

INDUCE REPRESENTATION OF THE POINCARÉ GROUP ON THE LATTICE: SPIN 1/2 AND 1 CASE

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Abstract

Following standard methods we explore the construction of the discrete Poincaré group, the semidirect product of discrete translations and integral Lorentz transformations, using the Wigner-Mackey construction restricted to the momentum and position space on the lattice. The orbit condition, irreducibility and asymptotic limit are discussed.

1 Introduction

In a previous paper of this Symposium [14] we have discussed the induced representations of the euclidean group on the lattice via duality and Fourier transform. In this paper we apply the same method to the induced representation of Poincaré group on the lattice. We explore the problem of irreducibility connected with the orbit condition, the asymptotic limit of the difference equation for the Klein-Gordon and Dirac field.

In section 2 we present the realization of the integral Lorentz transformation, that can be factorized completely with the help of Kac generators, both for the fundamental and spin representation. In section 3 we introduce several types of Fourier transform that will be used to go from momentum to position space. In section 4 we reproduce the properties of covariant bispinors in discrete momentum space in similar fashion to the continuous one. In section 5, we elaborate the induced representation of Poincaré group on the lattice following standard procedure and we compare the results derived from the different types of Fourier transform.

2 Integral Lorentz transformations

A Lorentz transformation is integral if leaves invariant the cubic lattice and at the same time keeps invariant the bilinear form

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 \quad (1)$$

According to Coxeter [1] all integral Lorentz transformations (including reflections) are obtained by combining the operations of permuting the spatial coordinates x_1, x_2, x_3 and changing the signs of any of the coordinates x_0, x_1, x_2, x_3 together with the operation of adding the quantity $x_0 - x_1 - x_2 - x_3$ to each of the four coordinates of a point.

These operations can be described geometrically by the reflections on the plans perpendicular to the vectors

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3, \quad \alpha_4 = -(e_0 + e_1 + e_2 + e_3)$$

where $\{e_0, e_1, e_2, e_3, \}$ is an orthonormal basis.

In matritial form these reflections are

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

These reflections constitute a Coxeter group, the Dynkin diagram of which is the following



Kac has proved [2] that S_1, S_2, S_3, S_4 generate all the integral Lorentz transformations that keep invariant the upper half of the light cone.

This result can be used to factorized any integral Lorentz transformation that belong to the proper orthochronous group. Let

$$L \equiv \begin{pmatrix} a & e & f & g \\ b & & & \\ c & & * & \\ d & & & \end{pmatrix} \quad (2)$$

be an integral matrix of determinant +1, satisfying

$$LgL^t = g, \quad g = \text{diag}(1, -1, -1, -1) \quad (3)$$

and also $a \geq 1$. From

$$a^2 - b^2 - c^2 - d^2 = 1 \quad (4)$$

it follows that only one of b, c, d can be zero. Suppose $a > 1$. Then we apply S_1, S_2, S_3 to L from the left until b, c, d become non-positive integers. To the resulting matrix we apply S_4 . We get

$$L' = \begin{pmatrix} a' & e' & f' & g' \\ b' & & & \\ c' & & * & \\ d' & & & \end{pmatrix}$$

with $a' = 2a + b + c + d$. Obviously

$$(a + b + c + d)(a - b - c - d) = 1 - 2bc - 2bd - 2cd < 0$$

therefore $a + b + c + d < 0$ or

$$2a + b + c + d = a' < a \quad (5)$$

By iteration of the same algorithm we get

$$a > a' > a'' > \dots > a^{(k)} \geq 1 \quad (6)$$

The last inequality is a consequence of the fact that L and S_4 belong to the complete Lorentz group. Following this process we get an integral matrix with $a^{(k)} = 1$ which is a combination of S_1, S_2, S_3 , giving all the 24 elements of the cubic group on the lattice.

Therefore a general integral Lorentz transformation of the proper orthochronous type L can be decomposed as

$$L = P_1^\eta P_2^\theta P_3^i S_4 \dots S_4 P_1^\delta P_2^\varepsilon P_3^\zeta S_4 \left\{ S_1^\alpha S_2^\beta S_3^\gamma \right\}_{\text{all permutations}} \quad (7)$$

where $P_1 = S_1 S_2 S_3 S_2 S_1$, $P_2 = S_2 S_3 S_2$, $P_3 = S_3$ are the matrix which change sign of b, c, d and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota \dots = 0, 1$

