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# Analytical Solving of Partial Differential Equations using Symbolic Computing

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## Abstract

This work presents a brief discussion and a plan towards the analytical solving of Partial Differential Equations (PDEs) using symbolic computing, as well as a implementation of part of this plan as the *PDEtools* software-package of commands.

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# Introduction

The *PDEtools* software-package here presented is a MapleV R.3 implementation of analytical methods for solving and working with PDEs. The main commands of the package are: **pdsolve**, a PDE-solver; **dchange**, for making changes of variables; **sdsolve**, for solving systems of Ordinary Differential Equations (ODEs); **splitsys**, for splitting up systems of ODEs; and **mapde**, for mapping PDEs into more convenient PDEs. New features, with respect to the first version of the package[1], are: the **sdsolve**, **splitsys** and **mapde** commands, as well as significant improvements made to **dchange**.

The solving methods implemented in this version are, mainly, standard methods such as the characteristic strip for first-order PDEs, reduction to canonical form of second-order PDEs, separation of variables, etc.[2, 3]. Furthermore, the user can optionally participate in the solving process by giving the solver an extra argument (the **HINT** option) indicating a *functional form* for the indeterminate function. Note that, besides the inherent restrictions of the methods used in this release, the *PDEtools* package cannot tackle systems of PDEs.

## 1 The strategy for a PDE-solver

### 1.1 The objectives of a PDE solving command

The first point to be discussed is what the program (**pdsolve**) should return as the result. Our idea was that, given a PDE, **pdsolve** should return the most *general solution* it could find. At least, it should try to separate all the variables, giving the user the option of asking for the integration of the resulting *uncoupled* ODEs in order to arrive at a *complete solution*. *General solutions* depend on  $n$  arbitrary functions of  $k - 1$  variables, where  $n$  is the differential order of the given PDE and  $k$  is the number of independent variables. *Complete solutions*, on the other hand, depend on (at least)  $\sum_{i=1}^k n_i$  arbitrary constants, where  $n_i$  represents the differential order w.r.t each of the  $k$  variables<sup>3</sup>.

### 1.2 The solving methods and the plan

Let us classify the methods which may lead to the desired types of solutions into *symmetry* methods - those relying on the use of the theory of invariance under Lie groups of transformations[4] -, and *standard* methods - the other ones[2, 3]. Though it seems clear to us that a PDE-solver should make use of the systematic *symmetry* methods, these methods usually map the task of solving a PDE into a task of the same type, requiring the solving of the system of PDEs that determines the generators of the invariance group. On the other hand, *standard* methods permit the mapping of the task of solving a PDE into the tasks of realizing algebraic manipulations, evaluating integrals and solving ODEs, all of which are already implemented in Maple. For these reasons, we split the general objective of solving PDEs analytically into: 1. the implementation of *standard* methods for solving a single PDE; 2. the implementation of *standard* methods for solving systems of coupled linear PDEs; 3. the implementation of *symmetry* methods. This version of the

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<sup>3</sup>Depending on the PDE, **pdsolve** may also return a solution in some sense “intermediate”, containing less than  $n$  arbitrary functions but more general than a *complete* one.

*PDEtools* package is concerned with the first step and does not make use of *symmetry* methods.

## 2 The PDE-solver: an approach using *standard methods*

*Standard* methods can be divided into those having a clear prescription of when and how to be applied, e.g. the characteristic strip method, and those which work in a more *heuristic* manner, such as separation of variables. Also, all methods map the original problem into another problem, and the latter still needs to be solved. Furthermore, there are PDEs which can be tackled by more than one method, as well as PDEs (the majority) for which we do not know any specific method of solution.

Taking all these facts into account, it became clear to us that a wide-range PDE-solver should be able to make decisions concerning the method to be employed in *any* case, the trial strategy, and what to do when the strategy followed at first fails (perhaps try another method...). Therefore, our idea was to develop routines implementing *deterministic* methods, to be used right at the beginning of the solving process; and invest hard on algorithms for separating variables, to be used when the implemented *deterministic* methods either are not applicable or map the original PDE into a problem that the system fails to solve.

