## Excited state TBA and functional relations in spinless Fermion model

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The excited state thermodynamic Bethe ansatz (TBA) equations for the spinless Fermion model are presented by the quantum transfer matrix (QTM) approach. We introduce a more general family called *T*-functions and explore functional relations among them (*T*-system) and their certain combinations (*Y*-system). From their analytical property, we derive a closed set of non-linear integral equations which characterize the correlation length of  $\langle c_j^{\dagger} c_i \rangle$  at any finite temperatures. Solving these equations numerically, we explicitly determine the correlation length, which coincides with earlier results with high accuracy.

Thermodynamics of 1D quantum integrable systems have been discussed by the thermodynamic Bethe ansatz (TBA) method<sup>1</sup> based on the string hypothesis.<sup>2-4</sup> Thermal quantities are determined by a set of non-linear integral equations called TBA equations. As an alternative method, the quantum transfer matrix (QTM) approach has been proposed. $^{5-26}$  It utilizes the Trotter formula and reduces the calculation of the free energy to the eigenvalue problem of the QTM on a fictitious system. A set of auxiliary functions, including the QTM itself, satisfy functional relations. Under suitable choice, these functions have a property called ANZC (Analytic, NonZero and Constant asymptotics) in certain strips. This admits a transformation of the functional relations to non-linear integral equations (NLIE) which characterize the free energy.

Though each method reproduces the correct free energy, there seems to be no relation between the QTM and the TBA. Recently, however, we recognize the NLIE are identical to the TBA equations by utilizing the fusion hierarchy.<sup>14,19,21,25</sup> The members of fusion hierarchy (*T*-functions) and the combinations of them (*Y*-functions) satisfy functional relations called *T*-system and *Y*-system, respectively.(See ref. 27,28 for general relations between *T* and *Y*-systems.) By selecting these functions such that they are ANZC in appropriate strips, we derive the NLIE which are identical to the TBA equations. Moreover considering the sub-leading eigenvalues, we derive systematically the "excited state TBA" equations, which is difficult within the string hypothesis.

We apply this procedure to the spinless Fermion model. The physical quantities such as the free energy and the correlation length have been already derived by more practical choice of the auxiliary functions in ref. 26. Our present interest, however, is the derivation of the TBA and excited state TBA equations by utilizing the fusion hierarchy and to confirm the consistency between the resultant equations and the earlier ones. The Hamiltonian on the periodic lattice of size L is

$$\mathcal{H} = \frac{t}{2} \sum_{j=1}^{L} \left\{ c_j^{\dagger} c_{j+1} + c_{j+1}^{\dagger} c_j + 2\Delta \left( n_j - \frac{1}{2} \right) \left( n_{j+1} - \frac{1}{2} \right) \right\},$$

where we consider the model in the repulsive critical region  $0 \le \Delta = \cos \theta < 1$ , 0 < t and introduce the parameter  $p_0$  as  $p_0 = \pi/\theta (p_0 \ge 2)$ . The commuting QTM on a fictitious system of size N (Trotter number,  $N \in 2\mathbb{Z}$ ) has been defined by the Fermionic R-operator and its supertransposition.<sup>26</sup> The eigenvalue is expressed as

$$T_{1}(u,v) = \phi_{+}(v)\phi_{-}(v-2i)\frac{Q(v+2i)}{Q(v)}$$
$$+\varepsilon\phi_{-}(v)\phi_{+}(v+2i)\frac{Q(v-2i)}{Q(v)}, \qquad (1)$$
$$\phi_{\pm}(v) = \left(\frac{\operatorname{sh}\frac{\theta}{2}(v\pm iu)}{\sin\theta}\right)^{\frac{N}{2}}, \quad Q(v) = \prod_{j=1}^{N_{e}}\operatorname{sh}\frac{\theta}{2}(v-w_{j}).$$

Here  $\varepsilon = (-1)^{N/2+N_{\rm e}}$  and  $N_{\rm e} = \{0, 1, \dots, N/2\}$  is the quantum number counting the holes on odd sites and particles on even sites.  $\{w_j\}$  is a set of solutions to the Bethe ansatz equation (BAE):

$$\left(\frac{\phi_{+}(v)\phi_{-}(v-2i)}{\phi_{-}(v)\phi_{+}(v+2i)}\right)^{\frac{N}{2}} = -\varepsilon \prod_{k=1}^{N_{e}} \frac{Q(w_{j}-2i)}{Q(w_{j}+2i)}.$$
 (2)

