Chiral symmetry on the lattice [1]

P. Hasenfratz
Institute for Theoretical Physics
University of Bern
Sidlerstrasse 5, CH-3012 Bern, Switzerland

Abstract

As a non-perturbative and gauge invariant regularization the lattice provides a tool for deeper understanding of the celebrated Yang-Mills theory, QCD and chiral gauge theories. For illustration, I discuss some analytic developments on the lattice related to chiral symmetry, chiral fermions and improvement programs. Chiral symmetry on the lattice has an amazing history, and it might influence our perception of a symmetry beyond this example.

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1 Introduction

After the introduction of non-Abelian gauge theories [1] it took a long time to understand how to set up perturbation theory [2] and control the ultraviolet fluctuations [3] in such models. Asymptotic freedom [4] raised the possibility to connect non-Abelian gauge theories with the phenomenology of strong interactions. It was not clear, however, what to do with the wild infrared divergences present in perturbation theory. A few courageous theorists with good intuition interpreted this problem as a virtue: objects with color are confined by strong forces and the infrared problem in perturbation theory is an indication of that. Quantum chromodynamics (QCD), the theory of quarks in interaction with gluons in an $SU(3)$ gauge theory became an attractive candidate to describe the physics of hadrons [5].

In 1974 Wilson [6] presented a $U(1)$ gauge theory with fermions on a hypercubic four dimensional Euclidean lattice which has confinement in the $g \to \infty$ limit, where g is the bare gauge coupling. Although in this limit the model lives on a coarse lattice far from the continuum this work introduced many of the ideas, tools and notions which lead to a new field in particle physics. The formulation is fully gauge invariant and independent of perturbation theory. It creates also a strong link to critical phenomena in statistical physics. The construction was generalized to $SU(N)$ in the same paper. The confinement problem can be formulated in this language the following way: do these models stay in the confining phase as the bare gauge coupling is tuned towards $g = 0$ where the continuum theory is defined? It took a few years until the new tools (mean field and strong coupling expansion techniques [7], Hamilton formulation [8], Monte Carlo simulations [9]) have been adapted to quantum field theories, in particular to QCD.

In 1980 Creutz demonstrated [10] in a Monte Carlo simulation that close to the continuum limit the string tension σ between a static quark-antiquark pair in an $SU(2)$ Yang-Mills theory behaves as expected from continuum renormalization group considerations. This was a spectacular result indicating that one can get close to the continuum with modest computing power in this non-Abelian gauge theory and confinement survives this limit. Of course, today's state-of-the-art calculations control the systematic and statistical errors of the static quark-antiquark potential both in $SU(2)$ and $SU(3)$ gauge theories much better. The potential is followed up to 4 fm distances, the fine details of the fluctuating color flux tube are seen and quantitatively determined [11]. Although it is not proven rigorously, there remains little doubt that in these gauge theories the static quarks are confined in the continuum limit.

QCD, the theory of strong interactions is, however, a Yang-Mills gauge theory coupled to dynamical quarks. It has an extremely rich phenomenology most of which is beyond the reach of perturbation theory. QCD is an exciting theory in itself. In addition it enters most of the processes in the standard model in a relevant way either in hadronic matrix elements, or as a background. For a recent general overview on the theoretical and numerical activities in lattice QCD I refer to ref. [12] and the references therein.

In this contribution I will mainly consider the history of chiral symmetry on the lattice which was long and troubled and had an amazingly nice upshot. This outcome might influence our perception on the realization of a symmetry in a quantum field theory in general. The following overview is non-technical. For further details and applications I refer to the summaries in [13, 14, 15].

2 QCD on the lattice

A hyper-cubic lattice is a natural regularization when path integrals are used to formulate a quantum field theory in $d = 4$ Euclidean space. The quark field ψ lives on the lattice points indexed by an integer vector $x \in Z_4$ while the gauge fields are associated with the links $(x, \hat{\mu}), \mu = 1, \ldots, 4$ of the lattice. Derivatives are replaced by finite difference operators using, for example the nearest neighbor forward and backward difference operators[2]

$$\partial^{\mu}_{x,y} = \delta_{x+\hat{\mu},y} - \delta_{x,y}, \qquad \partial^{\mu*}_{x,y} = \delta_{x,y} - \delta_{x-\hat{\mu},y} \,. \tag{1}$$

A simple guess for the action of a massless free fermion on the lattice might read as

$$\sum_{x,y} \overline{\psi}_x \frac{1}{2} \gamma^{\mu} \left(\partial^{\mu}_{x,y} + \partial^{\mu*}_{x,y} \right) \psi_y \,. \tag{2}$$

Switching on the gauge interaction a parallel transporter is included in the finite difference operators to assure gauge invariance

$$\nabla^{\mu}_{x,y} = U(x,\mu)\, \delta_{x+\hat{\mu},y} - \delta_{x,y} \,, \qquad \nabla^{\mu*}_{x,y} = \delta_{x,y} - U(x-\hat{\mu},\mu)^{\dagger}\, \delta_{x-\hat{\mu},y} \,, \tag{3}$$

where $U(x, \mu)$ is an element of the gauge group $SU(3)$. The gauge field $U(x, \mu)$ can be expressed in terms of the vector potential $A_{\mu}(x)$ used in the continuum formulation as

$$U(x,\mu) = P \exp[i \int_0^1 d\tau A^{\mu}(x + (1-\tau)\hat{\mu})] \,, \tag{4}$$

where P denotes τ-ordering and A_{μ} is an element of the Lie algebra. On the lattice, however, the group element $U(x, \mu)$ is the fundamental field variable which transforms under a gauge transformation as

$$\tilde{U}(x,\mu) = V(x) U(x,\mu) V^{\dagger}(x+\hat{\mu}) \,. \tag{5}$$

[2]Here and in most of the following expressions we suppress the lattice unit a. All the fields and parameters are made dimensionless by appropriate a powers: in the mass m a factor of a, in the fermion field ψ a factor of $a^{3/2}$, etc, is absorbed.