### 2.1 The *deterministic* methods

As mentioned above, we denominate *deterministic* a method having a clear prescription of when and how to be applied. The methods falling into this category which are implemented in this release of the *PDEtools* package are just for first and second order PDEs. The most relevant ones are: the characteristic strip method<sup>4</sup>, the mapping between PDEs (realized using **mapde**), and some other particular methods applicable when the given PDE matches a pattern recognized by **pdsolve**. A detailed list of the implemented methods can be seen in [1]. All other PDEs are first tackled through separation of variables.

### 2.2 The *heuristic* algorithm for separating the variables

The main idea of the method of separation of variables consists of building a system of *uncoupled* ODEs equivalent to the original PDE. To determine the appropriate submethod to separate the variables, we implemented a *heuristic* algorithm, i.e. one that elaborates an ansatz whose success cannot be determined *a priori*. Concerning the algorithm itself, it seemed reasonable to us to build the ansatz with selected terms of the general sum of possible products, given, for example, for an indeterminate function  $f(x, y, z)$ , by

$$f_1(x) + f_2(y) + f_3(z) + f_1(x)f_2(y) + f_1(x)f_3(z) + f_2(y)f_3(z) + f_1(x)f_2(y)f_3(z)$$

For instance, separation *by sum* and *by product* would be obtained by selecting only the first three terms or just the last term, respectively, from the above; the criteria for

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<sup>4</sup>A refinement was developed in order to transform the parameterized solution of the strip, when possible, into an explicit *general* solution using differential invariants.

selecting the terms depend on the structure of the given PDE. When the ansatz leads to a partial separation of variables, **pdsolve** was programmed to reenter itself with the part containing the non-separated variables as argument, applying the whole strategy once again to that part. This usually results in a mixed combination of separation of variables and *deterministic* methods to solve a single given PDE.

### 2.3 The HINT option of pdsolve

As mentioned before, there is no general, ever-efficient method to solve PDEs. To minimize this problem, our idea was to permit an active participation of the user in the solving process. This was obtained by designing **pdsolve** so as to always follow a specific instruction concerning how to tackle the received PDE. We called this instruction HINT, and it can be given by the user; otherwise, it will be automatically generated by one of the subroutines of **pdsolve**. In both cases, the HINT may be an indication of either a solving method already known to **pdsolve**, or a mathematical expression to be taken as starting point in searching for the solution. As for the user's specification of a HINT, a noteworthy possibility is that of proposing a *functional* HINT; that is, to propose the indeterminate function as an expression involving functions possibly depending on many variables each, mapping the original PDE into another PDE. Functional HINTs significantly increase **pdsolve**'s possibilities of finding a solution using users' advice.

## 3 The PDEtools package

A brief overview of the most relevant commands of the package together with a few simple examples, which can be checked by hand, is as follows<sup>5</sup> :

- **pdsolve** looks for the general solution or the complete separation of the variables of a given PDE.

Example:

$$\text{PDE: } \left( \frac{\partial}{\partial y} f(x, y, z) \right) + \frac{1}{xz} \frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial z} f(x, y, z)$$

$$\text{pdsolve's answer: } f(x, y, z) = \_f1(x^2 + 2i\pi + 2\ln(z), z + y)$$

That is, a general solution was returned in terms of an arbitrary function,  $\_f1$ , with two differential invariants as arguments.

- **dchange** performs changes of variables in PDEs and other algebraic objects (integro-differential equations, limits, multiple integrals, etc...). New feature of this version is that **dchange** can also be used to analyze the underlying invariance groups of a PDE, since it works with changes of both the independent and the dependent variables, automatically extending the transformation equations to any required differential order.

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<sup>5</sup>The *PDEtools* package can be found, at any of the Internet addresses of the Maple Share Library, while tutorial sessions and the version under development can be obtained in the anonymous FTP site of the Symbolic Computing Group at UERJ: 152.92.4.69.

