

A hidden Goldstone mechanism in the Kagomé lattice antiferromagnet.

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Abstract

In this paper, we study the phases of the Heisenberg model on the Kagomé lattice with antiferromagnetic nearest neighbour coupling J_1 and ferromagnetic next neighbour coupling J_2 . Analysing the long wavelength, low energy effective action that describes this model, we arrive at the phase diagram as a function of $\chi = \frac{J_2}{J_1}$. The interesting part of this phase diagram is that for small χ , which includes $\chi = 0$, there is a phase with no long range spin order and with gapless and spin zero low lying excitations. We discuss our results in the context of earlier, numerical and experimental work.

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1 Introduction

The Heisenberg antiferromagnet (HAF) in two dimensions has been widely studied in the last decade from several viewpoints. One main motivation for this study has been the possibility of encountering novel ordered and disordered groundstates. The nearest neighbour HAF on the Kagomé lattice (NNKLAF) is one system which is expected to show such interesting behaviour. This model has been studied experimentally, [1]- [4] and theoretically, [5] - [16], through several methods and is expected to have a spin disordered ground state. An interesting, recent study is an exact spectra analysis [12] of finite sized $J_1 - J_2$ model on the Kagomé lattice shows that as $J_2 \rightarrow 0$, there is a spin disordered groundstate with a gap for excitations with spin. This gap is filled with a large number of closely spaced singlet excitations which could collapse to the ground state in the limit of the system size tending to infinity.

An approach to the problem of two dimensional antiferromagnets which has yielded very good results is through a field theoretical sigma model description such as has been developed in [17, 18, 19]. The sigma model describes the large amplitude, long wavelength fluctuations of the spin system. It is therefore capable of modelling the ordered and the disordered phases of the system. Its validity only requires the existence of short range spin order. The application of this method to develop the field theory for the Kagomé antiferromagnet has been described in [15, 16]. This is the method that we use here to determine the phases of the $J_1 - J_2$ model on the Kagomé lattice and in particular to understand the ground state and low energy spectrum of the NNKLAF.

Among the several families of Kagomé lattice antiferromagnets studied experimentally are the jarosites and the magnetoplumbite like compound $SrCr_{8-x}Ga_{4+x}O_{19}$. In these compounds additional, next to nearest neighbour and interplanar, couplings seem to stabilise one or the other of the planar states. Because of this the compounds fall into two groups. In the iron jarosites, $KFe_3(OH)_6(XO_4)_2$ with $X = S$ or Cr [1] and in $KCr_3(OD)_6(SO_4)_2$ [2] which realise $S = \frac{5}{2}$ and $S = \frac{3}{2}$ Kagomé lattice antiferromagnets respectively, $\mathbf{q} = 0$ long range order has been observed. In the other group made up of deuterium jarosite, $D_3OF_3(OH)_6(SO_4)_2$ ($S = \frac{5}{2}$) [3] and $SrCr_{8-x}Ga_{4+x}O_{19}$ ($S = \frac{3}{2}$) [4], short ranged $\sqrt{3} \times \sqrt{3}$ order is found. In addition it is observed in these two compounds that the low temperature spe-

cific heat has a T^2 behaviour. The usual interpretation of such a behaviour is that there are gapless excitations in the low energy spectrum. Such a gapless excitation is usually a Goldstone mode resulting from the breaking of some continuous symmetry in the model. In this case the symmetry that can be broken is the $SO(3)$ spin rotational symmetry of the hamiltonian. However the neutron scattering experiments show that there is only short ranged $\sqrt{3} \times \sqrt{3}$ order in the groundstate thereby negating this possibility. Therefore there is no direct explanation for the low temperature specific heat data.

There is an explanation for this puzzle, which we have explored in [15], where we give the mechanism for getting a gapless mode even in systems where all the symmetries of the microscopic Hamiltonian are intact and there is no long range antiferromagnetic order. Namely that the low energy theory acquires an extra continuous symmetry which is not present in the microscopic model and this symmetry breaks, giving rise to a gapless boson. This is what we call a hidden Goldstone mechanism. We work out this mechanism in detail for the $J_1 - J_2$ model described below and analyse the phases as the ratio, $\chi \equiv \frac{J_2}{J_1}$, is varied.

This model is defined by the Hamiltonian,

$$H = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - J_2 \sum_{\{ij\}} \vec{S}_i \cdot \vec{S}_j \quad (1)$$

Where $\langle i, j \rangle$ implies that i and j belong to neighbouring sites and $\{i, j\}$ implies that i and j belong are next to nearest neighbours. For positive value of the next neighbour coupling J_2 the $\sqrt{3} \times \sqrt{3}$ state is picked out from among the numerous degenerate groundstates of the classical NNKLAFF. We write down a field theory in terms of the five relevant fields, which we identify from a preliminary spin wave analysis of this model. This theory is an improvement on the spin wave analysis since this allows for large amplitude fluctuations of these five parameters. The theory is symmetric under $SO(3)_R \times SO(2)_L$, where the $SO(3)_R$ is the spin rotation symmetry of the Hamiltonian and the $SO(2)_L$ is a special symmetry of the effective low energy action. We analyse this field theory using the large N expansion described in [20] and identify that there are two phase transitions as we move towards $J_2 = 0$, which is the case of the NNKLAFF. With reference to figure (4), for large χ , we find that the system is in the planar spiral phase, the ground state is the $\sqrt{3} \times \sqrt{3}$

