

Dynamics of the Compact, Ferromagnetic $\nu = 1$ Edge

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We consider the edge dynamics of a compact, fully spin polarized state at filling factor $\nu = 1$. We show that there are two sets of collective excitations localized near the edge: the much studied, gapless, edge magnetoplasmon but also an additional edge spin wave that splits off below the bulk spin wave continuum. We show that both of these excitations can soften at finite wave-vectors as the potential confining the system is softened, thereby leading to edge reconstruction by spin texture or charge density wave formation. We note that a commonly employed model of the edge confining potential is non-generic in that it systematically underestimates the texturing instability.

I. INTRODUCTION

Following the seminal work of Halperin¹ and especially of Wen², there has been a great deal of work building on the insight that there must be interesting low energy dynamics at the edges of incompressible quantum Hall systems. Much of this work has focused on the presence of Luttinger liquid correlations at the edges of model quantum Hall systems, leading to predictions that have considerable support in experiments albeit with the perplexing feature that the latter seem insensitive to the presence or absence of incompressibility in the bulk³.

In this paper we wish to revisit one of the simplest quantum Hall (QH) systems, that at filling factor $\nu = 1$. *Prima facie* this would seem to be an uninspiring choice as the edge ground state correlations are known to be those of a Fermi liquid and hence no interesting power laws are expected by theory, or seen in the experiments thus far. Our motivation however is different and stems from the identification of $\nu = 1$ as the archetypical QH ferromagnet with non-trivial bulk dynamics characterized by a density-topological density relation⁴. At issue here is whether this novel dynamics has a counterpart at the edges of the system. We have previously shown in a Hartree-Fock/effective-action study⁵ that this physics allows for a reconstruction of the $\nu = 1$ edge in which charge is moved outwards by texture formation as the confining potential is softened—the same phenomenon takes place also in quantum dots⁶. Here we will ask the logically prior question of how the low lying but not necessarily gapless, edge excitations inclusive of spin are affected in the phase in which the edge is still sharp. We note that many of these results were noted previously in conference contributions⁷ and in the interim there have appeared two publications^{8,9} with some overlapping content which we discuss in the main text. Edge reconstruction has also been studied within the Chern-Simon-Ginzburg-Landau theory¹⁰ and more work on the reconstruction in quantum dots¹¹ has appeared—the latter is not directly connected to our results and we will not discuss it further. Finally, Milovanovic¹² has attempted to deduce the edge physics of ferromagnetic quantum Hall states from general consistency principles applied to the bulk low energy effective action, but she finds a mode structure that is at odds with our, microscopic, results. Currently, we do not understand how her approach can be modified to remove this discrepancy.

The outline of the paper is as follows. In Section II we define the problem more precisely and write down a set of generalized random phase approximation or time dependent Hartree-Fock equations (TDHF) for the particle-hole pair dynamics. Next we check that in the absence of an edge these equations recover the well known results^{13,14} for spin-waves in the bulk. In section IV we show that for the problem with an “ideal” edge (to be defined below) the TDHF equations give rise to a novel non-chiral edge spin wave (ESW) in addition to the much studied chiral edge magnetoplasmon (EMP). In Section V we discuss these modes away from the “ideal” case. We find that they can go soft as the confining potential at the edge is weakened and lead, naturally, to a charge density wave reconstruction of the edge as well as the spin textured reconstruction we have considered previously. We also show that a commonly employed model for studying edge reconstruction is non-generic and artificially suppresses textured reconstruction. In Section VI we discuss in more detail how the ESW can be fitted into a continuum description via a novel boundary condition. In Section VII we comment on the issues involved in going beyond the TDHF approximation, which turns out to be essentially exact at long wavelengths, and then conclude with a brief summary and discussion of the experimental relevance of these results.

II. HAMILTONIAN AND TDHF EQUATIONS

We are interested in the problem of the $\nu = 1$ QH state with a boundary where the filling factor decreases to zero. To this end we consider a semi-infinite 2DEG with a single boundary (Figure 1a). To model the confinement necessary to produce such an edge we assume the existence of a background compensating charge *in* the plane which falls from the density of the bulk $\nu = 1$ state to zero.

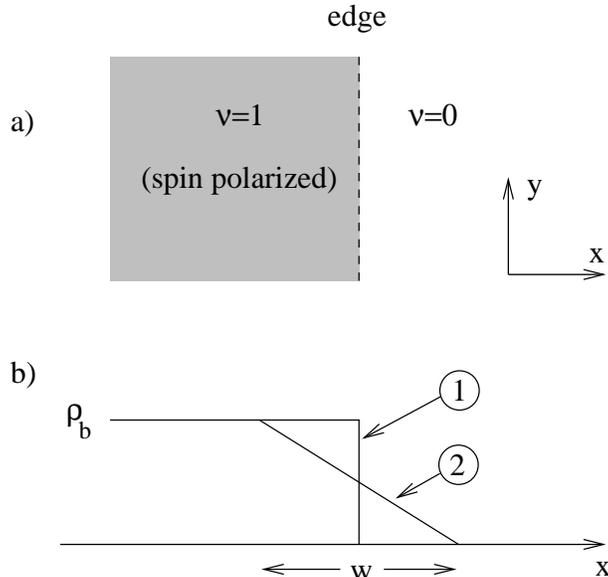


FIG. 1. (a) Geometry of studied quantum Hall system; (b) Examples of in-plane compensating background charges that confine the electrons: (1) the ideal edge and (2) a softer confinement.

The particular choice denoted 1 in Figure 1b, defined more precisely in (2.5) below, has the effect of *exactly* canceling the Hartree potential of the $\nu = 1$ state with a sharp edge at $x = 0$ which is thus favored. We shall refer to the edge *with this choice of confining potential* as the “ideal edge”. More generally we shall consider background densities that decrease more gradually but integrate to the same amount of charge as in the ideal edge, e.g. the piecewise linear background sketched as 2 (in fact we shall see that this choice is non-generic), as well as confining potentials independent of y that arise from charges not in the plane of the electrons; the latter will turn out to be essentially different in one important respect. These choices of confinement will tend to expand the area of the system and will compete with the tendency of the exchange to keep the electronic state spin polarized and compact. This softening of the confinement will eventually produce a reconstruction of the edge, but we will be interested here in the parameters for which the edge remains compact.