Eq. (5) implies that the trace of the product of gauge matrices along a closed path on the lattice is a gauge invariant quantity. Locality requires that the extension of the gauge action is $O(a)$. Taking the smallest loop, the plaquette, we get a discretized, local and gauge invariant action for QCD

$$S = -\frac{2}{g^2}\sum_p \mathrm{Re}\mathrm{Tr}\, U_p + \sum_{x,y} \overline{\psi}_x \, D_{x,y} \, \psi_y \, , \tag{6}$$

where the sum runs over the plaquettes, U_p is the product of four directed link matrices around the plaquette p and the Dirac operator has the form

$$D = \frac{1}{2}\gamma^\mu \left(\nabla^\mu + \nabla^{\mu *}\right) . \tag{7}$$

Indeed, taking the formal continuum limit $a \to 0$ of eq. (6) and using eqs. (3,4) we obtain the standard continuum form of the QCD action for a massless quark. Obviously, there are infinitely many local and gauge invariant lattice actions which have this property. The continuum predictions must not depend on this freedom, however. In an asymptotically free theory the scaling dimension of operators agrees with their engineering dimension up to logarithms (at least in perturbation theory) and it is easy to count the number of relevant operators. Perturbative renormalizability can be demonstrated on the lattice also [16].

In order to define the quantum theory we have to fix the integration measure of the path integral. Here we take the natural choice of the group invariant measure for every link variable $U(x,\mu)$ and integrate over the Grassmann variables $\overline{\psi}(x)$ and $\psi(x)$ according to the standard rules of Grassmann integration. This measure is also gauge invariant.

The action in eq. (6) has a problem, however. The Dirac operator in eq. (7) describes actually 16 massless fermion species - a problem called doubling. This is the first small surprise along a tortured way around chiral symmetry.

3 Doublers, chiral symmetry and a no-go theorem

The free fermion Dirac operator D in Fourier space has the form

$$D(p) = \sum_\mu \gamma^\mu \sin(p^\mu) \tag{8}$$

which has 16 zeros in the first Brillouin zone at $(0,0,0,0), \ldots, (\pi,\pi,\pi,\pi)$. It can be shown [17] that these species couple to the axial current with alternating sign and their total contribution to the $U(1)$ axial anomaly cancels [18, 19] (see also the contributions of Adler and Jackiw in this volume). This observation resolves a paradox: the theory is fully regularized, D anticommutes with a standardly defined γ_5, so we have a U(1) chiral symmetry without anomaly. Without the doublers this would be a contradiction.

The question remained whether there exists a clever lattice discretization such that the doublers are removed but (at least the non-singlet part of) chiral symmetry survives on the lattice. This question was answered by a no-go theorem [20] (see also ref. [17]). The no-go theorem states that for free fermions the following four conditions can not hold simultaneously:
1. $D(x)$ is local;
2. the Fourier transformed D behaves for $p \ll 1$ as $i\gamma^\mu p^\mu + O(p^2)$;
3. there are no doublers;
4. $\gamma_5 D + D\gamma_5 = 0$.

If we do not want to violate flavor symmetry for massless fermions then $D(x)$ is diagonal in the flavor indices and the statement is valid for each flavor separately. Locality is the most important property of a quantum field theory, so we can not give up this condition. The 2nd point just declares that we have a Dirac particle. If we do not want doublers (since there are no such copies in nature), there remains only the possibility to violate condition 4, i.e. to give up chiral symmetry on the lattice.

Accepting chiral symmetry breaking terms in the action there are different possibilities to follow. Wilson [21] suggested to add a dimension 5 operator to the action in eq. (6) which modifies the Dirac operator as

$$D_{\mathrm{W}} = \frac{1}{2}\left[\gamma^\mu(\nabla^\mu + \nabla^{\mu *}) - \nabla^\mu \nabla^{\mu *}\right] . \tag{9}$$

The new term (Wilson term) gives large $\propto$ cutoff masses to the doublers, but - at least classically - leaves the $p = 0$ pole unchanged. The Wilson term has an effect on the renormalization of bare parameters (in particular, for the bare fermion mass $m_q = 0$ is not protected), creates mixings between operators in different chiral multiplets and it also influences the way and speed the continuum is approached. The final physical predictions of the continuum theory are, however, independent of this term.

One might keep a part of the chiral symmetry by giving up flavor symmetry on the lattice [8, 22]. The Kogut-Susskind (or 'staggered') fermions describe 4 flavors and keep a (non-singlet) U(1) part of the original $U(4) \times U(4)$ symmetry. This formulation has many attractive properties, in particular for numerical simulations [23]. On the other hand, getting QCD with 3 light flavors is non-trivial with staggered fermions. I shall not follow this interesting possibility further in this contribution since staggered fermions are not directly related to the developments I want to discuss.

4 The Ginsparg-Wilson relation

The no-go theorem seemed to close all the ways towards a chiral symmetric lattice regularization. As the authors formulated [20]: 'The important consequence of our work is to *discourage any attempt to construct chiral invariant lattice models for QCD*'. No-go theorems, however, are frequently circumvented in an unexpected way. This is what happened with chiral symmetry on the lattice.