state and the low lying excitations about this state are the three gapless, magnons. In terms of the field theory this involves the symmetry breaking pattern $SO(3)_R \times SO(2)_L \rightarrow SO(2)_L$. Reducing J_2 , thereby reducing χ , takes us into a completely disordered phase where all the symmetries are intact. Further reducing J_2 takes us into the non-coplanar phase where the symmetry breaking pattern is $SO(3)_R \times SO(2)_L \rightarrow SO(3)_R$. In this phase, since the $SO(3)_R$ symmetry is intact, all correlations of vector and tensor operators constructed out of the spins are short ranged and because of the breaking of the $SO(2)_L$ symmetry, there is one gapless, spin singlet goldstone mode. For reasons that will be clear later we call this the non-coplanar phase. At the lattice level this $SO(2)_L$ symmetry manifests itself as a discrete symmetry of rotation by $2\pi/3$ followed by a translation by one lattice vector and at the level of the low energy long wavelength effective action this gets enhanced to a continuous symmetry. The theme of this paper is how this hidden Goldstone mechanism gives an explanation of the behaviour of the group two compounds.

The plan of the paper is as follows. In section (2), we describe the $\vec{K} = 0$ spinwave analysis and isolate the five relevant low energy modes. In section (3), we extend the description of these low lying modes to include large fluctuations so that the five parameters regroup into a $SU(2)$ matrix valued field and a unit vector field. In section (4), we describe the transformation of these fields under the symmetry operations described above. In section(5) we describe the effective field theory which describes the model and has been derived in [15] and show that there are three distinct phases in the $J_1 - J_2$ Kagomé lattice model. The details of these phases, in particular the non-coplanar phase at $\chi = 0$ are described in section (6). Section (7) contains details of the behaviour of the correlation functions in this important phase and section (8) contains a discussion of our results in the context of earlier numerical studies.

2 Ground state and low energy modes

The model that we consider in this paper is the Heisenberg hamiltonian on the Kagomé lattice with nearest neighbour antiferromagnetic coupling with strength J_1 and next nearest neighbour couplings with strength $J_2 = \chi J_1$. This is the model defined in equation (1). The two types of bonds are

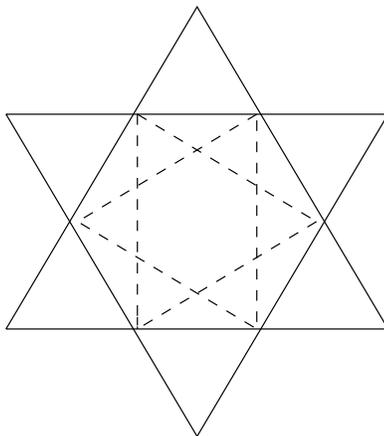


Figure 1: In the figure the dashed lines refer to bonds of strength $J_2 = \chi J_1$ and the bold lines to bonds of strength J_1 .

illustrated in figure (1). The hamiltonian can be rewritten (upto additive constants) as,

$$\frac{H}{J_1} = \frac{1}{2} \sum_{\Delta} (\sum_{it} \vec{S}_{it})^2 - \chi \frac{1}{2} \sum_{\Delta'} (\sum_{it'} \vec{S}_{it'})^2 \quad (2)$$

where, Δ denotes the nearest neighbour triangles and it their three vertices. Δ' denotes the next nearest neighbour triangles that lie in the hexagons and it' their vertices. It is clear from equation (2) that for all $\chi > 0$, the energy is minimised when the magnetisation of all the nearest neighbour triangles is zero and that of all the next nearest neighbour triangles is the maximum possible. This occurs for the $\sqrt{3} \times \sqrt{3}$ state shown in figure (2).

In order to identify the lowest lying excitations about this ground state, we do a spinwave analysis. In this analysis we treat the quantum fluctuations as rotations of the classical spin vectors, thereby mapping the spin variables \vec{S}_j on to bosonic variables P_j and Q_j .

In order to do this we rewrite the spins \vec{S}_j as,

$$\vec{S}_j = e^{\frac{-iw_j^a T^a}{\sqrt{s}}} \vec{S}_j^{cl} \quad (3)$$

where the T^a are the three generators of $SU(2)$ in the spin 1 representation. The $w_j^a = P_j E_j^a + Q_j Z^a$, P_j and Q_j being bosonic operators obeying the

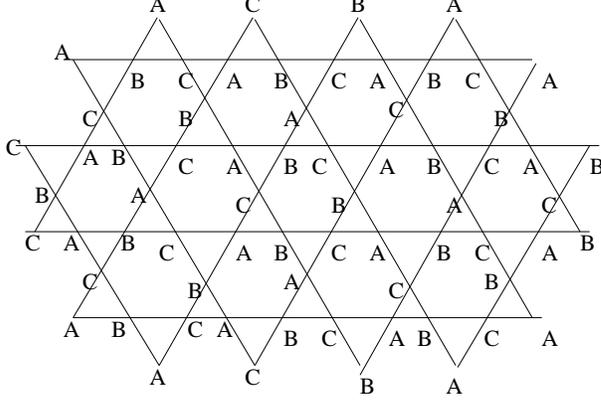


Figure 2: The figure shows the $\sqrt{3} \times \sqrt{3}$ state superposed on the Kagome lattice, where one choice for A, B and C is $\vec{S}_A = (1, 0, 0)$, $\vec{S}_B = (\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$, $\vec{S}_C = (\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0)$.