To formalize this we will restrict the Hilbert space to the lowest Landau level of either spin and choose Landau gauge $\mathbf{A} = Bx\hat{y}$ in which the single particle states are (setting the magnetic length $\ell = \sqrt{\hbar c/eB} = 1$)

$$u_k(x, y) = \frac{1}{\sqrt{\pi^{1/2}L}} e^{iky} e^{-(x-k)^2/2} \quad (2.1)$$

with $e^{ikL} = 1$ for a system of length L made periodic along the edge. Expanding the lowest Landau level electron field operator in this basis,

$$\Psi_\sigma(x, y) = \sum_k u_k(x, y) c_{k\sigma} \quad (2.2)$$

($\{c_k^\dagger, c_{k'}\} = \delta_{k,k'}$) we can write the many-body Hamiltonian as

$$\hat{H} = \frac{1}{2} \sum_{kpq\sigma\sigma'} V(p, q) c_{k+q, \sigma}^\dagger c_{k+p-q, \sigma'}^\dagger c_{k+p, \sigma'} c_{k, \sigma} - \sum_{kp\sigma} V(p, 0) \rho_b(k+p) c_{k, \sigma}^\dagger c_{k, \sigma}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{kp} V(p, 0) \rho_b(k) \rho_b(k+p) \\
& - \frac{1}{2} g \mu_B B \sum_k (c_{k\uparrow}^\dagger c_{k\uparrow} - c_{k\downarrow}^\dagger c_{k\downarrow}) .
\end{aligned} \tag{2.3}$$

Here $V(p, q) = \tilde{V}(q, p - q, p, 0)$ with $\tilde{V}(k_1, k_2, k_3, k_4)$ defined as $\int d^2 r d^2 r' U(\mathbf{r} - \mathbf{r}') \bar{u}_{k_1}(\mathbf{r}) \bar{u}_{k_2}(\mathbf{r}') u_{k_3}(\mathbf{r}') u_{k_4}(\mathbf{r})$, $\rho_b(k)$ is the occupation of the “background orbital” and defines the real space background charge density via $\rho_b(x) = \int \frac{dk}{2\pi} \rho_b(k) \frac{e^{-(x-k)^2}}{\sqrt{\pi}}$. The second term is the interaction of the electrons with the background charge, the third term is the c-number self interaction of the background charge density and in the fourth we will take the g-factor to be positive. We will take the interparticle potential to be of the unscreened Coulomb form $U(\mathbf{r}) = \frac{e^2}{|\mathbf{r}|}$, which leads to the explicit expression,

$$V(p, q) = \frac{e^2}{L} \sqrt{\frac{2}{\pi}} e^{-[\frac{q^2}{2} + \frac{(p-q)^2}{2}]} \int_{-\infty}^{\infty} dy K_0(|qy|) e^{-(y^2 + 2(p-q)y)/2} \tag{2.4}$$

for the matrix elements of interest.

The ideal edge is now defined by taking

$$\rho_b(k) = \Theta(-k) \tag{2.5}$$

which then exactly cancels the Hartree potential of the compact state

$$|G\rangle = \prod_{k \leq 0} c_{k\uparrow}^\dagger |0\rangle . \tag{2.6}$$

This is an exact eigenstate of \hat{H} for any y -invariant confining potential by virtue of minimizing the momentum along the y -direction. For the ideal edge Hamiltonian, this is the ground state and in this paper we will be interested only in those choices of confinement for which this continues to hold.

A first pass at the excitation spectrum is obtained by the Hartree-Fock (HF) decoupling, $\langle c_{k\uparrow}^\dagger c_{k\uparrow} \rangle = f_\uparrow(k) = \Theta(-k)$ which yields the eigenvalues,

$$\begin{aligned}
\epsilon_{HF}(k\sigma) = & \sum_p V(p, 0) [f_\uparrow(k+p) - \rho_b(k+p)] \quad \text{Hartree} \\
& - \delta_{\sigma\uparrow} \sum_p V(k-p, k-p) f_\uparrow(p) \quad \text{Exchange} .
\end{aligned} \tag{2.7}$$

These are sketched in Figure 2 for the ideal edge and for a linear background with (arbitrarily chosen) scale $w = 6.77$. The latter exhibits the beginnings of a minimum outside the region occupied by the electrons, which leads to a polarized reconstruction of the edge as discussed in the pioneering papers^{16,17}.

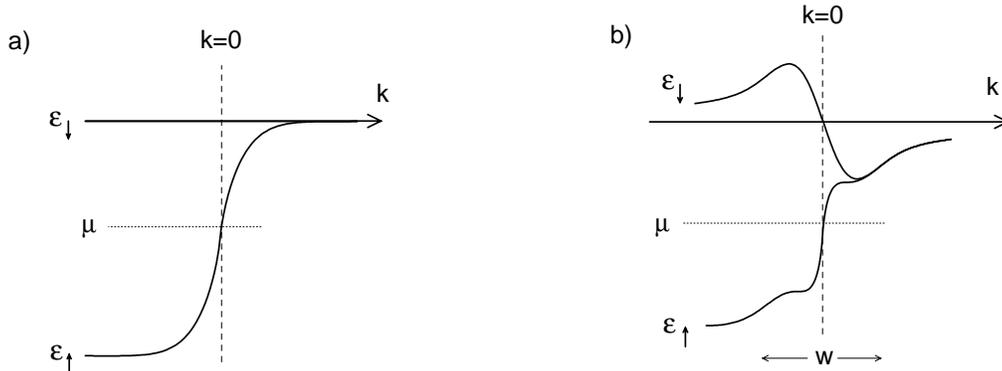


FIG. 2. Hartree-Fock eigenvalues $\epsilon_\sigma(k) \equiv \epsilon_{HF}(k\sigma)$ for the ideal edge (a) and for a linear background with $w = 6.77$. (μ is the chemical potential.)

A second pass at the (neutral) excitation spectrum requires that we go beyond HF to the time dependent Hartree-Fock approximation (TDHF) or the generalized random phase approximation. This consists of considering states which involve one particle-hole excitation of the Hartree-Fock ground states and diagonalizing the full Hamiltonian in this restricted subspace. A convenient way of deriving the TDHF equations is by the equations of motion method discussed, for example, in¹⁸. In our case it yields, for the operators $b_\sigma^\dagger(k, q) \equiv c_{k+q\sigma}^\dagger c_{k\uparrow}$, the equation,

$$-i\partial_t b_\sigma^\dagger(k, q) = [\epsilon_{HF}(k + q\sigma) - \epsilon_{HF}(k \uparrow)]b_\sigma^\dagger(k, q) + \sum_{k'} [f_\sigma(k + q) - f_\uparrow(k)] \\ \times [V(k - k' - q, k - k') - \delta_{\sigma\uparrow} V(k - k' + q, q)] b_\sigma^\dagger(k', q). \quad (2.8)$$

These describe the HF time evolution of the particle-hole states (first term) as well as the scattering between them (second term). At this level of approximation, there is no mixing between states with spin flips ($\sigma = \downarrow$, $\Delta S^z = -1$) and those without ($\sigma = \uparrow$, $\Delta S^z = 0$).

