MOTION AND TRAJECTORIES OF PARTICLES AROUND THREE-DIMENSIONAL BLACK HOLES

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Abstract. The motion of relativistic particles around three dimensional black holes following the Hamilton-Jacobi formalism is studied. It follows that the Hamilton-Jacobi equation can be separated and reduced by quadratures in analogy with the four-dimensional case. It is shown that: a) particles are trapped by the black hole independently of their energy and angular momentum, b) matter always falls to the centre of the black hole and cannot undertake a motion with stables orbits as in four dimensions. For the extreme values of the angular momentum of the black hole, we were able to find exact solutions for the equations of motion and trajectories of a test particle.

IPNO-TH 93/06

DFTUZ-93/01

February 1993

PACS numbers: 04.20.Jb, 97.60.Lf

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Since the discovery of three-dimensional gravity [**1**] many efforts have been performed in order to establish a closer analogy between three-dimensional gravity and the four-dimensional case. Following this perspective, it was shown that in three dimensions there are classical solutions [**2**] of the Einstein field equations that keep a narrow relation with the Schwarzschild or Kerr solutions. These "black hole" (BH) like solutions exhibit a behaviour similar to their four-dimensional homologous and keep also properties such as horizons or thermodynamic features [**3**]. However, in spite of these results, it seems interesting to investigate to what extent these analogies remain true. Following this research line, it was found recently [**4**] other BH-like solution for a three-dimensional gravity characterized by mass, charge and angular momentum. Hence, in many aspects, this solution is the natural analog of the four-dimensional Kerr-Newman solution.

The purpose of the present letter is to analyse the motion of relativistic test particles in the geometry found in [**4**] in order to understand some issues such as whether there is trapping of particles by this BH or not, or whether it makes sense to talk about cross section for capturing particles or not, etc.. This is an interesting point because some of the solutions found in the literature are solutions with cone-like singularities (exhibiting deficit angles) and in these cases, there is no trapping of particles for the same reason by which there is no trapping of particles by cosmic strings [**5**].

More precisely we will show the following issues:

a) The associated Hamilton-Jacobi equation is separable and trivially reducible by quadratures.

b) The two extreme cases, i.e., when the angular momentum of the BH is zero and maximum respectively, the equations of motion can be integrated exactly.

c) The only possible trajectories for test particles, are those that fall into the singularity.

In order to show these results we start by using the Hamilton-Jacobi formalism, applied originally by Kaplan and Carter for the Schwarzschild [**6**] and Kerr BH [**7**] respectively.

The authors in ref.[**4**] considered the action

$$I = \frac{1}{2\pi} \int d^2x \; dt \sqrt{-g} \left[R + 2l^{-2} \right] + B, \tag{1}$$

where B is a surface term and the radius l is given by $l = \frac{1}{\sqrt{-\Lambda}}$, with Λ being the (negative) cosmological constant. They found that the corresponding Einstein's equations are solved by the following BH field

$$ds^2 = -(N^2 - r^2 N_\phi) dt^2 + N^{-2} dr^2 + r^2 d\phi^2 + 2r^2 N_\phi dt\, d\phi, \tag{2}$$

where the lapse function N^2 and N_ϕ are defined as

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \tag{3a}$$

$$N_\phi = -\frac{J}{2r^2}. \tag{3b}$$

In (3) M and J are two constants of integration that can be understood as the mass and the angular momentum of the BH respectively.

As it was discussed in [**4**], the lapse function N^2 vanishes for

$$r_\pm = l\left[\frac{M}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}}\right)\right]^{\frac{1}{2}}. \tag{4}$$

The BH horizon is identified with r_+, and it will exist only if M and J satisfy the relations

$$M > 0, \qquad |J| \leq Ml. \tag{5}$$

Observe that in the extreme case $|J| = Ml$ both roots in (4) coincide.