The largest eigenvalue  $T_1^{(1)}(u, v)$  and the second largest eigenvalue  $T_1^{(2)}(u, v)$  lie in  $N_e = N/2$  and  $N_e = N/2 - 1$ , respectively. Hereafter we often omit the u variable in  $T_1(u, v)$ . The free energy per site f is represented by

$$f = -\frac{1}{\beta} \lim_{N \to \infty} \ln T_1^{(1)}(u_N, 0) - \frac{t}{4}\Delta, \ u_N = -\frac{\beta t \sin \theta}{\theta N}.$$
(3)

The correlation length  $\xi$  of  $\langle c_j^{\dagger} c_i \rangle$  can be also given by the ratio of  $T_1^{(1)}(v)$  to  $T_1^{(2)}(v)$ ,

$$\frac{1}{\xi} = -\lim_{N \to \infty} \ln \left| \frac{T_1^{(2)}(u_N, 0)}{T_1^{(1)}(u_N, 0)} \right|.$$
(4)

To evaluate  $T_1(v)$ , we consider a more general family called *T*-functions:

$$T_{n-1}(u,v) = \sum_{j=1}^{n} \varepsilon^{j-1} \phi_{-} \left( v - i(n+2-2j) \right) \\ \times \frac{\phi_{+} \left( v - i(n-2j) \right) Q(v+in) Q(v-in)}{Q(v+i(2j-n)) Q(v+i(2j-n-2))}, \quad (5)$$

where we set  $T_{-1}(x) = 0$ .

For any  $v \in \mathbb{C}$  and integers  $n \ge m \ge 1$ , the *T*-functions satisfy the following relations,

$$T_{n-1}(v + im)T_{n-1}(v - im) = T_{n+m-1}(v)T_{n-m-1}(v) +\varepsilon^{n-m}T_{m-1}(v + in)T_{m-1}(v - in).$$
(6)

From now on we consider the case  $p_0 \in \mathbb{Z}_{\geq 3}$ . In this case *T*-functions satisfy further functional relation,

$$T_{p_0}(v) = \varepsilon T_{p_0-2}(v) + (-1)^{N_e} (1 + \varepsilon^{p_0}) T_0(v + ip_0).$$
(7)

We call the above two functional relations the T-system. The proof of them are direct by using (5) and the periodicity of the T-functions

$$T_{n-1}(v) = T_{n-1}(v + 2p_0i).$$
 (8)

Let us define a set of combinations among T-functions as

$$Y_{j}(v) = \frac{T_{j+1}(v)T_{j-1}(v)}{T_{0}(v+i(j+1))T_{0}(v-i(j+1))},$$
  

$$\varepsilon^{j} + Y_{j}(v) = \frac{T_{j}(v+i)T_{j}(v-i)}{T_{0}(v+i(j+1))T_{0}(v-i(j+1))},$$
 (9)  

$$K(v) = (-1)^{N_{e}} \frac{T_{p_{0}-2}(v)}{T_{0}(v+ip_{0})}.$$

where  $1 \leq j \leq p_0 - 2$ ,  $Y_0(v) = 0$  and  $Y_{-1}(v) = \infty$ . We find that  $\{Y_j(v)\}_{j=1}^{p_0-1}$  and K(v) satisfy following finite set of functional relations, which we call the Y-system.

$$Y_{j}(v+i)Y_{j}(v-i) = (\varepsilon^{j-1} + Y_{j-1}(v))(\varepsilon^{j+1} + Y_{j+1}(v)),$$
  

$$\varepsilon^{p_{0}-1} + Y_{p_{0}-1}(v) = (\varepsilon + K(v))(\varepsilon^{p_{0}} + \varepsilon K(v)), \quad (10)$$
  

$$K(v+i)K(v-i) = \varepsilon^{p_{0}} + Y_{p_{0}-2}(v),$$

This can be proved by the *T*-system (6) and (7). Let  $T_{n-1}^{(k)}(v)$  denote  $T_{n-1}(v)$  constructed by the BAE roots which is relevant to the *k*-th largest eigenvalue and  $Y_j^{(k)}(v)$ ,  $K^{(k)}(v)$  be the *Y*-functions constructed from  $\{T_{n-1}^{(k)}\}$  as in (9). Using the property  $\phi_{\pm}(v \pm 2p_0i) = (-1)^{N/2}\phi_{\pm}(v)$  and  $Q(v + 2p_0i) = (-1)^{N_e/2}Q(v)$ , we can easily show for  $p_0 \in 2\mathbb{Z}$ 

$$T_{p_0-1}^{(2)}(v) = 0, \quad Y_{p_0-2}^{(2)}(v) = 0.$$
 (11)

