

Density matrix renormalization group approach to a two-dimensional bosonic model

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Density matrix renormalization group (DMRG) is applied to a (1+1)-dimensional $\lambda\phi^4$ model to study spontaneous breakdown of discrete Z_2 symmetry numerically. We obtain the critical coupling $(\lambda/\mu^2)_c = 59.89 \pm 0.01$ and the critical exponent $\beta = 0.1264 \pm 0.0073$, which are consistent with the Monte Carlo and the exact results, respectively. The results are based on extrapolation to the continuum limit with lattice sizes $L = 250, 500$, and 1000. We show that the lattice size $L = 500$ is sufficiently close to the the limit $L \rightarrow \infty$ [1].

1. Introduction

Hamiltonian diagonalization is a useful method for nonperturbative analysis of many-body quantum systems. If Hamiltonian is diagonalized, the system can be analyzed nonperturbatively at the amplitude level [2,3,4]. However, in general quantum field theories, the method does not work without reducing degrees of freedom because the dimension of Hamiltonian increases exponentially as the system size becomes large. We need to find a way to create a small number of optimum basis states [5,6]. S. White proposed a powerful method called density matrix renormalization group (DMRG) [7,8]. In DMRG, calculation accuracy of target states can be controlled systematically using density matrices. White calculated very accurately energy and wavefunctions of Heisenberg chains composed of more than 100 sites using a standard workstation. The calculation reproduced the exact value of ground-state energy in five digits or higher. DMRG has been applied to various one-dimensional models, such as Kondo, Hubbard, and t - J chain models, and achieved great success. In many cases, DMRG can give more accurate results than quantum Monte Carlo. A two-dimensional Hubbard model has also been studied with DMRG in both real- and momentum-space representation [9,10]. DMRG works well on small two-dimensional lattices and new techniques have been proposed for

solving larger lattices [11]. DMRG has also been extended to finite-temperature chain models using the transfer-matrix technique [12]. In particle physics, the massive Schwinger model has been studied using DMRG to confirm the well-known Coleman's picture of 'half-asymptotic' particles at a background field $\theta = \pi$ [13]. It would be interesting to seek a possibility of applying the method to QCD in order to study color confinement and spontaneous chiral symmetry breaking based on QCD vacuum wavefunctions.

DMRG was originally proposed as a method for spin and fermion chain models. In fermionic lattice models, the number of particles contained in each site is limited because of the Pauli principle. On the other hand, in bosonic lattice models, each site can contain infinite number of particles in principle. It is not evident whether Hilbert space can be described appropriately with a finite set of basis states in bosonic models. This point becomes crucial when DMRG is applied to gauge theories because gauge particles are bosons. Before working in lattice gauge theories like Kogut-Susskind Hamiltonian [14], we need to test DMRG in a simple bosonic model and recognize how many basis states are necessary for each site to reproduce accurate results. In this work, we apply DMRG to a $\lambda\phi^4$ model with (1+1) space-time dimensions. We construct a Hamiltonian model on a spatial lattice. The model has spontaneous breakdown of discrete Z_2

symmetry and the exact values of the critical exponents are known. We are going to justify the relevance of DMRG truncation of Hilbert space in the bosonic model by comparing our numerical results with the Monte Carlo and the exact results [15,16]. Our calculations are done with lattice sizes $L = 250, 500$ and 1000 . $L = 1000$ is about twice of the latest Monte Carlo one [15].

2. DMRG in the $\lambda\phi_{1+1}^4$ model

In the $\lambda\phi_{1+1}^4$ model, we divide a Hamiltonian

$$\tilde{H} = \sum_{n=1}^L h_n + \sum_{n=1}^{L-1} h_{n,n+1}, \quad (1)$$

into two parts

$$\begin{aligned} h_n &= \frac{1}{2}\pi_n^2 + \frac{\tilde{\mu}_0^2}{2}\phi_n^2 + \frac{\tilde{\lambda}}{4!}\phi_n^4, \\ h_{n,n+1} &= \frac{1}{2}(\phi_n - \phi_{n+1})^2. \end{aligned}$$

The field operator $\pi_n \equiv a\dot{\phi}_n$ is conjugate to ϕ_n , $[\phi_m(t), \pi_n(t)] = i\delta_{mn}$. The derivative has been replaced with a naive difference. (Errors associated with the difference can be discussed if necessary [17].) We rewrite Hamiltonian (1) using creation and annihilation operators a_n^\dagger and a_n .

$$\phi_n = \frac{1}{\sqrt{2}}(a_n^\dagger + a_n), \quad \pi_n = \frac{i}{\sqrt{2}}(a_n^\dagger - a_n), \quad (2)$$

where $[a_m, a_n^\dagger] = \delta_{mn}$ and $a_n|0\rangle = 0$. Note that a_n^\dagger and a_n are not creation and annihilation operators in Fock representation. The index n of the operators a_n^\dagger and a_n stands for the discretized spatial coordinate, not momentum. Real-space representation is better for our purpose because local interactions are useful for DMRG.