In order to use the Hamilton-Jacobi equation

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0, \tag{6}$$

we see that we need the contravariant components of the metric, which, after inverting $g_{\mu\nu}$, are given by

$$\begin{aligned} g^{00} &= -N^{-2}, \qquad g^{11} = N^2, \\ g^{22} &= \frac{1}{r^2}(1 - r^2 \frac{N_\phi^2}{N^2}), \qquad g^{02} = g^{20} = \frac{N_\phi}{N^2}. \end{aligned} \tag{7}$$

Substituting (7) in (6), the Hamilton-Jacobi equation for a relativistic test particle of mass m becomes

$$-N^{-2}\left(\frac{\partial S}{\partial t}\right)^2 + N^2\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(1 - r^2\frac{N_\phi^2}{N^2}\right)\left(\frac{\partial S}{\partial \phi}\right)^2 + 2\frac{N_\phi}{N^2}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi} + m^2 = 0. \quad (8)$$

The equation (8) can be easily separated with the aid of the following ansatz

$$S(r,\phi,t) = -Et + L\phi + S_1(r), \quad (9)$$

where E and L are the energy and the angular momentum of the particle respectively.

Substituting (9) into (8) and solving for S_1, we get

$$S_1 = \int \frac{dr}{N}\sqrt{\left(\frac{E}{N}\right)^2 - \frac{L^2}{r^2}\left[1 - r^2\left(\frac{N_\phi}{N}\right)^2\right] + 2\frac{N_\phi EL}{N^2} - m^2}, \quad (10)$$

and a solution of the Hamilton-Jacobi is obtained by quadratures.

The trajectory of the particle can be determined, as usual, by stating that $\frac{\partial S}{\partial E}$ and $\frac{\partial S}{\partial M}$ are constants respectively. However, we shall adopt another approach, and we shall obtain the equations of motion directly from the so called first integral of the geodesic equation **[8]**

$$P^\mu = m\frac{dX^\mu}{d\tau} = -g^{\mu\nu}\frac{\partial S}{\partial X^\nu}, \quad (11)$$

where τ is the proper-time of the test particle.

Substituting equations (7), (9) and (10) into (11), we obtain, after straightforward calculations, that

$$m\frac{dt}{d\tau} = -\frac{1}{N^2}(E + N_\phi L), \quad (12a)$$

$$m\frac{d\phi}{d\tau} = \frac{N_\phi}{N^2}(N_\phi L + E) - \frac{L}{r^2}, \quad (12b)$$

$$m^2\left(\frac{dr}{d\tau}\right)^2 = (E + N_\phi L)^2 - \left(\frac{LN}{r}\right)^2 - m^2N^2. \quad (12c)$$

Equations (12a-12c) describe the motion of a relativistic test particle with mass m in the geometry given by (2). Hence, we can analyse what kind of motion around the BH can be described by the test particle.

In order to see whether bounded orbits (which does not mean closed orbits) are allowed by this geometry or not, the first thing we must do is to search for turning points. With this goal, we just impose that the r.h.s. of (12c) must vanish. Hence, the turning points are given, in principle, by

$$\frac{\alpha}{r^2} + \beta r^2 + \gamma = 0, \tag{13}$$

where we defined

$$\alpha = L^2 M - \frac{1}{4}m^2 J^2 - JLE, \tag{14a}$$

$$\beta = -\frac{m^2}{l^2}, \tag{14b}$$

$$\gamma = -\frac{L^2}{l^2} + E^2 + m^2 M. \tag{14c}$$

Solving (13) we find the roots

$$R^2_{max} = -\frac{1}{2\beta}[\gamma + \Delta], \tag{15a}$$

$$R^2_{min} = -\frac{1}{2\beta}[\gamma - \Delta], \tag{15b}$$

where

$$\Delta = \sqrt{\gamma^2 - 4\alpha\beta}.$$

A necessary condition (but not sufficient) for R^2_{min} to exist is that $R^2_{min} \geq 0$. Observing that $\beta < 0$ by (14b), this leads to the following condition

$$\alpha \leq 0, \tag{16}$$

which must always hold.

