Functional Methods and Effective Potentials for Non-Linear Composites

Yves-Patrick Pellegrini* and Marc Barthélémy†
Service de Physique de la Matière Condensée,
Commissariat à l'Énergie Atomique,
BP12, 91680 Bruyres-le-Chtel, France

Gilles Perrin‡
Institut Français du Ptrole
1 et 4, avenue du Bois-Prau, 92852 Rueil-Malmaison Cedex, France.

(First submitted: August 21, 1998. Revised version: May 3, 1999. Printed: September 18, 2021)

A formulation of variational principles in terms of functional integrals is proposed for any type of local plastic potentials. The minimization problem is reduced to the computation of a path integral. This integral can be used as a starting point for different approximations. As a first application, it is shown how to compute to second-order the weak-disorder perturbative expansion of the effective potentials in random composite. The three-dimensional results of Suquet and Ponte-Castañeda (1993) for the plastic dissipation potential with uniform applied tractions are retrieved and extended to any space dimension, taking correlations into account. In addition, the viscoplastic potential is also computed for uniform strain rates.

Keywords: A.Variational principles; A.Inhomogeneous materials; B.Microstructure; B.Viscoplastic material; B.Constitutive behavior, principles.

To be published in the Journal of Mechanics and Physics of Solids.

I. INTRODUCTION

In the last decade, studies of disordered non-linear composites have forced new homogeneization methods in Mechanics. Most such methods rely on rigorous variational principles, from which exact optimal bounds can be deduced (Ponte-Castañeda and Suquet, 1998). These principles are either standard minimum energy principles, or more refined approaches using a linear reference material, be it homogeneous (Hashin and Shtrikman, 1962ab; Talbot and Willis, 1985), or even heterogeneous (Ponte-Castañeda, 1991; Suquet, 1993).

In the study of non-linear dielectric media as well, few exact results were previously available, save for the case where the non-linearity in the response was treated as a perturbation of the linear behavior (Stroud and Hui, 1988). In a recent pioneering work, Blumenfeld and Bergman (1989) computed the effective dielectric permittivity of a strongly non-linear random medium to second order in a weak-disorder expansion; cf. also Bergman and Lee (1998). The perturbative expansion was carried out by use of the Green function associated to the linear problem. An analogous approach was subsequently undertaken by Suquet and Ponte-Castañeda (1993) on strongly non-linear elastic composites. These results were then further extended to composites with inclusions of complex shapes (see, *eg.*, Ponte-Castañeda and Suquet (1998) for a review).

Perturbative results are important as testing-benches for self-consistent effective-medium theories. Linear self-consistent formulae (Budianski, 1965) are known to give qualitatively correct predictions even for a high contrast between the constituents of the random medium. They account for, e.g. the existence of a "rigidity threshold", i.e. a volumetric concentration of voids above which a porous linear elastic material loses its rigidity (Sahimi, 1998). In addition to the fulfillment of exact non-linear bounds (Gilormini, 1995), criteria of acceptability for a non-linear self-consistent formula include the recovery of second-order perturbative results, and also constraints on the critical behavior near the rigidity threshold (Levy and Bergman, 1994) where field fluctuations are enhanced. In Mechanics, a recent self-consistent effective-medium theory (Nebozhyn and Ponte-Castañeda, 1997) reproduces exact perturbative results to second order, but its critical fluctuations have not been studied yet. An equivalent theory has been proposed in electrostatics (Ponte-Castañeda and Kailasam, 1997). These result were derived from a variational principle.

*e-mail: `pellegri@bruyeres.cea.fr`
†e-mail: `mark@argento.bu.edu` until 08/31/99, then `barthele@bruyeres.cea.fr`
‡e-mail: `gilles.perrin@ifp.fr`

As a complement to classical means, functional methods originating from field theory (for a book on functional methods see, e.g., Kleinert, 1995) and from the statistical physics of disordered systems (Mézard *et al.*, 1987) were recently harnessed to contribute to effective-medium studies (Barthlmy and Orland, 1993; Parcollet *et al.*, 1996). They have not yet spread in the mechanics community. Their major interest is that they not only provide a convenient workhorse for implementing variational principles under various constraints, but that they also lend themselves to approximations resulting in self-consistent effective-medium formulae (Budianski, 1965; Parcollet *et al.*, 1996), some of them being of a new type (Pellegrini and Barthélémy, 1999). With such methods, the problem of the minimization of the potential becomes equivalent to the computation of a functional integral, as shown below. This single integral, which may be written down for all types of local potentials, can then be approximated using standard tools (saddle point methods, perturbative expansions, etc.).

The purpose of this article is to introduce in detail the specific mathematical apparatus needed to apply functional methods to variational principles in mechanics. As a first application, we re-derive the result of Suquet and Ponte-Castañeda (1993) for the weak-disorder expansion of a strongly non-linear viscoplastic composite assumed to be of the Norton type. We extend it to any space dimension, and show how to take correlations into account.

We adopt an approach (Barthélémy and Orland, 1998) which utilizes the so-called "replica method" (Edwards and Anderson, 1975) developed in the framework of spin glass theory (Mézard *et al.*, 1987). The replica method enables one to average over the disorder in a non-perturbative way, but is by no means compulsory in order to exploit the functional formulation. It is however a most natural route towards extensions to self-consistent formulae. Specific applications to self-consistent formulae for linear or non-linear materials (including the EMT) will be presented elsewhere.

The paper is self-contained, and is organized as follows. Section II introduces the variational formulations of the problem and explains how to cast them under a functional form, either starting from the local viscoplastic potential (expressed in terms of the stress tensor) or from the plastic dissipation potential (expressed in terms of the strain rate tensor). We show that minimizing a potential amounts to computing the statistical average of the logarithm of some suitably defined partition function. The computation is carried out with the replica method, explained in the text. In Sec.III, we specialize to the perturbative expansion, beginning with the plastic dissipation potential (Sec. III A 1). The results are then applied to the Norton law (Sec. III A 2). We retrieve the result of Suquet and Ponte-Castañeda (1993), generalized to any space dimension. The perturbative calculation of the viscoplastic potential, slightly more involved, is presented next, in Sec. III B. In both cases, the disorder is assumed to be *site-disorder*: the local potentials are statistically uncorrelated from point to point, for simplification purposes. This restriction is overcome in section IV, where spatial correlations (not necessarily isotropic) are introduced. Before concluding in Sec. VI, we comment on the range of applicability of the method (Sec. V).

Technical details are left to appendixes. Appendix A is a brief reminder of the use of fourth-rank tensors in mechanics, and defines various fundamental fourth-rank tensors employed hereafter. Various algebras and sub-algebras encountered in the paper are also defined in this appendix, which ought be read before the reader goes through our calculations. The determinant (det), trace (tr) and inversion ($^{-1}$) operators will always be indexed by a label indicating in which algebra or sub-algebra of operators they act. Useful formulae for restricted gaussian integrals over vectors or second-rank tensors, which we could not find in the literature, are derived in Appendix B. Inversion and determinant formulae for matrices in the replica space are given in Appendix C. Evaluations of functional integrals are relegated to Appendix D. The Legendre transform between the perturbative expansions of the viscoplastic and of the plastic dissipation potential is examined in Appendix E.

Before going on, we state our notational conventions: unless otherwise indicated, scalars are denoted by regular typefaces (eg. a or A); vectors are denoted by bold typefaces ($\mathbf{a}$); tensors of rank two by sans-serif typefaces (a) [the exceptions are $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ which appear in boldface], and tensors of rank four by capital Blackboard typefaces ($\mathbb{A}$). Tensor indices are denoted by roman letters (e.g. A_{ij}), whereas replica indices are Greek letters. The Einstein summation convention on repeated indices is used, except when the indices are underlined ($A_{ii} = \mathrm{tr}\,\mathsf{A}$, whereas $A_{\underline{ii}}$ denotes the i^{th} diagonal element of A). Dyads (i.e. tensor products of vectors) are denoted omitting the tensor product operator: $\mathsf{A} = \mathbf{ab}$ is the dyad with components $A_{ij} = a_i b_j$.

II. THE PROBLEM AND ITS FORMULATION WITH REPLICAS

A. Variational principles and effective potentials

We adopt here the presentation of Ponte-Castañeda and Suquet (1998). Let $\boldsymbol{\sigma}(\mathbf{x})$ be the local stress (tensor) field, and $\mathsf{d}(\mathbf{x})$ be the local strain rate (tensor) field deriving from the local velocity (vector) field $\mathbf{v}(\mathbf{x})$. The material domain, of volume V, is denoted by Ω. Then the equilibrium equations on $\boldsymbol{\sigma}$ read, for $\mathbf{x} \in \Omega$ (the upper index $^{\mathrm{t}}$ denotes the transpose):

$$\nabla.\boldsymbol{\sigma} = \mathbf{0}, \tag{2.1}$$
$$\boldsymbol{\sigma} = {}^{\mathrm{t}}\boldsymbol{\sigma}. \tag{2.2}$$

In addition,

$$\mathsf{d} = \frac{1}{2}\left[\nabla\mathbf{v} + {}^{\mathrm{t}}(\nabla\mathbf{v})\right]. \tag{2.3}$$

The local constitutive relations are expressed by means of either the viscoplastic potential $\psi_{\mathbf{x}}$ or the plastic dissipation potential $\phi_{\mathbf{x}}$, as:

$$\boldsymbol{\sigma} = \frac{\partial\phi_{\mathbf{x}}}{\partial\mathsf{d}}(\mathsf{d}), \qquad \mathsf{d} = \frac{\partial\psi_{\mathbf{x}}}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}). \tag{2.4}$$

The subscript $_{\mathbf{x}}$ indicates that the potentials may vary from point to point in the material, according to the local material properties. Both local potentials are usually convex functions of the fields, and are linked by the (Legendre) duality relation

$$\phi_{\mathbf{x}}(\mathsf{d}) = \max_{\boldsymbol{\sigma}}\left[\boldsymbol{\sigma}:\mathsf{d} - \psi_{\mathbf{x}}(\boldsymbol{\sigma})\right]. \tag{2.5}$$

Classically, two types of boundary conditions are considered:

1) for boundary conditions of *uniform traction* (ut) on the boundary $\partial\Omega$ of the material, the heterogeneous material is shown to be macroscopically described by a pair of effective potentials Ψ and Φ such that

$$\Psi_{\mathrm{ut}}(\boldsymbol{\Sigma}) = \min_{\boldsymbol{\sigma}/\left\{\substack{\nabla.\boldsymbol{\sigma}=\mathbf{0},\boldsymbol{\sigma}={}^{\mathrm{t}}\boldsymbol{\sigma} \\ \boldsymbol{\sigma}.\mathbf{n}=\boldsymbol{\Sigma}.\mathbf{n} \text{ on } \partial\Omega}\right.} \overline{\psi_{\mathbf{x}}(\boldsymbol{\sigma})}, \tag{2.6a}$$
$$\Phi_{\mathrm{ut}}(\mathsf{D}) = \min_{\mathbf{v}/\overline{\mathsf{d}}=\mathsf{D}} \overline{\phi_{\mathbf{x}}(\mathsf{d})}. \tag{2.6b}$$

(the overline denotes a volume average on Ω, and $\mathbf{n}$ is the outward normal to Ω).

2) for boundary conditions of *uniform strain rate* (us) D on the boundary, the effective potentials are:

$$\Psi_{\mathrm{us}}(\boldsymbol{\Sigma}) = \min_{\boldsymbol{\sigma}/\left\{\substack{\nabla.\boldsymbol{\sigma}=\mathbf{0},\boldsymbol{\sigma}={}^{\mathrm{t}}\boldsymbol{\sigma} \\ \overline{\boldsymbol{\sigma}}=\boldsymbol{\Sigma}}\right.} \overline{\psi_{\mathbf{x}}(\boldsymbol{\sigma})}, \tag{2.7a}$$
$$\Phi_{\mathrm{us}}(\mathsf{D}) = \min_{\mathbf{v}/\mathbf{v}=\mathsf{D}.\mathbf{x} \text{ on } \partial\Omega} \overline{\phi_{\mathbf{x}}(\mathsf{d})}. \tag{2.7b}$$

For both types of boundary conditions, the effective potentials Φ and Ψ are shown (Suquet, 1987; Willis, 1989) to be (Legendre) dual functions such that

$$\Psi(\boldsymbol{\Sigma}) + \Phi(\mathsf{D}) = \boldsymbol{\Sigma}:\mathsf{D}, \qquad \mathsf{D} = \frac{\partial\Psi(\boldsymbol{\Sigma})}{\partial\boldsymbol{\Sigma}}, \qquad \boldsymbol{\Sigma} = \frac{\partial\Phi(\mathsf{D})}{\partial\mathsf{D}}. \tag{2.8}$$

In both cases $\boldsymbol{\Sigma} = \overline{\boldsymbol{\sigma}}$ and $\mathsf{D} = \overline{\mathsf{d}}$. Depending on the boundary conditions considered, the latter equalities are either definitions, or theorems (Hill, 1963). Therefore only one equation in each pair (2.6) or (2.7) is sufficient to characterize the effective homogeneized material.