In the following, we will find it very useful, on account of the one dimensional nature of the Landau gauge description, to think of Eqn (2.8) as describing the quantum mechanics of a particle representing the particle hole pair that is located at k , carries quantum numbers q and σ , hops between different spatial locations with an effective kinetic energy $T_{\sigma q}(k - k') = L[f_\sigma(k + q) - f_\uparrow(k)][V(k - k' - q, k - k') - \delta_{\sigma\uparrow} V(k - k' + q, q)]$ and experiences a potential $V_{\sigma q}(k) = \epsilon_{HF}(k + q\sigma) - \epsilon_{HF}(k \uparrow)$. In the limit $L \rightarrow \infty$, the Schrödinger equation for its motion becomes the integral equation,

$$\int_{-\infty}^0 \frac{dk'}{2\pi} T_{\sigma q}(k - k') \psi_{\sigma q}^{(\alpha)}(k') + V_{\sigma q}(k) \psi_{\sigma q}^{(\alpha)}(k) = \epsilon_{\sigma q}^{(\alpha)} \psi_{\sigma q}^{(\alpha)}(k). \quad (2.9)$$

Its solutions then define the TDHF eigenstates through their action on the HF ground state $|G\rangle$,

$$|\alpha; \sigma q\rangle = \int_{-\infty}^0 \frac{dk}{2\pi} \psi_{\sigma q}^{(\alpha)}(k) b_\sigma^\dagger(k, q) |G\rangle. \quad (2.10)$$

III. BULK MODES

In the absence of edges, there are no polarized $\Delta S^z = 0$ particle-hole pair states. In Eqn (2.8) this is reflected in the trivial time evolution when we put $\sigma = \uparrow$ and $f_\uparrow(k) = 1 \forall k$. The non-trivial dynamics is in the $\Delta S^z = -1$ sector, where $\sigma = \downarrow$. In this sector we need to solve (suppressing the spin index),

$$\int_{-\infty}^{+\infty} \frac{dk'}{2\pi} T_q(k - k') \psi_q^{(\alpha)}(k') + V_q(k) \psi_q^{(\alpha)}(k) = \epsilon_q^{(\alpha)} \psi_q^{(\alpha)}(k), \quad (3.1)$$

with $T_q(k - k') = -LV(k - k' - q, k - k')$ and $V_q(k) = \Delta_Z + \int_{-\infty}^{+\infty} \frac{dp}{2\pi} L V(k - p, k - p) \equiv \Delta_Z + \Delta_{\text{exch}}$ independent of k and q , is the “exchange enhanced spin gap”. (We have included the bare Zeeman gap, Δ_Z , in $V_q(k)$.)

This is a translationally invariant problem with plane wave eigenstates, $\psi^{(\alpha)} \propto e^{i\alpha k}$, $-\infty < \alpha < +\infty$ in all q sectors, whose energies are given by,

$$\epsilon_q^{(\alpha)} = \Delta_Z + \Delta_{\text{exch}} + M(\alpha, q) \\ \text{with } M(\alpha, q) = - \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{i\alpha k} L V(k - q, k). \quad (3.2)$$

In our construction of these $\Delta S^z = 1$ states we have used orbitals with well defined y -momentum and so it is clear that $\epsilon_q^{(\alpha)}$ is the energy of a state with y -momentum q . The interpretation of α is a bit obscure in this derivation—it appears as the momentum “conjugate” to the index of the Landau gauge orbitals. However, one can check that $M(\alpha, q)$ is isotropic in the (α, q) plane, leading to the identification of α as the x -momentum of the spin wave state. The resulting dispersion relation is identical with the result obtained in^{13,14}. Note that S^z is a good quantum number and that the single spin flip states exhaust the $\Delta S^z = -1$ states in the lowest Landau level. Hence the spin wave states that we have obtained by diagonalizing the Hamiltonian in the basis of the single spin flip states are exact eigenstates in the lowest Landau level.

IV. MODES OF THE IDEAL EDGE

Returning to the problem of the ideal edge, we note that it is now possible to have single particle-hole excitations with $\Delta S^z = 0$, their wavefunctions $\psi_{\uparrow q}(k)$ having support on $-q \leq k \leq 0$. The second new feature is that the $\Delta S^z = -1$ sector no longer consists solely of single spin flip states, it being possible to excite additional particle-hole pairs within the spin up Landau level. Consequently our analysis will now be approximate for both sectors. We should note that the qualitative features of several of our results have also appeared in the numerical solution of the TDHF equations by Franco and Brey⁸.

The $\Delta S^z = 0$ excitation, the edge magnetoplasmon (EMP), is well known and has been much studied. Here we will derive its long wavelength dispersion in our particular choice of confining potential. To this end we use the asymptotic forms,

$$\begin{aligned} \frac{1}{e^2} T_{\uparrow q}(k - k') &= -\log\left(\frac{q}{k - k'}\right)^2 \\ \frac{1}{e^2} V_{\uparrow q}(k) &= \frac{k + q}{2\pi} \log\left(\frac{C}{(k + q)^2}\right) - \frac{k}{2\pi} \log\left(\frac{C}{k^2}\right) \end{aligned} \quad (4.1)$$

($C = 8e^{2-\gamma}$, $\gamma = 0.5772\dots$ is the Euler constant) and the change of variables, $k \rightarrow q(x - \frac{1}{2})$, $k' \rightarrow q(x' - \frac{1}{2})$ to rewrite the integral equation (2.9) in the dimensionless form,

$$\int_{-1/2}^{+1/2} \frac{dx'}{2\pi} \log(x - x')^2 \psi(x') + \phi(x) \psi(x) = \tilde{\epsilon} \psi(x) \quad (4.2)$$

where

$$\phi(x) = -\frac{x + \frac{1}{2}}{2\pi} \log\left(\frac{1}{2} + x\right)^2 + \frac{x - \frac{1}{2}}{2\pi} \log\left(\frac{1}{2} - x\right)^2 \quad (4.3)$$

and the dimensionless eigenvalue $\tilde{\epsilon}$ is related to the dimensionful eigenvalue by

$$\tilde{\epsilon} = \frac{\epsilon(q)}{e^2 q} - \frac{1}{2\pi} \log\left(\frac{C}{q^2}\right). \quad (4.4)$$