To proceed further, one clarifies the analyticity of the Y-functions by numerical studies on the T-functions, keeping the Trotter number N finite. From them, after suitable modification we confirm the Y-functions have the ANZC property in the strip  $\text{Im}v \in [-1,1]$ . We call this "physical strip". Then we can transform the Y-system to a closed set of NLIE in the following way. First we take the logarithmic derivative and perform the Fourier transformation on both side of the first and third equation in (10). Second by Cauchy's theorem, the Fourier mode for the logarithmic derivative of  $Y_j(v)$  (K(v)) is expressed by those of  $\varepsilon^{j+1} + Y_{j+1}(v)$ and  $\varepsilon^{j-1} + Y_{j-1}(v)$   $(\varepsilon^{p_0} + Y_{p_0-2}(v))$ . Finally performing the inverse Fourier transformation and integrating over v, we derive the desired NLIE. In these NLIE, the Trotter limit  $N \to \infty$  can be taken analytically. The eigenvalue  $T_1^{(k)}(v)$  are determined after performing such transformation on

$$T_1(v+i)T_1(v-i) = (\varepsilon + Y_1(v))T_0(v+2i)T_0(v-2i).$$
(12)

For the largest eigenvalue in the sector  $N_{\rm e} = N/2$ , all functions and functional relations are equivalent to those of the XXZ model in the critical regime.<sup>21</sup> Therefore one concludes that the resultant NLIE which characterize the free energy for the spinless fermion model are identical to the TBA equations for the XXZ model.<sup>4,21</sup>

Let us consider the second largest eigenvalue in the sector  $N_{\rm e} = N/2 - 1$ . As is mentioned in ref. 26, we find that two pure imaginary eigenvalues which are complex conjugate each other, are degenerate in magnitude. All the BAE roots for these two eigenvalues are real. The two distributions of the BAE roots are symmetrical with respect to the imaginary axis. Hereafter we consider the case  $\text{Im}T_1^{(2)}(0) > 0$  (the case  $\text{Im}T_1^{(2)}(0) < 0$  also can be discussed in similar way). According to the numerical studies, we have the following conjecture which is quite different from that in XXZ model.<sup>21</sup>

**Conjecture 1**  $T_{n-1}^{(2)}(v)$  has one real zero  $\zeta_{n-1}$  for  $\{n \in 2\mathbb{Z} | 2 \leq n \leq p_0 - 1\}$  and two real zeros  $\zeta_{n-1}$  and  $-\zeta'_{n-1}$  for  $\{n \in 2\mathbb{Z} + 1 | 2 < n < p_0/2\}$ . All other zeros are out of the physical strip.

Here  $\zeta_{n-1}, \zeta'_{n-1} > 0$ . For example we depict the location of zeros in Fig. 1 for  $p_0 = 10$ .

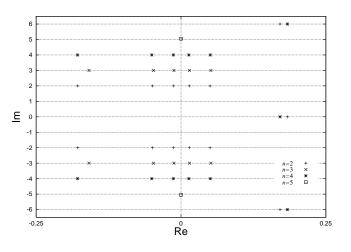


FIG. 1. Location of zeros for  $T_{n-1}^{(2)}(u,v)$  for u = -0.05,  $p_0 = 6$ , N = 12. There exists one real zero for n = 2 and n = 4, while for n = 3 two zeros are at  $\pm \infty$ . Zeros on the imaginary axis for n = 5 are sextuple roots.