A particular case of integral Lorentz transformations are the boost or integral transformations of two inertial systems with relative velocity. The general expression for these transformations can be obtained with the help of Cayley parameters [3]. Let us take $n = p = q = 0$ and m, r, s, t , integer numbers. We have two cases, corresponding to two diophantine equations:

$$\text{i) } m^2 - r^2 - s^2 - t^2 = 1$$

$$L = \begin{pmatrix} m^2 + r^2 + s^2 + t^2 & 2mr & 2ms & 2mt \\ 2mr & m^2 + r^2 - s^2 - t^2 & 2rs & 2rt \\ 2ms & 2rs & m^2 - r^2 + s^2 - t^2 & 2st \\ 2mt & 2rt & 2st & m^2 - r^2 - s^2 + t^2 \end{pmatrix} \quad (8)$$

$$\text{ii) } m^2 - r^2 - s^2 - t^2 = 2$$

$$L = \begin{pmatrix} m^2 - 1 & mr & ms & mt \\ mr & r^2 + 1 & rs & rt \\ ms & rs & s^2 + 1 & st \\ mt & rt & st & t^2 + 1 \end{pmatrix} \quad (9)$$

The solutions of the diophantine equation i) and ii) are obtained by applying all the Coxeter reflections to the vector $(1, 0, 0, 0)$ in case i) and to the vector $(2, 1, 1, 0)$ in case ii)

From the inspection of (7) if we take the quotient of L with respect the subgroup of all integral rotations or cubic group we are left with the coset representatives which

are not exhausted by the pure Lorentz transformations (8) or (9), because these are always symmetric matrices. Therefor we have to add all the integral Lorentz matrix that applied to the vector $(1, 0, 0, 0)$ gives all the integral vectors of the type (m, r, s, t) with $m^2 - r^2 - s^2 - t^2 = 1$. This is equivalent to say that we apply to $(1, 0, 0, 0)$ not only the matrix L given by (8) but also its square root matrix, namely,

$$\sqrt{L} = \begin{pmatrix} m & r & s & t \\ r & 1 + \frac{r^2}{m+1} & \frac{rs}{m+1} & \frac{rt}{m+1} \\ s & \frac{rs}{m+1} & 1 + \frac{s^2}{m+1} & \frac{st}{m+1} \\ t & \frac{rt}{m+1} & \frac{st}{m+1} & 1 + \frac{t^2}{m+1} \end{pmatrix} \quad (10)$$

In position space the space-time coordinates of the lattice x_μ are integer numbers. They transform under integral Lorentz transformations into integral coordinates. The same is true for the increments Δx_μ .

In momentum space the components of the four-momentum are not integer numbers but they can be constructed with the help of integral coordinates, namely,

$$p_\mu \equiv m_0 c \left(\frac{c\Delta t}{((c\Delta t)^2 - (\Delta\vec{x})^2)^{1/2}}, \frac{\Delta\vec{x}}{((c\Delta t)^2 - (\Delta\vec{x})^2)^{1/2}} \right)$$

If Δx_μ transform under integral Lorentz transformations the new p'_μ will be given in terms of integral $\Delta x'_\mu$.

Using the homomorphism between the groups $SO(3, 1)$ and $SL(2, \mathcal{C})$ we obtain the representation of integral Lorentz transformations in 2-dimensional complex matrices. From the knowledge of the Cayley parameters [5] we read off the matrix elements of $\alpha \in SL(2, \mathcal{C})$

$$\alpha = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} m + t + i(n - \lambda), & -p + r + i(q + s) \\ p + r + i(q - s), & m - t - i(n + \lambda) \end{pmatrix} \quad (11)$$

$$\Delta = \det \alpha = m^2 - r^2 - s^2 - t^2 + m^2 + p^2 + q^2 - \lambda^2 \quad (12)$$

For instance, we calculate the 2-dimensional representation of the Coxeter reflection S_i multiplied by the parity operator P in order to get an element of the proper Lorentz group) identifying its matrix elements with Lorentz matrix written in terms of Cayley parameters: Easy calculations give the unique solutions:

$$D(PS_1) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 - i \\ 1 - i & 0 \end{pmatrix} \quad (13)$$

$$D(PS_2) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \quad (14)$$

$$D(PS_3) = \pm \begin{pmatrix} i & \\ & -i \end{pmatrix} \quad (15)$$

$$D(PS_4) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 - i \\ -1 - i & 2i \end{pmatrix} \quad (16)$$

The integral Lorentz transformations without rotations as given in (8) and (9) have a 2-dimensional representation making $n = p = q = \lambda = 0$ in (11) and the choice $\Delta = m^2 - r^2 - s^2 - t^2 = 1$ or 2 in (12).