The problem considered in this article consists in computing the effective potentials for disordered materials. Eqs. (2.6), (2.7) refer to one sample (one particular realization of the disorder). A key assumption is that the effective potentials are *self-averaging*, i.e. that they do not depend on the sample at hand in the so-called "thermodynamic limit", where the size of the sample goes to infinity. In other words, the material contains in the various regions of its bulk all the possible realizations of the disorder. Therefore, rather than carrying out the calculation of the effective potentials for one particular configuration of the disorder (which is impossible), we arrive at the correct result by considering statistical averages. For instance, in the limit $V \to \infty$ we can as well write, instead of (2.6a):

$$\Psi_{\mathrm{ut}}(\boldsymbol{\Sigma}) = \langle\Psi_{\mathrm{ut}}(\boldsymbol{\Sigma})\rangle = \left\langle \min_{\sigma/\left\{\substack{\nabla.\boldsymbol{\sigma}=\mathbf{0},\boldsymbol{\sigma}={}^{\mathrm{t}}\boldsymbol{\sigma} \\ \boldsymbol{\sigma}.\mathbf{n}=\boldsymbol{\Sigma}.\mathbf{n} \text{ on } \partial\Omega}\right.} \overline{\psi_{\mathbf{x}}(\boldsymbol{\sigma})} \right\rangle. \tag{2.9}$$

Note that the statistical averaging operator $\langle . \rangle$ and the infimum operator *do not commute* (inverting the operators would lead to the incorrect result that the effective potential is the average of the local potentials).

B. Functional representations

The minimization problem is reformulated by extending to functionals the following straightforward result for scalar functions:

$$\min_{y\in[a,b]} f(y) = -\lim_{\beta\to\infty}\frac{1}{\beta}\log\int_a^b dx\, e^{-\beta f(x)}. \tag{2.10}$$

Note that this formula holds even if the minimum is met at several points in the interval.

Each infimum (2.6) or (2.7) is obtained for the state of the fields $\boldsymbol{\sigma}$ or d which minimize the potentials, among all the possible states fulfilling the equilibrium and boundary constraints. In statistical mechanics, this problem is analogous to that of finding the lowest energy state E_0 of some hamiltonian $H[s]$ functionally depending on some field $s(\mathbf{x})$. In this case, E_0 is given by

$$E_0 = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z, \tag{2.11a}$$

$$Z = \mathrm{tr}\; e^{-\beta H}, \tag{2.11b}$$

where the trace operator tr here denotes the sum over all the allowed states of the system, i.e. the configurations of s. Indeed, the partition function Z is obviously dominated by $\exp(-\beta E_0)$ in the limit $\beta\to\infty$, where $E_0\equiv H[s_0]$ and s_0 is the configuration of s that minimizes H (β is the reciprocal of the temperature in statistical physics).

The means of summing over all the possible states of a continuous field subjected to constraints is to use functional integrals (also termed *path integrals*) with suitably defined measures. With such a formalism, it is more convenient to use the variational formulations (2.6b) and (2.7a) where the boundary conditions are implemented through volume averages. We therefore write (with the additional statistical averages):

1) for uniform tractions [d is computed in terms of $\mathbf{v}$ by (2.3)],

$$\Phi_{\mathrm{ut}}(\mathsf{D}) = -\lim_{\beta\to\infty}\frac{1}{\beta}\left\langle\log Z_{\mathrm{ut}}\right\rangle, \qquad Z_{\mathrm{ut}} = \int \mathcal{D}\mathbf{v}\,\delta(\overline{\mathsf{d}}-\mathsf{D})\, e^{-\beta\overline{\phi_{\mathbf{x}}(\mathsf{d})}}, \tag{2.12}$$

where δ is the Dirac distribution and

$$\mathcal{D}\mathbf{v} = \prod_{\mathbf{x}\in\Omega} d\mathbf{v}(\mathbf{x}), \tag{2.13a}$$

$$\delta(\overline{\mathsf{d}}-\mathsf{D}) = \prod_{i\le j}\delta(\overline{d}_{ij}-D_{ij}); \tag{2.13b}$$

For brevity, the constrained measure in (2.12) will be denoted by

$$\mathcal{D}_{\mathrm{ut}}\mathbf{v} = \mathcal{D}\mathbf{v}\,\delta(\overline{\mathsf{d}}-\mathsf{D}). \tag{2.14}$$

2) for uniform strain rates,

$$\Psi_{\mathrm{us}}(\boldsymbol{\Sigma}) = -\lim_{\beta\to\infty}\frac{1}{\beta}\left\langle\log Z_{\mathrm{us}}\right\rangle, \qquad Z_{\mathrm{us}} = \int \mathcal{D}_s\boldsymbol{\sigma}\,\delta(\boldsymbol{\nabla}.\boldsymbol{\sigma})\delta(\overline{\boldsymbol{\sigma}}-\boldsymbol{\Sigma})\, e^{-\beta\overline{\psi_{\mathbf{x}}(\boldsymbol{\sigma})}}. \tag{2.15}$$

with

$$\mathcal{D}_s\boldsymbol{\sigma} = \prod_{\mathbf{x}\in\Omega} d_s\boldsymbol{\sigma}(\mathbf{x}), \tag{2.16a}$$

$$d_s\boldsymbol{\sigma}(\mathbf{x}) = 2^{d(d-1)/4}\Big[\prod_{i<j}\delta\big(\sigma_{ij}(\mathbf{x})-\sigma_{ji}(\mathbf{x})\big)\Big]\Big[\prod_{i,j} d\sigma_{ij}(\mathbf{x})\Big], \tag{2.16b}$$

$$\delta(\boldsymbol{\nabla}.\boldsymbol{\sigma}) = \prod_{i,\mathbf{x}\in\Omega}\delta\big(\partial_k\sigma_{ki}(\mathbf{x})\big), \tag{2.16c}$$

$$\delta(\overline{\boldsymbol{\sigma}}-\boldsymbol{\Sigma}) = \prod_{i\le j}\delta(\overline{\sigma}_{ij}-\Sigma_{ij}). \tag{2.16d}$$

The integration measure $d_s\boldsymbol{\sigma}(\mathbf{x})$ allows one to integrate over all *symmetric* stress tensors (cf. Appendix B). The normalization factor $2^{d(d-1)/4}$ in $d_s\boldsymbol{\sigma}(\mathbf{x})$, where d is the space dimension, is of no importance but has been included for consistency with Appendix B. We set:

$$\mathcal{D}_{\mathrm{us}}\boldsymbol{\sigma} = \mathcal{D}_s\boldsymbol{\sigma}\,\delta(\boldsymbol{\nabla}.\boldsymbol{\sigma})\delta(\overline{\boldsymbol{\sigma}}-\boldsymbol{\Sigma}). \tag{2.17}$$

C. The replica method

Hereafter, we focus on $Z_{\rm ut}$, but the following considerations apply *mutatis mutandis* to $Z_{\rm us}$. The statistical average of $\log Z_{\rm ut}$ is a complicated quantity to evaluate directly. The replica method consists in writing

$$\langle \log Z_{\rm ut}\rangle = \lim_{r\to 0}\frac{1}{r}(\langle Z_{\rm ut}^r\rangle - 1), \tag{2.18}$$

and in computing $\langle Z_{\rm ut}^r\rangle$ with r an integer (in the replica literature, the number of replicas, r, is usually denoted by n. We do not use the latter in order to avoid confusions with the Norton exponent). At the end of the calculation, we take the limit $r\to 0$, assuming the analytical continuation is legitimate, which we cannot prove in general. Therefore introducing r *replicas* $\mathbf{v}^\alpha$ of the field $\mathbf{v}$, the partition function

$$Z_{\rm ut}^r = \prod_{\alpha=1}^{r}\left(\int \mathcal{D}_{\rm ut}\mathbf{v}^\alpha\, e^{-\beta\overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}}\right) = \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\, e^{-\beta\sum_{\gamma=1}^{r}\overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}} \tag{2.19}$$

is that of a system made of r replicas of the initial system. The greek labels (which exclude β, this symbol being devoted to the reciprocal of the "temperature") are replica labels running from 1 to r. The statistical average therefore is

$$\langle Z_{\rm ut}^r\rangle = \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\,\left\langle e^{-\beta\sum_{\gamma=1}^{r}\overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}}\right\rangle. \tag{2.20}$$

For simplification purposes, we shall limit ourselves to *site-disordered* materials where the local potentials are statistically uncorrelated from point to point (we explain in Sec. IV how to overcome this restriction). By definition of site disorder, we have for any function F the property that

$$\left\langle F\left(\psi_{\mathbf{x}}(\boldsymbol{\sigma}(\mathbf{x}))\right) F\left(\psi_{\mathbf{y}}(\boldsymbol{\sigma}(\mathbf{y}))\right)\right\rangle = \left\langle F\left(\psi_{\mathbf{x}}(\boldsymbol{\sigma}(\mathbf{x}))\right)\right\rangle \left\langle F\left(\psi_{\mathbf{y}}(\boldsymbol{\sigma}(\mathbf{y}))\right)\right\rangle \text{ if } \mathbf{x}\neq\mathbf{y}. \tag{2.21}$$

For this type of disorder, the average in (2.20) is readily carried out. Replacing the volume integral by a Riemann sum $\int d\mathbf{x}\to v\sum_{\mathbf{x}}$, where v denotes the unit "infinitesimal" volume element of the theory (here the size of v also is the statistical correlation radius within which the potential is constant), one obtains

$$\begin{aligned}\langle Z_{\rm ut}^r\rangle &= \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\,\left\langle e^{-(v\beta/V)\sum_{\mathbf{x},\gamma=1}^{r}\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}\right\rangle\\ &= \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\,\left\langle \prod_{\mathbf{x}} e^{-(v\beta/V)\sum_{\gamma=1}^{r}\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}\right\rangle\\ &= \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\,\prod_{\mathbf{x}}\left\langle e^{-(v\beta/V)\sum_{\gamma=1}^{r}\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}\right\rangle.\end{aligned} \tag{2.22}$$

This finally yields:

$$\langle Z_{\rm ut}^r\rangle = \int\prod_{\alpha=1}^{r}\mathcal{D}_{\rm ut}\mathbf{v}^\alpha\,\exp\left[\int\frac{d\mathbf{x}}{v}\log\left\langle e^{-(v\beta/V)\sum_{\gamma=1}^{r}\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}\right\rangle\right], \tag{2.23}$$

where we reintroduce the integral notation in the exponent. This is the usual shorthand notation used in functional integral methods and has to be understood as the limit of a Riemann sum.

This expression, completed by (2.18) and (2.12), constitutes an exact functional representation of the variational problem (2.6b) for the particular type of disorder under study, and is valid for any form of $\phi_{\mathbf{x}}(\mathsf{d})$. In other words, we recast a minimization problem under a computational form. Computing this integral exactly is equivalent to solving the minimization problem. We note that before averaging, the replicas are independent from one another, but are coupled after the average is carried out. Here appears an important feature of the replica method: the use of replicated fields allows one to average over disorder by transforming the disordered problem into a non-disordered one, but with couplings (if there were no couplings, averaging the logarithm would be a trivial matter). We are now left with a complicated field-theory problem without disorder, consisting of r coupled fields, and we have to resort to approximations. In the next section, we present a perturbative calculation of this integral.

An important simplification arises from the fact that $\langle Z_{\rm ut}^r\rangle$ always shows up in the form $\langle Z_{\rm ut}^r\rangle = (C\beta^c)^r f(r,\beta)$, where $f(0,x)=1$ and is a non-homogeneous function of x, C is a β-independent constant, and c is an exponent linked to d, the space dimension. Therefore

$$-\frac{1}{\beta}\frac{1}{r}(\langle Z_{\rm ut}^r\rangle - 1) = -\frac{1}{\beta}\left[\log(C\beta^c) + \frac{1}{r}\log f(r,\beta) + O(r)\right], \tag{2.24}$$

and the multiplicative factor $(C\beta^c)^r$ does not contribute to the limit $\beta\to\infty$. Hereafter, such multiplicative factors will therefore systematically be dropped out of the calculations. This will be indicated by the proportionality symbol $\propto$ appearing instead of the equal sign in equations.

III. PERTURBATIVE CALCULATION

We detail in this section the perturbative calculation for Φ first, since it is simpler than that for Ψ. The latter is examined next, once the reader has become acquainted with the method. In the following, we assume that the d-dimensional material is rigid-viscoplastic, described by the dual [in the sense of (2.5)] Norton potentials ($1\le n\le\infty$, $m=1/n$).

$$\begin{cases} \phi_{\mathbf{x}}(\mathsf{d}) = \theta_m d_{\rm eq}^{m+1}/(m+1) \\ {\rm tr}_2(\mathsf{d}) = 0 \end{cases}, \qquad \psi_{\mathbf{x}}(\boldsymbol{\sigma}) = \omega_n \sigma_{\rm eq}^{n+1}/(n+1), \tag{3.1}$$

with

$$\theta_m = \sigma_0/\dot\varepsilon_0^m, \qquad \omega_n = \dot\varepsilon_0/\sigma_0^n. \tag{3.2}$$

The equivalent Mises norms $\sigma_{\rm eq}$ and $d_{\rm eq}$ are:

$$d_{\rm eq} = \left[\frac{d-1}{d}\,\underset{2}{\rm tr}(\mathsf{d}'^2)\right]^{1/2}, \qquad \sigma_{\rm eq} = \left[\frac{d}{d-1}\,\underset{2}{\rm tr}(\boldsymbol{\sigma}'^2)\right]^{1/2}. \tag{3.3}$$

These are the d-dimensional counterparts of the usual three-dimensional definitions (for $d=3$, we recover the factors $2/3$ and $3/2$). The deviatoric part of the stress or strain rate tensor has been denoted by a prime: $\mathsf{a}' = \mathsf{a} - {\rm tr}_2(\mathsf{a})\mathsf{I}/d$, where $\mathsf{a}=\boldsymbol{\sigma}$ or d.