 $Y_j^{\left(2\right)}$  and  $K^{\left(2\right)}$  have the asymptotic values

$$Y_j^{(2)} \to \begin{cases} -\frac{\sin(j+2)\theta \sin j\theta}{\cos^2 \theta} & \text{for } j \in 2\mathbb{Z} \\ \frac{\cos(j+2)\theta \cos j\theta}{\cos^2 \theta} & \text{for } j \in 2\mathbb{Z}+1 \end{cases},$$
(13a)

$$K^{(2)} \to \begin{cases} 1 & \text{for } p_0 \in 2\mathbb{Z} \\ i \tan \theta \ (v \to \infty) & \text{for } p_0 \in 2\mathbb{Z} + 1 \\ -i \tan \theta \ (v \to -\infty) & \text{for } p_0 \in 2\mathbb{Z} + 1 \end{cases}$$
(13b)

From conjecture 1 and above asymptotics, the factors of the rhs in (10) such as  $(-1)^{j+1} + Y_{j-1}^{(2)}(v)$  are ANZC in Im $v \in [-\epsilon, \epsilon]$  ( $\epsilon \ll 1$ ). For the lhs, we need to modify  $Y_j^{(2)}(v)$  ( $\{j \in 2\mathbb{Z} | 2 \leq j \leq p_0 - 3\}$  or  $\{j \in 2\mathbb{Z} + 1 | 1 \leq j \leq p_0/2\}$ ) and  $K^{(2)}(v)$  ( $p_0 \in 2\mathbb{Z} + 1$ ) such that they are ANZC in the physical strip. For example, we take the case  $p_0 = 6$ . The following modification,

$$\begin{split} \widetilde{Y}_{1}^{(2)}(v) &= \frac{Y_{1}^{(2)}(v)e^{\frac{\pi}{p_{0}}v \operatorname{th}\frac{\pi}{4}v}}{\left(\operatorname{th}\frac{\pi}{4}(v-\operatorname{i}(u+1))\operatorname{th}\frac{\pi}{4}(v+\operatorname{i}(u+1))\right)^{\frac{N}{2}}},\\ \widetilde{Y}_{2}^{(2)}(v) &= \frac{Y_{2}^{(2)}(v)}{\operatorname{th}\frac{\pi}{4}(v-\zeta_{1})\operatorname{th}\frac{\pi}{4}(v-\zeta_{3})},\\ \widetilde{Y}_{3}^{(2)}(v) &= Y_{3}^{(2)}(v)e^{\frac{\pi}{p_{0}}v \operatorname{th}\frac{\pi}{4}v}, \end{split}$$

is sufficient. Note that the factor  $e^{\pi v \operatorname{th}(\pi v/4)/p_0}$  is included to compensate the singularity caused by  $Y_1^{(2)}(v)$  and  $Y_3^{(2)}(v)$  tending to zero as  $e^{-\pi |v|/p_0}$  at  $v = \pm \infty$ . Let us rewrite the Y-functions in the Trotter limit as

$$\eta_j(v) = \lim_{N \to \infty} Y_j^{(2)}(u_N, v), \ \kappa(v) = \lim_{N \to \infty} K^{(2)}(u_N, v). \ (14)$$

After taking the logarithmic derivative and performing the Fourier transformation, we derive the NLIE obeyed by  $\eta_i$  and  $\kappa$ .