commutation rules $[P_j, Q_{j'}] = -i\hbar\delta_{j,j'}$. $\vec{S}_j^{cl} = S\hat{n}_j$, where the \hat{n}_j are arranged according to the $\sqrt{3} \times \sqrt{3}$ configuration and the set $\{\hat{n}_j, \hat{E}_j, \hat{Z}\}$ form an orthogonal triad at each site. Thereafter, for small fluctuations, putting these definitions into equation (3) and expanding to order $\frac{1}{S}$, we get,

$$\vec{S}_j = S\hat{n}_j(1 - \frac{P_j^2 + Q_j^2}{2S}) + \sqrt{S}\hat{E}_jP_j + \sqrt{S}\hat{Z}Q_j \quad (4)$$

Before proceeding, we notice that the lattice splits into magnetic unit cells consisting of 9 points each because of the periodicity of the lattice and the $\sqrt{3} \times \sqrt{3}$ groundstate. Therefore we expand our notation a bit, and replace \vec{S}_j equivalently by $\vec{S}_{Jj\beta}$. In this notation every lattice index j is replaced by one unit cell index such as J and two sublattice indices (j, β) . This labelling is shown in figure (3) which also shows the structure of each unit cell. We make such an expansion of $S_{Jj\beta}$ as in equation (4), keeping upto quadratic terms in the P and Q . We finally get the fluctuations hamiltonian in terms of the Fourier transformed variables $P_{Kj\beta}$ and $Q_{Kj\beta}$, which are defined as follows.

$$P_{Kj\beta} = \frac{1}{N} \sum_J P_{Jj\beta} e^{-i\vec{x}_J \cdot \vec{K}}$$

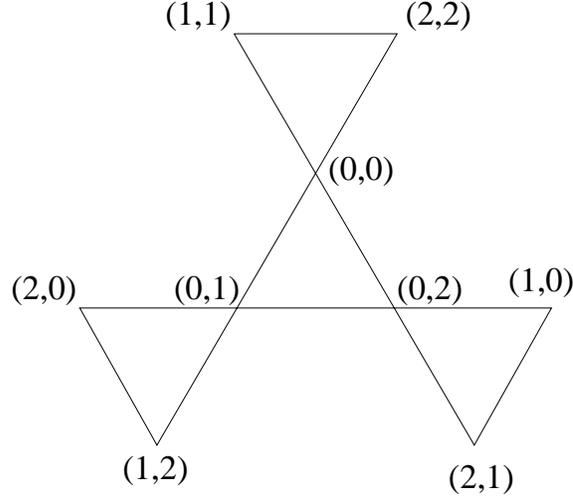


Figure 3:

$$Q_{Kj\beta} = \frac{1}{N} \sum_J Q_{Jj\beta} e^{-i\vec{x}_J \cdot \vec{K}} \quad (5)$$

In terms of the $\mathbf{P}_{\mathbf{K}}$ and $\mathbf{Q}_{\mathbf{K}}$ the hamiltonian reduces to a 9×9 block which is given by,

$$H_{sw} = \frac{1}{2} \sum_{\mathbf{K}} \mathbf{P}_{-\mathbf{K}}^T M^{-1} \mathbf{P}_{\mathbf{K}} + \frac{1}{2} \mathbf{Q}_{-\mathbf{K}}^T K \mathbf{Q}_{\mathbf{K}} \quad (6)$$

The eigenvalues and the eigenvectors of these matrix $\Omega^2 = M^{-1}K$ for $\vec{K} = 0$ give the gaps and normal modes of the system of oscillators. These matrices are given in the appendix (A). The 9 eigenvectors of Ω^2 , denoted by $\phi_{j\beta}$ are given by $\phi_{j\beta} = e_j \times e_\beta$, where,

$$\begin{aligned} e_0 &= \frac{1}{\sqrt{3}}(1, 1, 1) \\ e_1 &= \frac{1}{\sqrt{3}}(1, \alpha, \alpha^2) \\ e_2 &= \frac{1}{\sqrt{3}}(1, \alpha^2, \alpha) \end{aligned} \quad (7)$$

The eigenvalues corresponding to the eigenvectors $\phi_{j\beta}$ are $\omega_{j\beta}^2$. These fall

into three groups. $\omega_{00}^2 = \omega_{01}^2 = \omega_{02}^2 = 0$, $\omega_{11}^2 = \omega_{22}^2 = 18\chi(3\chi + 1)$ and $\omega_{10}^2 = \omega_{12}^2 = \omega_{20}^2 = \omega_{21}^2 = 36\chi^2 + 25\chi + 9/2$. Accordingly, we classify the first three modes as S-S modes since they are soft for all χ , the second pair as H-S modes since they are gapless for $\chi = 0$ and hard for $\chi \neq 0$ and the last four as the H-H modes as they are hard for all χ . In the large χ regime the S-S modes are the only relevant modes, while we need to take both the S-S modes and the H-S modes into consideration to describe low energy physics in the small χ domain.

3 Parametrisation of large amplitude fluctuations.

In this section, starting from the expression for the normal modes of the $\vec{K} = 0$ spinwave hamiltonian, we develop a parametrisation of the S-S and H-S modes for large amplitude fluctuations. These five modes, since they are gapless at the NNKLAF end, will govern the low energy physics, both at the NNKLAF end and close to it, i.e for small χ . For $\chi = 0$, some of these modes are dispersionless. This arises from the possibility of having local fluctuations about line defects and have been discussed in [13].