The additional incompressibility constraint ${\rm tr}_2(\mathsf{d}) = \boldsymbol{\nabla}.\mathbf{v} = 0$, which characterize incompressible materials, can as well be accounted for by adding a term $(K/2)\,{\rm tr}_2(\mathsf{d})^2$ to $\phi_{\mathbf{x}}(\mathsf{d})$, and by letting the constant $K\to\infty$ at the end of the calculation. Instead, we shall implement it directly by multiplying the measure $\mathcal{D}_{\rm ut}\mathbf{v}$ by a factor $\delta(\boldsymbol{\nabla}.\mathbf{v})$. Both ways actually amount to the same.

The constitutive parameters σ_0 and $\dot\varepsilon_0$ decribe the material and are random functions of the position variable $\mathbf{x}$, uncorrelated from point to point, according to our site-disorder hypothesis. The exponent n is taken to be constant in the material, for simplification purposes (but this is not a limitation of the method).

A. The plastic dissipation potential

1. General framework

Integral (2.23) is an exact expression for the effective potential. Its perturbation expansion is considered hereafter. The measure $\mathcal{D}_{ut}\mathbf{v}^\alpha$ contains the term $\delta(\overline{\mathsf{d}}-\mathsf{D})$. This means that the main contribution to this integral is coming from fields around D. Indeed, if we suppose that only the constant field $\mathsf{d}=\mathsf{D}$ (i.e. $\mathbf{v}=\mathsf{D}.\mathbf{x}$ such that ${\rm tr}_2(\mathsf{D})=0$) contributes to the integral, we find the trivial lowest order result

$$\Phi_{\rm ut}(\mathsf{D}) = \langle\phi_{\mathbf{x}}(\mathsf{D})\rangle. \tag{3.4}$$

We shall now seek the first corrective term, and compute (2.23) by integrating over fields $\mathbf{v}$ such that d differs little from D. The perturbation expansion is obtained with Laplace's method for asymptotic expansions of integrals (Bender and Orszag, 1978). We therefore write

$$\mathsf{d} = \mathsf{D} + \mathsf{e} \tag{3.5}$$

and expand the logarithm in (2.23) in powers of e. The linear and quadratic terms are then kept inside the main exponential, and the rest is further expanded in e. We start from the expansion:

$$\phi_{\mathbf{x}}(\mathsf{D}+\mathsf{e}) = \phi_{\mathbf{x}}(\mathsf{D}) + W'_{ij} e_{ji} + \frac{1}{2} W''_{ij,kl} e_{ji} e_{lk} + O(\mathsf{e}^3), \tag{3.6}$$

with

$$W'_{ij}(\mathbf{x}) = \left.\frac{\partial \phi_{\mathbf{x}}}{\partial d_{ij}(\mathbf{x})}\right|_{\mathsf{d}=\mathsf{D}}, \qquad W''_{ij,kl}(\mathbf{x}) = \left.\frac{\partial^2 \phi_{\mathbf{x}}}{\partial d_{ij}(\mathbf{x})\partial d_{kl}(\mathbf{x})}\right|_{\mathsf{d}=\mathsf{D}}. \tag{3.7}$$

With a change of integration variables such that $\mathbf{v}$ now denotes the correction to $\mathsf{D}.\mathbf{x}$, the expansion of $\langle Z^r_{\mathrm{ut}}\rangle$ takes the form (taking into account the incompressibility constraint):

$$\langle Z^r_{\mathrm{ut}}\rangle = \langle e^{-r\beta\frac{v}{V}\phi_{\mathbf{x}}(\mathsf{D})}\rangle^{\frac{V}{v}} \int \prod_{\alpha} \mathcal{D}_{\mathrm{ut}}\mathbf{v}^\alpha \delta(\boldsymbol{\nabla}.\mathbf{v}^\alpha)\left[1+O(\mathsf{e}^3)\right]$$
$$\times \exp\left[-\beta\langle \mathsf{W}'\rangle_e : \sum_\alpha \bar{\mathsf{e}}^\alpha - \frac{1}{2}\frac{\beta}{V}\sum_{\alpha\gamma}\int d\mathbf{x}\, \mathsf{e}^\alpha(\mathbf{x}) : \mathbb{M}^{\alpha\gamma} : \mathsf{e}^\gamma(\mathbf{x})\right]. \tag{3.8}$$

The functional measure now is $\mathcal{D}_{\mathrm{ut}}\mathbf{v} = \mathcal{D}\mathbf{v}\,\delta(\bar{\mathsf{e}})$, and $\mathsf{e} = [\boldsymbol{\nabla}\mathbf{v} + {}^{\mathrm{t}}(\boldsymbol{\nabla}\mathbf{v})]/2$. Besides, we have introduced the matrix $\mathbb{M}^{\alpha\gamma} = \mathbb{M}_1\delta_{\alpha\gamma} - \beta(v/V)\mathbb{M}_2$, with

$$\mathbb{M}_1 = \langle \mathbb{W}''\rangle_e \tag{3.9a}$$
$$\mathbb{M}_2 = \langle \mathsf{W}'\mathsf{W}'\rangle_e - \langle \mathsf{W}'\rangle_e\langle \mathsf{W}'\rangle_e. \tag{3.9b}$$

The notation $\langle\cdot\rangle_e$ stands for the weighted average

$$\langle A(\mathbf{x})\rangle_e \equiv \frac{\langle A(\mathbf{x})e^{-r\beta\frac{v}{V}\phi_{\mathbf{x}}(\mathsf{D})}\rangle}{\langle e^{-r\beta\frac{v}{V}\phi_{\mathbf{x}}(\mathsf{D})}\rangle} = \langle A(\mathbf{x})\rangle + O(r). \tag{3.10}$$

Only the leading term in r is needed in (3.10) [a handwaving argument for this is that the sums over replicas in (3.8) are at least proportional to r: the contribution is null for $r=0$. They therefore already provide the linear terms in r required for evaluating (2.18)], and we shall therefore in practice identify $\langle.\rangle_e$ with $\langle.\rangle$. Hereafter, square matrices in the replica space are denoted by a double overbar. The identity in the replica space is $I^{\alpha\gamma} = \delta_{\alpha\gamma}$, and we furthermore define the matrix $U^{\alpha\gamma} = 1$ for all α, γ, so that

$$\overline{\overline{\mathbb{M}}} = \mathbb{M}_1\overline{\overline{I}} - \beta(v/V)\mathbb{M}_2\overline{\overline{U}}. \tag{3.11}$$

The second-order term in the weak-disorder expansion of $\langle Z^r_{\mathrm{ut}}\rangle$ is provided by the gaussian integral (3.8) only, neglecting the corrections $O(\mathsf{e}^3)$ and higher. As in centered scalar gaussian integrals, the term proportional to e^3 vanishes upon integration, and the next non-zero correction in fact is an $O(\mathsf{e}^4)$. The aim of the paper being to explain the functional method, we shall ignore such higher-order terms hereafter but they could as well be computed by pursuing Laplace's expansion scheme.

The details of the integration over $\mathbf{v}$ with measure $\mathcal{D}_{\mathrm{ut}}\mathbf{v}$ are left to Appendix D. Taking into account that e is expressed in terms of the gradient of $\mathbf{v}$ involves a change from $\mathbf{v}(\mathbf{x})$ to its Fourier components $\mathbf{v}(\mathbf{k})$. One finds

$$\langle Z^r_{\mathrm{ut}}\rangle \propto \langle e^{-r\beta\frac{v}{V}\phi_{\mathbf{x}}(\mathsf{D})}\rangle^{\frac{V}{v}} \prod_{\mathbf{k}\neq\mathbf{0}}\left\{\det_{2[\perp\hat{\mathbf{k}}],\mathrm{rep}}\left[\mathsf{Q}_{\hat{\mathbf{k}}}.\left(\hat{\mathbf{k}}.\overline{\overline{\mathbb{M}}}.\hat{\mathbf{k}}\right).\mathsf{Q}_{\hat{\mathbf{k}}}\right]\right\}^{-1/2}, \tag{3.12}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$ and $\mathsf{Q}_{\hat{\mathbf{k}}} = \mathsf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$.

We now calculate the determinant. Defining

$$\mathsf{M}_1 = \mathsf{Q}_{\hat{\mathbf{k}}}.(\hat{\mathbf{k}}.\mathbb{M}_1.\hat{\mathbf{k}}).\mathsf{Q}_{\hat{\mathbf{k}}}, \tag{3.13a}$$
$$\mathsf{M}_2 = \mathsf{Q}_{\hat{\mathbf{k}}}.(\mathbf{k}.\mathbb{M}_2.\mathbf{k}).\mathsf{Q}_{\hat{\mathbf{k}}}, \tag{3.13b}$$

we obtain, with the help of formulae (C2) in Appendix C:

$$\det_{2[\perp\hat{\mathbf{k}}],\mathrm{rep}}\left[\mathsf{Q}_{\hat{\mathbf{k}}}.\left(\hat{\mathbf{k}}.\overline{\overline{\mathbb{M}}}.\hat{\mathbf{k}}\right).\mathsf{Q}_{\hat{\mathbf{k}}}\right] = 1 + \left[\log\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_1) - \beta\frac{v}{V}\operatorname*{tr}_2\left(\mathsf{M}_2.\mathsf{M}_1^{-1_{2[\perp\hat{\mathbf{k}}]}}\right)\right] r + O(r^2). \tag{3.14}$$

Carrying out as in (2.18) the limit $r \to 0$ in (3.12) it follows that

$$-\frac{1}{\beta}\langle \log Z_{\rm ut}\rangle = \langle \phi_{\mathbf{x}}(\mathsf{D})\rangle - \frac{1}{2}\frac{v}{V}\sum_{\mathbf{k}\neq\mathbf{0}} \mathop{\rm tr}_2 \left(\mathsf{M}_2.\mathsf{M}_1^{-1_{2[\perp\hat{\mathbf{k}}]}}\right) + o(\log(\beta)/\beta) + o(1/\beta). \tag{3.15}$$

The sum over $\mathbf{k}\neq\mathbf{0}\ldots$ (divided by V) can now be replaced by a d-dimensional integral [divided by $(2\pi)^d$] over all the Fourier modes $\mathbf{k}$ in the continuum limit, since the integrand is not singular at $k=0$. The finite size of the elementary cells of volume v is accounted for by an upper cut-off Λ in this integral. It is chosen such that

$$v\int_{k<\Lambda}\frac{d\mathbf{k}}{(2\pi)^d} = 1. \tag{3.16}$$

Hence $\Lambda\to\infty$ when $v\to 0$. We denote by $S_d = 2\pi^{d/2}/\Gamma(d/2)$ the area of the sphere of unit radius in $\mathbf{R}^d$, and by $d\Omega_{\hat{\mathbf{k}}}$ the integration measure over the solid angle in the direction $\hat{\mathbf{k}}$ ($d\mathbf{k} = k^{d-1}\,dk\,d\Omega_{\hat{\mathbf{k}}}$).

Letting $\beta\to\infty$ in (3.15) we deduce, by (2.12):

$$\Phi_{\rm ut}(\mathsf{D}) = \langle\phi_{\mathbf{x}}(\mathsf{D})\rangle - \frac{1}{2}\int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\mathop{\rm tr}_2\left(\mathsf{M}_2.\mathsf{M}_1^{-1_{2[\perp\hat{\mathbf{k}}]}}\right). \tag{3.17}$$

This result is the second-order perturbation expansion valid for any potential $\phi_{\mathbf{x}}(\mathsf{d})$ with the constraint $\mathrm{tr}_2\,\mathsf{d}=0$.