In order to complete the picture we have to add the 2-dimensional representation of the matrix \sqrt{L} given in (10) which turns out to be

$$\alpha = \frac{1}{\sqrt{2(m+1)}} \begin{pmatrix} m+1+t & r-is \\ r+is & m+1-t \end{pmatrix} \quad (17)$$

We give also the 2×2 matrix representation of the discrete momentum. Let p_μ a four momentum which is obtained by applying all the integral Lorentz transformations, given by (7) divided by the cubic group, to the vector $(m_0c, 0, 0, 0)$. In 2-dimensional matrix form this is equivalent to apply the matrix α given by (17) to the unit matrix multiply by m_0c :

$$\alpha \begin{pmatrix} m_0c & 0 \\ 0 & m_0c \end{pmatrix} \alpha^+ = m_0c \begin{pmatrix} m+t & r-is \\ r+is & m-t \end{pmatrix} \quad (18)$$

which is of the standard form if we identify the components of the 4-momentum as

$$p_\mu = m_0c(m, r, s, t) \quad (19)$$

with m, r, s, t integral numbers, satisfying $m^2 - r^2 - s^2 - t^2 = 1$.

3 Fourier transform on the lattice

In order to go from position space to momentum space on the lattice we can define several restrictions of the continuous variables of Fourier transform to the discrete variables on the lattice.

I type. Discrete position and momentum variables of finite rank

We construct an orthonormal basis [6]

$$f_j(p_m) = \left(\frac{1 + \frac{1}{2}i\varepsilon p_m}{1 - \frac{1}{2}\varepsilon p_m} \right)^j, \quad j = 0, 1, \dots, N-1 \quad (20)$$

$$p_m = \frac{2}{\varepsilon} tg \frac{\pi}{N} m, \quad m = 0, 1, \dots, N-1$$

that satisfy periodic boundary conditions:

$$f_0(p_m) = f_N(p_m), \quad p_{m+N} = p_m,$$

orthogonality relations:

$$\frac{1}{N} \sum_{j=0}^{N-1} f_j^*(p_m) f_j(p_{m'}) = \delta_{mm'} \quad (21)$$

and completeness relations:

$$\frac{1}{N\varepsilon} \sum_{m=0}^{N-1} f_j^*(p_m) f_{j'}(p_m) = \frac{1}{\varepsilon} \delta_{jj'} \quad (22)$$

The finite Fourier transform reads

$$\hat{F}_m = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j^*(p_m) F_j \quad (23)$$

for some periodic function F_j on the lattice

$$F_{j+N} = F_j$$

If we write $f_j(p_m) \equiv \exp(i\frac{2\pi}{N}mj)$ this transform coincide with standard finite Fourier transform [7].

II type. Discrete position and continuous momentum

When we restrict the position variables in continuous Fourier transform to discrete values we obtain the Fourier series.

$$\text{Orthonormal basis : } \left\{ \frac{1}{\sqrt{2\pi}} e^{ikj\varepsilon} \right\}_{j=-\infty}^{\infty} \equiv f_j(k) \quad (24)$$

$$\text{Orthogonality relations : } \int_{-\pi/\varepsilon}^{\pi/\varepsilon} f_j^*(k) f_{j'}(k) dk = \frac{1}{\varepsilon} \delta_{jj'} \quad (25)$$

$$\text{Completeness relation : } \sum_{j=-\infty}^{\infty} f_j^*(k) f_j(k') = \frac{1}{\varepsilon} \delta(k - k') \quad (26)$$

Fourier expansion: for a periodic function $F(k)$

$$F(k) = \sum_{j=-\infty}^{\infty} f_j^* F_j \quad (27)$$

$$F_j = \int_{-\pi/a}^{\pi/a} f_j(k) F(k) dk \quad (28)$$

Now we make the change of variable

$$P = \frac{2}{\varepsilon} tg \frac{1}{2} k\varepsilon \quad (29)$$

$$f_j(p) = \frac{1}{\sqrt{2\pi}} \left(\frac{1 + \frac{1}{2}i\varepsilon p}{1 - \frac{1}{2}i\varepsilon p} \right)^j \quad (30)$$

and the orthogonality relations become

$$\int_{-\infty}^{\infty} f_j^*(p) f_{j'}(p) \frac{dp}{1 + \frac{1}{4}\varepsilon^2 p^2} = \frac{1}{\varepsilon} \delta_{jj'} \quad (31)$$

and the completeness relation

$$\sum_{j=-\infty}^{\infty} f_j^*(p) f_j(p') = \left(1 + \frac{1}{4}\varepsilon^2 p^2 \right) \delta(p - p') \quad (32)$$

Notice that $f_j(p)$ may not be a periodic function.