The scaled equation is solved by the choice $\psi_o(x) \equiv 1$ which, being nodeless, is associated with the lowest eigenvalue by the Perron-Frobenius theorem. The corresponding eigenvalue is $\tilde{\epsilon}_o = -1/\pi$, whence we obtain the EMP dispersion,

$$\epsilon(q) = \frac{e^2 q}{2\pi} \log\left(\frac{8}{e^\gamma q^2}\right). \quad (4.5)$$

We now turn to the $\Delta S^z = -1$ sector, which will yield a new edge excitation, the edge spin wave (ESW), in addition to the bulk spin wave continuum found previously. The relevant integral equation is now (2.9) with the simplified notation,

$$\begin{aligned} T_{\downarrow q}(k - k') &\rightarrow T_q(k - k') \\ V_{\downarrow q}(k) &\rightarrow V(k) = -\int_{-\infty}^0 \frac{dk'}{2\pi} T_0(k - k'). \end{aligned} \quad (4.6)$$

(We have ignored the Zeeman energy which can be trivially added back to the various dispersions we will compute below.) Note that for the ideal edge, the potential energy is independent of q exactly as in the bulk case. Figure 3a shows a plot of $V(k)$ for this case which exhibits a minimum at the edge, as expected from the Hartree-Fock energy levels. This should lead us to expect one or more spin wave states bound to the edge. However, there is a competing effect — the kinetic energy is larger near the edge as well, for the “particle” can only hop to the left.

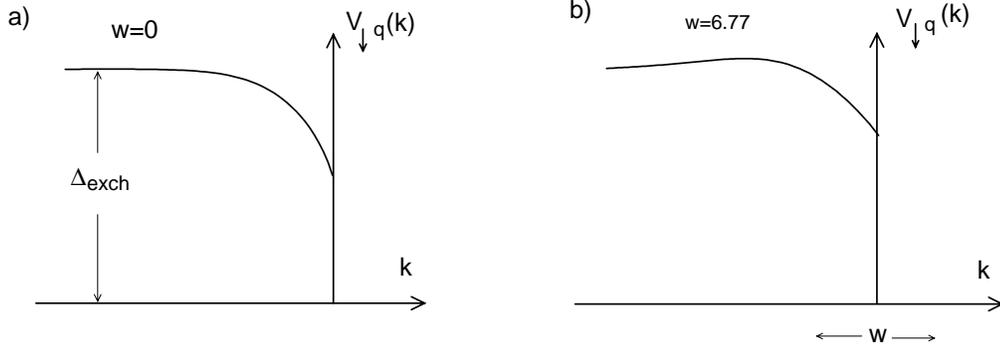


FIG. 3. Potential energy for spin flip particle hole pairs for (a) the ideal edge and (b) a linear background with $w = 6.77$ and $q = 0.817$ for illustration.

Exactly at $q = 0$, these effects compensate each other perfectly: the ground state in this sector is given by the eigenfunction $\psi(k) \equiv 1$ with eigenvalue zero. The eigenvalue can be checked by direct substitution and the Perron-Frobenius theorem again shows that it is the ground state. (The existence of this state is not really a surprise — it is just the statement that even for a geometry with an edge, S_{total}^- gives rise to a degenerate state when acting on the ferromagnetic ground state.) It is important to note though, that $\psi(k) = 1$ is really a *marginally bound* state as in the absence of the attraction, the wavefunction would be suppressed at the edge.

Consequently, at any finite q , where the matrix elements of the kinetic energy are reduced by terms of $O(q^2)$, while the potential energy is unchanged, the lowest energy state is indeed bound to the edge — albeit with a localization length that diverges as $q \rightarrow 0$. Although this is of academic interest, we note that $T_q(k - k') \rightarrow 0$ as $q \rightarrow \infty$ and in that limit an arbitrarily large number of states become bound to the edge. Figure 4a shows the first few states for a finite system with two edges, which clearly exhibits a state bound to each edge as well as the first few continuum states. The existence of a bound state *per edge* is a non-generic feature of the ideal edge problem which is not sensitive to the sign of q , i.e. whether the particle is moved inwards or outwards. Generically, as we shall see, they exist for only one sign of q and are therefore chiral. Figure 4b illustrates the presence of multiple bound states at larger values of q .

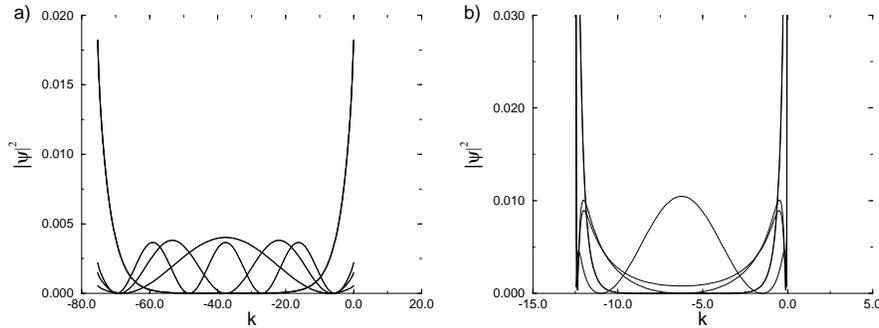


FIG. 4. Probability densities for the lowest energy spin wave states in a system with two (ideal) edges. In (a), $q = 0.628$ and one state is bound to each edge and in (b) $q = 1.885$ and two states are bound to each edge. (For numerical reasons, data are shown for Hall bars of different width in (a) and (b).)

At small q , the localization length¹⁹ of the ESWs is parametrically longer than the edge region which is $O(1)$ in our units. This enables us to get a variational estimate of their properties at small q by using the simple exponential form $\psi(k) = \sqrt{2\lambda}e^{\lambda k}$. Far from the edge, this form is dictated by the recovery of translation invariance and it has the feature that it correctly reproduces the exact answer at $q = 0$ in the limit $\lambda \rightarrow 0$. It is possible that this procedure gives asymptotically exact answers at small q for some choices of interaction, but we do not have a proof of this. We expect to discuss this question elsewhere²⁰.