2. Application to the Norton law

We evaluate the previous result (3.17) for the Norton potential (3.1). We write

$$\theta_m = \langle\theta_m\rangle + \delta\theta_m. \tag{3.18}$$

Defining $\hat{\mathsf{D}} = \mathsf{D}'/\sqrt{\mathrm{tr}_2(\mathsf{D}'^2)}$, so that $\mathrm{tr}_2(\hat{\mathsf{D}}.\hat{\mathsf{D}}) = \mathrm{tr}_4(\hat{\mathsf{D}}\hat{\mathsf{D}}) = 1$, (3.7) and (3.9) entail:

$$\mathbb{M}_1 = m_1\mathbb{J} + m_2\hat{\mathsf{D}}\hat{\mathsf{D}}, \tag{3.19a}$$

$$\mathbb{M}_2 = m_3\hat{\mathsf{D}}\hat{\mathsf{D}}, \tag{3.19b}$$

where

$$m_1 = \frac{d-1}{d}\langle\theta_m\rangle D_{\rm eq}^{m-1}, \tag{3.20a}$$

$$m_2 = (m-1)m_1, \tag{3.20b}$$

$$m_3 = \frac{d-1}{d}\left\langle\delta\theta_m^2\right\rangle D_{\rm eq}^{2m}. \tag{3.20c}$$

Setting

$$D_{\hat{\mathbf{k}}} = |\mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\mathsf{D}}.\hat{\mathbf{k}}|^2 = \hat{\mathbf{k}}.\hat{\mathsf{D}}^2.\hat{\mathbf{k}} - (\hat{\mathbf{k}}.\hat{\mathsf{D}}.\hat{\mathbf{k}})^2, \tag{3.21}$$

and introducing the unit vector $\hat{\mathbf{u}} = \mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\mathsf{D}}.\hat{\mathbf{k}}/|\mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\mathsf{D}}.\hat{\mathbf{k}}|$, we obtain

$$\mathsf{M}_1 = \frac{m_1}{2}\left(\mathsf{Q}_{\hat{\mathbf{k}}} - \hat{\mathbf{u}}\hat{\mathbf{u}}\right) + \left(m_2 D_{\hat{\mathbf{k}}} + \frac{m_1}{2}\right)\hat{\mathbf{u}}\hat{\mathbf{u}}, \tag{3.22a}$$

$$\mathsf{M}_2 = m_3 D_{\hat{\mathbf{k}}}\,\hat{\mathbf{u}}\hat{\mathbf{u}}. \tag{3.22b}$$

Since $\mathsf{Q}_{\hat{\mathbf{k}}} - \hat{\mathbf{u}}\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}\hat{\mathbf{u}}$, are a pair of complementary orthogonal projectors in $L(2[\perp\hat{\mathbf{k}}])$ (where $\mathsf{Q}_{\hat{\mathbf{k}}}$ plays the role of the identity), the inverse of M_1, its product with M_2 and the final trace in (3.17) are readily obtained, resulting in:

$$\Phi_{\rm ut}(\mathsf{D}) = \langle\phi_{\mathbf{x}}(\mathsf{D})\rangle - \int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\frac{m_3 D_{\hat{\mathbf{k}}}}{m_1 + 2m_2 D_{\hat{\mathbf{k}}}}. \tag{3.23}$$

Now replacing m_1, m_2 and m_3 by their values (3.20) finally yields

$$\Phi_{\rm ut}(\mathsf{D}) = \frac{\theta_{m\,\rm eff}(\mathsf{D})}{m+1}D_{\rm eq}^{m+1}, \tag{3.24a}$$

$$\theta_{m\,\rm eff}(\mathsf{D}) = \langle\theta_m\rangle\left[1 - \frac{\left\langle\delta\theta_m^2\right\rangle}{\langle\theta_m\rangle^2}\int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\frac{(m+1)D_{\hat{\mathbf{k}}}}{1+2(m-1)D_{\hat{\mathbf{k}}}} + O(\left\langle\delta\theta_m^3\right\rangle/\langle\theta_m\rangle^3)\right]. \tag{3.24b}$$

This result reduces to that of Suquet and Ponte-Castañeda (1993) for a space dimension $d=3$.

B. The viscoplastic potential

We undertake a similar calculation for the viscoplastic potential. It can be checked (cf. Appendix E) that the resulting perturbative expression (3.34) is linked by a Legendre transform to (3.24), the one for the dissipation potential, though different boundary conditions are considered. This is a consequence of our working in the thermodynamic limit where the volume of the system $V \to \infty$ (in the calculations, this shows up essentially in that the Fourier sums are carried out with $k > 0$, rather than with $k > (2\pi)/L$, where L would be the typical system size). As noticed by Hill and Mandel in elasticity, the homogeneized stiffness tensor obtained with uniform strains on $\partial\Omega$, and the homogeneized compliances tensor obtained with uniform stresses on $\partial\Omega$ are inverse of one another up to terms of order $O\big((\xi/L)^3\big)$ in three dimensions, where ξ is the typical size of the heterogeneities [26]. They therefore become exact inverses in the thermodynamic limit. The same effect is at play here. The viscoplastic potential can thus directly be obtained from the dissipation potential as done in Appendix E. However we prefer to compute it from scratch because differences with the previous section show up, and it may be useful to see what these are for future purposes: indeed, as soon as one leaves exact perturbative expansions and deals with self-consistent estimates, one approximation scheme worked out separately for dual potentials quite often generates two estimates which are not duals of one another (a notable exception is the self-dual EMT theory). Treating in a symmetrical way the two potentials is then a more sensible thing to do than working with only one.

1. General framework

We now have to compute

$$\langle Z^r_{\rm us}\rangle = \int \prod_{\alpha=1}^{r} \mathcal{D}_{\rm ut}\sigma^\alpha \, \exp\left[\frac{1}{v}\int d\mathbf{x} \log \left\langle e^{-\beta(v/V)\sum_{\gamma=1}^{r}\psi_{\mathbf{x}}(\sigma^\gamma)}\right\rangle\right]. \tag{3.25}$$

The perturbative expansion of the effective potential is carried out around the constant solution $\boldsymbol{\sigma} = \boldsymbol{\Sigma}$, writing $\boldsymbol{\sigma} = \boldsymbol{\Sigma} + \mathsf{s}$. The expansion of $\psi_{\mathbf{x}}(\boldsymbol{\Sigma}+\mathsf{s})$ has the same form as (3.6). We now denote by W' and $\mathbb{W}$ the first and second derivatives of $\psi_{\mathbf{x}}(\boldsymbol{\sigma})$ evaluated at $\boldsymbol{\sigma} = \boldsymbol{\Sigma}$, respectively. Using s as integration variables, the expansion of $\langle Z^r_{\rm us}\rangle$ is:

$$\begin{aligned}\langle Z^r_{\rm us}\rangle \propto \langle e^{-r\beta\frac{v}{V}\psi_{\mathbf{x}}(\boldsymbol{\Sigma})}\rangle^{\frac{V}{v}} \int \prod_\alpha \tilde{\mathcal{D}}_s \mathsf{s}^\alpha \left[1 + O(\mathsf{s}^3)\right] \\ \times \exp\left[-\beta\frac{v}{V}\langle \mathsf{W}'\rangle_e : \sum_\alpha \overline{\mathsf{s}}^\alpha - \frac{1}{2}\beta\frac{1}{V}\sum_{\alpha\gamma}\int d\mathbf{x}\, \mathsf{s}^\alpha(\mathbf{x}) : \mathbb{M}^{\alpha\gamma} : \mathsf{s}^\gamma(\mathbf{x})\right],\end{aligned} \tag{3.26}$$

where $\tilde{\mathcal{D}}_s\mathsf{s} = \mathcal{D}_s\mathsf{s}\,\delta(\boldsymbol{\nabla}.\mathsf{s})\delta(\overline{\mathsf{s}})$ and $\mathbb{M}^{\alpha\gamma}$ is defined as in Eqs. (3.9) and (3.11). Once again, the details of the integration over s with measure $\tilde{\mathcal{D}}_s\mathsf{s}$ are left to Appendix D. The implementation of the constraint $\boldsymbol{\nabla}.\mathsf{s} = 0$ involves a change from $\mathsf{s}(\mathbf{x})$ to its Fourier components $\mathsf{s}(\mathbf{k})$. The result is:

$$\langle Z^r_{\rm us}\rangle \propto \langle e^{-r\beta\frac{v}{V}\psi_{\mathbf{x}}(\boldsymbol{\Sigma})}\rangle^{\frac{V}{v}} \prod_{\mathbf{k}\neq\mathbf{0}} \left\{ \det_{4s',\rm rep}(\overline{\overline{\mathbb{M}}}) \det_{2[\perp\hat{\mathbf{k}}],\rm rep}\left[\mathsf{Q}_{\hat{\mathbf{k}}}.\left(\hat{\mathbf{k}}.\overline{\overline{\mathbb{M}}}^{-1_{4s',\rm rep}}.\hat{\mathbf{k}}\right).\mathsf{Q}_{\hat{\mathbf{k}}}\right]\right\}^{-1/2}. \tag{3.27}$$

The determinants are once again straightforwardly obtained, and the limits $r \to 0$ and $\beta \to \infty$ carried out as in the previous section. Setting now

$$\mathsf{M}_1 = \mathsf{Q}_{\hat{\mathbf{k}}}.(\hat{\mathbf{k}}.\mathbb{M}_1^{-1_{4s'}}.\hat{\mathbf{k}}).\mathsf{Q}_{\hat{\mathbf{k}}}, \tag{3.28a}$$
$$\mathsf{M}_2 = \mathsf{Q}_{\hat{\mathbf{k}}}.(\mathbf{k}.\mathbb{M}_1^{-1_{4s'}} : \mathbb{M}_2 : \mathbb{M}_1^{-1_{4s'}}.\mathbf{k}).\mathsf{Q}_{\hat{\mathbf{k}}}, \tag{3.28b}$$

the second-order perturbation expansion for the viscoplastic potential finally reads:

$$\Psi_{\rm us}(\boldsymbol{\Sigma}) = \langle\psi_{\mathbf{x}}(\boldsymbol{\Sigma})\rangle - \frac{1}{2}\int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\left[\operatorname*{tr}_4(\mathbb{M}_2 : \mathbb{M}_1^{-1_{4s'}}) - \operatorname*{tr}_2\left(\mathsf{M}_2.\mathsf{M}_1^{-1_{2[\perp\hat{\mathbf{k}}]}}\right)\right]. \tag{3.29}$$

Writing $\omega_n = \langle\omega_n\rangle + \delta\omega_n$, and $\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}'/\sqrt{\mathrm{tr}_2(\boldsymbol{\Sigma}'^2)}$, we have

$$\mathbb{M}_1 = m_1\mathbb{J} + m_2\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}} = m_1(\mathbb{J} - \hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}) + (m_1+m_2)\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}} \tag{3.30a}$$

$$\mathbb{M}_2 = m_3\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}, \tag{3.30b}$$

where

$$m_1 = \frac{d}{d-1}\langle\omega_n\rangle\Sigma_{\mathrm{eq}}^{n-1}, \tag{3.31a}$$

$$m_2 = (n-1)m_1 \tag{3.31b}$$

$$m_3 = \frac{d}{d-1}\left\langle\delta\omega_n^2\right\rangle\Sigma_{\mathrm{eq}}^{2n}. \tag{3.31c}$$

Noting that $\mathbb{J} - \hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}$ are complementary orthogonal projectors in $L(4s')$, we compute $\mathbb{M}_1^{-1_{4s'}}$, $\mathbb{M}_1^{-1_{4s'}}:\mathbb{M}_2$ and $\mathbb{M}_1^{-1_{4s'}}:\mathbb{M}_2:\mathbb{M}_1^{-1_{4s'}}$. Now introducing the unit vector $\hat{\mathbf{u}} = \mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\boldsymbol{\Sigma}}.\hat{\mathbf{k}}/|\mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\boldsymbol{\Sigma}}.\hat{\mathbf{k}}|$, we express M_1 and M_2 in terms of $\mathsf{Q}_{\hat{\mathbf{k}}} - \hat{\mathbf{u}}\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}\hat{\mathbf{u}}$ and compute the rest. With

$$\Sigma_{\hat{\mathbf{k}}} = 1 - 2|\mathsf{Q}_{\hat{\mathbf{k}}}.\hat{\boldsymbol{\Sigma}}.\hat{\mathbf{k}}|^2 = 1 - 2\left[\hat{\mathbf{k}}.\hat{\boldsymbol{\Sigma}}^2.\hat{\mathbf{k}} - (\hat{\mathbf{k}}.\hat{\boldsymbol{\Sigma}}.\hat{\mathbf{k}})^2\right], \tag{3.32}$$

we arrive at

$$\Psi_{\mathrm{us}}(\boldsymbol{\Sigma}) = \langle\psi_{\mathbf{x}}(\boldsymbol{\Sigma})\rangle - \frac{1}{2}\int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\frac{m_3\Sigma_{\hat{\mathbf{k}}}}{m_1 + m_2\Sigma_{\hat{\mathbf{k}}}}. \tag{3.33}$$

Finally replacing m_1, m_2, m_3 with their values (3.31) yields the desired expression for the effective Norton viscoplastic potential:

$$\Psi_{\mathrm{us}}(\boldsymbol{\Sigma}) = \frac{\omega_{n\,\mathrm{eff}}(\boldsymbol{\Sigma})}{n+1}\Sigma_{\mathrm{eq}}^{n+1}, \tag{3.34a}$$

$$\omega_{n\,\mathrm{eff}}(\boldsymbol{\Sigma}) = \langle\omega_n\rangle\left[1 - \frac{1}{2}\frac{\left\langle\delta\omega_n^2\right\rangle}{\langle\omega_n\rangle^2}\int\frac{d\Omega_{\hat{\mathbf{k}}}}{S_d}\frac{(n+1)\Sigma_{\hat{\mathbf{k}}}}{1+(n-1)\Sigma_{\hat{\mathbf{k}}}} + O(\left\langle\delta\omega_n^3\right\rangle/\langle\omega_n\rangle^3)\right]. \tag{3.34b}$$

IV. OTHER TYPES OF DISORDER

We briefly discuss here how to cope with non site-disordered models, with possibly anisotropic correlations. It suffices to write the average of the exponential in (2.20) simply as

$$\left\langle e^{-\beta\sum_{\gamma=1}^{r}\overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}}\right\rangle = \exp\left[\log\left\langle e^{-\beta\sum_{\gamma=1}^{r}\overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}}\right\rangle\right] \tag{4.1}$$

and to expand the logarithm as in Sec. III A 1, only keeping the quadratic terms in the outer exponential. Equ. (3.8) is unchanged, save for a slightly different (but equivalent) form for the first averaged prefactor, and for a modification in the quadratic term in the main exponential. This term becomes

$$-\frac{1}{2}\frac{\beta}{V}\sum_{\alpha,\gamma}\int d\mathbf{x}\,d\mathbf{y}\,\mathsf{e}^\alpha(\mathbf{x}):\mathbb{M}^{\alpha\gamma}(\mathbf{x}|\mathbf{y}):\mathsf{e}^\alpha(\mathbf{y}), \tag{4.2}$$

where

$$\mathbb{M}^{\alpha\gamma}(\mathbf{x}|\mathbf{y}) = \langle\mathbb{W}''\rangle\delta_{\alpha\gamma}\delta(\mathbf{x}-\mathbf{y}) - \frac{\beta}{V}\left[\langle\mathsf{W}'(\mathbf{x})\mathsf{W}'(\mathbf{y})\rangle - \langle\mathsf{W}'(\mathbf{x})\rangle\langle\mathsf{W}'(\mathbf{y})\rangle\right]. \tag{4.3}$$

This requires modifications in the calculations, but does not change their principle. One merely has to deal with summations over another pair of indices, namely the coordinates $\mathbf{x}$ and $\mathbf{y}$. Now the determinants, e.g. in (3.12), become functional determinants and must simultaneously be evaluated as determinants of operators in the Fourier space, in addition to being operators on finite-dimensional vector spaces. However, this is not a serious problem in the perturbative calculation considered here since only traces are required in the final second-order results.

