

# Some new results on an old controversy: is perturbation theory the correct asymptotic expansion in nonabelian models?

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Several years ago it was found that perturbation theory for two-dimensional  $O(N)$  models depends on boundary conditions even after the infinite volume limit has been taken termwise, provided  $N > 2$ . There ensued a discussion whether the boundary conditions introduced to show this phenomenon were somehow anomalous and there was a class of ‘reasonable’ boundary conditions not suffering from this ambiguity. Here we present the results of some computations that may be interpreted as giving some support for the correctness of perturbation theory with conventional boundary conditions; however the fundamental underlying question of the correctness of perturbation theory in these models and in particular the perturbative  $\beta$  function remain challenging problems of mathematical physics.

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## I. INTRODUCTION

Several years ago A. Patrascioiu and the second-named author [1] investigated the dependence of perturbation theory (PT) on boundary conditions (b.c.). Surprisingly they found that in two-dimensional nonlinear sigma-models with non-abelian symmetry the coefficients of the perturbative expansion depend on the boundary conditions (b.c.) used, even after the infinite volume limit is taken term by term. In particular it was found that the so-called superinstanton b.c. (s.i.b.c.) lead to results that differ from the standard ones (typically obtained using periodic b.c.). This is remarkable, because the full, nonperturbative thermodynamic limit should be independent of the b.c. used to approximate it.

A little later F. Niedermayer, M. Niedermaier and P. Weisz [2] on the one hand confirmed and sharpened the results of [1]: they gave an analytic expression for the second order term in the PT expansion of the energy and in addition showed that the third order term diverges in the thermodynamic limit, thereby in some sense exhibiting an even stronger dependence on the b.c. used. On the other hand they gave arguments why the s.i.b.c. are in some sense anomalous; using plausible but unproven correlation inequalities they argued that the thermodynamic limit of the invariant two-point function should be sandwiched between the finite volume two-point function with free (Neumann) and 0-Dirichlet b.c., whereas the s.i.b.c. two-point function lies outside that interval.

Assuming certain bounds on the remainder terms of the truncated PT expansion and also assuming the equality of the termwise thermodynamic limits with free and Dirichlet b.c. the authors of [2] conclude that with those b.c. one obtains indeed a correct asymptotic expansion

of the infinite volume two-point functions by taking the infinite volume limit term by term.

Patrascioiu and the second named author [3] wrote a comment on the paper [2] in which they stressed that the arguments put forward in that paper did not suffice to settle the issue of the correctness of conventional PT, since several unproven assumptions were made.

The present paper answers at least one of the question left over, by demonstrating that at least one of the assumptions made in [2] is indeed correct.

## II. THE PROBLEM

We are dealing with the  $O(N)$  vector models in  $2D$ ; these models describe ‘spins’ (unit vectors in  $\mathbb{R}^N$ ) assigned to the vertices of a (finite) lattice  $\Lambda \subset \mathbb{Z}^2$  and distributed with a Gibbs measure  $\exp(-S_\Lambda)d\mu_\Lambda$  defined in terms of the action

$$S_\Lambda \equiv \beta \sum_{\langle xy \rangle} (1 - \vec{s}_x \cdot \vec{s}_y) \quad (1)$$

and the  $O(N)$  invariant a priori measure  $d\mu_\Lambda$ . One is of course interested in the thermodynamic limit  $\Lambda \rightarrow \mathbb{Z}^2$ , in practice taken through square lattices of size  $L$ .

In a nutshell the problem is the following: consider the expectation value  $\langle \mathcal{O} \rangle_{L,bc}$  of an observable  $\mathcal{O}$  in a finite box of size  $L$  with boundary conditions  $bc$ . Then PT can be derived using standard theorems on asymptotic expansions of integrals and yields

$$|\langle \mathcal{O} \rangle_{L,bc} - \sum_{i=0}^k c_{i,bc}(L)\beta^{-i}| = R_k(\beta, L) \quad (2)$$

with

$$\lim_{\beta \rightarrow \infty} R_k(\beta, L)\beta^k = 0. \quad (3)$$

The problem is what happens to this asymptotic expansion in the thermodynamic limit  $L \rightarrow \infty$ . It may happen, and has been shown in the case of the  $O(N)$  models for  $O(N)$  invariant observables and periodic b.c. [4, 5], that

$$\lim_{\beta \rightarrow \infty} c_i(L) \equiv c_i(\infty) \quad (4)$$

exists for all  $i$ , but the unsolved problem is whether a  $R_k(\beta, \infty)$  with  $\lim_{\beta \rightarrow \infty} R_k(\beta, \infty)\beta^k = 0$  exists, such that the analogue of (2) holds with  $L$  replaced by  $\infty$ . A proof of this would require some uniform control of the  $R_k(\beta, L)$ , which so far has not been achieved, except for  $N = 2$  [6].