An equivalent procedure, which offers a different perspective, is to trade the integral equation for an approximate differential equation by the gradient expansion,

$$\psi(k') = \psi(k) + \psi'(k)(k' - k) + \frac{1}{2}\psi''(k)(k' - k)^2 + \dots \quad (4.7)$$

Upon inserting in (2.9) and integrating, we find, to $O(\psi'')$,

$$-\alpha_q(k)\psi''(k) + \beta_q(k)\psi'(k) + \gamma_q(k)\psi(k) = \epsilon(k)\psi(k) \quad (4.8)$$

where

$$\begin{aligned} -2\alpha_q(k) &= \int_{-\infty}^0 \frac{dk'}{2\pi} (k' - k)^2 T_q(k - k') \\ \beta_q(k) &= \int_{-\infty}^0 \frac{dk'}{2\pi} (k' - k) T_q(k - k') \\ \gamma_q(k) &= \int_{-\infty}^0 \frac{dk'}{2\pi} [T_q(k - k') - T_0(k - k')] . \end{aligned} \quad (4.9)$$

Away from the edge, $\beta_q(k)$ vanishes rapidly while $\alpha_q(k)$ and $\gamma_q(k)$ tend to constants α_q, γ_q , and we get an effective Schrödinger equation, valid for long wavelength bulk spin waves. A plane wave solution, $\psi(k) \sim e^{ik_x k}$ then has energy $\epsilon = \gamma_q + \alpha_q k_x^2$ which is easily seen to be the expansion of our earlier result (3.2) to $O(k_x^2)$. For the Coulomb interaction, $\alpha_q = \frac{1}{4}\sqrt{\frac{\pi}{2}} + O(q^2)$ and $\gamma_q = \frac{1}{4}\sqrt{\frac{\pi}{2}}q^2 + O(q^4)$.

In the same region, a bound state will exhibit purely exponential decay, $\psi(k) \sim e^{\lambda k}$, in this approximation, which fixes $\epsilon = \gamma_q - \alpha_q \lambda^2$. To complete the solution we need to solve,

$$[-\alpha_q(k)\psi''(k) + \alpha_q \lambda^2 \psi(k)] + \beta_q(k)\psi'(k) + [\gamma_q(k) - \gamma_q]\psi(k) = 0 . \quad (4.10)$$

At small q , $\gamma_q(k) - \gamma_q \sim O(q^2)$. Assuming that λ^2 , $\psi''/\psi \ll q^2$ which will be clear *a posteriori*, we can neglect the first term. Integrating the remaining terms,

$$\int_{-\infty}^0 dk \beta_q(k)\psi'(k) + \int_{-\infty}^0 dk [\gamma_q(k) - \gamma_q]\psi(k) = 0 . \quad (4.11)$$

If the $q \rightarrow 0$ limit is smooth in that $\psi'(k)$ vanishes uniformly in the edge region (which is really equivalent to the assumption of a purely exponential form for the bound state when combined with our earlier neglect of $\psi''(k)$) we find that

$$\begin{aligned} \frac{\psi'}{\psi} \Big|_{\text{edge}} &= \int_{-\infty}^0 dk [\gamma_q - \gamma_q(k)] / \int_{-\infty}^0 dk \beta_q(k) \\ &= cq^2 + O(q^4) \end{aligned} \quad (4.12)$$

where $c = \frac{2}{3}(\frac{2}{\pi})^{3/2} = 0.33863\dots$ for the Coulomb interaction. From this derivation it is clear that this will apply also to the continuum states at low energies for which ψ'' can be ignored as well.

This boundary condition then fixes $\lambda = cq^2 + O(q^4)$ which is indeed much smaller than q and yields the bound state energy $\epsilon(q) = \gamma_q - \alpha_q c^2 q^4 + \dots$ which shows that the bound state splits off below the spin wave continuum at $O(q^4)$. This means that if we try to take a formal continuum limit in which the magnetic length is taken to zero while the spin wave stiffness is held fixed, the edge bound states disappear ($c \sim \ell$). Consequently the ESWs are a matter of microscopic detail and cannot arise generally in a continuum/sigma-model treatment. We will revisit the boundary condition question below.

V. SOFT CONFINEMENT: MODES AND EDGE RECONSTRUCTION

We turn now to the effect of softening the confinement away from the limit of the ideal edge; a parallel discussion can be given for hardening the confinement. We will assume that the modified confinement does not alter the compact edge. Instead we study its effect on the edge excitation spectrum which will allow us to identify two instabilities towards reconstruction of the edge.

To this end we consider adding an extra potential $V_p(k)$ that exerts an outward force on the electrons near the edge. Previously, the Hartree potential vanished, but now it equals $V_p(k)$ for both the up and down spin bands. In Figure 5 we plot two examples of this. The first (5a) is uniformly repulsive near the edge and is qualitatively similar to the effect produced by moving the compensating charge out of the plane of the electrons. The second (5b) is the potential of the linear confinement with $w > 0$ which gives rise to a dipole *in* the plane at the edge. Note the special feature that it leaves the HF eigenvalues exactly at the edge unshifted, with the down spin eigenvalue degenerate with those deep in the bulk.

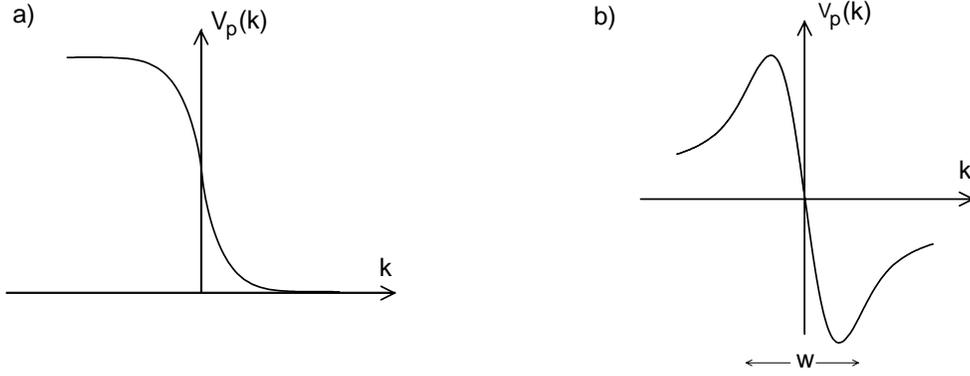


FIG. 5. Perturbing edge potentials $V_p(k)$: (a) uniformly repulsive potential due e.g. to background charge moved out of the plane of the electrons; (b) potential due to a linear in-plane background charge, $w = 6.77$.

This modification has the effect of modifying the HF eigenvalues,

$$\delta\epsilon_{HF}(k\sigma) = V_p(k) \quad (5.1)$$

and hence the potential energies for the particle-hole pairs by

$$\delta V_{\sigma q}(k) = V_p(k+q) - V_p(k) . \quad (5.2)$$

Consequently, the particle-hole pairs see an additional attraction near the edge that varies with q and typically is maximum for some $q > 0$.