In the case of translation-invariant disorder, the kernel $\mathbb{M}^{\alpha\gamma}(\mathbf{x}|\mathbf{y})$ is diagonal in the Fourier representation and (3.12) still holds, provided that one writes $\mathbb{M}(\mathbf{k})$ (for its Fourier transform with respect to $\mathbf{x}-\mathbf{y}$) instead of $\mathbb{M}$. The two-point correlation function $g(\mathbf{x})$ is defined by:

$$\langle\delta\theta_m(\mathbf{x})\delta\theta_m(\mathbf{y})\rangle = \left\langle\delta\theta_m^2\right\rangle g(\mathbf{x}-\mathbf{y}), \tag{4.4a}$$

$$\langle\delta\omega_n(\mathbf{x})\delta\omega_n(\mathbf{y})\rangle = \left\langle\delta\omega_n^2\right\rangle g(\mathbf{x}-\mathbf{y}), \tag{4.4b}$$

with

$$g(\mathbf{x}=\mathbf{0}) = \int \frac{d\mathbf{k}}{(2\pi)^d}\, g(\mathbf{k}) = 1. \tag{4.5}$$

The final results are then only modified as:

$$\theta_{m\,\mathrm{eff}}(\mathsf{D}) = \langle\theta_m\rangle \left[1 - \frac{\left\langle\delta\theta_m^2\right\rangle}{\langle\theta_m\rangle^2} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{(m+1)g(\mathbf{k})D_{\hat{\mathbf{k}}}}{1+2(m-1)D_{\hat{\mathbf{k}}}} + O(\left\langle\delta\theta_m^3\right\rangle / \langle\theta_m\rangle^3)\right], \tag{4.6a}$$

$$\omega_{n\,\mathrm{eff}}(\mathbf{\Sigma}) = \langle\omega_n\rangle \left[1 - \frac{1}{2}\frac{\left\langle\delta\omega_n^2\right\rangle}{\langle\omega_n\rangle^2} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{(n+1)g(\mathbf{k})\Sigma_{\hat{\mathbf{k}}}}{1+(n-1)\Sigma_{\hat{\mathbf{k}}}} + O(\left\langle\delta\omega_n^3\right\rangle / \langle\omega_n\rangle^3)\right]. \tag{4.6b}$$

The Legendre duality property once again applies, whatever the nature of the correlations, and the result is once again independent of the boundary conditions. The previous expressions for isotropic site-disorder are recovered using $g(\mathbf{x}) = v\delta(\mathbf{x})$. Ellipsoidal symmetry corresponds to a correlation function $g(\mathbf{x}) = g(|\mathsf{Z}.\mathbf{x}|)$, where Z is a symmetric constant tensor (Willis 1977; Ponte-Castañeda and Suquet, 1995).

V. DISCUSSION

In this section, we address the range of applicability of the functional method. Firstly, a major concern of studies on heterogeneous media is about bounds. In principle, the above functional formulation should be apt to deliver bounds, at least as far as the initial expressions (2.12) and (2.15) are concerned. One might think, e.g. to use the convexity properties of the exponential in order to obtain inequalities on approximations to the potentials. However, to the knowledge of these authors, the replica method is an inadequate tool to use for subsequent calculations, if these are to preserve inequalities. Indeed, a convexity equality of the type

$$\mu\left(e^{\sum_{\alpha=1}^r \mathcal{V}^\alpha}\right) \geq e^{\mu\left(\sum_{\alpha=1}^r \mathcal{V}^\alpha\right)}, \tag{5.1}$$

where μ is some normalized functional measure and $\mathcal{V}$ is some potential, valid as long as the potential is replicated r times, $r \geq 1$, may not survive the limit $r \to 0$. In the zero-replica limit indeed, quantities always positive when $r \geq 1$ may become negative, minima may transform into maxima (Parisi, 1984), and it is most often hard to conclude on inequalities. The replica method has been invented, and employed, to find estimates (which can be of very good quality) to free energies when disorder is present, not bounds.

Second, one must not confuse the initial functional formulation with the replica method, which we use as a tool in order to work out averages. These are distinct matters. Actually, there are two starting points for the calculations: either we transfer the statistical averages of the logarithm *inside* the functional integral by means of the replica method, and carry out approximations on the effective potential

$$-\frac{1}{\beta}\log\left\langle e^{-\beta\sum_\gamma \overline{\phi_{\mathbf{x}}(\mathsf{d}^\gamma)}}\right\rangle, \tag{5.2}$$

cf. (4.1). This is what we did in the paper. Either we carry out an approximation on the non-replicated initial functional integral (2.12), then take the logarithm, and average afterwards only. Save for perturbative expansions, the results will be different.

Third, all types of calculations feasible with classical methods (especially the perturbative ones) should translate into simple approximation schemes to the basic equations (2.12) and (2.15). But we believe that the compact starting point of the functional formulation might enable one to obtain non-perturbative results more easily, and of different nature, than by classical means. There resides its main interest.

Finally, the trick consisting in writing the minimum of some functional as the limit of a particular functional integral could be applied elsewhere, e.g., to compute variational expressions such as those presented in Ref. [23] (computing a maximum can of course be done by identical means).

VI. CONCLUSION

A field-theoretic method in order to compute the effective properties of highly non-linear composites has been introduced. The minimization problem was shown to be equivalent to the computation of a functional integral. We believe the main advantage of this formulation is that it is valid for all types of local potentials, and that it can be used as a convenient starting point for approximations. As a first application, a weak-disorder second-order perturbative calculation of this functional integral was presented. The second-order results of Suquet and Ponte-Castañeda (1993) for the effective plastic dissipation potential (with uniform traction boundary conditions) were recovered. In addition, the second-order correction to the viscoplastic potential was also obtained, and these results were extended to correlated disorder. With this method, minimizing over a vector velocity field, or a tensor stress field, is found to be equally feasible.

A natural extension of this work is to find a well-behaved non-perturbative approximation. Self-consistent calculations are currently under study.

ACKNOWLEDGMENTS

We thank P. Suquet and M. Garajeu for stimulating discussions, and for communicating to us very useful references, and J.-M. Diani for his interest in this work. Thanks are also due to Prof. J. R. Willis for useful comments.

APPENDIX A: FOURTH-RANK TENSORS IN MECHANICS

Before briefly reminding the reader about the main properties of fourth-rank tensors in mechanics, we need to set our notations about the various algebra and sub-algebras of linear operators that come into play.

We designate by $L(2)$ the algebra of linear operators (second-rank $d \times d$ square matrices) $\mathsf{M} : \mathbf{R}^d \to \mathbf{R}^d$. Next, fourth-rank tensors appear in mechanics as representations of linear operators $\mathbb{M} : L(2) \to L(2)$. They can therefore be considered as square matrices, the indices of which are pairs of indices, and their components are denoted by $M_{ij,kl}$, such that $i,\, j,\, k,\, l = 1, \ldots, d$. We denote the algebra they belong to by $L(4)$. The product in $L(4)$ is

$$(\mathbb{M}{:}\mathbb{N})_{ij,kl} = M_{ij,mn} N_{nm,kl}. \tag{A1}$$

Hence applying $\mathbb{M}$ to a second-rank tensor N, we have

$$(\mathbb{M}{:}\mathsf{N})_{ij} = M_{ij,kl} N_{lk}. \tag{A2}$$

The identity $\mathbb{I}$ in $L(4)$ is

$$I_{ij,kl} = \delta_{il}\delta_{jk}, \tag{A3}$$

and the trace is defined (and denoted) by

$$\mathrm{t}r_4\mathbb{M} = M_{ij,ji}. \tag{A4}$$

Hence $\mathrm{t}r_4\mathbb{I} = d^2$.

The algebra $L(2)$ can be split in the direct sum of that of symmetric traceless matrices [which we denote by $L(2s')$], diagonal matrices proportional to the identity [$L(2d)$], and anti-symmetric matrices [$L(2a)$]. The algebra of symmetric matrices is the direct sum $L(2s) = L(2s') \oplus L(2d)$. The algebra $L(4)$ therefore admits three remarkable sub-algebras, which we denote by $L(4s')$, $L(4d)$, $L(4a)$, generated by the mutually orthogonal projectors $\mathbb{J}$, $\mathbb{K}$ and $\mathbb{I}^a$ respectively, defined by

$$J_{ij,kl} = \frac{1}{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) - \frac{1}{d}\delta_{ij}\delta_{kl}, \tag{A5a}$$

$$K_{ij,kl} = \frac{1}{d}\delta_{ij}\delta_{kl}, \tag{A5b}$$

$$I^a_{ij,kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{A5c}$$

They obey $\mathbb{I} = \mathbb{I}^a + \mathbb{J} + \mathbb{K}$. Hence, the sub-algebras $L(4s')$, $L(4d)$, $L(4a)$ are that of the endomorphisms on $L(2s')$, $L(2d)$, $L(2a)$ respectively. The operators $\mathbb{J}$, $\mathbb{K}$, and $\mathbb{I}^a$ are the appropriate identity operators in each sub-algebra. We have

$$\mathrm{tr}_4 \mathbb{J} = d(d+1)/2 - 1, \tag{A6a}$$
$$\mathrm{tr}_4 \mathbb{K} = 1, \tag{A6b}$$
$$\mathrm{tr}_4 \mathbb{I}^a = d(d-1)/2. \tag{A6c}$$

These numbers count the number of eigenvalues of each operator. In $L(4s)$, the sub-algebra of the endomorphisms on $L(2s)$, the identity is $\mathbb{I}^s = \mathbb{J} + \mathbb{K}$:

$$I^s_{ij,kl} = \frac{1}{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \tag{A7}$$

and $\mathrm{tr}_4 \mathbb{I}^s = d(d+1)/2$. Note that $L(4s') \oplus L(4d) \subset L(4s)$, but the two sets are not equal.

Finally note that the inverse (resp. determinant) in $L(4)$ of an operator $\mathbb{M} \in L(4s') \oplus L(4d) \oplus L(4a)$ is the sum (resp. product) of the inverses (reps. determinants) of its constituents in their respective sub-algebra.

APPENDIX B: GAUSSIAN INTEGRALS

1. Gaussian integrals over vector fields

We start from the well-known d-dimensional gaussian integral over a vector field $\boldsymbol{\lambda}$ (Kleinert, 1995). If M is a $d \times d$ positive-definite symmetric matrix, and $\mathbf{b}$ any vector, then

$$\int d\boldsymbol{\lambda}\, \exp\left(-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}.\boldsymbol{\lambda} + \mathbf{b}.\boldsymbol{\lambda}\right) = \left[\frac{(2\pi)^d}{\det_2(\mathsf{M})}\right]^{1/2} \exp\left(\frac{1}{2}\mathbf{b}.\mathsf{M}^{-1_2}.\mathbf{b}\right), \tag{B1}$$

where the integration measure is $d\boldsymbol{\lambda} = \prod_i d\lambda_i$.