Once we got expressions for the turning points R_{min} and R_{max}, we could be suggested that both of them would always exist, or at least that they would exist for some values of the parameters M and J (which characterizes the BH) and L and E (which characterizes the orbit). This would lead us to the naive conclusion that bounded orbits are possible for the case at hand. However, in order to have bounded orbits, we must also be sure that R_{min} is greater than the BH horizon, and as we shall see, this will never occur. We shall establish the above result by using two different approaches.

The first one makes use of the corresponding effective potential, which can be obtained in the following way: we start in the same way as if we were interested in obtaining the turning points (let us denote them by R_{TP}), that is, putting the r.h.s. of (12c) equal to zero. This gives us an algebraic equation like $F(R_{TP}, E, L, M, J) = 0$, which, after solving for R_{TP}, gives us the turning points. However, instead of solving for R_{TP}, if we now solve for E, and if we consider fixed values for L, M and J, we will get $E = E(R_{TP})$. The effective potential $V_{eff}(r)$ will coincide precisely with this function, provided R_{TP} is thought as the variable r.

For the problem at hand a direct calculation of the effective potential yields

$$V_{eff}(r) = \frac{JL}{2r^2} + N(r)\sqrt{\left(\frac{L}{r}\right)^2 + m^2}, \tag{17}$$

and a lengthy, but straightforward analysis shows that there are neither minima nor maxima in the region of physical interest[1] (outside the BH horizon).

Hence, independently of its energy and angular momentum, a test particle always falls into the singularity $r = 0$ and there are no stable orbits. As already mentioned, this result does not have analog in four dimensions, where some stable orbits are allowed **[6,9]**.

The same result can be reached by analysing the equations of motion (12). In fact, we note that although the set of equations (12) is difficult of to solve, it is possible to extract physical information from them.

Actually, for any value of the BH parameters, the only equation that can be exactly integrated is (12c),

$$\frac{1}{r^2} = -\frac{2\beta}{\gamma + \Delta\ sin\, x}, \tag{18}$$

where $x = 2\sqrt{-\beta}\frac{\tau}{m}$.

From this equation we can infer some general consequences

(a) When $\Delta^2 > 0$ the motion of the particle can be bounded between two circles of radii R_{max} and R_{min} respectively; if R_{min} is smaller than the horizon such motion will exist only until the particle reachs the gravitational radius where it will be captured by the BH.

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[1] We have checked this result numerical and analitically.

(b) If $\Delta^2 = 0$ the test particle will describe, in principle, a circular motion and again, it will be possible only if its radius is greater than the horizon.

(c) If $\Delta^2 < 0$ the motion will not take place.

The conditions (a)-(b) imply relations between the parameters of the particle (m, L, E) and the BH ones (M, J). For a circular motion this relation is

$$J = \frac{(El - L)^2}{m^2 l} + Ml, \tag{19}$$

However, Ml is the maximum angular momentum of the BH which is physically allowed and, as a consequence, a circular orbit is possible only if both relations $J = Ml$ and $El = L$ are simultaneously satisfied.

On the other hand, the radius R_0 of this circular orbit is given by (it suffices to substitute $\Delta = 0$ into (15))

$$R_0{}^2 = \frac{-\gamma}{2\beta}|_{J=Ml=\frac{ML}{E}} = \frac{Ml^2}{2} = R_g{}^2, \tag{20}$$

which is just the horizon (4) in the extreme case $J = Ml$. Thus, we conclude that, although equation (18) gives an oscillating solution, the presence of the horizon implies that a circular motion can never occur, which means that there is no minimum of $V_{eff}(r)$ in the physical region.

Therefore, we have seen by using two different points of view that the three-dimensional BH just studied are objects that, although trap matter, never allow stable orbits for test particles moving around them.