By contrast, the $q < 0$ pairs which exist in the $\Delta S_z = -1$ sector alone, are repelled from the edge while the $q = 0$ problem is left unaffected for both spin sectors. In Figure 6a we show examples of this effect for the potential sketched in Figure 5a.

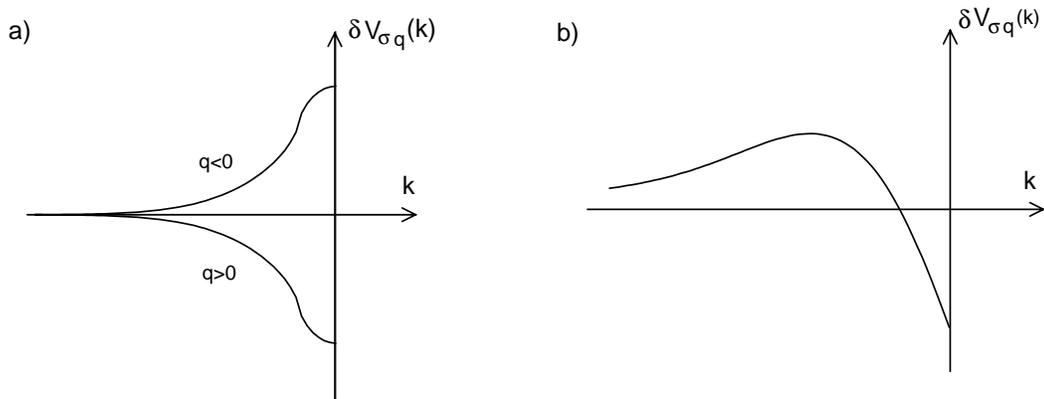


FIG. 6. Shift in potentials $V_{\sigma q}$ due to the perturbing potentials in Figure 5a and 5b respectively, $|q| = 0.817$.

The qualitative effect of this perturbation on the EMP spectrum is to localize the mode wavefunctions $\psi(k)$ closer to the edge and to decrease the corresponding eigenvalues. As the spectrum in a given q sector is discrete, the latter effect can be reliably calculated perturbatively in $V_p(k)$. In the small q limit considered previously $\psi(k) = 1/\sqrt{q}$ for $-q \leq k \leq 0$, whence,

$$\begin{aligned} \Delta\epsilon_{\text{EMP}}(q) &\sim \int_{-q}^0 dk |\psi(k)|^2 \delta V_{\uparrow q}(k) \\ &\sim qV_p'(0) \end{aligned} \quad (5.3)$$

to leading order in the perturbing potential. Note that this is just the energy gained by the particle-hole pair dipole in the edge electric field. At larger q and $V_p(k)$, the shift in the dispersion is a matter of more detailed computation and can lead to the development of a *minimum* (see Figure 7) that softens steadily as the confinement is weakened further.

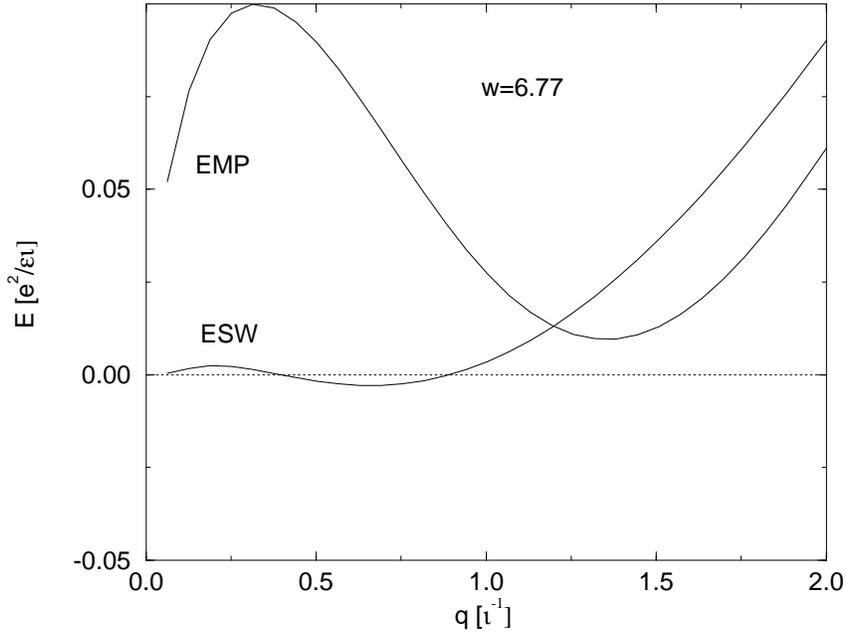


FIG. 7. EMP and ESW dispersion relations for a linear confinement with $w = 6.77$. Note the softening of both modes at positive wavevectors and that the edge is already unstable to textured reconstruction at small Zeeman energies.

If this new minimum dips below zero, the edge will reconstruct by macroscopically occupying it, i.e. by a condensation of the corresponding bosons. If q_0 is the wavevector at the minimum, then the corresponding boson creation operator is,

$$B^\dagger(q_0) = \int_{-q}^0 dk \psi_{q_0}(k) b^\dagger(k, q_0). \quad (5.4)$$

This leads us to consider the state with M condensed bosons,

$$[B^\dagger(q_0)]^M |G\rangle \quad (5.5)$$

which is, in turn, recognized as the “M boson” piece of the broken symmetry state,

$$\prod_{k \leq 0} (u_k c_{k\uparrow}^\dagger + v_k c_{k+q_0\uparrow}^\dagger) |0\rangle. \quad (5.6)$$

with the identification, $\psi_{q_0}(k) = v(k)/u(k)$ close to the transition. Evidently, (5.6) describes a charge density wave state with a modulation wavevector q_0 along the edge.

Returning to the $\Delta S_z = -1$ sector, we note that the perturbation has two important effects. First, much as in the EMP case, it causes the $q > 0$ ESWs to soften and develop a minimum at a $q > 0$. Second, generically, it pushes the $q < 0$ ESWs up into the bulk continuum where they cease to be bound states, although they may survive as resonances. In this fashion, the ESWs become chiral as well. (Note that for the case of a hardened confinement, the ESWs will be chiral in the opposite sense to the EMPs.)