III type: discrete position and discrete momentum of infinite rank

We construct an orthonormal basis

$$f_n(j) = \left(\frac{1 + \frac{1}{2}i\varepsilon k}{1 - \frac{1}{2}i\varepsilon p} \right)^n, \quad k, n \in \mathcal{Z} \quad (33)$$

satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N f_n(k) f_n(k') = \delta_{kk'} \quad (34)$$

(Proof: For $k \neq k'$ we use the identity

$$1 + \cos \theta + \cos 2\theta + \dots + \cos N\theta = \frac{1}{2} + \frac{\sin \left(N + \frac{1}{2}\right) \theta}{\sin \left(\frac{\theta}{2}\right)}$$

with $e^{i\theta} \equiv f_1^*(k) f_1(k')$).

The completeness relation is now:

$$\sum_{k=-\infty}^{\infty} f_n^*(k) f_{n'}(k) = \lim_{L \rightarrow \infty} \delta_L(n - n') \quad (35)$$

where

$$\delta_L(n - n') = \sum_{j=-L}^L f_n(k) f_{n'}^*(k) \quad (36)$$

is a δ sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \delta_L(n - n') = 1 \quad (37)$$

(Proof:

$$\begin{aligned} & \frac{1}{2N+1} \sum_{n=-N}^N \delta_L(n - n') = \\ &= 1 + \sum_{k=1}^L \frac{1}{2N+1} \sum_{n=-N}^N (f_n(k) f_{n'}^*(k) + c.c.) = \\ &= 1 + \sum_{k=1}^L \frac{1}{2N+1} \left(\frac{\sin \left(N + \frac{1}{2}\right) \theta}{\sin \frac{1}{2}\theta} f_{n'}^*(k) + c.c. \right) \xrightarrow{N \rightarrow \infty} 1 \end{aligned}$$

for all L , as required)

The Fourier transform becomes:

$$\hat{F}(k) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N f_n(k) F_n \quad (38)$$

where $F_n \rightarrow 0$ when $n \rightarrow \infty$ and $\frac{1}{2N+1} \sum_{n=-N}^N F_n \xrightarrow{N \rightarrow \infty} 0$

$$F_n = \sum_{k=-\infty}^{\infty} f_n(k) \hat{F}(k) \quad (39)$$

The Fourier transform of type III was introduced in [8]. When

$$\begin{aligned} n, n' &\rightarrow \infty, \quad \varepsilon \rightarrow 0 \quad m\varepsilon \rightarrow x \\ f_n(k) &\rightarrow e^{ikx} \end{aligned}$$

the orthogonality relations converges

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N f_n^*(k) f_n(k') = \int_{-\pi}^{\pi} e^{-ikx} e^{ik'x} dk$$

the completeness relations becomes

$$\sum_{j=-\infty}^{\infty} f_n^*(k) f_{n'}(k) \rightarrow \sum_{k=-\infty}^{\infty} e^{-ikx} e^{ikx'}$$

and the Fourier transform converges to:

$$\begin{aligned} \hat{F}(k) &= \int e^{ixk} F(x) dx \\ F(k) &= \sum_{k=-\infty}^{\infty} e^{ikx} \hat{F}(k) \end{aligned}$$

4 Dirac and vector representation of the Lorentz group and covariant wave equations

Let $L(\alpha)$ be an element of the proper Lorentz group corresponding to the element $\alpha \in SL(2, C)$ and I_s the parity operator. If the components of four-momentum are written as 2×2 matrix

$$\tilde{p} \equiv p^\mu \sigma_\mu = p^0 \sigma_0 + p^i \sigma_i \quad (40)$$

where $\sigma_0 = 1$ and σ_i are the Pauli matrices.

The transformations of \tilde{p} under parity and $SL(2, C)$ are

$$I_s : \tilde{p} \rightarrow \tilde{p}^s = p^0 \sigma_0 - p^i \sigma_i = (\det \tilde{p}) (\tilde{p})^{-1}$$

$$\alpha : \tilde{p} \rightarrow \alpha \tilde{p} \alpha^+$$

$$\tilde{p}^s \rightarrow (\alpha^+)^{-1} \tilde{p}^s \alpha^{-1} \quad (41)$$

It follows

$$I_s L(\alpha) I_s^{-1} = L((\alpha^+)^{-1}) \quad (42)$$

Therefore the matrix $(\alpha^+)^{-1}$ gives an other 2-dimensional representation of the Lorentz group non-equivalent to $\alpha \in SL(2, C)$. In order to enlarge the proper Lorentz group with space reflection we take both representations α and $(\alpha^+)^{-1}$.