From the expression in equation (3) for the spins, putting in the forms of the eigenmodes for the five relevant modes as expressions for the $P_{j\beta}$ and $Q_{j\beta}$, we observe that for $\vec{K} = 0$, all the modes satisfy $\vec{S}_1 + \vec{S}_2 + \vec{S}_3 = \Delta\vec{S} = 0$, for the spins $\vec{S}_1, \vec{S}_2, \vec{S}_3$ lying on a triangular plaquette. For the dispersionless H-S modes, this identity continues to hold for $\vec{K} \neq 0$.

At the spin wave level, the expansion of $\vec{S}_{j\beta}$ to order $\frac{1}{S}$ indicates that we are looking at small fluctuations about the classical ordered configuration. For the purposes of the field theory we need to parametrise large amplitude fluctuations of these relevant modes. We now proceed to do this.

Though an exact treatment would involve deriving this for $\vec{K} \neq 0$, we use the zero \vec{K} expressions as an approximation. This is valid because we are interested only in the long wavelength excitations.

If the 9 spins of the unit cell may be thought of as a rigid unit, the three S-S modes correspond to the rotation of this unit about the three co-ordinate axes. This set of three modes can therefore be parametrised by a unitary matrix U_J , which brings about this rotation from the ground state configuration

to the body fixed frame of the rigid body. This is the usual interpretation given to the gapless Goldstone modes that occur in antiferromagnetic models.

Since the H-S modes cost zero energy at $\chi = 0$, they must leave the relative angles between the spins on each triangle intact. They could however, distort the angles between neighbouring triangles. Hence they create non planar configurations within the unit cell. We can parameterise these configurations as follows,

$$\vec{S}_{Jj\beta} = V_{Jj}^\dagger \hat{n}_\beta \quad (8)$$

V_{Jj} are rotation matrices that rotate the three spins (labelled by β) belonging to each $\sqrt{7} \times \sqrt{7}$ triangle (labelled by the same j) rigidly. They are not independent of each other but are constrained by the fact that the inter-unit cell triangles are also not distorted. It is difficult to solve for these constraints exactly. However, we can use the eigenfunctions of the H-S modes to obtain the following, approximate, solution.

$$V_{Jj} = \exp^{i\frac{2\pi}{3}jT^3} \exp^{i\frac{\pi}{3}\hat{m}_J \cdot \vec{T}} \exp^{-i\frac{2\pi}{3}jT^3} \exp^{-i\frac{\pi}{3}T^3} \quad (9)$$

When $\hat{m} = \hat{z}$, V_{Jj} reduces to the identity matrix and the spin configuration is undistorted. When \hat{m} deviates from the z-axis, we can show that the configurations produced by equations (8) and (9) are, upto quadratic order in the deviations, zero energy configurations in the $\chi \rightarrow 0$ limit. We therefore use equations (8) and (9) to approximate the long wavelength, large amplitude fluctuations of the H-S modes.

If the spins on the unit cell form a rigid unit and if the ground state configuration is thought of as a space fixed frame, then the S-S modes, parametrised by the U_J , rotate the spins to the body fixed frame. further the V_{Jj} cause a rotation in this body fixed frame, whose magnitude and direction are decided by \hat{m} which is a vector in this body fixed frame. $\vec{S}_{Jj\beta}$ is therefore given by,

$$\vec{S}_{Jj\beta} = U_J^\dagger V_{Jj}^\dagger \hat{n}_\beta \quad (10)$$

4 Order parameters and symmetries.

Now that we have a parametrisation of the large amplitude fluctuations of the five relevant modes, it is necessary to look at the symmetries of the microscopic model and how they act on the fields U_J and \hat{m}_J . There are

two important groups of transformations which leave the hamiltonian invariant. The first is a global $SO(3)$ group of rotations of each spin under the transformation

$$S_{Jj\beta}^a \rightarrow \Omega_R^{\dagger ab} S_{Jj\beta}^b \quad (11)$$

Since this is an overall rotation of the spins with respect to the space fixed frame, this simply adds on to the matrix U_J and leaves \hat{m}_J invariant, thus,

$$\begin{aligned} U_J &\rightarrow U_J \Omega_R \\ \hat{m}_J &\rightarrow \hat{m}_J \end{aligned} \quad (12)$$

Secondly, the hamiltonian is invariant under the rotation of each spin by $\frac{2\pi}{3}$ followed by a unit translation. Under this transformation the spins transform as follows,

$$S_{Jj\beta}^a \rightarrow R^{\beta\beta'} S_{Jj\beta'}^a \quad (13)$$

In the continuum limit this shows up as a combination of $U(1)$ rotations in both the space fixed frame and in the body fixed frame, under which the fields transform as follows,

$$\begin{aligned} U_J &\rightarrow \Omega_L U_J \\ \hat{m}_J &\rightarrow \Omega_L \hat{m}_J \end{aligned} \quad (14)$$

Looking at the way these rotations act on U_J , we call them $SO(3)_R$ and $SO(2)_L$ rotations respectively.

As mentioned in the previous section, the H-S modes create non-coplanar configurations in the NNKLAF. This is contrary to the effect of the S-S modes which are rigid rotations of the spins of the unit cell. Hence the effect of the H-S modes may be measured by defining an order parameter that leaves the spins on the triangular plaquettes rigid, yet distorting the planarity of adjoining triangles. A suitable candidate is the scalar triple product of three spins lying in a row on each unit cell.