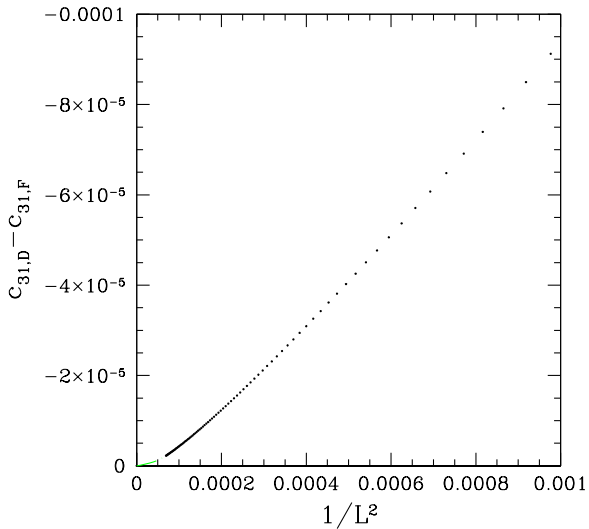


FIG. 1: Coefficient  $c_{31}$ : difference between free and Dirichlet b.c.

### III. NEW RESULTS

The idea of [2] is to sandwich infinite volume expectations between the finite volume ones with Dirichlet and free b.c.:

$$\langle \mathcal{O} \rangle_{L,F} \leq \langle \mathcal{O} \rangle_{\infty} \leq \langle \mathcal{O} \rangle_{L,D}, \quad (5)$$

for  $\mathcal{O}$  in a certain class containing invariant polynomials in the spins with positive coefficients (this includes in particular the invariant two-point functions). For  $N = 2$  (5) is a consequence of Ginibre's inequalities [7], whereas for  $N > 2$  it remains a plausible, but unproven conjecture.

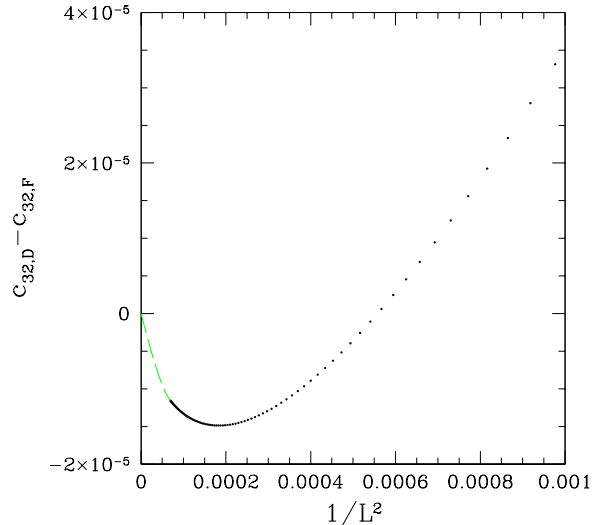


FIG. 2: Coefficient  $c_{32}$ : difference between free and Dirichlet b.c.

The arguments of [2] rely on two conjectures:

(i) equality of the thermodynamic limit of the PT coefficients, taken with free and Dirichlet b.c.:

$$\lim_{L \rightarrow \infty} c_{i,F}(L) = \lim_{L \rightarrow \infty} c_{i,D}(L) \quad (6)$$

(ii) a bound on the remainders  $R_k$  in a region in the  $(\beta, L)$  plane that can be considered as perturbative (for instance  $L < \exp(\ln^2 \beta)$ ). We have nothing to contribute to (ii), but some new computations confirming (i) beyond what was stated in [2]. All our computations, just as those of [1, 2] refer to a special observable, namely the nearest neighbor spin-spin scalar product located in the center of the lattice, i.e.  $\mathcal{O} = \vec{s}(0,0) \cdot \vec{s}(0,1)$ .