Soon after the no-go theorem was presented Ginsparg and Wilson [24] suggested a condition for the Dirac operator which, as the authors formulated, implies a 'remnant' chiral symmetry on the lattice. More than 15 years later this condition, called the Ginsparg-Wilson relation, became very important in the theoretical developments concerning global and local chiral symmetry on the lattice and beyond.

Ginsparg and Wilson treated free fermions and used some basic notions of Wilson's renormalization group theory. Consider a renormalization group transformation which blocks the fermion field ψ_x living on the fine lattice into the χ_{x_B} fermion field on the coarse lattice:

$$\begin{aligned}\exp(-S_B(\overline{\chi},\chi)) = \int D\overline{\psi}D\psi\, \exp[\,-S(\overline{\psi},\psi) \\ - \sum_{x_B,y_B}(\overline{\chi}_{x_B} - \sum_x \overline{\psi}_x\, \omega^\dagger_{x,x_B})\, R^{-1}_{x_B,y_B}(\chi_{y_B} - \sum_y \omega_{y_B,y}\,\psi_y)\,].\end{aligned} \tag{10}$$

Here S and S_B are the original and blocked actions, respectively, ω_{x,x_B} describes the averaging process in a local neighborhood of the coarse lattice point x_B, while R is an arbitrary local operator whose inverse is also local and diagonal in Dirac space[3]. Take the fine lattice infinitely fine, so the field ψ lives in the continuum and S is a chiral invariant action in the continuum. Since the block transformation explicitly breaks chiral symmetry (it has a $\overline{\chi}\ldots\chi$ structure), so does the action S_B on the lattice. The renormalization group transformation, however, does not change the long distance (in lattice units) behavior: S_B should remember that the starting action was chiral invariant. It is a few steps of algebra to dig out this information [24]. Writing

$$S_B = \sum_{x_B,y_B} \overline{\chi}_{x_B}\, D_{x_B,y_B}\, \chi_{y_B} \tag{11}$$

the Dirac operator is constrained by the relation

$$D\gamma^5 + \gamma^5 D = D\gamma^5 2RD\,. \tag{12}$$

[3]The simplest example is $R^{-1}_{x_B,y_B} = \frac{1}{2}\delta_{x_B,y_B}$

The physical implications of the Ginsparg-Wilson relation in eq. (12) which was obtained for free fermions go far beyond of what the derivation above might suggest. Assume there exists a local, gauge covariant solution of this equation in lattice QCD. The first trivial observation is that the inverse of D satisfies

$$D^{-1}\gamma^5 + \gamma^5 D^{-1} = \gamma^5 2R\,. \tag{13}$$

The inverse of D, the quark propagator, from which the physical correlators are constructed is a non-local quantity. The equation above shows, however, that its anticommutator with γ^5 is local, i.e. has an extension of $O(a)$ only. The violation of chiral symmetry is a contact term. It has been shown in ref. [24] also that a Dirac operator satisfying eq. (12) reproduces the $U(1)$ chiral anomaly.

The following simple consideration strongly suggests that lattice QCD with a Dirac operator satisfying eq. (12) gives chiral invariant answers for all the physical questions. Consider an arbitrary hadron correlator where the operators are separated by physical distances. Write down the corresponding Ward identity using the standard method in path integral formulation: introduce new Grassmann integration variables which are related to the original ones by an infinitesimal γ^5 chiral rotation. The action is not invariant and, according to eq. (12), gives an insertion $\propto \overline{\chi}D\gamma^5 2RD\chi$ to the correlation function. The fermion field χ ($\overline{\chi}$) will be paired with one of the antifermion (fermion) fields of the hadron operators. The two propagators which bind the breaking term to the hadron operators are canceled by the two D operators in the breaking term. What remains is $\gamma^5 2R_{x,y}$, where x and y are the positions of the hadron operators. Since R is local and the hadron operators are separated at physical distances, the contribution from the breaking term is zero.

Eq. (12) is a highly non-trivial equation. The chance of finding a local solution in the interactive case was rather dim. The paper and its results were hardly noticed and got completely forgotten[4].

5 Improvement programs

On a lattice with finite (in physical units) lattice spacing a the predictions systematically deviate from the final $a \to 0$ continuum values. To increase the resolution (decrease the cutoff effects) by a factor of two in a given physical volume requires at least 16 times more computing work and memory. Since the final continuum limit is universal but the cutoff effects are not, the search for lattice actions with reduced cutoff effects is an important issue since the beginnings. Although this question was raised from the numerical side, the

[4] According to SLAC Spires the paper was not cited at all over twelve years between 1986 and 1997 September when the first solution of the GW equation in the presence of gauge fields was identified.

theory behind the improvement programs reveal interesting aspects of quantum field theories and it is also part of the developments around chiral symmetry.

5.1 Symanzik improvement

To illustrate the basic idea consider a free massless scalar field with a simple nearest neighbor action on the lattice

$$S_0 = \sum_x \frac{1}{2}\partial_\mu\phi(x)\partial_\mu\phi(x)\,. \tag{14}$$

The pole of the propagator in Fourier space defines the energy as the function of the momentum. For small momenta the energy has the form

$$E^2 = \mathbf{p}^2 - \frac{a^2}{12}\left((\mathbf{p}^2)^2 + \sum_{i=1}^{3} p_i^4\right) + O(a^4)\,, \tag{15}$$

where for better visual understanding the lattice unit a is written out explicitly. The leading cutoff effect is $O(a^2)$. It is easy to see that by adding a dimension 6 ('irrelevant') term cS_1 to S_0, where

$$S_1 = \sum_{x,\mu} \frac{1}{2}\partial_\mu\partial_\mu\phi(x)\partial_\mu\partial_\mu\phi(x) \tag{16}$$

and $c_1 = 1/12$ the $O(a^2)$ cutoff effect cancels. The action $S = S_0 + c_1 S_1$ is an $O(a^2)$ improved lattice action which approaches continuum limit significantly faster than the original action S_0.