We show in section (7) which follows, that this is a suitable order parameter with which to describe the phases of the NNKLAF.

5 Field theory and phases.

In our previous work, [15], we have derived the long wavelength, low energy field theory describing the deformed triangular antiferromagnet close to the Kagomé lattice limit. Based on the same considerations the same field theory would also describe the $J_1 - J_2$ model on the Kagomé lattice for J_2 close to zero. This action is given by the expression,

$$S[\phi_r^a, \hat{m}] = \int d^3x \partial_\mu \phi_r^a \partial_\mu \phi_r^a + \frac{1}{g_2} \partial_\mu \hat{m} \cdot \partial_\mu \hat{m} + V(m^z) \quad (15)$$

Along with the constraint,

$$\sum_r \phi_r^a \phi_r^b = \frac{1}{g_1} (1 + f(m^z)) \delta_{ab} \quad (16)$$

The interaction is built into the constraint which puts $\phi_r^a = \sqrt{\frac{1}{g_1} (1 + f(m^z))} \Phi_r^a$, where the fields Φ_r^a make up the columns of the matrix U and the function $f(m^z) = \alpha(1 - (m^z)^2)$. The potential $V(m^z) = \lambda_0((m^z)^2 - \eta_0)^2$. The parameters $\alpha, \lambda_0, \eta_0$ and g are all functions of χ .

In [15], we had shown that when g_1 is in the strong coupling regime and g_2 in the weak coupling regime, there exists a phase where the $SO(3)_R$ symmetry is unbroken and the $SO(2)_L$ is broken. We now give a simple large N formalism where the physics of this regime can be analysed in a systematic $1/N$ expansion.

The large N formalism we use is of the standard type used to analyse disordered phases of non-linear σ models [20]. ϕ_r^a can be thought of as a set of three orthogonal 3 dimensional vectors. This is generalised to a set of three orthogonal N dimensional vectors. We denote them by ϕ_r^a , where $a = 1, 2, \dots, N$ and $r = 1, 2, 3$. The coupling constants in the model are defined to scale with N as follows. $g_{1(2)} \rightarrow g_{1(2)}/N$, $\alpha \rightarrow N\alpha$ and $\lambda_0 \rightarrow N\lambda_0$. This results in the RHS of the constraint in equation(16) to be multiplied by N. We then use a 3×3 matrix valued Lagrange multiplier field, $\mu^{ij}(x)$, to impose the constraint as is usual in this method. The ϕ fields can then be integrated out and the partition function is expressed as,

$$Z = \int_{\mu, \hat{m}} e^{-NS_{eff}[\mu, \hat{m}]} \quad (17)$$

where, $S_{eff}[\mu, \hat{m}]$ is given by,

$$S_{eff}[\mu, \hat{m}] = \frac{1}{2} \ln \det(-\partial_\mu^2 - i\mu) + \int_x \frac{1}{g_1} (1 + f(m^z)) \text{tr} \mu + \frac{1}{g_2} (\partial_\mu \hat{m})^2 + V(m^z) \quad (18)$$

It is now clear that using the saddle point method, a systematic expansion in $1/N$ can be developed for Z . The same procedure can easily be generalised for correlation functions as well.

The saddle point equations are obtained by setting the variation of the effective action in equation(18) with respect to $\mu^{ij}(x)$ and $\hat{m}(x)$ equal to zero. We look for translationally invariant (x independent) solutions. Putting $m^z = \cos(\theta)$ and $-i\mu^{ij} = M_a^2 \delta^{ij}$, the saddle point equations can be written as,

$$\frac{\delta S}{\delta \mu^{ij}} = 0 \quad (19)$$

$$\frac{\delta S}{\delta (m^z)^2} = 0 \quad (20)$$

Applying these conditions and putting in for μ_{ij} the ansatz, $i\mu_{ij} = M_a^2 \delta_{ij}$ we get the following solutions for m^z ,

$$\frac{1}{g_1} - \alpha(m^z)^2 = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{K^2 + M_a^2} \quad (21)$$

From the second condition (20), we get the two possible solutions for m^z of which the first one is,

$$m^z = \eta_0 - \frac{2\alpha M_a^2}{\lambda_0} = \cos \bar{\theta} \quad (22)$$

The other possible condition solution is $m^z = 1$.

If we define g_{crit} by the equation,

$$\frac{1}{g_{crit}} = \int \frac{d^3 K}{(2\pi)^3} \frac{1}{k^2} = \frac{\Lambda}{2\pi^2} \quad (23)$$

Then when

$$\frac{1}{g_1} - \alpha \geq \frac{1}{g_{crit}} \quad (24)$$

Then, $M_a^2 = 0$ and from equation (22), if $\eta_0 > 1$, the solution for m^z in this case is $m^z = 1$. Since in this phase $\frac{1}{M_a}$, which is the correlation length for the Φ fields, diverges, this describes an $SO(3)_R$ broken phase. In addition, in this phase the $SO(2)_L$ symmetry is unbroken because $m^z = 1$. The low lying excitations in this phase are the three Goldstone bosons coming from this symmetry breaking. They are the three spinwave modes with spin = 1.