Let $\pi \equiv \{I, I_s\}$ the space reflection group and $\alpha \in SL(2, C)$, then the semidirect product

$$SL(2, C) \otimes \pi$$

with the multiplication law

$$(\alpha, \pi) (\alpha', \pi') = (\alpha \alpha', \pi \pi') \text{ if } \pi = I \quad (43)$$

$$(\alpha, \pi) (\alpha', \pi') = \left(\alpha (\alpha'^+)^{-1}, \pi \pi' \right) \text{ if } \pi = I_s \quad (44)$$

form a group.

This group has a 4-dimensional representation, a particular elements of which is

$$\bar{D}(\alpha, I) = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^+)^{-1} \end{pmatrix}, \quad \bar{D}(e, I_s) = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \quad (45)$$

that satisfy

$$\bar{D}(e, I_s) \bar{D}(\alpha, I) \bar{D}(e, I_s^{-1}) =, \quad \bar{D}\left((\alpha^+)^{-1}, I\right) \quad (46)$$

With respect to this representation a-4-component spinor in momentum space $\psi(p)$ transform as follows

$$U(\alpha, I) \bar{\psi}(p) = \bar{D}(\alpha, I) \bar{\psi}(L^{-1}(\alpha)p) \quad (47)$$

$$U(e, I_s) \bar{\psi}(p) = \bar{D}(e, I_s) \bar{\psi}(I_s, p) \quad (48)$$

Using a similarity transformation we obtain an equivalent representation

$$D(\alpha, \pi) = M \bar{D}(\alpha, \pi) M^{-1}$$

with

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ -\sigma_0 & \sigma_0 \end{pmatrix}$$

In this representation

$$D(\alpha, I) = \frac{1}{2} \begin{pmatrix} \alpha + (\alpha^+)^{-1}, & -\alpha + (\alpha^+)^{-1} \\ -\alpha + (\alpha^+)^{-1}, & \alpha + (\alpha^+)^{-1} \end{pmatrix} \quad (49)$$

$$D(e, I_s) = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad (50)$$

The new four-spinor

$$\psi(p) = M \bar{\psi}(p)$$

transform as

$$U(\alpha, I) \psi(p) = D(\alpha, I) \psi(L^{-1}(\alpha)p) \quad (51)$$

$$U(e, I_s) \psi(p) = D(e, I_s) \psi(I_s, p) \quad (52)$$

The Dirac wave equation can be considered as a consequence of the relativistic invariance [9]

In the rest system we want a projection operator that selects one irreducible representation out of the Dirac representation. This is achieved in the rest system by

$$Q = \frac{1}{2}(I + \beta), \quad \beta \equiv \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad (53)$$

In order to get the projection operator in an arbitrary system we apply $D(\kappa)$ given by (49) and (17)

$$Q \rightarrow Q(\kappa) = D^{-1}(\kappa) Q D(\kappa) = \frac{1}{2}(I + W(\kappa)) \quad (54)$$

where

$$W(\kappa) = \frac{1}{2} \begin{pmatrix} \kappa^+ \kappa + (\kappa^+ \kappa)^{-1}, & -\kappa^+ \kappa + (\kappa^+ \kappa)^{-1} \\ \kappa^+ \kappa - (\kappa^+ \kappa)^{-1}, & -\kappa^+ \kappa - (\kappa^+ \kappa)^{-1} \end{pmatrix} \quad (55)$$

Using the identities

$$\begin{aligned} (\kappa^+ \kappa)^{-1} &= \frac{1}{m_0 c} \sum_{\mu=0}^3 \sigma^\mu p_\mu \\ \kappa^+ \kappa &= \frac{1}{m_0 c} \sigma^0 p_0 - \sum_{j=1}^3 \sigma^j p_j \end{aligned} \quad (56)$$

we find

$$W(\kappa) = \frac{1}{m_0 c} \sum_{\mu=0}^3 \gamma^\mu p_\mu \quad (57)$$

where γ^μ are Dirac matrices with the realization

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \\ \gamma^0 &= \gamma_0, \quad \gamma^j = -\gamma_j \end{aligned}$$

Collecting these result we obtain the Dirac equation in momentum space

$$Q(\kappa) \psi(p) = \frac{1}{2} (I + W(\kappa) \psi(p)) = \psi(p)$$

or

$$\sum_{\mu=0}^3 (\gamma^\mu p_\mu - m_0 c I) \psi(p) = 0 \quad (58)$$

(An equivalent method can be used applying to the projection operator the Foldy-Wouthuysen transformation [10])

