## <span id="page-0-0"></span>Superconductivity and Quantum Spin Disorder in Cuprates

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(April 26, 2024)

A fundamental connection between superconductivity and quantum spin fluctuations in underdoped cuprates, is revealed. A variational calculation shows that Cooper pair hopping strongly reduces the local magnetization  $m_0$ . This effect pertains to recent neutron scattering and muon spin rotation measurements in which  $m_0$  varies weakly with hole doping in the poorly conducting regime, but drops precipitously above the onset of superconductivity.

74.20.Mn, 75.10.Jm, 71.10.Fd

When holes are introduced into the copper oxide planes of high  $T_c$  cuprates, spin and charge correlations change dramatically. The *local* magnetization  $m_0$ , measured by $\mu$ SR [\[1](#page-3-0)] and elastic neutron scattering [[2\]](#page-3-0) on e.g.  $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ , reveals a qualitative difference between the insulating and superconducting phases:  $m_0$  is rather insensitive to doping in the poorly conducting regime  $0 \leq x \leq 0.06$ , but drops precipitously above the onset of superconductivity at  $x > 0.06$ , becoming undetectable at optimal doping  $x \approx 0.15$ . Theoretically, holes can cause dilution and frustration in the Heisenberg antiferromagnet, which create spin textures: either random ("spin glass") or with ordering wavevector away from  $(\pi, \pi)$  (sometimes called "stripes") [\[3](#page-3-0)]. However, the apparent reduction of local magnetization by the onset of superonductivity, is a novel and poorly understood effect. Theory must go beyond purely magnetic models, and involve the superconducting degrees of freedom.

We find that this problem is amenable to a variational approach, using hole-doped Resonating Valence Bonds (RVB) states, originally suggested by Anderson for the spin correlations of high  $T_c$  cuprates [\[5–7](#page-3-0)].

These RVB states are excellent trial states for doped Mott insulators, with large Hubbard repulsion U:

(i) Configurations with doubly occupied sites are excluded.

(ii) Marshall's sign criterion for the magnetic energy [\[8](#page-3-0)] is satisfied, and Heisenberg antiferromagnetism at zero doping is accurately recovered.

(iii) For doped systems, spin and charge correlations are parameterized independently, without explicit spin nor gauge symmetry breaking.

These are important advantages over commonly used Spin Density Wave, Hartree-Fock and BCS wavefunctions for the antiferromagnet, metal and superconducting phases respectively. RVB states permit an unbiased determination of ground state spin and charge correlations appropriate for the cuprates.

A phenomenological low energy effective Hamiltonian is used, with two major components: Heisenberg interaction for spins, and single or Cooper pair hoppingng kinetic energy for fermion holes.

Our key results are as follows:

(i) For the magnetic energy alone, the local magneti-

zation  $m_0$  is weakly dependent on doping concentration. This holds independently of inter-hole correlations for either randomly localized or extended states.

(ii) In contrast to (i),  $m_0$  is strongly reduced by the kinetic energy of Cooper pair hopping, which correlates the reduction of  $m_0$  with the rise of superconducting stiffness, andhence [[9\]](#page-3-0) the transition temperature  $T_c$ .

Our results agree with the experimentally reported correlation between  $m_0$  and  $T_c$  [\[1,2](#page-3-0)]. This relation appears to be weakly dependent on the precise hole density. A brief discussion concludes the paper.

The Wavefunctions: The hole-doped RVB states are compactly defined by

$$
\Psi[u, v; x] = \mathcal{P}_G(x)\overline{\psi}[u, v]
$$

$$
\overline{\psi}[u, v] \equiv \exp\left(\sum_{ij} \left(v_{ij} f_i^{\dagger} f_j^{\dagger} + u_{ij} (a_i^{\dagger} b_j^{\dagger} - b_i^{\dagger} a_j^{\dagger})\right)|0\rangle,
$$

$$
\mathcal{P}_G(x) \equiv \delta\left(\sum_i n_i^f - xL^2\right) \prod_i \delta\left(n_i^a + n_i^b + n_i^f - 1\right). (1)
$$

Where  $a_i^{\dagger}, b_i^{\dagger}$  and  $f_i^{\dagger}$  are Schwinger bosons and hole fermions respectively [\[10\]](#page-3-0),  $i = 1, \dots L^2$  is a site index on the square lattice, and  $\mathcal{P}_G(x)$  is the Gutzwiller projection onto states with no double occupancies. As a result of the projection,  $\Psi$  can be written as a sum over bond configuations of singlets and hole pairs which cover the lattice as depicted in Fig. 1.