When

$$\frac{1}{g_1} - \alpha < \frac{1}{g_{crit}} \quad (25)$$

Then $M_a \neq 0$ (which follows from (21)) and this is a phase in which the $SO(3)_R$ symmetry is unbroken. In this disordered regime, the fields Φ have a finite correlation length which can be calculated by solving equation (21) to get M_a .

Now there are two possible solutions for m^z . If

$$\eta_0 - \frac{\alpha M_a^2}{\lambda_0} > 1 \quad (26)$$

Then this offers no solution to equation (22) for $m^z = \cos \bar{\theta}$ which is always less than or equal to 1. Hence the other solution, $m^z = 1$ is picked out. This implies that this is also a phase in which the $SO(2)_L$ symmetry is unbroken. The groundstate, in this phase shows no long range order and all low lying excitations about this state are gapped.

The third possibility is when

$$\eta_0 - \frac{\alpha M_a^2}{\lambda_0} < 1 \quad (27)$$

In this case there is a consistent alternate solution to equation (22) for $m^z = \eta_0 - \frac{\alpha M_a^2}{\lambda_0} = \cos \bar{\theta}$. Since \hat{m} no longer points in the \hat{z} direction, this spoils the axial symmetry that existed earlier. In this phase, the $SO(3)_R$ symmetry is, as earlier, unbroken but the $SO(2)_L$ symmetry is broken and there is one massless particle which is the angular variable ϕ_m . Since \hat{m} is a spin singlet under $SO(3)_R$ rotations this is a spinless excitation. The other field θ_m acquires a gap, which can be calculated.

In this $SO(2)_L$ broken phase the fluctuations in θ_m are gapped and those of ϕ_m are gapless. We obtain the mass gap for the θ_m fluctuations and rewrite

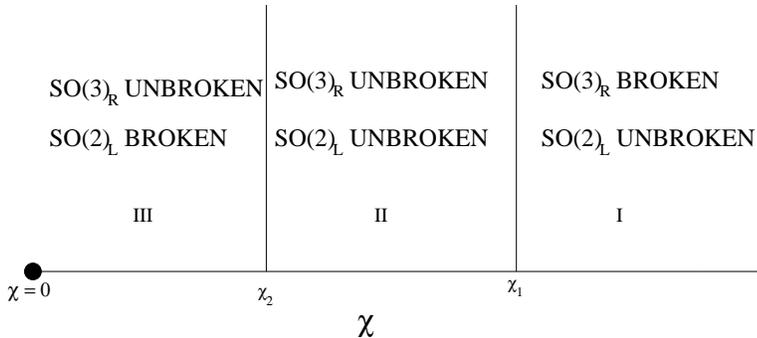


Figure 4: phase diagram of the $J_1 - J_2$ model on the Kagomé lattice. Here $\chi = \frac{J_2}{J_1}$ and the point $\chi = 0$ is the NNKLAf.

the part of the action S_m , which involves just the fields \hat{m} , in terms of the variables θ_m and ϕ_m we get, up to quadratic order in the fields,

$$S_m = \int d^3x \sin^2(\bar{\theta}_m) \partial_\mu \phi_m \partial_\mu \phi_m + \partial_\mu \theta_m \partial_\mu \theta_m + \frac{M_\theta^2}{2} (\theta_m)^2 \quad (28)$$

Where M_θ is got by expanding S_m about the average value of $m_3 = \cos \bar{\theta}$ given in equation (22). This is given by the expression,

$$M_\theta^2 = \frac{\lambda_0}{g_2} [(2\eta_0 - 3) \cos(2\bar{\theta}) - \cos(4\bar{\theta})] \quad (29)$$

So far the discussion has been restricted to a regime where the \hat{m} fields are approximated by their classical values. We still need to ascertain that including fluctuations in \hat{m} does not destroy the ordered state. This has been established in [15] within a one loop R.G calculation. This calculation showed that while the fluctuations of \hat{m} tend to destroy the order, there exists a region of parameter space where they do not succeed in doing so. This ensures the stability of the $SO(2)_L$ broken phase over the effect of quantum fluctuations.

6 Phases of the Kagomé lattice.

In the previous section we have analysed the longwavelength field theory (15) and shown that the system undergoes two phase transitions as the coupling

constants are tuned. To relate these results to the phases of Kagomé lattice model that we are considering, we need to relate the coupling constant of the lattice model χ to the coupling constants of the field theory. It is very difficult to reliably calculate this relation. However, based on some general features and the numerical results obtained by Lecheminant et. al. [12], it is possible to make some qualitative statements.

Firstly it is possible to estimate the potential, $V(m_z)$ and hence the parameters η_0 and λ_0 by substituting equation (9) into the hamiltonian and taking \hat{m}_I to be independent of I . By doing so we get, $\eta_0 = 1 + 4\chi$ and $\lambda_0 = \frac{27}{4}J_1$ [16]. While this will be modified by the fluctuations, we assume that the qualitative fact that η_0 is an increasing function of χ and approaches $\eta_0 = 1$ as $\chi \rightarrow 0$, remains true. This amounts to assuming that the gap of the singlet excitations (the H-S modes) decreases as χ decreases and approaches 0 as $\chi \rightarrow 0$. This is consistent with our spinwave spectrum and the results in reference [12]

Next, as χ decreases and the corresponding bonds become weaker, we can expect large amplitude fluctuations of the spins to cost less energy and therefore for the spins to disorder. The results of reference [12] strongly support this scenario. We therefore assume that g_1 increases as χ decreases.