We apply the operator $\sum (\gamma^\mu p_\mu - m_0 c)$ from the left to (59) and obtain

$$(p^\mu p_\mu - m_0^2 c^2) \psi(p) = 0 \quad (59)$$

The Dirac equation is invariant under the group (α, π) defined before. In other words, if $\psi(p)$ is a solution of the Dirac equation, so is $U(\alpha, \pi) \psi(p)$. Put $\pi = I$. Then

$$U(\alpha, I) \psi(p) = D(\alpha, I) \psi(L^{-1}p)$$

$$D^{-1}(\alpha) Q(\kappa) D(\alpha) = Q(x\alpha)$$

$$Q(\kappa\alpha) \psi(L^{-1}p) = \psi(L^{-1}p)$$

Then

$$\begin{aligned} Q(\kappa) U(\alpha, I) \psi(p) &= Q(\kappa) D(\alpha, I) \psi(L^{-1}(\alpha)p) = D(\alpha, I) Q(\kappa\alpha) \psi(L^{-1}(\alpha)p) = \\ &= D(\alpha, I) \psi(L^{-1}(\alpha)p) = U(\alpha, I) \psi(p) \end{aligned}$$

as required. For the space reflection

$$Q(I_s \kappa) \psi(I_s p) = \psi(I_s p)$$

$$D^{-1}(I_s) Q(\kappa) D(I_s) = Q(I_s p)$$

$$U(e, I_s) \psi(p) = D(I_s) \psi(I_s p)$$

we get

$$\begin{aligned} Q(\kappa) U(e, I_s) \psi(p) &= Q(\kappa) D(I_s) \psi(I_s p) = D(I_s) Q(I_s \kappa) \psi(I_s p) = \\ &= D(I_s) \psi(I_s p) = U(e, I_s) \psi(p) \end{aligned}$$

as required. Notice that all properties of Dirac representation in continuous momentum space are carried out to the discrete momentum space without modification.

For the vector representation of the Lorentz group we take an element $L(p)$ such that takes the momentum in the rest system to an arbitrary system.

In the rest system the projection operator is

$$Q = \frac{1}{2}(1 - g) = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (60)$$

Then

$$L^{-1}(p) Q L(p) \equiv Q(p) = g_{\mu\nu} - \frac{p_\mu p_\nu}{m_0^2} \quad (61)$$

And the wave equation in momentum space becomes $Q(p) \psi(p) = \psi(p)$ or

$$\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) \psi_\mu(p) = \psi_\nu(p) \quad (62)$$

where $\psi_\mu(p)$ is a 4-component vector function.

In order to construct wave equation in configuration space for the Dirac and vector representation we use the Fourier transform given in section 3. We take the three types

Type I. Boundary condition imposes finite values for the momentum p_μ

$$p_\mu = \frac{2}{\varepsilon_\mu} t g \frac{\pi}{N} m_\mu \quad m_\mu = 0, 1, \dots, N-1 \quad (63)$$

Define the difference operator [8]

$$\delta_\mu^+ = \frac{1}{\varepsilon_\mu} \Delta_\mu \prod_{\nu \neq \mu} \tilde{\Delta}_\nu, \quad \delta_\mu^- = \frac{1}{\varepsilon_\mu} \nabla_\mu \prod_{\nu \neq \mu} \tilde{\nabla}_\nu, \quad \mu, \nu = 1, 2, 3 \quad (64)$$

$$\eta^+ = \prod_{\mu=0}^3 \tilde{\Delta}_\mu, \quad \eta^- = \prod_{\mu=0}^3 \tilde{\nabla}_\mu \quad (65)$$

Multiply the Dirac equation (60) by

$$\eta^+ f(n_\mu, p_\mu)$$

with

$$f(n_\mu, p_\mu) = \prod_{\mu=0}^3 \left(\frac{1 + \frac{i}{2}\varepsilon_\mu p_\mu}{1 - \frac{i}{2}\varepsilon_\mu p_\mu} \right)^{n_\mu} \quad m_\mu, n_\mu \in \mathcal{Z} \quad (66)$$

the plane waves on the lattice, and using the properties of these functions

$$\frac{1}{\varepsilon_\mu} \Delta_\mu f(n_\mu, p_\mu) = ip_\mu \tilde{\Delta}_\mu f(n_\mu, p_\mu) \quad (67)$$

we obtain

$$(i\gamma^\mu \delta_\mu^+ - m_0 c \eta^+) \psi(p_\mu) f(n_\mu, p_\mu) = 0$$

Adding for $m = 0, 1 \dots N-1$, we get the inverse Fourier transform

$$\psi(\eta_\mu) = \sum_{m=0}^{N-1} \psi(p_\mu) f(n_\mu, p_\mu) \quad (68)$$

that satisfies the Dirac equation on the lattice

$$(i\gamma^\mu \delta_\mu^+ - m_0 c \eta^+) \psi(n_\mu) = 0 \quad (69)$$

Type II. The momentum is continuous and has the range $-\infty < p < \infty$

The plane waves appearing in the Fourier transform are

$$f(n_\mu, p_\mu) = \frac{1}{\sqrt{2\pi}} \prod_{\mu=0}^3 \left(\frac{1 + \frac{1}{2}i\varepsilon_\mu p_\mu}{1 - \frac{1}{2}i\varepsilon_\mu p_\mu} \right)^{n_\mu} \quad (70)$$