$$\begin{split} \ln \eta_{1}(v) &= -\frac{\beta \pi t \sin \theta}{2\theta \mathrm{ch}(\frac{\pi v}{2})} + s_{1} * \ln((1+\eta_{2})h_{1})(v) + \begin{cases} \pi \mathrm{i} & \text{for } p_{0} \leq 5\\ \pi \mathrm{i} - \frac{\pi}{6} \operatorname{vth} \frac{\pi v}{4} & \text{for } p_{0} = 6\\ \ln\{\mathrm{th} \frac{\pi}{4}(v-\zeta_{2})\mathrm{th} \frac{\pi}{4}(v+\zeta_{2}')\} & \text{for } p_{0} \geq 7 \end{cases},\\ \ln \eta_{j}(v) &= s_{1} * \ln(1-\eta_{j-1})(1-\eta_{j+1})(v) + \ln\{\mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v-\zeta_{j+1})\} + \pi \mathrm{i} & \text{for } \{j \in 2\mathbb{Z} | 2 \leq j \leq p_{0} - 3\},\\ \ln \eta_{j}(v) &= s_{1} * \ln((1+\eta_{j-1})(1+\eta_{j+1})h_{j})(v) \\ &+ \begin{cases} \ln\{\mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j-1}')\mathrm{th} \frac{\pi}{4}(v-\zeta_{j+1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j+1}')\mathrm{th} \frac{\pi}{4}(v+\zeta_{j+1}')\mathrm{th} \\ \ln\{\mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j-1}')\mathrm{th} \frac{\pi}{4}(v-\zeta_{j+1})\mathrm{th} \frac{\pi}{4}v & \text{for } \{j \in 2\mathbb{Z} + 1 | 3 \leq j < \frac{p_{0}-4}{2}\mathrm{ch} + 1 \mathrm{th} \mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j-1}')\mathrm{th} \frac{\pi}{4}v & \text{for } \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\mathrm{ch} + 1 \mathrm{th} \mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j-1}')\mathrm{th} \mathrm{th} \frac{\pi}{4}v & \text{for } \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\mathrm{ch} + 1 \mathrm{th} \mathrm{th} \frac{\pi}{4}(v-\zeta_{j-1})\mathrm{th} \frac{\pi}{4}(v+\zeta_{j-1}')\mathrm{th} \mathrm{th} \mathrm{th} v + \zeta_{j-1}')\mathrm{th} \mathrm{th} \mathrm{th} v + \zeta_{j-1}'\mathrm{th} \mathrm{th} \mathrm{th} \mathrm{th} v + \zeta_{j-1}'\mathrm{th} \mathrm{th} \mathrm{th} \mathrm{th} \mathrm{th} v + \zeta_{j-1}'\mathrm{th} \mathrm{th} \mathrm{th} \mathrm{th} \mathrm{th} v + \zeta_{j-1}'\mathrm{th} \mathrm{th} \mathrm{$$

i.

where

$$h_j(v) = \begin{cases} \exp\left(\frac{2\pi}{p_0}\left(\frac{v \operatorname{sh} \frac{\pi}{2} v - 1}{\operatorname{ch} \frac{\pi}{2} v}\right)\right) \text{ for } \{j \in 2\mathbb{Z} + 1 | j = \frac{p_0 - 4}{2}, \frac{p_0}{2}\}\\ 1 & \text{otherwise.} \end{cases}$$

The symbol \* denotes the convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy,$$

and  $s_1(v) = 1/4 \operatorname{ch} \frac{\pi}{2} v$ . Here the integration constants have been fixed from the asymptotic values (13a) and (13b). In addition to above equations, we need to impose the consistency condition coming from  $T_j^{(2)}(\zeta_j) = 0$ and  $T_j^{(2)}(-\zeta'_j) = 0$ . From (9) and (14), this leads to  $\eta_j(\zeta_j \pm \mathbf{i}) = 1$  for  $\{j \in 2\mathbb{Z} + 1 | 1 \leq j \leq p_0 - 2\}$  and  $\eta_j(\zeta_j \pm \mathbf{i}) = \eta_j(\zeta'_j \pm \mathbf{i}) = -1$  for  $\{j \in 2\mathbb{Z} | 1 < j < p_0/2 - 1\}$ . Explicitly they read

$$0 = i \frac{\beta \pi t \sin \theta}{2\theta \operatorname{ch}(\frac{\pi \zeta_1}{2})} + s_1 * \ln((1+\eta_2)h_1)(\zeta_1+i) + \begin{cases} -\pi i & \text{for } p_0 \leq 5\\ -\pi i - \frac{\pi}{6}(\zeta_1+i)\operatorname{th}\frac{\pi(\zeta_1+i)}{4} & \text{for } p_0 = 6\\ \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_1-\zeta_2+i)\} + \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_1+\zeta_2'+i)\} - 2\pi i & \text{for } p_0 \geq 7 \end{cases}$$

$$0 = \begin{cases} s_1 * \ln(1-\eta_{j-1})(1-\eta_{j+1})(\zeta_j+i) + \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_j-\zeta_{j-1}+i)\} + \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_j-\zeta_{j+1}+i)\}\\ s_1 * \ln(1-\eta_{j-1})(1-\eta_{j+1})(\zeta_j'+i) + \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_j'-\zeta_{j-1}+i)\} + \ln\{\operatorname{th}\frac{\pi}{4}(\zeta_j'-\zeta_{j+1}+i)\} - 2\pi i \\ \operatorname{for } \{j \in 2\mathbb{Z} | 2 \leq j < \frac{p_0}{2} - 1\}, \end{cases}$$