Example: The most general 1st. order ODE admitting the rotation group.

$$ODE: \frac{-y + x \left( \frac{\partial}{\partial x} y \right)}{x + y \left( \frac{\partial}{\partial x} y \right)} = H(\sqrt{x^2 + y^2})$$

$$transformation: \{ y = \cos(\varepsilon) y^* - x^* \sin(\varepsilon), x = y^* \sin(\varepsilon) + x^* \cos(\varepsilon) \}$$

$$dchange's answer: \frac{-y^* + x^* \left( \frac{\partial}{\partial x^*} y^* \right)}{x^* + y^* \left( \frac{\partial}{\partial x^*} y^* \right)} = H(\sqrt{y^{*2} + x^{*2}})$$

Introducing canonical coordinates  $(r, \theta)$ , the ODE is reduced to a quadrature:

$$transformation: \{ x = r \cos(\theta), y = r \sin(\theta) \}$$

$$dchange's answer: r \left( \frac{\partial}{\partial r} \theta \right) = H(r)$$

• **sdsolve** looks for a complete (or partial) solution of a *coupled* system of ODEs (linear or not), by mixing linear algebra techniques with a *sequential* approach:

Example:

$$ODEs system: \left\{ \frac{\partial^2}{\partial x^2} h(x) = \frac{g(x)}{f(x)}, \quad \frac{\partial}{\partial x} g(x) = -f(x), \quad \frac{\partial}{\partial x} f(x) = e^{g(x)} \right\}$$

$$sdsolve's answer: \left[ \left\{ f(x) = \tanh \left( \frac{1}{2} \sqrt{-C3} (x + C4) \sqrt{2} \right) \sqrt{2} \sqrt{-C3} \right\}, \right. \\ \left. \left\{ g(x) = \ln \left( \frac{\partial}{\partial x} f(x) \right) \right\}, \left\{ h(x) = - \int \frac{g(x)x}{f(x)} dx + \int \frac{g(x)}{f(x)} dx x + C1 + C2x \right\} \right]$$

• **mapde** maps PDEs into other PDEs, more convenient in some cases. The mappings implemented and tested up to now are:

- PDEs that explicitly depend on the indeterminate function into PDEs which do not (i.e. only depend on it through derivatives).
- linear  $2^{nd}$  order PDEs, with two differentiation variables, into PDEs in canonical form. Though the success depends on the proposed problem, in principle this program works both with PDEs with constant or variable coefficients<sup>6</sup>.

Example: mapping of a  $2^{nd}$  order PDE into canonical form.

$$PDE: x^2 \left( \frac{\partial^2}{\partial x^2} f(x, y) \right) + 2yx \left( \frac{\partial^2}{\partial y \partial x} f(x, y) \right) + y^2 \left( \frac{\partial^2}{\partial y^2} f(x, y) \right) = 0$$

$$mapde's answer: \left( \frac{\partial^2}{\partial \xi_1^2} f(-\xi_1, \xi_2) \right) - \xi_1^2 = 0, \quad \text{where } \left\{ -\xi_2 = -\frac{y}{x}, -\xi_1 = x \right\}$$

- **strip** evaluates the *characteristic strip* associated to a first-order PDE.
- **pdtest** tests a solution found by **pdsolve** for a given PDE by making a careful simplification of the PDE with respect to this solution.

<sup>6</sup>In the case of variable coefficients, this kind of mapping involves the solving of auxiliary PDEs.

## Conclusions

Taking into account the algorithmic character of many of the existing exact solving methods for PDEs, it is clear that computational approaches can play an important role in the progress in this area. In this context, the goal of the material here presented is to contribute both to the discussion of what would be a computational strategy for the problem and to the implementation of that strategy using the mathematical methods and the algebraic computing resources available nowadays. In particular, considering there is still no general method able to solve *all* possible PDEs, great emphasis was put in the *interactive* character of the package (the HINT option and the **dchange** command). We believe this to be a basic feature any PDE-solving package should have.

Finally, routines for the analytical solving of systems of PDEs are now under development. We expect to report them in the near future.

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