As before we multiply the Dirac equation in momentum space by

$$\eta^+ f(n_\mu, p_\mu)$$

and using the property

$$\frac{1}{\varepsilon_\mu} \Delta_\mu f(n_\mu, p_\mu) = ip_\mu \tilde{\Delta}_\mu f(n_\mu, p_\mu) \quad (71)$$

we get

$$(i\gamma^\mu \delta_\mu^+ - m_0 c \eta^+) \psi(p_\mu) f(n_\mu, p_\mu) = 0 \quad (72)$$

Integrating for p and putting

$$\psi(n_\mu) = \int_{-\infty}^{\infty} \psi(p_\mu) f^*(n_\mu, p_\mu) \prod_{\mu=0}^3 \left(\frac{dp_\mu}{1 + \frac{1}{4}\varepsilon_\mu^2 p_\mu^2} \right) \quad (73)$$

we obtain the desired Dirac equation in configuration space.

Type III. The momentum is discrete and was given in section 2, namely

$$k_\mu = m_0 c(m, r, s, t) \quad , \quad m^2 - r^2 - s^2 - t^2 = 1 \quad , \quad m, r, s, t \in \mathcal{Z} \quad (74)$$

The plane waves are

$$f(n_\mu, k_\mu) = \prod_{\mu=0}^3 \left(\frac{1 + \frac{1}{2}i\varepsilon_\mu k_\mu}{1 - \frac{1}{2}i\varepsilon_\mu k_\mu} \right)^{n_\mu} \quad (75)$$

Notice that we do not impose boundary conditions therefore the parameters k_μ are discrete and infinite. We multiply the Dirac equation in momentum space by

$$\eta^+ f(n_\mu, k_\mu)$$

and using

$$\frac{1}{\varepsilon_\mu} \Delta_\mu f(n_\mu, k_\mu) = ik_\mu \tilde{\Delta}_\mu f(n_\mu, k_\mu) \quad (76)$$

we get as before

$$(i\gamma^\mu \delta_\mu^+ - m_0 c \eta^+) \psi(k_\mu) f(n_\mu, k_\mu) = 0. \quad (77)$$

Summing over all values of Cayley parameter that define a four-momentum we get

$$(i\gamma^\mu \delta_\mu^+ - m_0 c \eta^+) \psi(n_\mu) = 0 \quad (78)$$

where

$$\psi(n_\mu) = \sum_{m,r,s,t} \psi(k_\mu) f(n_\mu, k_\mu). \quad (79)$$

Notice that $\psi(k_\mu)$ should satisfy

$$\psi(k_\mu) \xrightarrow{K_\mu \rightarrow \infty} 0 \quad \sum_{m,r,s,t} \psi(k_\mu) < \infty$$

5 Induced representations of the discrete Poincaré groups

Let $\mathcal{P}_+^\uparrow = T_4 \times_5 SO(3, 1)$ be the Poincaré group restricted to the integral Lorentz transformations and discrete translations on the lattice with the group composition

$$(a, \wedge) (a' \wedge') = (a + \wedge a', \wedge \wedge') \quad (80)$$

In order to construct irreducible representations we follow standard method

- (1) Choose an *UIR* $D^{\mathring{k}}(a)$ of the translation group T_4
- (2) Define a little group $H \in SO(3, 1)$ by the stability condition

$$h \in H \quad : \quad D^{\mathring{k}}(h^{-1}a) = D^{\mathring{k}}(a) \quad (81)$$

- (3) This condition leads to the following decomposition of the *UIR* of $T_4 \times_s H$

$$D^{\mathring{k}, \alpha}(a, h) = D^{\mathring{k}}(a) \otimes D^\alpha(h) \quad (82)$$

- (4) Choose coset generators c of $T_4 \times_s H$ constructed from the group action

$$(\tilde{a}, \tilde{\wedge}) c = c'(a, h) \quad (83)$$

(5) Then the induced representations is:

$$D^{\overset{\circ}{k},\alpha}(\tilde{a}, \tilde{\lambda}) = D^{\overset{\circ}{k},\alpha}(a, h) \delta\left((c')^{-1}(\tilde{a}, \tilde{\lambda})c, (a, h)\right) \quad (84)$$

this is an *UIR* of \mathcal{P}_+^\uparrow

In the case of the discrete Pincaré group for the massive case, $\overset{\circ}{k} = m_0c(1, 0, 0, 0)$ and the little group is the cubic group that satisfies condition (81).