As already remarked, the perturbing potential for the $\Delta S_z = -1$ particle-hole pairs vanishes at $q = 0$ and hence the ground state wavefunction of that problem is still $\psi(k) = 1$; indeed, this is still a consequence of the ferromagnetic ground state being an eigenstate of the squared total spin operator. Consequently, the bound states must again be localized on a length scale that diverges at small q . For a generic potential, we can again employ the variational choice $\psi(k) = \sqrt{2\lambda}e^{\lambda k}$ to obtain,

$$\begin{aligned}\lambda &= \sqrt{\frac{32}{\pi}} \left[V_p(-\infty) - V_p(0) \right] q \\ \epsilon(q) &= \gamma_q - \sqrt{\frac{32}{\pi}} \left[V_p(-\infty) - V_p(0) \right]^2 q^2.\end{aligned}\tag{5.7}$$

Evidently, bound states exist only for $q > 0$ and the ESWs are now chiral, as advertised. Note that the localization length is much smaller, $O(\frac{1}{q})$ as against $O(\frac{1}{q^2})$ for the ideal edge, and that the dispersion is accordingly much softer as it now departs at $O(q^2)$ from that of bulk spin waves.

It is clear from (5.7) that the case of the linear confinement is special, in that $V_p(-\infty) = V_p(0)$. In this case the particle-hole pair potential (see Figure 6b) is both attractive and repulsive. At small q , $\delta V_{\downarrow q}(k) \sim qV'_p(k)$ and hence its integral $\int_{-\infty}^0 dk \delta V_{\downarrow q}(k) = q[V_p(0) - V_p(-\infty)]$, vanishes to leading order in q . As a consequence, we obtain the modified variational results,

$$\begin{aligned}\lambda &= \sqrt{\frac{2}{\pi}} \left[\frac{4}{3\pi} + 2|V'_p(0)| \right] q^2 \\ \epsilon(q) &= \gamma_q - \sqrt{\frac{2}{\pi}} \left[\frac{2}{3\pi} + |V'_p(0)| \right]^2 q^4\end{aligned}\tag{5.8}$$

which differ from those for the ideal edge only in the increase in the coefficient c . In particular, there are still ESWs of *both* chiralities at small q whose localization length is still $O(\frac{1}{q^2})$.

As with the EMPs, softening can lead to the development of an ESW minimum at a finite $q_0 > 0$ (see Figure 7), which can, again, dip below zero energy. If that happens, in complete analogy to our treatment of the EMP softening, one can see that the edge reconstructs to a state of the form,

$$\prod_{k \leq 0} (u_k c_{k\uparrow}^\dagger + v_k c_{k+q_0\downarrow}^\dagger) |0\rangle\tag{5.9}$$

which is the spin textured edge studied previously.

One important point emerges from (5.8). For a linear confinement, ESW softening can take place only at $O(q^4)$, which is a much less efficient process than the generic softening at $O(q^2)$ detailed in (5.7). On tracking the relative softenings of the EMP and ESW minima, we find that the phase diagram for the linear confinement is dominated by the charge density wave instability²¹; this has also been noted by Franco and Brey from the completely equivalent Hartree-Fock analysis of the reconstructed edge⁸. A generic confinement however, will lead to much softer ESWs and hence should lead to a much stronger tendency to textured reconstruction.

VI. MORE ON BOUNDARY CONDITIONS

In our analysis of the ideal edge spin wave problem, we noted that our variational calculation is equivalent, at long wavelengths perpendicular *and* parallel to the edge, to imposing the boundary condition $\psi'(0) = cq^2\psi(0)$ on the differential equation $-\alpha_q\psi'' = (\epsilon - \gamma_q)\psi$. A similar reformulation is possible for the non-ideal edge as well, where the bound states can be obtained from the modified boundary conditions $\psi'(0) = \lambda\psi(0)$ with λ given by equations (5.7) and (5.8). At least for the ideal edge, where we have studied the wavefunctions in some detail, the boundary condition seems to work fairly well at moderately small wavenumbers (there is a finite size problem that we have not overcome at really small q) and hence we would like to explore this notion a bit further. We should note that the existence of *some* boundary condition reformulation of the problem is quite likely at small q where the bound state size must

diverge. One can simply rescale in units of this size and shrink the boundary region to zero. If the boundary by itself does not induce subexponential corrections to $\psi(k)$, it will end up giving rise to an effective boundary condition.

The Landau gauge analysis identifies the state,

$$|\psi_q\rangle = \int \frac{dk}{2\pi} \psi_q(k) c_{k+q\downarrow}^\dagger c_{k\uparrow} |G\rangle \quad (6.1)$$

with the *one-dimensional* spin wave eigenfunction $\psi_q(k)$. At long wavelengths, this can be written in the suggestive form,

$$|\psi_q^{(2)}\rangle = \int dx \int dy e^{iqy} \psi_q(x) \Psi_\downarrow^\dagger(x, y) \Psi_\uparrow(x, y) |G\rangle \quad (6.2)$$

in terms of the *two-dimensional* spin wave eigenfunction $e^{iqy} \psi_q(x)$. This indicates that in more general geometries, we can obtain spin wave states by solving for the appropriate eigenfunctions of $-\alpha_q \nabla^2$ and identifying them with states via (6.2). It also suggests that we replace our boundary condition by the two-dimensional form $\partial_x \psi = -c \partial_y^2 \psi$ which can also be generalized, along with its analogs for non-ideal edges, to other shapes for the edge. In physical units, this is $\partial_x \psi = -c \ell \partial_y^2 \psi$ wherein the explicit ℓ leads to a “trivial”, Neumann boundary condition in the continuum limit. For the generic non-ideal edge however, we get, according to (5.7), $\partial_x \psi \propto \partial_y \psi$ which does survive the continuum limit.

This prescription for constructing spin wave states enables us to make contact with the interesting work of Oaknin et al⁹, who independently arrived at it in their study of spin waves on a disc with one important difference—reasoning directly in the continuum, they concluded that a Neumann boundary condition was appropriate regardless of the details of the confinement. We have seen that this is not “sufficiently” correct. The boundary conditions needed to reproduce the ESWs are different, although they *do* approach the Neumann condition for asymptotically long wavelengths along the edge. This is clear in the semi-infinite geometry that we have studied in the bulk of this paper but it is less clear in the case of the disc. Here Oaknin et al obtained extremely impressive overlaps between the lowest energy exact eigenstates for a system of $N = 30$ electrons and spin waves with Δ (the net angular momentum of the spin flip pair which replaces q in our geometry) ranging from 1 to 30, and the states generated by their prescription.

