equiv n^{-1}\sum_j G_{jj}^2$. Repeating computations of previous proof, we obtain the following relation

$$R_n = \frac{1}{z'n}\sum_{j,m}\mathbf{E}\{G_2^C G'_{jj}G'_{jm}H_{mj}\}.$$

Comparing again the right-hand side of this equality with those of (B.2) and (V.69) and repeating the corresponding computation, we conclude that

$$R_n = -\frac{w^2}{z'n^2}\sum_{j,m}\mathbf{E}\{G_2^C G'_{jj}G'_{jj}G'_{mm}\} + \frac{w^2}{z'n^2}\sum_{j,m}\mathbf{E}\{G_2^C G'_{jj}G'_{jm}G'_{mm}\} + \Phi''_n, \qquad \text{(B.5)}$$

where $\Phi''_n = o(1)$ as $n \to \infty$ for $z' = z^\dagger$ and $|\Im z| > 0$. Taking into account (V.72), we derive from (B.5) that for $z \in U_0$

$$C_3 R_n - C_3 R_n - n^{-1} C_4 |R_n|^{1/2} + o(1) \leq 0,$$

where C_3 and C_4 are some absolute constants (cf.(II.28)). This inequality implies (B.4). Lemma is proved.

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