The finite system algorithm of DMRG is applied to the Hamiltonian. A superblock Hamiltonian H_S is composed of two blocks and one site:

$$H_S = \bar{H}_L + h_{n-1,n} + h_n + h_{n,n+1} + \bar{H}_R, \quad (3)$$

where \bar{H}_L and \bar{H}_R are effective Hamiltonian for the left and right blocks, respectively. $h_{n-1,n}$ ($h_{n,n+1}$) is an interaction between the left (right) block and the inserted n -th bare site. The target

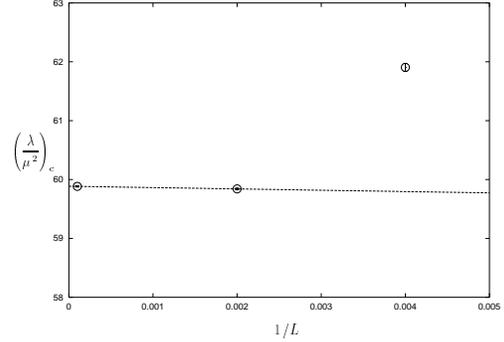


Figure 1. $(\lambda/\mu^2)_c$ is plotted as a function of $1/L$ for $L = 250, 500$, and 1000 . Extrapolation to the limit $L \rightarrow \infty$ gives $(\lambda/\mu^2)_c = 59.89 \pm 0.01$.

state is expanded as

$$|\Psi\rangle = \sum_{i=1}^M \sum_{j=1}^N \sum_{k=1}^M \Psi_{ijk} |i, j, k\rangle_n. \quad (4)$$

where the index j is for the inserted bare site and i and k are for the renormalized blocks. The relevance of truncation with M and N is checked numerically by seeing convergence of energy and wavefunction. Parameter values $(M, N) = (10, 10)$, which give good convergence, are used in all calculations. In Fig. 1 and 2, we extrapolate the results to the continuum limit and obtain the critical coupling $(\lambda/\mu^2)_c = 59.89 \pm 0.01$ and the critical exponent $\beta = 0.1264 \pm 0.0073$, which are consistent with the Monte Carlo $(\lambda/\mu^2)_c = 61.56_{-0.24}^{+0.48}$ and the exact $\beta = 0.125$ results, respectively.

3. Conclusion

We have determined the critical coupling constant $(\lambda/\mu^2)_c$ and the critical exponent β of the model by extrapolating the numerical results for finite but sufficiently large lattices to the continuum limit. DMRG truncation works well also in the bosonic model. The lattice with $L = 500$ can give results sufficiently close to the limit $L \rightarrow \infty$.

The numerical calculations were carried on RIKEN VPP and Yukawa Institute SX5.

Table 1
Various results for the critical coupling constant $(\lambda/\mu^2)_c$ are listed.

Method	Result	Reference
DMRG	59.89 ± 0.01	This work
Monte Carlo	$61.56^{+0.48}_{-0.24}$	[15]
Gaussian effective potential	61.266	[18]
Gaussian effective potential	61.632	[22]
Connected Green function	58.704	[22]
Coupled cluster expansion	$22.8 < (\lambda/\mu^2)_c < 51.6$	[19]
Non-Gaussian variational	41.28	[21]
Discretized light cone	43.896, 33.000	[20]
Discretized light cone	42.948, 46.26	[3]

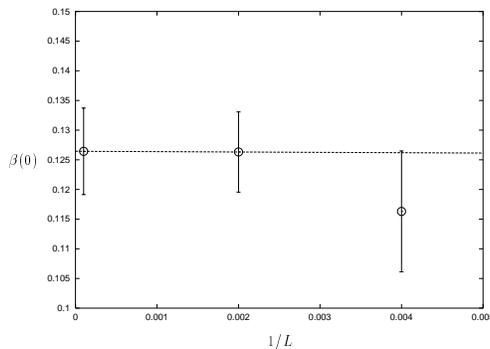


Figure 2. β is plotted as a function of $1/L$ for $L = 250, 500$, and 1000 . Extrapolation to the limit $L \rightarrow \infty$ gives $\beta = 0.1264 \pm 0.0073$.

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