To check that our boundary conditions are more appropriate than theirs even for a disc, we have carried out a similar numerical test, also for a system of 30 electrons. To simplify the computations, we have used a hard core potential $V(\mathbf{x}) = \delta^2(\mathbf{x})$ and the disc analog of our ideal edge with only the exchange potential entering the Hartree-Fock description. By hypothesis, this should not matter to Oaknin et al. We extract the value $c = 1/\sqrt{2\pi}$ from an analysis on the semi-infinite plane and then postulate the boundary condition,

$$\left. \frac{\partial \psi}{\partial r} \right|_{r=R} = -\frac{c}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} \quad (6.3)$$

at the radius R of the disc. The eigenfunctions of the Laplacian on the disc are

$$\psi_{k\Delta}(r, \theta) = J_\Delta(kr) e^{i\Delta\theta} \quad (6.4)$$

which then must satisfy $k J'_\Delta(k_\Delta R) = c \frac{\Delta^2}{R^2} J_\Delta(k_\Delta R)$. (The other Bessel function, $I_\Delta(kr)$, leads to new solutions only at large values of Δ , $\Delta > R/c$, which are beyond the validity of the long wavelength approximation and so we do not consider it here.) On comparing the exact eigenstates in the single spin flip sector with our ansatz (first entry) as well as the Neumann ansatz (second entry), we find that the overlaps for the ground state wavefunction range from (0.999901, 0.99981) for $\Delta = 1$, via (0.997695, 0.99436) for $\Delta = 3$, to (0.905964, 0.734507) for $\Delta = 10$. Evidently our ansatz does better, but the improvement is barely perceptible at small Δ where one has reason to trust this procedure! This is due partly to the smallness of the right hand term in (6.3) which causes it to mimic the Neumann condition for the small δ values where a continuum description is appropriate, and partly to the nodeless character of the ground state wavefunction which makes all overlaps large. In sum, while it is possible to detect the requirement of a non-trivial boundary condition for a small system such as the one considered by Oaknin et al, the effect is quite small and was naturally overlooked by them. As the system size is increased, the effect will increase, for in the infinite radius limit we approach the problem of the linear edge—and in that case, the lowest energy eigenfunctions with the Neumann condition which are not bound to the edge, have zero overlap with the true lowest energy eigenfunctions. Our boundary condition, by contrast, is designed to reproduce exactly those states.

A point of explanation is on order. Readers may be puzzled by our unwillingness to make more of the substantial discrepancy in overlaps at $\Delta = 10$ and other readers familiar with the Oaknin et al paper might be puzzled that the overlap they report for the same value of Δ is much larger. In reporting overlaps we have literally used the prescription embodied in (6.2). This requires evaluating an integral involving two Landau orbitals and the Bessel function over

the entire real line. This becomes meaningless at large Δ once one of the Landau orbitals is well outside the disc for at least some locations of the spin-flip pair. Indeed this causes the Neumann states to develop nodes and their overlap with the true ground state to plummet. Our boundary condition does better as it pushes the nodes of the Bessel functions further out, but clearly the whole prescription is no longer well motivated in this limit. The results reported by Oaknin et al are *not* obtained by the literal application of (6.2); instead they are obtained by a procedure that is equivalent to approximating the Bessel functions by polynomials which no longer exactly obey the Neumann condition at the boundary but are then able to avoid the sign problem coming from integrating outside the disc²². While it is interesting that their final prescription does as well as it does, it is still a somewhat *ad hoc* fix for the problems of the formalism at large Δ which does not appear susceptible to generalization to other geometries, e.g. the semi-infinite plane. Consequently we will not pursue it further in this paper.

Finally, we note that the low energy spin physics of the $\nu = 1$ state is expected to be governed by the ferromagnetic $O(3)$ sigma model appropriate to all isotropic ferromagnets, e.g. the magnetization of the $\nu = 1$ state at finite temperatures has been analyzed in this fashion by Read and Sachdev²³. For the ideal edge, the boundary condition on the spin flip pair wavefunction translates into the boundary condition,

$$\mathbf{n} \cdot (\partial_x \mathbf{n} \times c \partial_y^2 \mathbf{n}) = 0 \quad (6.5)$$

where $\mathbf{n}(x)$ is the vector order parameter field. Indeed one can find, in this particular case, a similar phenomenon in a lattice ferromagnet in which the bonds between the spins on the edge are weakened from their bulk values. If the bonds along the edge have strength J' while those in the bulk have strength J , there is a set of edge spin waves with localization length $\lambda = (J - J')aq^2/J$ at momentum q parallel to the edge. In this case one can arrive at the boundary condition (6.5) by directly considering the equations of motion for the edge spins. This boundary condition, and all the others we have considered in this paper, have the feature that they lead to currents even at the boundary which would be unnatural in a purely continuum approach. We emphasize that allowing for these microscopic currents (which represent the motion of the spins at the edge) is necessary in order to capture the edge spin waves within the continuum formulation, the latter in turn are interesting for their role in textured reconstruction.

VII. CONCLUDING REMARKS

The foregoing analysis has been confined to the TDHF approximation and we have only kept track of single particle hole pair states. The states in the $\Delta S_z = 0$ sector will only mix among themselves and the resulting interacting problem of many pairs is still described by the theory of a single non-interacting chiral boson at low energies, this is the simplest instance of Wen's bosonic description of quantum Hall edges²⁴. The dispersion of the bosonic mode is given *exactly* by our calculation at long wavelengths. The $\Delta S_z = -1$ sector will involve a mixing between the ESWs and the EMPs which will renormalize the ESW dispersion. However it is easy to see that energy and momentum conservation rule out decay of the ESWs by EMP emission at low momenta (for $q < \hbar v/\rho_s$ where v is the EMP velocity and ρ_s is the ESW stiffness, e.g. given by our expressions (5.7) and (5.8)) and they continue to be well defined excitations in that region. Our arguments are less robust at the larger wavevectors that are more germane to actual reconstruction instabilities and while we feel quite confident that the qualitative physics uncovered in this paper is equally applicable in that region, we do not have a sense of the quantitative error involved in restricting calculations to the TDHF approximation.

To summarize: We have shown that the edge dynamics of a polarized compact $\nu = 1$ state exhibits two modes, the EMP and an ESW. The existence of the latter is a new result, and this mode has several unusual properties, most notably its lack of a consistent chirality for all confinements. Both modes can soften with softening confinement and cause the edge to reconstruct by charge density wave or spin texture formation, respectively. We expect that this dual physics generalizes *mutatis mutandis* to other quantum Hall ferromagnets.

We expect that the most likely experimental detection of the physics described in this paper is through the observation of the resulting edge reconstructions and their sensitivity to various edge parameters, e.g. the Zeeman energy in the case of textured reconstruction. Direct detection of the ESW for a compact edge appears to be a challenge.

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