In the interacting case the situation is more involved. As it is well known the coefficients of the higher dimensional operators enter the relation between the renormalized and the bare parameters. This relation contains diverging powers and logs of the cutoff. Expressed in terms of the renormalized parameters, however, the continuum predictions show no sign of the presence of higher dimensional operators (universality). They have a role, however, in the improvement.

In spite of the technical complications, the final statement on improvement is bold and simple: by adding to the original interactive action a linear combination of the dimension 6 operators with properly chosen coefficients, the $O(a^2)$ cutoff effects can be eliminated in all physical quantities to all orders of perturbation theory. This has been shown by Symanzik [25] for the ϕ^4 scalar model and in the asymptotically free $d = 2$ non-linear σ-model. This is a highly non-trivial result which has been obtained with the help of Callan-Symanzik type of renormalization group equations and local effective lagrangians.

The improvement technique can be extended to gauge theories and QCD [26]. During the last two decades large effort was invested to calculate the improvement coefficients not only for the action but for currents and other densities also. These coefficients can be calculated in perturbation theory, or by simulations non-perturbatively. The Symanzik improvement has an important role in controlling the cutoff effects in different applications [27].

In QCD, if chiral symmetry is broken on the lattice, the leading cutoff effects are $O(a)$ which makes improvement even more important. $O(a)$ improvement can be achieved by adding a lattice version of the dimension 5 operator $\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$ with a tuned coefficient [26]. This term breaks chiral symmetry and has no place in a chiral symmetric formulation. A chiral symmetric action is automatically $O(a)$ improved making this possibility even more attractive.

The $U(1)$ (non-singlet) chiral symmetry preserved by the 4-flavor staggered fermions assures $O(a)$ improvement. Nevertheless, as painfully realized, this action has large cutoff effects on lattices which are typically simulated today [28]. Although the problem could be alleviated by canceling $O(a^2)$ flavor symmetry breaking effects [29], the message remains. Close to the continuum limit $O(a)$ ($O(a^2)$) cutoff effects dominate in bosonic (fermionic) theories. Whether in today's simulations this is the case depends on the theory, the space-time dimension, the form of the leading action and on the form of the higher dimensional operators chosen. Accidentally, just the $d = 2$ asymptotically free $O(3)$ non-linear σ-model, which was used by Symanzik to illustrate how to extend his method to theories with constraints, defies compliance with the Symanzik program in numerical simulations [30].

5.2 Classically perfect actions

There exist lattice actions which have no cutoff effects whatsoever in the classical theory. Take the $d = 2$ asymptotically free $O(3)$ non-linear σ-model mentioned above. One can put this theory on a quadratic lattice with a local, classically perfect action. The corresponding Euler-Lagrange equations have exact scale invariant instanton solutions, they carry a topological charge Q which is an integer, on any configuration the value of the action $S \geq 4\pi|Q|$ and $S = 4\pi|Q|$ on solutions. If the interaction is switched off the particles have the exact continuum dispersion relation $E = |\mathbf{p}|$. The existence of such an action follows from Wilson's renormalization group (RG) theory: the classically perfect action is the fixed point (FP) of a RG transformation and is determined by *classical* saddle point equations [31].

The same is true for QCD in $d = 4$ [32, 33]. Write the QCD action in the form $\beta S_g + S_f$, where S_g and S_f denote the gauge and fermion parts, $\beta \propto 1/g^2$

and consider the RG transformation in QCD

$$\exp\left[-\left(\beta' S'_g(V) + S'_f(\bar{\chi},\chi,V)\right)\right] =$$
$$\int D\bar{\psi} D\psi DU \exp\left[-\beta\left(S_g(U)+T_g(V,U)\right)+\left(S_f(\bar{\psi},\psi,U)+T_f(\bar{\chi},\chi;\bar{\psi},\psi,U)\right)\right] . \tag{17}$$

Here $T_g(V,U)$ defines the gauge invariant averaging process from the fine U to the coarse field V. The averaging function for the fermions $T_f(\bar{\chi},\chi;\bar{\psi},\psi,U)$ can be chosen in the form like in eq. (10) after introducing parallel transporters when adding fermion fields in different lattice points in the averaging: $\omega \to \omega(U)$ and similarly $R^{-1} \to R^{-1}(U)$. There is a considerable freedom here, the details are not important. In an asymptotically free theory the FP is at $\beta \to \infty$ where the path integral above is reduced to classical saddle point equations. For the gauge action this reads as

$$S_g^{\mathrm{FP}}(V) = \min_{\{U\}}\left[S_g^{\mathrm{FP}}(U) + T_g(V,U)\right] , \tag{18}$$

while for the fixed point Dirac operator in S_f^{FP} one obtains

$$D^{\mathrm{FP}}(V)^{-1}_{x_B,y_B} = R(V)_{x_B,y_B} + \sum_{n,n'} \omega(U)_{x_B,x} D^{\mathrm{FP}}(U)^{-1}_{x,y} \omega(U)^{\dagger}_{y,y_B} . \tag{19}$$

Using the FP equation eq. (18) for the gauge action S_g^{FP} one can derive results similar to that discussed above for the $O(3)$ σ-model on classical solutions, scale invariant instantons, etc. at finite lattice resolution. The FP Dirac operator from eq. (19) has exact dispersion relation in the free field limit and gives the correct continuum magnetic moment (g-factor) independently of the lattice resolution.