Now we look at equation (25) which determines the first phase boundary for the transition from phase I to phase II (see fig.(4)). The above assumptions imply that for large χ the system is in phase I and as χ is decreased and we move from phase I to phase II the correlations of the Φ fields become short ranged and the corresponding mass gap M_a increases from the value zero (at χ_1). η_0 takes the value one for the NNKLAFF and increases as χ is increased. As χ is further decreased, M_a increases and from equation (27), for the second phase boundary, we see that at some point the inequality is saturated. This is the second critical point χ_2 .

This forms the argument for the scenario depicted in figure (4). For $\chi > \chi_1$, the system is in the Néel ordered spiral phase. In this phase, marked I in the diagram, the $SO(3)_R$ symmetry is broken down to nothing and the $SO(2)_L$ symmetry is unbroken. The value of χ_1 is determined by the saturation of the inequality (24). As χ is reduced below χ_1 the system undergoes a transition into the phase marked II. This is a phase with all symmetries intact. Further down, there is a second phase transition at $\chi = \chi_2$, which is determined by the condition (27). The phase III is one in which the $SO(3)_R$ symmetry continues to be unbroken and the $SO(2)_L$

symmetry is broken.

It has been seen in reference [12] that for $J_2 = 0$ the system is disordered and a large number of low lying singlet excitations that are observed. In our analysis of the continuum model, the breaking of the $SO(2)_L$ symmetry would result in the collapse of such states onto the ground state as the system size is increased to infinity. Our analysis predicts an intermediate phase II, where both the symmetries are intact. But since it seems that the ordering of the ϕ_m field is driven by the disordering of the U field and vice-versa, it is possible that in the real system, fluctuations could cause χ_1 and χ_2 to coincide, thereby causing a direct jump from phase I to phase III.

This is our picture of the phases of the Kagomé lattice as a function of χ . In the next section we describe the behaviour of the correlation functions in the phase III and give an expression for the relevant order parameter.

7 Correlation functions and order parameter.

We will now construct a suitable local order parameter, in terms of the spins, to describe the phase III discussed in the previous section. This is the phase in which the $SO(3)_R$ spin symmetry is unbroken and the $SO(2)_L$ symmetry is broken. The order parameter should therefore be a spin singlet and transform non-trivially under the $SO(2)_L$ symmetry. In terms of the field theory variables, the transverse components of the \hat{m} field, i.e (m_x, m_y) , is such an order parameter. It will have a non-zero value in phase III and its correlation functions will be long ranged.

Since the $SO(2)_L$ symmetry is not present in the lattice spin model, the identification of such an order parameter in terms of the spins is not straightforward. As discussed in section(3), the \hat{m} fields represent spin configurations where the elementary triangles are left intact but neighbouring triangles are not coplanar. Now consider the scalar triple product of three spins lying on the same line in a unit cell. e.g. labelled by (11), (02) and (00) (see figure (5)).

$$C(X, 11, 02, 00) = \vec{S}_{11} \cdot \vec{S}_{02} \times \vec{S}_{00} \quad (30)$$

This operator is a spin singlet. When the two neighbouring triangles that they belong to are coplanar, then so are these three spins and consequently the their scalar triple product is zero. When the triangles are not coplanar, then neither are the three spins. Since $\vec{S}_{02} \times \vec{S}_{00}$ is the vector normal to the

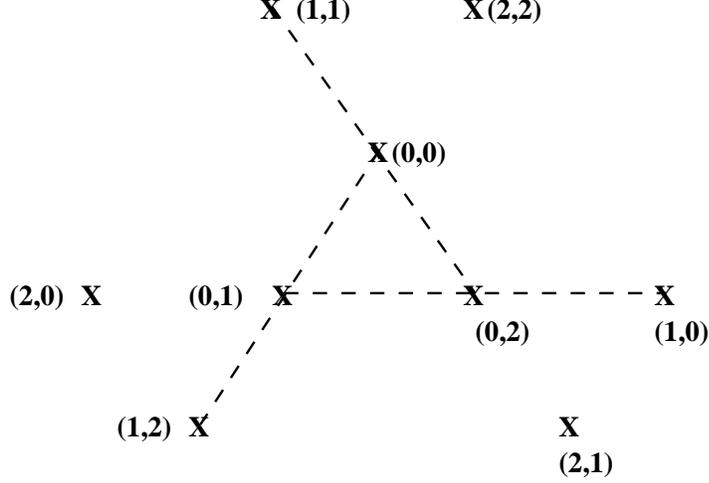


Figure 5: The crosses refer to various sites on the unit cell and the dashed lines connect sites forming the order parameters $C(X,11,02,00)$, $C(X,12,00,01)$ and $C(X,10,01,02)$.

plane of the central triangle and \vec{S}_{11} is a vector lying in the plane of the neighbouring triangle, therefore the scalar triple product defined in equation (30) is a measure of the angle between the planes of the two adjoining triangles. We can therefore expect it to reflect the behaviour of the \hat{m} field.