In [2] the authors state that for this observable they checked (6) for  $i = 2$ . We confirmed this, and found that the difference  $c_{i,D}(L) - c_{i,F}(L)$  goes to zero like  $L^{-2}$ .

The main new result we wish to report here is the computation of the third order coefficients  $c_3$ . For general  $N$  we can write

$$c_{3,bc} = (N-1)c_{31,bc} + (N-1)^2 c_{32,bc} + (N-1)^3 c_{33,bc}. \quad (7)$$

We evaluated the coefficients  $c_{31,bc}$ ,  $c_{32,bc}$  and  $c_{33,bc}$  from  $L = 4$  up to  $L = 120$ , where  $bc$  stands with 'bc' standing either for 'F' (free) or 'D' (Dirichlet). The differences  $c_{3k,D} - c_{3k,F}$  ( $k = 1, 2, 3$ ) are shown in the three figures below. In the figures the dots are the actual values, whereas the dashed lines are a spline through the points forced to hit the origin. The data strongly suggest that these differences go to zero like  $O(L^{-2})$  for  $L \rightarrow \infty$ , even though the nonmonotonicity observed in  $c_{32}$  is a little surprising.

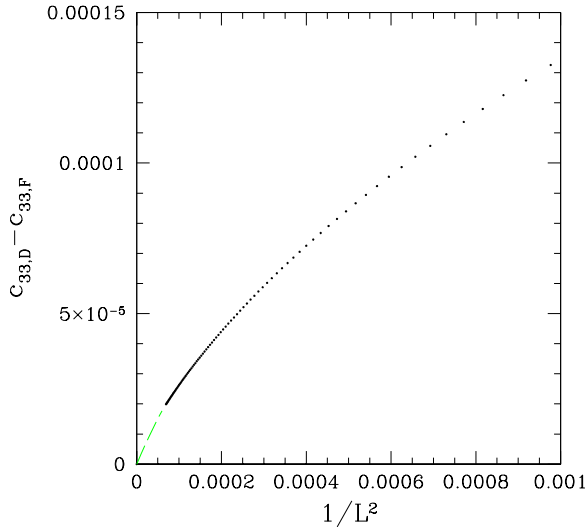


FIG. 3: Coefficient  $c_{33}$ : difference between free and Dirichlet b.c.

In this brief report we cannot describe in detail how the

PT coefficients presented here were obtained; in fact the computations are quite complex, involving many lattice Feynman graphs that have to be evaluated with complicated non-translation invariant propagators. The details of these computations, as well as a computation of the third order PT coefficients with s.i.b.c., confirming their logarithmic divergence asserted by [2] can be found in [8].

#### IV. DISCUSSION

By pushing the perturbative calculation of the nearest neighbor two-point function for the  $O(N)$  model to order  $\beta^{-3}$  we have provided some support for the arguments of [2] in favor of the conventional view that the termwise thermodynamic limit of PT provides the correct asymptotic expansion in the thermodynamic limit. It should, however, be stressed that the main problem, namely control over the remainder terms  $R_k$  remains open, and so does the important question of the status of PT in perturbatively asymptotically free theories.

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- [1] A. Patrascioiu and E. Seiler, *Phys. Rev. Lett.* **74** (1995) 1920.
  - [2] F. Niedermayer, M. Niedermaier and P. Weisz, *Phys. Rev.* **D 56** (1996) 2555.
  - [3] A. Patrascioiu and E. Seiler, *Phys. Rev.* **D57** (1996) 1394.
  - [4] S. Elitzur, *Nucl.Phys.* B212 (1983) 501.
  - [5] F. David, *Phys. Lett.* **B 96** (1980) 371-374.
  - [6] J.Bricmont, J.-R.Fontaine, J.L.Lebowitz, E.H.Lieb and T.Spencer, *Commun.Math.Phys.* **78** (1981) 545.
  - [7] J.Ginibre *Commun.Math.Phys.* **16** (1970) 310.
  - [8] M. Aguado, Technical notes on a 2-d lattice  $O(N)$  model problem, MPP-2004-118, hep-lat/0409155.