The action S_g^{FP} is defined if for any configuration V the value $S_g^{\mathrm{FP}}(V)$ (a real number) can be calculated. Take a configuration V on an L^4 lattice and assume that the blocking is from $L' = 2^k L$ to L. Considering any valid (gauge invariant, local) gauge action on the r.h.s. of eq. (18) the minimum gives the value of $S_g^{\mathrm{FP}}(V)$ with an error which goes to zero as $\propto 4^{-k}$. If we insert an approximate solution for the FP gauge action on the r.h.s., the error is reduced further. The convergence of the solution of the FP Dirac operator in eq. (19) is $\propto 2^{-k}$.

Considering the action $\beta S_g^{\mathrm{FP}} + \overline{\psi} D^{\mathrm{FP}} \psi$ for *finite* β we are off the renormalized trajectory (where the action would be quantum perfect). It is expected nevertheless that a classically perfect action will perform well in quantum simulations also. This is demonstrated in different $d = 2$ and 4 models including $SU(3)$ Yang-Mills theory and in (quenched) QCD [34].

In a numerical simulation, where we need the value the gauge action and the $D^{\mathrm{FP}} v$ matrix-vector product frequently, there is no way to calculate them using the FP equations above. One can, however, parametrize the solutions in terms of gauge loops for S_g and a certain number of fermion offsets and paths for D, fit the coefficients to eqs. (18,19) and obtain an approximate solution.

This improvement program is less systematic than Symanzik improvement. Very close to the continuum limit the Symanzik program with non-perturbatively determined coefficients is the best procedure. On today's lattices, however, the approximate FP actions in many cases perform better than Symanzik improved actions [34]. They are, however, also more expensive.

As I shall discuss, the FP Dirac operator played a relevant role in the quest for a chiral symmetric lattice regularizations also.

6 Domain-wall and overlap fermions

In 1992 Kaplan [35] suggested a new approach for lattice fermions allowing them to move in a five-dimensional space. Setting up a four-dimensional domain-wall some light modes become localized to the wall, the other, non-localized modes remain heavy on the cutoff level. The idea is related to earlier considerations in the continuum [36].

Consider a $d=5$ free Dirac operator in the continuum with an x_5 dependent mass term

$$D = \gamma^\mu \partial^\mu + \gamma^5 \partial^5 - M(x_5)\,, \tag{20}$$

where $M(x_5 \to \pm\infty) = \pm m_0$ and forms a kink between these values at $x_5 = 0$. The Dirac equation has a massless ($ip_4 = |\mathbf{p}|, \mathbf{p} = (p_1, p_2, p_3)$) left handed solution which is localised to the $d=4$ kink at $x_5 = 0$. All the other modes live on the cutoff level m_0. Kaplan realized that this construction can be taken over to the lattice and the mechanism works in the presence of gauge fields also. Actually, the five-dimensional model is not a genuine gauge-fermion theory: the gauge fields know nothing about the fifth direction. It might be better to consider this extra dimension as an internal 'flavor' space and the associated new fermion fields as regulators which are there to keep chiral symmetry [37]. In this approach, which is related to other descriptions [38] also, the effect of the massless chiral fermion can be represented as an overlap of two fermionic states [37].

The domain-wall approach for vector theories like QCD has been streamlined by Shamir [39]. The domain-wall was replaced by Dirichlet boundary conditions. The left and right handed components of the Dirac fermion are separated with one chirality bound exponentially on one wall, and the other on the opposite wall. Using the notations in eqs. (1,3) the five-dimensional Dirac operator has the form

$$D_{\rm DW} = \frac{1}{2}\left[\gamma^5(\partial^5 + \partial^{*5}) - \partial^5\partial^{*5}\right] + D_{\rm W} - m_0\,, \tag{21}$$

where $D_{\rm W}$ is the four-dimensional Dirac operator as defined in eq. (9), and $0 < m_0 < 2$.[5] The Dirichlet boundary conditions set up in $x_5 = 0$ and $x_5 = N$ read

$$P_R\chi(x_1, x_2, x_3, x_4, 0) = 0\,, \qquad P_L\chi(x_1, x_2, x_3, x_4, N) = 0\,, \tag{22}$$

where $P_{R/L} = (1 \pm \gamma^5)/2$ and χ is the five-dimensional fermion field. In the limit $N \to \infty$ the resulting effective four-dimensional theory is chiral symmetric on the lattice. I postpone the discussion on this most important point. Let me mention that, at the time of its discovery, it was not an issue whether the domain-wall idea had anything to do with the Ginsparg-Wilson relation.

This was a real breakthrough. Although $D_{\rm DW}$ has an extra dimension (or 'flavor' space), its structure is quite similar to that of the four-dimensional Wilson action which raised the hope for a relatively simple generalization of the numerical procedures. The computational cost is increased by a factor $O(N)$, where N should be large, but in problems where chiral symmetry plays an essential role there might be no other way to proceed. Concerning the present status of applications with domain-wall fermions I refer to the recent paper in ref. [40], while for an application with overlap fermions see ref. [41].