$$0 = s_1 * \ln((1+\eta_{j-1})(1+\eta_{j+1})h_j)(\zeta_j+i) \tag{16}$$

$$+ \begin{cases} \ln\{th\frac{\pi}{4}(\zeta_{j}-\zeta_{j-1}+i)\} + \ln\{th\frac{\pi}{4}(\zeta_{j}+\zeta'_{j-1}+i)\} \\ + \ln\{th\frac{\pi}{4}(\zeta_{j}-\zeta_{j+1}+i)\} + \ln\{th\frac{\pi}{4}(\zeta_{j}+\zeta'_{j+1}+i)\} - 2\pi i \\ \ln\{th\frac{\pi}{4}(\zeta_{j}-\zeta_{j-1}+i)\} + \ln\{th\frac{\pi}{4}(\zeta_{j}+\zeta'_{j-1}+i)\} - \pi i - \frac{\pi}{p_{0}}(\zeta_{j}+i)th\frac{\pi}{4}(\zeta_{j}+i) \\ \ln\{th\frac{\pi}{4}(\zeta_{j}-\zeta_{j-1}+i)\} + \ln\{th\frac{\pi}{4}(\zeta_{j}+\zeta'_{j-1}+i)\} - \pi i \\ -\frac{\pi}{p_{0}}(\zeta_{j}+i)th\frac{\pi}{4}(\zeta_{j}+i) \\ 0 \end{cases} for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-4}{2}\} \\ for \{j \in 2\mathbb{Z} + 1 | j = \frac{p_{0}-$$

where the convolutions should be interpreted as

$$s_1 * g(\zeta + \mathbf{i}) = \text{p.v.}\left(\int_{-\infty}^{\infty} \frac{g(x)}{4\mathrm{ish}\frac{\pi}{2}(\zeta - x)} dx\right) + \frac{1}{2}g(\zeta).$$

Here p.v. means the principal value. Then the Trotter limit of  $T_1^{(2)}(v)$  can be expressed from (12) as

$$\lim_{N \to \infty} T_1^{(2)}(u_N, 0) = \ln th \frac{\pi}{4} \zeta_1 + \int_{-\infty}^{\infty} dv s_1(v) \ln(1 - \eta_1(v)) + t\beta \int_{-\infty}^{\infty} dv s_1(v) \frac{\sin^2 \theta}{ch\theta v - \cos \theta} - \frac{t}{2}\beta \cos \theta + \frac{\pi}{2}i.$$

Especially for  $p_0 = 4$ , the zero  $\zeta_1$  and Y-functions  $\eta_1(v)$  are explicitly determined by using the fact (11), which is a novel feature in the present approach.<sup>26</sup>

$$\eta_1(v) = -\exp\left(-\frac{\sqrt{2}\beta t}{\operatorname{ch}\left(\frac{\pi}{2}v\right)}\right), \quad \zeta_1 = \frac{2}{\pi}\operatorname{ch}^{-1}\left(\frac{\sqrt{2}\beta t}{\pi}\right)$$

Finally we obtain the correlation length (4) as

$$\frac{1}{\xi} = -\ln th \frac{\pi}{4} \zeta_1 - \int_{-\infty}^{\infty} dv s_1(v) \ln \left(\frac{1 - \eta_1(v)}{1 + \eta_1^{(1)}(v)}\right), \quad (17)$$

where the function  $\eta_1^{(1)}(v)$  characterizing the free energy has been calculated in ref. 4,21. We evaluate (15) and (16) numerically and explicitly determine the correlation length in Fig. 2.

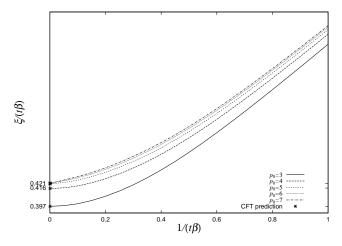


FIG. 2. Ratio of the correlation length and inverse temperature for  $p_0 = 3, 4, 5, 6$  and 7.

These results are consistent with those in ref. 26. Especially in the low temperature limit they agree with known expression from CFT

$$\lim_{\beta \to \infty} \xi_2(\beta) / \beta = \frac{t \sin \theta}{2\theta} \left( \frac{\pi - \theta}{\pi} + \frac{\pi}{4(\pi - \theta)} \right)^{-1}.$$
 (18)

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