Now let us discuss some particular cases where we were able to obtain exact solutions of the equations (12).

a) *Case J = 0*

This situation corresponds to have an anti- de Sitter metric provided we perform the transformation $t \to \frac{t}{\sqrt{-M}}$ in the metric (2). It is interesting to note that for this case (16) imply $M < 0$ and, as a consequence, the limit $J \to 0$ corresponds precisely the solution discussed in [**1**]. The solutions of the corresponding equations of motion can be written as

$$\frac{1}{r^2} = -\frac{2\beta}{\gamma + \Delta' sin\ x}, \tag{21a}$$

$$\Delta' + \gamma\ tan\frac{x}{2} = -2\sqrt{\rho\ \beta}\ tan(\frac{\sqrt{-\rho}}{L}\phi), \tag{21b}$$

while the trajectory is given by

$$\frac{\gamma+2\beta\, r^2}{\Delta'} = 2\gamma\, \frac{\Delta^{'}+2\sqrt{\rho\,\beta}\;\tan\,(\frac{\sqrt{-\rho}}{L}\phi)}{\gamma^2+\left[\Delta^{'}+2\sqrt{\rho\,\beta}\;\tan\,(\frac{\sqrt{-\rho}}{L}\phi)\right]^2}, \tag{22}$$

with $\Delta^{'}=\sqrt{\gamma^2-4\rho\beta}$ and $\rho = ML^2$.

Some geometrical properties of this solution are discussed, for instance, in **[10-11]**.

b) <u>*Case $J = Ml$*</u>

This extreme case is interesting because the horizon still remains and the equations (12) are notably simplified.

The solutions of the equations of motion are given by

$$\begin{aligned}
\frac{1}{r^2} &= -\frac{2\beta}{\gamma+\Delta\; sin\, x}\\
\frac{1}{\sqrt{-\beta}}\phi &= \frac{1}{a^2-b^2}\left[\frac{(Ab-Ba)\;\cos\; x}{a+b\;\sin\; x}+(Aa-Bb)\mathcal{J}\right],\\
\frac{1}{\sqrt{-\beta}}\;t &= \frac{1}{a^2-b^2}\left[\frac{(Ab-aB)\;\cos\; x}{a+b\;\sin\; x}+(A^{'}a-B^{'}b)\mathcal{J}\right],
\end{aligned} \tag{23}$$

where the coefficients are defined as

$$\begin{aligned}
&A = 2\beta LMl^2 - Ml^3E\beta + L\gamma, \quad B = L\Delta,\\
&A^{'} = l^2E\gamma + Ml^3\beta, \quad B^{'} = l^2E\Delta,\\
&a = \gamma + \beta Ml^2, \quad b = \Delta,
\end{aligned}$$

and

$$\mathcal{J} = \begin{cases} \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right] & \text{if } a^2>b^2\\ \frac{1}{\sqrt{b^2-a^2}}\ln\left[\frac{a\;\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\;\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}\right] & \text{if } a^2<b^2\;. \end{cases}$$

The equation of the trajectory is tedious to write, so we avoid this calculation since we shall not need it for future applications.

For the case of a charged BH, our results can be extended straightforwardly. In fact, as it was discussed in **[4]** the electromagnetic coupling to the metric (2) corresponds to perform the change

$$N^2 \to N^2 + \frac{Q^2}{2}\ln\left(\frac{r}{r_0}\right), \tag{24}$$

being Q the electric charge of the BH and r_0, a constant.

The new effective potential V_{eff}^{Q} for this case has the same structure of that given by (17), provided the replacement (24) is performed. Again, V_{eff}^{Q} does not have neither maxima nor minima and the same previous conclusions for the uncharged BH can be reached. However, for the charged BH, the equations of motion become very complicated to be solved analytically. This last example shows that the effective potential approach may be sometimes more convenient than trying to solve directly the corresponding equations of motion.

We would like to thank M. Asorey by discussions a critical reading of the manuscript. C.F. would like to thank to F.M. de Almeida, R. Donangelo and L.S. de Paula for some help with internet problems and also to the Department of Theoretical Physics of Zaragoza where part of this work was done. J.G. thanks J. Zanelli by discussions and correspondence concerning this work. This research was partially supported by CNPq-Brazil (C.F) by CNRS and FONDECYT grant 0867/91 (J.G) and by DGICYT (Spain) grant PB90-0916 (A.J. S.-S).

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