7 The fixed-point Dirac operator in QCD solves the Ginsparg-Wilson equation

It was noticed in 1997 that the fixed point Dirac operator in QCD satisfies the Ginsparg-Wilson relation [33]. This conclusion follows directly from the saddle-point equation eq. (19). Considering the r.h.s. of eq. (19) on a very fine lattice the fermion propagates over a very fine gauge field configuration U. In this case the fermion propagator goes to its continuum limit in this classical equation and the only term which is not anticommuting with γ^5 is $R(V)_{x_B,y_B}$. This leads to the equation eq. (13) (or eq. (12)) with a local R. In addition to being classically perfect, the fixed point QCD action has this important quantum property. The locality of the fixed-point action follows from general arguments of Wilson renormalization group theory[6]

In retrospect it is difficult to understand why this trivial observation was not made earlier. The free fixed point Dirac operator was explicitly known for

[5]For $m_0 < 0$ there exist no massless modes, for $m_0 > 2$ doublers appear.

[6]Actually, one can optimize the averaging function ω in eq. (19) to make D not only local but decaying very fast [13].

different block transformations [42][7] Its physical content is just the statement that D_{FP} is local, the energy-momentum dispersion relation is exactly linear and the energy runs in $(0, \infty)$ (in the first Brillouin zone) like in a chiral symmetric continuum formulation.

It was also illustrated in [33] that the Ginsparg-Wilson relation implies Ward identities from which the standard physical conclusions of chiral symmetry follow.

8 Index theorem on the lattice

It is a simple exercise to show that the solutions of the Ginsparg-Wilson relation satisfy the index theorem on the lattice [43]: the zero modes of D are chiral and the associated index is a topological invariant which represents the topological charge on the lattice.

For notational simplicity consider the case $2R = 1$ in eq. (12) and assume the hermiticity property in Euclidean space $D^\dagger = \gamma^5 D \gamma^5$. It is then easy to show that the spectrum $\{\lambda\}$ of D lies on a unit circle around the point $z = 1$ in the complex plane with the following properties. The real modes $\lambda = 0$ and $\lambda = 2$ are chiral (i.e. the corresponding eigenvectors are also eigenvectors of γ^5 with ± 1 eigenvalues). If ϕ_λ is an eigenvector with a complex eigenvalue λ then $\phi_{\lambda^*} = \gamma^5 \phi_\lambda$ and they are orthogonal: $(\phi_\lambda, \gamma^5 \phi_\lambda) = 0$

Define the density

$$q(x) = \frac{1}{2} \mathrm{tr}(\gamma^5 D(x,x)) \,, \tag{23}$$

where the trace is taken in color and Dirac space. Consider now

$$\begin{aligned} \sum_x q(x) = \frac{1}{2} \mathrm{Tr}\big(\gamma^5 D\big) = -\frac{1}{2} \mathrm{Tr}\big[\gamma^5 (2 - D)\big] & \\ = -\frac{1}{2} \sum_\lambda (2 - \lambda)(\phi_\lambda, \gamma^5 \phi_\lambda) \,, & \end{aligned} \tag{24}$$

where Tr denotes trace in all the indices and in the first step we subtracted zero: $\mathrm{Tr}\gamma^5 = 0$. Using the orthogonality properties discussed above only the $\lambda = 0$ modes contribute giving

$$\sum_x q(x) = n_- - n_+ \,, \tag{25}$$

[7] An example was already presented in the Appendix of ref. [24] refering to the unpublished PhD thesis of M. Peskin. In 1982 the community was not yet ripe to appreciate the Ginsparg-Wilson paper and pieces of its content were independently rediscovered later.

where n_- (n_+) is the number of left (right) handed zero modes of the Dirac operator. The index of D is, therefore, a sum over x of the pseudoscalar density $q(x)$. It can be shown [44] that on smooth configurations $q(x)$ is the continuum topological charge density

$$q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}^c(F_{\mu\nu}F_{\rho\sigma}) + O(a^2)\,, \tag{26}$$

where the trace is taken in color space. In the case of the fixed point Dirac operator one can show in addition that $\sum_x q(x)$ is the fixed-point topological charge Q_{FP} which takes integer values on any gauge configuration, i.e. the topological charge from the gauge and from the fermion sector is always consistent on the lattice if the FP action is used.

9 Ginsparg-Wilson fermions from the domain-wall construction

The simplicity of the index theorem demonstrated the power of the Ginsparg-Wilson relation. It was obvious that this relation is relevant. In a short time a second solution, the overlap Dirac operator, was identified [45]. Unlike the fixed point Dirac operator this solution has a simple explicit structure which allows to represent it in a computer to machine precision. In addition, the overlap operator binds the Ginsparg-Wilson approach to the domain-wall idea.

I present here this connection in a form as it occured after several simplifying steps [46] and summarized in ref. [47]. As eq. (22) shows the $d = 4$ fermion fields can be naturally identified with

$$\psi_x = P_L\chi(x,1) + P_R\chi(x,N)\,, \qquad \overline{\psi}_x = \overline{\chi}(x,1)P_R + \overline{\chi}(x,N)P_L\,, \tag{27}$$

where $x \in Z_4$. The inverse of $\langle\psi_x\overline{\psi}_x\rangle$ defines an effective four-dimensional Dirac operator $D^N_{x,y}$. In the $N\to\infty$ limit $D^N \to D$ can be written as

$$aD = 1 - \mathcal{A}(\mathcal{A}^\dagger\mathcal{A})^{-\frac{1}{2}}\,, \tag{28}$$

where

$$\mathcal{A} = -a_5(D_{\mathrm{W}} - m_0)\left(1 - \frac{1}{2}a_5(D_{\mathrm{W}} - m_0)\right)^{-1} \tag{29}$$

and a and a_5 are the lattice units in the four-dimensional space and in the fifth direction, respectively. Taking the limit $a_5 \to 0$ one obtains the overlap operator

$$D_{\mathrm{ov}} = 1 - A(A^\dagger A)^{-\frac{1}{2}}\,, \tag{30}$$

where $A = -(D_{\mathrm{W}} - m_0)$. Both D_{ov} and the Dirac operator D in eq. (28) satisfy the Ginsparg-Wilson relation with $2R = 1$. This can be shown easily by observing that both operators have the form $1 - V$, where V is unitary.