To confirm this we express the right hand side of equation(30) in terms of the U and \hat{m} fields using the parameterisation of the spins given in equations (8) and (9). Since it is a spin singlet, it is independent of Φ_r^a and it turns out to be,

$$\begin{aligned}
 C(X, 11, 02, 00) &= \frac{3}{4} \text{Cos} \phi_m(X) \text{Sin} \theta_m(X) \\
 &= \frac{3}{4} \hat{m}(X) \cdot \hat{E}_1
 \end{aligned} \tag{31}$$

where $\hat{E}_1 = (1, 0, 0)$, a vector in the body fixed frame. Thus $C(X,11,02,00)$ is proportional to m_x . Similarly we can define the operators

$$\begin{aligned}
 C(X, 12, 00, 01) &= \frac{3}{4} \text{Cos}(\phi_m(X) + \frac{2\pi}{3}) \text{Sin} \theta_m(X) \\
 &= \frac{3}{4} \hat{m}(X) \cdot \hat{E}_3
 \end{aligned} \tag{32}$$

and

$$\begin{aligned}
C(X, 10, 01, 02) &= \frac{3}{4} \text{Cos}(\phi_m(X) - \frac{2\pi}{3}) \text{Sin}\theta_m(X) \\
&= \frac{3}{4} \hat{m}(X) \cdot \hat{E}_2
\end{aligned} \tag{33}$$

where $\hat{E}_2 = (\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $\hat{E}_3 = (\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0)$. By taking other appropriate linear combinations of these, we can construct an operator which is proportional to m_y and we see that operators of the type written down in equations (31, 32 and 33) are good order parameters to characterise the $SO(2)_L$ broken phase.

8 Conclusions and discussion

In this paper, we have described the $J_1 - J_2$ model on the Kagomé lattice by a field theory of the low energy long wavelength excitations. Analysis of this theory shows that there are three phases in this model. Based on a comparison with earlier , exact diagonalization studies, we relate the coupling constants of the field theory to the parameter χ of the hamiltonian. Thereby, we depict our results as a phase diagram on the χ axis as shown in figure (4). Accordingly, the NNKLAF lies in the phase III which has been described earlier. In this phase, the ground state is disordered and there are massless singlet excitations over the ground state. At large and small χ our description of the $J_1 - J_2$ model matches with the numerical studies. In addition, at intermediate χ , we see a completely disordered phase (II). We have described the transition from phase II to phase III by a suitable singlet operator constructed out of the spins.

Further, the field theoretic approach explains the origin of the gapless excitation in the disordered phase III, by giving a new mechanism. This mechanism of obtaining a Goldstone mode is likely to be operative in the model for the group 2 compounds and provides a means of getting a gapless bosonic excitation which lead to a T^2 behaviour of the specific heat. This is interesting because all symmetries of the microscopic model are apparently intact and hence there seems to be no reason for the existence of such a gapless mode.

In their exact spectra analysis of the $J_1 - J_2$ model on the Kagomé lattice, Lecheminant et al [12], see a trend that is supportive of the above picture

as far as the NNKLAF is concerned. Namely they see that while there is no long range spin order at the $J_2 = 0$ end, there is a proliferation of spin singlet excited states with a small gap which could collapse to the ground state in the limit of an infinite lattice. Comparing our results to this, it seems likely that this collapse is a signal of the breaking of the $SO(2)_L$ symmetry in the limit of infinite lattice size.

Appendix A.

The fluctuations hamiltonian is given by the expression,

$$H = \frac{1}{2} \sum P_{-K} M^{-1} P_K + Q_{-K} K Q_K \quad (34)$$

where the M_0^{-1} and The K_0 are given by,

$$(M_0^{-1})_{i\alpha j\beta} = \begin{bmatrix} (I_0)_{\alpha\beta} & (I_1)_{\alpha\beta} & (I_1^T)_{\alpha\beta} \\ (I_1^T)_{\alpha\beta} & (I_0)_{\alpha\beta} & (I_1)_{\alpha\beta} \\ (I_1)_{\alpha\beta} & (I_1^T)_{\alpha\beta} & (I_0)_{\alpha\beta} \end{bmatrix}_{ij}$$

where,

$$(I_0)_{\alpha\beta} = \begin{bmatrix} 2(1-2\chi) & 1 & 1 \\ 1 & 2(1-2\chi) & 1 \\ 1 & 1 & 2(1-2\chi) \end{bmatrix}_{\alpha\beta} \quad (I_1)_{\alpha\beta} = \begin{bmatrix} 2\chi & 0 & 1 \\ 1 & 2\chi & 0 \\ 0 & 1 & 2\chi \end{bmatrix}_{\alpha\beta}$$

and

$$(K_0)_{i\alpha j\beta} = \begin{bmatrix} (\bar{I}_0)_{\alpha\beta} & (\bar{I}_1)_{\alpha\beta} & (\bar{I}_1^T)_{\alpha\beta} \\ (\bar{I}_1^T)_{\alpha\beta} & (\bar{I}_0)_{\alpha\beta} & (\bar{I}_1)_{\alpha\beta} \\ (\bar{I}_1)_{\alpha\beta} & (\bar{I}_1^T)_{\alpha\beta} & (\bar{I}_0)_{\alpha\beta} \end{bmatrix}_{ij}$$

where,

$$(\bar{I}_0)_{\alpha\beta} = \begin{bmatrix} 2(1-2\chi) & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 2(1-2\chi) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 2(1-2\chi) \end{bmatrix}_{\alpha\beta} \quad (\bar{I}_1)_{\alpha\beta} = \begin{bmatrix} 2\chi & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2\chi & 0 \\ 0 & -\frac{1}{2} & 2\chi \end{bmatrix}_{\alpha\beta}$$

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