This formula is meaningless if M is not invertible in $L(2)$. However, let $\hat{\mathbf{k}}$ be a unit vector, $\mathbf{b}$ now be a real vector, and $\mathsf{M} = \mathsf{M}_\perp + \epsilon^2 \hat{\mathbf{k}}\hat{\mathbf{k}}$, where $\mathsf{M}_\perp$ is invertible only in $L(2[\perp \hat{\mathbf{k}}])$, the set of the endomorphisms which operate in the subspace of vectors *orthogonal* to $\hat{\mathbf{k}}$. Then $\mathsf{M}^{-1_2} = \mathsf{M}_\perp^{-1_{2[\perp\hat{\mathbf{k}}]}} + \epsilon^{-2}\hat{\mathbf{k}}\hat{\mathbf{k}}$, $\det_2(\mathsf{M}) = \epsilon^2 \det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp)$, and

$$\int d\boldsymbol{\lambda}\, \exp\left(-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}.\boldsymbol{\lambda} + i\mathbf{b}.\boldsymbol{\lambda}\right) = \left[\frac{(2\pi)^d}{\epsilon^2 \det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp)}\right]^{1/2} \exp\left[-\frac{1}{2}\mathbf{b}.\mathsf{M}^{-1_{2[\perp\hat{\mathbf{k}}]}}.\mathbf{b} - \frac{1}{2\epsilon^2}(\mathbf{b}.\hat{\mathbf{k}})^2\right]. \tag{B2}$$

Letting $\epsilon \to 0$ and using the representation of the Dirac distribution

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi\epsilon^2}} e^{-\frac{1}{2}(x/\epsilon)^2}, \tag{B3}$$

we obtain

$$\int d\boldsymbol{\lambda}\, \exp\left(-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}_\perp.\boldsymbol{\lambda} + i\mathbf{b}.\boldsymbol{\lambda}\right) = \left[\frac{(2\pi)^{d+1}}{\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp)}\right]^{1/2} \exp\left(-\frac{1}{2}\mathbf{b}.\mathsf{M}^{-1_{2[\perp\hat{\mathbf{k}}]}}.\mathbf{b}\right)\delta(\mathbf{b}.\hat{\mathbf{k}}). \tag{B4}$$

We define the integration measure over *the vectors orthogonal to* $\hat{\mathbf{k}}$ to be

$$\underset{[\perp\hat{\mathbf{k}}]}{d\boldsymbol{\lambda}} = \delta(\boldsymbol{\lambda}.\hat{\mathbf{k}})d\boldsymbol{\lambda}. \tag{B5}$$

Multiplying both sides of (B4) by $\exp(i\mathbf{b}.\mathbf{a})$, where $\mathbf{a}$ is a real vector, and integrating over $\mathbf{b}$ yields

$$\int_{[\perp\hat{\mathbf{k}}]} d\mathbf{b}\, \exp\left(-\frac{1}{2}\mathbf{b}.\mathsf{M}^{-1_{2[\perp\hat{\mathbf{k}}]}}.\mathbf{b} + i\mathbf{b}.\mathbf{a}\right)$$

$$= \left[\frac{(2\pi)^{-(d+1)}}{\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp^{-1_{2[\perp\hat{\mathbf{k}}]}})}\right]^{1/2} \int d\boldsymbol{\lambda} d\mathbf{b}\, \exp\left[-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}_\perp.\boldsymbol{\lambda} + i\mathbf{b}.(\boldsymbol{\lambda}+\mathbf{a})\right]$$

$$= (2\pi)^d \left[\frac{(2\pi)^{-(d+1)}}{\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp^{-1_{2[\perp\hat{\mathbf{k}}]}})}\right]^{1/2} \int d\boldsymbol{\lambda}\, \exp\left(-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}_\perp.\boldsymbol{\lambda}\right)\delta(\mathbf{a}+\boldsymbol{\lambda})$$

$$= \left[\frac{(2\pi)^{d-1}}{\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{M}_\perp^{-1_{2[\perp\hat{\mathbf{k}}]}})}\right]^{1/2} \exp\left(-\frac{1}{2}\mathsf{a}.\mathsf{M}_\perp.\mathsf{a}\right), \tag{B6}$$

where $\delta(\mathbf{x}) = \prod_i \delta(x_i)$. Note that we can exchange $\mathsf{M}_\perp^{-1_{2[\perp\hat{\mathbf{k}}]}}$ and $\mathsf{M}_\perp$ in (B6). Moreover, $\mathsf{M}_\perp$ can be written $\mathsf{M}_\perp = \mathsf{Q}_{\hat{\mathbf{k}}}.\mathsf{M}.\mathsf{Q}_{\hat{\mathbf{k}}}$, where $\mathsf{Q}_{\hat{\mathbf{k}}}$ is the projector $\mathsf{Q}_{\hat{\mathbf{k}}} = \mathsf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$, for some $\mathsf{M} \in L(2)$. Besides, the $\mathbf{b}$ are now orthogonal to $\hat{\mathbf{k}}$ (because of the integration measure) so that $\mathbf{b}.\mathsf{Q}_{\hat{\mathbf{k}}}.\mathsf{M}.\mathsf{Q}_{\hat{\mathbf{k}}}.\mathbf{b} = \mathbf{b}.\mathsf{M}.\mathbf{b}$. Finally, the result can be analytically continued to complex values of $\mathbf{a}$. Changing the names of the variables, we therefore arrive at the formula

$$\int_{[\perp\hat{\mathbf{k}}]} d\boldsymbol{\lambda}\, \exp\left(-\frac{1}{2}\boldsymbol{\lambda}.\mathsf{M}.\boldsymbol{\lambda} + \mathbf{b}.\boldsymbol{\lambda}\right) = \left[\frac{(2\pi)^{d-1}}{\det_{2[\perp\hat{\mathbf{k}}]}(\mathsf{Q}_{\hat{\mathbf{k}}}.\mathsf{M}.\mathsf{Q}_{\hat{\mathbf{k}}})}\right]^{1/2} \exp\left[\frac{1}{2}\mathbf{b}.(\mathsf{Q}_{\hat{\mathbf{k}}}.\mathsf{M}.\mathsf{Q}_{\hat{\mathbf{k}}})^{-1_{2[\perp\hat{\mathbf{k}}]}}.\mathbf{b}\right]. \tag{B7}$$

2. Gaussian integrals over tensors of rank 2

The generic gaussian integral over all matrices $\mathsf{s} \in L(2)$ is

$$\int d\mathsf{s}\, \exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s} + \mathsf{b}:\mathsf{s}\right) = \left[\frac{(2\pi)^{d^2}}{\det_4(\mathbb{M})}\right]^{1/2} \exp\left(\frac{1}{2}\mathsf{b}:\mathbb{M}^{-1_4}:\mathsf{b}\right), \tag{B8}$$

where $d\mathsf{s} = \prod_{ij} ds_{ij}$ is the appropriate measure, $\mathbb{M}$ is a symmetric matrix of $L(4)$, i.e. is such that $M_{ij,kl} = M_{kl,ij}$. This formula is a direct consequence of (B1) via a mapping of s onto a column vector in $\mathbf{R}^{d^2}$. Once again, the integral is meaningless in general if $\mathbb{M}$ is not invertible in $L(4)$. We apply the same method as in Sec. B 1.

a. Gaussian integrals over symmetric matrices

Let $\mathsf{b} \in L(2)$ be real. We consider $\mathbb{M} = \mathbb{M}_s + (\epsilon^2/2)\mathbb{I}^a$, where $\mathbb{M}_s \in L(4s)$ and is invertible in $L(4s)$. Then $\mathbb{M}^{-1_4} = \mathbb{M}_s^{-1_{4s}} + (2/\epsilon^2)\mathbb{I}^a$ and $\det_4(\mathbb{M}) = \det_{4s}(\mathbb{M})(\epsilon^2/2)^{d(d-1)/2}$. Hence from (B8),

$$\int d\mathsf{s}\, \exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s} + i\mathsf{b}:\mathsf{s}\right)$$

$$= \left[\frac{(2\pi)^{d^2}}{\det_{4s}(\mathbb{M}_s)}\right]^{1/2} \frac{1}{(\epsilon^2/2)^{d(d-1)/4}} \exp\left(-\frac{1}{2}\mathsf{b}:\mathbb{M}_s^{-1_{4s}}:\mathsf{b} - \frac{1}{\epsilon^2}\mathsf{b}:\mathbb{I}^a:\mathsf{b}\right). \tag{B9}$$

But $\mathsf{b}:\mathbb{I}^a:\mathsf{b} = (1/2)\sum_{i<j}(b_{ij} - b_{ji})^2$. Using (B3) and going to the limit $\epsilon \to 0$, we find

$$\int d\mathsf{s}\, \exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}_s:\mathsf{s} + i\mathsf{b}:\mathsf{s}\right)$$

$$= \left[\frac{(2\pi)^{d^2}(4\pi)^{d(d-1)/2}}{\det_{4s}(\mathbb{M}_s)}\right]^{1/2} \exp\left(-\frac{1}{2}\mathsf{b}:\mathbb{M}_s^{-1_{4s}}:\mathsf{b}\right) \prod_{i<j}\delta(b_{ij} - b_{ji}). \tag{B10}$$

We define the measure over the *symmetric* tensors of rank two as:

$$d_s\mathsf{s} = 2^{d(d-1)/4}\Big[\prod_{i,j<i}\delta(s_{ij} - s_{ji})\Big]\, d\mathsf{s}. \tag{B11}$$

Multiplying both sides of (B10) by $\exp(i\mathsf{b}:\mathsf{a})$, and integrating with respect to b with $d\mathsf{b}$ yields

$$\int d_s\mathsf{b}\,\exp\left(-\frac{1}{2}\mathsf{b}:\mathbb{M}_s^{-1_{4s}}:\mathsf{b}+i\mathsf{a}:\mathsf{b}\right)$$
$$=\left[\frac{(2\pi)^{-d^2-d(d-1)/2}}{\det_{4s}(\mathbb{M}_s^{-1_{4s}})}\right]^{1/2}\int d\mathsf{b}\,d\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}_s:\mathsf{s}+i(\mathsf{s}+\mathsf{a}):\mathsf{b}\right)$$
$$=(2\pi)^{d^2}\left[\frac{(2\pi)^{-d^2-d(d-1)/2}}{\det_{4s}(\mathbb{M}_s^{-1_{4s}})}\right]^{1/2}\int d\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}_s:\mathsf{s}\right)\delta(\mathsf{s}+\mathsf{a})$$
$$=\left[\frac{(2\pi)^{d(d+1)/2}}{\det_{4s}(\mathbb{M}_s^{-1_{4s}})}\right]^{1/2}\exp\left(-\frac{1}{2}\mathsf{a}:\mathbb{M}_s:\mathsf{a}\right), \tag{B12}$$

where $\delta(\mathsf{x})=\prod_{ij}\delta(x_{ij})$. Whence the generic result for gaussian integrals over *symmetric* matrices, with $\mathbb{M}\in L(4)$:

$$\int d_s\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s}+\mathsf{b}:\mathsf{s}\right)=\left[\frac{(2\pi)^{d(d+1)/2}}{\det_{4s}(\mathbb{I}^s:\mathbb{M}:\mathbb{I}^s)}\right]^{1/2}\exp\left(\frac{1}{2}\mathsf{b}:(\mathbb{I}^s:\mathbb{M}:\mathbb{I}^s)^{-1_{4s}}:\mathsf{b}\right). \tag{B13}$$

b. Gaussian integrals over traceless symmetric matrices

If $\mathbb{I}^s:\mathbb{M}:\mathbb{I}^s$ is not invertible in $L(4s)$, but only in $L(4s')$ for instance, the procedure can be repeated: let us assume that $\mathbb{M}=\mathbb{M}_{s'}+\epsilon^2\mathbb{K}$, where $\mathbb{M}_{s'}\in L(4s')$. Starting from (B13) with $\mathsf{b}\to i\mathsf{b}$, using $(\mathbb{I}^s:\mathbb{M}:\mathbb{I}^s)^{-1_{4s}}=\mathbb{M}_{s'}^{-1_{4s'}}+\mathbb{K}/\epsilon^2$, $\det_{4s}(\mathbb{I}^s:\mathbb{M}:\mathbb{I}^s)=\det_{4s'}(\mathbb{M}_{s'})\epsilon^2$ and letting $\epsilon\to 0$, one finds:

$$\int d_s\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}_{s'}:\mathsf{s}+i\mathsf{b}:\mathsf{s}\right)=\left[\frac{d(2\pi)^{d(d+1)/2+1}}{\det_{4s'}(\mathbb{M}_{s'})}\right]^{1/2}\exp\left(-\frac{1}{2}\mathsf{b}:\mathbb{M}_{s'}^{-1_{4s'}}:\mathsf{b}\right)\delta(tr_2\mathsf{b}). \tag{B14}$$

We define the integration measure over *symmetric traceless* tensors to be

$$d_{s'}\mathsf{s}=d^{1/2}\delta(tr_2\mathsf{s})\,d_s\mathsf{s}. \tag{B15}$$

As one easily checks, we have, with (B11)

$$\int d_s\mathsf{x}\,\exp(i\mathsf{s}:\mathsf{x})=2^{d(d-1)/4}(2\pi)^{d(d+1)/2}[\prod_i\delta(s_{\underline{ii}})][\prod_{i<j}\delta(s_{ij}+s_{ji})]. \tag{B16}$$

Multiplying (B14) by $\exp(i\mathsf{a}:\mathsf{b})$, and integrating over b with measure $d_s\mathsf{b}$, we deduce, using $\mathbb{M}_{s'}=\mathbb{J}:\mathbb{M}:\mathbb{J}$, the result for gaussian integrals over *symmetric traceless* tensors:

$$\int d_{s'}\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s}+\mathsf{b}:\mathsf{s}\right)=\left[\frac{(2\pi)^{d(d+1)/2-1}}{\det_{4s'}(\mathbb{J}:\mathbb{M}:\mathbb{J})}\right]^{1/2}\exp\left[\frac{1}{2}\mathsf{b}:(\mathbb{J}:\mathbb{M}:\mathbb{J})^{-1_{4s'}}:\mathsf{b}\right]. \tag{B17}$$

c. Other gaussian integrals

For the sake of completeness, we finally give without demonstration the results for gaussian integrals over *antisymmetric tensors*, with measure

$$d_a\mathsf{s}=2^{d(d-1)/4}[\prod_i\delta(s_{\underline{ii}})][\prod_{i<j}\delta(s_{ij}+s_{ji})]d\mathsf{s}, \tag{B18}$$

and over *diagonal tensors proportional to the identity*, of the type $s\mathsf{I}/d$, with measure

$$d_d\mathsf{s}=d^{-1/2}d\mathsf{s}\int ds\,\delta(\mathsf{s}-s\mathsf{I}/d). \tag{B19}$$

One finds

$$\int d_a\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s}+\mathsf{b}:\mathsf{s}\right) = \left[\frac{(2\pi)^{d(d-1)/2}}{\det_{4a}(\mathbb{I}^a:\mathbb{M}:\mathbb{I}^a)}\right]^{1/2}\exp\left[\frac{1}{2}\mathsf{b}:(\mathbb{I}^a:\mathbb{M}:\mathbb{I}^a)^{-1_{4a}}:\mathsf{b}\right] \tag{B20}$$

$$\int d_d\mathsf{s}\,\exp\left(-\frac{1}{2}\mathsf{s}:\mathbb{M}:\mathsf{s}+\mathsf{b}:\mathsf{s}\right) = \left[\frac{2\pi}{\det_{4d}(\mathbb{K}:\mathbb{M}:\mathbb{K})}\right]^{1/2}\exp\left[\frac{1}{2}\mathsf{b}:(\mathbb{K}:\mathbb{M}:\mathbb{K})^{-1_{4d}}:\mathsf{b}\right]. \tag{B21}$$

The last identity is trivial (scalar gaussian integral).