A valid Dirac operator should be local, i.e. it should have an extension of $O(a)$ (inverse cutoff). It has been shown [48] that a solution of the Ginsparg-Wilson equation can not be ultralocal[8]. Ultralocality is, however, not necessary. Locality (and so, universality) requires that the couplings in the action decay with the distance faster than any physical signal. An exponential decay of the couplings like $\exp(-\kappa r/a)$, where κ is $O(1)$ defines a local operator. It has been shown [49] that the overlap Dirac operator satisfies this bound if the theory on the lattice is close to the continuum limit.

10 Exact chiral symmetry transformation

As I discussed before, one can raise arguments that the Ginsparg-Wilson relation implies the physical consequences of chiral symmetry. One can derive Ward identities also and demonstrate that they lead to the same physical predictions as those with exact chiral symmetry [50]. This approach is, however, cumbersome. In 1998 Lüscher has found an exact symmetry transformation which could be identified as the chiral symmetry transformation on the lattice [51]. This was a theoretical breakthrough. It gave an elegant technical tool to derive Ward identities, study anomalies and to handle the problem of chiral gauge theories. Chiral symmetry is realized on the lattice in an unusual, but beautiful way.

Consider a vector gauge theory with N_f flavors and take $2R = 1$ in eq. (12) for simplicity. If the Dirac operator D satisfies the Ginsparg-Wilson relation then the infinitesimal non-singlet transformation

$$\psi' = \psi + i\epsilon T\gamma^5\left(1 - \frac{1}{2}D\right)\psi\,, \qquad \overline{\psi}' = \overline{\psi} + i\epsilon\overline{\psi}\left(1 - \frac{1}{2}D\right)\gamma^5 T \tag{31}$$

and the singlet transformation

$$\psi' = \psi + i\epsilon\gamma^5\left(1 - \frac{1}{2}D\right)\psi\,, \qquad \overline{\psi}' = \overline{\psi} + i\epsilon\overline{\psi}\left(1 - \frac{1}{2}D\right)\gamma^5 \tag{32}$$

leave the fermion action $\overline{\psi}D\psi$ invariant. In eq. (31) T is an $SU(N_f)$ generator. It is trivial to demonstrate this statement. Less trivial is the way this symmetry transformation avoids the Nielsen-Ninomiya no-go theorem: the chiral transformation is not a simple γ^5 rotation as the no-go theorem assumed. The transformation depends on D and so it depends on the gauge field over which the fermions propagate. In the formal continuum limit the transformation goes over to the standard γ^5 rotation[9].

[8] An ultralocal Dirac operator has a finite number of fermionic offsets. The Wilson Dirac operator $D_{\rm W}$, for example, is ultralocal having 9 offsets.

[9] In eqs. (31,32) D is multiplied by the lattice unit a as it is obvious from dimensional analysis.

The action is invariant under the singlet transformation also, but the fermion integration measure is not [51]. A non-trivial Jacobi determinant is generated depending on the gauge field

$$\prod_x d\overline{\psi}'_x d\psi'_x = \big(1 + i\epsilon \mathrm{Tr}(\gamma^5 D)\big) \prod_x d\overline{\psi}_x d\psi_x \,. \tag{33}$$

For a non-singlet transformation the Jacobi determinant is 1 due to $\mathrm{tr}(T) = 0$. According to the index theorem on the lattice, a factor of $1 + i\epsilon 2N_f Q_{\mathrm{top}}$ is produced by the measure for a singlet transformation, where Q_{top} is the topological charge of the gauge configuration as defined by the index of the Dirac operator. Consider the transformation in eq. (32) as a change of variable in the fermionic path integral. The expectation value $\langle \mathcal{O} \rangle$ of an arbitrary operator remains unchanged which leads to the correct anomalous Ward identity on the lattice

$$\langle \delta\mathcal{O} \rangle + 2N_f \langle Q_{\mathrm{top}} \mathcal{O} \rangle = 0 \,, \tag{34}$$

where $\delta\mathcal{O} = \big[\mathcal{O}(\overline{\psi}', \psi') - \mathcal{O}(\overline{\psi}, \psi)\big]/i\epsilon$ is the variation of the operator $\mathcal{O}$ under the transformation.

11 Left and right handed fermions, the mass term and the θ parameter

In the formal continuum the massless fermion action of a vector theory falls into left handed and right handed parts

$$\overline{\psi} D \psi = \overline{\psi}_L D \psi_L + \overline{\psi}_R D \psi_R \,. \tag{35}$$

This is also possible on the lattice although in this case the projectors on the ψ field depend on the gauge configuration [52, 53, 13]:

$$\psi_L = \hat{P}_L \psi \,, \qquad \psi_R = \hat{P}_R \psi \,, \qquad \overline{\psi}_L = \overline{\psi} P_R \,, \qquad \overline{\psi}_R = \overline{\psi} P_L \,, \tag{36}$$

where

$$\hat{P}_{R/L} = \frac{1}{2}(1 \pm \hat{\gamma}_5) \,, \qquad P_{R/L} = \frac{1}{2}(1 \pm \gamma_5) \,. \tag{37}$$

Here $\hat{\gamma}_5$ depends on the gauge field through the Dirac operator

$$\hat{\gamma}_5 = \gamma_5 (1 - D) \tag{38}$$

and satisfies $(\hat{\gamma}_5)^2 = 1$ due to the Ginsparg-Wilson relation (I consider $2R = 1$ for simplicity). Using these definitions one obtains the decomposition in eq. (35) on the lattice.