For the coset representative $c \equiv (0, \Lambda)$ we can choose the integral Lorentz transformations $\Lambda \equiv L(k)$ that take $\overset{\circ}{k}$ into an arbitrary discrete momentum. The Dirac delta function in (84) is zero unless

$$(a, h) = (0, L^{-1}(k'))(\tilde{a}, \tilde{\lambda})(0, L(k)) = (L^{-1}(k')\tilde{a}, L^{-1}(k')\tilde{\lambda}L(k))$$

Substituting in (84) and using (82) we get

$$D_{k'k}^{\overset{\circ}{k},\alpha}(\tilde{a}, \tilde{\lambda}) = D_{k'k}^{\overset{\circ}{k}}(L^{-1}(k')\tilde{a}) D_{k'k}^{\alpha}(L^{-1}(k')\tilde{\lambda}L(k)) \quad (85)$$

The spinor representation of the second factor is given with respect to the element $L^{-1}(k')\tilde{\lambda}L(k)$ that belongs to the little group, $SU(2)$, and this group becomes irreducible when restricted to the spin 1/2 or spin 1 representation [12]. The first factor can be written:

$$D_{k'k}^{\overset{\circ}{k}}(L^{-1}(k')\tilde{a}) = D^{k'}(\tilde{a}) \quad (86)$$

where $k' = (L^{-1}(k'))^T \overset{\circ}{k}$ are all the points that defined the *UIR* of translation group and belong to the orbit.

We want to characterized the discrete orbits by functions in the momentum space that vanish on the orbit points and only in these points. A natural way to construct these functions is through the Dirac equation in momentum space (59). Multiplying this equation from the left by $(\gamma_\mu p_\mu + m_c)$ we get

$$(p_\mu p_\mu - m_c^2) \psi(p) = 0 \quad (87)$$

The p_μ have different physical meaning as we stressed in the realization of Dirac equation in position space. We have to check whether the constraints (88) satisfy the following conditions:

- (1) they should be polynomials or infinite product of polynomials that vanish on the orbit points,
- (2) the constraints should admit a periodic extension of the momentum space,
- (3) the constraints should vanish on a orbit and they must be Lorentz invariant,
- (4) the constraints should vanish only on the points of the orbit,
- (5) when the lattice spacing goes to zero, the difference equation should go to the continuous Minkowski limit.

In the Fourier transform of type I and II the p variable are related to the physical momentum k by the expresion

$$p_\mu = \frac{2}{\varepsilon} tg \frac{\pi}{N} m_\mu \quad m_\mu = 0, 1, \dots, N-1$$

or

$$p_\mu = \frac{2}{\varepsilon} tg \pi \varepsilon k_\mu \quad k_\mu \in \mathcal{Z}$$

Due to the special trigonometric function this expression is periodic in momentum space. Nevertheless when Lorentz transformations are applied to the components of physical momentum the new momentum do not satisfies the constraint equation and the constraints vanish at different points. Therefore conditions (3) and (4) are violated, but they can be recovered in the asymptotic limit when $\varepsilon \rightarrow 0$

$$((\pi\varepsilon k_\mu) (\pi\varepsilon k_\mu) - m_0^2) \psi(k_\mu) = 0$$

In the case of Fourier transform of type III the p_μ coincide whit the physical momentum whose discrete values are the points of the orbit given by $\wedge_\mu \overset{\circ}{p}_\mu$ and therefore condition (3) and (4) are fullfild, although (2) is violated. In all the three cases the wave equation on position space, as described in section 4 gives in the asymptotic limit, the continuous Dirac equation.

6 Concluding remarks

We have attempted a new program for introducing Poincaré symmetry on the lattice, that was considered broken, as many authors have claimed [13]. So far some points of this program have been achieved; such as, realization of all the integral Lorentz transformation and is representation in 2-dimensional space. Several *UIR* of the translation group on the lattice, some versions of the Fourier transform with discrete position and discrete or continuous momentum, the Dirac equation on the lattice in position space. As far as the induced representation of the Poincaré group, following similar technic as in the continuous case, leads to some inconsistensy: for the type I and II of the *UIR* of translation group the scheme is irreducible and invariant only in the asymptotic limit, but they satisfy all the requirement for the induced representation. For the type III the representation is Lorentz invariant and irreducible, but the requirement of the orbit condition necessary for the induced representations is kept only in the asymptotic limit.

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