APPENDIX C: INVERSE AND DETERMINANT IN REPLICA SPACE

In replica space, A and B being replica-independent operators pertaining to some algebra $\mathcal{A}$, the inverse and determinant of a matrix of the type

$$\overline{\overline{M}} = A\overline{\overline{I}} + B\overline{\overline{U}}, \tag{C1}$$

read

$$\overline{\overline{M}}^{-1_{\mathcal{A},\mathrm{rep}}} = A^{-1_{\mathcal{A}}}\overline{\overline{I}} - (A+rB)^{-1_{\mathcal{A}}}BA^{-1_{\mathcal{A}}}\overline{\overline{U}}, \tag{C2a}$$

$$\det_{\mathcal{A},\mathrm{rep}}(\overline{\overline{M}}) = \det_{\mathcal{A}}(A)^{r-1}\det_{\mathcal{A}}(A+rB) = 1 + r\left[\log\det_{\mathcal{A}}(A) + \mathrm{tr}_{\mathcal{A}}(BA^{-1_{\mathcal{A}}})\right] + O(r^2) \tag{C2b}$$

The expansion derives from the operator identity (Kleinert, 1995) $\log\det = \mathrm{tr}\log$.

APPENDIX D: GAUSSIAN INTEGRALS OVER THE VELOCITY AND STRESS FIELDS

1. Integration over the velocity field

We detail here the steps leading to (3.12).

The Dirac distribution implementing the constraint $\mathbf{\nabla}.\mathbf{v} = 0$ in (3.8) is exponentiated first: we introduce a scalar field λ and write:

$$\delta(\mathbf{\nabla}.\mathbf{v}) \propto \int \mathcal{D}\lambda\,\exp\left[i\int d\mathbf{x}\,\lambda(\mathbf{x})\mathbf{\nabla}.\mathbf{v}(\mathbf{x})\right]. \tag{D1}$$

Then we Fourier transform the integrand. By definition, for any function f (which will be λ or s), the Fourier transform is:

$$f(\mathbf{k}) = \int d\mathbf{x}\,e^{i\mathbf{k}.\mathbf{x}}f(\mathbf{x}), \quad f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d}\,e^{-i\mathbf{k}.\mathbf{x}}f(\mathbf{k}). \tag{D2}$$

Since $f(\mathbf{x})$ is real, we write for $\mathbf{k}\neq\mathbf{0}$:

$$f(\mathbf{k}) = \frac{1}{\sqrt{2}}[f^{(1)}(\mathbf{k}) + if^{(2)}(\mathbf{k})], \tag{D3a}$$

$$f(-\mathbf{k}) = \frac{1}{\sqrt{2}}[f^{(1)}(\mathbf{k}) - if^{(2)}(\mathbf{k})], \tag{D3b}$$

where $f^{(1)}$ and $f^{(2)}$ are real functions of the wavevector; also, we set $f(\mathbf{k}=\mathbf{0}) = f^{(1)}(\mathbf{0})$. This decomposition is carried out for all $\lambda^\alpha(\mathbf{k})$ and $\mathbf{v}^\alpha(\mathbf{k})$.

Introducing the set $\mathcal{K}^+ = \{\mathbf{k}/k_1 > 0\} \cup \{\mathbf{k}/k_1 = 0, k_2 > 0\} \cup \ldots \cup \{\mathbf{k}/k_1 = 0, \ldots, k_{d-1} = 0, k_d > 0\}$, a functional measure $\mathcal{D}f$ over $f(\mathbf{x})$ thus becomes in the Fourier representation

$$\mathcal{D}f = \prod_{\mathbf{x}} df(\mathbf{x}) \propto df^{(1)}(\mathbf{0})\prod_{\mathbf{k}\in\mathcal{K}^+}[df^{(1)}(\mathbf{k})df^{(2)}(\mathbf{k})]. \tag{D4}$$

Note that $f^{(1)}$ and $f^{(2)}$ are defined only for $\mathbf{k} \in \mathcal{K}^+$. By Parseval's identity, we have

$$\int d\mathbf{x}\, \lambda(\mathbf{x}) \boldsymbol{\nabla}.\mathbf{v}(\mathbf{x}) = \int_{\mathcal{K}^+} \frac{d\mathbf{k}}{(2\pi)^d} \left[\lambda^{(2)}(\mathbf{k})\mathbf{v}^{(1)}(\mathbf{k}).\mathbf{k} - \lambda^{(1)}(\mathbf{k})\mathbf{v}^{(2)}(\mathbf{k}).\mathbf{k} \right]. \tag{D5}$$

As usual in functional methods, the latter integral is to be understood as a Riemann discrete sum by applying the correspondence:

$$\int \frac{d\mathbf{k}}{(2\pi)^d} \to \frac{1}{V} \sum_{\mathbf{k}}. \tag{D6}$$

In particular, this allows one to separate the contribution of the mode $\mathbf{k} = 0$. The argument of the exponential in (3.8) is likewise transformed into a sum of Fourier modes via

$$\begin{aligned} &\int d\mathbf{x}\, \mathsf{e}^{\alpha}(\mathbf{x}) \!:\! \mathbb{M}^{\alpha\gamma} \!:\! \mathsf{e}^{\gamma}(\mathbf{x}) \\ &= \frac{1}{V}\, \mathsf{e}^{\alpha(1)}(\mathbf{0}) \!:\! \mathbb{M}^{\alpha\gamma} \!:\! \mathsf{e}^{\gamma(1)}(\mathbf{0}) + \frac{1}{V} \sum_{\mathbf{k}\in\mathcal{K}^+} \left[\mathsf{e}^{\alpha(1)}(\mathbf{k}) \!:\! \mathbb{M}^{\alpha\gamma} \!:\! \mathsf{e}^{\gamma(1)}(\mathbf{k}) + \mathsf{e}^{\alpha(2)}(\mathbf{k}) \!:\! \mathbb{M}^{\alpha\gamma} \!:\! \mathsf{e}^{\gamma(2)}(\mathbf{k}) \right] \\ &= \frac{1}{V} \sum_{\mathbf{k}\in\mathcal{K}^+} \left[\mathbf{v}^{\alpha(1)}(\mathbf{k}).(\mathbf{k}.\mathbb{M}^{\alpha\gamma}.\mathbf{k}).\mathbf{v}^{\gamma(1)}(\mathbf{k}) + \mathbf{v}^{\alpha(2)}(\mathbf{k}).(\mathbf{k}.\mathbb{M}^{\alpha\gamma}.\mathbf{k}).\mathbf{v}^{\gamma(2)}(\mathbf{k}) \right]. \end{aligned} \tag{D7}$$

In the last equality, we have used the symmetry of $\mathbb{M}^{\alpha\gamma}$ with respect to its tensor indices. Note that $\mathsf{e}^{(1)} = -[\mathbf{k}\mathbf{v}^{(2)} + \mathbf{v}^{(2)}\mathbf{k}]/2$ and $\mathsf{e}^{(2)} = [\mathbf{k}\mathbf{v}^{(1)} + \mathbf{v}^{(1)}\mathbf{k}]/2$.

Because of the constraints $\delta(\overline{\mathsf{e}}^{\alpha})$, the linear terms $\overline{\mathsf{e}}^{\alpha}$ disappear from the exponential. There is no dependence of the integrand with respect to $\lambda^{(1)}(\mathbf{0})$ nor to $\mathbf{v}^{(1)}(\mathbf{0})$, so that integration over these variables only yields harmless β-independent infinite multiplicative factors which we drop out, since they do not contribute to the final result in the limit $\beta \to \infty$. The reason is the same as that invoked in conjunction with Equ. (2.24). Such physically irrelevant (because multiplicative) infinities are often encountered when dealing with functional integrals. They are related to (here) unimportant normalization questions.

We therefore see that for each $\mathbf{k} \neq \mathbf{0}$, mutually complex conjugate partial integrals $A(\mathbf{k}) = \int d\lambda^{\alpha(2)}(\mathbf{k})\, d\mathbf{v}^{\alpha(1)}(\mathbf{k})\, [\ldots]$ and $A^*(\mathbf{k}) = \int d\lambda^{\alpha(1)}(\mathbf{k}) d\mathbf{v}^{\alpha(2)}(\mathbf{k})[\ldots]$ show up in pairs, and can be evaluated independently from one another. Renaming, in $A(\mathbf{k})$, the integration variables into $\mathbf{k}$-independent $d\lambda^{\alpha}$ and $d\mathbf{v}^{\alpha}$, we arrive at

$$\langle Z^r_{\mathrm{ut}} \rangle \propto \langle e^{-r\beta \frac{v}{V} \phi_{\mathbf{x}}(\mathsf{D})} \rangle^{\frac{V}{v}} \prod_{\mathbf{k}\in\mathcal{K}^+} |A(\mathbf{k})|^2, \tag{D8}$$

where

$$A(\mathbf{k}) = \int \left(\prod_{\alpha} d\lambda^{\alpha} d\mathbf{v}^{\alpha} \right) \exp\left[-\frac{1}{2}\frac{\beta}{V^2} \sum_{\alpha\gamma} \mathbf{v}^{\alpha}.(\mathbf{k}.\mathbb{M}^{\alpha\gamma}.\mathbf{k}).\mathbf{v}^{\gamma} + \frac{i}{V} \sum_{\alpha} \lambda^{\alpha}(\mathbf{v}^{\alpha}.\mathbf{k}) \right] \tag{D9a}$$

$$\propto \int \left[\prod_{\alpha} d\mathbf{v}^{\alpha} \delta(\mathbf{v}^{\alpha}.\mathbf{k}) \right] \exp\left[-\frac{1}{2}\frac{\beta}{V^2} \sum_{\alpha\gamma} \mathbf{v}^{\alpha}.(\mathbf{k}.\mathbb{M}^{\alpha\gamma}.\mathbf{k}).\mathbf{v}^{\gamma} \right] \tag{D9b}$$

$$\propto \det_{2[\perp\hat{\mathbf{k}}],\mathrm{rep}} \left[\mathsf{Q}_{\hat{\mathbf{k}}}.\left(\hat{\mathbf{k}}.\overline{\overline{\mathbb{M}}}.\hat{\mathbf{k}} \right).\mathsf{Q}_{\hat{\mathbf{k}}} \right]^{-1/2}, \tag{D9c}$$

and $\mathsf{Q}_{\hat{\mathbf{k}}} = \mathsf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$ with $\hat{\mathbf{k}} = \mathbf{k}/k$. In the step from (D9b) to (D9c) we used formula (B7), trivially generalized to the replica space, and discarded, among other irrelevant factors, a power of the modulus k. Equ. (3.12) finally follows from the fact that $A(\mathbf{k}) = A(-\mathbf{k})$.

2. Integration over the stress field

We now detail the steps leading to (3.27), which differ little from above. Because of our considering the Norton viscoplastic potential (which does not depend on σ_{m}) for explicit applications, the matrix $\mathbb{M}_1$ given in (3.30) is not invertible in $L(4s)$, but only in $L(4s')$. As a consequence [cf. (C2a)], $\overline{\overline{\mathbb{M}}}$ is not invertible in $L(4s, \mathrm{rep})$, but only in $L(4s', \text{rep})$. This leads to peculiarities that have to be taken into account but does not change the principle of the calculation. The equivalent of formula (3.27) for $\overline{\overline{\mathbb{M}}}$ invertible in $L(4s, \text{rep})$ is provided hereafter for the sake of completeness.