 $u(\mathbf{r}_{ij})$  and  $v(\mathbf{r}_{ij})$  are *independent* spin and hole bond parameters respectively.  $u_{ij} \geq 0$  connects i on sublattice A to  $j \in B$  respectively, which ensures Marshall's sign.



FIG. 1. A bond configuration in the doped RVB states  $\Psi[u, v]$ . Solid (empty) circles represent spins (holes) with bond correlations  $u_{ij}$  ( $v_{kl}$ ).

<span id="page-1-0"></span>The expectation value of an observable  $O$  is computed by a sum over loop coverings:

$$
\langle O \rangle = \sum_{\gamma, \Lambda_{\gamma}} W_{\Lambda, \gamma} O_{\Lambda, \gamma},
$$
  

$$
W_{\Lambda, \gamma} = \frac{1}{\langle \Psi[u, v] | \Psi[u, v] \rangle} \det_{ij \in \gamma}^{2} |v_{ij}| \prod_{\lambda \in \Lambda_{\gamma}} \left( 2 \prod_{(i,j) \in \lambda} u_{ij} \right). (2)
$$

 $\Lambda_{\gamma}$  denotes a list of directed loops  $\{\lambda_{\alpha}\}\$  which cover the lattice except for subset  $\gamma$  of  $xL^2$  hole sites. W are positive Boltzmann weights, with which the Monte Carlo Metropolis step is defined following Refs. [\[6,7](#page-3-0)]. Ergodicity and convergence of our program was checked against precise transfer matrix results[[13\]](#page-3-0).

The Gutzwiller Approximation (GA) amounts to dropping the projector  $\mathcal{P}(x)$  in state [\(1](#page-0-0)) and setting  $\Psi \to \psi$ , after adjusting the overall normalization of  $u$  and  $v$  to satisfy the global constraints  $\langle n_i^a + n_i^b \rangle = (1 - x)$ , and  $\langle n_i^f \rangle = x$ .  $\bar{\psi}$  is a Fock state of decoupled spins and holes, with easily computable correlations [\[11,12](#page-3-0)].

For the spin correlations we use power law decaying functions  $u_p(r) \equiv 1/r^p$ . The single variational parameter p determines the long range spin correlations and local magnetization [\[14](#page-3-0)]. (Other forms for  $u(r)$ , with qualitatively similar results, will be described elsewhere  $[12]$  $[12]$ ).

We discuss four cases of inter-hole correlations:

$$
\begin{array}{rcl}\nv_{ins}^{\gamma}(\mathbf{r}_{ij}) &=& \begin{cases} 1 & (i,j) \in \gamma \\ 0 & (i,j) \notin \gamma \end{cases} \\
v_{met}(\mathbf{r}) &=& 1/L^2 \sum_{\mathbf{k} \in \Sigma} e^{-i\mathbf{k} \cdot \mathbf{r}} \\
v_{\alpha}(\mathbf{r}) &=& \sum_{\hat{\eta}} c_{\alpha}(\hat{\eta}) \delta_{\mathbf{r}, \hat{\eta}}, \quad \alpha = s, d\n\end{array} \tag{3}
$$

where  $\hat{\eta}$  are nearest neighbor vectors on the square lattice,  $c_s = 1$  and  $c_d = \hat{\eta}_x^2 - \hat{\eta}_y^2$ .

 $v_{ins}^{\gamma}$  puts the  $xL^2$  holes on random sites. This state describes an insulator with disordered localized charges.

 $v_{met}$  has filled Fermi pockets  $\Sigma$ , containing  $xL^2$  occupied **k**-states centered around  $(\pm \pi/2, \pm \pi/2)$  in the Brillouin zone [\[15](#page-3-0)]. It describes weakly interacting holes in a "metallic" state. In real space,  $v_{met}(\mathbf{r}_{ij})$  decays slowly as  $\sim r^{-3/2}$ . Correlations in this state were previously computed by Bonesteel and Wilkins [\[7](#page-3-0)].

 $v_s$  and  $v_d$  describe tightly bound hole pairs in relative s and d-wave symmetry respectively.



FIG. 2. (a) The local magnetization squared of doped RVB wavefunctions  $\Psi[u_p, v]$  versus the variational power p, defined by the bond parameters  $u_p(r) = 1/r^p$ . Lattice size is  $40 \times 40$ , and hole concentration is 10%. Results are agree well with the Gutzwiller approximation (solid line). The hole bond parameters v are defined in Eq.3. Note that  $m_0^2$  is weakly dependent on v; the data for  $v = v_s$ , overlaps that of  $v_d$ . (b) Finite size scaling of  $m_0(L)$  for  $p = 3.3$  which indicates vanishing local magnetization at  $L \to \infty$ .