The relations above are striking in many ways. The decomposition $\psi(x) = \psi_L(x) + \psi_R(x)$ depends on the gauge field in the neighborhood of x. Even for free fermions, the projectors depend on the Dirac operator. Beyond that, there is an asymmetry between the fermions and antifermions: the projectors on $\overline{\psi}$ are standard, they are identical to those in the continuum. This asymmetry is deeply related to the fermion number anomaly in chiral gauge theories [54].

The scalar and pseudo-scalar densities are introduced as usual

$$S = \overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L = \overline{\psi} \left(1 - \frac{1}{2} D \right) \psi \,, \tag{39}$$

$$P = \overline{\psi}_L \psi_R - \overline{\psi}_R \psi_L = \overline{\psi} \gamma_5 \left(1 - \frac{1}{2} D \right) \psi \,. \tag{40}$$

These densities behave correctly under chiral transformations. A mass term in the action of a vector theory like QCD can be introduced as

$$S = S_g(U) + \int dx \left[\overline{\psi} D \psi + \overline{\psi}_L m \psi_R + \overline{\psi}_R m^\dagger \psi_L - i\theta q(x) \right] \,, \tag{41}$$

where a CP breaking θ-term is added also and a sum over flavors is assumed. For a real mass m the massive Dirac operator can be written as $(1 - m/2)(D + m)$. The mass shifts the eigenvalue circle by m and modifies the radius in such a way that the point $\lambda = 2$, where the ultraviolet eigenvalues are concentrated remains fixed. Starting from eq. (41) the known Ward identities can be derived, but this time in a controlled environment. The Ward identities are valid at any finite value of the cutoff (and so also in the continuum limit) as far as the Dirac operator is a local solution of the Ginsparg-Wilson relation. Conserved vector, axial vector and chiral currents can be defined [55, 56]. It became possible to clarify theoretically opaque quantities like the topological susceptibility [58, 59] and to make progress on the related Witten-Veneziano relation [60]. The Ginsparg-Wilson relation has also been used to construct actions with improved chiral and scaling properties, like the chirally improved (CI) action[57].

12 Chiral gauge theories

Standard perturbatively defined regularizations have problems with chiral gauge theories. Of course, in principle one can give up gauge symmetry and introduce all the counter terms needed to get gauge invariant renormalized predictions in perturbation theory. This might not be the best way to proceed, not even practically. In addition, our ultimate goal is to understand these theories (like the standard model) beyond perturbation theory.

The decomposition eq. (35) is the first step towards a gauge invariant formulation of chiral gauge theories on the lattice. The projectors in eq. (36) are gauge field dependent, so the dividing surface between the left and right handed fermions in the space of fermion degrees of freedom is moving as the gauge configuration is changing. Even the number of fermions with a certain chirality, say left handed, is changing. Since the number of left handed antifermions is constant (their projectors are gauge field independent), the total fermion number is changing as we are flying over the gauge configuration space. Indeed, the difference between the fermion and antifermion degrees of freedom is

$$\mathrm{Tr}\hat{P}_L - \mathrm{Tr}P_R = \frac{1}{2}\mathrm{Tr}(\gamma_5 D)\,, \tag{42}$$

which, according to the index theorem discussed before, takes different values in the different topological sectors. On one hand, eq. (42) shows that fermion number violation occurs naturally in this formulation and this a welcome feature. On the other hand, having different degrees of freedom in the different topological sectors might imply difficulties to connect these sectors with each other in establishing the theory.

In the last few years important results were obtained in this field. Lüscher has demonstrated that a U(1) chiral gauge theory with fermions in anomaly free representation can be fully defined on the lattice so that gauge symmetry is exactly preserved [61]. The same could be established for a general compact group in every order of perturbation theory[62]. Beyond the $U(1)$ case it is not known whether and under which conditions these theories are free of non-perturbative obstructions (anomalies). Such obstructions might exist as has been shown in an $SU(2)$ chiral theory with a single left handed fermion in the fundamental representation [63]. This was also demonstrated on the lattice [64]. A different approach which is pursued on the lattice since a long time is to give up explicit gauge symmetry. I refer here to a recent paper [65] and to the references therein.

13 Closing remarks

The Metropolis algorithm also celebrated its fiftieth anniversary recently [66]. Nobody thought fifty years ago that Yang-Mills theory, this beautiful theoretical construction, will have so much to do with a stochastic algorithm as it is the case since almost three decades. These simulations delivered a lot of non-perturbative results on the glueball spectrum, static potential and the related fluctuating flux tube and thermodynamics. The results are quantitative and some of them are quite precise. The numerical results can be connected to analytic predictions in some corners of the theory (for a highly non-trivial example I refer to the contribution of van Baal in this volume).

QCD is, of course, more difficult and much more exciting. If the doubts concerning the validity of the running staggered fermion simulations [23] are positively clarified we shall certainly see a large number of interesting non-perturbative QCD predictions during the next few years. I think, this will happen in any case soon.

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