The Dirac distribution implementing the constraint $\boldsymbol{\nabla}.\mathsf{s} = \mathbf{0}$ in the measure $\tilde{\mathcal{D}}_s\mathsf{s}$ is exponentiated by the introduction of a vector field $\boldsymbol{\lambda}$ as:

$$\begin{aligned} \delta(\boldsymbol{\nabla}.\mathsf{s}) &\propto \int \mathcal{D}\boldsymbol{\lambda}\, \exp\left(i \int d\mathbf{x}\, \boldsymbol{\lambda}(\mathbf{x}).[\boldsymbol{\nabla}.\mathsf{s}(\mathbf{x})]\right) \\ &= \int \mathcal{D}\boldsymbol{\lambda}\, \exp\left(i \frac{1}{V} \sum_{\mathbf{k}\in\mathcal{K}+} \left\{\mathsf{s}^{(1)}(\mathbf{k}) : [\mathbf{k}\, \boldsymbol{\lambda}^{(2)}(\mathbf{k})] - \mathsf{s}^{(2)}(\mathbf{k}) : [\mathbf{k}\, \boldsymbol{\lambda}^{(1)}(\mathbf{k})]\right\}\right). \end{aligned} \tag{D10}$$

A reasoning paralleling that in Sec. D 1 yields:

$$\langle Z^r_{\text{us}} \rangle \propto \langle e^{-r\beta \frac{v}{V} \psi_{\mathbf{x}}(\boldsymbol{\Sigma})} \rangle^{\frac{V}{v}} \prod_{\mathbf{k}\in\mathcal{K}+} |A(\mathbf{k})|^2, \tag{D11}$$

where now

$$A(\mathbf{k}) = \int \left(\prod_\alpha d\boldsymbol{\lambda}^\alpha d_s\mathsf{s}^\alpha\right) \exp\left[-\frac{1}{2}\frac{\beta}{V^2} \sum_{\alpha\gamma} \mathsf{s}^\alpha : \mathbb{M}^{\alpha\gamma} : \mathsf{s}^\gamma + \frac{i}{V} \sum_\alpha \mathsf{s}^\alpha : (\mathbf{k}\, \boldsymbol{\lambda}^\alpha)\right] \tag{D12a}$$

$$\propto \det_{4s',\text{rep}} (\overline{\overline{\mathbb{M}}})^{-1/2} \int \left(\prod_\alpha d\boldsymbol{\lambda}^\alpha_{[\perp\hat{\mathbf{k}}]}\right) \exp\left\{-\frac{1}{2}\frac{k^2}{\beta} \sum_{\alpha\gamma} \boldsymbol{\lambda}^\alpha . \left[\hat{\mathbf{k}} . \left(\mathbb{M}^{-1_{4s',\text{rep}}}\right)^{\alpha\gamma} . \hat{\mathbf{k}}\right] . \boldsymbol{\lambda}^\gamma\right\} \tag{D12b}$$

$$\propto \det_{4s',\text{rep}} (\overline{\overline{\mathbb{M}}})^{-1/2} \det_{2[\perp\hat{\mathbf{k}}],\text{rep}} \left[\mathsf{Q}_{\hat{\mathbf{k}}} . \left(\hat{\mathbf{k}} . \overline{\overline{\mathbb{M}}}^{-1_{4s',\text{rep}}} . \hat{\mathbf{k}}\right) . \mathsf{Q}_{\hat{\mathbf{k}}}\right]^{-1/2}, \tag{D12c}$$

and $\mathsf{Q}_{\hat{\mathbf{k}}} = \mathsf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}$, and $\hat{\mathbf{k}} = \mathbf{k}/k$. In the first step from (D12a) to (D12b), we used the formula (B14) generalized to the replica space. The resulting factor $\delta(\text{tr}_2\, \mathbf{k}\boldsymbol{\lambda}) \propto \delta(\hat{\mathbf{k}}.\boldsymbol{\lambda})$ was absorbed in the measure $d\boldsymbol{\lambda}^\alpha_{[\perp\hat{\mathbf{k}}]}$ defined in (B5). Next, we appealed to (B7), also extended to replica space. Equ. (3.27) follows. Had $\overline{\overline{\mathbb{M}}}$ been invertible in $L(4s, \text{rep})$, one would have found

$$\langle Z^r \rangle \propto \langle e^{-r\beta \frac{v}{V} \psi_{\mathbf{x}}(\boldsymbol{\Sigma})} \rangle^{\frac{V}{v}} \prod_{\mathbf{k}\neq\mathbf{0}} \left\{\det_{4s,\text{rep}} (\overline{\overline{\mathbb{M}}}) \det_{2,\text{rep}} \left[\left(\hat{\mathbf{k}} . \overline{\overline{\mathbb{M}}}^{-1_{4s,\text{rep}}} . \hat{\mathbf{k}}\right)\right]\right\}^{-1/2} \tag{D13}$$

instead.

APPENDIX E: THE LEGENDRE TRANSFORM

Though the plastic dissipation potential (3.24) and the viscoplastic potential (3.34) have been obtained for different boundary conditions, we show here that they are linked by the Legendre transform (2.8).

Let us deduce (3.24) from (3.34), for instance, assuming that (2.8) holds between both. We write $\Phi(\mathsf{D}) = \Phi_0(\mathsf{D}) + \delta\Phi(\mathsf{D})$, $\Psi(\boldsymbol{\Sigma}) = \Psi_0(\boldsymbol{\Sigma}) + \delta\Psi(\boldsymbol{\Sigma})$, where $\Phi_0(\mathsf{D}) = \langle \phi_{\mathbf{x}}(\mathsf{D}) \rangle$, $\Psi_0(\boldsymbol{\Sigma}) = \langle \psi_{\mathbf{x}}(\boldsymbol{\Sigma}) \rangle$ are the leading terms in the perturbative expansions, and $\delta\Phi(\mathsf{D})$ and $\delta\Psi(\boldsymbol{\Sigma})$ are the corrective terms. It is straightforward to check that $\Phi_0(\mathsf{D})$ and $\Psi_0(\boldsymbol{\Sigma})$ are Legendre duals: $\Phi_0(\mathsf{D}) + \Psi_0(\boldsymbol{\Sigma}) = \boldsymbol{\Sigma} : \mathsf{D}$. Hence

$$\Phi(\mathsf{D}) = \boldsymbol{\Sigma} : \mathsf{D} - \Psi_0(\boldsymbol{\Sigma}) - \delta\Psi(\boldsymbol{\Sigma}) = \Phi_0(\mathsf{D}) - \delta\Psi(\boldsymbol{\Sigma}), \tag{E1}$$

so that $\delta\Phi(\mathsf{D}) = -\delta\Psi(\boldsymbol{\Sigma})$. Moreover, since $\boldsymbol{\Sigma} = \partial\Phi(\mathsf{D})/\partial\mathsf{D} = \partial\Phi_0(\mathsf{D})/\partial\mathsf{D} + \partial\delta\Phi(\mathsf{D})/\partial\mathsf{D}$, we obtain to second order

$$\Phi(\mathsf{D}) \simeq \Phi_0(\mathsf{D}) - \delta\Psi\big(\partial\Phi_0(\mathsf{D})/\partial\mathsf{D}\big). \tag{E2}$$

The Legendre transform of $\Psi_0(\boldsymbol{\Sigma}) = \langle \omega_n \rangle \Sigma_{\text{eq}}^{n+1}/(n+1)$ reads, with $m = 1/n$:

$$\Phi_0(\mathsf{D}) = \langle \omega_n \rangle^{-m} D_{\text{eq}}^{m+1}/(m+1), \quad \underset{2}{\text{tr}}(\mathsf{D}) = 0. \tag{E3}$$

In addition,

$$\mathbf{\Sigma}' \simeq \frac{d-1}{d}\langle\omega_n\rangle^{-m} D_{\rm eq}^{m-1}\mathsf{D}, \qquad \Sigma_{\rm eq} \simeq \langle\omega_n\rangle^{-m} D_{\rm eq}^{m}. \tag{E4}$$

These expressions for $\mathbf{\Sigma}'$ and $\Sigma_{\rm eq}$ are correct to first order only. However, symmetry considerations show that $\mathbf{\Sigma}'$ is always proportional to D. Thus $\hat{\mathbf{\Sigma}} = \hat{\mathsf{D}}$, whence from the definitions (3.21) and (3.32),

$$\Sigma_{\hat{\mathbf{k}}} = 1 - 2D_{\hat{\mathbf{k}}}. \tag{E5}$$

The weak-disorder expansion of $\Phi(\mathsf{D})$ has to be expressed in terms of $\theta_m = \omega_n^{-m} = \langle\theta_m\rangle + \delta\theta_m$. To second order, we have

$$\langle\omega_n\rangle \simeq \langle\theta_m\rangle^{-1/m}\left[1 + \frac{1}{2}\frac{m+1}{m^2}\frac{\langle\delta\theta_m^2\rangle}{\langle\theta_m\rangle^2}\right], \qquad \frac{\langle\delta\omega_n^2\rangle}{\langle\omega_n\rangle^2} \simeq \frac{1}{m^2}\frac{\langle\delta\theta_m^2\rangle}{\langle\theta_m\rangle^2}. \tag{E6}$$

Gathering these results into the perturbative expansion (E2) of $\Phi(\mathsf{D})$ finally yields (3.24) back, as announced.

---

[1] Barthlmy, M. and Orland, H. (1993) Replica field theory for composite media. *J. Phys. I (France)* **3** 2171–2177.
[2] Barthlmy, M. and Orland, H. (1998) A path-integral approach to effective non-linear medium *Eur. Phys. J. B* **6** 537-541.
[3] Bender, C. M. and Orszag, S. A. (1978) *Advanced mathematical methods for scientists and engineers*, McGraw-Hill, Singapore.
[4] Bergman, D. J. and Lee, H.-C. (1998) Comment on "Strongly nonlinear composite dielectrics: A perturbation method for finding the potential field and bulk effective properties". Unpublished.
[5] Blumenfeld, R. and Bergman, D. J. (1989) Exact calculation to second order of the effective dielectric constant of a strongly nonlinear inhomogeneous composite. *Phys. Rev. B* **40**, 1987–1989.
[6] Blumenfeld, R. and Bergman, D. J. (1991) Strongly nonlinear composite dielectrics: A perturbation method for finding the potential field and bulk effective properties. *Phys. Rev. B* **44**, 7378–7386.
[7] Budianski, B. (1965) On the elastic moduli of some heterogeneous materials. *J. Mech. Phys. Solids* **13** 223-227.
[8] Edwards, S. F. and Anderson, P. W. (1975) Theory of Spin Glasses. *J. Phys. F* **5** 965–974.
[9] Gilormini, P. (1995) Insuffisance de l'extension classique du modèle autocohérent au comportement non-linéaire. C. R. Acad. Sci. Paris, Série IIb 320 115-122.
[10] Hashin, Z. and Shtrikman, S. (1962a) On some variational principles in anisotropic and nonhomogeneous elasticity. *J. Mech. Phys. Sol.* **10**, 335–342.
[11] Hashin, Z. and Shtrikman, S. (1962b) A variational approach to the theory of the elastic behavior of polycrystals. *J. Mech. Phys. Sol.* **10**, 343–352.
[12] Hill, R. (1963) Elastic Properties of Reinforced Solids: Some Theoretical Principles. *J. Mech. Phys. Sol.* **11**, 357–372.
[13] Kleinert, H. (1995) *Path Integrals*, 2nd Edition, World Scientific, Singapore.
[14] Levy, O. and Bergman, D. J. (1994) Critical behavior of the weakly nonlinear conductivity and flicker noise of two-component composites, *Phys. Rev. B* **50**, 3652–3660.
[15] Mzard, M., Parisi, G. and Virasoro, M. (1987) *Spin Glass Theory and Beyond*, World Scientific, Singapore.
[16] Parcollet, O., Barthlmy, M. and Zrah, G. (1996) Replica treatment of the effective elastic bahavior of a composite. *Phys. Rev. B* **53**, 3161–3170.
[17] Parisi, G. (1984) An introduction to the statistical mechanics of amorphous systems. In *Les Houches, Session XXXIX , 1982 – Développements Récents en Théorie des Champs et Mécanique Statistique / Recent Advances in Field Theory and Statistical Mechanics* (J.-B. Zuber and R. Stora, eds.). North Holland, Amsterdam, pp. 473–523.
[18] Pellegrini, Y.-P., Barthélémy, M. (1999) Self-Consistent Effective-Medium Approximations with Path Integrals. Submitted.
[19] Ponte-Castañeda, P. (1991) The effective mechanical properties of Nonlinear Isotropic Composites. *J. Mech. Phys. Sol.* **39**, 45–71.
[20] Ponte-Casatñeda, P. and Suquet, P. (1995) On the effective mechanical behavior of weakly inhomogeneous nonlinear materials. *Eur. J. Mech., A/Solids* **14**, 205–236.
[21] Ponte-Casatñeda, P. and Kailasam, M. (1997) Nonlinear electrical conductivity in heterogeneous media. *Proc. R. Soc. London* **453**, 793–816.
[22] Nebozhyn, M. V. and Ponte-Castañeda, P. (1998) Exact second-order estimates of the self-sonsistent type for nonlinear composite materials. *Mechanics of Materials* **28**, 9–22.
[23] Ponte-Castañeda, P. and Suquet, P. (1998) Nonlinear Composites. *Adv. Appl. Mech.* **34**, 171–302, and references therein.

[24] Sahimi, M. (1998) Non-linear and non-local transport processes in heterogeneous media: from long-range correlated percolation to fracture and materials breakdown. *Phys. Rep.* **306** 213–395, and references therein.

[25] Stroud, D. and Hui, P. M. (1988) Nonlinear susceptibilities of granular matter. *Phys. Rev. B* **37** 8719–8724.

[26] Suquet, P. (1987) Elements of homogeneization for inelastic solid mechanics. In *Homogeneization techniques for composite media* (E. Sanchez-Palencia and A. Zaoui, eds.). Lecture Notes in Physics **272**, Springer Verlag, Berlin, pp. 193–278.

[27] Suquet, P. and Ponte-Castañeda, P. (1993) Small-contrast perturbation expansions for the effective properties of nonlinear composites. *C. R. Acad. Sci. (Paris) Ser. II* **317**, 1515–1522.

[28] Suquet, P. (1993) Overall potentials and extremal surfaces of power-law or ideally plastic materials. *J. Mech. Phys. Sol* **41**, 981–1002.

[29] Talbot, D. R. S. and Willis, J. R. (1985) Variational Principles for Inhomogeneous Nonlinear Media. *IMA J. Appl. Math.* **35** 39–54.

[30] Willis, J. R. (1977) Bounds and self-consistent estimetes for the overall properties of anisotropic composites. *J. Mech. Phys. Solids* **25** 185-202.

[31] Willis, J. R. (1989) The structure of overall constitutive relations for a class of nonlinear composites. *IMA J. Appl. Math.* **43** 231–242.