Order parameters: The local magnetization  $m_0(L)$  is defined by  $m_0^2 = 1/L^2 \sum_j \langle \mathbf{S}_i \mathbf{S}_j \rangle_p e^{-i(\pi,\pi)\mathbf{r}_{ij}}$ . In Fig.2(a),  $m_0^2(p)$  for  $\Psi[u_p, v; x = 0.1]$  is plotted for various choices of v. The GA (solid line) seems to work well for  $m_0(p)$ . Finite size scaling in Fig.2(b) indicates vanishing long range order  $m_0 \rightarrow 0$  at  $p_c = 3.3$ , which lowers the bound given previously by Ref. [\[6](#page-3-0)]: at  $p_c \leq 5$ . The GA approximation at $L \to \infty$  suggests that  $p_c \leq 3$  [[12\]](#page-3-0).

The superconducting singlet order parameters are

$$
\Delta_i^{s,d} = \sum_{\hat{\eta}} c_{s,d}(\hat{\eta}) \Delta_{i,i+\hat{\eta}}
$$

$$
\Delta_{ij} = f_i^{\dagger} f_j^{\dagger} (a_i b_j - b_i a_j) / \sqrt{2}
$$
(4)

By gauge invariance imposed by the Gutzwiller projector,  $\langle \Delta_{s,d} \rangle = 0$ . However,  $\Psi[u, v_{s(d)}; x > 0]$  describe true s  $(d)$ -wave superconductors as seen by the (off-diagonal) long range order in  $\Delta_{s,d}$  [[12\]](#page-3-0).

In contrast, the insulator states  $\Psi[u, v_{ins}, x]$  and the "metallic" states  $\Psi[u, v_{met}, x]$  have no long range superconducting order of either symmetry[[12\]](#page-3-0).

Effective Hamiltonians: Magnetic order is driven by the diluted Heisenberg model [\[14](#page-3-0)],

$$
\mathcal{H}^J = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j (1 - f_i^{\dagger} f_i)(1 - f_j^{\dagger} f_j)
$$
(5)

where e.g.  $S^x + iS^y \equiv a^{\dagger}b$ . In Fig. [3](#page-2-0) the expectation value  $E_{mag}(p) = \langle \mathcal{H}^J \rangle$  is plotted as a function of  $m_0^2(p)$  for  $x = 0.1$  and various choices of v from (3). Within numerical errors all states minimize  $\mathcal{H}^J$  at around <span id="page-2-0"></span> $p_{min} \approx 2.7$ , which by Fig. [2\(](#page-1-0)a) yields local magnetization of  $m_0^2 = 0.08 \approx m_0^2(0)(1 - 2x)$ , where  $m_0(0)$  agrees with the ground state local magnetization of the undoped Heisenberg model [\[16](#page-3-0)]. We have found that  $p_{min} \approx 2.7$ appears to be independent of x in for  $0 \leq x \leq 0.15$ . Thus we conclude that aside from the trivial kinematical constraints, the hole density and correlations have little effect on the magnetic energy at low doping.

A single hole hopping in the antiferromagnetic background has been shown by semiclassical arguments [[17,10\]](#page-3-0), to be effectively restricted at low energies to hopping between sites on the same sublattice: Next we consider the single hole hopping process

$$
\mathcal{H}^{t'} = \sum_{\langle ik \rangle \in A, B} t'_{ik} f_i^{\dagger} f_k (a_k^{\dagger} a_i + b_k^{\dagger} b_i)
$$
(6)

where  $i, k$  are removed by two adjacent lattice steps, and  $t' > 0$ . Unconstrained, the single hole ground state of  $H^{t'}$  has momentum on the edge of the magnetic Brillouin zone, in agreement with exact diagonalization of  $t-J$ clusters [[18\]](#page-3-0). (Note: *inter*-sublattice hopping (the t-term in the t-J model) is a high energy process in the antiferromagnetic background. It primarily renormalizes  $t'$ and the quasiparticle's short distance structure [[17,10\]](#page-3-0).

The single holes hopping (6) prefers the metallic state  $v = v_{met}$  over the superconductor  $v = v_s, v_d$  [\[12\]](#page-3-0). It also prefers longer range  $u(r)$  and thus actually *enhances* magnetic order at low doping. This is a type of a Nagaoka effect, where mobile holes seperately polarize each of the sublattices ferromagnetically.

Now we consider Cooper pairs hopping terms

$$
\mathcal{H}^{J'} = -J'\left(\sum_{ijk}\Delta_{ij}^{\dagger}\Delta_{ik} + \sum_{\langle ij\rangle,\tilde{i}\tilde{j}}\Delta_{ij}^{\dagger}\Delta_{\tilde{i}\tilde{j}}\right) \tag{7}
$$

The first term is derived from the large U Hubbard model to order  $J' = t^2/U$  [\[10](#page-3-0)]. It includes a *rotation* of the singlet pair, which prefers  $v_d$  over  $v_s$ . The second term is a parallel translation of singlets. It prefers superconductivity with  $v = v_d$  over metallic states with  $v = v_{met}$ [[19\]](#page-3-0).

In Fig. 3 the ground state energy  $E_{ph}$  of (7) is plotted for  $v = v_d$ ,  $x=0.1$  and  $L=40$ . For  $v = v_s$ ,  $E_{ph} > 0$ . The variational energy is minimized at  $p = 3.35$ , which by the finite size scaling of Fig[.2](#page-1-0)(b) indicates vanishing  $m_0$  at large L. Note the striking difference between the minima of  $E_{mag}$  and  $E_{ph}$ . Thus, Cooper pair hopping drives the groundstate toward a spin liquid phase!



FIG. 3. The magnetic energy  $E_{mag}$  ([5](#page-1-0)) and Cooper pair hopping energy  $E_{ph}$  (7) versus local magnetization squared  $m_0^2$ , used as a variational parameter. The density of holes is 0.1 and lattice size is  $L = 40$ . The magnetic energy is minimized at  $m_0^2 \approx 0.08$ , consistent with a diluted quantum Heisenberg antiferromagnet, and is weakly dependent on inter-hole correlations. The points of  $E_{mag}(v_s)$  overlaps  $E_{mag}(v_d)$ . In contrast, Cooper pair hopping prefers vanishing  $m_0$  at  $L \rightarrow \infty$ .

A simple explanation is that pairs can hop with greater overlap when parallel bonds have maximum singlet components. When  $u(r)$  is longer ranged, triplet contributions are larger, which inhibits pair delocalization. Incidentally, the Gutzwiller approximation (GA) fails to predict this effect since it decouples the local correlations between spins and hole pairs.

Since  $\mathcal{H}^{J'}$  is the effective model which drives superconductivity it produces phase stiffness, which in the continuum approximation is given by

$$
\mathcal{H}^{J'} \approx \frac{V_0}{2} \int d^2x (\nabla \phi_i)^2 \tag{8}
$$

The stiffness constant  $V_0$  can be determined variationally from the RVB states, by imposing a uniform gauge field twist on the bond parameters  $v_{i,j} \rightarrow v_{i,j} exp(i(x_i +$  $(x_j)\phi/2L$ ) and measuring  $E_{ph}(\phi)$  to find  $V_0 = d^2E_{ph}/d\phi^2$ . Following Ref. [\[9](#page-3-0)], at low doping for the square lattice  $V_0$  is roughly equal to  $T_c$ .

results In Fig.([4\)](#page-3-0) we show our main result: The staggered magnetization  $m_0$  for  $\mathcal{H}^J + \mathcal{H}^{J'}$  is plotted against the superconducting to magnetic stiffness ratio  $V_0/J$  for different doping concentrations  $x = 0.05, 0.1, 0.15$ .

Two primary observations are made: (i) The local magnetization is sharply reduced at relatively low superconducting stiffness (and  $T_c/J$ ). (ii) The relation between  $m_0$  and  $V_0/J$  appears to be weakly dependent on the precise hole concentration.

<span id="page-3-0"></span>

FIG. 4. The relation between thermodynamic local magnetization  $m_0^{L=\infty}$  and superconducting phase stiffness  $V_0$  (related to  $T_c$ , see text). J is the Heisenberg exchange energy. The points are considered upper bounds on  $m_0$ , which may even vanish for  $V_0/J \geq 0.2$ .

Discussion: Because of finite size uncertainty,  $m_0$  in Fig.(4) is an upper bound on the thermodynamic local magnetization. The GA extrapolation suggests that  $m_0$ may actually vanish already  $V_0/J \geq 0.2$ . This is in qualitative agreement with the doping dependent of the local magnetization measured by Refs. [2,1], which diminishes rapidly above the onset of superconductivity.

In a quantized theory of stripes [20], mechanisms for diminishing  $m_0$  assume anisotropic magnetic couplings, or fluctuating anti-phase domain walls. A direct connection between superconductivity and  $m_0$  is not obvious in these approaches.

In a recent *projected*  $SO(5)$  theory [21], spins and holepairs dynamics have been considered with excluded double occupancies. A variational relation is obtained between superconducting stiffness and the magnetic order parameter, which resembles the results of this paper.

Acknowledgments: Useful conversations with A. Aharony, O. Entin, C. Henley, S. Kivelson and S-C. Zhang, are gratefully acknowledged. MH thanks Taub computing center for support. AA is supported by the Israel Science Foundation and the Fund for Promotion of Research at Technion.

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